

WALL-CROSSING FOR QUASIMAPS TO GIT STACK BUNDLES

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ABSTRACT. We define the notion of ϵ -stable quasimaps to a GIT stack bundle, and study the wall-crossing behavior of the resulting ϵ -quasimap theory as ϵ varies.

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0. Introduction

We work over \mathbb{C} .

This paper considers GIT stack bundles over a nonsingular projective variety Y . Roughly speaking, a GIT stack bundle over Y is a fiber bundle over Y ,

$$[E^{\text{ss}}/G] \rightarrow Y,$$

whose fibers are isomorphic to a given GIT stack quotient $[V^{\text{ss}}(\theta)/G]$. A precise definition of GIT stack bundles is given in equation (1.3) below. The notion of GIT stack bundles generalizes the notion of GIT bundles, which was previously studied in [Oh21] and [CLS22]. Toric stack bundles are examples of GIT stack bundles which are previously studied, see [Jia08] and [JTY17].

A recent advance in Gromov-Witten theory of GIT (stack) quotients is the wall-crossing formula for ϵ -quasimaps, proven in increasing generality in [CFK14], [CFK17], [CJR17], [CFK20] and in full generality in [Zho22]. For a rational number $\epsilon > 0$, the notion of ϵ -quasimaps to a GIT quotient is introduced in [CFKM14] (which extends the case of target Grassmannians in [Tod11]) and extended to GIT stack quotient in [CCFK15]. ϵ -quasimaps can be used to define Gromov-Witten type theories by integrations against virtual fundamental classes of moduli spaces of stable ϵ -quasimaps. For ϵ large, ϵ -quasimaps coincide with stable maps and ϵ -quasimap theory recovers Gromov-Witten theory in this case. For $\epsilon > 0$ very small, ϵ -quasimaps are simpler and the associated theory can often be explicitly analyzed. Therefore it is reasonable to hope for explicit results in Gromov-Witten theory by studying how ϵ -quasimap theory changes as ϵ varies. It turns out that ϵ -quasimap theory only changes when ϵ passes through certain values, called the walls. The wall-crossing formula expresses such changes explicitly.

In this paper, we define the notion of ϵ -quasimaps to GIT stack bundles, generalizing the notion of (0^+) -quasimaps to GIT bundles defined in [Oh21]. Using this notion, we construct ϵ -quasimap theory of a GIT stack bundle. We then establish a wall-crossing formula for ϵ -quasimap theory of a GIT stack bundle, closely following the approach of [Zho22]. Our main theorem is Theorem 5.5.

The rest of the paper is organized as follows. In Section 1 we present some basic definitions, such as presentations of GIT stack bundles in Section 1.1 and quasimap I -functions in Section 1.2. In Section 2 we discuss the notion of ϵ -quasimaps to a GIT stack bundle and their moduli spaces. In Section 2.1 we define ϵ -quasimaps. In Section 2.2 we prove that moduli stacks of ϵ -quasimaps to a GIT stack bundle are proper. In Section 2.3 we construct obstruction theories for moduli stacks of ϵ -quasimaps. In Section 2.5 we consider ϵ -quasimaps to a specific GIT stack bundle. In Section 3 we consider the master spaces, which extend the objects introduced in [Zho22] to GIT stack quotients. After introducing various required notions, the definition of master spaces is given in Section 3.5. Properness of master spaces is proven in Section 3.6. In Section 4, we work out in

detail the virtual localization formula for a \mathbb{C}^* -action on master spaces. Finally, in Section 5, we establish the wall-crossing formula.

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1. Preliminaries

Throughout this paper, we consider the following set-up. Let G be a reductive complex algebraic group. Let Y be a non-singular projective variety and let

$$\pi : E \rightarrow Y$$

be a fiber bundle over Y with a fiberwise G -action such that the fiber V is an affine variety with at worst local complete intersection singularities. Fix a character $\theta : G \rightarrow \mathbb{C}^*$ in $\chi(G)$ and

$$L_\theta := E \times \mathbb{C}_\theta \in \text{Pic}^G(E),$$

where \mathbb{C}_θ is the one dimensional representation of G determined by character θ . Let

$$V^s(G, \theta), V^{ss}(G, \theta) \subset V$$

be the θ -stable and θ -semistable loci respectively.

1.1. Presentations of GIT stack bundles. The notion considered in this Section is adopted from [Oh21]. We make the following

Assumption 1.1.

- (1) There is a morphism of varieties $\psi : Y \rightarrow \prod_{j=1}^r \mathbb{P}^{n_j-1}$ for some $r, n_j \in \mathbb{Z}_{>0}$.
- (2) There is a $S := (\mathbb{C}^*)^r$ action on V which commutes with the G -action, where r is same as in the assumption above.

Definition 1.2. A *presentation* of (E, G, θ) is the triple

$$(\psi : Y \rightarrow \prod_{j=1}^r \mathbb{P}^{n_j-1}, (S \times G) \text{ - action on } V, m \in \mathbb{Z}_{>0})$$

which satisfies the following conditions:

- (i) E is the pullback of a vector bundle $[(\prod_{j=1}^r \mathbb{C}^{n_j}) \times V/S]$ on $[\prod_{j=1}^r \mathbb{C}^{n_j}/S]$. That is, we have a fiber product

$$(1.1) \quad \begin{array}{ccc} E & \longrightarrow & [(\prod_{j=1}^r \mathbb{C}^{n_j}) \times V/S] \\ \downarrow & & \downarrow \\ Y & \longrightarrow & [\prod_{j=1}^r \mathbb{C}^{n_j}/S]. \end{array}$$

Here the bottom map is given by

$$Y \xrightarrow{\psi} \prod_{j=1}^r \mathbb{P}^{n_j-1} \hookrightarrow \prod_{j=1}^r [\mathbb{C}^{n_j}/\mathbb{C}^*] \cong \left[\prod_{j=1}^r \mathbb{C}^{n_j}/S \right];$$

- (ii) The morphism $E \rightarrow [(\prod_{j=1}^r \mathbb{C}^{n_j}) \times V/S]$ in diagram (1.1) is G -equivariant;
- (iii) Let A be the coordinate ring of the affine space $(\prod_{j=1}^r \mathbb{C}^{n_j}) \times V$. Then, $A^{S \times G} \cong \mathbb{C}$ as \mathbb{C} -algebras;
- (iv) $V^s(G, \theta) = V^{ss}(G, \theta) \neq \emptyset$ and is nonsingular. Moreover, the G -action on $V^s(G, \theta)$ has finite stabilizers;
- (v) Denote $W := \prod_{j=1}^r \mathbb{C}^{n_j}$ and $\tilde{V} := W \times V$. Then

$$\tilde{V}^{ss}(S \times G, \tilde{\theta}) = W^{ss}(S, \eta) \times V^{ss}(G, \theta).$$

Here $\eta = (1, \dots, 1) \in \mathbb{Z}^r \cong \chi(S)$ and $\tilde{\theta} = m\eta + \theta \in \chi(S) \oplus \chi(G) \cong \chi(S \times G)$.

From conditions (i) and (ii) above, we have a fiber diagram

$$(1.2) \quad \begin{array}{ccccc} [E/G] & \longrightarrow & [\tilde{V}/(S \times G)] & \longrightarrow & [V/(S \times G)] \\ \downarrow \pi & & \downarrow & & \downarrow \\ Y & \longrightarrow & [W/S] & \longrightarrow & BS. \end{array}$$

Here \tilde{V} and W are as defined in condition (v) and $BS := [\text{Spec } \mathbb{C}/S]$ is the classifying space of S .

Definition 1.3. $[E/G] \rightarrow Y$ is a fiber bundle over a base scheme Y with stack quotients $[V/G]$ as its fibers. From diagram (1.2) we have

$$[E/G] := [\tilde{V}/(S \times G)] \times_{[W/S]} Y.$$

From conditions (iii) and (v), the GIT stack quotient of (semi-)stable locus $[\tilde{V}^{ss}(\tilde{\theta})/(S \times G)]$ is a nonsingular, projective and open substack of $[\tilde{V}/(S \times G)]$.

Definition 1.4. We define a *GIT stack bundle* over Y

$$(1.3) \quad [E^{ss}/G] := [\tilde{V}^{ss}(\tilde{\theta})/(S \times G)] \times_{[W^{ss}/S]} Y$$

as a fiber bundle over Y with GIT stack quotient $[V^{ss}(\theta)/G]$ as its generic fibers.

From the definition, we have fibered diagram

$$\begin{array}{ccc} [E^{ss}/G] & \longrightarrow & [\tilde{V}^{ss}(\tilde{\theta})/(S \times G)] \\ \downarrow \pi & & \downarrow \\ Y & \longrightarrow & [W^{ss}(\eta)/S]. \end{array}$$

Hence, $[E^{\text{ss}}/G]$ is nonsingular, projective and open substack of $[E/G]$. Moreover, analogous to [CCFK15, Diagram 2.1], we have a diagram of natural morphisms for various GIT fiber bundles

$$(1.4) \quad \begin{array}{ccc} [E^{\text{ss}}/G] & \hookrightarrow & [E/G] \\ \downarrow & & \downarrow \\ E//_{\theta}G & \longrightarrow & Y, \end{array}$$

where $E//_{\theta}G := \tilde{V}//_{\tilde{\theta}}(S \times G) \times_{W//_{\eta}S} Y$ is a fiber bundle over Y with the GIT scheme quotient $V//_{\theta}G$ as its generic fibers.

1.2. Graph Quasimaps and I -functions. We recall some constructions for quasimaps to GIT (stack) quotients, see e.g. [CFK14].

Let W be an affine variety with at worst local complete intersection singularities. Let G be a reductive complex algebraic group acting on W such that the (semi-)stable locus ($W^{\text{ss}} = W^{\text{s}}$) is smooth and non-empty. Denote

$$\mathcal{X} := [W^{\text{ss}}/G].$$

Let $\beta : \text{Pic}([W/G]) \rightarrow \mathbb{Q}$ be an effective curve class and fix a $\epsilon \in \mathbb{Q}_{>0}$.

A prestable quasimap from a twisted marked curve (C, x) [AGV08, Section 4] to \mathcal{X} consists of a tuple $((C, x), u)$ such that $u : C \rightarrow [W/G]$ is a representable morphism with finitely many base points disjoint from nodes and marked points.

Definition 1.5. The length $l(p)$ at a point p of a prestable quasimap $((C, x_1, \dots, x_n), [u])$ to a quotient stack target \mathcal{X} is defined by

$$l(x) := \min \left\{ \frac{(u^*s)_p}{m} \mid s \in H^0(W, L_{m\theta})^G, u^*s \neq 0, m > 0 \right\}.$$

Remark 1.6. We refer to [CFKM14] for the definition of length at a point of a quasimap to a general GIT target. From the definition, $l(x)$ is non-zero if and only if x is a base point. By prestability, these points are away from stacky nodes and markings. Hence, we can use the same notion of stability for orbifold target.

Let $\varphi : (C, x_1, \dots, x_n) \rightarrow (\underline{C}, \underline{x}_1, \dots, \underline{x}_n)$ be the coarse moduli space for marked twisted curve (C, x_1, \dots, x_n) .

Definition 1.7. [CCFK15, Section 2.3] A prestable quasimap $((C, x_1, \dots, x_k), [u])$ to a quotient stack \mathcal{X} is said to be ϵ -stable if the following two conditions hold:

- (1) The \mathbb{Q} line bundle

$$\omega_{\underline{C}} \left(\sum_{i=1}^n x_i \right) \otimes (\varphi_*([u]^*L_{\theta}))^{\otimes \epsilon}$$

on the coarse curve \underline{C} is ample;

- (2) $\epsilon l(x) \leq 1$ for every point x in C .

A moduli of genus- g , ϵ -stable quasimaps to \mathcal{X} of curve class β with n -marked points is denoted by $Q_{g,n}^\epsilon(\mathcal{X}, \beta)$.

Definition 1.8. A single marked, genus-0 (ϵ -)quasimap graph space is defined to be a special moduli space of genus-0, (ϵ -)stable quasimaps:

$$QG_{0,1}(\mathcal{X}, \beta) := Q_{0,1}^\epsilon(\mathcal{X} \times \mathbb{P}^1, \beta \times [\mathbb{P}^1]).$$

Thus the underlying curve of an object in this graph space has a unique rational tail with its coarse moduli mapped isomorphically to \mathbb{P}^1 . Consider a \mathbb{C}^* -action on \mathbb{P}^1 given by

$$\lambda \cdot [x, y] = [\lambda x, y], \quad \lambda \in \mathbb{C}^*.$$

This induces a \mathbb{C}^* -action on $QG_{0,1}(\mathcal{X}, \beta)$. Let

$$F_{\star, \beta} \subset QG_{0,1}(\mathcal{X}, \beta)^{\mathbb{C}^*}$$

be a \mathbb{C}^* -fixed component such that the unique marked point is mapped to $\infty \in \mathbb{P}^1$ and $0 \in \mathbb{P}^1$ is a base point of length $\deg(\beta)$. Let $[F_{\star, \beta}]^{\text{vir}}$ be its virtual fundamental class and let $N_{F_{\star, \beta}/QG_{0,1}(\mathcal{X}, \beta)}^{\text{vir}}$ be the virtual normal bundle in the sense of [GP99]. We denote the \mathbb{C}^* -equivariant parameter to be z with the Euler class of standard representation as

$$e_{\mathbb{C}^*}(\mathbb{C}_{\text{std}}) = -z.$$

Let

$$I_\mu(\mathcal{X}) = \coprod_r I_{\mu_r}(\mathcal{X})$$

denote the cyclotomic inertia stack of the stack \mathcal{X} (c.f. [AGV08, Section 3.1], [Zho22, Section 1.5]). Define

$$\hat{e}\nu : QG_{0,1}(\mathcal{X}, \beta) \rightarrow I_\mu(\mathcal{X})$$

to be the composition of evaluation map at the unique point with the band-inverting involution on $I_\mu(\mathcal{X})$.

Definition 1.9. We define an I -function as

$$(1.5) \quad I(q, z) := 1 + \sum_{\beta > 0} q^\beta I_\beta(z)$$

where the sum is over all effective curve classes $\beta \in \text{Pic}([W/G])^\vee$ and

$$I_\beta(z) := (-z\mathfrak{t}^2)(\hat{e}\nu)_* \left(\frac{[F_{\star, \beta}]^{\text{vir}}}{e_{\mathbb{C}^*}(N_{F_{\star, \beta}/QG_{0,1}(\mathcal{X}, \beta)}^{\text{vir}})} \right).$$

Here \mathfrak{t} is a locally constant function on $I_\mu(\mathcal{X})$ that takes the value r on $I_{\mu_r}(\mathcal{X})$.

1.3. Inflated Projective bundle. We recall the definition of inflated projective bundle and refer the reader to [Zho22, Appendix A] for a more detailed treatment.

Let X be any algebraic stack. Let L_1, \dots, L_k be line bundles on X . Consider a projective bundle

$$P = \mathbb{P}(L_1 \oplus \dots \oplus L_k) \rightarrow X,$$

and coordinate hyperplanes

$$H_i = \mathbb{P}(L_1 \oplus \dots \oplus \{0\} \oplus \dots \oplus L_k),$$

where $\{0\}$ is in the i -th place. Set $P_{k-1} = P$ and inductively for $i = k-1, \dots, 1$ define union of codimension- i coordinate subspaces

$$Z_i = \bigcup (H_{j_1} \cap \dots \cap H_{j_i}),$$

where j_1, \dots, j_i runs through all subsets of $\{1, \dots, k\}$ of size i . Let $Z_{(i)} \subset P_i$ be the proper transform of Z_i and let

$$P_{i-1} = \text{Bl}_{Z_{(i)}} P_i \rightarrow P_i$$

be the blowup along $Z_{(i)}$. Then

$$\mathcal{P}(L_1, \dots, L_k) := P_0 \rightarrow X$$

is called *inflated projective bundle* associated to line bundles L_1, \dots, L_k .

2. ϵ -Stable Quasimaps

Throughout the paper we use twisted curves with balanced nodes and trivialized gerbe markings [AGV08, Section 4]. A marking on a family of curve $\pi : C \rightarrow R$ is a closed substack $\Sigma \subset C$ in the (relative) smooth locus, together with sections $R \rightarrow \Sigma$ of $\pi|_{\Sigma}$, such that $\Sigma \rightarrow R$ is a gerbe banded by some μ_r .

2.1. Definitions.

2.1.1. Prestable quasimaps. Let (E, G, θ) be as in Section 1.1, with

$$(\psi : Y \rightarrow \prod_{j=1}^r \mathbb{P}^{n_j-1}, (S \times G) \text{ - action on } V, m \in \mathbb{Z}_{>0})$$

as its presentation. Choose a class

$$\beta := (\beta', \tilde{\beta}) \in \text{Ker}(\text{Pic}(Y)^\vee \oplus \text{Pic}^{S \times G}(\tilde{V})^\vee \rightarrow \chi(S)^\vee).$$

The map $\text{Pic}(Y)^\vee \oplus \text{Pic}^{S \times G}(\tilde{V})^\vee \rightarrow \chi(S)^\vee$ is the one induced from diagram (1.2).

Definition 2.1. We define a *prestable morphism* $C \rightarrow [E/G]$ of type (g, n, β) to be a pair of morphisms

$$(f : (C, x_1, \dots, x_n) \rightarrow Y, u : C \rightarrow [\tilde{V}/(S \times G)])$$

such that,

- (C, x_1, \dots, x_n) is a genus g twisted curve with n markings;
- f is a morphism of degree β' ;

- u is a representable morphism of stacks. Moreover, the group homomorphism

$$\tilde{\beta} : \text{Pic}([\tilde{V}/(S \times G)]) \rightarrow \mathbb{Q}, \quad \tilde{\beta}(L) := \deg(u^*(L))$$

is the class of map u ;

- $Q_f \cong Q_u$ as principal S -bundles over C_0 .

To define C_0, Q_f and Q_u , we first note that by [AV02, Proposition 9.1.1], $u : C \rightarrow [\tilde{V}/(S \times G)]$ factors as

$$(2.1) \quad C \xrightarrow{\phi} C_0 \xrightarrow{[u]} [\tilde{V}/(S \times G)],$$

where $\phi : C \rightarrow C_0$ is a contraction of u -constant unmarked rational trees [AV02, Lemma 9.2.1] (connected tree of rational curves attached to rest of the curve at single point) and $[u]$ is a representable morphism of class $\tilde{\beta}$. Note that factorization in [AV02, Proposition 9.1.1], contracts both unstable rational trees and unstable rational bridges. In order to avoid contraction of rational bridges we stabilize them by adding sections, apply contraction and forget the added sections. Composition of morphisms

$$(2.2) \quad C_0 \xrightarrow{[u]} [\tilde{V}/(S \times G)] \rightarrow [W/S].$$

provides a principal S -bundle on C_0 which we denote by Q_u .

We can similarly construct a principal S -bundle starting with the map f . To see this, consider a composition of morphisms

$$C \xrightarrow{f} Y \xrightarrow{\psi} \prod_{j=1}^r \mathbb{P}^{n_j-1} \rightarrow \mathbb{P}^{n_{j_0}-1}$$

for each $1 \leq j_0 \leq r$. This is equivalent to a surjective morphism of sheaves on C ,

$$\mathcal{O}_C^{\oplus n_{j_0}} \xrightarrow{\varphi_{j_0}} \mathcal{L}_{j_0} \rightarrow 0,$$

where \mathcal{L}_{j_0} is the pullback of $\mathcal{O}_{\mathbb{P}^{n_{j_0}-1}}(1)$ on C . Denote by T_1, \dots, T_l the rational tails on each fiber C contracted under ϕ . Let t_1, \dots, t_l be the sections of C_0 corresponding to contraction points of T_1, \dots, T_l on C_0 . Consider morphisms of sheaves on C_0

$$\mathcal{O}_{C_0}^{\oplus n_{j_0}} \xrightarrow{\varphi_{j_0}|_{C_0}} \mathcal{L}_{j_0}|_{C_0} \hookrightarrow \mathcal{L}_{j_0}|_{C_0} \otimes \mathcal{O}_{C_0}(\sum_{i=1}^l \deg(\mathcal{L}_{j_0}|_{T_i}) \cdot t_i).$$

This gives a morphism of stacks

$$C_0 \rightarrow \mathbb{P}^{n_{j_0}-1} \hookrightarrow [\mathbb{C}^{n_{j_0}}/\mathbb{C}^*],$$

with degree $\deg(\mathcal{L}_{j_0})$ (see [CFK14, Section 3.2.2]). So we have a morphism of stacks

$$(2.3) \quad C_0 \rightarrow \prod_{j=1}^r [\mathbb{C}^{n_j}/\mathbb{C}^*] = [W/S],$$

of degree $d' := (\deg(\mathcal{L}_1), \dots, \deg(\mathcal{L}_r))$. This map comes with a principal S -bundle over C_0 , which we denote by Q_f .

