The general solutions to some systems of Sylvester-type quaternion matrix equations with an application[1]

Qing-Wen Wang[a,b,*], **Long-Sheng Liu**[a,c]

a Department of Mathematics, Shanghai University, Shanghai 200444, P. R. China
b Collaborative Innovation Center for the Marine Artificial Intelligence, Shanghai 200444, P. R. China
c School of Mathematics and Computational Science, Anqing Normal University, Anqing 246011, People's Republic of China

Abstract: Sylvester-type matrix equations have applications in areas including control theory, neural networks, and image processing. In this paper, we establish the necessary and sufficient conditions for the system of Sylvester-type quaternion matrix equations to be consistent and derive an expression of its general solution (when it is solvable). As an application, we investigate the necessary and sufficient conditions for quaternion matrix equations to be consistent and derive a formula for its general solution involving η-Hermicity. As a special case, we also present the necessary and sufficient conditions for the system of two-sided Sylvester-type quaternion matrix equations to have a solution and derive a formula for its general solution (when it is solvable). Finally, we present an algorithm and an example to illustrate the main results of this paper.

Keywords: matrix equation; Matrix equation; Quaternion; η-Hermitian; Moore-Penrose; Rank
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1. **Introduction**

Throughout this paper, The field of real numbers is denoted by $\mathbb{R}$. $\mathbb{H}^{m\times n}$ represents the space of all $m\times n$ matrices over $\mathbb{H}$,

$$\mathbb{H}=\{v_0+v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}|\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{ijk}=-1,v_0,v_1,v_2,v_3\in\mathbb{R}\}.$$

Here, the rank of A is denoted by $r(A)$, while I and 0 represent an identity matrix and a zero matrix of appropriate sizes, respectively. Term A^* represents the conjugate transpose of A. The Moore-Penrose (M-P) inverse of $A\in\mathbb{H}^{l\times k}$, $A^\dagger$, is defined as the solution of $AYA=A$, $YAY=Y$, $(AY)^*=AY$ and $(YA)^*=YA$. Moreover, $L_A=I-A^\dagger A$ and $R_A=I-AA^\dagger$ represent two projectors along A. Recall that a quaternion matrix A is called η-Hermitian if $A=A^{\eta^*}$, where $A^{\eta^*}=-\eta A^*\eta$ and $\eta\in\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ [31]. It is well-known that $(L_A)^{\eta^*}=R_{A^{\eta^*}}$, $(R_A)^{\eta^*}=L_{A^{\eta^*}}$.

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* Corresponding author.
Email address: wqw@t.shu.edu.cn, wqw369@yahoo.com (Q.W. Wang).

Since Roth [28] first considered the following one-sided Sylvester-type matrix equation in 1952:

$$AX + YB = C, \tag{1.1}$$

which has applications in control theory and singular system control [8], neural networks [43], there have been extensive studies of equation (1.1). For example, Baksalary and Kala [1] investigated the necessary and sufficient conditions for the solvability of equation (1.1) by using the generalized inverses of the matrices involved. Further, Flanders and Wimmer [9] provided an invariant proof of Roth's theorem, while Baksalary and Kala [1] established the necessary and sufficient conditions for the following Sylvester-type matrix equation to be consistent:

$$C_3X_3D_3 + C_4X_4D_4 = E_1. \tag{1.2}$$

Özgüler [25] studied the necessary and sufficient conditions for the solvability of equation (1.2) over a principal ideal domain. Furthermore, Wang [32] provided some necessary and sufficient conditions for equation (1.2) to enable a solution over an arbitrary regular ring with identity and obtained an expression for its general solution.

In 1843, Irish mathematician sir William Rowan Hamilton introduced quaternions. It is well known that the quaternion algebra, $\mathbb{H}$, is an associative noncommutative division algebra over $\mathbb{R}$, which has applications in computer science, orbital mechanics, signal and color image processing, and control theory ([2], [4], [15], [26], [27] [30]).

Based on the wide applications of quaternions, interest in Sylvester-type matrix equations has expanded to $\mathbb{H}$, finding many applications, including signal processing, color-image processing and so on (see, e.g., [16], [29], [40], [41]). Many researchers have studied the Sylvester-type matrix equations over $\mathbb{H}$ ([10]-[20], [22], [23], [33]-[39]). For example, He et al. [11] investigated some necessary and sufficient conditions for Sylvester-type quaternion matrix equations and derived an expression for their general solution. In addition, Solvability conditions and the general solution for a system of constrained two-sided Sylvester-type quaternion matrix equations were established by Wang [37]. Moreover, Wang et al. [38] presented some necessary and sufficient conditions for the Sylvester-type matrix equations

$$\begin{aligned} &A_1X = C_1,\ XB_1 = C_2,\\ &A_2Y = C_3,\ YB_2 = C_4,\\ &A_3XB_3 + A_4YB_4 = C_c \end{aligned} \tag{1.3}$$

to provide a common solution and an expression for a general solution to equations (1.3) over $\mathbb{H}$. In 2022, Liu, et al. [19] derived some necessary and sufficient conditions to solve the following Sylvester-type quaternion matrix equation by using ranks of coefficient matrices and M-P inverses, respectively:

$$A_1X_1 + X_2B_1 + A_2Y_1B_2 + A_3Y_2B_3 + A_4Y_3B_4 = B. \tag{1.4}$$

They also given an expression for a general solution (when it is solvable). Moreover, He and Wang [14] studied the solvability conditions for the following Sylvester quaternion matrix equations to be consistent using matrix decomposition:

$$\begin{aligned} A_1Y_1 = A_2,\ Y_1B_1 = B_2, \\ A_{11}Y_1B_{11} + A_{22}Y_2B_{22} + A_{33}Y_3B_{33} = B. \end{aligned} \tag{1.5}$$

However, to our knowledge, there is no additional information to extend equations (1.5) and investigate the necessary and sufficient conditions for equations (1.5) to be consistent in terms of M-P inverses and derive an expression for its general solution using these inverses. Motivated by the worked mentioned above and keeping the interest and wide application of matrix equations, in this paper, we extend equations (1.5), i.e., the following Sylvester-type matrix equations:

$$\begin{aligned} &A_1U = C_1,\ VB_1 = D_1, \\ &A_2X = C_2,\ XB_2 = D_2, \\ &A_3Y = C_3,\ YB_3 = D_3, \\ &A_4Z = C_4,\ ZB_4 = D_4, \\ E_1U + VF_1 + &E_2XF_2 + E_3YF_3 + E_4ZF_4 = C_c. \end{aligned} \tag{1.6}$$

This is achieved using rank equalities and M-P inverses of some coefficients quaternion matrices in equations (1.6) and derive a formula for its general solution (when it is solvable), where A_i, B_i, C_i, D_i, E_i, F_i $(i = \overline{1,4})$ and C_c are given matrices, while X, Y, Z are unknown. It is obvious that the system of matrix equations (1.6) is an extension of the other equations (1.1), (1.2), (1.3), (1.4) and (1.5). As a special case of equations (1.6), we present some necessary and sufficient conditions for the following system of two-sided Sylvester-type matrix equations to provide a solution and derive an expression of its general solution (when it is solvable):

$$\begin{aligned} &A_1X = C_1,\ XB_1 = D_1, \\ &A_2Y = C_2,\ YB_2 = D_2, \\ &A_3Z = C_3,\ ZB_3 = D_3, \\ E_1XF_1 + &E_2YF_2 + E_3ZF_3 = C. \end{aligned} \tag{1.7}$$

We known that η-Hermitian matrices have some applications, such as in linear modeling (e.g., [12], [13], [30]). Many researchers have studied matrix equations involving η-Hermicity. For instance, He and Wang [10] established the necessary and sufficient conditions for a solution to the following matrix equation:

$$B_1XB_1^{\eta^*} + C_1YC_1^{\eta^*} = D_1, \tag{1.8}$$

where X and Y are η-Hermitian. Zhang and Wang [44] presented the solvability conditions and the general solution of the following matrix equations:

$$\begin{aligned} A_1X = C_1,\ YB_1 = D_1, \\ A_2XA_2^{\eta^*} + A_3YA_3^{\eta^*} = D_3, \end{aligned} \tag{1.9}$$

where X and Y are η-Hermitian.

Furthermore, as an application of equations (1.6), we investigate some necessary and sufficient conditions for the following matrix equations to be consistent and derive an expression for its general solution:

$$\begin{aligned} A_1U &= C_1, \\ A_2X &= C_2,\ X = X^{\eta^*}, \\ A_3Y &= C_3,\ Y = Y^{\eta^*}, \\ A_4Z &= C_4,\ Z = Y^{\eta^*}, \\ E_1U + (E_1U)^{\eta^*} + E_2XE_2^{\eta^*} &+ E_3YE_3^{\eta^*} + E_4ZE_4^{\eta^*} = C_c. \end{aligned} \tag{1.10}$$

As a special case of equations (1.10), we establish the necessary and sufficient conditions for the following system of matrix equations to provide a solution and a formula for its general solution, which is an η-Hermitian solution:

$$\begin{aligned} A_1X &= C_1,\ X = X^{\eta^*}, \\ A_2Y &= C_2,\ Y = Y^{\eta^*}, \\ A_3Z &= C_3,\ Z = Y^{\eta^*}, \\ E_1XE_1^{\eta^*} + E_2YE_2^{\eta^*} &+ E_3ZE_3^{\eta^*} = C. \end{aligned} \tag{1.11}$$

Clearly, equation (1.8) and equations (1.9) are special case of equations (1.10).

The remainder of this article is built up as follows. In Section 2, we present the preliminaries. In Section 3, we establish some necessary and sufficient conditions for the system of matrix equations (1.6) to have a solution by using the M-P inverses and rank equalities of the quaternion matrices involved. In addition, we provide a formula for its general solution (when it is solvable). As a special case of equations (1.6), we also present the solvability conditions and a formula for the general solution of equations (1.7) (when it is solvable). In Section 4, as an application of equations (1.6), we investigate some solvability conditions and the general solution to equations (1.10), where X, Y, Z are η-Hermitian. Moreover, as a special case of equations (1.10), we also investigate some solvability conditions and the general solution to equations (1.11), which is an η-Hermitian solution. In section 5, we present an algorithm and an example to illustrate the main results. Finally, we provide a brief conclusions to close the paper in Section 6.

2. Preliminaries

The following lemma is due to Marsaglia and Styan [24], which can be generalized to $\mathbb{H}$.

Lemma 2.1. [24] *Let $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{m\times k}$, $C \in \mathbb{H}^{l\times n}$, $D \in \mathbb{H}^{j\times k}$ and $E \in \mathbb{H}^{l\times i}$ be given. Then we have the following rank equality:*

$$r\begin{pmatrix} A & BL_D \\ R_EC & 0 \end{pmatrix} = r\begin{pmatrix} A & B & 0 \\ C & 0 & E \\ 0 & D & 0 \end{pmatrix} - r(D) - r(E).$$

Lemma 2.2. [3] *Let A_1 and A_2 be given matrices $\mathbb{H}$. Then $A_1X = A_2$ is solvable if and only if $A_2 = A_1A_1^{\dagger}A_2$. In this case, the general solution to this equation can be expressed as*

$$X = A_1^{\dagger}A_2 + L_{A_1}U_1,$$

where U_1 is an any matrix with conformable size over $\mathbb{H}$.

Lemma 2.3. [3] *Let* A_1 *and* A_2 *be given matrices with adequate shapes over* $\mathbb{H}$. *Then* $XA_1 = A_2$ *is solvable if and only if* $A_2 = A_2A_1^{\dagger}A_1$. *In this case, the general solution to this equation can be expressed as*

$$X = A_2A_1^{\dagger} + U_1R_{A_1},$$

where U_1 *is an any matrix with conformable size over* $\mathbb{H}$.

Lemma 2.4. [3] *Let* $A_1 \in \mathbb{H}^{m_1\times n_1}, B_1 \in \mathbb{H}^{r_1\times s_1}, C_1 \in \mathbb{H}^{m_1\times r_1}$ *and* $C_2 \in \mathbb{H}^{n_1\times s_1}$ *be given matrices. Then the system*

$$A_1X_1 = C_1, \quad X_1B_1 = C_2 \tag{2.1}$$

is consistent if and only if

$$R_{A_1}C_1 = 0, \quad C_2L_{B_1} = 0, \quad A_1C_2 = C_1B_1.$$

Under these conditions, a general solution to equations (2.1) *can be expressed as*

$$X_1 = A_1^{\dagger}C_1 + L_{A_1}C_2B_1^{\dagger} + L_{A_1}U_1R_{B_1},$$

where U_1 *is an arbitrary matrix of appropriate shape over* $\mathbb{H}$.

Lemma 2.5. [19] *Let* A_i, B_i *and* B $(i = \overline{1,4})$ *be given quaternion matrices with appropriate sizes. Put*

$R_{A_1}A_{i+1} = A_{ii},\ B_{i+1}L_{B_1} = B_{ii}(i = \overline{1,3}),\ T_1 = R_{A_1}BL_{B_1},\ B_{22}L_{B_{11}} = N_1,\ R_{A_{11}}A_{22} = M_1,$
$S_1 = A_{22}L_{M_1},\ C = R_{M_1}R_{A_{11}}, C_1 = CA_{33},\ C_2 = R_{A_{11}}A_{33},\ C_3 = R_{A_{22}}A_{33},\ C_4 = A_{33},\ D = L_{B_{11}}L_{N_1},$
$D_1 = B_{33},\ D_2 = B_{33}L_{B_{22}},\ D_3 = B_{33}L_{B_{11}},\ D_4 = B_{33}D,\ E_1 = CT_1,\ E_2 = R_{A_{11}}T_1L_{B_{22}},$
$E_3 = R_{A_{22}}T_1L_{B_{11}},\ E_4 = T_1D,\ C_{11} = (L_{C_2},\ L_{C_4}),\ D_{11} = \begin{pmatrix} R_{D_1} \\ R_{D_3} \end{pmatrix},\ C_{22} = L_{C_1},\ D_{22} = R_{D_2},$
$C_{33} = L_{C_3},\ D_{33} = R_{D_4},\ F_1 = C_1^{\dagger}E_1D_1^{\dagger} + L_{C_1}C_2^{\dagger}E_2D_2^{\dagger},\ E_{11} = R_{C_{11}}C_{22},\ E_{22} = R_{C_{11}}C_{33},\ E_{33} = D_{22}L_{D_{11}},$
$E_{44} = D_{33}L_{D_{11}},\ F_2 = C_3^{\dagger}E_3D_3^{\dagger} + L_{C_3}C_4^{\dagger}E_4D_4^{\dagger},\ M = R_{E_{11}}E_{22},\ N = E_{44}L_{E_{33}},\ F = F_2 - F_1,\ E = R_{C_{11}}FL_{D_{11}},$
$S = E_{22}L_M,\ G_1 = E_2 - C_2C_1^{\dagger}E_1D_1^{\dagger}D_2,\ F_{11} = C_2L_{C_1},\ F_{22} = C_4L_{C_3},\ G_2 = E_4 - C_4C_3^{\dagger}E_3D_3^{\dagger}D_4.$

Then following statements are equivalent:

(1) *Equation* (1.4) *is consistent.*

(2)

$$R_{C_i}E_i = 0,\ E_iL_{D_i} = 0\ (i = \overline{1,4}),\ R_{E_{22}}EL_{E_{33}} = 0.$$

