A note on regular sets in Cayley graphs

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Abstract

A subset R of the vertex set of a graph Γ is said to be (κ, τ)-regular if R induces a κ-regular subgraph and every vertex outside R is adjacent to exactly τ vertices in R. In particular, if R is a (κ, τ)-regular set of some Cayley graph on a finite group G, then R is called a (κ, τ)-regular set of G. Let H be a non-trivial normal subgroup of G, and κ and τ a pair of integers satisfying $0 \le \kappa \le |H| - 1$, $1 \le \tau \le |H|$ and $\gcd(2, |H| - 1) \mid \kappa$. It is proved that (i) if τ is even, then H is a (κ, τ)-regular set of G; (ii) if τ is odd, then H is a (κ, τ)-regular set of G if and only if it is a $(0, 1)$-regular set of G.

Keywords: regular set; perfect code; Cayley graph

AMS Subject Classification (2020): 05C25, 05E18, 94B25

1 Introduction

In the paper, all groups considered are finite groups with identity element denoted as 1, and all graphs considered are finite, undirected and simple. Let R be a subset of the vertex set of a graph Γ, and κ and τ a pair of nonnegative integers. We call R a (κ, τ)-*regular set* (or *regular set* for short if there is no need to emphasize the parameters κ and τ in the context) of Γ if every vertex in R is adjacent to exactly κ vertices in R and every vertex outside R is adjacent to exactly τ vertices in R. In particular, we call R a perfect code of Γ if $(\kappa, \tau) = (0, 1)$ and a total perfect code of Γ if $(\kappa, \tau) = (1, 1)$. The concept of (κ, τ)-*regular set* was introduced in [3] and further studied in [1, 2, 4, 5]. Very recently, regular sets in Cayley Graphs was studied in [8, 9].

Let G be a group and X an inverse closed subset of $G \setminus \{1\}$. The Cayley graph $\mathrm{Cay}(G, X)$ on G with connection set X is the graph with vertex set G and edge set $\{\{g, gx\} \mid g \in G, x \in X\}$. A subset R of G is called a (κ, τ)-*regular set* of G if there is a Cayley graph Γ on G such that R is a (κ, τ)-regular set of Γ. Regular sets of Cayley graphs are closely related to codes of groups. Let C and Y be two subsets of G and λ a

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positive integer. If for every $g \in G$ there exist precisely λ pairs $(c, y) \in C \times Y$ such that $g = cy$, then C is called a *code* of G with respect to Y [6]. In particular, if $\lambda = 1$ and Y is an inverse closed subset of G containing 1, then C is called a *perfect code* of G [7]. Let H be a subgroup of G. It is straightforward to check that H is a $(0, \tau)$-regular set of G if and only if H is a code of G with respect to some inverse closed subset of G. Actually, if H is a $(0, \tau)$-regular set of the Cayley graph $\mathrm{Cay}(G, X)$, then H is a code of G with respect to $Y := X \cup Z$ for any inverse closed subset Z of H with cardinality τ; and on the other hand, if H is a code of G with respect to Y, then H is a $(0, \tau)$-regular set of the Cayley graph $\mathrm{Cay}(G, X)$ where $X = Y \setminus H$ and $\tau = \frac{|H||Y|}{|G|}$.

It is a natural question when a normal subgroup of a group is a regular set. This question was studied by Wang et al in [9]. They proved that, for any finite group G, if a non-trivial normal subgroup H of G is a perfect code of G, then for any pair of integers κ and τ with $0 \le \kappa \le |H| - 1$, $1 \le \tau \le |H|$ and $\gcd(2, |H| - 1) \mid \kappa$, H is also a (κ, τ)-regular set of G. It was also shown in [9] that there exists normal subgroups of some group which are (κ, τ)-regular sets for some pair of integers κ and τ but not perfect codes of the group. In this paper, we extend the mains results in [9] by proving the following theorem.

Theorem 1.1. *Let G be a group and H a non-trivial normal subgroup of G. Let κ and τ be a pair of integers satisfying $0 \le \kappa \le |H| - 1$, $1 \le \tau \le |H|$ and $\gcd(2, |H| - 1) \mid \kappa$. The following two statements hold:*

(i) *if τ is even, then H is a (κ, τ)-regular set of G;*

(ii) *if τ is odd, then H is a (κ, τ)-regular set of G if and only if it is a perfect code of G.*

It was proved in [7, Theorem 2.2] that a normal subgroup H of G is a perfect code of G if and only if

for any $g \in G$ with $g^2 \in H$, there exists $h \in H$ such that $(gh)^2 = 1$.

Note that condition # always holds if H is of odd order or odd index [7, Corollary 2.3]. Therefore, Theorem 1.1 has a direct corollary as follows.