Remark 2.2. Using the factorization (2.1), we denote a prestable morphism $(f, u) : C \rightarrow [E/G]$ as a triple

$$(\phi : (C, x) \rightarrow (C_0, x), f : C \rightarrow Y, [u] : C_0 \rightarrow [\tilde{V}/(S \times G)]),$$

for rest of the paper.

Definition 2.3. We call a prestable morphism $(\phi, f, [u]) : C \rightarrow [E/G]$ a *prestable quasimap* to $[E^{ss}/G]$ of type (g, n, β) if

- (1) $(\phi, f, [u]) : C \rightarrow [E/G]$ is a prestable morphism of type (g, n, β) ;
- (2) f is non-constant on each component of a rational tree of C if and only if the rational tree is contracted under ϕ ;
- (3) $[u]$ is a representable morphism such that $[u]^{-1}[\tilde{V}^{\text{us}}(\tilde{\theta})/(S \times G)]$ is finite and disjoint from all the nodes and marked points.

Here

$$[\tilde{V}^{\text{us}}(\tilde{\theta})/(S \times G)] := [\tilde{V}/(S \times G)] \setminus [\tilde{V}^{ss}(\tilde{\theta})/(S \times G)]$$

is the base locus and $y \in [\tilde{V}^{\text{us}}(\tilde{\theta})/(S \times G)]$ is called a *base point*. We denote the stack of prestable quasimaps to $[E^{ss}/G]$ of type (g, k, β) by

$$Q_{g,k}^{\text{pre}}([E^{ss}/G], \beta).$$

2.1.2. Stability condition. To simplify the notation we introduce

$$X := [\tilde{V}/(S \times G)], \quad X^{ss} := [\tilde{V}^{ss}/(S \times G)].$$

Definition 2.4. A prestable quasimap to $[E^{ss}/G]$,

$$(\phi : (C, x_1, \dots, x_n) \rightarrow (C_0, x_1, \dots, x_n), f : C \rightarrow Y, [u] : C_0 \rightarrow X),$$

is ϵ -stable if for each irreducible component $C' \subset C_0$, $f|_{C'}$ is a stable map to Y or its associated quasimap $((C_0, x), [u])|_{C'}$ to X^{ss} is ϵ -stable in the sense of Definition 1.7.

Definition 2.5. An isomorphism between two quasimaps (over \mathbb{C})

$$(\phi : (C, x) \rightarrow (C_0, x), f : C \rightarrow Y, [u])$$

and

$$(\phi' : (C', x') \rightarrow (C'_0, x'), f' : C' \rightarrow Y, [v])$$

is a pair of isomorphism and 2-isomorphism

$$(\theta : (C, x) \xrightarrow{\sim} (C', x'), \delta : [u] \circ \phi \xrightarrow{\sim} [v] \circ \phi' \circ \theta)$$

such that

$$f' \circ \theta = f, \quad \theta(x_i) = x'_i \quad \text{for all } i = 1, \dots, n.$$

2.1.3. Moduli stack. The moduli space of ϵ -stable quasimaps to $[E^{ss}/G]$ of type (g, n, β) is denoted by

$$Q_{g,n}^{\epsilon}([E^{ss}/G], \beta).$$

As the stability condition is a union of two open conditions, $Q_{g,n}^\epsilon([E^{ss}/G], \beta)$ is an open substack of $Q_{g,n}^{\text{pre}}([E^{ss}/G], \beta)$.

Proposition 2.6. *For $\epsilon > 0$ fixed, the automorphism group of an ϵ -stable quasimap is finite and reduced.*

Proof. It is enough to prove this for rational components $C' \subset C$. For C' such that $f|_{C'}$ is constant, $(C_0, [u])|_{C'}$ is ϵ -stable and the result follows from [CCFK15, Section 2.4.2]. Furthermore, if $f|_{C'}$ is non-constant for $C' \subset C$ then $(C', f_{C'})$ is a stable map to Y and by prestability condition, $\phi(C')$ cannot be a rational tail. Hence $(C', [u]|_{C'})$ is a rational bridge which is either 0^+ -stable or is constant under $[u]$. In either cases the proposition follows by [AV02, Theorem 1.4.1] and [CCFK15, Section 2.4.2]. \square

Theorem 2.7. *$Q_{g,n}^\epsilon([E^{ss}/G], \beta)$ is a Deligne–Mumford stack, locally of finite type over \mathbb{C} .*

Proof. By Proposition 2.6, it suffices to prove that the stack is Artin. Moreover, as ϵ -stability condition is an open condition, $Q_{g,n}^\epsilon([E^{ss}/G], \beta)$ is an open substack of $Q_{g,n}^{\text{pre}}([E^{ss}/G], \beta)$. Hence it is enough to show $Q_{g,n}^{\text{pre}}([E^{ss}/G], \beta)$ is an Artin stack, locally of finite type. Indeed, $Q_{g,n}^{\text{pre}}([E^{ss}/G], \beta)$ fits into the following fiber diagram of Artin stacks, locally of finite type over \mathbb{C} ([CCFK15, Section 2.4.1],[AV02, Theorem 1.4.1]):

$$(2.4) \quad \begin{array}{ccc} Q_{g,n}^{\text{pre}}([E^{ss}/G], \beta) & \longrightarrow & Q_{g,n}^{\text{pre}}(X^{ss}, (d', \beta'')) \\ \downarrow & & \downarrow \\ \mathfrak{M}_{g,n}^{\text{tw,pre}}(Y, \beta') & \longrightarrow & Q_{g,n}^{\text{pre}}([W^{ss}/S], d'), \end{array}$$

where we write $\tilde{\beta} = (\tilde{d}, \beta'')$ and $\mathfrak{M}_{g,n}^{\text{tw,pre}}(Y, \beta')$ is a stack of degree β' prestable maps from genus g , n -pointed twisted curve to Y . Over geometric points, the above diagram looks as follows:

$$\begin{array}{ccc} (\phi : (C, x) \rightarrow (C_0, x), f, [u]) & \longmapsto & ((C_0, x), [u]) \\ \downarrow & & \downarrow \\ ((C, x), f) & \longmapsto & ((C_0, x), [u']). \end{array}$$

Here, $[u']$ is the map obtained in (2.2). Hence, the vertical map is given by projection. The lower map is constructed as in (2.3). Note here that the contraction map $C \rightarrow C_0$ is completely determined by the curve C and map f , see conditions (2) in Definition 2.3. Hence, the lower map is well-defined. The diagram is commutative by the definition of prestable morphism to $[E^{ss}/G]$, specifically we need $Q_f \cong Q_u$.

To see that the diagram is cartesian, consider the morphism

$$\mathfrak{M}_{g,n}^{\text{tw,pre}}(Y, \beta') \times_{Q_{g,n}^{\text{pre}}([W^{ss}/S], d')} Q_{g,n}^{\text{pre}}(X^{ss}, (d', \beta'')) \rightarrow Q_{g,n}^{\text{pre}}([E^{ss}/G], \beta)$$

which maps

$$(((C, x), f), ((C_0, x), [u])) \mapsto (\phi : (C, x) \rightarrow (C_0, x), f, [u]).$$

Therefore, we have the required result. \square

2.2. Properness of moduli stack. For a class $\beta = (\beta', \tilde{\beta})$ and line bundle $L' \times \tilde{L}$ on $[E/G]$, define

$$\beta(L' \times \tilde{L}) := \beta'(L') + \tilde{\beta}(\tilde{L}).$$

Let $\mathcal{O}(1)$ be an ample line bundle on Y . Then,

$$(2.5) \quad L := \pi^*(\mathcal{O}(1) \otimes \psi^*(\times_{j=1}^r \mathcal{O}_{\mathbb{P}^{n_j-1}}(m))) \otimes [E \times \mathbb{C}_\theta/G]$$

is an ample line bundle on $[E/G]$. By [CFKM14, Lemma 3.2.1], we have $\beta(L) \geq 0$ and $\beta(L) = 0$ if and only if $\beta = 0$, if and only if f and $[u]$ are constant. This gives the following result on boundedness.

Lemma 2.8. *The number of irreducible components of the underlying curve of an ϵ -stable quasimap to $[E^{ss}/G]$ of type (g, n, β) is bounded.*

Proof. Irreducible components with positive genus or marked points are bounded by g, n . Rational tails ($g = 0, n = 0$) contracted under ϕ are bounded because $\beta(L) > 0$ on these components, since by definition f is non-constant. Similarly, non-contracted rational components are bounded by applying [CFKM14, Corollary 3.2.3] (see [CCFK15, Section 2.4.3]) on quasimap $(C_0, [u])$ to X^{ss} . \square

Corollary 2.9. $Q_{g,n}^\epsilon([E^{ss}/G], \beta)$ is finite type over \mathbb{C} .

Proof. This follows from Lemma 2.8 and Theorem 2.7. \square

Proposition 2.10. $Q_{g,n}^\epsilon([E^{ss}/G], \beta)$ is proper over \mathbb{C} .

Before we begin the proof, we start with an essential lemma.

Lemma 2.11. $Q_{g,n}^{pre}([E^{ss}/G], \beta)$ fits into the following fibered diagram,

$$(2.6) \quad \begin{array}{ccc} Q_{g,n}^{pre}([E^{ss}/G], \beta) & \longrightarrow & Q_{g,n}^{pre}(X^{ss}, (d', \beta'')) \\ \downarrow & & \downarrow \\ \mathfrak{M}_{g,n}^{pre}(Y, \beta') & \longrightarrow & Q_{g,n}^{pre}(W//S, d'), \end{array}$$

where $\mathfrak{M}_{g,n}^{pre}(Y, \beta')$ is the stack of degree β' prestable maps to Y from n -pointed, genus g non-twisted curves.

Proof. Over geometric points, the above diagram looks as follows

$$\begin{array}{ccc} (\phi : (C, x) \rightarrow (C_0, x), f, [u]) & \longmapsto & ((C_0, x), [u]) \\ \downarrow & & \downarrow \\ ((\underline{C}, x), \underline{f}) & \longmapsto & ((\underline{C}_0, x), \underline{[u]}), \end{array}$$

where \underline{C} and \underline{C}_0 are the coarse curves of C and C_0 respectively. $((\underline{C}_0, x), \underline{[u]}) \in Q_{g,n}^{pre}(W//S, d')$ is the coarse image of $((C_0, x), [u]) \in Q_{g,n}^{pre}([W^{ss}/S], d')$ from (2.4). The vertical map on the right is

given by forgetting the orbifold structure while the bottom map is the contraction of rational tails on which \underline{f} is non-constant, see [Oh21, diagram 4.4.47]. Then the lemma follows by an argument similar to the proof of (2.4). The only non-trivial claim is $C \cong \underline{C} \times_{\underline{C}_0} C_0$. This follows by noting that rational trees contracted under ϕ do not carry any orbifold structure [AV02, Lemma 9.2.1]. \square

Now we have all the ingredients to prove properness.

Proof of Proposition 2.10. We use the valuative criteria. Let R be a discrete valuation ring over \mathbb{C} with quotient field K . Denote $\Delta := \text{Spec}(R)$ and let $\bullet \in \text{Spec}(R)$ be its unique closed point. Set $\Delta_0 := \Delta \setminus \{\bullet\} = \text{Spec}(K)$.

We first prove separatedness. From Lemma 2.11,

$$Q_{g,n}^\epsilon([E^{ss}/G], \beta) \subset \mathfrak{M}_{g,n}^{\text{pre}}(Y, \beta') \times_{Q_{g,n}^{\text{pre}}(W//S, d')} Q_{g,n}^{\text{pre}}(X^{ss}, (d', \beta'')).$$

Let $\{((\phi_i, \underline{C}_i, f_i), ((C_0)_i, [u'_i]))\}_{i=1,2} \in Q_{g,n}^\epsilon([E^{ss}/G], \beta)(\Delta)$ be two objects which are isomorphic over Δ_0 . We will show that they are isomorphic over Δ .

Recall that stability (Definition 2.4) allows certain components of the underlying curve to be ϵ -unstable if (C, f) restricted to these components is a stable map to Y . Hence, even though our objects are ϵ -stable, $((C_0)_i, [u'_i])$ may not be ϵ -stable. Specifically, such a component $C'_0 \subset C_0$ may fail ϵ -stability in one (or more) of the following ways:

- (1) C'_0 is an unmarked rational tail connected to the rest of the curve at one point with degree less than $1/\epsilon$,
- (2) C'_0 is an unmarked rational bridge (connected to the rest of the curve at two points) with degree 0,
- (3) C'_0 has a base point of length greater than $1/\epsilon$.

As ϕ contracts rational tails on which f is non-constant, by stability condition all rational tails that survive the contraction ϕ are ϵ -stable. Hence situation (1) above does not occur. We add l sections $s_i : \Delta \rightarrow C_i \xrightarrow{\phi_i} (C_0)_i$ to these surviving rational tails and rational bridges so they become 0^+ -stable. Therefore we get $((\phi_i, \underline{C}_i, x_i, s_i, f_i), ((C_0)_i, x_i, s_i, [u'_i])) \in Q_{g,n+l}^{0^+}([E^{ss}/G], \beta)(\Delta)$.

Now we note that $Q_{g,n+l}^{0^+}([E^{ss}/G], \beta)$ is separated. This follows by following the proof of [Oh21] and noting that the stable (quasi)map moduli $\overline{M}_{g,n}(Y, \beta')$ and $Q_{g,n}^{0^+}(X^{ss}, (d', \beta''))$ are proper. Hence, $\{((\phi_i, \underline{C}_i, x_i, s_i, f_i), ((C_0)_i, x_i, s_i, [u'_i]))\}_{i=1,2}$ are isomorphic over Δ . Finally, forgetting the added extra sections, we see $\{((\phi_i, C_i, f_i), ((C_0)_i, [u'_i]))\}_{i=1,2}$ are isomorphic over Δ , proving separatedness.

Next, to show completeness, we carefully modify the argument of [Oh21] to make it work for ϵ -stable case. We take an object

$$((\phi, (C, x), f), ((C_0, x), [u])) \in Q_{g,n}^\epsilon([E^{ss}/G], \beta)(\Delta_0)$$

over Δ_0 and our aim is to construct its extension over Δ .

We first construct a family of curves over Δ by extending stabilization of C_0 . Note that $((C_0, x), [u]) \in Q_{g,n}^{\text{pre}}(X^{ss}, (d', \beta''))(\Delta_0)$ can be made 0^+ -stable by adding sections on the rational tails and contracting rational bridges which map constantly under $[u]$. As $Q_{g,n+l}^{0^+}(X^{ss}, (d', \beta''))$

is proper, we can extend

$$((C'_0, x, r), [u']) \in Q_{g,n+l}^{0+}(X^{ss}, (d', \beta''))(\Delta_0)$$

to get

$$((\overline{C}'_0, \overline{x}, \overline{r}), [\overline{u}']) \in Q_{g,n+l}^{0+}(X^{ss}, (d', \beta''))(\Delta).$$

Next we use the curve \overline{C}'_0 over Δ to add sections on C . Let $q_i : \Delta \rightarrow \overline{C}'_0$ be distinct sections such that $p_i : \Delta_0 \xrightarrow{q_i|_{\Delta_0}} \overline{C}'_0|_{\Delta_0} \hookrightarrow C$ makes $((C, x, p), f)$ a family of stable maps over Δ_0 . As $\overline{M}_{g,n+l}(Y, \beta')$ is proper, we can extend to get a family of stable maps over Δ . That is, we get,

$$((\overline{C}, \overline{x}, \overline{p}), \overline{f}) \in \overline{M}_{g,n+l}(Y, \beta')(\Delta).$$

Forgetting the sections p_i and contracting f -nonconstant rational tails, we get a map $\overline{\phi} : (\overline{C}, x) \rightarrow (\overline{C}_0, x)$. Moreover, this map extends the contraction map over Δ_0 since $(\overline{C}_0, x)|_{\Delta_0} = (C_0, x)$.

Now that we have a hold over curve \overline{C}_0 over Δ which extends C_0 , we can start adding sections to stabilize $[u]$. Recall from the discussion above, all the surviving rational tails in C_0 are ϵ -stable. However, C_0 may have base points of length greater than $1/\epsilon$. Let m be the maximum length of base points. If $m < 1/\epsilon$, we take $\epsilon' = \epsilon$. If $m \geq 1/\epsilon$, take $\epsilon' \in \mathbb{Q}$ such that $1/(m+1) < \epsilon' < 1/m$. Our aim is to make $[u] : C_0 \rightarrow X$ ϵ' -stable. By definition, lengths of all base points are less than $1/\epsilon'$. If the degree d of any rational tail satisfies $1/\epsilon < d \leq 1/\epsilon'$, we add a section on that component. Similarly, we add a section on any $[u]$ -constant rational bridge.

Following the discussion above, let $q : \Delta \rightarrow \overline{C}_0$ be the added distinct sections such that,

$$((C_0, x, q), [u]) \in Q_{g,n'}^{\epsilon'}(X^{ss}, (d', \beta''))(\Delta_0).$$

As $Q_{g,n'}^{\epsilon'}(X^{ss}, (d', \beta''))(\Delta_0)$ is proper, we get its extension

$$((\overline{C}_0, \overline{x}, \overline{q}), [\overline{u}_0]) \in Q_{g,n'}^{\epsilon'}(X^{ss}, (d', \beta''))(\Delta).$$

The data $((\overline{\phi}, (\overline{C}, \overline{x}), \overline{f}), ((\overline{C}_0, \overline{x}), [\overline{u}_0]))$ form a candidate for the extension over Δ . However, the central fiber may still have some problematic rational components which may map constantly under both \overline{f} and $[\overline{u}_0]$. More importantly, as $((\overline{C}_0, \overline{x}, \overline{q}), [\overline{u}_0])$ is an ϵ' -stable extension, we may have base points of length greater than $1/\epsilon$ on \overline{f} -constant rational components.

We first contract the problematic rational tails. To be precise, contract rational tails C'' (if any) in the central fiber such that map $\overline{f}|_{C''}$ and $[[\overline{u}_0] \circ \overline{\phi}]|_{C''}$ are constant (using the line bundle (2.5) to stabilize). Let \tilde{C} and \tilde{C}_0 be the respective curves after such contraction and $\tilde{\phi}$ the contraction map between them. Define \tilde{f} and $[\tilde{u}_0]$ such that the following diagrams commute,

$$\begin{array}{ccc} \overline{C} & \longrightarrow & \tilde{C} \\ \overline{f} \downarrow & \swarrow \tilde{f} & \\ Y & & \end{array} \quad \begin{array}{ccc} \overline{C}_0 & \longrightarrow & \tilde{C}_0 \\ [\overline{u}_0] \downarrow & \swarrow [\tilde{u}_0] & \\ X & & \end{array}$$

Finally we tweak the map $[\tilde{u}_0]$ (on the central fiber) to make the extension ϵ -stable. Let $((\tilde{\phi}, (\tilde{C}, \overline{x}), \tilde{f}), ((\tilde{C}_0, \overline{x}), [\tilde{u}_0]))$ be the object obtained from the above procedure. By prestability,

base points are away from nodes and marked points. Hence we can assume the underlying curve $\tilde{C}(\Delta_0)$ to be smooth over Δ_0 (shrinking Δ if necessary). After extension (and forgetting the stabilizing sections), curve over the central fiber $\tilde{C}(\bullet)$ may be nodal.

Now we divide the argument into two cases.

Case 1: If the map $(\tilde{C}(\bullet), \tilde{f}(\bullet))$ over the closed point \bullet is stable, define

$$((\hat{\phi}, (\hat{C}, \hat{x}), \hat{f}), ((\hat{C}_0, \hat{x}), [\hat{u}]]) := ((\tilde{\phi}, (\tilde{C}, \tilde{x}), \tilde{f}), ((\tilde{C}_0, \tilde{x}), [\tilde{u}_0])).$$

Case 2: If the map $(\tilde{C}(\bullet), \tilde{f}(\bullet))$ from the central fiber is unstable, properness of $\overline{M}_{g,n}(Y, \beta')$ implies $(\tilde{C}(\Delta_0), \tilde{f}(\Delta_0))$ is also not stable. By stability, this means that the contraction map $\tilde{\phi}(\Delta_0)$ is an isomorphism and $(\tilde{C}_0(\Delta_0), [\tilde{u}_0](\Delta_0))$ is ϵ -stable. As $Q_{g,n}^\epsilon(X^{ss}, (d', \beta''))$ is proper, by extending we get $(\hat{C}_0, [\hat{u}]) \in Q_{g,n}^\epsilon(X^{ss}, (d', \beta''))(\Delta)$. Define $\hat{\phi} : \hat{C} \cong \hat{C}_0$ (coarse curve of \hat{C}_0). Note that \hat{C}_0 may have extra components not present in \tilde{C} (extension may add rational component of degree $1/\epsilon < d < 1/\epsilon'$ over central fiber). Define $\hat{f} : \hat{C} \rightarrow Y$ such that the following diagram commutes,

$$\begin{array}{ccc} \hat{C} & \longrightarrow & \tilde{C} \\ \hat{f} \downarrow & \swarrow \tilde{f} & \\ Y, & & \end{array}$$

where the top map is the contraction of added rational tails over the closed point \bullet .