(3)

$$r\begin{pmatrix} B & A_2 & A_3 & A_4 & A_1 \\ B_1 & 0 & 0 & 0 & 0 \end{pmatrix} = r(B_1) + r(A_2,\ A_3,\ A_4,\ A_1),$$

$$r\begin{pmatrix} B & A_2 & A_4 & A_1 \\ B_3 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 \end{pmatrix} = r(A_2,\ A_4,\ A_1) + r\begin{pmatrix} B_3 \\ B_1 \end{pmatrix},$$

$$r\begin{pmatrix} B & A_3 & A_4 & A_1 \\ B_2 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 \end{pmatrix} = r(A_3,\ A_4,\ A_1) + r\begin{pmatrix} B_2 \\ B_1 \end{pmatrix},$$

$$r\begin{pmatrix} B & A_4 & A_1 \\ B_2 & 0 & 0 \\ B_3 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_2 \\ B_3 \\ B_1 \end{pmatrix} + r(A_4,\ A_1),$$

$$r\begin{pmatrix} B & A_2 & A_3 & A_1 \\ B_4 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 \end{pmatrix} = r(A_2,\ A_3,\ A_1) + r\begin{pmatrix} B_4 \\ B_1 \end{pmatrix},$$

$$r\begin{pmatrix} B & A_2 & A_1 \\ B_3 & 0 & 0 \\ B_4 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_3 \\ B_4 \\ B_1 \end{pmatrix} + r(A_2,\ A_1),$$

$$r\begin{pmatrix} B & A_3 & A_1 \\ B_2 & 0 & 0 \\ B_4 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_2 \\ B_4 \\ B_1 \end{pmatrix} + r(A_3,\ A_1),$$

$$r\begin{pmatrix} B & A_1 \\ B_2 & 0 \\ B_3 & 0 \\ B_4 & 0 \\ B_1 & 0 \end{pmatrix} = r\begin{pmatrix} B_2 \\ B_3 \\ B_4 \\ B_1 \end{pmatrix} + r(A_1),$$

$$r\begin{pmatrix} B & A_2 & A_1 & 0 & 0 & 0 & A_4 \\ B_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B & A_3 & A_1 & A_4 \\ 0 & 0 & 0 & B_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 & 0 & 0 & 0 \\ B_4 & 0 & 0 & B_4 & 0 & 0 & 0 \end{pmatrix}$$

$$= r\begin{pmatrix} B_3 & 0 \\ B_1 & 0 \\ 0 & B_2 \\ 0 & B_1 \\ B_4 & B_4 \end{pmatrix} + r\begin{pmatrix} A_2 & A_1 & 0 & 0 & A_4 \\ 0 & 0 & A_3 & A_1 & A_4 \end{pmatrix}.$$

In this case, the solution of equation (1.4) *can be expressed as*

$$
\begin{aligned}
&X_1 = A_1^\dagger(B - A_2Y_1B_2 - A_3Y_2B_3 - A_4Y_3B_4) - A_1^\dagger U_1B_1 + L_{A_1}U_2,\\
&X_2 = R_{A_1}(B - A_2Y_1B_2 - A_3Y_2B_3 - A_4Y_3B_4)B_1^\dagger + A_1A_1^\dagger U_1 + U_3R_{B_1},\\
&Y_1 = A_{11}^\dagger TB_{11}^\dagger - A_{11}^\dagger A_{22}M_1^\dagger TB_{11}^\dagger - A_{11}^\dagger S_1A_{22}^\dagger TN_1^\dagger B_{22}B_{11}^\dagger\\
&- A_{11}^\dagger S_1U_4R_{N_1}B_{22}B_{11}^\dagger + L_{A_{11}}U_5 + U_6R_{B_{11}},\\
&Y_2 = M_1^\dagger TB_{22}^\dagger + S_1^\dagger S_1A_{22}^\dagger TN_1^\dagger + L_{M_1}L_{S_1}U_7 + U_8R_{B_{22}} + L_{M_1}U_4R_{N_1},\\
&Y_3 = F_1 + L_{C_2}V_1 + V_2R_{D_1} + L_{C_1}V_3R_{D_2},\\
&\textit{or}\\
&Y_3 = F_2 - L_{C_4}W_1 - W_2R_{D_3} - L_{C_3}W_3R_{D_4},
\end{aligned}
$$

where $T = T_1 - A_{33}Y_3B_{33}$, $U_i(i = \overline{1,8})$ *are arbitrary matrices with appropriate sizes over* $\mathbb{H}$,

$$
\begin{aligned}
&V_1 = (I_m,\ 0)\left[C_{11}^\dagger(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})\right] - (I_m,\ 0)\left[C_{11}^\dagger U_{11}D_{11} - L_{C_{11}}U_{12}\right],\\
&W_1 = (0,\ I_m)\left[C_{11}^\dagger(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})\right] - (0,\ I_m)\left[C_{11}^\dagger U_{11}D_{11} - L_{C_{11}}U_{12}\right],\\
&W_2 = \left[R_{C_{11}}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^\dagger\right]\begin{pmatrix} 0\\ I_n\end{pmatrix} + \left[C_{11}C_{11}^\dagger U_{11} + U_{21}R_{D_{11}}\right]\begin{pmatrix} 0\\ I_n\end{pmatrix},\\
&V_2 = \left[R_{C_{11}}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^\dagger\right]\begin{pmatrix} I_n\\ 0\end{pmatrix} + \left[C_{11}C_{11}^\dagger U_{11} + U_{21}R_{D_{11}}\right]\begin{pmatrix} I_n\\ 0\end{pmatrix},\\
&V_3 = E_{11}^\dagger FE_{33}^\dagger - E_{11}^\dagger E_{22}M^\dagger FE_{33}^\dagger - E_{11}^\dagger SE_{22}^\dagger FN^\dagger E_{44}E_{33}^\dagger - E_{11}^\dagger SU_{31}R_NE_{44}E_{33}^\dagger + L_{E_{11}}U_{32} + U_{33}R_{E_{33}},\\
&W_3 = M^\dagger FE_{44}^\dagger + S^\dagger SE_{22}^\dagger FN^\dagger + L_ML_SU_{41} + L_MU_{31}R_N - U_{42}R_{E_{44}},
\end{aligned}
$$

$U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41}$ *and* U_{42} *are arbitrary matrices of appropriate sizes over* $\mathbb{H}$. m *is the column number of* A_4 *and* n *is the row number of* B_4.

3. Necessary and sufficient conditions for the existence of a solution to equations (1.6)

The goal of this section is to establish the solvability conditions and a formula of its general solution to equations (1.6).

For the convenience, we define some notations the follows: Let $A_i, B_i, C_i, D_i, E_i, F_i$ $(i = \overline{1,4})$, and C_c be given matrices of appropriate sizes over $\mathbb{H}$. Put

$$
\begin{aligned}
&E_iL_{A_i} = A_{ii},\ R_{B_i}F_i = B_{ii}(i = \overline{1,4}),\ B_{j1} = B_{jj}L_{B_{11}}(j = \overline{2,4}),\ T_1 = C_c - E_1A_1^\dagger C_1\\
&- D_1B_1^\dagger F_1 - \left(\sum_{i=2}^{4} E_i(A_i^\dagger C_i + L_{A_i}D_iB_i^\dagger)F_i\right),\ A_{1j} = R_{A_{11}}A_{jj},\ B_{31}L_{B_{21}} = N_1,\ R_{A_{12}}A_{13} = M_1,\\
&S_1 = A_{13}L_{M_1},\ R_{A_{11}}T_1L_{B_{11}} = T_2,
\end{aligned}
\tag{3.1}
$$

$$G = R_{M_1}R_{A_{12}},\ G_1 = GA_{14},\ G_2 = R_{A_{12}}A_{14},\ G_3 = R_{A_{13}}A_{14},\ G_4 = A_{14},\ H = L_{B_{21}}L_{N_1},$$
$$H_1 = B_{41},\ H_2 = B_{41}L_{B_{31}},\ H_3 = B_{41}L_{B_{21}},\ H_4 = B_{41}H,\ L_1 = GT_2,\ L_2 = R_{A_{12}}T_2L_{B_{31}}, \tag{3.2}$$
$$L_3 = R_{A_{13}}T_2L_{B_{21}},\ L_4 = T_2H,$$

$$C_{11} = (L_{G_2},\ L_{G_4}),\ D_{11} = \begin{pmatrix} R_{H_1} \\ R_{H_3} \end{pmatrix},\ C_{22} = L_{G_1},\ D_{22} = R_{H_2},\ C_{33} = L_{G_3},\ D_{33} = R_{H_4},$$
$$E_{11} = R_{C_{11}}C_{22}\ E_{22} = R_{C_{11}}C_{33},\ E_{33} = D_{22}L_{D_{11}},\ E_{44} = D_{33}L_{D_{11}},\ M = R_{E_{11}}E_{22}, \tag{3.3}$$
$$N = E_{44}L_{E_{33}},\ F = F_{44} - F_{33},\ E = R_{C_{11}}FL_{D_{11}},\ S = E_{22}L_M,\ F_{11} = G_2L_{G_1},$$

$$G_{11} = L_2 - G_2G_1^{\dagger}L_1H_1^{\dagger}H_2,\ F_{22} = G_4L_{G_3},\ G_{22} = L_4 - G_4G_3^{\dagger}L_3H_3^{\dagger}H_4,$$
$$F_{33} = G_1^{\dagger}L_1H_1^{\dagger} + L_{G_1}G_2^{\dagger}L_2H_2^{\dagger},\ F_{44} = G_3^{\dagger}L_3H_3^{\dagger} + L_{G_3}G_4^{\dagger}L_4H_4^{\dagger}. \tag{3.4}$$

Theorem 3.1. *Consider* (1.6) *with the notation in* (3.1) *to* (3.4)*. The following statements are equivalent:*

(1) *System* (1.6) *has a solution.*

(2)

$$A_iD_i = C_iB_i,\ (i = \overline{2,4}) \tag{3.5}$$

and

$$R_{A_j}C_j = 0,\ D_jL_{B_j} = 0,\ R_{G_j}L_j = 0,\ L_jL_{H_j} = 0(j = \overline{1,4}),\ R_{E_{22}}EL_{E_{33}} = 0. \tag{3.6}$$

(3) (3.5) *holds and*

$$r(C_i,\ A_i) = r(A_i),\ r\begin{pmatrix} D_i \\ B_i \end{pmatrix} = r(B_i)\ (i = \overline{1,4}), \tag{3.7}$$

$$r\begin{pmatrix} C_c & E_1 & E_2 & E_3 & E_4 & D_1 \\ F_1 & 0 & 0 & 0 & 0 & B_1 \\ C_1 & A_1 & 0 & 0 & 0 & 0 \\ C_2F_2 & 0 & A_2 & 0 & 0 & 0 \\ C_3F_3 & 0 & 0 & A_3 & 0 & 0 \\ C_4F_4 & 0 & 0 & 0 & A_4 & 0 \end{pmatrix} = r\begin{pmatrix} E_1 & E_2 & E_3 & E_4 \\ A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & A_4 \end{pmatrix} + r(F_1,\ B_1), \tag{3.8}$$

$$r\begin{pmatrix} C_c & E_1 & E_2 & E_4 & E_3D_3 & D_1 \\ C_1 & A_1 & 0 & 0 & 0 & 0 \\ C_2F_2 & 0 & A_2 & 0 & 0 & 0 \\ C_4F_4 & 0 & 0 & A_4 & 0 & 0 \\ F_3 & 0 & 0 & 0 & B_3 & 0 \\ F_1 & 0 & 0 & 0 & 0 & B_1 \end{pmatrix} = r\begin{pmatrix} E_1 & E_2 & E_4 \\ A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_4 \end{pmatrix} + r\begin{pmatrix} F_3 & B_3 & 0 \\ F_1 & 0 & B_1 \end{pmatrix}, \tag{3.9}$$

$$r\begin{pmatrix} C_c & E_1 & E_3 & E_4 & E_2D_2 & D_1 \\ C_1 & A_1 & 0 & 0 & 0 & 0 \\ C_3F_3 & 0 & A_3 & 0 & 0 & 0 \\ C_4F_4 & 0 & 0 & A_4 & 0 & 0 \\ F_2 & 0 & 0 & 0 & B_2 & 0 \\ F_1 & 0 & 0 & 0 & 0 & B_1 \end{pmatrix} = r\begin{pmatrix} E_1 & E_3 & E_4 \\ A_1 & 0 & 0 \\ 0 & A_3 & 0 \\ 0 & 0 & A_4 \end{pmatrix} + r\begin{pmatrix} F_2 & B_2 & 0 \\ F_1 & 0 & B_1 \end{pmatrix}, \qquad (3.10)$$

$$r\begin{pmatrix} C_c & E_4 & E_1 & E_2D_2 & E_3D_3 & D_1 \\ F_2 & 0 & 0 & B_2 & 0 & 0 \\ F_3 & 0 & 0 & 0 & B_3 & 0 \\ F_1 & 0 & 0 & 0 & 0 & B_1 \\ C_4F_4 & A_4 & 0 & 0 & 0 & 0 \\ C_1 & 0 & A_1 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} F_2 & B_2 & 0 & 0 \\ F_3 & 0 & B_3 & 0 \\ F_1 & 0 & 0 & B_1 \end{pmatrix} + r\begin{pmatrix} E_4 & E_1 \\ A_4 & 0 \\ 0 & A_1 \end{pmatrix}, \qquad (3.11)$$

$$r\begin{pmatrix} C_c & E_1 & E_2 & E_3 & E_4D_4 & D_1 \\ C_1 & A_1 & 0 & 0 & 0 & 0 \\ C_2F_2 & 0 & A_2 & 0 & 0 & 0 \\ C_3F_3 & 0 & 0 & A_3 & 0 & 0 \\ F_4 & 0 & 0 & 0 & B_4 & 0 \\ F_1 & 0 & 0 & 0 & 0 & B_1 \end{pmatrix} = r\begin{pmatrix} E_1 & E_2 & E_3 \\ A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix} + r\begin{pmatrix} F_4 & B_4 & 0 \\ F_1 & 0 & B_1 \end{pmatrix}, \qquad (3.12)$$

$$r\begin{pmatrix} C_c & E_2 & E_1 & E_3D_3 & E_4D_4 & D_1 \\ F_3 & 0 & 0 & B_3 & 0 & 0 \\ F_4 & 0 & 0 & 0 & B_4 & 0 \\ F_1 & 0 & 0 & 0 & 0 & B_1 \\ C_2F_2 & A_2 & 0 & 0 & 0 & 0 \\ C_1 & 0 & A_1 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} F_3 & B_3 & 0 & 0 \\ F_4 & 0 & B_4 & 0 \\ F_1 & 0 & 0 & B_1 \end{pmatrix} + r\begin{pmatrix} E_2 & E_1 \\ A_2 & 0 \\ 0 & A_1 \end{pmatrix}, \qquad (3.13)$$

$$r\begin{pmatrix} C_c & E_3 & E_1 & E_2D_2 & E_4D_4 & D_1 \\ F_2 & 0 & 0 & B_2 & 0 & 0 \\ F_4 & 0 & 0 & 0 & B_4 & 0 \\ F_1 & 0 & 0 & 0 & 0 & B_1 \\ C_3F_3 & A_3 & 0 & 0 & 0 & 0 \\ C_1 & 0 & A_1 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} F_2 & B_2 & 0 & 0 \\ F_4 & 0 & B_4 & 0 \\ F_1 & 0 & 0 & B_1 \end{pmatrix} + r\begin{pmatrix} E_3 & E_1 \\ A_3 & 0 \\ 0 & A_1 \end{pmatrix}, \qquad (3.14)$$

