# Symmetric deformed 2D/3D Hurwitz-Kontsevich model and affine Yangian of $\mathfrak{gl}(1)$

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#### Abstract

Since the  $(\beta$ -deformed) Hurwitz Kontsevich model corresponds to the special case of affine Yangian of  $\mathfrak{gl}(1)$ . In this paper, we construct two general cases of the  $\beta$ -deformed Hurwitz Kontsevich model. We find that the W-operators of these two models can be represented by the generators  $e_k$ ,  $f_k$ ,  $\psi_k$  of the affine Yangian of  $\mathfrak{gl}(1)$ , and the eigenstates (the symmetric functions  $Y_\lambda$  and 3-Jack polynomials) can be obtained from the 3D Young diagram representation of affine Yangian of  $\mathfrak{gl}(1)$ . Then we can see that the W-operators and eigenstates are symmetric about the permutations of coordinate axes.

**Keywords:** Hurwitz-Kontsevich model, Affine Yangian, W operators, Jack polynomials, 3-Jack polynomials.

### 1 Introduction

The affine Yangian of  $\mathfrak{gl}(1)$  appears independently in the work of Maulik-Okounkov [1] and Schiffmann-Vasserot [2] in connection with the AGT conjecture. The affine Yangian of  $\mathfrak{gl}(1)$  is isomorphic to the universal enveloping algebra of  $W_{1+\infty}[3]$ . They can be obtained from the Miura transformation and Yang-Baxter equation[4]. The Miura transformation shows that one can think of  $W_N$ -algebra as being a quantization of the space of N-th order differential operators[5]. The specific free field embedding of  $W_N$  in the Fock space of N free bosons depends on the ordering of the Boson fields, we find this ordering corresponds to the ordering  $j=1,2,\cdots,N$  of the slices of a 3D Young diagram on the plane z=j[6]. The Yang-Baxter equation show that the R-matrix  $R_{12}$  satisfies[5, 7]

$$R_{12}(\alpha_0 \partial_z + J_1(z))(\alpha_0 \partial_z + J_2(z)) = (\alpha_0 \partial_z + J_2(z))(\alpha_0 \partial_z + J_1(z))R_{12}$$

where the OPE of Boson field  $J_i(z)$  is

$$J_i(z)J_j(w) \sim \frac{\delta_{i,j}}{(z-w)^2},$$

The monodromy operator is  $\mathcal{T}_A = \mathcal{R}_{A1}\mathcal{R}_{A2}\cdots\mathcal{R}_{AN}$ . The Hamiltonian operator  $\mathcal{H}$  equals the vacuum expectation of  $\mathcal{T}_A$  in the Bosonic Fock space  $\mathcal{F}_A$ . The eigenstates of  $\mathcal{H}$  are 3-Jack polynomials[8].

3-Jack polynomials are symmetric functions defined on 3D Young diagrams [8]. In special case  $h_1 = h, h_2 = -h^{-1}$ , where  $h_1, h_2$  are parameters in the affine Yangian of

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 $\mathfrak{gl}(1)$ , 3-Jack polynomials become Jack polynomials on 2D Young diagrams[8], and when h=1, Jack polynomials become Schur functions[9]. Jack polynomials are the eigenstates of the Hurwitz operator  $W_0$  in the  $\beta$ -deformed Hurwitz-Kontsevich model[10]. The  $\beta$ -deformed Hurwitz-Kontsevich model, as the deformation of the Kontsevich model (the eigenstates are Schur functions), describes the Hurwitz numbers and Hodge integrals over the moduli space of complex curves [11, 12, 13]. W-representations give the dual expressions for the partition functions through differentiation rather than integration [11]. The W operators  $W_n$ ,  $n \in \mathbb{Z}$  can be represented by Bosons. The matrix models are generated by the W-operators. The Hurwitz-Kontsevich model (58) is a matrix model[11]

$$Z_0\{p\} = \int_{N \times N} \sqrt{\det\left(\frac{\sinh(\frac{\phi \otimes I - I \otimes \phi}{2})}{\frac{\phi \otimes I - I \otimes \phi}{2}}\right)} d\phi e^{-\frac{1}{2t}\operatorname{Tr}\phi^2 - \frac{N}{2}\operatorname{Tr}\phi - \frac{1}{6}tN^3 + \frac{1}{24}tN + \operatorname{Tr}(e^{\phi}\psi)},$$

where  $\psi$  is an  $N \times N$  matrix and the time variables  $p_k = \text{Tr}\psi^k$ , which can be represented by the exponent of  $W_0$  acting on the function  $e^{p_1/e^{tN}}$ . The  $N \times N$  complex matrix model [14, 15] and the Gaussian hermitian one-matrix model [11, 15] are generated by  $e^{W_{-n}/n} \cdot 1$  for n = 1, 2 respectively. Similarly for the  $\beta$ -deformed case.

In [16], we find that the W operators in Hurwitz-Kontsevich model can be represented by the generators  $e_k$ ,  $f_k$ ,  $\psi_k$  of the affine Yangian of  $\mathfrak{gl}(1)$  in the special case  $h_1=1,h_2=-1$ 

$$W_0 = \frac{1}{6}\psi_3 + \frac{N}{2}\psi_2, \ E_1 = e_1 + Ne_0, \ E_{-1} = -f_1 - Nf_0.$$
 (1)

Then their representation space are the space of Schur functions. The W operators in  $\beta$ -deformed Hurwitz-Kontsevich model can be represented by the generators  $e_k$ ,  $f_k$ ,  $\psi_k$  of the affine Yangian of  $\mathfrak{gl}(1)$  in the special case  $h_1 = h$ ,  $h_2 = -h^{-1}$ 

$$W_0 = \frac{1}{6}\sqrt{\beta}\psi_3 + \frac{1}{2}(\beta N - \frac{1}{3}(1-\beta))\psi_2, \ E_1 = e_1 + \sqrt{\beta}Ne_0, \ E_{-1} = -f_1 - \sqrt{\beta}Nf_0, \ (2)$$

where  $\beta = h^{-2}$ . Then their representation space are the space of Jack polynomials. Other W operators can be obtained from  $W_0$ ,  $E_1$ ,  $E_{-1}$ .

Clearly, there should be a general Hurwitz-Kontsevich model which corresponds to the affine Yangian of  $\mathfrak{gl}(1)$ . In this paper, we construct two general cases of the Hurwitz-Kontsevich model. One is called the symmetric deformed 2D Hurwitz-Kontsevich model, the operators and the eigenstates are symmetric about x-axis and y-axis in coordinate system xOy. The other one is called the symmetric deformed 3D Hurwitz-Kontsevich model, the operators and the eigenstates are symmetric about x-axis, y-axis and z-axis in coordinate system O-xyz. Since the  $\beta$ -deformed Hurwitz-Kontsevich model describes the Hurwitz numbers and Hodge integrals over the moduli space of complex curves, the symmetric deformed 2D/3D Hurwitz-Kontsevich model should describe the Hurwitz numbers and Hodge integrals over a more general case.

The paper is organized as follows. In section 2, we recall the definition of affine Yangian of  $\mathfrak{gl}(1)$  from Yang-Baxter equation, which is copied from section 2 of paper [16]. In section 3, we construct the symmetric deformed 2D Hurwitz-Kontsevich model, and represent the W operators and eigenstates by the affine Yangian of  $\mathfrak{gl}(1)$ . In section 4, we construct the symmetric deformed 3D Hurwitz-Kontsevich model, and also represent the W operators and eigenstates by the affine Yangian of  $\mathfrak{gl}(1)$ .