Claim 2.12. $((\hat{\phi}, (\hat{C}, \hat{x}), \hat{f}), ((\hat{C}_0, \hat{x}), [\hat{u}]]) \in Q_{g,n}^\epsilon([E^{ss}/G], \beta)(\Delta)$ is the required extension.

Indeed, by construction $((\hat{\phi}, (\hat{C}, \hat{x}), \hat{f}), ((\hat{C}_0, \hat{x}), [\hat{u}]])|_{\Delta_0} = ((\phi, (C, x), f), (C_0, x), [u])$. We start by checking that $\hat{\phi}$ is the contraction of \hat{f} -nonconstant rational tails. Let $C''(\Delta)$ be a component of $\hat{C}(\Delta)$ such that over the central point $C''(\bullet)$ is a $\hat{f}(\bullet)$ -nonconstant tail. This implies $C''(\Delta_0)$ is $\hat{f}(\Delta_0)$ -nonconstant (shrinking if necessary) and $\hat{\phi}(\Delta_0)$ contracts the tail. Hence, by extension $\hat{\phi}(\bullet)$ contracts $C''(\bullet)$ too. Likewise, if the central fiber $C''(\bullet)$ is $\hat{f}(\bullet)$ -constant and $C''(\Delta_0)$ is $\hat{f}(\Delta_0)$ -constant, we get that $\hat{\phi}(\Delta)|_{C''}$ is an isomorphism. The only non-trivial case is when $C''(\Delta_0)$ is $\hat{f}(\Delta_0)$ -nonconstant but its limit $C''(\bullet)$ is $\hat{f}(\bullet)$ -constant. Here as $C''(\Delta_0)$ is $\hat{f}(\Delta_0)$ -nonconstant, $\hat{\phi}(\Delta_0)$ contracts $C''(\Delta_0)$ and by extension $\hat{\phi}(\bullet)$ contracts $C''(\bullet)$. Hence, $C''(\bullet)$ is $[[\hat{u}] \circ \hat{\phi}](\bullet)$ -constant. By assumption, $C''(\bullet)$ is also $\hat{f}(\bullet)$ -constant. Such tails cannot occur as these are exactly the rational tails that we contract to get \tilde{C} from \overline{C} . Therefore we have

$$((\hat{\phi}, (\hat{C}, \hat{x}), \hat{f}), ((\hat{C}_0, \hat{x}), [\hat{u}]]) \in \mathfrak{M}_{g,n}^{\text{pre}}(Y, \beta') \times_{Q_{g,n}^{\text{pre}}(W//S, d')} Q_{g,n}^{\text{pre}}(X^{ss}, (d', \beta''))(\Delta).$$

It remains to check ϵ -stability. Let C'' be an irreducible component of \hat{C} and let C''_0 be its corresponding image under $\hat{\phi}$. If the curve over central fiber $C''(\bullet)$ is $\hat{f}(\bullet)$ -stable, we are done. So let us assume that $C''(\bullet)$ is $\hat{f}(\bullet)$ -unstable. We want to show that $C''_0(\bullet)$ is ϵ -stable. As $C''(\bullet)$ is $\hat{f}(\bullet)$ -unstable, $C''(\Delta_0)$ is $\hat{f}(\Delta_0)$ -unstable, then $C''_0(\Delta_0)$ is ϵ -stable and by completeness of $Q_{g,n}^\epsilon(X^{ss}, (d', \beta''))$, we get that the central fiber $C''_0(\bullet)$ is ϵ -stable. \square

2.3. Obstruction Theory. Let $\mathfrak{M}_{g,n}^{tw}$ be the stack of n -pointed genus g twisted curves. It is a smooth Artin stack, locally of finite type [Ols07, Theorem 1.10]. Let $\mathfrak{C} \rightarrow \mathfrak{M}_{g,n}^{tw}$ be its universal

curve. Define

$$\mathfrak{Bun}_G^{tw} := \text{Hom}_{\mathfrak{M}_{g,n}^{tw}}(\mathfrak{C}, BG \times \mathfrak{M}_{g,n}^{tw}),$$

which is smooth over $\mathfrak{M}_{g,n}^{tw}$ [CCFK15, Section 2.4.5]. We have a fiber diagram of forgetful morphisms,

$$\begin{array}{ccc} Q_{g,n}^{\text{pre}}(X^{ss}, (d', \beta'')) & \longrightarrow & Q_{g,n}^{\text{pre}}([V^{ss}/(S \times G)], \beta'') \\ \downarrow & & \downarrow \\ Q_{g,n}^{\text{pre}}([W^{ss}/S], d') & \longrightarrow & \mathfrak{Bun}_S^{tw}. \end{array}$$

Combining this with diagram (2.4), we get the following fibered diagram

$$(2.7) \quad \begin{array}{ccc} Q_{g,n}^{\text{pre}}([E^{ss}/G], \beta) & \xrightarrow{p_1} & Q_{g,n}^{\text{pre}}([V^{ss}/(S \times G)], \beta'') \\ \downarrow p_2 & & \downarrow \mu_1 \\ \mathfrak{M}_{g,n}^{tw}(Y, \beta') & \xrightarrow{\mu_2} & \mathfrak{Bun}_S^{tw}. \end{array}$$

The map μ_1 can be factored as

$$Q_{g,n}^{\text{pre}}([V^{ss}/(S \times G)], \beta'') \xrightarrow{\mu_1^1} \mathfrak{Bun}_{S \times G}^{tw} \xrightarrow{\mu_1^2} \mathfrak{Bun}_S^{tw}.$$

Let the complex $E_1^\bullet \in D^b(Q_{g,n}^{\text{pre}}([V^{ss}/(S \times G)], \beta''))$ be the μ_1^1 -relative perfect obstruction theory, see [CCFK15, Section 2.4.5] for its construction. Define a complex (see e.g. [CFKM14, Remark 4.5.3])

$$E_{\mu_1}^\bullet := \text{Cone}(E_1^\bullet[-1] \rightarrow \mathbb{L}_{\mu_1^1}[-1] \rightarrow (\mu_1^1)^* \mathbb{L}_{\mu_1^2}) \in D^b(Q_{g,n}^{\text{pre}}([V^{ss}/(S \times G)], \beta'')),$$

where $\mathbb{L}_{\mu_1^1}, \mathbb{L}_{\mu_1^2}$ are cotangent complexes for μ_1^1, μ_1^2 respectively.

Next consider the composition

$$\mu' : \mathfrak{M}_{g,k}^{tw}(Y, \beta') \xrightarrow{\mu_2} \mathfrak{Bun}_S^{tw} \xrightarrow{\mu_3} \text{Spec}(\mathbb{C}).$$

This composition factors through the stack of twisted curves via a forgetting map,

$$\mathfrak{M}_{g,k}^{tw}(Y, \beta') \xrightarrow{\mu_2'} \mathfrak{M}_{g,k}^{tw} \xrightarrow{\mu_3'} \text{Spec}(\mathbb{C}).$$

Let $E_2^\bullet \in D^b(\mathfrak{M}_{g,k}^{tw}(Y, \beta'))$ be the relative perfect obstruction theory for μ_2' . Such an obstruction theory exists by [AGV08, Section 4.5]. Define

$$E_{\mu'}^\bullet := \text{Cone}(E_2^\bullet[-1] \rightarrow \mathbb{L}_{\mu_2'}[-1] \rightarrow (\mu_2')^* \mathbb{L}_{\mathfrak{M}_{g,k}^{tw}}) \in D^b(\mathfrak{M}_{g,k}^{tw}(Y, \beta')).$$

Using this, a perfect obstruction theory $E_{\mu_2}^\bullet$ relative to \mathfrak{Bun}_S^{tw} can be constructed using an argument similar to [Oh21]. We present the details in our situation.

Let \mathfrak{C} be the universal curve on $\mathfrak{M}_{g,k}^{tw}(Y, \beta')$. Let $\Phi : \mathfrak{C} \rightarrow \mathfrak{C}_\circ$ be the contraction of f -nonconstant rational tails. Similar to [Oh21], we have a commutative diagram

$$\begin{array}{ccc} T_{\mathfrak{C}}(-\sum_i p_i) & \longrightarrow & f^*T_Y \\ \nu_1 \downarrow & & \downarrow \nu_2 \\ \Phi^*T_{\mathfrak{C}_\circ}(-\sum_i p_i) & \longrightarrow & \Phi^*T_{BS}|_{\mathfrak{C}_\circ}. \end{array}$$

Define the complex E'^\bullet to be the dual derived pushforward of

$$\text{Cone}(\nu_1)[-1] \rightarrow f^*T_Y.$$

As the composition $\text{Cone}(\nu_1)[-1] \rightarrow f^*T_Y \xrightarrow{\nu_2} \Phi^*T_{BS}|_{\mathfrak{C}_\circ}$ is zero in the derived category, we have a morphism

$$\text{Cone}(\text{Cone}(\nu_1)[-1] \rightarrow f^*T_Y) \rightarrow \Phi^*T_{BS}|_{\mathfrak{C}_\circ}.$$

This induces the dual morphism $\mu_2^*\mathbb{L}_{\mathfrak{Bun}_S^{tw}} \rightarrow E'^\bullet$ such that the following diagram commutes,

$$\begin{array}{ccc} \mu_2^*\mathbb{L}_{\mathfrak{Bun}_S^{tw}} & \longrightarrow & E'^\bullet \\ \downarrow & \swarrow & \\ \mathbb{L}_{\mathfrak{M}_{g,k}^{tw}(Y, \beta')} & & \end{array}$$

Define a complex

$$E_{\mu_2}^\bullet := \text{Cone}(\mu_2^*\mathbb{L}_{\mathfrak{Bun}_S^{tw}} \rightarrow E'^\bullet) \in D^b(\mathfrak{M}_{g,k}^{tw}(Y, \beta')).$$

We define

$$E^\bullet := (p_1^*E_{\mu_1}^\bullet \oplus p_2^*E_{\mu_2}^\bullet)|_{Q_{g,k}^\epsilon([E/G], \beta)} \in D^b(Q_{g,k}^\epsilon([E^{ss}/G], \beta))$$

as the relative perfect obstruction theory for $Q_{g,k}^\epsilon([E^{ss}/G], \beta)$ over \mathfrak{Bun}_S^{tw} .

2.4. Evaluation map. Let $I_\mu(X^{ss})$ be the cyclotomic inertia stack of X . Define

$$I_\mu([E^{ss}/G]) := Y \times_{[W^{ss}/S]} I_\mu(X^{ss})$$

to be the corresponding stack for our target space. To define an evaluation map from $Q_{g,k}^\epsilon([E^{ss}/G], \beta)$ to $I_\mu([E^{ss}/G])$, consider the universal curve \mathcal{C} over $Q_{g,k}^\epsilon([E^{ss}/G], \beta)$ and the universal morphism $(\mathcal{F}, \mathcal{U}) : \mathcal{C} \rightarrow [E/G]$. Restriction of these maps to the marked points gives the desired evaluation map, which we represent as a pair

$$ev := (ev', \tilde{ev}) : Q_{g,k}^\epsilon([E^{ss}/G], \beta) \rightarrow I_\mu([E^{ss}/G]).$$

Note that since marked points are away from base points, evaluation maps take values in $I_\mu([E^{ss}/G])$.

2.5. Quasimaps to $\tilde{\mathbb{P}}$. We end this section with moduli of ϵ -stable quasimaps to a space with \mathbb{P}^N as its fibers and construct contraction maps on them.

Let Y be a nonsingular projective variety as before. For $N \geq 0$, consider an example where $V = \mathbb{C}^{N+1}$ and $G = \mathbb{C}^*$. Then $E \rightarrow Y$ is a total space with \mathbb{C}^{N+1} as its fibers. Let $(\psi : Y \rightarrow \prod_{j=1}^r \mathbb{P}^{n_j}, (\mathbb{C}^*)^r \times \mathbb{C}^*$ -action on $\mathbb{C}^{N+1}, m \in \mathbb{Z}_{>0})$ be its presentation [Oh21, Section 4.2]. Let

$W := \prod_{j=1}^r \mathbb{C}^{n_j+1}$ and $\tilde{V} = W \times \mathbb{C}^{N+1}$ be as before. Define

$$\tilde{\mathbb{P}} := [E^{ss}/G] \rightarrow Y.$$

The fibers of this map are $[(\mathbb{C}^{N+1} \setminus \{\vec{0}\})/\mathbb{C}^*]$.

Consider the moduli stack $Q_{g,n}^\epsilon(\tilde{\mathbb{P}}, \beta)$ of ϵ -stable quasimaps to $\tilde{\mathbb{P}}$ of type (g, n, β) . From above, it is a proper Deligne–Mumford stack of finite type with a virtual fundamental class denoted by $[Q_{g,n}^\epsilon(\tilde{\mathbb{P}}, \beta)]^{\text{vir}}$.

For a positive integer d_0 , let $\epsilon_0 = 1/d_0$ be a wall. Denote the chamber to the right as $\epsilon_+ := (1/d_0, 1/(d_0 - 1))$ and similarly the chamber to the left as $\epsilon_- := (1/(d_0 + 1), 1/d_0)$. Consider a closed point in $Q_{g,n}^{\epsilon_+}(\tilde{\mathbb{P}}, \beta)$,

$$(\phi : C \rightarrow C_0, f : C \rightarrow Y, [u] : C_0 \rightarrow [\tilde{V}/((\mathbb{C}^*)^r \times \mathbb{C}^*)]).$$

Note that by condition (v) from the definition of presentation for $\tilde{\mathbb{P}}$ in Section 1.1,

$$\tilde{V}^{ss}((\mathbb{C}^*)^r \times \mathbb{C}^*, \tilde{\theta}) = \prod_{j=1}^r (\mathbb{C}^{(n_j+1)} \setminus \{0\}) \times (\mathbb{C}^{N+1} \setminus \{0\}).$$

Thus we get a map from (semi-)stable locus to product of \mathbb{P}^n 's;

$$[\tilde{V}^{ss}/(S \times G)] \rightarrow \prod_{j=1}^r \mathbb{P}^{n_j} \times \mathbb{P}^N.$$

From the stability condition, the map $[u]$ restricted to any rational tail of C_0 is ϵ_+ -stable. Moreover, f is constant on the preimage of these tails in C . Consider contraction $C_0 \rightarrow C'_0$ of rational tails having degree $(0, d_0)$ [CFK14, Section 3.2.2], [Tod11], [MM07]. The image of such a contraction map

$$(\phi' : C \rightarrow C'_0, f : C \rightarrow Y, [u'] : C'_0 \rightarrow [W \times \mathbb{C}^{N+1}/((\mathbb{C}^*)^r \times \mathbb{C}^*)])$$

lies in $Q_{g,n}^{\epsilon_-}(\tilde{\mathbb{P}}, \beta)$. This can be extended functorially over families of quasimaps to give a map

$$(2.8) \quad c : Q_{g,n}^{\epsilon_+}(\tilde{\mathbb{P}}, \beta) \rightarrow Q_{g,n}^{\epsilon_-}(\tilde{\mathbb{P}}, \beta)$$

By taking composition of such morphism, we have

$$(2.9) \quad c_\epsilon : Q_{g,n}^\epsilon(\tilde{\mathbb{P}}, \beta) \rightarrow Q_{g,n}^{0+}(\tilde{\mathbb{P}}, \beta).$$

for every $\epsilon > 0$.

Similarly, we can replace the last k marked points by base points of length d_0 [CFK14, Section 3.2.3]. This map may convert certain rational components into rational tails. Hence, we take the 0^+ -stabilization of the obtained quasimaps. Let $C' \subset C$ be such a rational tail connected to the rest of the curve at p . Let $C'_0 \subset C_0$ be its corresponding image under the contraction map ϕ . If $f|_{C'}$ is nonconstant, then we contract C'_0 from C_0 and replace p by a base point of length equal to the degree of $C'_0 \subset C_0$. If $f|_{C'}$ is constant, we contract both $C' \subset C$ and $C'_0 \subset C_0$. Latter contraction

gives a base point of length equal to the degree of C'_0 . In short, for every $\epsilon > 0$, we have a map

$$(2.10) \quad b_{k,\epsilon} : Q_{g,n+k}^\epsilon(\tilde{\mathbb{P}}, \beta) \rightarrow Q_{g,n}^{0+}(\tilde{\mathbb{P}}, \beta + k(0, d_0)).$$

3. Master Spaces

The purpose of this Section is to construct master spaces.

3.1. Weighted Twisted Curves. Let $d_0, \epsilon_+, \epsilon_-$ be as above. Fix non-negative integers g, n and a pair of non-negative integers $d := (d', d'')$, such that $2g - 2 + n + \epsilon_0 d'' \geq 0$. When $g = n = 0$, we require the inequality to be strict, i.e. $\epsilon_0 d'' > 2$. For n -marked twisted curves of genus g , we assign to each irreducible component C a pair of nonnegative integers (d'_C, d''_C) such that $\sum_C (d'_C, d''_C) = (d', d'')$. We call (d', d'') as the degree of the curve. We will see that such an assignment is continuous in a sense that over a family of curves it is locally given by the degree of some line bundle. Such a decorated twisted curve is referred as n -marked weighted twisted curve of genus g and degree d [Cos06], [HL10]. The moduli stack of such weighted curves is denoted by $\mathfrak{M}_{g,n,d}^{\text{wt}}$.

Definition 3.1. We define the stack of (ϵ_0) -semistable weighted curves to be the open substack $\mathfrak{M}_{g,n,d}^{\text{wt,ss}} \subset \mathfrak{M}_{g,n,d}^{\text{wt}}$ defined by the following conditions:

- the curve has no degree- $(0, 0)$ rational bridge,
- the curve has no rational tail of degree strictly less than $(0, d_0)$ or strictly greater than $(0, \infty)$ in dictionary (lexicographic) order.

3.2. Entangled tails. Here we discuss entangled tails briefly and refer readers to [Zho22, Section 2.2] for a more detailed treatment. Let $m := \lfloor d''/d_0 \rfloor$ be the maximum number of degree- $(0, d_0)$ rational tails. Set

$$\mathfrak{U}_m = \mathfrak{M}_{g,n,d}^{\text{wt,ss}}$$

Let $\mathfrak{Z}_i \subset \mathfrak{U}_m$ be the reduced substack parameterizing curves with at least i rational tails of degree $(0, d_0)$. \mathfrak{Z}_m is the deepest stratum and hence smooth. Define

$$\mathfrak{U}_{m-1} \rightarrow \mathfrak{U}_m$$

to be the blowup along \mathfrak{Z}_m and let $\mathfrak{E}_{m-1} \subset \mathfrak{U}_{m-1}$ be the exceptional divisor. Inductively let $\mathfrak{Z}_{(i)} \subset \mathfrak{U}_i$ be the proper transform of \mathfrak{Z}_i and let

$$\mathfrak{U}_{i-1} \rightarrow \mathfrak{U}_i$$

be the blowup along $\mathfrak{Z}_{(i)}$. Set $\tilde{\mathfrak{M}}_{g,n,d} = \mathfrak{U}_0$.

Definition 3.2. $\tilde{\mathfrak{M}}_{g,n,d}$ is called the moduli stack of *genus g , n -marked, ϵ_0 -semistable curves of degree d with entangled tails*.

For a scheme R , an R -family of semistable curves with entangled tails corresponding to a morphism $e : R \rightarrow \tilde{\mathfrak{M}}_{g,n,d}$ is denoted by

$$(\pi : C \rightarrow R, x, e),$$

where $(\pi : C \rightarrow R, x)$ is the family of n -marked weighted twisted curves associated with $R \xrightarrow{e} \tilde{\mathfrak{M}}_{g,n,d} \rightarrow \mathfrak{M}_{g,n,d}^{\text{wt,ss}}$.

Let

$$\xi = (\pi : C \rightarrow \text{Spec}K, x, e)$$

be a geometric point of $\tilde{\mathfrak{M}}_{g,n,d}$. Let E_1, \dots, E_l be the degree $(0, d_0)$ rational tails in C . Locally let \mathfrak{h}_i be the locus where E_i remains a rational tail. Then near the image of ξ , the analytic branches of $\mathfrak{Z}_1 \subset \mathfrak{M}_{g,n,d}^{\text{wt,ss}}$ are

$$\mathfrak{h}_1, \dots, \mathfrak{h}_l.$$

Let $\mathfrak{h}_{i,k}$ be the proper transform of \mathfrak{h}_i in \mathfrak{U}_k , where

$$k = \min\{i \mid \text{the image of } \xi \text{ in } \mathfrak{U}_i \text{ lies in } \mathfrak{Z}_{(i)}\}.$$

There exists a unique subset $\{i_1, \dots, i_k\} \subset \{1, \dots, l\}$ such that the image of ξ in \mathfrak{U}_k lies in the intersection of $\mathfrak{h}_{i_1,k}, \dots, \mathfrak{h}_{i_k,k}$.