$$r\begin{pmatrix} C_c & E_1 & E_4D_4 & E_2D_2 & E_3D_3 & D_1 \\ F_4 & 0 & B_4 & 0 & 0 & 0 \\ F_2 & 0 & 0 & B_2 & 0 & 0 \\ F_3 & 0 & 0 & 0 & B_3 & 0 \\ F_1 & 0 & 0 & 0 & 0 & B_1 \\ C_1 & A_1 & 0 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} F_4 & B_4 & 0 & 0 & 0 \\ F_2 & 0 & B_2 & 0 & 0 \\ F_3 & 0 & 0 & B_3 & 0 \\ F_1 & 0 & 0 & 0 & B_1 \end{pmatrix} + r\begin{pmatrix} E_1 \\ A_1 \end{pmatrix}, \qquad (3.15)$$

$$
r\begin{pmatrix}
C_c & E_2 & E_1 & 0 & 0 & 0 & E_4 & E_3D_3 & D_1 & 0 & 0 & E_4D_4\\
F_3 & 0 & 0 & 0 & 0 & 0 & 0 & B_3 & 0 & 0 & 0 & 0\\
F_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_1 & 0 & 0 & 0\\
0 & 0 & 0 & C_c & E_3 & E_1 & E_4 & 0 & 0 & E_2D_2 & D_1 & 0\\
0 & 0 & 0 & F_2 & 0 & 0 & 0 & 0 & 0 & B_2 & 0 & 0\\
0 & 0 & 0 & F_1 & 0 & 0 & 0 & 0 & 0 & 0 & B_1 & 0\\
F_4 & 0 & 0 & -F_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_4\\
C_2F_2 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
C_1 & 0 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & C_3F_3 & A_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & C_1 & 0 & A_1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & C_4F_4 & 0 & 0 & A_4 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
= r\begin{pmatrix}
F_3 & 0 & B_3 & 0 & 0 & 0 & 0\\
F_1 & 0 & 0 & B_1 & 0 & 0 & 0\\
0 & F_2 & 0 & 0 & B_2 & 0 & 0\\
0 & F_1 & 0 & 0 & 0 & B_1 & 0\\
F_4 & F_4 & 0 & 0 & 0 & 0 & B_4
\end{pmatrix}
+ r\begin{pmatrix}
E_2 & E_1 & 0 & 0 & E_4\\
0 & 0 & E_3 & E_1 & E_4\\
A_2 & 0 & 0 & 0 & 0\\
0 & A_1 & 0 & 0 & 0\\
0 & 0 & A_3 & 0 & 0\\
0 & 0 & 0 & A_1 & 0\\
0 & 0 & 0 & 0 & A_4
\end{pmatrix}. \tag{3.16}
$$

In this case, the general solution to system (1.6) *is*

$$
\begin{aligned}
&U = A_1^{\dagger}C_1 + L_{A_1}S_1,\ V = D_1B_1^{\dagger} + S_2R_{A_1},\ X = A_2^{\dagger}C_2 + L_{A_2}D_2B_2^{\dagger} + L_{A_2}U_1R_{B_2},\\
&Y = A_3^{\dagger}C_3 + L_{A_3}D_3B_3^{\dagger} + L_{A_3}U_2R_{B_3},\ Z = A_4^{\dagger}C_4 + L_{A_4}D_4B_4^{\dagger} + L_{A_4}U_3R_{B_4},
\end{aligned} \tag{3.17}
$$

where

$$
\begin{aligned}
&S_1 = A_{11}^{\dagger}(T_1 - A_{22}XB_{22} - A_{33}YB_{33} - A_{44}ZB_{44}) - A_{11}^{\dagger}W_{11}B_{11} + L_{A_{11}}W_{12},\\
&S_2 = R_{A_{11}}(T_1 - A_{22}XB_{22} - A_{33}YB_{33} - A_{44}ZB_{44})B_{11}^{\dagger} + A_{11}A_{11}^{\dagger}W_{11} + W_{13}R_{B_{11}},\\
&U_1 = A_{12}^{\dagger}TB_{21}^{\dagger} - A_{12}^{\dagger}A_{13}M_1^{\dagger}TB_{21}^{\dagger} - A_{12}^{\dagger}S_1A_{13}^{\dagger}TN_1^{\dagger}B_{31}B_{21}^{\dagger}\\
&- A_{12}^{\dagger}S_1U_4R_{N_1}B_{31}B_{21}^{\dagger} + L_{A_{21}}U_5 + U_6R_{B_{21}},\\
&U_2 = M_1^{\dagger}TB_{31}^{\dagger} + S_1^{\dagger}S_1A_{22}^{\dagger}TN_1^{\dagger} + L_{M_1}L_{S_1}U_7 + U_8R_{B_{22}} + L_{M_1}U_4R_{N_1},\\
&U_3 = F_{10} + L_{G_2}V_1 + V_2R_{H_1} + L_{G_1}V_3R_{H_2},\ or U_3 = F_{20} - L_{G_4}W_1 - W_2R_{H_3} - L_{G_3}W_3R_{H_4},\\
&V_1 = (I_m,\ 0)\left[C_{11}^{\dagger}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})\right] - (I_m,\ 0)\left[C_{11}^{\dagger}U_{11}D_{11} - L_{C_{11}}U_{12}\right],\\
&W_1 = (0,\ I_m)\left[C_{11}^{\dagger}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})\right] - (0,\ I_m)\left[C_{11}^{\dagger}U_{11}D_{11} - L_{C_{11}}U_{12}\right],\\
&W_2 = \left[R_{C_{11}}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^{\dagger}\right]\begin{pmatrix}0\\ I_n\end{pmatrix} + \left[C_{11}C_{11}^{\dagger}U_{11} + U_{21}R_{D_{11}}\right]\begin{pmatrix}0\\ I_n\end{pmatrix},
\end{aligned}
$$

$$V_2=\Big[R_{C_{11}}(F-C_{22}V_3D_{22}-C_{33}W_3D_{33})D_{11}^{\dagger}\Big]\begin{pmatrix}I_n\\0\end{pmatrix}+\Big[C_{11}C_{11}^{\dagger}U_{11}+U_{21}R_{D_{11}}\Big]\begin{pmatrix}I_n\\0\end{pmatrix},$$
$$V_3=E_{11}^{\dagger}FE_{33}^{\dagger}-E_{11}^{\dagger}E_{22}M^{\dagger}FE_{33}^{\dagger}-E_{11}^{\dagger}SE_{22}^{\dagger}FN^{\dagger}E_{44}E_{33}^{\dagger}$$
$$-E_{11}^{\dagger}SU_{31}R_NE_{44}E_{33}^{\dagger}+L_{E_{11}}U_{32}+U_{33}R_{E_{33}},$$
$$W_3=M^{\dagger}FE_{44}^{\dagger}+S^{\dagger}SE_{22}^{\dagger}FN^{\dagger}+L_ML_SU_{41}+L_MU_{31}R_N+U_{42}R_{E_{44}},$$

where $T=T_1-A_{33}U_3B_{33}$, $U_j(j=\overline{4,6})$, $U_{i1}(i=\overline{1,4})$, U_{12}, U_{32}, U_{33} *and* U_{42} *are arbitrary matrices with appropriate shapes over* $\mathbb{H}$. m *is the column number of* A_4 *and* n *is the row number of* B_4.

Proof. (1) $\Leftrightarrow$ (2) It is clear that the system of matrix equations (1.6) is solvable if and only if both

$$\begin{aligned}&A_1U=C_1,\ VB_1=D_1,\\&A_2X=C_2,\ XB_2=D_2,\\&A_3Y=C_3,\ YB_3=D_3,\\&A_4Z=C_4,\ ZB_4=D_4\end{aligned}\tag{3.18}$$

and

$$E_1U+VF_1+E_2XF_2+E_3YF_3+E_4ZF_4=C_c\tag{3.19}$$

are solvable. It follows from Lemma 2.2, Lemma 2.3, and Lemma 2.4 that the system of matrix equations (3.18) has a solution if and only if (3.5) holds and

$$R_{A_i}C_i=0,\ B_iL_{D_i}=0\ (i=\overline{1,4}).\tag{3.20}$$

In this case, the general solution of equations (3.18) can be expressed as

$$\begin{aligned}&U=A_1^{\dagger}C_1+L_{A_1}S_1,\ V=D_1B_1^{\dagger}+S_2R_{A_1},\\&X=A_2^{\dagger}C_2+L_{A_2}D_2B_2^{\dagger}+L_{A_2}U_1R_{B_2},\\&Y=A_3^{\dagger}C_3+L_{A_3}D_3B_3^{\dagger}+L_{A_3}U_2R_{B_3},\\&Z=A_4^{\dagger}C_4+L_{A_4}D_4B_4^{\dagger}+L_{A_4}U_3R_{B_4}.\end{aligned}\tag{3.21}$$

By substituting U,V,X,Y,Z from (3.21) into (3.19) yields

$$A_{11}S_1+S_2B_{11}+A_{22}U_1B_{22}+A_{33}U_2B_{33}+A_{44}U_3B_{44}=T_1,\tag{3.22}$$

where A_{ii}, $B_{ii}(i=\overline{1,4})$, and T_1 are defined by (3.1). By Lemma 2.5, we obtain that equation (3.22) has a solution if and only if

$$R_{G_i}L_i=0,\ L_iL_{H_i}=0\ (i=\overline{1,4}),\ R_{E_{22}}EL_{E_{33}}=0.\tag{3.23}$$

Under these conditions, the general solution to the matrix equation (3.22) can be expressed as

$$
\begin{aligned}
&S_1 = A_{11}^{\dagger}(T_1 - A_{22}XB_{22} - A_{33}YB_{33} - A_{44}ZB_{44}) - A_{11}^{\dagger}W_{11}B_{11} + L_{A_{11}}W_{12},\\
&S_2 = R_{A_{11}}(T_1 - A_{22}XB_{22} - A_{33}YB_{33} - A_{44}ZB_{44})B_{11}^{\dagger} + A_{11}A_{11}^{\dagger}W_{11} + W_{13}R_{B_{11}},\\
&U_1 = A_{12}^{\dagger}TB_{21}^{\dagger} - A_{12}^{\dagger}A_{13}M_1^{\dagger}TB_{21}^{\dagger} - A_{12}^{\dagger}S_1A_{13}^{\dagger}TN_1^{\dagger}B_{31}B_{21}^{\dagger}\\
&- A_{12}^{\dagger}S_1U_4R_{N_1}B_{31}B_{21}^{\dagger} + L_{A_{21}}U_5 + U_6R_{B_{21}},\\
&U_2 = M_1^{\dagger}TB_{31}^{\dagger} + S_1^{\dagger}S_1A_{22}^{\dagger}TN_1^{\dagger} + L_{M_1}L_{S_1}U_7 + U_8R_{B_{22}} + L_{M_1}U_4R_{N_1},\\
&U_3 = F_{10} + L_{G_2}V_1 + V_2R_{H_1} + L_{G_1}V_3R_{H_2},\ or U_3 = F_{20} - L_{G_4}W_1 - W_2R_{H_3} - L_{G_3}W_3R_{H_4},\\
&V_1 = (I_m,\ 0)\left[C_{11}^{\dagger}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})\right] - (I_m,\ 0)\left[C_{11}^{\dagger}U_{11}D_{11} - L_{C_{11}}U_{12}\right],\\
&W_1 = (0,\ I_m)\left[C_{11}^{\dagger}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})\right] - (0,\ I_m)\left[C_{11}^{\dagger}U_{11}D_{11} - L_{C_{11}}U_{12}\right],\\
&W_2 = \left[R_{C_{11}}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^{\dagger}\right]\begin{pmatrix} 0 \\ I_n \end{pmatrix} + \left[C_{11}C_{11}^{\dagger}U_{11} + U_{21}R_{D_{11}}\right]\begin{pmatrix} 0 \\ I_n \end{pmatrix},\\
&V_2 = \left[R_{C_{11}}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^{\dagger}\right]\begin{pmatrix} I_n \\ 0 \end{pmatrix} + \left[C_{11}C_{11}^{\dagger}U_{11} + U_{21}R_{D_{11}}\right]\begin{pmatrix} I_n \\ 0 \end{pmatrix},\\
&V_3 = E_{11}^{\dagger}FE_{33}^{\dagger} - E_{11}^{\dagger}E_{22}M^{\dagger}FE_{33}^{\dagger} - E_{11}^{\dagger}SE_{22}^{\dagger}FN^{\dagger}E_{44}E_{33}^{\dagger}\\
&- E_{11}^{\dagger}SU_{31}R_NE_{44}E_{33}^{\dagger} + L_{E_{11}}U_{32} + U_{33}R_{E_{33}},\\
&W_3 = M^{\dagger}FE_{44}^{\dagger} + S^{\dagger}SE_{22}^{\dagger}FN^{\dagger} + L_ML_SU_{41} + L_MU_{31}R_N + U_{42}R_{E_{44}},
\end{aligned}
$$

where $T = T_1 - A_{33}U_3B_{33}$, $U_j(j = \overline{4,6})$, $U_{i1}(i = \overline{1,4})$, U_{12}, U_{32}, U_{33} and U_{42} are arbitrary matrices with appropriate shapes over $\mathbb{H}$. m is the column number of A_4 and n is the row number of B_4.

To sum up, both the equations (3.18) and the equation (3.19) are solvable if and only if conditions (3.5), (3.20) and (3.23) hold, i.e, the system of matrix equations (1.6) has a solution if and only if (3.5) and (3.6) hold. Under these conditions, the general solution to equations (1.6) can be expressed as (3.17).

(2) $\Leftrightarrow$ (3) We first show that (3.20) $\Leftrightarrow$ (3.7). According to Lemma 2.1, it follows that

$$
\begin{aligned}
&R_{A_i}C_i = 0 \Leftrightarrow r(R_{A_i}C_i) = 0 \Leftrightarrow r(C_i,\ A_i) = r(A_i)(i = 1,2,3) \Leftrightarrow (3.7),\\
&D_jL_{B_j} = 0 \Leftrightarrow r(D_jL_{B_j}) = 0 \Leftrightarrow r\begin{pmatrix} D_j \\ B_j \end{pmatrix} = r(B_j)(j = 1,2,3) \Leftrightarrow (3.7).
\end{aligned}
\tag{3.24}
$$

It follows from (3.24) that (3.20) $\Leftrightarrow$ (3.7).