## 2 Affine Yangian of $\mathfrak{gl}(1)$

In this section, we review affine Yangian of  $\mathfrak{gl}(1)$  from the Yang-Baxter equation, which is copied from section 2 of [16]. Introduce three complex numbers  $h_1, h_2, h_3$  which satisfy

 $h_1 + h_2 + h_3 = 0$ , and

$$\sigma_2 = h_1 h_2 + h_1 h_3 + h_2 h_3, 
\sigma_3 = h_1 h_2 h_3.$$

Let Bosons  $a_{j,m}$  satisfy

$$[a_{i,n}, a_{j,m}] = -\frac{h_{\sigma}}{\sigma_3} \delta_{i,j} m \delta_{n+m,0}, \tag{3}$$

where  $\sigma=1,2,3$ . Then the OPE of current  $J_j(z)=\sum_{m\in\mathbb{Z}}a_{j,m}z^{-m-1}$  is

$$J_j(z)J_k(w) \sim -\frac{h_\sigma}{\sigma_3} \frac{\delta_{jk}}{(z-w)^2}.$$
 (4)

We say  $J_j(z)$  is of type  $\sigma$ . Remark that here  $h_{\sigma}$ ,  $\sigma = 1, 2, 3$  associate to coordinate axes x-axis, y-axis, z-axis, our results are always symmetric about these three axes, then we always choose  $\sigma = 3$  to express our results, which means that we choose z-axis as the preferred one.

 $\mathcal{R}$ -matrix satisfies the equation[5]

$$\mathcal{R}_{12}(\alpha_0 \partial + J_1(z))(\alpha_0 \partial + J_2(z)) = (\alpha_0 \partial + J_2(z))(\alpha_0 \partial + J_1(z))\mathcal{R}_{12}. \tag{5}$$

where  $J_1(z)$  and  $J_2(z)$  are of type  $\sigma$  and  $\tau$  respectively. Let

$$J_{-}(z) = h_{\tau} J_{1}(z) - h_{\sigma} J_{2}(z) = \sum_{j \in \mathbb{Z}} a_{12,m} z^{-m-1},$$

where 1 and 2 in  $a_{12,m}$  correspond to that in  $J_1(z)$  and  $J_2(z)$ . When there is no ambiguity, we omit them. The OPE of  $J_-(z)$  is

$$J_{-}(z)J_{-}(w) \sim \rho \frac{\delta_{jk}}{(z-w)^2}$$
 (6)

with

$$\rho = -\frac{h_{\tau}h_{\sigma}(h_{\tau} + h_{\sigma})}{\sigma_3}.$$

Let

$$\mathcal{R} = 1 + \frac{R^{(1)}}{a_0} + \frac{R^{(2)}}{a_0^2} + \cdots, \tag{7}$$

From (5), the expression of  $R^{(n)}$  can be obtained. One can find that of the first five terms in [5]. We list the first two terms

$$R^{(1)} = -\sum_{k>0} a_{-k} a_k, \tag{8}$$

$$R^{(2)} = \frac{1}{2} \left( \sum_{i,k>0} (a_{-j-k} a_j a_k + a_{-j} a_{-k} a_{j+k} + a_{-j} a_{-k} a_j a_k) + \rho \sum_{i>0} j a_{-j} a_i \right), \quad (9)$$

here  $a_j$  is  $a_{12,j}$  defined above.

The R-matrix satisfies the Yang-baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.\tag{10}$$

We consider N Fock spaces  $\mathcal{F}_j$  and one additional auxiliary space  $\mathcal{F}_A$  as in paper [5]. Every current  $J_j(z)$  for  $j = 1, 2, \dots, N$  is of type  $\sigma$  which acts on  $\mathcal{F}_j$ , and  $J_A(z)$  is of type  $\tau$  which acts on  $\mathcal{F}_A$ . The monodromy operator

$$\mathcal{T}_A = \mathcal{R}_{A1} \mathcal{R}_{A2} \cdots \mathcal{R}_{AN} \tag{11}$$

satisfies

$$\mathcal{R}_{AB}\mathcal{T}_{A}\mathcal{T}_{B} = \mathcal{T}_{B}\mathcal{T}_{A}\mathcal{R}_{AB},\tag{12}$$

which means that  $\mathcal{T}_A$  satisfies the Yang-Baxter equation again. Define

$$\mathcal{H} \equiv \langle 0|_A \mathcal{T}_A |0\rangle_A,\tag{13}$$

$$\mathcal{E} \equiv \langle 0|_A \mathcal{T}_A a_{A,-1} |0\rangle_A,\tag{14}$$

$$\mathcal{F} \equiv \langle 0|_A a_{A,1} \mathcal{T}_A |0\rangle_A,\tag{15}$$

$$\mathcal{H}_{\square} \equiv \langle 0|_A a_{A,1} \mathcal{T}_A a_{A,-1} |0\rangle_A. \tag{16}$$

The parameters in these currents are  $a_{Aj,0}$ . We change the parameters to u by the relation

$$-\frac{a_{Aj,0}\sigma_3}{h_{\tau}h_{\sigma}} = u - q_j - \frac{h_{\tau} - h_{\sigma}}{2},\tag{17}$$

For convenience, we let  $q_j = 0$ , which means that the zero mode  $a_{j,0} = 0$ . Define

$$e(u) = h_{\tau}^{-1}(\mathcal{H}(u))^{-1}\mathcal{E}(u),$$
 (18)

$$f(u) = -h_{\tau}^{-1} \mathcal{F}(u) (\mathcal{H}(u))^{-1}, \tag{19}$$

$$\psi(u) = h_{\tau}^{-1}(\mathcal{H}_{\square}(u - h_{\tau}) - \mathcal{E}(u - h_{\tau})(\mathcal{H}(u - h_{\tau}))^{-1}\mathcal{F}(u - h_{\tau}))(\mathcal{H}(u - h_{\tau}))^{-1}(20)$$

They generate affine Yangian of  $\mathfrak{gl}(1)$ . Introduce the generating functions:

$$e(u) = \sum_{j=0}^{\infty} \frac{e_j}{u^{j+1}}, \ f(u) = \sum_{j=0}^{\infty} \frac{f_j}{u^{j+1}}, \ \psi(u) = 1 + \sigma_3 \sum_{j=0}^{\infty} \frac{\psi_j}{u^{j+1}}.$$
 (21)

The mode operators satisfy[17, 18]

$$[\psi_i, \psi_k] = 0, \tag{22}$$

$$[e_{j+3}, e_k] - 3[e_{j+2}, e_{k+1}] + 3[e_{j+1}, e_{k+2}] - [e_j, e_{k+3}]$$

$$+\sigma_2[e_{i+1}, e_k] - \sigma_2[e_i, e_{k+1}] - \sigma_3\{e_i, e_k\} = 0, \tag{23}$$

$$[f_{j+3}, f_k] - 3[f_{j+2}, f_{k+1}] + 3[f_{j+1}, f_{k+2}] - [f_j, f_{k+3}]$$

$$+\sigma_2[f_{i+1}, f_k] - \sigma_2[f_i, f_{k+1}] + \sigma_3\{f_i, f_k\} = 0, \tag{24}$$

$$[e_j, f_k] = \psi_{j+k},\tag{25}$$

$$[\psi_{j+3}, e_k] - 3[\psi_{j+2}, e_{k+1}] + 3[\psi_{j+1}, e_{k+2}] - [\psi_j, e_{k+3}]$$

$$+\sigma_2 \left[ \psi_{j+1}, e_k \right] - \sigma_2 \left[ \psi_j, e_{k+1} \right] - \sigma_3 \left\{ \psi_j, e_k \right\} = 0, \tag{26}$$

$$[\psi_{j+3}, f_k] - 3[\psi_{j+2}, f_{k+1}] + 3[\psi_{j+1}, f_{k+2}] - [\psi_j, f_{k+3}]$$

$$+\sigma_2 \left[ \psi_{i+1}, f_k \right] - \sigma_2 \left[ \psi_i, f_{k+1} \right] + \sigma_3 \left\{ \psi_i, f_k \right\} = 0, \tag{27}$$

The affine Yangian  $\mathcal{Y}$  of  $\mathfrak{gl}(1)$  is the associative algebra with generators  $e_j$ ,  $f_j$  and  $\psi_j$ ,  $j = 0, 1, \ldots$  satisfying the above relations[7, 19] and boundary conditions

$$[\psi_0, e_i] = 0, [\psi_1, e_i] = 0, [\psi_2, e_i] = 2e_i, \tag{28}$$

$$[\psi_0, f_i] = 0, [\psi_1, f_i] = 0, [\psi_2, f_i] = -2f_i, \tag{29}$$

and a generalization of Serre relations

$$\operatorname{Sym}_{(j_1, j_2, j_3)} [e_{j_1}, [e_{j_2}, e_{j_3+1}]] = 0, \tag{30}$$

$$Sym_{(j_1,j_2,j_3)}[f_{j_1},[f_{j_2},f_{j_3+1}]] = 0, (31)$$

where Sym is the complete symmetrization over all indicated indices which include 6 terms.