Definition 3.3. With the notation above, we call E_{i_1}, \dots, E_{i_k} the *entangled tails* of ξ .

Next we define a gluing morphism which will be useful later. Fix $k \geq 1$ and let $\mathfrak{M}_{g,n+k,d-kd_0}^{\text{wt,ss}}$ and $\mathfrak{M}_{0,1,d_0}^{\text{wt,ss}}$ be stacks of ϵ_0 -semistable weighted curves. Note that we denote the pair $(0, d_0)$ as d_0 to lighten the notation and hereafter we will use $d_0 = (0, d_0)$ wherever the context makes it clear. Define a gluing morphism

$$\mathfrak{gl}_k : \mathfrak{M}_{g,n+k,d-kd_0}^{\text{wt,ss}} \times' (\mathfrak{M}_{0,1,d_0}^{\text{wt,ss}})^k \rightarrow \mathfrak{Z}_k \subset \mathfrak{M}_{g,n,d}^{\text{wt,ss}}$$

where the product \times' is formed by matching the sizes of automorphism groups at the last k markings. These orders of automorphism groups at the $(n+i)$ -th marked point define locally constant morphisms for each $i = 1, \dots, k$ denoted by

$$\mathfrak{r}_i : \mathfrak{M}_{g,n+k,d-kd_0}^{\text{wt,ss}} \times' (\mathfrak{M}_{0,1,d_0}^{\text{wt,ss}})^k \rightarrow \mathbb{N}.$$

As $\mathfrak{M}_{g,n+k,d-kd_0}^{\text{wt,ss}} \times' (\mathfrak{M}_{0,1,d_0}^{\text{wt,ss}})^k$ is smooth, \mathfrak{gl}_k factors through $\mathfrak{Z}_k^{\text{nor}}$. This induces a morphism

$$(3.1) \quad \tilde{\mathfrak{gl}}_k : \tilde{\mathfrak{M}}_{g,n+k,d-kd_0} \times' (\mathfrak{M}_{0,1,d_0}^{\text{wt,ss}})^k \rightarrow \mathfrak{Z}_{(k)},$$

which is étale of degree $k! / \prod_{i=1}^{i=k} \mathfrak{r}_i$ ([Zho22, Lemma 2.4.2]).

3.3. Calibration bundle. Recall that $\mathfrak{Z}_1 \subset \mathfrak{M}_{g,n,d}^{\text{wt,ss}}$ is the reduced substack parametrizing curves with at least one degree $(0, d_0)$ rational tail.

Definition 3.4. When $(g, n, d) \neq (0, 1, d_0)$, the *universal calibration bundle* is defined to be the line bundle $\mathcal{O}_{\mathfrak{M}_{g,n,d}^{\text{wt,ss}}}(-\mathfrak{Z}_1)$; when $(g, n, d) = (0, 1, d_0)$, the universal calibration bundle is the cotangent bundle at the unique marked point of rational tail.

The calibration bundle of $\tilde{\mathfrak{M}}_{g,n,d}$, denoted by

$$\mathbb{M}_{\tilde{\mathfrak{M}}_{g,n,d}},$$

is the pullback of the universal calibration bundle of $\mathfrak{M}_{g,n,d}^{\text{wt,ss}}$ via the forgetfull map $\tilde{\mathfrak{M}}_{g,n,d} \rightarrow \mathfrak{M}_{g,n,d}^{\text{wt,ss}}$.

Definition 3.5. The moduli stack of ϵ_0 -semistable curves with calibrated tails is defined as

$$M\tilde{\mathfrak{M}}_{g,n,d} := \mathbb{P}_{\tilde{\mathfrak{M}}_{g,n,d}}(\mathbb{M}_{\tilde{\mathfrak{M}}_{g,n,d}} \oplus \mathcal{O}_{\tilde{\mathfrak{M}}_{g,n,d}}).$$

An R -point of $M\tilde{\mathfrak{M}}_{g,n,d}$ consists of

$$(\pi : C \rightarrow R, x, e, N, v_1, v_2),$$

where

- $(\pi : C \rightarrow R, x, e) \in \tilde{\mathfrak{M}}_{g,n,d}(R)$;
- N is a line bundle on R ;
- $v_1 \in \Gamma(R, \mathbb{M}_R \otimes N)$, $v_2 \in \Gamma(R, N)$ have no common zeros, where \mathbb{M}_R is the calibration bundle for the family of curves $\pi : C \rightarrow R$, defined by pulling back $\mathbb{M}_{\tilde{\mathfrak{M}}_{g,n,d}}$ via the map $R \rightarrow \tilde{\mathfrak{M}}_{g,n,d}$.

3.4. Quasimap with entangled tails. Let \tilde{V}, S, G be as defined in Section 1.1. Recall $X = [\tilde{V}/(S \times G)]$, $X^{ss} = [\tilde{V}^{ss}/(S \times G)]$ and fix a curve class $\tilde{\beta} = (d', \beta'')$. Let $Q_{g,n}^{\text{pre}}(X^{ss}, \tilde{\beta})$ be the stack of genus- g , n -marked quasimaps to X^{ss} with curve class $\tilde{\beta}$. For each irreducible component $C'_0 \subset C_0$, assign a pair $(d'_{C'_0}, d''_{C'_0})$ given by the degree of class (i.e. $d'_{C'_0} := d'|_{C'_0}$ and $d''_{C'_0} := \beta''_{C'_0}(L_\theta)$). We call such a pair of non-negative numbers the degree of the curve and give it the dictionary order. Define

$$Q_{g,n}^{ss}(X^{ss}, \tilde{\beta}) \subset Q_{g,n}^{\text{pre}}(X^{ss}, \tilde{\beta})$$

to be the open substack where the quasimaps have no

- rational tails of degree $< (0, d_0)$ or $> (0, \infty)$ or rational bridges of degree $(0, 0)$,
- base points of length $> d_0$ on rational components (bridges and tails).

Let $\beta = (\beta', \tilde{\beta})$ be as before. Using degrees of quasimaps, we assign weights to irreducible components of the underlying curve. This defines a forgetful morphism $Q_{g,n}^{ss}(X^{ss}, \tilde{\beta}) \rightarrow \mathfrak{M}_{g,n,d}^{\text{wt,ss}}$ where $d = \deg(\tilde{\beta})$. Define the moduli stack of ϵ_0 -semistable quasimaps to $[E^{ss}/G]$ as the fiber product

$$Q_{g,n}^{ss}([E^{ss}/G], \beta) := \mathfrak{M}_{g,n}^{\text{tw,pre}}(Y, \beta') \times_{Q_{g,n}^{\text{pre}}([W^{ss}/S], d')} Q_{g,n}^{ss}(X^{ss}, (d', \beta'')).$$

Composing with projection from above, we have

$$Q_{g,n}^{ss}([E^{ss}/G], \beta) \rightarrow Q_{g,n}^{ss}(X^{ss}, (d', \beta'')) \rightarrow \mathfrak{M}_{g,n,d}^{\text{wt,ss}}.$$

Definition 3.6. We define the stack of n -pointed genus g , ϵ_0 -semistable quasimaps with entangled tails to $[E^{ss}/G]$ with curve class β to be

$$Q_{g,n}^{\sim}([E^{ss}/G], \beta) := Q_{g,n}^{ss}([E^{ss}/G], \beta) \times_{\mathfrak{M}_{g,n,d}^{\text{wt,ss}}} \tilde{\mathfrak{M}}_{g,n,d}.$$

The construction above can be summarized in the following fibered diagram,

$$(3.2) \quad \begin{array}{ccccc} Q_{g,n}^{\sim}([E^{ss}/G], \beta) & \longrightarrow & Q_{g,n}^{\sim}(X^{ss}, (d', \beta'')) & \longrightarrow & \tilde{\mathfrak{M}}_{g,n,d} \\ \downarrow & & \downarrow & & \downarrow \\ Q_{g,n}^{ss}([E^{ss}/G], \beta) & \longrightarrow & Q_{g,n}^{ss}(X^{ss}, (d', \beta'')) & \longrightarrow & \mathfrak{M}_{g,n,d}^{\text{wt},ss} \\ \downarrow & & \downarrow & & \\ \mathfrak{M}_{g,n}^{\text{tw},\text{pre}}(Y, \beta') & \longrightarrow & Q_{g,n}^{\text{pre}}([W^{ss}/S], d') & & \end{array}$$

For a scheme R , an R -point of $Q_{g,n}^{\sim}([E^{ss}/G], \beta)$ is given by

$$((\phi : C \rightarrow C_0, f, x), (C_0, x, [u], e)),$$

where $((\phi : C \rightarrow C_0, f, x), (C_0, x, [u])) \in Q_{g,n}^{ss}([E^{ss}/G], \beta)(R)$ and $(C_0, x, e) \in \tilde{\mathfrak{M}}_{g,n,d}(R)$.

Lemma 3.7. *The stack $Q_{g,n}^{\sim}([E^{ss}/G], \beta)$ is an Artin stack of finite type.*

Proof. Recall from Theorem 2.7, $Q_{g,n}^{\text{pre}}([E^{ss}/G], \beta)$ is an Artin stack locally of finite type. Hence, so is $Q_{g,n}^{ss}([E^{ss}/G], \beta)$. As the morphism $\tilde{\mathfrak{M}}_{g,n,d} \rightarrow \mathfrak{M}_{g,n,d}^{\text{wt},ss}$ is projective, by Definition 3.6 we get that $Q_{g,n}^{\sim}([E^{ss}/G], \beta)$ is an Artin stack of locally finite type.

To show it is of finite type, it is enough to show that $Q_{g,n}^{ss}([E^{ss}/G], \beta)$ is bounded. We only need to show that the underlying curve has finitely many unmarked rational components, as the other components are bounded by g, n . Such components with restriction of f being non-constant are bounded by β' . Hence it is enough to show that f -constant rational components with no marked points are finite. This in turn follows from the definition and noting that $Q_{g,n}^{ss}(X^{ss}, (d', \tilde{\beta}))$ is bounded, see [Zho22, proof of Lemma 3.1.2]. \square

Definition 3.8. An R -family of ϵ_0 -semistable quasimaps with entangled tails is called ϵ_+ -stable if the underlying family of quasimaps to $[E^{ss}/G]$ is ϵ_+ -stable. We denote the moduli of n -marked, genus g , ϵ_+ -stable quasimaps of degree β to $[E^{ss}/G]$ by $\tilde{Q}_{g,n}^{\epsilon_+}([E^{ss}/G], \beta)$.

From the definition, we have a natural isomorphism

$$\tilde{Q}_{g,n}^{\epsilon_+}([E^{ss}/G], \beta) \cong Q_{g,n}^{\epsilon_+}([E^{ss}/G], \beta) \times_{\mathfrak{M}_{g,n,d}^{\text{wt},ss}} \tilde{\mathfrak{M}}_{g,n,d}.$$

As $Q_{g,n}^{\epsilon_+}([E^{ss}/G], \beta)$ is a proper Deligne–Mumford stack, we have

Lemma 3.9. *The stack $\tilde{Q}_{g,n}^{\epsilon_+}([E^{ss}/G], \beta)$ is a proper Deligne–Mumford stack.*

Similar to the obstruction theory of $Q_{g,n}^{\epsilon_+}([E^{ss}/G], \beta)$ in Section 2.3, we can construct a perfect obstruction theory for $\tilde{Q}_{g,n}^{\epsilon_+}([E^{ss}/G], \beta)$ relative to $\mathfrak{Bun}_S^{\text{tw}}$. In fact, the virtual classes induced by these two obstruction theories are related by the following lemma.

Lemma 3.10. *Under the forgetful morphism $\tilde{Q}_{g,n}^{\epsilon_+}([E^{ss}/G], \beta) \rightarrow Q_{g,n}^{\epsilon_+}([E^{ss}/G], \beta)$, the pushforward of $[\tilde{Q}_{g,n}^{\epsilon_+}([E^{ss}/G], \beta)]^{\text{vir}}$ is equal to $[Q_{g,n}^{\epsilon_+}([E^{ss}/G], \beta)]^{\text{vir}}$.*

Proof. This follows from [Cos06, Theorem 5.0.1], see also [HW21]. \square

3.4.1. Splitting off entangled tails. Let

$$\mathfrak{C}_k^* \subset \tilde{\mathfrak{M}}_{g,n,d}$$

be the locus where there are exactly k entangled tails. This is a locally closed smooth substack of codimension one. Define the boundary divisor

$$\mathcal{D}_k \subset \tilde{\mathfrak{M}}_{g,n,d}$$

as the closure of \mathfrak{C}_k^* . It is a smooth divisor with natural maps

$$\mathcal{D}_k \rightarrow \mathfrak{C}_k \rightarrow \mathfrak{Z}_{(k)}.$$

Write $\tilde{\mathfrak{gl}}_k^* \mathcal{D}_k$ for the pullback along the gluing morphism (3.1).

Next we define the restriction $\tilde{Q}_{g,n}^{\epsilon_+}([E^{ss}/G], \beta)|_{\tilde{\mathfrak{gl}}_k^* \mathcal{D}_k}$ by the following fibered diagram,

$$\begin{array}{ccccc} \tilde{Q}_{g,n}^{\epsilon_+}([E^{ss}/G], \beta)|_{\tilde{\mathfrak{gl}}_k^* \mathcal{D}_k} & \longrightarrow & \tilde{Q}_{g,n}^{\epsilon_+}([E^{ss}/G], \beta) & \longrightarrow & Q_{g,n}^{\epsilon_+}([E^{ss}/G], \beta) \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{\mathfrak{gl}}_k^* \mathcal{D}_k & \xrightarrow{i_{\mathcal{D}}} & \tilde{\mathfrak{M}}_{g,n,d} & \longrightarrow & \mathfrak{M}_{g,n,d}^{\text{wt,ss}} \end{array}$$

Define $[\tilde{Q}_{g,n}^{\epsilon_+}([E^{ss}/G], \beta)|_{\tilde{\mathfrak{gl}}_k^* \mathcal{D}_k}]^{\text{vir}} := i_{\mathcal{D}}^! [\tilde{Q}_{g,n}^{\epsilon_+}([E^{ss}/G], \beta)]^{\text{vir}}$.

Moreover, the map $\tilde{\mathfrak{gl}}_k^* \mathcal{D}_k \rightarrow \mathfrak{M}_{g,n,d}^{\text{wt,ss}}$ factors as the composition

$$\tilde{\mathfrak{gl}}_k^* \mathcal{D}_k \rightarrow \mathfrak{M}_{g,n+k,d-kd_0}^{\text{wt,ss}} \times' (\mathfrak{M}_{0,1,d_0}^{\text{wt,ss}})^k \rightarrow \mathfrak{M}_{g,n,d}^{\text{wt,ss}}$$

Completing the analogous fibered diagram, we get

$$(3.3) \quad \begin{array}{ccccc} \tilde{Q}_{g,n}^{\epsilon_+}([E^{ss}/G], \beta)|_{\tilde{\mathfrak{gl}}_k^* \mathcal{D}_k} & \xrightarrow{p} & \bigsqcup_{\vec{\beta}} Q_{\vec{\beta}} & \longrightarrow & Q_{g,n}^{\epsilon_+}([E^{ss}/G], \beta) \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{\mathfrak{gl}}_k^* \mathcal{D}_k & \longrightarrow & \mathfrak{M}_{g,n+k,d-kd_0}^{\text{wt,ss}} \times' (\mathfrak{M}_{0,1,d_0}^{\text{wt,ss}})^k & \longrightarrow & \mathfrak{M}_{g,n,d}^{\text{wt,ss}} \end{array}$$

where $\vec{\beta} = (\beta_0, \beta_1, \dots, \beta_k)$ runs through all $(k+1)$ -tuples of effective curve classes such that $\deg(\beta_i) = (0, d_0)$ for each $i = 1, \dots, k$ along with

$$\beta = \beta_0 + \dots + \beta_k,$$

and we denote

$$Q_{\vec{\beta}} := \tilde{Q}_{g,n+k}^{\epsilon_+}([E^{ss}/G], \beta_0) \times_{(I_{\mu}[E^{ss}/G])^k} \prod_{i=1}^k Q_{0,1}^{\epsilon_+}([E^{ss}/G], \beta_i).$$

Lemma 3.11.

$$(3.4) \quad \begin{aligned} & [\tilde{Q}_{g,n}^{\epsilon_+}([E^{ss}/G], \beta)|_{\tilde{\mathfrak{gl}}_k^* \mathcal{D}_k}]^{\text{vir}} = \\ & p^* \left(\sum_{\vec{\beta}} \Delta_{(I_{\mu}[E^{ss}/G])^k}^! [\tilde{Q}_{g,n+k}^{\epsilon_+}([E^{ss}/G], \beta_0)]^{\text{vir}} \boxtimes \prod_{i=1}^k [Q_{0,1}^{\epsilon_+}([E^{ss}/G], \beta_i)]^{\text{vir}} \right), \end{aligned}$$

where $\Delta_{(I_\mu[E^{ss}/G])^k} : (I_\mu[E^{ss}/G])^k \rightarrow (I_\mu[E^{ss}/G])^k \times (I_\mu[E^{ss}/G])^k$ is the diagonal morphism.

Proof. This follows by an argument similar to the one for [Zho22, Lemma 3.2.1]. \square

3.5. Definition of the master space. Let $g, n, d, d_0 = 1/\epsilon_0$ be as above. As in Section 3.3, let $\mathbb{M}_{\tilde{\mathfrak{M}}_{g,n,d}}$ be the calibration bundle of $\tilde{\mathfrak{M}}_{g,n,d}$ and let $M\tilde{\mathfrak{M}}_{g,n,d}$ be the moduli of curves with calibrated tails.

Definition 3.12. The *moduli stack of genus g , n -marked ϵ_0 -semistable quasimaps with calibrated tails to $[E^{ss}/G]$ of curve class β* is defined to be

$$MQ_{g,n}^\sim([E^{ss}/G], \beta) := Q_{g,n}^\sim([E^{ss}/G], \beta) \times_{\tilde{\mathfrak{M}}_{g,n,d}} M\tilde{\mathfrak{M}}_{g,n,d}.$$

For a scheme R , an R -point of $MQ_{g,n}^\sim([E^{ss}/G], \beta)$ can be expressed as

$$((\phi : C \rightarrow C_0, f, x), (C_0, x, [u], e, N, v_1, v_2)),$$

where $((\phi : C \rightarrow C_0, f, x), (C_0, x, [u], e)) \in Q_{g,n}^\sim([E^{ss}/G], \beta)(R)$ and $(C_0, x, e, N, v_1, v_2) \in M\tilde{\mathfrak{M}}_{g,n,d}(R)$.

From diagram (3.2), we have the following natural correspondence

$$(3.5) \quad MQ_{g,n}^\sim([E^{ss}/G], \beta) \cong \mathfrak{M}_{g,n}^{tw,pre}(Y, \beta') \times_{Q_{g,k}^{pre}([W^{ss}/S], d')} MQ_{g,n}^\sim(X^{ss}, \tilde{\beta}),$$

where

$$MQ_{g,n}^\sim(X^{ss}, \tilde{\beta}) := Q_{g,n}^\sim(X^{ss}, \tilde{\beta}) \times_{\tilde{\mathfrak{M}}_{g,n,d}} M\tilde{\mathfrak{M}}_{g,n,d}.$$

Let $((\phi : C \rightarrow C_0, f, x), (C_0, x, [u], e)) \in Q_{g,n}^\sim([E^{ss}/G], \beta)(\mathbb{C})$ be a geometric point of ϵ_0 -semistable quasimaps with entangled tails. Recall from the definition in Section 3.4 that the curve C_0 may have a rational tail of degree $(0, d_0)$ and base points of length d_0 .

Definition 3.13. We call a base point a *relevant base point* if it is contained in a rational component.

Definition 3.14. A degree- $(0, d_0)$ rational tail $E \subset C_0$ is called a *constant tail* if E contains a (relevant) base point of length d_0 .

We now describe the stability condition.

Definition 3.15. An R -family of ϵ_0 -semistable quasimaps with calibrated tails

$$((\phi : C \rightarrow C_0, f, x), (C_0, x, [u], e, N, v_1, v_2))$$

is ϵ_0 -stable if over every geometric point r of R ,

- (1) any constant tail is an entangled tail;
- (2) if the geometric fiber $C_0(r)$ of C_0 contains rational tails of degree $(0, d_0)$, then all the length d_0 relevant base points (in the sense of Definition 3.13) are contained in these tails;
- (3) if $v_1(r) = 0$, then $((\phi : C \rightarrow C_0, f, x), (C_0, x, [u]))(r)$ is an ϵ_+ -stable quasimap to $[E^{ss}/G]$;
- (4) if $v_2(r) = 0$, then $((\phi : C \rightarrow C_0, f, x), (C_0, x, [u]))(r)$ is an ϵ_- -stable quasimap to $[E^{ss}/G]$.