Corollary 1.2. *Let G be a group and H a non-trivial normal subgroup of G. If either $|H|$ or $|G/H|$ is odd, then H is a (κ, τ)-regular set of G for every pair of integers κ and τ satisfying $0 \le \kappa \le |H| - 1$, $1 \le \tau \le |H|$ and $\gcd(2, |H| - 1) \mid \kappa$.*

Remark 1. It is a challenging question whether Theorem 1.1 and Corollary 1.2 can be generalized to non-normal subgroups H of G.

Remark 2. Let H be a nontrivial normal subgroup of G of even order not satisfying condition #. Let κ and τ be a pair of integers satisfying $0 \le \kappa \le |H| - 1$, $2 \le \tau \le |H|$ and $2 \mid \tau$. Then Theorem 1.1 (i) and [7, Theorem 2.2] imply that H is a (κ, τ)-regular set but not a perfect code of G.

2 Proof of Theorem 1.1

Throughout this section, we use $\dot\bigcup_{i=1}^{n} S_i$ to denote the union of the pair-wise disjoint sets $S_1, S_2, \ldots, S_n$. Let G be a group and H a non-trivial normal subgroup of G. Let κ and τ be a pair of integers satisfying $0 \le \kappa \le |H|-1$, $1 \le \tau \le |H|$ and $\gcd(2, |H|-1) \mid \kappa$. We firstly prove three lemmas and then complete the proof of Theorem 1.1.

Lemma 2.1. *If τ is even, then H is a $(0,\tau)$-regular set of G.*

Proof. Let $A := \{1, a_1, \ldots, a_s\}$ be a left transversal of H in G. Assume that the number of involutions contained in a_iH is n_i for every $1 \le i \le s$. Let σ be a permutation on $\{1, \ldots, s\}$ such that $a_i^{-1}H = a_{\sigma(i)}H$. Since H is normal in G, we have

$$a_{\sigma^2(i)}H = a_{\sigma(i)}^{-1}H = Ha_{\sigma(i)}^{-1} = (a_{\sigma(i)}H)^{-1} = (a_i^{-1}H)^{-1} = Ha_i = a_iH.$$

It follows that σ is the identity permutation or an involution. Assume that σ fixes t integers in $\{1, \ldots, s\}$. Then $0 \le t \le s$ and $s-t$ is even. Set $\ell := \frac{s-t}{2}$. Without loss of generality, we assume that

$$\sigma(i) = \begin{cases} i, & \text{if } i \le t; \\ i+\ell, & \text{if } t < i \le t+\ell; \\ i-\ell, & \text{if } t+\ell < i \le s. \end{cases}$$

Then a_iH is inverse closed if $i \le t$ and $(a_{t+j}H)^{-1} = a_{t+j+\ell}H$ for every positive integer j not greater than ℓ. In particular, $n_i = 0$ if $i > t$. For every $i \in \{1, \ldots, s\}$, take a subset X_i of a_iH of cardinality τ by the following rules:

- if $i \le t$ and $n_i \ge \tau$, then X_i consists of exactly τ involutions;
- if $i \le t$, $n_i < \tau$ and $\tau - n_i$ is even, then X_i consists of n_i involutions and $\frac{\tau-n_i}{2}$ pairs of mutually inverse elements of order grater than 2;
- if $i \le t$, $n_i < \tau$ and $\tau - n_i$ is odd, then X_i consists of $n_i - 1$ involutions and $\frac{\tau+1-n_i}{2}$ pairs of mutually inverse elements of order grater than 2;
- if $t < i \le t+\ell$, then X_i consists of exactly τ elements of order greater than 2;
- if $i > t+\ell$, then set $X_i = X_{i-\ell}^{-1}$.

Note that $X_1, \ldots, X_s$ are pair-wise disjoint. Set $X = \dot\bigcup_{i=1}^{s} X_i$. Then X is an inverse closed subset of G satisfying $X \cap H = \emptyset$ and $|X \cap gH| = \tau$ for every $g \in G \setminus H$. It follows that H is a $(0,\tau)$-regular set of the Cayley graph $\mathrm{Cay}(G, X)$ and therefore a $(0,\tau)$-regular set of G. □

Lemma 2.2. *If τ is odd, then H is a $(0,\tau)$-regular set of G if and only if it is a perfect code of G.*