The affine yangian  $\mathcal{Y}$  has a representation on the 3D Young diagrams. In order to describe the representation, as in our paper [20], we use the following notations. For a 3D Young diagram  $\pi$ , the notation  $\square \in \pi^+$  means that this box is not in  $\pi$  and can be added to  $\pi$ . Here "can be added" means that when this box is added, it is still a 3D Young diagram. The notation  $\square \in \pi^-$  means that this box is in  $\pi$  and can be removed from  $\pi$ . Here "can be removed" means that when this box is removed, it is still a 3D Young diagram. For a box  $\square$ , we let

$$h_{\square} = h_1 y_{\square} + h_2 x_{\square} + h_3 z_{\square}, \tag{32}$$

where  $(x_{\square}, y_{\square}, z_{\square})$  is the coordinate of box  $\square$  in coordinate system O - xyz. Here we use the order  $y_{\square}, x_{\square}, z_{\square}$  to match that in paper [7].

Following [7, 19], introduce

$$\psi_0(u) = \frac{u + \sigma_3 \psi_0}{u} \tag{33}$$

and

$$\varphi(u) = \frac{(u+h_1)(u+h_2)(u+h_3)}{(u-h_1)(u-h_2)(u-h_3)}.$$
(34)

For a 3D Young diagram  $\pi$ , define  $\psi_{\pi}(u)$  by

$$\psi_{\pi}(u) = \psi_0(u) \prod_{\square \in \pi} \varphi(u - h_{\square}). \tag{35}$$

The representation of affine Yangian on 3D Young diagrams is given by

$$\psi(u)|\pi\rangle = \psi_{\pi}(u)|\pi\rangle, \tag{36}$$

$$e(u)|\pi\rangle = \sum_{\square \subset \pi^{+}} \frac{E(\pi \to \pi + \square)}{u - h_{\square}} |\pi + \square\rangle,$$
 (37)

$$f(u)|\pi\rangle = \sum_{\square \in \pi^{-}} \frac{F(\pi \to \pi - \square)}{u - h_{\square}} |\pi - \square\rangle$$
 (38)

where  $|\pi\rangle$  means the state characterized by the 3D Young diagram  $\pi$  and the coefficients

$$E(\pi \to \pi + \Box) = -F(\pi + \Box \to \pi) = \sqrt{\frac{1}{\sigma_3} \operatorname{res}_{u \to h_\Box} \psi_\pi(u)}.$$
 (39)

Equations (37) and (38) mean generators  $e_j$ ,  $f_j$  acting on the 3D Young diagram  $\pi$  by

$$e_j|\pi\rangle = \sum_{\square \in \pi^+} h_{\square}^j E(\pi \to \pi + \square)|\pi + \square\rangle,$$
 (40)

$$f_j|\pi\rangle = \sum_{\square \in \pi^-} h_{\square}^j F(\pi \to \pi - \square)|\pi - \square\rangle.$$
 (41)

The orthogonality of 3D Young diagrams is  $\langle \pi | \pi' \rangle = \delta_{\pi,\pi'}$ . In the following, we treat  $E(\pi \to \pi + \Box) | \pi + \Box \rangle$  as one element, still denoted by  $|\pi + \Box \rangle$ . Then  $\langle \pi | \pi \rangle$  does not equal 1, but it can be calculated by

$$\langle \pi + \Box | \pi + \Box \rangle = E^2(\pi \to \pi + \Box) \langle \pi | \pi \rangle \text{ and } \langle 0 | 0 \rangle = 1.$$
 (42)

The result above means that the symmetric functions associated to 3D Young diagrams are related to the growth processes of 3D Young diagrams. A 3D Young diagram with two different growth processes are linearly related. Remark that the orthogonality (42) become that of Jack polynomials calculated in [21] in the special case  $h_1 = h, h_2 = -h^{-1}$ .

For the use in the following, we calculate the action of  $\psi_k$  on 3D Young diagram  $|\pi\rangle$ 

$$\psi_{\pi}(u) = \frac{u + \sigma_3 \psi_0}{u} \prod_{\square \in \pi} \frac{(u - h_{\square} + h_1)(u - h_{\square} + h_2)(u - h_{\square} + h_3)}{(u - h_{\square} - h_1)(u - h_{\square} - h_2)(u - h_{\square} - h_3)}$$
$$= (1 + \frac{\sigma_3 \psi_0}{u}) \prod_{\square \in \pi} (1 + \frac{2\sigma_3}{u^3} + \frac{6h_{\square}\sigma_3}{u^4} + o(u^{-4}))$$

then we obtain

$$\psi_1|\pi\rangle = 0, \ \psi_2|\pi\rangle = 2|\pi||\pi\rangle, \ \psi_3|\pi\rangle = \sum_{\square \in \pi} (6h_{\square} + 2\psi_0\sigma_3)|\pi\rangle,$$
 (43)

where  $|\pi|$  is the box number of  $\pi$ . Other  $\psi_k|\pi\rangle$  can also be calculated this way, we only list this three since we will use them to describe the Hurwitz-Kontsevich model.

### 3 The symmetric deformed 2D Hurwitz-Kontsevich model

We associate the complex numbers  $h_1$ ,  $h_2$ ,  $h_3$  to y-axis, x-axis, z-axis respectively. In this section, we consider 3D Young diagrams which have one layer in z-axis direction and treat them as 2D Young diagrams. We want our results are symmetric about x-axis and y-axis, which means that the results are symmetric about  $h_1$  and  $h_2$ .

Let  $\psi_0 = -1/h_1h_2$  and  $p = (p_1, p_2, \cdots)$  with  $p_n$  being power sum, we have  $\langle \pi | \pi \rangle = 0$  (defined in (42)) unless 3D Young diagram  $\pi$  has one layer in z-axis direction, and N = 1 in (11). The eigenstates of Hamiltonian  $\mathcal{H}$  are symmetric functions which we denoted by  $Y_{\lambda}(p)$  in [6]. For 2D Young diagrams (n), the symmetric functions  $Y_{(n)} = Y_{(n)}(p)$  are determined by

$$\sum_{n>0} \binom{h_2/h_1}{n} Y_{(n)} z^n = \exp(\sum_{n=1}^{\infty} (-1)^{n-1} \frac{p_n}{n} \frac{h_2}{h_1^n} z^n).$$
 (44)

Define the symmetric operator  $\hat{Y}_{(n)}$  by

$$\sum_{n\geq 0} \binom{h_2/h_1}{n} \hat{Y}_{(n)} z^n = \exp(\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\operatorname{ad}_{e_1}^{n-1} e_0}{n!} \frac{h_2}{h_1^n} z^n). \tag{45}$$

The Pieri formula  $Y_{(n)}Y_{\lambda}$  is defined by

$$Y_{(n)}Y_{\lambda} := \hat{Y}_{(n)} \cdot Y_{\lambda}. \tag{46}$$

Note that the actions of the generators  $e_k$ ,  $f_k$ ,  $\psi_k$  of affine Yangian of  $\mathfrak{gl}(1)$  are the same with that in (36)-(38).

From (44) and (46), we can obtain all the expressions of  $Y_{\lambda}$ . For example, from (44), we get

$$Y_{\square} = p_1, \tag{47}$$

$$Y_{\square \square} = \frac{1}{h_1 - h_2} p_2 - \frac{h_2}{h_1 - h_2} p_1^2, \tag{48}$$

$$Y_{\square\square} = \frac{1}{2h_1 - h_2} \frac{1}{h_1 - h_2} (2p_3 - 3h_2p_1p_2 + h_2^2p_1^3). \tag{49}$$

From (46), we know  $Y_{\square}Y_{\square}=Y_{\square\square}+Y_{\square},$  then

$$Y_{ \bigsqcup} = \frac{1}{h_2 - h_1} p_2 - \frac{h_1}{h_2 - h_1} p_1^2.$$

Note that the symmetric functions are symmetric about x-axis and y-axis ( $h_1$  and  $h_2$ ), which means that  $Y_{\lambda}$  becomes  $Y_{\lambda'}$  when  $h_1$  and  $h_2$  exchange  $h_1 \leftrightarrow h_2$ , where  $\lambda'$  is the conjugate of  $\lambda$ . For example, the expressions of  $Y_{\square}$  and  $Y_{\square}$  exchange when  $h_1$  and  $h_2$  exchange.