We denote by

$$MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta)$$

the category of *genus- g , n -marked, ϵ_0 -stable quasimaps with calibrated tails to $[E^{ss}/G]$ of curve class β .*

Lemma 3.16. *The stability condition in Definition 3.15 is an open condition.*

Proof. Conditions (3) and (4) are open conditions since ϵ -stability is an open condition in the space of prestable quasimaps. Conditions (1) and (2) are open: notice that they are open conditions in $MQ_{g,n}^{\sim}(X^{ss}, \tilde{\beta})$ by [Zho22, Lemma 4.1.4] and hence open in $MQ_{g,n}^{\sim}([E^{ss}/G], \beta)$ by equation (3.5). \square

Let

$$\xi = ((\phi : C \rightarrow C_0, f, x), (C_0, x, [u], e, N, v_1, v_2)) \in MQ_{g,n}^{\sim}([E^{ss}/G], \beta)(\mathbb{C})$$

be an ϵ_0 -semistable quasimap with calibrated tails. Let $E \subset C_0$ be a degree- $(0, d_0)$ rational tail with $y \in E$ as its unique node (or marking).

Definition 3.17. E is said to be a *fixed tail* if $\text{Aut}(E, y, u|_E)$ is infinite.

Lemma 3.18. *The family of quasimaps to $[E^{ss}/G]$ with entangled tails*

$$\eta = ((\phi : C \rightarrow C_0, f, x), (C_0, x, [u], e))$$

obtained from ξ has infinitely many automorphisms if and only if

- (1) *there is at least one degree $(0, d_0)$ rational tail, and*
- (2) *each entangled tail is a fixed tail.*

Moreover, when $\text{Aut}(\eta)$ is infinite, its identity component $\Gamma \subset \text{Aut}(\eta)$ is isomorphic to \mathbb{C}^ and acts on each T_y, E_i by the same nonzero weight for $i = 1, \dots, k$.*

Proof. First we show the “only if” part. Consider the union of f -nonconstant irreducible components $C'' \subset C$. By prestability condition of quasimaps, for any irreducible rational component $C' \subset C''$ in this union ($f|_{C'}$ is non-constant), C' gets contracted under the contraction map ϕ . Hence, $(\phi(C''), x, [u]|_{C''})$ is a 0^+ -stable quasimap to X and therefore has finitely many automorphisms. This means $\eta|_{C'}$ has infinitely many automorphism only if $f|_{C'}$ is constant.

So we can assume that f is constant on the underlying curve C . In particular, the contraction map ϕ is an isomorphism and η has infinitely many automorphisms only if $(C, x, [u], e)$ has infinitely many automorphisms. Then we get “only if” implication by the “only if” part of [Zho22, Lemma 4.1.8].

The “if” part and rest of the lemma follows directly from the “if” part of [Zho22, Lemma 4.1.8]. \square

Lemma 3.19. *If ξ is ϵ_0 -stable, then ξ has finitely many automorphisms.*

Proof. This follows from Lemma 3.18 and an argument similar to [Zho22, Lemma 4.1.10]. \square

Proposition 3.20. $MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta)$ is a Deligne–Mumford stack of finite type over \mathbb{C} .

Proof. By Lemma 3.7, $Q_{g,n}^{\sim}([E^{ss}/G], \beta)$ is an Artin stack of finite type. By Definition 3.12, so is $MQ_{g,n}^{\sim}([E^{ss}/G], \beta)$. Finally by Lemmas 3.16 and 3.19, $MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta)$ is a Deligne–Mumford stack of finite type. \square

3.5.1. Virtual fundamental class. Define a moduli stack

$$\mathcal{G}_{g,n,\beta''}^{\text{tw},d'}$$

parametrizing objects of the form (C_0^{wt}, x, u) where C_0^{wt} is a weighted twisted curve such that all degrees add up to d' and (C_0, x, u) is an n -marked genus g , quasimap to X^{ss} of degree β'' . Here C_0 is the underlying twisted curve without the weights.

The map $Q_{g,n}^{\text{ss}}(X^{ss}, (d', \beta'')) \rightarrow \mathfrak{M}_{g,n,d}^{\text{wt,ss}}$ factors through $\mathcal{G}_{g,n,\beta''}^{\text{tw},d'}$. Moreover, we have a fibered diagram

$$\begin{array}{ccccc} Q_{g,n}^{\text{ss}}(X^{ss}, (d', \beta'')) & \longrightarrow & \mathcal{G}_{g,n,\beta''}^{\text{tw},d'} & \longrightarrow & \mathfrak{M}_{g,n,d}^{\text{wt,ss}} \\ \downarrow & & \downarrow & & \\ Q_{g,n}^{\text{pre}}([W^{ss}/S], d') & \longrightarrow & \mathfrak{Bun}_S^{\text{tw,wt}} & & \end{array}$$

Here $\mathfrak{Bun}_S^{\text{tw,wt}} := \mathfrak{Bun}_S^{\text{tw}} \times_{\mathfrak{M}_{g,n,d}^{\text{wt,ss}}} \mathfrak{M}_{g,n,d}^{\text{wt,ss}}$ is a stack of principal S -bundles over weighted twisted curves. The maps originating from $\mathcal{G}_{g,n,\beta''}^{\text{tw},d'}$ are appropriate forgetful morphisms. $Q_{g,n}^{\text{ss}}(X^{ss}, (d', \beta'')) \rightarrow \mathcal{G}_{g,n,\beta''}^{\text{tw},d'}$ is obtained by forgetting the data of quasimap to $[W^{ss}/S]$ along with assigning degree of its class on first coordinate as weight.

Define

$$\tilde{M}\mathcal{G}_{g,n,\beta''}^{\text{tw},d'} := \mathcal{G}_{g,n,\beta''}^{\text{tw},d'} \times_{\mathfrak{M}_{g,n,d}^{\text{wt,ss}}} M\tilde{\mathfrak{M}}_{g,n,d}.$$

Then by diagram (3.2) and the definition of $MQ_{g,n}^{\sim}([E^{ss}/G], \beta)$, we have

$$MQ_{g,n}^{\sim}([E^{ss}/G], \beta) = \mathfrak{M}_{g,n}^{\text{tw,pre}}(Y, \beta') \times_{\mathfrak{Bun}_S^{\text{tw,wt}}} \tilde{M}\mathcal{G}_{g,n,\beta''}^{\text{tw},d'}.$$

Let $\pi : \mathfrak{C}_o \rightarrow \tilde{M}\mathcal{G}_{g,n,\beta''}^{\text{tw},d'}$ be the universal curve and let $u : \mathfrak{C}_o \rightarrow [V/(S \times G)]$ be the universal map. Similar to [Zho22, Section 4.2], the forgetful morphism $\tilde{M}\mathcal{G}_{g,n,\beta''}^{\text{tw},d'} \rightarrow M\tilde{\mathfrak{M}}_{g,n,d}$ admits a relative perfect obstruction theory

$$(R\pi_*(u^*\mathbb{T}_{[V/(S \times G)]}))^\vee \rightarrow \mathbb{L}_{\tilde{M}\mathcal{G}_{g,n,\beta''}^{\text{tw},d'}/M\tilde{\mathfrak{M}}_{g,n,d}}.$$

Let $\mathbb{E}_{\tilde{M}\mathcal{G}_{g,n,\beta''}^{\text{tw},d'}}$ be the absolute perfect obstruction theory on $\tilde{M}\mathcal{G}_{g,n,\beta''}^{\text{tw},d'}$. Now as $\mathfrak{Bun}_S^{\text{tw}}$ is a smooth Artin stack, we have a perfect obstruction theory \mathbb{E}_{μ_1} relative to the forgetful morphism

$$\mu_1 : \tilde{M}\mathcal{G}_{g,n,\beta''}^{\text{tw},d'} \rightarrow \mathfrak{Bun}_S^{\text{tw}},$$

defined as

$$\mathbb{E}_{\mu_1} := \text{Cone}(\mu_1^* \mathbb{L}_{\mathfrak{Bun}_S^{\text{tw}}} \rightarrow \mathbb{E}_{\tilde{M}\mathcal{G}_{g,n,\beta''}^{\text{tw},d'}}).$$

As it is defined by forgetting the weights, $\mathfrak{Bun}_S^{\text{tw,wt}} \rightarrow \mathfrak{Bun}_S^{\text{tw}}$ is étale, and we obtain a perfect obstruction theory relative to $\mathfrak{Bun}_S^{\text{tw,wt}}$ which we again denote by \mathbb{E}_{μ_1} .

Let \mathbb{E}_{μ_2} be the perfect relative obstruction theory of $\mathfrak{M}_{g,n}^{\text{tw,pre}}(Y, \beta')$ relative to

$$\mu_2 : \mathfrak{M}_{g,n}^{\text{tw,pre}}(Y, \beta') \rightarrow \mathfrak{Bun}_S^{\text{tw,wt}}.$$

Note that we get a perfect obstruction theory relative to $\mathfrak{Bun}_S^{\text{tw}}$ from Section 2.3, which again can be lifted to $\mathfrak{Bun}_S^{\text{tw,wt}}$.

We define (suppressing obvious pullbacks)

$$\mathbb{E}_\mu := (\mathbb{E}_{\mu_1} \oplus \mathbb{E}_{\mu_2})|_{MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta)} \in D^b(MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta))$$

as the perfect relative obstruction theory of $MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta)$ over $\mathfrak{Bun}_S^{\text{tw,wt}}$. This defines a virtual fundamental class

$$[MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta)]^{\text{vir}} \in A_*(MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta)).$$

From the relative obstruction theory, we obtain an absolute one defined as

$$\mathbb{E}_{MQ} := \text{Cone}(\mathbb{E}_\mu(-1) \rightarrow \mathbb{L}_\mu(-1) \rightarrow \mu^* \mathbb{L}_{\mathfrak{Bun}_S^{\text{tw,wt}}}),$$

see [CFKM14, Remark 4.5.3].

3.6. Properness of master space. The purpose of this Subsection is to prove the following

Proposition 3.21. *$MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta)$ is proper over \mathbb{C} .*

We will use valuative criteria for properness. Let (R, \bullet) be a complete discrete valuation \mathbb{C} -algebra with K as its fraction field and residue field \mathbb{C} . We will prove that given any

$$\xi^* = ((\phi^* : C^* \rightarrow C_0^*, f^*, x^*), (C_0^*, x^*, [u^*], e^*, N^*, v_1^*, v_2^*)) \in MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta)(K),$$

it has a unique (up to finite base change) extension to

$$\xi = ((\phi : C \rightarrow C_0, f, x), (C_0, x, [u], e, N, v_1, v_2)) \in MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta)(R).$$

3.6.1. Case 1: Assume $(g, n, d) \neq (0, 1, d_0)$ and ξ^* does not have degree- $(0, d_0)$ rational tails. Then

$$\eta^* = ((\phi^* : C^* \rightarrow C_0^*, f^*, x^*), (C_0^*, x^*, [u^*]))$$

is a K -family of ϵ_- -stable quasimaps to $[E^{ss}/G]$. As $Q_{g,n}^{\epsilon_-}([E^{ss}/G], \beta)$ is proper, it uniquely extends to an R -family of ϵ_- -stable quasimaps

$$\eta_- = ((\phi_- : C_- \rightarrow C_{0-}, f_-, x_-), ((C_{0-}, x_-, [u_-])).$$

As the representable morphism $M\tilde{\mathfrak{M}}_{g,n,d} \rightarrow \tilde{\mathfrak{M}}_{g,n,d} \rightarrow \mathfrak{M}_{g,n,d}^{\text{wt,ss}}$ is proper, we have an ϵ_0 -semistable extension of ξ^*

$$\xi_- = ((\phi_- : C_- \rightarrow C_{0-}, f_-, x_-), (C_{0-}, x_-, [u_-], e_-, N_-, v_{1-}, v_{2-})).$$

We need to show that it is ϵ_0 -stable.

First we start with a definition. Consider a totally ramified finite base change $\mathrm{Spec}R' \rightarrow \mathrm{Spec}R$ of degree r . Let K' be the fraction field of R' .

Definition 3.22. A modification of η_- of degree r is the family of ϵ_0 -semistable quasimaps over R'

$$\tilde{\eta} = ((\tilde{\phi} : \tilde{C} \rightarrow \tilde{C}_0, \tilde{f}, \tilde{x}), (\tilde{C}_0, \tilde{x}, [\tilde{u}])),$$

together with an isomorphism

$$\tilde{\eta}|_{\mathrm{Spec}K'} \cong \eta_-|_{\mathrm{Spec}K'}.$$

Let $\eta' = ((\phi' : C' \rightarrow C'_0, f', x'), (C'_0, x', [u']))$ be the pullback of η_- to $\mathrm{Spec}R'$.

Lemma 3.23. *The family of curves C'_0 is obtained from \tilde{C}_0 by contracting the rational tails of degree $(0, d_0)$ on the special fiber.*

Proof. Using separatedness of $Q_{g,n}^{\epsilon_-}([E^{ss}/G], \beta)$, this follows by a similar argument given in [Zho22, Lemma 5.2.2]. \square

As the extension $R \subset R'$ is totally ramified, $C'_0(\bullet) \cong C_{0-}(\bullet)$. Let p_1, \dots, p_k be the length d_0 relevant base points in the special fiber. For the modifications to be ϵ_0 -stable, rational tails of degree $(0, d_0)$ should contract to length d_0 relevant base points on $C'_0(\bullet)$ (Condition (2) of Definition 3.15). Hence, over each p_i we have a contracted tail E_i of degree $(0, d_0)$. Let each E_i intersect the other component of $\tilde{C}_0(\bullet)$ at some A_{a_i-1} -singularity. Such a modification $\tilde{\eta}$ is said to have singularity type $(a_1/r, \dots, a_k/r)$.

Singularity types of all such modifications are bounded from above by $(b_1, \dots, b_k) \in \mathbb{Q}_{>0} \cup \{\infty\}$ which we describe briefly. Let $B^* \subset C_{0-}(K)$ be the locus of length d_0 relevant base points and let $B \subset C_{0-}$ be its closure. Let p_{l+1}, \dots, p_k be the length d_0 relevant base points in the special fiber contained in B . Assign $b_i = \infty$ for $i = l+1, \dots, k$. Replace B by additional orbifold markings [Zho22, Lemma 5.2.4] so that the generic fiber of η^* is an ϵ_+ -stable quasimap. Its ϵ_+ -extension to R will have l degree- $(0, d_0)$ rational tails E_1, \dots, E_l corresponding to p_1, \dots, p_l . We define (b_1, \dots, b_l) to be the singularity type of this ϵ_+ -extension.

To see the bound, we record [Zho22, Lemma 5.2.5]. Even though the lemma was proven for quasimaps to $[V^{ss}/G]$ with affine V , the same proof works for quasimaps to $[E^{ss}/G]$.

Lemma 3.24 (c.f. [Zho22], Lemma 5.2.5). *For $a := (a_1, \dots, a_k) \in \mathbb{Q}_{>0}^k$ and sufficiently divisible r , the following are equivalent:*

- (1) η_- has a modification $\tilde{\eta}$ of degree r and singularity type a ;
- (2) $a_i \leq b_i$ for each i .

Moreover, assuming $\tilde{\eta}$ exists, then

- $\tilde{\eta}$ is uniquely determined by a and r .
- for each i , E_i contains no length- d_0 relevant base point if and only if $a_i = b_i$, where E_i is a rational tail of $\tilde{\eta}$ lying over p_i .

Recall from the definition of ϵ_0 -semistable calibrated bundle (Definition 3.5), v_1 and v_2 have no common zeros. Denote their vanishing order at the closed point by $\text{ord}(v_1)$ and $\text{ord}(v_2)$. When $v_{1-}(\bullet) \neq 0$, the stability condition (see Definition 3.15) is trivially true (as η_- is ϵ_- -stable quasimap, it does not have degree d_0 rational tails), and we can take $\xi := \xi_-$. To show that this is the unique stable extension, consider a degree r modification $\tilde{\eta}$ with singularity type (a_1, \dots, a_k) . If $a_i > 0$ for some i , it means that there is a degree- $(0, d_0)$ -rational tail in the special fiber. Moreover, applying [Zho22, Lemma 2.10.1] to extensions C_- and \tilde{C} , we have the following relation,

$$(3.6) \quad \text{ord}(\tilde{v}_1) - \text{ord}(\tilde{v}_2) = r(\text{ord}(v_{1-}) - \text{ord}(v_{2-})) - r \sum_{i=1}^k a_i.$$

As $\text{ord}(v_{1-}) = 0$, we have $0 \leq \text{ord}(\tilde{v}_1) < \text{ord}(\tilde{v}_2)$. Hence $\tilde{v}_2(\bullet) = 0$ but also contains degree $(0, d_0)$ -rational tail, contradicting stability. Similarly, if there are no length- d_0 base points in the special fiber, ξ_- is ϵ_0 -stable and we are done.

So let us assume $v_{1-}(\bullet) = 0$ and let $p_1, \dots, p_k \in C_{0-}(\bullet)$ be length- d_0 relevant base points in the special fiber. Let $\tilde{\eta}$ be a non-trivial modification of η_- with singularity type (a_1, \dots, a_k) . Put $\delta := \text{ord}(v_{1-}) - \text{ord}(v_{2-})$ and $|a| := \sum_{i=1}^k (a_i)$. By Lemma 3.24, such a modification is stable if and only if

$$(3.7) \quad \left\{ \begin{array}{l} |a| \leq \delta; \\ 0 < a_i \leq b_i, \text{ for all } i = 1, \dots, k; \\ \text{if } |a| < \delta \text{ then } a_i = b_i \text{ for all } i = 1, \dots, k; \\ \text{if } a_i < b_i \text{ then } a_i \text{ is maximal among } a_i, \dots, a_k, \text{ for all } i = 1, \dots, k. \end{array} \right.$$

There exists a unique solution of (a_1, \dots, a_k) up to a totally ramified base change. By Lemma 3.24, (a_1, \dots, a_k) and r completely determine the modification $\tilde{\eta}$ which in turn gives a unique ϵ_0 -stable extension $\tilde{\xi}$. We take $\xi = \tilde{\xi}$ as the required unique (up to base change) extension.

3.6.2. Case 2: We assume $(g, n, d) = (0, 1, d_0)$. Let $\beta = (0, \beta_0)$, so that $d_0 = \beta_0(L_\theta)$. Recall that the curve C_0 cannot have rational tails with degrees less than d_0 . Hence, C_0 is an irreducible rational tail with one special point. The map $\phi : C \rightarrow C_0$ is an isomorphism and $f : C \rightarrow Y$ is a constant map.

In this case, $Q_{0,1}^\sim([E^{ss}/G], \beta) \cong Y \times_{[W^{ss}/S]} Q_{0,1}^\sim(X^{ss}, (0, \beta_0))$. This implies the following isomorphism for ϵ_0 -semistable quasimaps with calibrated tails,

$$MQ_{0,1}^\sim([E^{ss}/G], \beta) \cong Y \times_{[W^{ss}/S]} MQ_{0,1}^\sim(X^{ss}, (0, \beta_0)).$$

Now properness follows from that of $MQ_{0,1}^{\epsilon_0}(X^{ss}, (0, \beta_0))$ [Zho22, Section 5.3].

3.6.3. Case 3: Assume that the generic fiber contains degree- $(0, d_0)$ rational tails and let $\mathcal{E}_1^*, \dots, \mathcal{E}_l^* \subset C_0^*$ be the entangled rational tails. Let $C_{0\sim}^*$ be the union of other components along with non-entangled rational tails. Viewing nodes on $C_{0\sim}^*$ as new markings y_{\sim}^* , we obtain a family of quasimaps

over $\text{Spec}K$

$$\eta_{\sim}^* = ((\phi_{\sim}^* : C_{\sim}^* \rightarrow C_{0\sim}^*, f_{\sim}^* = f^*|_{C_{0\sim}^*}, x_{\sim}^*, y_{\sim}^*), (C_{0\sim}^*, [u_{\sim}^*] = [u^*]|_{C_{0\sim}^*}, x_{\sim}^*, y_{\sim}^*)).$$

Similarly, consider the node on entangled tails as marking z_i^* to obtain quasimaps of type $(g, n, \beta) = (0, 1, (0, d_0))$ for $i = 1, \dots, l$,

$$\eta_i^* = ((\phi_i^* : \mathcal{E}_i^* \cong \mathcal{E}_i^*, f_i^*, z_i^*), (\mathcal{E}_i^*, [u_i^*], z_i^*)).$$

η_{\sim}^* does not contain any entangled tails. Hence by stability condition (Definition 3.15 conditions (1)-(2)), there are no constant tails and no length d_0 base points. Hence η_{\sim}^* is ϵ_+ -stable. In general, any stable extension can be obtained by gluing ϵ_+ -stable η_{\sim} and ϵ_0 -semistable η_i of type $(g, n, \beta) = (0, 1, (0, d_0))$ for $i = 1, \dots, l$ at markings z_i, y_i .