Proof. The sufficiency follows from [9, Theorem 1.2]. Now we prove the necessity. Let H be a $(0,\tau)$-regular set of the Cayley graph $\mathrm{Cay}(G,X)$. Then $X = X^{-1}$, $X \cap H = \emptyset$ and $|X \cap gH| = \tau$ for every $g \in G \setminus H$. Let $A := \{1, a_1, \ldots, a_s\}$ be a left transversal of H in G and set $X_i = X \cap a_iH$ for every $i \in \{1, 2, \ldots, s\}$. Then $X = \dot\cup_{i=1}^{s} X_i$. If X_i contains an involution for each $i \in \{1, \ldots, s\}$, then H is a perfect code of G with respect to $\{1, y_1, \ldots, y_s\}$ where y_i is an involution in X_i, $i = 1, \ldots, s$. Now we assume that there exists at least one integer $k \in \{1, \ldots, s\}$ such that X_k contains no involution. Then $x^{-1} \neq x$ for every element $x \in X_k$. It follows that $|X_k \cup X_k^{-1}|$ is even. Since $|X_k| = \tau$ and τ is odd, we get $X_k \neq X_k^{-1}$. Since H is normal in G, we obtain $(a_kH)^{-1} = (Ha_k)^{-1} = a_k^{-1}H$. Assume that $a_k^{-1}H = a_jH$ for some $j \in \{1, \ldots, s\}$. Then $X_k^{-1} \subseteq a_jH$. Since $X = \dot\cup_{i=1}^{s} X_i$ and $X^{-1} = X$, we conclude that $X_k^{-1} = X_j$. Therefore, without loss of generality, we can assume that $X_i^{-1} = X_{i+\ell}$ if $1 \leq i \leq \ell$ and $X_i^{-1} = X_i$ if $2\ell < i \leq s$ where ℓ is a positive integer not greater than $\frac{s}{2}$. Note that X_i contains at least one involution if $X_i^{-1} = X_i$ (as it is of odd cardinality). For every $i \in \{1, \ldots, s\}$, take an element $y_i \in X_i$ by the following rules:

- y_i is an arbitrary element in X_i if $i \leq \ell$;
- $y_i = y_{i-\ell}^{-1}$ if $\ell < i \leq 2\ell$;
- y_i is an involution if $i > 2\ell$.

Then H is a perfect code of G with respect to $\{1, y_1, \ldots, y_s\}$. □

Lemma 2.3. *H is a (κ,τ)-regular set of G if and only if H is a $(0,\tau)$-regular set of G.*

Proof. $\Rightarrow$) Let H be a (κ,τ)-regular set of the Cayley graph $\mathrm{Cay}(G,X)$. Then $|H \cap X| = \kappa$ and $|gH \cap X| = \tau$ for every $g \in G \setminus H$. Set $Y = X \setminus H$. Then $|H \cap Y| = 0$ and $|gH \cap Y| = \tau$ for every $g \in G \setminus H$. Since $X^{-1} = X$ and $H^{-1} = H$, we get $Y^{-1} = Y$. It follows that H is a $(0,\tau)$-regular set of the Cayley graph $\mathrm{Cay}(G,Y)$ and therefore a $(0,\tau)$-regular set of G.

$\Leftarrow$) Let H be a $(0,\tau)$-regular set of the Cayley graph $\mathrm{Cay}(G,Y)$. Then $|H \cap Y| = 0$ and $|gH \cap Y| = \tau$ for every $g \in G \setminus H$. Let m be the number of elements contained in H of order greater than 2. Then m is even and the number of involutions contained in H is $|H| - 1 - m$. Recall that $0 \leq \kappa \leq |H| - 1$ and $\gcd(2, |H| - 1) \mid \kappa$. If κ is odd, then $|H|$ is even and therefore contains at least one involution. Take an inverse closed subset Z of H of cardinality κ by the following rules:

- if $m \geq \kappa$ and κ is even, then Z consists of exactly $\frac{\kappa}{2}$ pairs of mutually inverse elements of order grater than 2;
- if $m \geq \kappa$ and κ is odd, then Z consists of $\frac{\kappa-1}{2}$ pairs of mutually inverse elements of order grater than 2 and one involution;
- if $m < \kappa$, then Z consists of $\frac{m}{2}$ pairs of mutually inverse elements of order grater than 2 and $\kappa - m$ involutions.

Set $X = Y \cup Z$. Then $|H \cap X| = \kappa$ and $|gH \cap X| = \tau$ for every $g \in G \setminus H$. Therefore H is a (κ, τ)-regular set of the Cayley graph $\mathrm{Cay}(G, X)$ and therefore a (κ, τ)-regular set of G. □

Proof of Theorem 1.1. Lemma 2.1 and Lemma 2.3 imply that H is a (κ, τ)-regular set of G if τ is even. Now assume τ is odd. By Lemma 2.2 and Lemma 2.3, H is a (κ, τ)-regular set of G if and only if it is a perfect code of G. □

Acknowledgements The first author was supported by Natural Science Foundation of Chongqing (cstc2019jcyj-msxmX0146) and the Foundation of Chongqing Normal University (21XLB006).

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