Since  $Y_{\lambda}$  corresponds to 3D Young diagram  $\pi$  with property (42), then  $Y_{\lambda}$  is dependent on the box growth of  $\lambda$ . Denote  $\square$  obtained from  $\square$  by adding one box by  $\square_{h_1h_2}$ , while denote  $\square$  obtained from  $\square$  by adding one box by  $\square_{h_2h_1}$ , then we know that

$$Y_{\square_{h_1h_2}} = \varphi(h_1 - h_2)Y_{\square_{h_2h_1}}.$$
(50)

Generally, we have

$$Y_{\lambda+B+A}(p) = \varphi(h_A - h_B)Y_{\lambda+A+B}(p), \tag{51}$$

where A or B denotes a box.

From (46), we know

$$Y_{\square}Y_{\square}=Y_{\square \square}+Y_{\square h_1h_2}, \quad Y_{\square}Y_{\square}=Y_{\square h_2h_1}+Y_{\square},$$

then we have

$$Y_{\square_{h_1h_2}} = \frac{1}{h_2 - 2h_1} \frac{1}{h_1 - h_2} (2p_3 - 2(h_1 + h_2)p_1p_2 + 2h_1h_2p_1^3),$$

$$Y_{\square_{h_2h_1}} = \frac{1}{h_1 - 2h_2} \frac{1}{h_2 - h_1} (2p_3 - 2(h_1 + h_2)p_1p_2 + 2h_1h_2p_1^3),$$

$$Y_{\square_{h_2h_1}} = \frac{1}{2h_2 - h_1} \frac{1}{h_2 - h_1} (2p_3 - 3h_1p_1p_2 + h_1^2p_1^3).$$

Others can be obtained this way.

The power sum operators can be represented by the generators  $e_k$  of affine Yangian,

$$p_n = \frac{1}{(n-1)!} \operatorname{ad}_{e_1}^{n-1} e_0 = \frac{1}{n-1} [e_1, p_{n-1}]$$
(52)

where

$$\operatorname{ad}_{A}^{n}B = \underbrace{[A, [A, \cdots [A, B] \cdots]]}_{n}.$$

Let

$$\widetilde{\operatorname{ad}}_{A}^{n}B = [\cdots [[B, \underbrace{A], \cdots A], A]}_{n},$$

we have that

$$\langle p_n, p_n \rangle = \frac{1}{((n-1)!)^2} \langle 0 | (-1)^n \widetilde{\mathrm{ad}}_{f_0}^{n-1} f_1 \mathrm{ad}_{e_1}^{n-1} e_0 | 0 \rangle.$$
 (53)

In order to calculate  $\langle p_n, p_n \rangle$ , we calculate the bracket  $[\widetilde{\operatorname{ad}}_{f_0}^{n-1} f_1, \operatorname{ad}_{e_1}^{n-1} e_0]$ . When n = 1,  $[f_0, e_0] = -\psi_0$ , we have

$$\langle p_1, p_1 \rangle = \psi_0$$
, and  $\langle p_1^m, p_1^m \rangle = m! \psi_0^m$ , (54)

for any  $m = 1, 2, 3, \cdots$ . Generally,

$$\begin{split} & [\widetilde{\operatorname{ad}}_{f_0}^{n-1}f_1,\operatorname{ad}_{e_1}^{n-1}e_0] = [[\widetilde{\operatorname{ad}}_{f_0}^{n-2}f_1,f_1],[e_1,\operatorname{ad}_{e_1}^{n-2}e_0]] \\ = & -[[f_1,[e_1,\operatorname{ad}_{e_1}^{n-2}e_0]],\widetilde{\operatorname{ad}}_{f_0}^{n-2}f_1] - [[[e_1,\operatorname{ad}_{e_1}^{n-2}e_0],\widetilde{\operatorname{ad}}_{f_0}^{n-2}f_1,f_1],f_1] \end{split}$$

the second term equal zero, the first term equals  $n(n-1)[\widetilde{\mathrm{ad}}_{f_0}^{n-2}f_1, \mathrm{ad}_{e_1}^{n-2}e_0]$ , then we have

$$\langle p_n, p_n \rangle = \frac{n}{n-1} \langle p_{n-1}, p_{n-1} \rangle,$$

which means

$$\langle p_n, p_n \rangle = n\psi_0, \text{ and } \langle p_n^m, p_n^m \rangle = n^m m! \psi_0^m.$$
 (55)

The conditions in the equation above show that the symmetric functions satisfy

$$\langle Y_{\lambda}, Y_{\mu} \rangle = \delta_{\lambda,\mu} E^2(\phi \to \Box \to \cdots \to \lambda).$$

We take an example to explain the notations. Let  $\lambda = \mu = \square$ ,

$$\langle Y_{\square \square}, Y_{\square \square} \rangle = \langle \frac{1}{h_1 - h_2} p_2 - \frac{h_2}{h_1 - h_2} p_1^2, \frac{1}{h_1 - h_2} p_2 - \frac{h_2}{h_1 - h_2} p_1^2 \rangle = \frac{2\psi_0}{h_1(h_1 - h_2)},$$

and

$$E^{2}(\phi \to \Box \to \Box \Box) = E^{2}(\phi \to \Box)E^{2}(\Box \to \Box \Box) = \psi_{0} \frac{2}{h_{1}(h_{1} - h_{2})},$$

that is,

$$\langle Y_{\square \square}, Y_{\square \square} \rangle = E^2(\phi \to \square \to \square).$$

From the property (55), we have

$$e^{\sum_{n=1}^{\infty} \frac{p_k \bar{p}_k}{n\psi_0}} = \sum_{\lambda} \frac{1}{\langle Y_{\lambda}, Y_{\lambda} \rangle} Y_{\lambda} \{p\} Y_{\lambda} \{\bar{p}\}.$$
 (56)

Note that the summation of the right hand side is over all Young diagrams, we know that there are many box growth processes of  $\lambda$ , in the summation we only choose one of them. Then

$$e^{-h_1 h_2 p_1/e^{tN}} = e^{p_1/e^{tN} \psi_0} = \sum_{\lambda} \frac{1}{\langle Y_{\lambda}, Y_{\lambda} \rangle} Y_{\lambda} \{ p_k = e^{-tN} \delta_{k,1} \} Y_{\lambda} \{ \bar{p} \}.$$
 (57)

The symmetric deformed 2D Hurwitz-Kontsevich model is generated by

$$Z_0\{p\} = e^{tW_0}e^{p_1/\psi_0e^{tN}} = \sum_{\lambda} \frac{e^{tc_{\lambda}}}{\langle Y_{\lambda}, Y_{\lambda} \rangle} Y_{\lambda}\{p_k = e^{-tN}\delta_{k,1}\} Y_{\lambda}\{p\}$$
 (58)

with

$$c_{\lambda} = \sum_{\square \subseteq N} (h_{\square} + \psi_0 \sqrt{\beta} N),$$

where  $h_{\square}$  is defined in (32) with  $z_{\square} = 1$  and t is a deformation parameter, and the operator  $W_0$  equals

$$W_{0} = \frac{1}{2} \sum_{k,l=1}^{\infty} \left( k l p_{k+l} \frac{\partial}{\partial p_{k}} \frac{\partial}{\partial p_{l}} - h_{1} h_{2}(k+l) p_{k} p_{l} \frac{\partial}{\partial p_{k+l}} \right)$$

$$+ \frac{1}{2} \sum_{k=1}^{\infty} ((h_{1} + h_{2})(k-1) + 2\psi_{0} \sqrt{\beta} N) k p_{k} \frac{\partial}{\partial p_{k}}.$$

$$(59)$$

The symmetric functions  $Y_{\lambda}\{p\}$  will be recalled below. The parameter  $\beta$  equals  $1/h_1$ , or  $-h_2$ , or their multiplications, such as,  $\sqrt{-h_2/h_1}$ . We will see that the value of  $\beta$  does not affect the eigenstates of  $W_0$ .