Let η_{\sim} and η_i be ϵ_0 -stable extensions (if they exist) of η_{\sim}^* and η_i^* respectively. By properness of $Q_{g,n+l}^{\epsilon_+}([E^{ss}/G], \beta - ld_0)$, we can take η_{\sim} to be the ϵ_+ -extension of η_{\sim}^* . However η_i^* is only ϵ_0 -semistable and does not have a unique extension. Still, there is a unique choice of η_1, \dots, η_l such that ξ is stable. To see this, note that f_i^* is a constant morphism. Hence we can consider each η_i^* to be an element of $Q_{0,1}^{ss}(X^{ss}, (0, d_0))$. Therefore, uniqueness follows by an argument similar to the one in [Zho22, Section 5.4].

4. Localization on Master Space

Consider the \mathbb{C}^* -action on $MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta)$ given by scaling the section v_1 : for $\lambda \in \mathbb{C}^*$,

$$\begin{aligned} \lambda \cdot ((\phi : C \rightarrow C_0, f, x), (C_0, x, [u], e, N, v_1, v_2)) \\ := ((\phi : C \rightarrow C_0, f, x), (C_0, x, [u], e, N, \lambda v_1, v_2)). \end{aligned}$$

4.1. Fixed components. We describe \mathbb{C}^* -fixed components of $MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta)$.

4.1.1. $v_1 = 0$. Let

$$F^+ \subset MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta)$$

denote the fixed component of $MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta)$ defined by $v_1 = 0$. By stability condition and forgetting the trivial data N, v_1, v_2 , we have

$$(4.1) \quad F^+ \cong \tilde{Q}_{g,n}^{\epsilon_+}([E^{ss}/G], \beta), \text{ under which } [F^+]^{\text{vir}} = [\tilde{Q}_{g,n}^{\epsilon_+}([E^{ss}/G], \beta)]^{\text{vir}}.$$

The calibration bundle will be the virtual normal bundle with a \mathbb{C}^* -action of weight 1.

4.1.2. $v_2 = 0$. Similarly when $v_2 = 0$, we have a fixed component denoted by

$$F^- \subset MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta).$$

By the stability condition, the underlying quasimap is ϵ_- -stable and hence does not have degree- $(0, d_0)$ rational tails. Therefore we have

$$(4.2) \quad F^- \cong Q_{g,n}^{\epsilon_-}([E^{ss}/G], \beta), \text{ under which } [F^-]^{\text{vir}} = [Q_{g,n}^{\epsilon_-}([E^{ss}/G], \beta)]^{\text{vir}}.$$

Note that for $g = 0, n = 1$ and $\deg(\beta) = (0, d_0)$, $Q_{g,n}^{\epsilon_-}([E^{ss}/G], \beta)$ is empty and $v_2 \neq 0$. When it is nonempty, the virtual normal bundle is the dual of the calibration bundle with \mathbb{C}^* -action of weight (-1) .

4.1.3. $v_1, v_2 \neq 0$, *Case I.* When $g = 0, n = 1, \deg(\beta) = (0, d_0)$, the curve is irreducible. The \mathbb{C}^* -fixed component F_β is given by

$$F_\beta = \{\xi \mid \text{the domain curve is a single fixed tail, } v_1 \neq 0, v_2 \neq 0\}.$$

Let C be the domain curve with the unique marked point x_\star . As $\deg(\beta) = (0, d_0)$, $f : C \rightarrow Y$ is a constant map (which means that the contraction map is an isomorphism), and $[u] : C \rightarrow X$ is a genus 0, 1-pointed quasimap to X^{ss} with degree β_0 such that $\deg(\beta_0) = (0, d_0)$. Let \mathfrak{r}_\star be the restriction of \mathfrak{r} at the unique point x_\star .

Fix a nonzero tangent vector v_∞ at $\infty \in \mathbb{P}^1$. There is a unique isomorphism $C \rightarrow \mathbb{P}^1$ mapping the marked point to ∞ , the base point to 0, and sending $(v_2/v_1)^{\otimes \mathfrak{r}_\star}$ to v_∞ . This along with the morphism $[u]$ determine a point in quasimap graph space $QG_{0,1}(X^{ss}, \beta_0)$ (whose definition is recalled in Section 1.2). As C is a fixed tail, this maps into the fixed component F_{\star, β_0} . Including constant map f to Y and taking the data over an arbitrary scheme, we have a morphism

$$F_\beta \rightarrow Y \times_{[W/S]} F_{\star, \beta_0}.$$

Denote $F_{\star, \beta} := Y \times_{[W/S]} F_{\star, \beta_0}$.

Lemma 4.1. *The morphism $F_\beta \rightarrow F_{\star, \beta}$ is étale of degree \mathfrak{r}_\star .*

Proof. Lifting a morphism $S \rightarrow Y \times_{[W/S]} F_{\star, \beta_0}$ to $S \rightarrow F_\beta$ is the same as choosing an \mathfrak{r}_\star -th root of v_∞ . Note that the point in Y completely determines the constant map f . \square

Lemma 4.2. *The pullback of $[F_{\star, \beta}]^{vir}$ along the above morphism is equal to $[F_\beta]^{vir}$ and*

$$\frac{1}{e_{\mathbb{C}^*}(N_{F_\beta/MQ_{0,1}^{\epsilon_0}([E^{ss}/G], \beta)}^{vir})} = (\mathfrak{r}_\star z) \cdot \mathbb{I}_{\beta_0}(\mathfrak{r}_\star z),$$

where

$$\mathbb{I}_{\tilde{\beta}_0}(z) := \frac{1}{e_{\mathbb{C}^*}(N_{F_{\star, \beta_0}/QG_{0,1}(X^{ss}, \beta_0)}^{vir})}.$$

Proof. Define F_{β_0} to be the image of the forgetful morphism $F_\beta \rightarrow F_{\beta_0}$ forgetting the data of constant map $f : C \rightarrow Y$. Then we have

$$F_\beta = Y \times_{[W/S]} F_{\beta_0}.$$

Note that F_{β_0} is the fixed component of the master space $MQ_{0,1}^{\epsilon_0}(X^{ss}, \beta_0)$ with target X^{ss} . Then the first result follows from [Zho22, Lemma 6.4.2].

The second expression follows by $MQ_{0,1}^{\epsilon_0}([E^{ss}/G], \beta) \cong Y \times_{[W/S]} MQ_{0,1}^{\epsilon_0}(X^{ss}, \beta_0)$ and [Zho22, Lemma 6.4.2]. \square

4.1.4. $v_1, v_2 \neq 0$, *Case II.* When $2g - 2 + n + \epsilon_0 d > 0$, the following lemma gives the \mathbb{C}^* -fixed components.

Lemma 4.3. *Let*

$$\xi = ((\phi : C \rightarrow C_0, f, x), (C_0, x, [u], e, N, v_1, v_2)) \in MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta)(\mathbb{C})$$

be an ϵ_0 -stable quasimap to $[E^{ss}/G]$ with calibrated tails such that $v_1 \neq 0$ and $v_2 \neq 0$. Then ξ is \mathbb{C}^ -fixed if and only if*

- (1) *there is at least one degree-(0, d_0) rational tail, and*
- (2) *each entangled tail is a fixed tail.*

Proof. As v_1, v_2 are nonzero, v_1/v_2 is a non-vanishing section of the calibration bundle. Let

$$\eta^\sim = ((\phi : C \rightarrow C_0, f, x), (C_0, x, [u], e))$$

be the underlying quasimap with entangled tails. By the construction of calibration bundle, ξ is fixed if and only if the action of $\text{Aut}(\eta^\sim)$ on the calibration bundle induces a surjection $\text{Aut}(\eta^\sim) \twoheadrightarrow \mathbb{C}^*$. The result then follows by Lemma 3.18. \square

For $k \geq 1$, let $\vec{d} := ((d', d''_0), (0, d''_1), \dots, (0, d''_k))$ be a $(k+1)$ -tuple of pairs of non-negative integers such that $\deg(\vec{\beta}) = (d', d''_0) + (0, d''_1) + \dots + (0, d''_k)$ and $d''_i = d_0$ for $i = 1, \dots, k$. Define $F_{\vec{d}} \subset MQ_{g,n}^\sim(X^{ss}, \vec{\beta})$ by

$$F_{\vec{d}} = \{\eta \mid \eta \text{ has exactly } k \text{ entangled tails}$$

$$\text{which are fixed tails with degrees } (0, d''_1), \dots, (0, d''_k)\}.$$

From equation (3.5), we have a projection map $MQ_{g,n}^\sim([E^{ss}/G], \beta) \xrightarrow{\text{pr}_2} MQ_{g,n}^\sim(X^{ss}, (d', \beta''))$. Pulling back $F_{\vec{d}}$ along pr_2 and restricting it to ϵ_0 -stable master space, we have the fixed components

$$F_{\vec{\beta}} := \text{pr}_2^*(F_{\vec{d}})|_{MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta)},$$

indexed by $(k+1)$ -tuple of effective curves $\vec{\beta} := (\bar{\beta}, \beta_1, \dots, \beta_k)$ such that $\beta = \bar{\beta} + \beta_1 + \dots + \beta_k$ with $\deg(\beta_i) = (0, d_0)$ for each i and $\deg(\bar{\beta}) = (d', d''_0)$.

Lemma 4.4. $F_{\vec{\beta}} \subset MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta)^{\mathbb{C}^*}$ *is an closed and open substack.*

Proof. This follows by noting that $F_{\vec{d}} \subset MQ_{g,n}^{\epsilon_0}(X^{ss}, (d', \vec{\beta}))^{\mathbb{C}^*}$ is closed and open substack [Zho22, Lemma 6.5.2]. \square

Let $\mathfrak{C}_k^* \subset \tilde{\mathfrak{M}}_{g,n,d}$ be the locally closed smooth substack with exactly k entangled tails. By [Zho22, Lemma 6.5.3], $F_{\vec{d}} \rightarrow \tilde{\mathfrak{M}}_{g,n,d}$ factors through \mathfrak{C}_k^* . Hence we have the following sequence of morphisms,

$$F_{\vec{\beta}} \rightarrow F_{\vec{d}} \rightarrow \mathfrak{C}_k^* \rightarrow \mathfrak{Z}_{(k)}.$$

Moreover, recall the gluing morphism (3.1) given by

$$\tilde{\mathfrak{gl}}_k : \tilde{\mathfrak{M}}_{g,n+k,d-kd_0} \times' (\mathfrak{M}_{0,1,d_0}^{\text{wt,ss}})^k \rightarrow \mathfrak{Z}_{(k)}.$$

Using the above two morphisms, we form a fibered diagram.

$$(4.3) \quad \begin{array}{ccccccc} \mathfrak{gl}_k^* F_{\vec{\beta}} & \longrightarrow & \mathfrak{gl}_k^* F_{\vec{d}} & \longrightarrow & \mathfrak{gl}_k^* \mathfrak{C}_{k-1}^* & \longrightarrow & \tilde{\mathfrak{M}}_{g,n+k,d-kd_0} \times' (\mathfrak{M}_{0,1,d_0}^{\text{wt,ss}})^k \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \mathfrak{gl}_k \\ F_{\vec{\beta}} & \longrightarrow & F_{\vec{d}} & \longrightarrow & \mathfrak{C}_k^* & \longrightarrow & \mathfrak{Z}_{(k)}. \end{array}$$

Let $\mathcal{C}_{\mathfrak{gl}_k^* F_{\vec{\beta}}}$ be the universal curve over $\mathfrak{gl}_k^* F_{\vec{\beta}}$. Then $\mathcal{C}_{\mathfrak{gl}_k^* F_{\vec{\beta}}}$ can be obtained by gluing $\mathcal{E}_1, \dots, \mathcal{E}_k$ to the last k -markings of $\mathcal{C}_{\vec{\beta}}$, where $\mathcal{E}_1, \dots, \mathcal{E}_k$ is the pullback of universal curve of $(\mathfrak{M}_{0,1,d_0}^{\text{wt,ss}})^k$ and $\mathcal{C}_{\vec{\beta}}$ is the pullback of $\tilde{\mathfrak{M}}_{g,n+k,d-kd_0}$. Let $p_i \in \mathcal{C}_{\mathfrak{gl}_k^* F_{\vec{\beta}}}$ be the node on \mathcal{E}_i . Let $T_{p_i} \mathcal{E}_i$ be the line bundle formed by the relative tangent bundle along p_i and denote $\Theta_i := T_{p_i} \mathcal{E}_i \otimes T_{p_i} \mathcal{C}_{\vec{\beta}}$. Then by [Zho22, Lemma 2.5.5] we have

$$\Theta_1 \cong \dots \cong \Theta_k =: \Theta.$$

Let $\mathbb{M}_{\vec{\beta}}$ be the calibration bundle on $\tilde{\mathfrak{M}}_{g,n+k,d-kd_0}$ and let $\mathbb{M}_{\vec{\beta}}$ be the calibration bundle on $F_{\vec{\beta}}$. Using a suitable pullback, we have a canonical isomorphism of line bundles on $\mathfrak{gl}_k^* F_{\vec{\beta}}$,

$$\mathbb{M}_{\vec{\beta}}^{\vee} \otimes \Theta_1 \otimes \dots \otimes \Theta_k \cong \mathbb{M}_{\vec{\beta}}^{\vee} \cong \mathcal{O}_{\mathfrak{gl}_k^* F_{\vec{\beta}}}^{v_2/v_1}.$$

Hence we have a canonical isomorphism on $\mathfrak{gl}_k^* F_{\vec{\beta}}$:

$$(4.4) \quad \Theta^{\otimes k} \cong \mathbb{M}_{\vec{\beta}}^{\vee}.$$

Note that $\mathcal{C}_{\vec{\beta}}$ does not have any length- d_0 relevant base points. Therefore by restricting universal quasimaps to $\mathcal{C}_{\vec{\beta}}$, we have

$$\mathfrak{gl}_k^* F_{\vec{\beta}} \rightarrow \tilde{Q}_{g,n+k}^{\epsilon+}([E^{ss}/G], \bar{\beta}).$$

Let $Z \rightarrow \tilde{Q}_{g,n+k}^{\epsilon+}([E^{ss}/G], \bar{\beta})$ be the stack of k -th roots of the pullback to $\tilde{Q}_{g,n+k}^{\epsilon+}([E^{ss}/G], \bar{\beta})$ of the line bundle $\mathbb{M}_{\vec{\beta}}^{\vee}$ and let $L \rightarrow Z$ be the universal k -th root. Then (4.4) gives rise to

$$(4.5) \quad \mathfrak{gl}_k^* F_{\vec{\beta}} \rightarrow Z.$$

Note that $\tilde{Q}_{g,n+k}^{\epsilon+}([E^{ss}/G], \bar{\beta}) \rightarrow \tilde{\mathfrak{M}}_{g,n+k,d-kd_0}$ factors through

$$\tilde{\mathcal{G}}_{g,n+k,\beta_0}^{\text{tw},d'} := \mathcal{G}_{g,n+k,\beta_0}^{\text{tw},d'} \times_{\mathfrak{M}_{g,n+k,\bar{d}}^{\text{wt,ss}}} \tilde{\mathfrak{M}}_{g,n+k,\bar{d}}.$$

Define $Z_V \rightarrow \tilde{\mathcal{G}}_{g,n+k,\beta_0}^{\text{tw},d'}$ to be the stack of k -th roots of the pullback of $\mathbb{M}_{\vec{\beta}}^{\vee}$. Then, there is a natural isomorphism

$$(4.6) \quad Z \cong Z_V \times_{\mathfrak{Bun}_S^{\text{tw,wt}}} \mathfrak{M}_{g,n+k}^{\text{tw,pre}}(Y, \beta').$$

Next we restrict our quasimaps to \mathcal{E}_i . By definition, it consists of a constant map $f_i : \mathcal{E}_i \rightarrow Y$ and a quasimap with calibrated tails $[u_i] : \mathcal{E}_i \rightarrow X$ of degree $(0, d_0)$ and a base point of length d_0 .

This allows us to express $\mathfrak{gl}_k^* F_{\vec{\beta}}|_{\mathcal{E}_i}$ as a space over Y with $\mathfrak{gl}_k^* F_{\vec{d}}|_{\mathcal{E}_i}$ as its fibers,

$$\begin{aligned} \mathfrak{gl}_k^* F_{\vec{\beta}}|_{\mathcal{E}_i} &\cong Y \times_{[W/S]} \mathfrak{gl}_k^* F_{\vec{d}}|_{\mathcal{E}_i}, \\ &\cong (Y \times_{[W/S]} I_{\mu}(X^{ss})) \times_{I_{\mu}(X^{ss})} \mathfrak{gl}_k^* F_{\vec{d}}|_{\mathcal{E}_i} \\ &\cong I_{\mu}([E^{ss}/G]) \times_{I_{\mu}(X)} \mathfrak{gl}_k^* F_{\vec{d}}|_{\mathcal{E}_i}. \end{aligned}$$

Let $V_i^* \rightarrow Z$ be the total space of $L^{\otimes r_i} \otimes (T_{p_i} C_{\vec{\beta}}^{\otimes -r_i})$ minus its zero section. Define

$$Z' := V_1^* \times_Z \cdots \times_Z V_k^*,$$

and twist \mathcal{E}_i with it to get

$$\mathcal{E}'_i := \mathcal{E}_i \times_Z Z' = \mathcal{E}_i \times_{\mathfrak{gl}_k^* F_{\vec{\beta}}} (\mathfrak{gl}_k^* F_{\vec{\beta}} \times_Z Z').$$

An R -point of $\mathfrak{gl}_k^* F_{\vec{\beta}} \times_Z Z'$ consists of a morphism $g : R \rightarrow \mathfrak{gl}_k^* F_{\vec{\beta}}$ along with nonvanishing sections $s_i \in H^0(R, g^*(T_{p_i} \mathcal{E}_i)^{\otimes r_i})$ for $i = 1, \dots, k$. Consider a $\mathbb{C}^* \times (\mathbb{C}^*)^k$ -action on \mathcal{E}'_i given by \mathbb{C}^* -action on \mathcal{E}_i induced by its action on master space (and trivial on Z') while $(\mathbb{C}^*)^k$ acts by scaling sections (s_1, \dots, s_k) (and acts trivially on \mathcal{E}_i).

There is a unique morphism (for each i in $1, \dots, k$)

$$g_i : \mathcal{E}'_i \rightarrow \mathbb{P}^1,$$

which maps the marking p_i to ∞ , the unique base point to 0, and the vector s_i to v_{∞} (a fixed nonzero tangent vector to \mathbb{P}^1 at ∞). Moreover, the $\mathbb{C}^* \times (\mathbb{C}^*)^k$ -action on \mathcal{E}'_i is compatible with the standard \mathbb{C}^* -action on \mathbb{P}^1 ([Zho22, Lemma 6.5.4]): Let $(\lambda, t) = (\lambda, t_1, \dots, t_k) \in \mathbb{C}^* \times (\mathbb{C}^*)^k$, then the diagram

$$(4.7) \quad \begin{array}{ccc} \mathcal{E}'_i & \xrightarrow{g_i} & \mathbb{P}^1 \\ (\lambda^k, t) \downarrow & & \downarrow \lambda^{r_i} t_i^{-1} \\ \mathcal{E}'_i & \xrightarrow{g_i} & \mathbb{P}^1 \end{array}$$

is commutative.

Recall the definition of quasimap graph space $QG_{0,1}(X^{ss}, (0, d_i))$ and let

$$QG_{0,1}^*(X^{ss}, (0, d_i)) \subset QG_{0,1}(X^{ss}, (0, d_i))$$

be an open substack where the domain curve is irreducible. Define

$$F_{*,d_i} \subset QG_{0,1}^*(X^{ss}, (0, d_i))$$

to be the fixed (with respect to canonical \mathbb{C}^* -action) component where the marking is at ∞ and a base point of length d_0 is at 0.