We now turn to show that (3.23) holds if and only if (3.8) to (3.16) hold. By Lemma 2.5, (3.23) is equivalent to

$$r\begin{pmatrix} T_1 & A_{22} & A_{33} & A_{44} & A_{11} \\ B_{11} & 0 & 0 & 0 & 0 \end{pmatrix} = r(B_{11}) + r(A_{22},\ A_{33},\ A_{44},\ A_{11}), \tag{3.25}$$

$$r\begin{pmatrix} T_1 & A_{22} & A_{44} & A_{11} \\ B_{33} & 0 & 0 & 0 \\ B_{11} & 0 & 0 & 0 \end{pmatrix} = r(A_{22},\ A_{44},\ A_{11}) + r\begin{pmatrix} B_{33} \\ B_{11} \end{pmatrix}, \tag{3.26}$$

$$r\begin{pmatrix} T_1 & A_{33} & A_{44} & A_{11} \\ B_{22} & 0 & 0 & 0 \\ B_{11} & 0 & 0 & 0 \end{pmatrix} = r(A_{33},\ A_{44},\ A_{11}) + r\begin{pmatrix} B_{22} \\ B_{11} \end{pmatrix}, \tag{3.27}$$

$$r\begin{pmatrix} T_1 & A_{44} & A_{11} \\ B_{22} & 0 & 0 \\ B_{33} & 0 & 0 \\ B_{11} & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_{22} \\ B_{33} \\ B_{11} \end{pmatrix} + r(A_{44},\ A_{11}), \tag{3.28}$$

$$r\begin{pmatrix} T_1 & A_{22} & A_{33} & A_{11} \\ B_{44} & 0 & 0 & 0 \\ B_{11} & 0 & 0 & 0 \end{pmatrix} = r(A_{22},\ A_{33},\ A_{11}) + r\begin{pmatrix} B_{44} \\ B_{11} \end{pmatrix}, \tag{3.29}$$

$$r\begin{pmatrix} T_1 & A_{22} & A_{11} \\ B_{33} & 0 & 0 \\ B_{44} & 0 & 0 \\ B_{11} & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_{33} \\ B_{44} \\ B_{11} \end{pmatrix} + r(A_{22},\ A_{11}), \tag{3.30}$$

$$r\begin{pmatrix} T_1 & A_{33} & A_{11} \\ B_{22} & 0 & 0 \\ B_{44} & 0 & 0 \\ B_{11} & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_{22} \\ B_{44} \\ B_{11} \end{pmatrix} + r(A_{33},\ A_{11}), \tag{3.31}$$

$$r\begin{pmatrix} T_1 & A_{11} \\ B_{22} & 0 \\ B_{33} & 0 \\ B_{44} & 0 \\ B_{11} & 0 \end{pmatrix} = r\begin{pmatrix} B_{22} \\ B_{33} \\ B_{44} \\ B_{11} \end{pmatrix} + r(A_{11}), \tag{3.32}$$

$$r\begin{pmatrix} T_1 & A_{22} & A_{11} & 0 & 0 & 0 & A_{44} \\ B_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ B_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -T_1 & A_{33} & A_{11} & A_{44} \\ 0 & 0 & 0 & B_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{11} & 0 & 0 & 0 \\ B_{44} & 0 & 0 & B_{44} & 0 & 0 & 0 \end{pmatrix}$$

$$= r\begin{pmatrix} B_{33} & 0 \\ B_{11} & 0 \\ 0 & B_{22} \\ 0 & B_{11} \\ B_{44} & B_{44} \end{pmatrix} + r\begin{pmatrix} A_{22} & A_{11} & 0 & 0 & A_{44} \\ 0 & 0 & A_{33} & A_{11} & A_{44} \end{pmatrix}, \tag{3.33}$$

respectively. Hence, we only show that

$$(19+i) \Leftrightarrow (36+i)\ (i=\overline{1,9}),$$

respectively. When we show that (3.23) holds if and only if (3.8) to (3.16) hold, respectively. It is easy to know that there exist the $U_0, V_0,\ X_0, Y_0$ and Z_0 of the equations (1.6) such that

$$\begin{aligned} &A_1U_0 = C_1,\ V_0B_1 = D_1, \\ &A_2X_0 = C_2,\ X_0B_2 = D_2, \\ &A_3Y_0 = C_3,\ Y_0B_3 = D_3, \\ &A_4Z_0 = C_4,\ Z_0B_4 = D_4, \end{aligned} \tag{3.34}$$

where

$$\begin{aligned} &U_0 = A_1^{\dagger}C_1,\ V_0 = D_1B_1^{\dagger},\ X_0 = A_1^{\dagger}C_1 + L_{A_1}D_1B_1^{\dagger}, \\ &Y_0 = A_2^{\dagger}C_2 + L_{A_2}D_2B_2^{\dagger},\ Z_0 = A_3^{\dagger}C_3 + L_{A_3}D_3B_3^{\dagger}, \end{aligned}$$

It follows from Lemma 2.1, (3.34) and elementary transformations that

$$\begin{aligned} &(3.25) \Leftrightarrow r(T_1,\ E_1L_{A_1},\ E_2L_{A_2},\ E_3L_{A_3}) = r(E_1L_{A_1},\ E_2L_{A_2},\ E_3L_{A_3}) \\ &\Leftrightarrow r\begin{pmatrix} C & E_1 & E_2 & E_3 \\ C_1F_1 & A_1 & 0 & 0 \\ C_2F_2 & 0 & A_2 & 0 \\ C_3F_3 & 0 & 0 & A_3 \end{pmatrix} = r\begin{pmatrix} E_1 & E_2 & E_3 \\ A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix} \Leftrightarrow (3.8). \end{aligned}$$

Similarly, we can show that (3.26) $\Leftrightarrow$ (3.9), (3.27) $\Leftrightarrow$ (3.10), (3.28) $\Leftrightarrow$ (3.11), (3.29) $\Leftrightarrow$ (3.12), (3.30) $\Leftrightarrow$ (3.13), (3.31) $\Leftrightarrow$ (3.14), (3.32) $\Leftrightarrow$ (3.15),

$$\begin{aligned} &(3.33) \Leftrightarrow \\ &r\begin{pmatrix} T_1 & E_2L_{A_2} & E_1L_{A_1} & 0 & 0 & 0 & E_4L_{A_4} \\ R_{B_3}F_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ R_{B_1}F_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -T_1 & E_3L_{A_3} & E_1L_{A_1} & E_4L_{A_4} \\ 0 & 0 & 0 & R_{B_2}F_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_{B_1}F_1 & 0 & 0 & 0 \\ R_{B_4}F_4 & 0 & 0 & R_{B_4}F_4 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

$$= r\begin{pmatrix} R_{B_3}F_3 & 0 \\ R_{B_1}F_1 & 0 \\ 0 & R_{B_2}F_2 \\ 0 & R_{B_1}F_1 \\ R_{B_4}F_4 & R_{B_4}F_4 \end{pmatrix} + r\begin{pmatrix} E_2L_{A_2} & E_1L_{A_1} & 0 & 0 & E_4L_{A_4} \\ 0 & 0 & E_3L_{A_3} & E_1L_{A_1} & E_4L_{A_4} \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} C_c & E_2 & E_1 & 0 & 0 & 0 & E_4 & E_3D_3 & D_1 & 0 & 0 & E_4D_4 \\ F_3 & 0 & 0 & 0 & 0 & 0 & 0 & B_3 & 0 & 0 & 0 & 0 \\ F_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_c & E_3 & E_1 & E_4 & 0 & 0 & E_2D_2 & D_1 & 0 \\ 0 & 0 & 0 & F_2 & 0 & 0 & 0 & 0 & 0 & B_2 & 0 & 0 \\ 0 & 0 & 0 & F_1 & 0 & 0 & 0 & 0 & 0 & 0 & B_1 & 0 \\ F_4 & 0 & 0 & -F_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_4 \\ C_2F_2 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_1 & 0 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_3F_3 & A_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_1 & 0 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_4F_4 & 0 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= r\begin{pmatrix} F_3 & 0 & B_3 & 0 & 0 & 0 & 0 \\ F_1 & 0 & 0 & B_1 & 0 & 0 & 0 \\ 0 & F_2 & 0 & 0 & B_2 & 0 & 0 \\ 0 & F_1 & 0 & 0 & 0 & B_1 & 0 \\ F_4 & F_4 & 0 & 0 & 0 & 0 & B_4 \end{pmatrix} + r\begin{pmatrix} E_2 & E_1 & 0 & 0 & E_4 \\ 0 & 0 & E_3 & E_1 & E_4 \\ A_2 & 0 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 & 0 \\ 0 & 0 & A_3 & 0 & 0 \\ 0 & 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & 0 & A_4 \end{pmatrix} \Leftrightarrow (3.16).$$

We have thus proved the theorem. □

Remark 3.2. Chu et al. gave potential applications of the maximal and minimal ranks in the discipline of control theory(e.g., [5], [6], [7]). We may consider the rank bounds of the general solution of the equation (1.6).

Next, we discuss the special case of (1.6). Let A_i, B_i, C_i, D_i, E_i, F_i $(i = \overline{1,3})$ and C be given matrices of appropriate sizes over $\mathbb{H}$.

$$E_iL_{A_i} = A_{ii},\ R_{B_i}F_i = B_{ii}(i = \overline{1,3}),\ M_1 = R_{A_{11}}A_{22},\ N_1 = B_{22}L_{B_{11}},\ S_1 = A_{22}L_{M_1},$$

$$G = R_{M_1}R_{A_{11}},\ T_1 = C - \left[\sum_{i=1}^{3} E_i(A_i^{\dagger}C_i + L_{A_i}D_iB_i^{\dagger})F_i\right],\ G_1 = GA_{33},\ G_2 = R_{A_{11}}A_{33},$$

$$G_3 = R_{A_{22}}A_{33},\ G_4 = A_{33},\ H = L_{B_{11}}L_{N_1},\ H_1 = B_{33},\ L_1 = GT_1,\ H_2 = B_{33}L_{B_{22}},$$

$$H_3 = B_{33}L_{B_{11}},\ H_4 = B_{33}D,\ L_2 = R_{A_{11}}T_1L_{B_{22}},\ L_3 = R_{A_{22}}T_1L_{B_{11}},\ L_4 = T_1H,$$

$$C_{11} = (L_{G_2},\ L_{G_4}),\ D_{11} = \begin{pmatrix} R_{H_1} \\ R_{H_3} \end{pmatrix},\ C_{22} = L_{G_1},\ D_{22} = R_{H_2},\ C_{33} = L_{G_3},\ D_{33} = R_{H_4},$$

$E_{11}=R_{C_{11}}C_{22},\ E_{22}=R_{C_{11}}C_{33},\ E_{33}=D_{22}L_{D_{11}},\ E_{44}=D_{33}L_{D_{11}},\ M=R_{E_{11}}E_{22},\ N=E_{44}L_{E_{33}},$
$F=F_{20}-F_{10},\ E=R_{C_{11}}FL_{D_{11}},\ S=E_{22}L_M,\ F_{11}=G_2L_{G_1},$
$G_5=L_2-G_2G_1^{\dagger}L_1H_1^{\dagger}H_2,\ F_{22}=G_4L_{G_3},\ G_6=L_4-G_4G_3^{\dagger}L_3H_3^{\dagger}H_4,$
$F_{10}=G_1^{\dagger}L_1H_1^{\dagger}+L_{G_1}G_2^{\dagger}L_2H_2^{\dagger},\ F_{20}=G_3^{\dagger}L_3H_3^{\dagger}+L_{G_3}G_4^{\dagger}L_4H_4^{\dagger}.$

Theorem 3.2. *The following statements are equivalent:*

(1) *system* (1.7) *has a solution.*

(2)

$$A_iD_i=C_iB_i,\ (i=\overline{1,3}) \tag{3.35}$$

and

$$R_{A_i}C_i=0,\ D_iL_{B_i}=0,\ R_{G_j}L_j=0,\ L_jL_{H_j}=0\ (i=\overline{1,3},j=\overline{1,4}),\ R_{E_{22}}EL_{E_{33}}=0.$$

(3) (3.35) *holds and for* $i=\overline{1,3}$.

$$r(C_i,\ A_i)=r(A_i),\ r\begin{pmatrix}D_i\\B_i\end{pmatrix}=r(B_i),$$

$$r\begin{pmatrix}C & E_1 & E_2 & E_3\\ C_1F_1 & A_1 & 0 & 0\\ C_2F_2 & 0 & A_2 & 0\\ C_5F_3 & 0 & 0 & A_3\end{pmatrix}=r\begin{pmatrix}E_1 & E_2 & E_3\\ A_1 & 0 & 0\\ 0 & A_2 & 0\\ 0 & 0 & A_3\end{pmatrix},$$

$$r\begin{pmatrix}C & E_1 & E_3 & E_2D_2\\ F_2 & 0 & 0 & B_2\\ C_1F_1 & A_1 & 0 & 0\\ C_3F_3 & 0 & A_3 & 0\end{pmatrix}=r\begin{pmatrix}E_1 & E_3\\ A_1 & 0\\ 0 & A_3\end{pmatrix}+r(F_2,\ B_2),$$

$$r\begin{pmatrix}C & E_3 & E_2 & E_1D_1\\ F_1 & 0 & 0 & B_1\\ C_3F_3 & A_3 & 0 & 0\\ C_2F_2 & 0 & A_2 & 0\end{pmatrix}=r\begin{pmatrix}E_3 & E_2\\ A_1 & 0\\ 0 & A_3\end{pmatrix}+r(F_1,\ B_1),$$

$$r\begin{pmatrix}C & E_3 & E_1D_1 & E_2D_2\\ F_1 & 0 & B_1 & 0\\ F_2 & 0 & 0 & B_2\\ C_3F_3 & A_3 & 0 & 0\end{pmatrix}=r\begin{pmatrix}F_1 & B_1 & 0\\ F_2 & 0 & B_2\end{pmatrix}+r\begin{pmatrix}E_3\\ A_3\end{pmatrix},$$

$$r\begin{pmatrix}C & E_1 & E_2 & E_3D_3\\ F_3 & 0 & 0 & B_3\\ C_1F_1 & A_1 & 0 & 0\\ C_2F_2 & 0 & A_2 & 0\end{pmatrix}=r\begin{pmatrix}E_1 & E_2\\ A_1 & 0\\ 0 & A_2\end{pmatrix}+r(F_3,\ B_3),$$

$$r\begin{pmatrix} C & E_1 & E_3D_3 & E_2D_2 \\ F_3 & 0 & B_3 & 0 \\ F_2 & 0 & 0 & B_2 \\ C_1F_1 & A_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} F_3 & B_3 & 0 \\ F_2 & 0 & B_2 \end{pmatrix} + r\begin{pmatrix} E_1 \\ A_1 \end{pmatrix},$$

$$r\begin{pmatrix} C & E_2 & E_1D_1 & E_3D_3 \\ F_1 & 0 & B_1 & 0 \\ F_3 & 0 & 0 & B_3 \\ C_2F_2 & A_2 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} F_1 & B_1 & 0 \\ F_3 & 0 & B_3 \end{pmatrix} + r\begin{pmatrix} E_2 \\ A_2 \end{pmatrix},$$

$$r\begin{pmatrix} C & E_1D_1 & E_2D_2 & E_3D_3 \\ F_1 & B_1 & 0 & 0 \\ F_2 & 0 & B_2 & 0 \\ F_3 & 0 & 0 & B_3 \end{pmatrix} = r\begin{pmatrix} F_1 & B_1 & 0 & 0 \\ F_2 & 0 & B_2 & 0 \\ F_3 & 0 & 0 & B_3 \end{pmatrix},$$

$$r\begin{pmatrix} C & 0 & E_1 & 0 & E_3 & E_2D_2 & 0 & E_3D_3 \\ 0 & -C & 0 & E_2 & E_3 & 0 & -E_1D_1 & 0 \\ F_2 & 0 & 0 & 0 & 0 & B_2 & 0 & 0 \\ 0 & F_1 & 0 & 0 & 0 & 0 & B_1 & 0 \\ F_3 & F_3 & 0 & 0 & 0 & 0 & 0 & B_3 \\ C_1F_1 & 0 & A_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -C_2F_2 & 0 & A_2 & 0 & 0 & 0 & 0 \\ 0 & -C_3F_3 & 0 & 0 & A_3 & 0 & 0 & 0 \end{pmatrix}$$

$$= r\begin{pmatrix} F_2 & 0 & B_2 & 0 & 0 \\ 0 & F_1 & 0 & B_1 & 0 \\ F_3 & F_3 & 0 & 0 & B_3 \end{pmatrix} + r\begin{pmatrix} E_1 & 0 & E_3 \\ 0 & E_2 & E_3 \\ A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix}.$$