Note that when  $h_1 = h$ ,  $h_2 = -h^{-1}$ , the symmetric deformed 2D Hurwitz-Kontsevich model become the  $\beta$ -deformed Hurwitz-Kontsevich model in [10, 16] different by  $\sqrt{\beta}$ . When h = 1, the symmetric deformed 2D Hurwitz-Kontsevich model (58) become the Hurwitz-Kontsevich model.

We define the operator

$$E_1 = [W_0, p_1] = \sum_{n=1}^{\infty} n p_{n+1} \frac{\partial}{\partial p_n} + \psi_0 \sqrt{\beta} N p_1.$$

$$(60)$$

Then acting on  $Y_{\lambda}$ 

$$E_1 Y_{\lambda} = \sum_{\square \in \lambda^+} (h_{\square} + \psi_0 \sqrt{\beta} N) Y_{\lambda + \square}. \tag{61}$$

The operator  $W_{-1}$  is defined to be  $W_{-1} = [W_0, E_1]$ . Acting on symmetric functions  $Y_{\lambda}$ ,

$$W_{-1}Y_{\lambda} = \sum_{\square \in \lambda^{+}} (h_{\square} + \psi_{0}\sqrt{\beta}N)^{2}Y_{\lambda+\square}.$$
 (62)

We define a series of operators

$$W_{-n} = \frac{1}{(n-1)!} \operatorname{ad}_{W_{-1}}^{n-1} E_1, \quad n \ge 2.$$
(63)

When  $h_1 = h$ ,  $h_2 = -h^{-1}$ ,  $\sqrt{\beta} = h^{-1}$ , operator  $W_{-2}$  gives the W-operator in the W-representations of  $\beta$ -deformed Gaussian hermitian matrix model [22]. Similarly to [10], the partition function hierarchy with W-representations is given by

$$Z_{-n}\{p\} = e^{W_{-n}/n} \cdot 1, \quad n \ge 1.$$
(64)

When  $h_1 = h$ ,  $h_2 = -h^{-1}$ ,  $\sqrt{\beta} = h^{-1}$ , it becomes the partition function hierarchy in eq.(48) in [10], and specially when n = 1 and n = 2,  $Z_{-n}\{p\}$  give the  $\beta$ -deformed rectangular complex (with  $N_1 = N_2$ ) and Gaussian hermitian matrix models [22] respectively. These results show that the operators and the symmetric functions in this paper are the generalization of that in [10], and have the property that they are symmetric about x-axis and y-axis. The annihilation operators  $E_{-1}$  and  $W_n$  can be defined similarly.

In the following of this section, we use the generators of affine Yangian of  $\mathfrak{gl}(1)$  to represent the symmetric deformed 2D Hurwitz-Kontsevich model.

In Yang-Baxter equation (12), since N = 1, we denote the Bosons  $a_{1,m}$  by  $b_m$ , which satisfy

$$[b_n, b_m] = \psi_0 n \delta_{n+m,0} = -\frac{1}{h_1 h_2} n \delta_{n+m,0}.$$
(65)

Their actions on symmetric functions  $Y_{\lambda}$  are

$$b_{-n} = p_n, \ b_n = -\frac{1}{h_1 h_2} n \frac{\partial}{\partial p_n}, \tag{66}$$

for n > 0, and

$$b_{-n} = \frac{1}{(n-1)!} \operatorname{ad}_{e_1}^{n-1} e_0, \ b_n = -\frac{1}{(n-1)!} \operatorname{ad}_{f_1}^{n-1} f_0.$$
 (67)

The generators of affine Yangian can be represented by Boson  $b_m$ . We list some of them [6].

$$\psi_1 = 0, \tag{68}$$

$$\psi_2 = -2h_1 h_2 \sum_{j>0} b_{-j} b_j = 2 \sum_{j>0} j p_j \frac{\partial}{\partial p_j}, \tag{69}$$

$$\psi_3 = 3h_1^2 h_2^2 \sum_{j,k>0} (b_{-j-k}b_j b_k + b_{-j}b_{-k}b_{j+k}) + 3\sigma_3 \sum_{j>0} jb_{-j}b_j - \sigma_3 \sum_{j>0} b_{-j}b_j, \quad (70)$$

and

$$e_0 = b_{-1}, \ e_1 = -h_1 h_2 \sum_{j>0} b_{-j-1} b_j,$$
 (71)

From the relations (26) and (27), we have

$$[\psi_3, e_k] = 6e_{k+1} + 2\psi_0 \sigma_3 e_k, \ [\psi_3, f_k] = -6f_{k+1} - 2\psi_0 \sigma_3 f_k. \tag{72}$$

From these results, we describe the symmetric deformed 2D Hurwitz-Kontsevich model by using the generators of affine Yangian. The operator  $W_0$  can be expressed by

$$W_0 = \frac{1}{6}\psi_3 + \frac{1}{2}(\psi_0\sqrt{\beta}N - \frac{1}{3}\psi_0\sigma_3)\psi_2. \tag{73}$$

From (43),

$$\psi_2 Y_{\lambda} = 2|\lambda| Y_{\lambda}, \ \psi_3 Y_{\lambda} = \sum_{\square \in \lambda} (6h_{\square} + 2\psi_0 \sigma_3) Y_{\lambda},$$

then

$$W_0 Y_{\lambda} = \sum_{\square \in \lambda} (h_{\square} + \psi_0 \sqrt{\beta} N) Y_{\lambda},$$

which matches (58).

The operator

$$E_{1} = [W_{0}, p_{1}] = \left[\frac{1}{6}\psi_{3} + \frac{1}{2}(\psi_{0}\sqrt{\beta}N - \frac{1}{3}\psi_{0}\sigma_{3})\psi_{2}, e_{0}\right]$$

$$= e_{1} + \psi_{0}\sqrt{\beta}Ne_{0}, \tag{74}$$

we can check that the action of  $E_1$  on  $Y_{\lambda}$  matches that of  $e_k$  on  $Y_{\lambda}$ . Then all the operators  $W_{-n}$  can be represented by  $e_k$ .

The annihilation operator

$$E_{-1} = [W_0, \frac{1}{h_1 h_2} \frac{\partial}{\partial p_1}] = [\frac{1}{6} \psi_3 + \frac{1}{2} (\psi_0 \sqrt{\beta} N - \frac{1}{3} \psi_0 \sigma_3) \psi_2, f_0]$$

$$= -f_1 - \psi_0 \sqrt{\beta} N f_0. \tag{75}$$

Then the operator

$$W_1 = [W_0, E_{-1}],$$

$$W_n = \frac{(-1)^n}{(n-1)!} \operatorname{ad}_{W_1}^{n-1} E_{-1}, \quad n \ge 2,$$

can be represented by the annihilation operators  $f_k$  of affine Yangian of  $\mathfrak{gl}(1)$ .

#### 4 The symmetric deformed 3D Hurwitz-Kontsevich model

In this section, the symmetric deformed 3D Hurwitz-Kontsevich model will be constructed. We will see the symmetric deformed 3D Hurwitz-Kontsevich model (the operators and the eigenstates) are symmetric about x-axis, y-axis and z-axis. This model corresponds to the general case of Yang-Baxter equation (12). The number N in monodromy matrix (11) means that the 3D Young diagrams have at most N layers in z-axis direction. The Yang-Baxter equation (12) cut a 3D Young diagram into a series of 2D Young diagrams by plane z = j for  $j = 1, 2, \dots, N$ . Then z-axis is a preferred one.