Now consider a family of curves $\mathcal{E}'_i \rightarrow \mathfrak{gl}_k^* F_{\vec{\beta}} \times_Z Z'$ along with $g_i : \mathcal{E}'_i \rightarrow \mathbb{P}^1$. This gives a morphism

$$\mathfrak{gl}_k^* F_{\vec{\beta}} \times_Z Z' \rightarrow QG_{0,1}(X^{ss}, \vec{\beta}_i),$$

where $\deg(\tilde{\beta}_i) = (0, d_i'')$. As \mathcal{E}'_i is a fixed tail, the morphism maps into \mathbb{C}^* -fixed locus $F_{*,\tilde{\beta}_i}$. As $F_{*,\tilde{\beta}_i}$ is \mathbb{C}^* -fixed, by diagram (4.7) we get that the morphism is invariant under the action of $(\mathbb{C}^*)^k$. Hence the map descends to

$$\tilde{\mathfrak{gl}}_k^* F_{\tilde{\beta}} \rightarrow F_{*,\tilde{\beta}_i}.$$

Combining the above map with the evaluation map at marked point p_i corresponding to \mathcal{E}_i , we define a morphism

$$(4.8) \quad \tilde{\mathfrak{gl}}_k^* F_{\tilde{\beta}} \rightarrow I_\mu([E^{ss}/G]) \times_{I_\mu(X^{ss})} F_{*,\tilde{\beta}_i},$$

where the map for the fiber product is given by evaluation map at special point $F_{*,\tilde{\beta}_i} \xrightarrow{ev} I_\mu(X^{ss})$ and induced map $I_\mu([E^{ss}/G]) \rightarrow I_\mu(X^{ss})$ on cyclotomic inertia stacks. Denote

$$F_{*,\beta_i} := I_\mu([E^{ss}/G]) \times_{I_\mu(X^{ss})} F_{*,\tilde{\beta}_i}.$$

Consider the evaluation map at the last k markings

$$ev_Z : Z \rightarrow (I_\mu[E^{ss}/G])^k$$

and composition

$$ev_{*,\beta_i} : F_{*,\beta_i} \rightarrow I_\mu[E^{ss}/G] \rightarrow I_\mu[E^{ss}/G],$$

where $F_{*,\beta_i} \rightarrow I_\mu[E^{ss}/G]$ is the projection map and $I_\mu[E^{ss}/G] \rightarrow I_\mu[E^{ss}/G]$ is the involution inverting the band. Using the above two maps, we can construct a fiber product

$$Z \times_{(I_\mu[E^{ss}/G])^k} \prod_{i=1}^k F_{*,\beta_i}.$$

By combining equations (4.5) and (4.8) we have a morphism

$$(4.9) \quad \varphi : \tilde{\mathfrak{gl}}_k^* F_{\tilde{\beta}} \rightarrow Z \times_{(I_\mu[E^{ss}/G])^k} \prod_{i=1}^k F_{*,\beta_i}.$$

Remark 4.5. Note that, by definition of F_{*,β_i} , we can rewrite

$$Z \times_{(I_\mu[E^{ss}/G])^k} \prod_{i=1}^k F_{*,\beta_i} = Z \times_{(I_\mu(X^{ss}))^k} \prod_{i=1}^k F_{*,\tilde{\beta}_i}$$

in terms of fixed locus data on graph quasimap to X .

Lemma 4.6. *The morphism φ is representable, finite, étale, of degree $\prod_{i=1}^k \mathfrak{r}_i$.*

Proof. Consider a faithfully flat cover $Z'' \rightarrow Z$ over which L and $T_{p_i}\mathcal{C}_{\tilde{\beta}}$ are trivialized. Given an R -point ξ of $Z'' \times_{(I_\mu[E^{ss}/G])^k} \prod_{i=1}^k F_{*,\beta_i}$, we want to show that its lifting to $Z'' \times_Z \tilde{\mathfrak{gl}}_k^* F_{\tilde{\beta}}$ is equivalent to the choice of an \mathfrak{r}_i -th root of a nonzero section of certain line bundle.

Let $\xi \in Z''(R) \times_{(I_\mu[E^{ss}/G])^k} \prod_{i=1}^k F_{*,\beta_i}(R)$. Glue the special point of rational tail \mathcal{E}''_i from $F_{*,\beta_i}(R)$ to the $(n+i)$ -th marked point of the underlying curve $\mathcal{C}''_{\tilde{\beta}}$ from $Z''(R)$. Similarly glue the quasimaps from both spaces. To get an R -point of $Z'' \times_Z \tilde{\mathfrak{gl}}_k^* F_{\tilde{\beta}}$, we need to choose the entanglement (i.e. a map $R \rightarrow \tilde{\mathfrak{M}}_{g,n,d}$) and define calibration bundle (N, v_1, v_2) .

Note that the entanglement and calibration bundle data (by definition) depend completely on the underlying curve of the quasimap to X^{ss} . Hence the lemma follows from a similar result [Zho22, Lemma 6.5.5] for quasimaps to X^{ss} . \square

Next we extend the group actions in diagram (4.7) over corresponding universal curves. Consider the fibered diagram

$$(4.10) \quad \begin{array}{ccc} Z' \times_Z \mathfrak{gl}_k^* F_{\vec{\beta}} & \xrightarrow{\varphi'} & Z' \times_{(I_\mu(X^{ss}))^k} \prod_{i=1}^k F_{*,\tilde{\beta}_i} \\ \downarrow p_1 & & \downarrow p_2 \\ \mathfrak{gl}_k^* F_{\vec{\beta}} & \xrightarrow{\varphi} & Z \times_{(I_\mu(X^{ss}))^k} \prod_{i=1}^k F_{*,\tilde{\beta}_i}. \end{array}$$

Let $\mathcal{C}_1 \rightarrow \mathfrak{gl}_k^* F_{\vec{\beta}}$ and $\mathcal{C}_2 \rightarrow Z \times_{(I_\mu(X^{ss}))^k} \prod_{i=1}^k F_{*,\tilde{\beta}_i}$ be the universal curves. Pulling back, we get an isomorphism of universal curves

$$(4.11) \quad \tilde{\varphi} : p_1^* \mathcal{C}_1 \rightarrow p_2^* \mathcal{C}_2.$$

We can glue $\mathbb{C}^* \times (\mathbb{C}^*)^k$ -action on \mathcal{E}_i for each $i = 1, \dots, k$ to get a $\mathbb{C}^* \times (\mathbb{C}^*)^k$ -action on $p_1^* \mathcal{C}_1$. Moreover, analogous to diagram (4.7), this action gives following commutative diagram: For any $(\lambda, t) = (\lambda, t_1, \dots, t_k) \in \mathbb{C}^* \times (\mathbb{C}^*)^k$, we have

$$(4.12) \quad \begin{array}{ccc} p_1^* \mathcal{C}_1 & \xrightarrow{\tilde{\varphi}} & p_2^* \mathcal{C}_2 \\ (\lambda^k, t) \downarrow & & \downarrow (\lambda^{r_1} t_1^{-1}, \dots, \lambda^{r_k} t_k^{-1}, t) \\ p_1^* \mathcal{C}_1 & \xrightarrow{\tilde{\varphi}} & p_2^* \mathcal{C}_2. \end{array}$$

This defines a $\mathbb{C}^* \times (\mathbb{C}^*)^k$ -action on $p_2^* \mathcal{C}_2$. To be precise, each $\lambda^{r_i} t_i^{-1}$ acts by the standard \mathbb{C}^* -action on \mathbb{P}^1 -component corresponding to \mathcal{E}_i and $t \in (\mathbb{C}^*)^k$ acts by the standard scaling of sections of Z' . In conclusion, $\tilde{\varphi} : p_1^* \mathcal{C}_1 \rightarrow p_2^* \mathcal{C}_2$ is a $\mathbb{C}^* \times (\mathbb{C}^*)^k$ -equivariant map, where the $\mathbb{C}^* \times (\mathbb{C}^*)^k$ -action on $p_2^* \mathcal{C}_2$ is given by vertical right arrow in the above diagram.

Now we have all we need to prove an important lemma. Let $[Z]^{\text{vir}} \in A_*(Z)$ be the flat pullback of $[\tilde{Q}_{g,n+k}^{\epsilon+}([E^{ss}/G], \vec{\beta})]^{\text{vir}}$ and let $[\mathfrak{gl}_k^* F_{\vec{\beta}}]^{\text{vir}}$ be the flat pullback of $[F_{\vec{\beta}}]^{\text{vir}}$. Let $\tilde{\psi}(\mathcal{E}_i)$ be the orbifold ψ -class of the rational tail \mathcal{E}_i at the unique node and let $\tilde{\psi}_{n+i}$ be the orbifold ψ -class of $\tilde{Q}_{g,n+k}^{\epsilon+}([E^{ss}/G], \vec{\beta})$ at the $(n+i)$ -th marking. Let $\psi(\mathcal{E}_i)$ and ψ_{n+i} be the corresponding coarse ψ -classes. Let $\mathcal{D}_i \subset \tilde{\mathfrak{M}}_{g,n,d}$ be the divisor defined by the closure of the locus where there are exactly i entangled tails. Let $I_{\tilde{\beta}_i}(z)$ be as defined in Section 1.2 for the quasimap graph space $QG_{0,1}(X^{ss}, \tilde{\beta}_i)$. Moreover, to simplify notation, we define

$$\mathbb{I}_{\tilde{\beta}_i}(z) := \frac{1}{e_{\mathbb{C}^*}(N_{F_{*,\beta}/QG_{0,1}(X^{ss},\beta)}^{\text{vir}})}.$$

Note that $I_{\tilde{\beta}_i}(z) = \mathbf{r}^2(e\hat{\nu})_*(\mathbb{I}_{\tilde{\beta}_i}(z) \cap [F_{*,\beta}]^{\text{vir}})$.

Lemma 4.7. *Via the morphism φ in (4.9), we have*

$$[\mathfrak{gl}_k^* F_{\vec{\beta}}]^{vir} = \varphi^*([Z]^{vir} \times_{(I_\mu[E^{ss}/G])^k} \prod_{i=1}^k [F_{*,\beta_i}]^{vir}),$$

and

$$\frac{1}{e_{\mathbb{C}^*}(N_{F_{\vec{\beta}}/MQ_{g,n}^{\epsilon_0}([E^{ss}/G],\beta)}|_{\mathfrak{gl}_k^* F_{\vec{\beta}}})} = \frac{\prod_{i=1}^k (\frac{\mathfrak{r}_i}{k} z + \psi(\mathcal{E}_i))}{-\frac{z}{k} - \tilde{\psi}(\mathcal{E}_1) - \tilde{\psi}_{n+1} - \sum_{i=k}^{\infty} |\mathcal{D}_i|} \cdot \boxtimes_{i=1}^k \mathbb{I}_{\tilde{\beta}_i}(\frac{\mathfrak{r}_i}{k} z + \psi(\mathcal{E}_i)).$$

Proof. Let \mathbb{E}_{MQ} be the absolute perfect obstruction theory on $MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta)$ (see Section 3.5.1). We have a distinguished triangle induced by $MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta) \rightarrow M\tilde{\mathfrak{M}}_{g,n,d}$:

$$\mathbb{L}_{M\tilde{\mathfrak{M}}_{g,n,d}} \rightarrow \mathbb{E}_{MQ} \rightarrow \mathbb{E}_1 \xrightarrow{+1}.$$

Restricting to $\mathfrak{gl}_k^* F_{\vec{\beta}}$ and taking \mathbb{C}^* -fixed components yield a distinguished triangle on $\mathfrak{gl}_k^* F_{\vec{\beta}}$,

$$(\mathbb{L}_{M\tilde{\mathfrak{M}}_{g,n,d}}|_{\mathfrak{gl}_k^* F_{\vec{\beta}}})^f \rightarrow (\mathbb{E}_{MQ}|_{\mathfrak{gl}_k^* F_{\vec{\beta}}})^f \rightarrow (\mathbb{E}_1)^f \xrightarrow{+1}.$$

By an argument similar to [Zho22, Lemma 6.5.6], we have

$$(\mathbb{L}_{M\tilde{\mathfrak{M}}_{g,n,d}}|_{\mathfrak{gl}_k^* F_{\vec{\beta}}})^f \cong \mathbb{L}_{\tilde{\mathfrak{M}}_{g,n+k,d-d_0k}}|_{\mathfrak{gl}_k^* F_{\vec{\beta}}}.$$

Thus $[\mathfrak{gl}_k^* F_{\vec{\beta}}]^{vir}$ is defined by the relative perfect obstruction theory $(\mathbb{E}_1)^f$ over $\tilde{\mathfrak{M}}_{g,n+k,d-d_0k}$. Moreover, similar to [Zho22, Equation (6.20)] we have

$$(4.13) \quad \frac{1}{e_{\mathbb{C}^*}((\mathbb{L}_{M\tilde{\mathfrak{M}}_{g,n,d}}^{\vee}|_{\mathfrak{gl}_k^* F_{\vec{\beta}}})^{mv})} = \frac{\prod_{i=1}^k (\frac{\mathfrak{r}_i}{k} z + \psi(\mathcal{E}_i))}{-\frac{z}{k} - \tilde{\psi}(\mathcal{E}_i) - \tilde{\psi}_{n+1} - \sum_{i=k}^{\infty} |\mathcal{D}_i|}.$$

Now we describe the obstruction theory for the right hand side. By Remark 4.5, we have $Z \times_{(I_\mu([E^{ss}/G])^k} \prod_{i=1}^k F_{*,\beta_i} \cong Z \times_{(I_\mu(X^{ss}))^k} \prod_{i=1}^k F_{*,\tilde{\beta}_i}$. Using equation (4.6), a construction similar to Section 3.5.1 gives an obstruction theory for Z . By [Zho22, Section 6.3] and a standard splitting-node argument, we have a perfect obstruction theory \mathbb{E}_2 for $Z \times_{(I_\mu([E^{ss}/G])^k} \prod_{i=1}^k F_{*,\beta_i}$ relative to $\tilde{\mathfrak{M}}_{g,n+k,d-d_0k}$. Finally, the virtual cycle is given by the relative perfect obstruction theory $(\mathbb{E}_2)^f$.

To relate \mathbb{E}_1 and \mathbb{E}_2 , we pull back the morphism φ to φ' as in diagram (4.10). By noting that the isomorphism $\tilde{\varphi}$ between universal curves (4.11) commutes with maps to Y and X , we have an isomorphism of obstruction theories

$$(4.14) \quad \alpha : p_1^* \mathbb{E}_1 \cong p_1^* \varphi^* \mathbb{E}_2.$$

As $\tilde{\varphi}$ is $\mathbb{C}^* \times (\mathbb{C}^*)^k$ -equivariant, so is α . As \mathbb{C}^* acts trivially on $Z' \times_Z \mathfrak{gl}_k^* F_{\vec{\beta}}$, we obtain an isomorphism of $(\mathbb{C}^*)^k$ -equivariant objects:

$$\alpha^{\mathbb{C}^*} : (p_1^* \mathbb{E}_1)^{\mathbb{C}^*} \cong (p_1^* \varphi^* \mathbb{E}_2)^{\mathbb{C}^*}.$$

As $(p_1^*\mathbb{E}_1)^{\mathbb{C}^*} = p_1^*\mathbb{E}_1^f$ and $(p_1^*\varphi^*\mathbb{E}_2)^{\mathbb{C}^*} = p_1^*\varphi^*\mathbb{E}_2^f$, we get an isomorphism between fixed parts

$$\mathbb{E}_1^f \cong \varphi^*\mathbb{E}_2^f.$$

This shows that they induce the same virtual fundamental class.

Next we relate the moving parts of \mathbb{E}_1 and \mathbb{E}_2 . Let

$$(z_1, \dots, z_k, w_1, \dots, w_k)$$

be the $(\mathbb{C}^*)^k \times (\mathbb{C}^*)^k$ -equivariant parameters. We obtain

$$\frac{1}{e_{(\mathbb{C}^*)^k \times (\mathbb{C}^*)^k}(p_2^*(\mathbb{E}_2^{\vee, \text{mv}}))} = \mathbb{I}_{\tilde{\beta}_1}(z_1) \boxtimes \dots \boxtimes \mathbb{I}_{\tilde{\beta}_k}(z_k).$$

Relating the $(\mathbb{C}^*)^k$ -action on $p_2^*\mathbb{E}_2$ to the \mathbb{C}^* -action on $p_1^*\mathbb{E}_1$ (see (4.12)), we have

$$\frac{1}{e_{\mathbb{C}^* \times (\mathbb{C}^*)^k}(p_1^*(\mathbb{E}_1^{\vee, \text{mv}}))} = \mathbb{I}_{\tilde{\beta}_1}\left(\frac{\mathbf{r}_1}{k}z - w_1\right) \boxtimes \dots \boxtimes \mathbb{I}_{\tilde{\beta}_k}\left(\frac{\mathbf{r}_k}{k}z - w_k\right).$$

By the canonical isomorphism between \mathbb{C}^* -equivariant intersection theory of $\mathfrak{gl}_k^* F_{\vec{\beta}}$ and the $\mathbb{C}^* \times (\mathbb{C}^*)^k$ -equivariant intersection theory of $Z' \times_Z \mathfrak{gl}_k^* F_{\vec{\beta}}$, we have

$$\frac{1}{e_{\mathbb{C}^*}(\mathbb{E}_1^{\vee, \text{mv}})} = \mathbb{I}_{\tilde{\beta}_1}\left(\frac{\mathbf{r}_1}{k}z - \psi(\mathcal{E}_1)\right) \boxtimes \dots \boxtimes \mathbb{I}_{\tilde{\beta}_k}\left(\frac{\mathbf{r}_k}{k}z - \psi(\mathcal{E}_k)\right).$$

This along with equation (4.13) gives us the result. \square

5. Wall-Crossing Formula

5.1. Outline. Recall that $X = [\tilde{V}/(S \times G)] = [(W \times V)/(S \times G)]$. We assume the ring

$$\bigoplus_{m=0}^{m=\infty} H^0(V, \mathcal{O}_V(m\theta))^{S \times G}$$

of $(S \times G)$ -invariants is generated by $H^0(V, \mathcal{O}_V(\theta))^{S \times G}$ as an $H^0(V, \mathcal{O}_V)^{S \times G}$ -algebra. This gives a map

$$(5.1) \quad X \rightarrow [(W \times \mathbb{C}^{N+1})/(S \times \mathbb{C}^*)],$$

where $N+1$ is the size of a basis for $H^0(V, \mathcal{O}_V(\theta))^{S \times G}$. By composing (5.1) with quasimaps to X , we have

$$Q_{g,n}^{\text{pre}}([E^{ss}/G], \beta) \rightarrow Q_{g,n}^{\text{pre}}(\tilde{\mathbb{P}}, (d', d'')),$$

where $\tilde{\mathbb{P}}$ is a fiber bundle over Y with \mathbb{P}^N as its fiber (see Section 2.5). This along with diagram (2.6) induce a map on ϵ_+ -stable quasimaps

$$(5.2) \quad i : Q_{g,n}^{\epsilon_+}([E^{ss}/G], \beta) \rightarrow Q_{g,n}^{\epsilon_+}(\tilde{\mathbb{P}}, (d', d'')).$$

Since fibers of $\tilde{\mathbb{P}}$ are isomorphic to \mathbb{P}^N and $f : C \rightarrow Y$ is constant on relevant rational tails, we can define a morphism (see Section 2.5 for construction)

$$c : Q_{g,n}^{\epsilon_+}(\tilde{\mathbb{P}}, (d', d'')) \rightarrow Q_{g,n}^{\epsilon_-}(\tilde{\mathbb{P}}, (d', d''))$$

by contracting all the degree-(0, d_0) relevant rational tails to length- d_0 base points. Using these morphism, we have

$$(5.3) \quad \begin{array}{ccc} Q_{g,n}^{\epsilon_+}([E^{ss}/G], \beta) & & Q_{g,n}^{\epsilon_-}([E^{ss}/G], \beta) \\ \downarrow i & & \downarrow i \\ Q_{g,n}^{\epsilon_+}(\tilde{\mathbb{P}}, (d', d'')) & \xrightarrow{c} & Q_{g,n}^{\epsilon_-}(\tilde{\mathbb{P}}, (d', d'')) \xrightarrow{c_{\epsilon_-}} Q_{g,n}^{0+}(\tilde{\mathbb{P}}, (d', d'')), \end{array}$$

where c_{ϵ_-} is defined as in equation (2.10).

The wall-crossing formula should relate two virtual fundamental classes $[Q_{g,n}^{\epsilon_+}([E^{ss}/G], \beta)]^{\text{vir}}$ and $[Q_{g,n}^{\epsilon_-}([E^{ss}/G], \beta)]^{\text{vir}}$. Since there is no natural morphism between these two moduli spaces, we use (5.3) to push forward both classes to $Q_{g,n}^{0+}(\tilde{\mathbb{P}}, (d', d''))$ and compare them there.

By virtual localization formula [GP99], [CKL17], we have

$$(5.4) \quad [MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta)]^{\text{vir}} = \sum_* (i_{F_*})_* \left(\frac{[F_*]^{\text{vir}}}{e_{\mathbb{C}^*}(N_{F_*/MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta)}^{\text{vir}})} \right),$$

where the sum is over all fixed components (F_+ , F_- and $F_{\vec{\beta}}$) and i_{F_*} is the inclusion of corresponding component into $MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta)$.