In this case, the general solution to system (1.7) *is*

$$X = A_1^{\dagger}C_1 + L_{A_1}D_1B_1^{\dagger} + L_{A_1}U_1R_{B_1},\ Y = A_2^{\dagger}C_2 + L_{A_2}D_2B_2^{\dagger} + L_{A_2}U_2R_{B_2},$$
$$Z = A_3^{\dagger}C_3 + L_{A_3}D_3B_3^{\dagger} + L_{A_3}U_3R_{B_3},$$

where

$$U_1 = A_{11}^{\dagger}TB_{11}^{\dagger} - A_{11}^{\dagger}A_{22}M_1^{\dagger}TB_{11}^{\dagger} - A_{11}^{\dagger}S_1A_{22}^{\dagger}TN_1^{\dagger}B_{22}B_{11}^{\dagger} - A_{11}^{\dagger}S_1U_4R_{N_1}B_{22}B_{11}^{\dagger} + L_{A_{11}}U_5 + U_6R_{B_{11}},$$
$$U_2 = M_1^{\dagger}TB_{22}^{\dagger} + S_1^{\dagger}S_1A_{22}^{\dagger}TN_1^{\dagger} + L_{M_1}L_{S_1}U_7 + U_8R_{B_{22}} + L_{M_1}U_4R_{N_1},$$
$$U_3 = F_{10} + L_{G_2}V_1 + V_2R_{H_1} + L_{G_1}V_3R_{H_2},$$

or

$$U_3 = F_{20} - L_{G_4}W_1 - W_2R_{H_3} - L_{G_3}W_3R_{H_4},$$
$$V_1 = (I_m,\ 0)\left[C_{11}^{\dagger}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})\right] - (I_m,\ 0)\left[C_{11}^{\dagger}U_{11}D_{11} - L_{C_{11}}U_{12}\right],$$

$$W_1 = (0,\ I_m)\left[C_{11}^{\dagger}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})\right]$$
$$- (0,\ I_m)\left[C_{11}^{\dagger}U_{11}D_{11} - L_{C_{11}}U_{12}\right],$$
$$W_2 = \left[R_{C_{11}}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^{\dagger}\right]\begin{pmatrix} 0 \\ I_n \end{pmatrix}$$
$$+ \left[C_{11}C_{11}^{\dagger}U_{11} + U_{21}R_{D_{11}}\right]\begin{pmatrix} 0 \\ I_n \end{pmatrix},$$
$$V_2 = \left[R_{C_{11}}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^{\dagger}\right]\begin{pmatrix} I_n \\ 0 \end{pmatrix}$$
$$+ \left[C_{11}C_{11}^{\dagger}U_{11} + U_{21}R_{D_{11}}\right]\begin{pmatrix} I_n \\ 0 \end{pmatrix},$$
$$V_3 = E_{11}^{\dagger}FE_{33}^{\dagger} - E_{11}^{\dagger}E_{22}M^{\dagger}FE_{33}^{\dagger} - E_{11}^{\dagger}SE_{22}^{\dagger}FN^{\dagger}E_{44}E_{33}^{\dagger}$$
$$- E_{11}^{\dagger}SU_{31}R_NE_{44}E_{33}^{\dagger} + L_{E_{11}}U_{32} + U_{33}R_{E_{33}},$$
$$W_3 = M^{\dagger}FE_{44}^{\dagger} + S^{\dagger}SE_{22}^{\dagger}FN^{\dagger} + L_ML_SU_{41} + L_MU_{31}R_N$$
$$+ U_{42}R_{E_{44}},$$

where $T = T_1 - A_{33}U_3B_{33}$, $U_j(j = \overline{4,6})$, $U_{i1}(i = \overline{1,4})$, U_{12}, U_{32}, U_{33} *and* U_{42} *are arbitrary matrices with appropriate shapes over* $\mathbb{H}$. m *is the column number of* A_3 *and* n *is the row number of* B_3.

Proof. It follows from Theorem 3.1 that this holds when $A_1, C_1, B_1, D_1, E_1, F_1$ vanish in Theorem 3.1 □

Lettiing $A_i,\ B_i,\ C_i,\ D_i(i = \overline{1,3})$, E_3 and F_3 vanish in Theorem 3.1, it yields to the following result:

Corollary 3.3. *Let* C_3, D_3, C_4, D_4 *and* E_1 *be given matrices with adequate shapes. Let* $M_1 = R_{C_3}C_4, N_1 = D_4L_{D_3}, S_1 = C_4L_{M_1}$. *Then the matrix equation* (1.2) *is consistent if and only if the following rank equalities hold:*

$$r(C_3\ E_1\ C_4) = r(C_3\ C_4), r\begin{pmatrix} D_3 \\ E_1 \\ D_4 \end{pmatrix} = r\begin{pmatrix} D_3 \\ D_4 \end{pmatrix},$$
$$r\begin{pmatrix} C_3 & E_1 \\ 0 & D_4 \end{pmatrix} = r(C_3) + r(D_4),\ r\begin{pmatrix} D_3 & 0 \\ E_1 & C_4 \end{pmatrix} = r(D_3) + r(C_4).$$

In this case, the general solution to equation (1.2) *can be expressed as*

$$X_3 = C_3^{\dagger}E_1D_3^{\dagger} - C_3^{\dagger}C_4M_1^{\dagger}E_1D_3^{\dagger} - C_3^{\dagger}S_1C_4^{\dagger}E_1N_1^{\dagger}D_4D_3^{\dagger}$$
$$- C_3^{\dagger}S_1Y_{11}R_{N_1}D_4D_4^{\dagger} + L_{C_3}Y_{12} + Y_{13}R_{D_3},$$
$$X_4 = M_1^{\dagger}E_1D_4^{\dagger} + S_1^{\dagger}S_1C_4^{\dagger}E_1N_1^{\dagger} + L_{M_1}L_{S_1}Y_{14} + Y_{15}R_{D_4}$$
$$+ L_{M_1}Y_{11}R_{N_1},$$

where $Y_{1i}(i=\overline{1,5})$ are any matrices with appropriate sizes over $\mathbb{H}$.

Remark 3.3. The above corollary has the main findings of [1].

Lettiing $A_3,\ B_3,\ C_3,\ D_3,\ E_3$ and F_3 vanish in Theorem 3.1, it yields to the following result:

Corollary 3.4. *Let $A_i, B_i, C_i(i=\overline{1,4})$ and C_c be given with appropriate sizes over $\mathbb{H}$. Set*

$$\begin{aligned}&A=A_3L_{A_1},\ B=R_{B_1}B_3,\ C=A_4L_{A_2},\ D=R_{B_2}B_4,\\&M=R_AC,\ N=DL_B,\ S=CL_M,\ E=C_c-A_3A_1^{\dagger}C_1B_3\\&-AC_2B_1^{\dagger}B_3-A_4A_2^{\dagger}C_3B_4-CC_4B_2^{\dagger}B_4.\end{aligned}$$

Then the following statements are equivalent:

(1) *the system of matrix equations* (1.3) *is solvable.*

(2)

$$\begin{aligned}&A_1C_2=C_1B_1,\ A_2C_4=C_3B_2,\ R_{A_1}C_1=0,\\&R_{A_2}C_3=0,\ C_2L_{B_1}=0,\ C_4L_{B_2}=0,\\&R_MR_AE=0,\ R_AEL_D=0,\ EL_BL_N=0,\ R_CEL_B=0.\end{aligned}$$

(3)

$$\begin{aligned}&A_1C_2=C_1B_1,\ A_2C_4=C_3B_2,\ r(A_1,\ C_1)=r(A_1),\\&r(A_2,\ C_3)=r(A_2),\ r\begin{pmatrix}C_2\\B_1\end{pmatrix}=r(B_1),\ r\begin{pmatrix}C_4\\B_2\end{pmatrix}=r(B_2),\\&r\begin{pmatrix}A_1&0&C_1B_3\\A_3&A_4C_4&C_c\\0&B_2&B_4\end{pmatrix}=r\begin{pmatrix}A_1&0&0\\A_3&0&0\\0&B_2&B_4\end{pmatrix},\\&r\begin{pmatrix}A_2&0&C_3B_4\\A_4&A_3C_2&C_c\\0&B_1&B_3\end{pmatrix}=r\begin{pmatrix}A_2&0&0\\A_4&0&0\\0&B_1&B_3\end{pmatrix},\\&r\begin{pmatrix}B_1&0&B_3\\0&B_2&B_4\\A_3C_2&A_4C_4&C_c\end{pmatrix}=r\begin{pmatrix}B_1&0&B_3\\0&B_2&B_4\end{pmatrix},\\&r\begin{pmatrix}C_1B_3&A_1&0\\C_3B_4&0&A_2\\C_c&A_3&A_4\end{pmatrix}=r\begin{pmatrix}A_1&0\\0&A_2\\A_3&A_4\end{pmatrix}.\end{aligned}$$

In this case, the general solution to the system (1.3) *can be expressed as*

$$\begin{aligned}X_1&=A_1^{\dagger}C_1+L_{A_1}C_2B_1^{\dagger}+L_{A_1}A^{\dagger}EB^{\dagger}R_{B_1}-L_{A_1}A^{\dagger}CM^{\dagger}R_AEB^{\dagger}R_{B_1}-L_{A_1}A^{\dagger}SC^{\dagger}EL_BN^{\dagger}DB^{\dagger}R_{B_1}\\&-L_{A_1}A^{\dagger}SVR_NDB^{\dagger}R_{B_1}+L_{A_1}(L_AU+ZR_B)R_{B_1},\\X_2&=A_2^{\dagger}C_3+L_{A_2}C_4B_2^{\dagger}+L_{A_2}M^{\dagger}R_AED^{\dagger}R_{B_2}+L_AL_MS^{\dagger}SC^{\dagger}EL_BN^{\dagger}R_{B_2}\\&+L_{A_2}L_M(V-S^{\dagger}SVNN^{\dagger})R_{B_2}+L_{A_2}WR_DR_{B_2},\end{aligned}$$

where U,V,W and Z are arbitrary matrices with appropriate sizes over $\mathbb{H}$.

Remark 3.4. The above corollary has the main findings of [38].

4. The general solution to equations (1.10) with η-Hermicity

In this section, as an application of equations (1.6), we establish some necessary and sufficient conditions for the system of matrix equations (1.10) to have a solution, and derive a formula for its general solution, where X, Y, Z are η-Hermitian. Let A_i, B_i, E_i $(i=\overline{1,4})$ and C_c be given with appropriate sizes over $\mathbb{H}$. Set

$$E_iL_{A_i} = A_{ii}(i=\overline{1,4}),\ R_{A_{11}}A_{jj} = A_{1j}(j=\overline{2,4}),\ R_{A_{12}}A_{13} = M_1,\ S_1 = A_{22}L_{M_1},$$

$$T_1 = C_c - E_1A_1^{\dagger}C_1 - C_1^{\eta^*}(A_1^{\eta^*})^{\dagger}E_1^{\eta^*} - \left[\sum_{i=1}^{3} E_i\left(A_i^{\dagger}B_i + L_{A_i}B_i^{\eta^*}(A_i^{\eta^*})^{\dagger}\right)E_i^{\eta^*}\right],$$

$$T_2 = R_{A_{11}}T_1(R_{A_{11}})^{\eta^*},\ G = R_{M_1}R_{A_{12}},\ G_1 = GA_{14},\ G_2 = R_{A_{12}}A_{14},\ G_3 = R_{A_{13}}A_{14},$$

$$G_4 = A_{14},\ L_1 = GT_2,\ L_2 = R_{A_{12}}T_2(R_{A_{13}})^{\eta^*},$$

$$L_3 = R_{A_{13}}T_2(R_{A_{12}})^{\eta^*},\ L_4 = T_2G^{\eta^*},\ C_{11} = (L_{G_2},\ L_{G_4}),\ E_{11} = R_{C_{11}}C_{22},\ C_{22} = L_{G_1},\ C_{33} = L_{G_3},$$

$$E_{22} = R_{C_{11}}C_{33},\ M = R_{E_{11}}E_{22},\ N = (R_{E_{22}}E_{11})^{\eta^*},\ F = F_{44} - F_{33},\ E = R_{C_{11}}F(R_{C_{11}})^{\eta^*},\ S = E_{22}L_M,$$

$$F_{11} = G_2L_{G_1},\ G_1 = L_2 - G_2G_1^{\dagger}L_1(G_4^{\eta^*})^{\dagger}G_3^{\eta^*},\ F_{22} = G_4L_{G_3},\ G_2 = L_4 - G_4G_3^{\dagger}L_3(G_2^{\eta^*})^{\dagger}G_1^{\eta^*},$$

$$F_1 = G_1^{\dagger}G_1(G_4^{\eta^*})^{\dagger} + L_{G_1}G_2^{\dagger}L_2(G_3^{\eta^*})^{\dagger},\ F_2 = G_3^{\dagger}L_3(G_2^{\eta^*})^{\dagger} + L_{G_3}G_4^{\dagger}L_4(G_1^{\eta^*})^{\dagger}.$$

Then we have the following theorem.

Theorem 4.1. *Consider* (1.10)*. The following statements are equivalent:*
(1) *The system of matrix equations* (1.10) *has a solution.*
(2)

$$R_{E_{22}}E(R_{E_{22}})^{\eta^*} = 0,\ R_{A_i}B_i = 0,\ R_{G_i}L_i = 0\ (i=\overline{1,4}).$$