The number N shows

$$\psi_0 = -\frac{N}{h_1 h_2} \tag{76}$$

in this section. The power sum variables associated to the 2D Young diagram on the plane z = j are denoted by  $p_{j,n}$ , which satisfy

$$\langle p_{j,n}, p_{i,m} \rangle = \delta_{j,i} \delta_{n,m} n \psi_0.$$
 (77)

Then the 3-Jack polynomials recalled in Appendix satisfy

$$\langle \tilde{J}_{\pi}, \tilde{J}_{\pi'} \rangle = \delta_{\pi, \pi'} E^2(\phi \to \Box \to \cdots \to \pi).$$
 (78)

From Yang-Baxter equation (12), the generators of affine Yangian of  $\mathfrak{gl}(1)$  can be represented by the Bosons  $a_{j,m}$  defined in (3) with  $\sigma = 3$ . By replacing  $a_{j,m}$  by its representation

$$a_{j,-m} = p_{j,m}, \quad a_{j,m} = -\frac{1}{h_1 h_2} m \frac{\partial}{\partial p_{j,m}}, \quad m \ge 0$$
 (79)

we list some of them [6]

$$\psi_{1} = 0,$$

$$\psi_{2} = 2 \sum_{j=1}^{N} \sum_{k>0} k p_{j,k} \frac{\partial}{\partial p_{j,k}},$$

$$\psi_{3} = 3 \sum_{i=1}^{N} \left( \sum_{j,k>0} (j k p_{i,j+k} \frac{\partial}{\partial p_{i,j}} \frac{\partial}{\partial p_{i,k}} - h_{1} h_{2} (j+k) p_{i,j} p_{i,k} \frac{\partial}{\partial p_{i,j+k}} \right)$$

$$-6 h_{3} \sum_{i,1< i_{2}} \sum_{k>0} k^{2} p_{i_{1},k} \frac{\partial}{\partial p_{i_{2},k}} - (-4N + 6j - 3) h_{3} \sum_{j=1}^{N} \sum_{k>0} k p_{j,k} \frac{\partial}{\partial p_{j,k}}$$

$$-3 h_{3} \sum_{i,1} \sum_{k>0} k^{2} p_{j,k} \frac{\partial}{\partial p_{j,k}},$$
(82)

and

$$e_0 = \sum_{j=1}^{N} p_{j,1}, \tag{83}$$

$$e_1 = \sum_{j=1}^{N} \sum_{k>0} p_{j,k+1} \frac{\partial}{\partial p_{j,k}}.$$
 (84)

Introduce the variables  $P_{n,j}$ ,  $n=1,2,\cdots$  and  $j=1,2,\cdots,n$ . From [23], the variables  $P_{n,j}$  are one to one correspondence with 3D Young diagrams. Then 3-Jack polynomials

should be polynomials of variables  $P_{n,j}$ . Let

$$P_{1,1} = p_{1,1} + p_{2,1} + \dots + p_{N,1},$$

$$P_{2,1} = p_{1,2} + p_{2,2} + \dots + p_{N,2},$$

$$P_{2,2} = -h_1 h_2 \sum_{j=1}^{N} p_{j,1}^2 + \frac{h_1 h_2}{N} (\sum_{j=1}^{N} p_{j,1})^2 - \sum_{j=1}^{N} (N - 2j + 1) h_3 p_{j,2}.$$

Then 3-Jack polynomials in equations (109,110,111) become

$$\tilde{J}_{\square} = \frac{1}{(h_1 - h_2)(h_1 - h_3)} (\psi_0^{-1} (1 + h_2 h_3 \psi_0) P_{1,1}^2 + (1 + h_2 h_3 \psi_0) h_1 P_{2,1} + P_{2,2}),$$

$$\tilde{J}_{\square} = \frac{1}{(h_2 - h_1)(h_2 - h_3)} (\psi_0^{-1} (1 + h_1 h_3 \psi_0) P_{1,1}^2 + (1 + h_1 h_3 \psi_0) h_2 P_{2,1} + P_{2,2}),$$

$$\tilde{J}_{\square} = \frac{1}{(h_3 - h_1)(h_3 - h_2)} (\psi_0^{-1} (1 + h_1 h_2 \psi_0) P_{1,1}^2 + (1 + h_1 h_2 \psi_0) h_3 P_{2,1} + P_{2,2}).$$

We can see that 3-Jack polynomials are symmetric about x-axis, y-axis and z-axis after changing the variables  $\{p_{j,n}\}$  to  $\{P_{n,j}\}$ .

The general relations between  $P_{n,j}$  and  $p_{j,n}$  can be obtained from

$$P_{n,1} = \frac{1}{(n-1)!} \operatorname{ad}_{e_1}^{n-1} e_0 \tag{85}$$

and the orthogonality  $\langle P_{n,j}, P_{m,i} \rangle = 0$  unless n = m, j = i. Then we have

$$e^{\sum_{j\leq n} \frac{P_{n,j}\bar{P}_{n,j}}{\langle \bar{P}_{n,j}, P_{n,j}\rangle}} = \sum_{\pi} \frac{1}{\langle \tilde{J}_{\pi}, \tilde{J}_{\pi}\rangle} \tilde{J}_{\pi} \{P_{n,j}\} \tilde{J}_{\pi} \{\bar{P}_{n,j}\}. \tag{86}$$

From the formula above, we obtain

$$e^{\frac{P_{1,1}}{\psi_0 e^{tM}}} = \sum_{\pi} \frac{1}{\langle \tilde{J}_{\pi}, \tilde{J}_{\pi} \rangle} \tilde{J}_{\pi} \{ \bar{P}_{n,j} = \delta_{n,1} \delta_{j,1} e^{-tM} \} \tilde{J}_{\pi} \{ P_{n,j} \}.$$
 (87)

The symmetric deformed 3D Hurwitz-Kontsevich model is generated by

$$\mathcal{Z}_0\{p\} = e^{t\mathcal{W}_0} \cdot e^{\frac{P_{1,1}}{\psi_0 e^{tM}}},\tag{88}$$

where

$$\mathcal{W}_{0} = \frac{1}{2} \sum_{i=1}^{N} \sum_{k,l=1}^{\infty} \left( k l p_{i,k+l} \frac{\partial}{\partial p_{i,k}} \frac{\partial}{\partial p_{i,l}} - h_{1} h_{2}(k+l) p_{i,k} p_{i,l} \frac{\partial}{\partial p_{i,k+l}} \right) 
+ (h_{1} + h_{2}) \sum_{i_{1} < i_{2}} \sum_{k>0} k^{2} p_{i_{1},k} \frac{\partial}{\partial p_{i_{2},k}} 
+ \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{\infty} \left( (h_{1} + h_{2})(k - 2N + 2j - 1) + 2\psi_{0} \sqrt{\beta} N \right) k p_{j,k} \frac{\partial}{\partial p_{j,k}}.$$
(89)

We can see that when N=1, the symmetric deformed 3D Hurwitz-Kontsevich model becomes the symmetric deformed 2D Hurwitz-Kontsevich model, which corresponds to that the 3D Young diagrams which have one layer in z-axis direction become 2D Young diagrams.