Define a morphism

$$\tau : MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta) \rightarrow Q_{g,n}^{0+}(\tilde{\mathbb{P}}, (d', d''))$$

by

- composing the underlying quasimap $Q_{g,n}^{\sim}([E/G], \beta) \rightarrow Q_{g,n}^{\sim}(X, (d', \beta'')) \rightarrow X$ with (5.1),
- taking the coarse moduli of the domain curve,
- taking the 0^+ -stabilization of the obtained quasimaps to $\tilde{\mathbb{P}}$.

Consider the trivial \mathbb{C}^* -action on $Q_{g,n}^{0+}(\tilde{\mathbb{P}}, (d', d''))$. Then τ is \mathbb{C}^* -equivariant. Pushing forward (5.4), we have

$$(5.5) \quad \sum_* \tau_* (i_{F_*})_* \left(\frac{[F_*]^{\text{vir}}}{e_{\mathbb{C}^*}(N_{F_*/MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta)}^{\text{vir}})} \right) = \tau_* [MQ_{g,n}^{\epsilon_0}([E^{ss}/G], \beta)]^{\text{vir}}.$$

The right hand side lies in $A_*(Q_{g,n}^{0+}(\tilde{\mathbb{P}}, (d', d''))) \otimes_{\mathbb{Q}} \mathbb{Q}[z]$, so the residue at $z = 0$ of the left hand side is zero.

To simplify the notation, we suppress the obvious pushforward. In particular, for any moduli stack M and morphism $\tau : M \rightarrow Q_{g,n}^{0+}(\tilde{\mathbb{P}}, (d', d''))$, we write

$$\int_{\beta} \alpha := \tau_*(\alpha \cap \beta), \quad \text{for } \alpha, \beta \in A_*(M).$$

5.2. Case $g = 0, n = 1$ and $d = d_0$. For the cyclotomic inertia stack

$$I_\mu([E^{ss}/G]) = \prod_r I_{\mu_r}([E^{ss}/G]),$$

we have a locally constant function that takes value r on $I_{\mu_r}([E^{ss}/G])$. Set $\mathbf{r}_1 := \text{ev}_1^*(r)$.

Lemma 5.1. For $l = 0, 1, 2, \dots$,

$$\int_{[Q_{0,1}^{\epsilon_+}([E^{ss}/G], \beta)]^{\text{vir}}} \mathbf{r}_1^2 \psi_1^l = \text{Res}_{z=0}(z^{l+1} I_{\beta_0}(z)).$$

Proof. Let $\tilde{\psi}_1$ be the corresponding orbifold ψ -class on the master space $MQ^{\epsilon_+}([E^{ss}/G], \beta)$. Applying localization formula to the master space, we have

$$\begin{aligned} \int_{[MQ_{0,1}^{\epsilon_0}([E^{ss}/G], \beta)]^{\text{vir}}} \tilde{\psi}_1^l &= \int_{[Q_{0,1}^{\epsilon_+}([E^{ss}/G], \beta)]^{\text{vir}}} \frac{\tilde{\psi}_1^l|_{Q_{0,1}^{\epsilon_+}([E^{ss}/G], \beta)}}{e_{\mathbb{C}^*}(N_{Q_{0,1}^{\epsilon_+}([E^{ss}/G], \beta)/MQ_{0,1}^{\epsilon_0}([E^{ss}/G], \beta)}^{\text{vir}})} \\ &\quad + \int_{[F_\beta]^{\text{vir}}} \frac{\tilde{\psi}_1^l|_{F_\beta}}{e_{\mathbb{C}^*}(N_{F_\beta/MQ_{0,1}^{\epsilon_0}([E^{ss}/G], \beta)}^{\text{vir}})}. \end{aligned}$$

Note that for $g = 0, n = 1, d = d_0$, $Q_{0,1}^{\epsilon_-}([E^{ss}/G], \beta)$ is empty. By Section 4.1.1, Lemmas 4.1, and 4.2, we have

$$\begin{aligned} \int_{[MQ_{0,1}^{\epsilon_0}([E^{ss}/G], \beta)]^{\text{vir}}} \tilde{\psi}_1^l &= \int_{[Q_{0,1}^{\epsilon_+}([E^{ss}/G], \beta)]^{\text{vir}}} \frac{(\psi_1^l/\mathbf{r}_1)^l}{-z + \alpha} + \int_{[F_{*,\beta}]^{\text{vir}}} \mathbf{r}_1^2 z^{l+1} \cdot (\mathbb{I}_{\beta_0}(\mathbf{r}_1 z)) \\ &= \int_{[Q_{0,1}^{\epsilon_+}([E^{ss}/G], \beta)]^{\text{vir}}} \frac{(\psi_1^l/\mathbf{r}_1)^l}{-z + \alpha} + z^{l+1} I_{\beta_0}(\mathbf{r}_1 z). \end{aligned}$$

Here α is the first Chern class of the calibration bundle on $Q_{0,1}^{\epsilon_+}([E^{ss}/G], \beta)$. Taking the residues on both sides gives

$$\int_{[Q_{0,1}^{\epsilon_+}([E^{ss}/G], \beta)]^{\text{vir}}} (\psi_1^l/\mathbf{r}_1)^l = \text{Res}_{z=0}(z^{l+1} I_{\beta_0}(\mathbf{r}_1 z)).$$

Applying the change of variable $z \mapsto z/\mathbf{r}_1$, we get the desired result. \square

5.3. Case $2g - 2 + n + \epsilon_0 d > 0$. By Lemma 4.7, contribution of $[F_{\tilde{\beta}}^-]$ in the residue of the left hand side of equation (5.5) is given by

$$(5.6) \quad \int_{[\tilde{\mathfrak{g}}_k^* F_{\tilde{\beta}}^-]^{\text{vir}}} \frac{\prod_{i=1}^k \mathbf{r}_i}{k!} \text{Res}_{z=0} \left(\frac{\prod_{i=1}^k (\frac{\mathbf{r}_i}{k} z + \psi(\mathcal{E}_i))}{-\frac{z}{k} - \tilde{\psi}(\mathcal{E}_1) - \tilde{\psi}_{n+1} - \sum_{i=k}^{\infty} [\mathcal{D}_i]} \cdot \boxtimes_{i=1}^k \mathbb{I}_{\tilde{\beta}_i}(\frac{\mathbf{r}_i}{k} z + \psi(\mathcal{E}_i)) \right).$$

Applying the change of variables

$$z \mapsto k(z - \tilde{\psi}(\mathcal{E}_1) - \tilde{\psi}_{n+1}) = \dots = k(z - \tilde{\psi}(\mathcal{E}_k) - \tilde{\psi}_{n+k})$$

and using $\mathbf{r}_i \tilde{\psi}_{n+i} = \psi_{n+i}$, (5.6) becomes

$$\int_{[\tilde{\mathbf{g}}_k^* F_{\vec{\beta}}]_{\text{vir}}} \frac{\prod_{i=1}^k \mathbf{r}_i}{(k-1)!} \text{Res}_{z=0} \left(\frac{\prod_{i=1}^k (\mathbf{r}_i z - \psi_{n+i})}{-z - \sum_{i=k}^{\infty} [\mathcal{D}_i]} \cdot \boxtimes_{i=1}^k \mathbb{I}_{\tilde{\beta}_i}(\mathbf{r}_i z - \psi_{n+i}) \right).$$

We push forward the expression along

$$\tilde{\mathbf{g}}_k^* F_{\vec{\beta}} \xrightarrow{\varphi} Z \times_{(I_{\mu}(X^{ss}))^k} \prod_{i=1}^k F_{*,d_i} \xrightarrow{pr_Z} Z \rightarrow \tilde{Q}_{n+k}^{\epsilon_+}([E^{ss}/G], \bar{\beta}).$$

Note that $\tilde{\mathbf{g}}_k^* F_{\vec{\beta}} \xrightarrow{\varphi} Z \times_{(I_{\mu}(X^{ss}))^k} \prod_{i=1}^k F_{*,d_i}$ has degree $\prod_{i=1}^k \mathbf{r}_i$ by Lemma 4.6 and $Z \rightarrow \tilde{Q}_{n+k}^{\epsilon_+}([E^{ss}/G], \bar{\beta})$ has degree $1/k$. By [Zho22, Lemma 2.7.3], the pullback of \mathcal{D}_i to $\tilde{\mathbf{g}}_k^* F_{\vec{\beta}}$ is equal to the pullback of boundary divisors \mathcal{D}'_{i-k} of $\tilde{\mathfrak{M}}_{g,n+k,d-kd_0}$. By Lemma 4.7 and the definition of I -coefficient, equation (5.6) becomes

$$(5.7) \quad \int_{[\tilde{Q}_{n+k}^{\epsilon_+}([E^{ss}/G], \bar{\beta})]_{\text{vir}}} \frac{1}{k!} \text{Res}_{z=0} \left(\frac{\prod_{i=1}^k \text{ev}_{n+1}^*((\mathbf{r}_i z - \psi_{n+i}) I_{\tilde{\beta}_i}(\mathbf{r}_i z - \psi_{n+i}))}{-z - \sum_{i=0}^{\infty} [\mathcal{D}'_i]} \right).$$

Remark 5.2. Recall that we define the integral by pushing it forward to $Q_{g,n}^{0+}(\tilde{\mathbb{P}}, (d', d''))$. For $M = \tilde{Q}_{n+k}^{\epsilon_+}([E^{ss}/G], \bar{\beta})$, we push it forward by composing

$$\begin{aligned} \tilde{Q}_{n+k}^{\epsilon_+}([E^{ss}/G], \bar{\beta}) &\rightarrow Q_{n+k}^{\epsilon_+}([E^{ss}/G], \bar{\beta}) \xrightarrow{i} Q_{n+k}^{\epsilon_+}(\tilde{\mathbb{P}}, (d', d'' - kd_0)) \xrightarrow{b_{\epsilon_+,k}} \\ &Q_n^{\epsilon_+}(\tilde{\mathbb{P}}, (d', d'')) \xrightarrow{c_{\epsilon_+}} Q_n^{0+}(\tilde{\mathbb{P}}, (d', d'')). \end{aligned}$$

Here $b_{\epsilon_+,k}$, c_{ϵ_+} are defined in Section 2.5.

Next we prove a lemma which will be used to integrate \mathcal{D}'_i .

Lemma 5.3. *For $s \geq 1$, $r = 1, 2, \dots$, we have*

$$(5.8) \quad \begin{aligned} &\int_{[\tilde{Q}_{n+k}^{\epsilon_+}([E^{ss}/G], \bar{\beta})]_{\text{vir}}} [\mathcal{D}'_r] \left(\sum_{i=0}^{\infty} [\mathcal{D}'_i] \right)^{s-1} \\ &= \sum_{\vec{\beta}'} \sum_J \int_{[\tilde{Q}_{n+k+r}^{\epsilon_+}([E^{ss}/G], \bar{\beta}'')]_{\text{vir}}} \frac{(-1)^{s-r}}{r!} \cdot \prod_{a=1}^r [\text{ev}_{n+k+a}^*((\mathbf{r}_{k+a} z - \psi_{n+k+a}) I_{\vec{\beta}'_a}(\mathbf{r}_{k+a} z - \psi_{n+k+a}))]_{-j_a-1}, \end{aligned}$$

where $\vec{\beta}' = (\vec{\beta}'', \vec{\beta}'_1, \dots, \vec{\beta}'_r)$ runs through all $(r+1)$ -tuples of effective curve classes such that $\bar{\beta} = \vec{\beta}'' + \sum_{a=1}^r \vec{\beta}'_a$, $\text{deg}(\vec{\beta}'_a) = (0, d_0)$ for $a = 1, \dots, r$; $J = (j_1, \dots, j_r)$ runs through all r -tuples of non-negative integers such that $\sum_{a=1}^r j_a = s - r$.

Proof. By Lemma 3.11 we have

$$\int_{[\tilde{Q}_{n+k}^{\epsilon_+}([E^{ss}/G], \bar{\beta})]_{\text{vir}}} [\mathcal{D}'_r] \left(\sum_{i=0}^{\infty} [\mathcal{D}'_i] \right)^{s-1} = \sum_{\vec{\beta}'} \frac{\prod_{a=1}^r \mathbf{r}_{n+k+a}}{r!} \int_{Q_{\vec{\beta}'}} p_* \left(\left(\sum_{i=0}^{\infty} [\mathcal{D}'_i] \right)^{s-1} \right),$$

where $Q_{\vec{\beta}'} := [\tilde{Q}_{g,n+k+r}^{\epsilon+}([E^{ss}/G], \vec{\beta}')]^{\text{vir}} \times_{(I_\mu[E^{ss}/G])^r} \prod_{a=1}^r [Q_{0,1}^{\epsilon+}([E^{ss}/G], \vec{\beta}'_a)]^{\text{vir}}$ and

$$p : \tilde{\mathfrak{g}}_r^* \mathcal{D}'_r \times_{\tilde{\mathfrak{m}}_{g,n+k,d-kd_0}} \tilde{Q}_{g,n+k}^{\epsilon+}([E^{ss}/G], \vec{\beta}) \rightarrow \bigsqcup_{\vec{\beta}'} Q_{\vec{\beta}'}$$

is the inflated projective bundle in the sense of Section 1.3 (by [Zho22, Lemma 2.7.2] and Diagram (3.3)). Now by [Zho22, Lemma 2.7.4], [Zho22, Lemma A.0.1], and the projection formula, the left hand side of equation (5.8) is equal to

$$\sum_{\vec{\beta}'} \sum_J \frac{\prod_{a=1}^r \mathfrak{t}_{n+k+a}}{r!} \int_{Q_{\vec{\beta}'}} (-1)^{s-r} \prod_{a=1}^r (\tilde{\psi}_{n+k+a} + \tilde{\psi}'_a)^{j_a},$$

where $\tilde{\psi}_{n+k+a}$ is the orbifold ψ -class of $\tilde{Q}_{g,n+k+r}^{\epsilon+}([E^{ss}/G], \vec{\beta}')$ at the $(n+k+a)$ -th marking, and $\tilde{\psi}'_a$ is the orbifold ψ -class of $Q_{0,1}^{\epsilon+}([E^{ss}/G], \vec{\beta}'_a)$.

Integrating the powers of $\tilde{\psi}'_a$ against $[Q_{0,1}^{\epsilon+}([E^{ss}/G], \vec{\beta}'_a)]^{\text{vir}}$ using Lemma 4.2, the expression on the left hand side of (5.8) becomes

$$\sum_{\vec{\beta}'} \sum_J \int_{[\tilde{Q}_{g,n+k+r}^{\epsilon+}([E^{ss}/G], \vec{\beta}')]^{\text{vir}}} \frac{(-1)^{s-r}}{r!} \prod_{a=1}^r \sum_{b=0}^{j_a} \binom{j_a}{b} \tilde{\psi}_{n+k+a}^b \text{ev}_{n+k+a}^* [\mathfrak{t}z I_{\vec{\beta}'_a}(\mathfrak{t}z)]_{b-j_a-1}.$$

Here we have suppressed the subscripts of \mathfrak{t} . Finally applying the change of variables $z \mapsto z - \tilde{\psi}$, we get the result. \square

Expanding

$$\frac{1}{-z - \sum_{i=0}^{\infty} [\mathcal{D}'_i]} = \frac{-1}{z} + \sum_{s \geq 1} \sum_{r=1}^{\infty} (-z)^{-s-1} [\mathcal{D}'_r] \left(\sum_{i=1}^{\infty} [\mathcal{D}'_i] \right)^{s-1}$$

and using Lemma 5.3, we get

Corollary 5.4. *The contribution to the left hand side of (5.5) from $F_{\vec{\beta}}$ is*

$$(5.9) \quad -\frac{1}{k!} \sum_{r=0}^{\infty} \sum_{\vec{\beta}'} \sum_{\vec{b}} \frac{(-1)^r}{r!} \int_{[\tilde{Q}_{n+k+r}^{\epsilon+}([E^{ss}/G], \vec{\beta}')]^{\text{vir}}} \left[\prod_{i=1}^k \text{ev}_{n+i}^* ((\mathfrak{t}_i z - \psi_{n+i}) I_{\vec{\beta}'_i}(\mathfrak{t}_i z - \psi_{n+i})) \right]_{b_0} \cdot \prod_{a=1}^r \text{ev}_{n+k+a}^* [(\mathfrak{t}_{k+a} z - \psi_{n+k+a}) I_{\vec{\beta}'_a}(\mathfrak{t}_{k+a} z - \psi_{n+k+a})]_{b_a},$$

where $\vec{\beta}'$ is similar to Lemma 5.3 and $\vec{b} = (b_0, \dots, b_r)$ runs through a $(r+1)$ -tuple of integers such that $b_0 + \dots + b_r = 0$ and $b_1, \dots, b_r < 0$.

Now we show the main theorem.

Theorem 5.5 (Wall-crossing formula).

$$\begin{aligned} & \int_{[Q_{g,n}^{\epsilon_-}([E^{ss}/G],\beta)]^{\text{vir}}} 1 - \int_{[Q_{g,n}^{\epsilon_+}([E^{ss}/G],\beta)]^{\text{vir}}} 1 \\ &= \sum_{k \geq 1} \sum_{\bar{\beta}} \frac{1}{k!} \int_{[Q_{g,n+k}^{\epsilon_+}([E^{ss}/G],\bar{\beta})]^{\text{vir}}} \prod_{i=1}^k \text{ev}_{n+i}^* [(z - \psi_{n+i}) I_{\bar{\beta}_i}(z - \psi_{n+i})]_0 \end{aligned}$$

where $\vec{\beta} = (\bar{\beta}, \beta_1, \dots, \beta_k)$ runs through all the $(k+1)$ -tuples of effective curve classes such that $\beta = \bar{\beta} + \beta_1 + \dots + \beta_k$ and $\deg(\beta_i) = (0, d_0)$ for $i = 1, \dots, k$.

Proof. The sum of residues at $z = 0$ of the left hand side of equation (5.5) is zero. Contribution of each fixed term is given by Section 4.1.1 (and Lemma 3.10), Section 4.1.2, and Corollary 5.4. Replace $k+r$ by k in equation (5.9) and rearrange the coefficient to get

$$\begin{aligned} & \int_{[Q_{g,n}^{\epsilon_-}([E^{ss}/G],\beta)]^{\text{vir}}} 1 - \int_{[Q_{g,n}^{\epsilon_+}([E^{ss}/G],\beta)]^{\text{vir}}} 1 \\ & - \sum_{k \geq 1} \sum_{\bar{\beta}} \sum_{r=0}^{k-1} \sum_{\bar{b}} \frac{(-1)^r}{r!(k-r)!} \int_{[Q_{g,n+k}^{\epsilon_+}([E^{ss}/G],\bar{\beta})]^{\text{vir}}} \prod_{i=1}^k \text{ev}_{n+i}^* [(\mathbf{r}_i z - \psi_{n+i}) I_{\bar{\beta}_i}(\mathbf{r}_i z - \psi_{n+i})]_{b_i} = 0, \end{aligned}$$

where $\bar{\beta}$ is as above and $\bar{b} = (b_1, \dots, b_k)$ runs through all k -tuples such that $b_1 + \dots + b_k = 0$ and $b_{k-r+1}, \dots, b_k < 0$. Using the symmetry of the last k -markings, we can rewrite the expression as

$$\begin{aligned} & \int_{[Q_{g,n}^{\epsilon_-}([E^{ss}/G],\beta)]^{\text{vir}}} 1 - \int_{[Q_{g,n}^{\epsilon_+}([E^{ss}/G],\beta)]^{\text{vir}}} 1 = \\ & \sum_{k \geq 1} \sum_{\bar{\beta}} \sum_{N \subsetneq \{1, \dots, k\}} \sum_{\bar{b}} \frac{(-1)^{\#N}}{k!} \int_{[Q_{g,n+k}^{\epsilon_+}([E^{ss}/G],\bar{\beta})]^{\text{vir}}} \prod_{i=1}^k \text{ev}_{n+i}^* [(\mathbf{r}_i z - \psi_{n+i}) I_{\bar{\beta}_i}(\mathbf{r}_i z - \psi_{n+i})]_{b_i} = 0. \end{aligned}$$

Here $b_1 + \dots + b_k = 0$ with $b_i < 0$ for each $i \in N$. For each fixed k, \bar{b} , note that

$$\sum_N (-1)^{\#N} = \begin{cases} 1, & \text{if } b_i \geq 0 \text{ for all } i = 1, \dots, k; \\ 0, & \text{otherwise,} \end{cases}$$

where the sum is over all $N \subsetneq \{1, \dots, k\}$ such that $b_i < 0$ for each $i \in N$. This along with $b_1 + \dots + b_k = 0$ forces each b_i that appear in the expression to be zero. Finally applying the change of variables $\mathbf{r}_i z \mapsto z$ (which does not change the degree-0 term) we get the desired result. \square

Remark 5.6. Theorem 5.5 may be extended to include insertions by working with moduli spaces with additional marked points, c.f. [Zho22, Remark 1.11.2]. We leave the precise formulation of this extension to the interested readers.

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