(3)

$$r(B_i,\ A_i) = r(A_i)\ (i=\overline{1,4}),$$

$$r\begin{pmatrix} C_c & E_1 & E_2 & E_3 & E_4 & (C_1)^{\eta^*} \\ (E_1)^{\eta^*} & 0 & 0 & 0 & 0 & (A_1)^{\eta^*} \\ C_1 & A_1 & 0 & 0 & 0 & 0 \\ C_2E_2^{\eta^*} & 0 & A_2 & 0 & 0 & 0 \\ C_3E_3^{\eta^*} & 0 & 0 & A_3 & 0 & 0 \\ C_4E_4^{\eta^*} & 0 & 0 & 0 & A_4 & 0 \end{pmatrix} = r\begin{pmatrix} E_1 & E_2 & E_3 & E_4 \\ A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & A_4 \end{pmatrix} + r\begin{pmatrix} E_1 \\ A_1 \end{pmatrix},$$

$$r\begin{pmatrix} C_c & E_4 & E_2 & E_1 & (C_3)^{\eta^*} & (C_1)^{\eta^*} \\ (E_3)^{\eta^*} & 0 & 0 & 0 & (A_3)^{\eta^*} & 0 \\ (E_1)^{\eta^*} & 0 & 0 & 0 & 0 & (A_1)^{\eta^*} \\ C_4E_4^{\eta^*} & A_4 & 0 & 0 & 0 & 0 \\ C_2E_2^{\eta^*} & 0 & A_2 & 0 & 0 & 0 \\ C_1 & 0 & 0 & A_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} E_4 & E_2 & E_1 \\ A_4 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_1 \end{pmatrix} + r\begin{pmatrix} E_3 & E_1 \\ A_3 & 0 \\ 0 & A_1 \end{pmatrix},$$

$$
r\begin{pmatrix} C_c & E_4 & E_3 & E_1 & (C_2)^{\eta^*} & (C_1)^{\eta^*} \\ (E_2)^{\eta^*} & 0 & 0 & 0 & (A_2)^{\eta^*} & 0 \\ (E_1)^{\eta^*} & 0 & 0 & 0 & 0 & (A_1)^{\eta^*} \\ C_4E_4^{\eta^*} & A_4 & 0 & 0 & 0 & 0 \\ C_3E_3^{\eta^*} & 0 & A_3 & 0 & 0 & 0 \\ C_1 & 0 & 0 & A_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} E_4 & E_3 & E_1 \\ A_4 & 0 & 0 \\ 0 & A_3 & 0 \\ 0 & 0 & A_1 \end{pmatrix} + r\begin{pmatrix} E_2 & E_1 \\ A_2 & 0 \\ 0 & A_1 \end{pmatrix},
$$

$$
r\begin{pmatrix} C_c & E_4 & E_1 & (C_3)^{\eta^*} & (C_2)^{\eta^*} & (C_1)^{\eta^*} \\ (E_3)^{\eta^*} & 0 & 0 & (A_3)^{\eta^*} & 0 & 0 \\ (E_2)^{\eta^*} & 0 & 0 & 0 & (A_2)^{\eta^*} & 0 \\ (E_1)^{\eta^*} & 0 & 0 & 0 & 0 & (A_2)^{\eta^*} \\ C_4E_4^{\eta^*} & A_4 & 0 & 0 & 0 & 0 \\ C_1 & 0 & 0 & A_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} E_3 & E_2 & E_1 \\ A_3 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_1 \end{pmatrix} + r\begin{pmatrix} E_4 & E_1 \\ A_4 & 0 \\ 0 & A_1 \end{pmatrix},
$$

$$
r\begin{pmatrix} C_c & E_2 & E_1 & 0 & 0 & 0 & E_4 & P_3 & C_1^{\eta^*} & 0 & 0 & P_4 \\ E_3^{\eta^*} & 0 & 0 & 0 & 0 & 0 & 0 & A_3^{\eta^*} & 0 & 0 & 0 & 0 \\ E_1^{\eta^*} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_1^{\eta^*} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_c & E_3 & E_1 & E_4 & 0 & 0 & P_2 & C_1^{\eta^*} & 0 \\ 0 & 0 & 0 & E_2^{\eta^*} & 0 & 0 & 0 & 0 & 0 & A_2^{\eta^*} & 0 & 0 \\ 0 & 0 & 0 & E_1^{\eta^*} & 0 & 0 & 0 & 0 & 0 & 0 & A_1^{\eta^*} & 0 \\ E_4^{\eta^*} & 0 & 0 & -E_4^{\eta^*} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_4^{\eta^*} \\ P_2 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_1 & 0 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_3 & A_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_1 & 0 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_4 & 0 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
= 2r\begin{pmatrix} E_2 & E_1 & 0 & 0 & E_4 \\ 0 & 0 & E_3 & E_1 & E_4 \\ A_2 & 0 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 & 0 \\ 0 & 0 & A_3 & 0 & 0 \\ 0 & 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & 0 & A_4 \end{pmatrix},
$$

where $P_i = C_iE_i^{\eta^*} (i = \overline{2,4})$. *In this case, the general solution to the system* (1.10) *can be expressed as*

$$
\begin{aligned}
&U = \frac{U_1 + (U_2)^{\eta^*}}{2},\ X = \frac{\tilde{X} + (\tilde{X})^{\eta^*}}{2}, \\
&Y = \frac{\tilde{Y} + (\tilde{Y})^{\eta^*}}{2},\ Z = \frac{\tilde{Z} + (\tilde{Z})^{\eta^*}}{2}, \\
&U_1 = A_1^{\dagger}C_1 + L_{A_1}S_1,\ U_2 = U_1^{\eta^*},
\end{aligned}
$$

$$\tilde{X} = A_1^{\dagger}B_1 + L_{A_1}B_1^{\eta^*}(A_1^{\eta^*})^{\dagger} + L_{A_1}U_1(L_{A_1})^{\eta^*},$$
$$\tilde{Y} = A_2^{\dagger}B_2 + L_{A_2}B_2^{\eta^*}(A_2^{\eta^*})^{\dagger} + L_{A_2}U_2(L_{A_2})^{\eta^*},$$
$$\tilde{Z} = A_3^{\dagger}B_3 + L_{A_3}B_3^{\eta^*}(A_3^{\eta^*})^{\dagger} + L_{A_3}U_3(L_{A_3})^{\eta^*},$$

where

$$S_1 = A_{11}^{\dagger}(T_1 - A_{22}XA_{22}^{\eta^*} - A_{33}YA_{33}^{\eta^*} - A_{44}ZA_{44}^{\eta^*}) - A_{11}^{\dagger}W_{11}A_{11}^{\eta^*} + L_{A_{11}}W_{12},$$
$$U_1 = A_{12}^{\dagger}T(A_{12}^{\dagger})^{\eta^*} - A_{12}^{\dagger}A_{13}M_1^{\dagger}T(A_{12}^{\dagger})^{\eta^*} - A_{12}^{\dagger}S_1U_4R_{M_1^{\eta^*}}A_{13}^{\dagger}(A_{12}^{\dagger})^{\eta^*} - A_{12}^{\dagger}S_1A_{13}^{\dagger}T(M_1^{\eta^*})^{\dagger}A_{13}^{\dagger}(A_{12}^{\dagger})^{\eta^*}$$
$$+ L_{A_{12}}U_5 + U_6R_{A_{12}^{\eta^*}},$$
$$U_2 = M_1^{\dagger}T(A_{13}^{\dagger})^{\eta^*} + S_1^{\dagger}S_1A_{13}^{\dagger}T(M_1^{\dagger})^{\eta^*} + L_{M_1}L_{S_1}U_7 + U_8R_{A_{13}^{\eta^*}} + L_{M_1}U_4R_{M_1^{\eta^*}},$$
$$U_3 = F_1 + L_{G_2}V_1 + V_2R_{G_4^{\eta^*}} + L_{G_1}V_3R_{G_3^{\eta^*}},\ \textit{or}\ U_3 = F_2 - L_{G_4}W_1 - W_2R_{G_2^{\eta^*}} - L_{G_3}W_3R_{G_1^{\eta^*}},$$
$$V_1 = (I_m,\ 0)\left[C_{11}^{\dagger}(F - C_{22}V_3C_{33}^{\eta^*} - C_{33}W_3C_{22}^{\eta^*})\right] - (I_m,\ 0)\left[C_{11}^{\dagger}U_{11}C_{11}^{\eta^*} + L_{C_{11}}U_{12}\right],$$
$$W_1 = (0,\ I_m)\left[C_{11}^{\dagger}(F - C_{22}V_3C_{33}^{\eta^*} - C_{33}W_3C_{22}^{\eta^*})\right] - (0,\ I_m)\left[C_{11}^{\dagger}U_{11}C_{11}^{\eta^*} + L_{C_{11}}U_{12}\right],$$
$$W_2 = \left[R_{C_{11}}(F - C_{22}V_3C_{33}^{\eta^*} - C_{33}W_3C_{22}^{\eta^*})(C_{11}^{\eta^*})^{\dagger}\right]\begin{pmatrix}0\\ I_n\end{pmatrix} + \left[C_{11}C_{11}^{\dagger}U_{11} + U_{21}L_{C_{11}}^{\eta^*}\right]\begin{pmatrix}0\\ I_n\end{pmatrix},$$
$$V_2 = R_{C_{11}}(F - C_{22}V_3C_{33}^{\eta^*} - C_{33}W_3C_{22}^{\eta^*})(C_{11}^{\eta^*})^{\dagger}\begin{pmatrix}0\\ I_n\end{pmatrix} + \left[C_{11}C_{11}^{\dagger}U_{11} + U_{21}L_{C_{11}}^{\eta^*}\right]\begin{pmatrix}I_n\\ 0\end{pmatrix},$$
$$V_3 = E_{11}^{\dagger}F(E_{22}^{\eta^*})^{\dagger} - E_{11}^{\dagger}E_{22}M^{\dagger}F(E_{22}^{\eta^*})^{\dagger} - E_{11}^{\dagger}SE_{22}^{\dagger}FN^{\dagger}E_{11}^{\eta^*}(E_{22}^{\eta^*})^{\dagger} - E_{11}^{\dagger}SU_{31}R_NE_{11}^{\eta^*}(E_{22}^{\eta^*})^{\dagger}$$
$$+ L_{E_{11}}U_{32} + U_{33}L_{E_{22}}^{\eta^*},$$
$$W_3 = M^{\dagger}F(E_{11}^{\eta^*})^{\dagger} + S^{\dagger}SE_{22}^{\dagger}FN^{\dagger} + L_ML_SU_{41} + L_MU_{31}R_N - U_{42}L_{E_{11}}^{\eta^*},$$

where $T = T_1 - A_{33}U_3(A_{33})^{\eta^*}$, $U_j(j = 4, 5, 6)$, $U_{i1}(i = 1, 2, 3, 4)$, U_{12}, U_{32}, U_{33}, *and* U_{42} *are any matrices with suitable dimensions over* $\mathbb{H}$.

Proof. Since the solvability of the system (1.10) is equivalent to system

$$\begin{gathered}
A_1U_1 = B_1,\ U_2(A_1)^{\eta^*} = B_1^{\eta^*},\ U_2 = (U_1)^{\eta^*},\\
A_2\tilde{X} = B_2,\ \tilde{X}(A_2)^{\eta^*} = B_2^{\eta^*},\ \tilde{X} = \tilde{X}^{\eta^*},\\
A_3\tilde{Y} = B_3,\ \tilde{Y}(A_3)^{\eta^*} = B_3^{\eta^*},\ \tilde{Y} = \tilde{Y}^{\eta^*},\\
A_4\tilde{Z} = B_4,\ \tilde{Z}(A_4)^{\eta^*} = B_4^{\eta^*},\ \tilde{Z} = \tilde{Z}^{\eta^*},\\
E_1U_1 + U_2E_1^{\eta^*} + E_2\tilde{X}E_2^{\eta^*} + E_3\tilde{Y}E_3^{\eta^*} + E_4\tilde{Z}E_4^{\eta^*} = C_c.
\end{gathered} \tag{4.1}$$

If the system (1.10) has a solution, say, $(U,\ X,\ Y,\ Z)$, then

$$(U_1,\ U_2,\ \tilde{X},\ \tilde{Y},\ \tilde{Z}) := (U,\ U^{\eta^*},\ X,\ Y,\ Z)$$

is a solution to the system of matrix equations (4.1). Conversely, if the system (4.1) has a solution, say

$$(U_1,\ U_2,\ \tilde{X},\ \tilde{Y},\ \tilde{Z}),$$

then equations (1.10) clearly has a solution

$$(U,\ X,\ Y,\ Z):\\ =(\frac{U_1+(U_2)^{\eta^*}}{2}\frac{\tilde{X}+(\tilde{X})^{\eta^*}}{2},\ \frac{\tilde{Y}+(\tilde{Y})^{\eta^*}}{2},\ \frac{\tilde{Z}+(\tilde{Z})^{\eta^*}}{2}).$$

□

Next, we study the special case (1.11) of the matrix equations (1.10).

Theorem 4.2. *Let A_i, C_i, E_i $(i=\overline{1,3})$ and C be given with appropriate size. Set*

$$\begin{aligned}
&E_1L_{A_1}=A_{11},\ E_2L_{A_2}=A_{22},\ E_3L_{A_3}=A_{33},\ R_{A_{11}}A_{22}=M_1,\ S_1=A_{22}L_{M_1},\\
&T_1=C-E_1(A_1^{\dagger}B_1+L_{A_1}B_1^{\eta^*}(A_1^{\eta^*})^{\dagger})E_1^{\eta^*}-E_2(A_2^{\dagger}B_2+L_{A_2}B_2^{\eta^*}(A_2^{\eta^*})^{\dagger})E_2^{\eta^*}\\
&-E_3(A_3^{\dagger}B_3+L_{A_3}B_3^{\eta^*}(A_3^{\eta^*})^{\dagger})E_3^{\eta^*},\ G=R_{M_1}R_{A_{11}},\\
&G_1=GA_{33},\ G_2=R_{A_{11}}A_{33},\ G_3=R_{A_{22}}A_{33},\ G_4=A_{33},\\
&L_1=GT_1,\ L_2=R_{A_{11}}T_1(R_{A_{22}})^{\eta^*},\ L_3=R_{A_{22}}T_1(R_{A_{11}})^{\eta^*},\\
&L_4=T_1G^{\eta^*},\ C_{11}=(L_{C_2},\ L_{C_4}),\ C_{22}=L_{C_1},\ C_{33}=L_{C_3},\\
&E_{11}=R_{C_{11}}C_{22},\ E_{22}=R_{C_{11}}C_{33},\ M=R_{E_{11}}E_{22},\\
&N=(R_{E_{22}}E_{11})^{\eta^*},\ F=F_2-F_1,\ E=R_{C_{11}}F(R_{C_{11}})^{\eta^*},\\
&S=E_{22}L_M,\ F_{11}=G_2L_{G_1},\ H_1=L_2-G_2G_1^{\dagger}L_1(G_4^{\eta^*})^{\dagger}G_3^{\eta^*},\\
&F_{22}=G_4L_{G_3},\ H_2=L_4-G_4G_3^{\dagger}L_3(G_2^{\eta^*})^{\dagger}G_1^{\eta^*},\\
&F_1=G_1^{\dagger}L_1(G_4^{\eta^*})^{\dagger}+L_{G_1}G_2^{\dagger}L_2(G_3^{\eta^*})^{\dagger},\ F_2=G_3^{\dagger}L_3(G_2^{\eta^*})^{\dagger}+L_{G_3}G_4^{\dagger}L_4(G_1^{\eta^*})^{\dagger}.
\end{aligned}$$

Then the following statements are equivalent:

(1) *The system of the matrix equations* (1.11) *is consistent.*

(2)

$$\begin{aligned}
&R_{A_j}B_j=0,\ R_{G_i}L_i=0(i=\overline{1,4},j=\overline{1,3}),\\
&R_{E_{22}}E(R_{E_{22}})^{\eta^*}=0.
\end{aligned}$$