From

$$W_0 \tilde{J}_{\pi} \{ P_{n,j} \} = c_{\pi} \tilde{J}_{\pi} \{ P_{n,j} \}$$
(90)

with

$$c_{\pi} = \sum_{\square \in \pi} (h_{\square} + \psi_0 \sqrt{\beta} N), \tag{91}$$

where  $h_{\square}$  is defined in (32) and  $\psi_0 = -N/h_1h_2$ , the partition function  $\mathcal{Z}_0\{p\}$  equals

$$\mathcal{Z}_{0}\{p\} = \sum_{\pi} \frac{e^{tc_{\pi}}}{\langle \tilde{J}_{\pi}, \tilde{J}_{\pi} \rangle} \tilde{J}_{\pi}\{\bar{P}_{n,j} = \delta_{n,1}\delta_{j,1}e^{-tM}\} \tilde{J}_{\pi}\{P_{n,j}\}.$$
(92)

Define the creation operator

$$\bar{E}_1 = [\mathcal{W}_0, p_1] = \sum_{j=1}^{N} \sum_{k=1}^{\infty} k p_{j,k+1} \frac{\partial}{\partial p_{j,k}} + \psi_0 \sqrt{\beta} N \sum_{j=1}^{N} p_{j,1}.$$
 (93)

Acting on 3-Jack polynomials,

$$\bar{E}_1 \tilde{J}_{\pi} = \sum_{\square \in \pi^+} (h_{\square} + \psi_0 \sqrt{\beta} N) \tilde{J}_{\pi+\square}. \tag{94}$$

Define the creation operator

$$\mathcal{W}_{-1} = [\mathcal{W}_0, \bar{E}_1],\tag{95}$$

its action on 3-Jack polynomials can be obtained from the actions of  $W_0$  and  $\bar{E}_1$  on 3-Jack polynomials

$$\mathcal{W}_{-1}\tilde{J}_{\pi} = (\mathcal{W}_{0}\bar{E}_{1} - \bar{E}_{1}\mathcal{W}_{0})\tilde{J}_{\pi} 
= \sum_{\square \in \pi^{+}} (h_{\square} + \psi_{0}\sqrt{\beta}N) \sum_{\square' \in \pi^{+}\square} (h_{\square'} + \psi_{0}\sqrt{\beta}N)\tilde{J}_{\pi^{+}\square} 
- \sum_{\square \in \pi^{+}} (h_{\square} + \psi_{0}\sqrt{\beta}N) \sum_{\square' \in \pi} (h_{\square'} + \psi_{0}\sqrt{\beta}N)\tilde{J}_{\pi^{+}\square} 
= \sum_{\square \in \pi^{+}} (h_{\square} + \psi_{0}\sqrt{\beta}N)^{2}\tilde{J}_{\pi^{+}\square}.$$
(96)

We define a series of operators

$$W_{-n} = \frac{1}{(n-1)!} \operatorname{ad}_{W_{-1}}^{n-1} \bar{E}_1, \quad n \ge 2.$$
(97)

Their actions on 3-Jack polynomials are determined by the actions of  $W_{-1}$  and  $\bar{E}_1$  on 3-Jack polynomials.

We introduce the partition function hierarchy with W-representations

$$\mathcal{Z}_{-n}\{p\} = e^{\mathcal{W}_{-n}/n} \cdot 1, \quad n \ge 1.$$
 (98)

From the discussion in the last section, we know that when n=1 and n=2, the partition function  $\mathcal{Z}_{-n}\{p\}$  is the generalization of the  $\beta$ -deformed rectangular complex (with  $N_1=N_2$ ) and Gaussian hermitian matrix models [22] respectively to the three dimensional case.

We construct the annihilation operator

$$\bar{E}_{-1} = [\mathcal{W}_0, -\psi_0 \frac{\partial}{\partial P_{1,1}}] = [\mathcal{W}_0, \frac{1}{h_1 h_2} \sum_{j=1}^N \frac{\partial}{\partial p_{j,1}}], \tag{99}$$

and

$$\mathcal{W}_1 = [\mathcal{W}_0, \bar{E}_{-1}],\tag{100}$$

then we introduce a series of the annihilation operators

$$W_n = \frac{(-1)^n}{(n-1)!} \operatorname{ad}_{W_1}^{n-1} \bar{E}_{-1}, \quad n \ge 2.$$
(101)

From the results above, we see that the symmetric deformed 3D Hurwitz-Kontsevich model (the operators and the eigenstates) is symmetric about x-axis and y-axis, but not symmetric about x-axis and z-axis, or y-axis and z-axis. In the following, we show that the symmetric deformed 3D Hurwitz-Kontsevich model can be represented by affine Yangian of  $\mathfrak{gl}(1)$ , then we will see that this model is symmetric about x-axis, y-axis and z-axis.

From (81) and (82), we have

$$W_0 = \frac{1}{6}\psi_3 + \frac{1}{2}(\psi_0\sqrt{\beta}N - \frac{1}{3}\psi_0\sigma_3)\psi_2. \tag{102}$$

Acting on 3-Jack polynomials,

$$\mathcal{W}_0 \tilde{J}_{\pi} = \frac{1}{6} \psi_3 \tilde{J}_{\lambda} + \frac{1}{2} (\psi_0 \sqrt{\beta} N - \frac{1}{3} \psi_0 \sigma_3) \psi_2 \tilde{J}_{\lambda} = \sum_{\square \in \pi} (h_{\square} + \psi_0 \sqrt{\beta} N) \tilde{J}_{\lambda}, \tag{103}$$

which matches that in (90).

The creation operator

$$\bar{E}_1 = [\mathcal{W}_0, p_1] = \left[\frac{1}{6}\psi_3 + \frac{1}{2}(\psi_0\sqrt{\beta}N - \frac{1}{3}\psi_0\sigma_3)\psi_2, e_0\right] = e_1 + \psi_0\sqrt{\beta}Ne_0.$$
 (104)

From the representation of affine Yangian of  $\mathfrak{gl}(1)$  on 3D Young diagrams, the action of  $\bar{E}_1$  on 3-Jack polynomials is

$$\bar{E}_1 \tilde{J}_{\pi} = (e_1 + \psi_0 \sqrt{\beta} N e_0) \tilde{J}_{\pi} = \sum_{\square \in \pi^+} (h_{\square} + \psi_0 \sqrt{\beta} N) \tilde{J}_{\pi + \square}.$$
 (105)

The creation operator

$$\mathcal{W}_{-1} = \left[ \frac{1}{6} \psi_3 + \frac{1}{2} (\psi_0 \sqrt{\beta} N - \frac{1}{3} \psi_0 \sigma_3) \psi_2, e_1 + \psi_0 \sqrt{\beta} N e_0 \right]$$

$$= e_2 + 2\psi_0 \sqrt{\beta} N e_1 + \psi_0 \beta N^2 e_0, \tag{106}$$

then  $W_{-n}$ ,  $n \ge 1$  can be represented by the generators  $e_k$  of affine Yangian of  $\mathfrak{gl}(1)$ . The annihilation operator

$$\bar{E}_1 = \left[\frac{1}{6}\psi_3 + \frac{1}{2}(\psi_0\sqrt{\beta}N - \frac{1}{3}\psi_0\sigma_3)\psi_2, f_0\right] = -f_1 - \psi_0\sqrt{\beta}Nf_0.$$
 (107)

Then  $W_n$ ,  $n \geq 1$  can be represented by the generators  $f_k$  of affine Yangian of  $\mathfrak{gl}(1)$ .

## Appendix: 3-Jack polynomials

3-Jack polynomials are eigenstates of the Hamiltonian  $\mathcal{H}$  defined in section 2. 3-Jack polynomials are the generalization of the symmetric functions  $Y_{\lambda}$  to the three dimensional case, which are also the generalization of Jack polynomials. Following the name and the notations we defined in paper [8], we denote 3-Jack polynomials by  $\tilde{J}_{\pi}$ , where

 $\pi$  is a 3D Young diagram. When N in monodromy matrix (11) equals 1, 3-Jack polynomials become  $Y_{\lambda}$ , and when N=1 and  $h_1=h, h_2=-h^{-1}, \alpha=1/\beta=h^2$ , 3-Jack polynomials become Jack polynomials  $\tilde{J}_{\lambda}$ . Jack polynomials  $\tilde{J}_{\lambda}$  are defined in [21], which equal  $J_{\lambda}$  (defined in book [9]) multiplied by a constant.