(3)

$$r(A_j,\ B_j)=r(A_j)(j=1,2,3),$$

$$r\begin{pmatrix} C & E_3 & E_1 & E_2\\ B_3E_3^{\eta^*} & A_3 & 0 & 0\\ B_1E_1^{\eta^*} & 0 & A_1 & 0\\ B_2E_2^{\eta^*} & 0 & 0 & A_2\end{pmatrix}=r\begin{pmatrix} E_3 & E_1 & E_2\\ A_3 & 0 & 0\\ 0 & A_1 & 0\\ 0 & 0 & A_2\end{pmatrix},$$

$$r\begin{pmatrix} C & E_3 & E_1 & E_2B_2^{\eta^*} \\ E_2^{\eta^*} & 0 & 0 & A_2^{\eta^*} \\ B_3E_3^{\eta^*} & A_3 & 0 & 0 \\ B_1E_1^{\eta^*} & 0 & A_1 & 0 \end{pmatrix} = r\begin{pmatrix} E_3 & E_1 \\ A_3 & 0 \\ 0 & A_1 \end{pmatrix} + r\begin{pmatrix} E_2 \\ A_2 \end{pmatrix},$$

$$r\begin{pmatrix} C & E_3 & E_2 & E_1B_1^{\eta^*} \\ E_1^{\eta^*} & 0 & 0 & A_1^{\eta^*} \\ B_3E_3^{\eta^*} & A_3 & 0 & 0 \\ B_2E_2^{\eta^*} & 0 & A_2 & 0 \end{pmatrix} = r\begin{pmatrix} E_3 & E_2 \\ A_3 & 0 \\ 0 & A_2 \end{pmatrix} + r\begin{pmatrix} E_1 \\ A_1 \end{pmatrix},$$

$$r\begin{pmatrix} C & E_3 & E_1B_1^{\eta^*} & E_2B_2^{\eta^*} \\ E_1^{\eta^*} & 0 & A_1^{\eta^*} & 0 \\ E_2^{\eta^*} & 0 & 0 & A_2^{\eta^*} \\ B_3E_3^{\eta^*} & A_3 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} E_1 & E_2 \\ A_1 & 0 \\ 0 & A_2 \end{pmatrix} + r\begin{pmatrix} E_3 \\ A_3 \end{pmatrix},$$

$$r\begin{pmatrix} C & 0 & E_1 & 0 & E_3 & E_2B_2^{\eta^*} & 0 & E_3B_3^{\eta^*} \\ 0 & -C & 0 & E_2 & E_3 & 0 & -E_1B_1^{\eta^*} & 0 \\ E_2^{\eta^*} & 0 & 0 & 0 & 0 & A_2^{\eta^*} & 0 & 0 \\ 0 & E_1^{\eta^*} & 0 & 0 & 0 & 0 & A_1^{\eta^*} & 0 \\ E_3^{\eta^*} & E_3^{\eta^*} & 0 & 0 & 0 & 0 & 0 & A_3^{\eta^*} \\ B_1E_1^{\eta^*} & 0 & A_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -B_2E_2^{\eta^*} & 0 & A_2 & 0 & 0 & 0 & 0 \\ 0 & -B_3E_3^{\eta^*} & 0 & 0 & A_3 & 0 & 0 & 0 \end{pmatrix}$$

$$= 2r\begin{pmatrix} E_1 & 0 & E_3 \\ 0 & E_2 & E_3 \\ A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix}.$$

In this case, the general solution of matrix equation (1.11) *can be expressed as*

$$X = \frac{\tilde{X} + (\tilde{X})^{\eta^*}}{2},\ Y = \frac{\tilde{Y} + (\tilde{Y})^{\eta^*}}{2},\ Z = \frac{\tilde{Z} + (\tilde{Z})^{\eta^*}}{2},$$
$$\tilde{X} = A_1^{\dagger}B_1 + L_{A_1}B_1^{\eta^*}(A_1^{\eta^*})^{\dagger} + L_{A_1}U_1(L_{A_1})^{\eta^*},$$
$$\tilde{Y} = A_2^{\dagger}B_2 + L_{A_2}B_2^{\eta^*}(A_2^{\eta^*})^{\dagger} + L_{A_2}U_2(L_{A_2})^{\eta^*},$$
$$\tilde{Z} = A_3^{\dagger}B_3 + L_{A_3}B_3^{\eta^*}(A_3^{\eta^*})^{\dagger} + L_{A_3}U_3(L_{A_3})^{\eta^*},$$

where

$$U_1 = A_{11}^{\dagger}T(A_{11}^{\dagger})^{\eta^*} - A_{11}^{\dagger}A_{22}M_1^{\dagger}T(A_{11}^{\dagger})^{\eta^*} - A_{11}^{\dagger}U_4A_{22}^{\dagger}T(M_1^{\dagger})^{\eta^*}(A_{22}^{\dagger})^{\eta^*} + L_{A_{11}}U_5 + U_6R_{A_{11}^{\eta^*}},$$
$$U_2 = M_1^{\dagger}T(A_{22}^{\dagger})^{\eta^*} + S_1^{\dagger}S_1A_{22}^{\dagger}T(M_1^{\dagger})^{\eta^*} + L_{M_1}L_{S_1}U_7 + U_8R_{A_{22}^{\eta^*}} + L_{M_1}U_4R_{M_1^{\eta^*}},$$
$$U_3 = F_1^0 + L_{G_2}V_1 + V_2R_{G_4^{\eta^*}} + L_{G_1}V_3R_{G_3^{\eta^*}},\ or\ U_3 = F_2^0 - L_{G_4}W_1 - W_2R_{G_2^{\eta^*}} - L_{G_3}W_3R_{G_1^{\eta^*}},$$

$$V_1 = (I_m,\ 0)\left[C_{11}^{\dagger}(F - C_{22}V_3C_{33}^{\eta^*} - C_{33}W_3C_{22}^{\eta^*})\right] - (I_m,\ 0)\left[C_{11}^{\dagger}U_{11}C_{11}^{\eta^*} + L_{C_{11}}U_{12}\right],$$

$$W_1 = (0,\ I_m)\left[C_{11}^{\dagger}(F - C_{22}V_3C_{33}^{\eta^*} - C_{33}W_3C_{22}^{\eta^*})\right] - (0,\ I_m)\left[C_{11}^{\dagger}U_{11}C_{11}^{\eta^*} + L_{C_{11}}U_{12}\right],$$

$$W_2 = \left[R_{C_{11}}(F - C_{22}V_3C_{33}^{\eta^*} - C_{33}W_3C_{22}^{\eta^*})(C_{11}^{\eta^*})^{\dagger}\right]\begin{pmatrix} 0 \\ I_n \end{pmatrix} + \left[C_{11}C_{11}^{\dagger}U_{11} + U_{21}L_{C_{11}}^{\eta^*}\right]\begin{pmatrix} 0 \\ I_n \end{pmatrix},$$

$$V_2 = R_{C_{11}}(F - C_{22}V_3C_{33}^{\eta^*} - C_{33}W_3C_{22}^{\eta^*})(C_{11}^{\eta^*})^{\dagger}\begin{pmatrix} 0 \\ I_n \end{pmatrix} + \left[C_{11}C_{11}^{\dagger}U_{11} + U_{21}L_{C_{11}}^{\eta^*}\right]\begin{pmatrix} I_n \\ 0 \end{pmatrix},$$

$$V_3 = E_{11}^{\dagger}F(E_{22}^{\eta^*})^{\dagger} - E_{11}^{\dagger}E_{22}M^{\dagger}F(E_{22}^{\eta^*})^{\dagger} - E_{11}^{\dagger}SE_{22}^{\dagger}FN^{\dagger}E_{11}^{\eta^*}(E_{22}^{\eta^*})^{\dagger} - E_{11}^{\dagger}SU_{31}R_NE_{11}^{\eta^*}(E_{22}^{\eta^*})^{\dagger}$$
$$+ L_{E_{11}}U_{32} + U_{33}L_{E_{22}}^{\eta^*},$$

$$W_3 = M^{\dagger}F(E_{11}^{\eta^*})^{\dagger} + S^{\dagger}SE_{22}^{\dagger}FN^{\dagger} + L_ML_SU_{41} + L_MU_{31}R_N - U_{42}L_{E_{11}}^{\eta^*},$$

where $T = T_1 - A_{33}U_3(A_{33})^{\eta^*}$, $U_j (j = 4, 5, 6)$, $U_{i1} (i = 1, 2, 3, 4)$, U_{12}, U_{32}, U_{33} *and* U_{42} *are any matrices with appropriate dimensions.*

Proof. It follows from Theorem 4.1 that this theorem holds when A_1, C_1, and E_1 vanish in Theorem 4.1. □

In Theorem 4.2, let $A_i,\ C_i,\ B_i,\ D_i (i = \overline{1,3})$, E_3 and F_3 be vanish. Then we can get the η-Hermitian solution of the matrix equation (1.8).

Corollary 4.3. *Let* B_1, C_1 *and* $D_1 = D_1^{\eta^*}$ *be given. Set* $M = R_{B_1}C_1, S = C_1L_M$. *Then the following statements are equivalent:*

(1) *Matrix equation* (1.8) *has a pair of* η*-Hermitian solutions* Y *and* Z.

(2)

$$R_MR_{B_1}D_1 = 0, \quad R_{B_1}D_1\left(R_{C_1}\right)^{\eta^*} = 0.$$

(3)

$$r\begin{pmatrix} B_1 & D_1 \\ 0 & C_1^{\eta^*} \end{pmatrix} = r\left(B_1\right) + r\left(C_1\right),$$
$$r\begin{pmatrix} B_1 & C_1 & D_1 \end{pmatrix} = r\begin{pmatrix} B_1 & C_1 \end{pmatrix}.$$

In this case, the η*-Hermitian solution to matrix equation* (1.8) *can be expressed as*

$$Y = B_1^{\dagger}D_1\left(B_1^{\dagger}\right)^{\eta^*} - \frac{1}{2}B_1^{\dagger}C_1M^{\dagger}D_1\left[I + \left(C_1^{\dagger}\right)^{\eta^*}S^{\eta^*}\right]\left(B_1^{\dagger}\right)^{\eta^*}$$
$$- \frac{1}{2}B_1^{\dagger}\left(I + SC_1^{\dagger}\right)D_1\left(M^{\dagger}\right)^{\eta^*}C_1^{\eta^*}\left(B_1^{\dagger}\right)^{\eta^*} - B_1^{\dagger}SW_2S^{\eta^*}\left(B_1^{\dagger}\right)^{\eta^*} + L_{B_1}U + U^{\eta^*}\left(L_{B_1}\right)^{\eta^*},$$
$$Z = \frac{1}{2}M^{\dagger}D_1\left(C_1^{\dagger}\right)^{\eta^*}\left[I + \left(S^{\dagger}S\right)^{\eta^*}\right] + \frac{1}{2}\left(I + S^{\dagger}S\right)C_1^{\dagger}D_1\left(M^{\dagger}\right)^{\eta^*}$$
$$+ L_MW_2\left(L_M\right)^{\eta^*} + VL_{C_1}^{\eta^*} + L_{C_1}V^{\eta^*} + L_ML_SW_1 + W_1^{\eta^*}\left(L_S\right)^{\eta^*}\left(L_M\right)^{\eta^*},$$

where W_1, U, V *and* $W_2 = W_2^{\eta^*}$ *are arbitrary matrices over* $\mathbb{H}$ *with appropriate sizes.*

Remark 4.3. The above corollary has the main findings of [10].

Corollary 4.4. *Let* $A_1, C_1, A_2, A_3, B_1, D_1, D_3$ *and* $D_3 = D_3^{\eta^*}$ *be coefficient matrices in* (1.9). *Define some new matrices as follows:*

$$\begin{aligned}
&B_4 = A_2 L_{A_1},\ C_4 = A_3 (R_{B_1})^{\eta^*},\\
&D_4 = D_3 - A_2 \left[A_1^{\dagger} C_1 + \left(A_1^{\dagger} C_1\right)^{\eta *} - A_1^{\dagger} A_1 C_1^{\eta^*} \left(A_1^{\dagger}\right)^{\eta *}\right] A_2^{\eta^*}\\
&- A_3 \left[D_1 B_1^{\dagger} + \left(D_1 B_1^{\dagger}\right)^{\eta^*} - \left(B_1^{\dagger}\right)^{\eta^*} B_1^{\eta^*} D_1 B_1^{\dagger}\right] A_3^{\eta^*},\\
&M = R_{B_4} C_4, \quad S = C_4 L_M.
\end{aligned}$$

Then the following statements are equivalent:

(1) *The system* (1.9) *has a solution* (X, Y, Z), *where* Y *and* Z *are* η*-Hermitian.*

(2) *The coefficient matrices in equations* (1.9) *satisfy*

$$\begin{aligned}
&A_1 C_1^{\eta^*} = C_1 A_1^{\eta^*}, \quad B_1^{\eta^*} D_1 = D_1^{\eta^*} B_1,\\
&R_{A_1} C_1 = 0, \quad D_{21} L_{B_1} = 0, \quad R_M R_{B_4} D_4 = 0,\\
&R_{B_4} D_4 (R_{C_4})^{\eta^*} = 0.
\end{aligned}$$

(3) *The coefficient matrices in equations* (1.9) *and their ranks satisfy*

$$\begin{aligned}
&A_1 C_1^{\eta^*} = C_1 A_1^{\eta^*}, \quad B_1^{\eta^*} D_1 = D_1^{\eta^*} B_1,\\
&r\begin{pmatrix} A_1 & C_1 \end{pmatrix} = r(A_1), \quad r\begin{pmatrix} D_1 \\ B_1 \end{pmatrix} = r(B_1),\\
&r\begin{pmatrix} D_3 & A_3 & A_2 \\ D_1^{\eta^*} A_3^{\eta^*} & B_1^{\eta^*} & 0 \\ C_1 A_2^{\eta^*} & 0 & A_1 \\ C_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_3 & A_2 \\ B_1^{\eta *} & 0 \\ 0 & A_1 \end{pmatrix} + r\begin{pmatrix} A_1 \end{pmatrix},\\
&r\begin{pmatrix} D_3 & A_2 & A_3 D_1 \\ A_3^{\eta^*} & 0 & B_1 \\ C_1 A_2^{\eta *} & A_1 & 0 \end{pmatrix} = r\begin{pmatrix} A_2 \\ A_1 \end{pmatrix} + r\begin{pmatrix} A_3^{\eta^*} & B_1 \end{pmatrix}.
\end{aligned}$$

In this case, the general solution to the system of matrix equations (1.9) *can be expressed as*

$$\begin{aligned}
&Y = Y^{\eta^*} = A_1^{\dagger} C_1 + \left(A_1^{\dagger} C_1\right)^{\eta^*} - A_1^{\dagger} A_1 C_1^{\eta^*} \left(A_1^{\dagger}\right)^{\eta^*}\\
&+ L_{A_1} V (L_{A_1})^{\eta^*},\\
&Z = Z^{\eta^*} = D_1 B_1^{\dagger} + \left(D_1 B_1^{\dagger}\right)^{\eta^*} - \left(B_1^{\dagger}\right)^{\eta^*} B_1^{\eta^*} D_1 B_1^{\dagger}\\
&+ (R_{B_1})^{\eta^*} W R_{B_1},\\
&V = V^{\eta^*} = B_4^{\dagger} D_4 \left(B_4^{\dagger}\right)^{\eta^*}\\
&- \frac{1}{2} B_4^{\dagger} C_4 M^{\dagger} D_4 \left[I + \left(C_4^{\dagger}\right)^{\eta^*} S^{\eta *}\right] \left(B_4^{\dagger}\right)^{\eta^*}
\end{aligned}$$

$$-\frac{1}{2}B_4^{\dagger}\left(I+SC_4^{\dagger}\right)D_4\left(M^{\dagger}\right)^{\eta^*}C_4^{\eta^*}\left(B_4^{\dagger}\right)^{\eta^*}$$
$$-B_4^{\dagger}SU_6S^{\eta^*}\left(B_4^{\dagger}\right)^{\eta^*}+L_{B_4}U_4+U_4^{\eta^*}\left(L_{B_4}\right)^{\eta^*},$$
$$W=W^{\eta^*}=\frac{1}{2}M^{\dagger}D_4\left(B^{\dagger}\right)^{\eta^*}\left[I+\left(S^{\dagger}S\right)^{\eta}\right]$$
$$+\frac{1}{2}\left(I+S^{\dagger}S\right)C_4^{\dagger}D_4\left(M^{\dagger}\right)^{\eta*}+L_MU_6\left(L_M\right)^{\eta}+U_5L_{C_4}^{\eta}$$
$$+L_{C_4}U_5^{\eta^*}+L_ML_5U_3+U_3^{\eta*}\left(L_S\right)^{\eta}\left(L_M\right)^{\eta^*},$$

where $U_3,\ U_4\ ,U_5$ *and* $U_6=U_6^{\eta^*}$ *are arbitrary matrices over* $\mathbb{H}$ *with appropriate sizes.*

5. Algorithm with a numerical example

In this section, we present an algorithm and an example to illustrate Theorem 3.1.