We recall the expressions of 3-Jack polynomials for some of 3D Young diagrams[8].

$$\tilde{J}_{\square} = p_{1,1} + \dots + p_{N,1}, \qquad (108)$$

$$\tilde{J}_{\square} = \frac{1}{(h_1 - h_2)(h_1 - h_3)} (-h_2(h_1 - h_3) \sum_{i=1}^{N} p_{i,1}^2 + \sum_{i=1}^{N} (h_1 - (2N - 2i + 1)h_3) p_{i,2}$$

$$+2h_2 h_3 \sum_{i_1 < i_2} p_{i_1,1} p_{i_2,1}), \qquad (109)$$

$$\tilde{J}_{\square} = \frac{1}{(h_2 - h_1)(h_2 - h_3)} (-h_1(h_2 - h_3) \sum_{i=1}^{N} p_{i,1}^2 + \sum_{i=1}^{N} (h_2 - (2N - 2i + 1)h_3) p_{i,2}$$

$$+2h_1 h_3 \sum_{i_1 < i_2} p_{i_1,1} p_{i_2,1}), \qquad (110)$$

 $\tilde{J}_{\square} = \frac{1}{(h_3 - h_1)(h_3 - h_2)} \left(-2 \sum_{i=1}^{N-1} (N - i)h_3 p_{i,2} + 2h_1 h_2 \sum_{i_1 < i_2} p_{i_1,1} p_{i_2,1}\right), \tag{111}$ 

and

$$\tilde{J}_{\text{HID}} = \frac{1}{(h_1 - h_2)(h_1 - h_3)(2h_1 - h_2)(2h_1 - h_3)} (h_2^2(6h_1^2 + 5h_1h_2 + h_2^2) \sum_{j=1}^{N} p_{j,1}^3, 
-3h_2(2h_1 + h_2) \sum_{j=1}^{N} ((2(N - j) + 3)h_1 + (2(N - j) + 1)h_2)p_{j,1}p_{j,2} 
+2 \sum_{j=1}^{N} (3(N - j + 1)(N - j + 2)h_1^2 + (6(N - j)^2 + 12(N - j) + 5)h_1h_2 
+(3(N - j)(N - j + 1) + 1)h_2^2)p_{j,3} - 3h_2^2h_3(2h_1 + h_2) \sum_{i < j} (p_{i,1}^2 p_{j,1} + p_{i,1}p_{j,1}^2) 
+3h_2h_3 \sum_{i < j} ((2N - 2j + 2)h_1 + (2N - 2j + 1)h_2)p_{i,1}p_{j,2} 
+3h_2h_3 \sum_{i < j} ((2N - 2i + 4)h_1 + (2N - 2i + 1)h_2)p_{i,2}p_{j,1} 
+6h_2^2h_3^2 \sum_{j_1 < j_2 < j_2} p_{j_1,1}p_{j_2,1}p_{j_3,1})$$
(112)

$$\tilde{J}_{h_1,h_2} = \frac{1}{(h_1 - h_2)(h_1 - h_3)(h_2 - 2h_1)(h_2 - h_3)} (2h_1h_2(2h_1^2 + 5h_1h_2 + 2h_2^2) \sum_{j=1}^{N} p_{j,1}^3 
+2h_3 \sum_{j=1}^{N} ((2N - 2j + 2)h_1^2 + (7N - 7j + 5)h_1h_2 + (2N - 2j + 2)h_2^2)p_{j,1}p_{j,2} 
+2 \sum_{j=1}^{N} ((3(N - j)^2 + 5(N - j) + 2)h_1^2 + (6(N - j)^2 + 10(N - j) + 5)h_1h_2 
+(3(N - j)^2 + 5(N - j) + 2)h_2^2)p_{j,3} + 8h_1h_2h_3^2 \sum_{i < j} (p_{i,1}^2p_{j,1} + p_{i,1}p_{j,1}^2) 
+2h_3 \sum_{i < j} ((2N - 2j + 2)h_1^2 + (4N - 4j + 1)h_1h_2 + (2N - 2j + 2)h_2^2)p_{i,1}p_{j,2} 
+4h_3^3 \sum_{i < j} (N - i + 1)p_{i,2}p_{j,1} + 12h_1h_2h_3^2 \sum_{j_1 < j_2 < j_3} p_{j_1,1}p_{j_2,1}p_{j_3,1})$$
(113)

$$\tilde{J}_{h_1,h_3} = \frac{1}{(h_1 - h_2)(h_1 - h_3)(h_3 - 2h_1)(h_3 - h_2)} (2h_2h_3(3h_1 + 2h_2) \sum_{j=1}^{N} (N - j)p_{j,1}p_{j,2} 
-2h_3 \sum_{j=1}^{N} (3(N - j)(N - j + 1)h_1 + (N - j)(3N - 3j + 1)h_2)p_{j,3} 
-2h_1h_2^2(3h_1 + 2h_2) \sum_{i < j} (p_{i,1}^2 p_{j,1} + p_{i,1}p_{j,1}^2) 
+2h_2 \sum_{i < j} (3h_1^2 - 2(N - j - 1)h_1h_2 - (2N - 2j)h_2^2)p_{i,1}p_{j,2} 
-2h_2^2 \sum_{i < j} ((2N - 2i + 1)h_1 + (2N - 2i)h_2)p_{i,2}p_{j,1} 
+12h_1h_3h_2^2 \sum_{j_1 < j_2 < j_3} p_{j_1,-1}p_{j_2,-1}p_{j_3,-1})$$
(114)

$$\tilde{J}_{h_{2},h_{3}} = \frac{1}{(h_{2} - h_{1})(h_{2} - h_{3})(h_{3} - 2h_{2})(h_{3} - h_{1})} (2h_{1}h_{3}(3h_{2} + 2h_{1}) \sum_{j=1}^{N} (N - j)p_{j,1}p_{j,2} 
-2h_{3} \sum_{j=1}^{N} (3(N - j)(N - j + 1)h_{2} + (N - j)(3N - 3j + 1)h_{1})p_{j,3} 
-2h_{2}h_{1}^{2}(3h_{2} + 2h_{1}) \sum_{i < j} (p_{i,1}^{2}p_{j,1} + p_{i,1}p_{j,1}^{2}) 
+2h_{1} \sum_{i < j} (3h_{2}^{2} - 2(N - j - 1)h_{1}h_{2} - (2N - 2j)h_{1}^{2})p_{i,1}p_{j,2} 
-2h_{1}^{2} \sum_{i < j} ((2N - 2i + 1)h_{2} + (2N - 2i)h_{1})p_{i,2}p_{j,1} 
+12h_{1}h_{3}h_{1}^{2} \sum_{j_{1} < j_{2} < j_{3}} p_{j_{1},1}p_{j_{2},1}p_{j_{3},1})$$
(115)

$$\tilde{J}_{\text{op}} = \frac{1}{(h_2 - h_1)(h_2 - h_3)(2h_2 - h_1)(2h_2 - h_3)} (h_1^2(6h_2^2 + 5h_1h_2 + h_1^2) \sum_{j=1}^{N} p_{j,1}^3 
-2h_1(2h_2 + h_1) \sum_{j=1}^{N} ((2(N - j) + 1)h_2 + (2(N - j) + 1)h_1)p_{j,1}p_{j,2} 
+2 \sum_{j=1}^{N} (3(N - j + 1)(N - j + 2)h_2^2 + (6(N - j)^2 + 12(N - j) + 5)h_1h_2 
+(3(N - j)(N - j + 1) + 1)h_1^2)p_{j,3} - 3h_1^2h_3(2h_2 + h_1) \sum_{i < j} (p_{i,1}^2 p_{j,1} + p_{i,1}p_{j,1}^2) 
+3h_1h_3 \sum_{i < j} ((2N - 2j + 2)h_2 + (2N - 2j + 1)h_1)p_{i,1}p_{j,2} 
+3h_1h_3 \sum_{i < j} ((2N - 2i + 4)h_2 + (2N - 2i + 1)h_1)p_{i,2}p_{j,1} 
+6h_1^2h_3^2 \sum_{j_1 < j_2 < j_3} p_{j_1,1}p_{j_2,1}p_{j_3,1})$$
(116)

$$\tilde{J}_{\text{B}} = \frac{1}{(h_3 - h_1)(h_3 - h_2)(2h_3 - h_1)(2h_3 - h_2)} (6h_3^2 \sum_{j=1}^{N} (N - j)(N - j - 1)p_{j,3} 
-6h_1h_2h_3 \sum_{i < j} (N - j)p_{i,1}p_{j,2} - 6h_1h_2h_3 \sum_{i < j} (N - i - 1)p_{i,2}p_{j,1} 
+6h_1^2h_2^2 \sum_{j_1 < j_2 < j_3} p_{j_1,1}p_{j_2,1}p_{j_3,1}).$$
(117)

## Data availability statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

#### Declaration of interest statement

The authors declare that we have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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