Algorithm 5.1

(1) Feed the values of $A_i, B_i\ C_i,\ D_i,\ E_i,\ F_i(i=\overline{1,4})$ and C_c with conformable shapes over $\mathbb{H}$.

(2) Compute the symbols in (3.1) to (3.4).

(3) Check (2) in Theorem 3.1 or (3.7) to (3.16). If no, it returns "inconsisten".

(4) Else, compute $U\ V,\ X,\ Y,\ Z$.

Example 5.1 Let

$$A_1=\begin{pmatrix}1 & 0 & \mathbf{i}\end{pmatrix},\ B_1=\begin{pmatrix}0\\1\\\mathbf{j}\end{pmatrix},\ A_2=\begin{pmatrix}0 & \mathbf{i}\\ \mathbf{j} & \mathbf{k}\end{pmatrix},\ B_2=\begin{pmatrix}\mathbf{i}\\1\end{pmatrix},$$
$$A_3=\begin{pmatrix}0 & 1\\ \mathbf{i} & \mathbf{j}\end{pmatrix},\ B_3=\begin{pmatrix}\mathbf{j}\\1\end{pmatrix},\ A_4=\begin{pmatrix}\mathbf{i} & 1\\ 0 & \mathbf{k}\end{pmatrix},\ B_4=\begin{pmatrix}1\\ \mathbf{k}\end{pmatrix},$$
$$C_1=1+2\mathbf{i},\ D_1=\begin{pmatrix}\mathbf{j}\\ -\mathbf{i}+\mathbf{j}\end{pmatrix},\ C_2=\begin{pmatrix}-1 & 3\mathbf{i}\\ 3\mathbf{j} & 3\mathbf{k}\end{pmatrix},\ D_2=\begin{pmatrix}2\mathbf{i}\\2\end{pmatrix},$$
$$C_3=\begin{pmatrix}0 & 2\\ \mathbf{i} & -1+2\mathbf{j}\end{pmatrix},\ C_4=\begin{pmatrix}-1 & 1+\mathbf{k}\\ 0 & \mathbf{k}\end{pmatrix},\ D_3=\begin{pmatrix}\mathbf{i}+\mathbf{j}\\2\end{pmatrix},$$
$$D_4=\begin{pmatrix}2\mathbf{i}\\ \mathbf{k}\end{pmatrix},\ C_3=\begin{pmatrix}-1 & 1+\mathbf{k}\\ 0 & \mathbf{k}\end{pmatrix},\ D_1=\begin{pmatrix}2\mathbf{i}\\2\end{pmatrix},$$
$$D_2=\begin{pmatrix}\mathbf{i}+\mathbf{j}\\2\end{pmatrix},\ D_3=\begin{pmatrix}2\mathbf{i}\\ \mathbf{k}\end{pmatrix},\ E_1=\begin{pmatrix}2 & \mathbf{i}\\ 0 & \mathbf{k}\end{pmatrix},$$
$$E_2=\begin{pmatrix}\mathbf{i} & \mathbf{j}\\ 0 & \mathbf{k}\end{pmatrix},\ E_3=\begin{pmatrix}0 & 1\\ \mathbf{j} & \mathbf{k}\end{pmatrix},\ F_1=\begin{pmatrix}\mathbf{i}\\ \mathbf{j}\end{pmatrix},$$
$$F_2=\begin{pmatrix}\mathbf{j}\\ \mathbf{i}\end{pmatrix},\ F_3=\begin{pmatrix}\mathbf{k}\\ \mathbf{i}\end{pmatrix},\ C=\begin{pmatrix}3\mathbf{i}+2\mathbf{k}\\ 1-4\mathbf{i}+3\mathbf{j}-\mathbf{k}\end{pmatrix}.$$

Computation directly yields

$$A_1D_1 = C_1B_1 = \begin{pmatrix} 2\mathbf{i} \\ 0 \end{pmatrix},$$

$$A_2D_2 = C_2B_2 = \begin{pmatrix} 2 \\ -1+2\mathbf{j}+\mathbf{k} \end{pmatrix},$$

$$A_3D_3 = C_3B_3 = \begin{pmatrix} -2+\mathbf{k} \\ -\mathbf{i} \end{pmatrix},$$

$$r(C_i,\ A_i) = r(A_i) = 2,\ r\begin{pmatrix} D_i \\ B_i \end{pmatrix} = r(B_i) = 1(i=\overline{1,3}),$$

$$(3.8) = 11,\ (3.9) = 8,\ (3.10) = 10,\ (3.11) = 9,$$

$$(3.12) = 10,\ (3.13) = 9,\ (3.14) = 9,\ (3.15) = 8,\ (3.16) = 19.$$

All the rank equalities in (3.7) to (3.16) hold. Hence, according to Theorem 3.1, the system of matrix equations (1.6) has a solution, and the general solution to matrix equations (1.6) can be expressed as

$$U = \begin{pmatrix} 0.5000+1.0000\mathbf{i} \\ 0 \\ 1-0.5000\mathbf{i} \end{pmatrix} + \begin{pmatrix} 0.5000 & 0 & -0.5000\mathbf{i} \\ 0 & 1.000 & 0 \\ 0.50000\mathbf{i} & 0 & 0.5000 \end{pmatrix} S_1,$$

$$V = \begin{pmatrix} 0 & 0.5000\mathbf{j} & 0.5000 \\ 0 & -0.5000\mathbf{i}+0.5000\mathbf{j} & 0.5000+0.5000\mathbf{k} \end{pmatrix},$$

$$X = \begin{pmatrix} 2.000 & 0 \\ 1.000\mathbf{i} & 3.000 \end{pmatrix},$$

$$Y = \begin{pmatrix} 1.000 & 1.000\mathbf{i} \\ 0 & 2.000 \end{pmatrix},\ Z = \begin{pmatrix} 1.000\mathbf{i} & 1.000\mathbf{j} \\ 0 & 1.000 \end{pmatrix},$$

where

$$S_1 = \begin{pmatrix} -0.5000+2.000\mathbf{i}-1.0000\mathbf{k} \\ 1.0000-3.0000\mathbf{i}+1.0000\mathbf{j}+1.0000\mathbf{k} \\ -2.0000-0.5000\mathbf{i}+1.0000\mathbf{j} \end{pmatrix}$$

$$- \begin{pmatrix} 1.0000 & -1.0000\mathbf{i} \\ -1.0000 & 1.0000\mathbf{i}+1.0000 \\ 1.0000\mathbf{i} & 1.0000 \end{pmatrix} W_{11} \begin{pmatrix} 2.0000 \\ -1.0000\mathbf{k} \\ 1.0000\mathbf{i} \end{pmatrix},$$

W_{11} is a any matrix equation with suitable size over $\mathbb{H}$.

Finally, we give the following conclusion that summarizes the work of this paper.

6. Conclusions

We have established the solvability conditions and a formula for the general solution to the Sylvester-type quaternion matrix equations (1.6). As an application of equations (1.6), we also have established some necessary and sufficient conditions for the system of quaternion matrix equations (1.10) to provide a solution and derived an exact expression of its general solution involving η-Hermicity. As a special case of equations (1.6), we have presented the necessary and sufficient conditions for the system of two-sided Sylvester-type quaternion matrix equations (1.7) to be consistent and derived a formula for its general solution (when it is solvable). As a special case of equations (1.10), we have investigated the necessary and sufficient conditions for the system of matrix equations (1.11) to have a solution and provided a general solution, which is an η-Hermitian.

It is noteworthy that the main results of (1.6) are available over $\mathbb{R}$ and $\mathbb{C}$ and for any division ring. Furthermore, motivated by [21], we can investigate equations (1.6) in tensor form.

References

[1] J.K. Baksalary, R. Kala. The matrix equations $AXB + CYD = E$. Linear Algebra Appl., 30:141-147, 1979.

[2] S. Brahma, B. Datta. An optimization approach for minimum norm and robust partial quadratic eigenvalue assignment problems for vibrating structures. J. Sound Vibration, 324(3):471-489, 2009.

[3] J.N. Buxton, R.F. Churchouse, and A.B. Tayler. Matrices Methods and Applications. Clarendon Press, Oxford, 1990.

[4] N.L. Bihan, J. Mars. Singular value decomposition of quaternion matrices: A new tool for vector-sensor signal processing. Signal Processing, 84(7):1177-1199, 2004.

[5] D.L. Chu, H. Chan, and D.W.C. Ho. Regularization of singular systems by derivative and proportional output feedback. SIAM J. Math. Anal., 19:21-38, 1998.

[6] D.L. Chu, L. De Lathauwer, and B. Moor. On the computation of restricted singular value decomposition via cosine-sine decomposition. SIAM J. Math. Anal., 22:550-601, 2000.

[7] D.L. Chu, Y.S. Hung, and H.J. Woerdeman. Inertia and rank characterizations of some matrix expressions. SIAM J. Math. Anal., 31:1187-1226, 2009.

[8] E.B. Castelan, V.G. da Silva. On the solution of a Sylvester equation appearing in descriptor systems control theory. Systems Control Lett., 54:109-117, 2005.

[9] H. Flandersm, H.K. Wimmer. On the matrix equations $AX - XB = C$ and $AX - YB = C$. SIAM J. Appl Math., 32:707-710, 1977.

[10] Z.H. He, Q.W. Wang. A real quaternion matrix equation with with applications. Linear Multilinear Algebra, 61:725-740, 2013.

[11] Z.H. He, Q.W. Wang, and Y. Zhang. A system of quaternary coupled Sylvester-type real quaternion matrix equations. Automatica, 87:25-31, 2018.

[12] Z.H. He, Q.W. Wang. The η-bihermitian solution to a system of real quaternion matrix equations. Linear Multilinear Algebra, 62:1509-1528, 2018.

[13] Z.H. He, Q.W. Wang, and Y. Zhang. Simultaneous decomposition of quaternion matrices involving η-Hermicity with applications. Appl. Math. Comput., 298:13-35, 2017.

[14] Z.H. He, M. Wang. A quaternion matrix equation with two different restrictions. Advances in Applied Clifford Algebras, 31:25, 2021.

[15] Z.H. He, O.M. Agudelo, Q. W. Wang, and B. De Moor. Two-sided coupled generalized sylvester matrix equations solving using a simultaneous decomposition for fifteen matrices. Linear Algebra Appl., 496:549-593, 2016.

[16] Z.G. Jia, M.K. Ng, and G.J. Song. Robust quaternion matrix completion with applications to image inpainting. Numer. Linear Algebra Appl., 26:e2245, 2019.

[17] I. Kyrchei. Explicit representation formulas for the minimum norm least squares solutions of some quaternion matrix equations. Linear Algebra Appl., 438:136-152, 2018.

[18] I. Kyrchei. Determinantal representations of the Drazin inverse over the quaternion skew field with applications to some matrix equations. Appl. Math. Comput., 238:193-207, 2014.

[19] L.S. Liu, Q.W. Wang, and M.S. Mehany. A Sylvester-type matrix equation over the Hamilton quaternions with an application. Mathematics, 10:1758, 2022.

[20] L.S. Liu, Q.W. Wang, J.F. Chen, and Y.Z. Xie. An exact solution to a quaternion matrix equation with an application. Symmetry, 14(2):375, 2022.

[21] T. Li, Q.W. Wang, and X.F. Zhang, A modified conjugate residual method and nearest kronecker product preconditioner for the generalized coupled Sylvester tensor equations. Mathematics, 10:1730, 2022.

[22] X. Liu, Y. Zhang, Consistency of split quaternion matrix equations $AX^{\star} - XB = CY + D$ and $X - AX^{\star}B = CY + D$. Adv. Appl. Clifford Algebras. 64:1-20, 2019.

[23] M.S. Mehany, Q.W. Wang. Three symmetrical systems of coupled Sylvester-like quaternion matrix equations. Symmetry, 14:550, 2022.

[24] G. Marsaglia, G.P. Styan. Equalities and inequalities for ranks of matrices. Linear Multilinear Algebra., 2:269-292, 1974.

[25] A.B. Özgüler. The matrix equation $AXB + CYD = E$ over a principal ideal domain. SIAM J. Matrix Anal. Appl., 12:581-591, 1991.

[26] S.C. Pei, C.M. Cheng. Color image processing by using binary quaternion-moment-preserving thresholding techniqe. IEEE Trans. Image Process, 8:614-628, 1999.

[27] L. Qi, Z.Y. Luo, Q.W. Wang, and X.Z. Zhang. Quaternion matrix optimization: motivation and analysis. J. Optim. Theory Appl., https://doi.org/10.1007/s10957-021-01906-y.

[28] W.E. Roth. The equations $AX - YB = C$ and $AX - XB = C$ in matrices. Proc. Amer. Math. Soc., 3:392-396, 1952.

[29] L. Rodman. Topics in Quaternion Linear Algebra. Princeton University Press, Princeton, NJ, USA, 2014.

[30] C.C. Took, D.P. Mandic. Augmented second-order statistics of quaternion random signals. Signal Process, 91:214-224, 2011.

[31] C.C. Took, D.P. Mandic, and F.Z. Zhang. On the unitary diagonalization of a special class of quaternion matrices. Appl. Math. Lett., 24:1806-1809, 2011.

[32] Q.W. Wang. The decomposition of pairwise matrices and matrix equations over an arbitrary skew field. Acta Math. Sinica., 39:396-403, 1991.

[33] R.N. Wang, Q.W. Wang, and L.S. Liu. Solving a system of Sylvester-like quaternion matrix equations. Symmetry, 14:1056, 2022.

[34] Q.W. Wang, A. Rehman, Z.H. He, and Y. Zhang. Constraint generalized Sylvester matrix equations. Automatica, 69:60-64, 2016.

[35] Q.W. Wang, Z.H. He. Systems of coupled generalized Sylvester matrix equations. Automatica, 50:2840-2844, 2014.

[36] Q.W. Wang, Z.H. He. Solvability conditions and general solution for mixed Sylvester equations. Automatica, 49:2713-2719, 2013.

[37] Q.W. Wang, Z.H. He, and Y. Zhang. Constrained two-side coupled Sylvester-type quaternion matrix equations. Automatica, 101:207-213, 2019.

[38] Q.W. Wang, J.W. Vander Woude, and H.X. Chang. A system of real quaternion matrix equations with application. Linear Algebra Appl., 431:2291-2303, 2009.

[39] Y.F. Xu, Q.W. Wang, L.S. Liu, and M. S. Mehany. A constrained system of matrix equations. Comp. Appl. Math., 41:166, 2022.

[40] S.W. Yu, Z.H. He, T.C. Qi, and X.X. Wang. The equivalence canonical form of five quaternion matrices with applications to imaging and Sylvester-type equations. J. Comput. Appl. Math., 393:113494, 2021.

[41] S.F. Yuan, Q.W. Wang, and X.F. Duan. On solutions of the quaternion matrix equation $AX = B$ and their applications in color image restoration. J. Comput. Appl. Math., 221:10-20, 2013.

[42] X. Zhang. A system of generalized Sylvester quaternion matrix equations and its applications. Appl. Math. Comput., 273:74-81, 2016.

[43] Y.N. Zhang, D.C. Jiang, and J. Wang. A recurrent neural network for solving Sylvester equation with time-varying coefficients. IEEE Trans Neural Networks, 13:1053-1063, 2002.

[44] Y. Zhang, R.H. Wang. The exact solution of a system of quaternion matrix equations involving η-Hermicity. Appl. Math. Comput., 222:201-209, 2013.