

Characterizing Riesz Bases via Biorthogonal Riesz-Fischer sequences

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Abstract

In this note we prove that if two Riesz-Fischer sequences in a separable Hilbert space H are biorthogonal and one of them is complete in H , then both sequences are Riesz bases for H . This complements a recent result by D. T. Stoeva where the same conclusion holds if one replaces the phrase “Riesz-Fischer sequences” by “Bessel sequences”.

Keywords: Riesz-Fischer sequences, Bessel sequences, Riesz sequences, Riesz bases, Biorthogonal sequences, Completeness.

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1 Introduction

Let H be a separable Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\| \cdot \|$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of vectors in H . We say that $\{f_n\}_{n=1}^{\infty}$ is a **Riesz basis** for H if $f_n = V(e_n)$ where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis for H and V is a bounded bijective operator from H onto H .

One of the many equivalences of Riesz bases [6, Theorem 1.1] states that

- A sequence is a Riesz basis for H , if and only if it is a complete Bessel sequence having a complete biorthogonal Bessel sequence in H .

Recall that $\{f_n\}_{n=1}^{\infty}$ is a **Bessel** sequence if there is a positive constant B so that

$$\sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 < B \cdot \|f\|^2 \quad \forall f \in H,$$

and $\{f_n\}_{n=1}^{\infty}$ is **complete** if its closed span in H is equal to H . **Biorthogonality** between two sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ means that

$$\langle f_n, g_m \rangle = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases}$$

Recently Stoeva [6] improved the above equivalence by assuming completeness on just one sequence.

Theorem A. [6, Theorem 2.5] *Let two sequences in H be biorthogonal. If both of them are Bessel sequences and one of them is complete in H , then they are Riesz bases for H .*

Our goal in this note is to complement Theorem A by replacing the phrase “Bessel sequences” by “Riesz-Fischer sequences”. Following Young [7, Chapter 4, Section 2], $\{f_n\}_{n=1}^{\infty}$ is a **Riesz – Fischer** sequence in H if the moment problem

$$\langle f, f_n \rangle = c_n$$

has at least one solution $f \in H$ for every sequence $\{c_n\}_{n=1}^{\infty}$ in the space $l^2(\mathbb{N})$. We prove the following.

Theorem 1.1. *Let two sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ in H be biorthogonal. If both of them are Riesz-Fischer sequences and one of them is complete in H , then they are Riesz bases for H .*

The proof is given in Section 3, once we present below some properties of Bessel and Riesz-Fischer sequences and a nice result connecting the two notions (Proposition A).

2 Riesz-Fischer sequences and Bessel sequences

In [7, Chapter 4, Section 2, Theorem 3] we find the following two theorems which provide a necessary and sufficient condition so that a sequence in H is either a Riesz-Fischer sequence or a Bessel sequence. Both results are attributed to **Nina Bari**.

- $\{f_n\}_{n=1}^{\infty}$ is a Riesz-Fischer sequence in H if and only if there exists a positive number A so that for any finite scalar sequence $\{\beta_n\}$ we have

$$A \sum |\beta_n|^2 \leq \left\| \sum \beta_n f_n \right\|^2. \quad (2.1)$$

- $\{f_n\}_{n=1}^{\infty}$ is a Bessel sequence in H if and only if there exists a positive number B so that for any finite scalar sequence $\{\beta_n\}$ we have

$$\left\| \sum \beta_n f_n \right\|^2 \leq B \sum |\beta_n|^2. \quad (2.2)$$

If a sequence is both a Bessel sequence and a Riesz-Fischer sequence, then it is called a **Riesz** sequence (see Seip [5, Lemma 3.2]). That is, $\{f_n\}_{n=1}^{\infty}$ is a Riesz sequence if there are some positive constants A and B , $A \leq B$, so that for any finite scalar sequence $\{\beta_n\}$ we have

$$A \sum |\beta_n|^2 \leq \left\| \sum \beta_n f_n \right\|^2 \leq B \sum |\beta_n|^2.$$

Remark 2.1. *A Riesz sequence is also a Riesz basis for the closure of its linear span in H (see [2, p. 68]). Therefore, a complete Riesz sequence in H is a Riesz basis for H .*

Now, it easily follows from (2.1) that a Riesz-Fischer sequence is also a **minimal** sequence, that is each f_n does not belong to the closed span of $\{f_k\}_{k \neq n}$ in H .

Remark 2.2. *It is well known that a sequence is minimal if and only if it has a biorthogonal sequence. A complete and minimal sequence is called **exact** and it has a unique biorthogonal sequence.*

Clearly now a Riesz-Fischer sequence has at least one biorthogonal sequence. As stated in Casazza et al. [1], one of them is a Bessel sequence (see Green [3, Proposition 1.2.4] for a proof).

Proposition A. [1, Proposition 2.3, (ii)]

The Riesz-Fischer sequences in H are precisely the families for which a biorthogonal Bessel sequence exists. In other words

(Part a) Suppose that a Bessel sequence $\{f_n\}$ is biorthogonal to a sequence $\{g_n\}$ in H . Then $\{g_n\}$ is a Riesz-Fischer sequence.

(Part b) If $\{f_n\}$ is a Riesz-Fischer sequence, then it has a biorthogonal Bessel sequence.

3 Proof of Theorem 1.1

First we prove the following result.

Lemma 3.1. *Let two sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ in H be biorthogonal. If $\{g_n\}_{n=1}^{\infty}$ is a Riesz sequence and $\{f_n\}_{n=1}^{\infty}$ is complete in H , then $\{g_n\}_{n=1}^{\infty}$ is also complete in H , hence both sequences are Riesz bases for \mathcal{H} .*

Proof. Denote by U the closed span of $\{g_n\}_{n=1}^{\infty}$ in H . Since $\{g_n\}_{n=1}^{\infty}$ is a Riesz sequence then it is a Riesz basis for U , therefore it has a dual biorthogonal Riesz basis for U , call it $\{h_n\}_{n=1}^{\infty}$. Thus

$$U = \overline{\text{span}}\{h_n\}_{n=1}^{\infty} = \overline{\text{span}}\{g_n\}_{n=1}^{\infty}.$$

Let U^\perp be the orthogonal complement of U in H , that is

$$U^\perp = \{f \in H : \langle f, g \rangle = 0 \text{ for all } g \in U\}.$$

Hence if $f \in U$, then $\langle f, g_n \rangle = 0$ and $\langle f, h_n \rangle = 0$ for all $n \in \mathbb{N}$.

Now, due to biorthogonality, for fixed $n \in \mathbb{N}$ we have $\langle f_n, g_n \rangle = 1$ and $\langle h_n, g_n \rangle = 1$. We also have $\langle f_n, g_m \rangle = 0$ and $\langle h_n, g_m \rangle = 0$ for all $m \neq n$. Therefore, $\langle (f_n - h_n), g_k \rangle = 0$ for all $k \in \mathbb{N}$, thus, $(f_n - h_n)$ belongs to U^\perp .

Suppose that $(f_n - h_n) \neq 0$. Then $(f_n - h_n)$ does not belong to $U = \overline{\text{span}}\{h_n\}_{n=1}^\infty$. It readily follows that f_n does not belong to U either (if $f_n \in U$ then $(f_n - h_n) \in U$ as well). Hence f_n belongs to U^\perp , so $\langle f_n, g_k \rangle = 0$ for all $k \in \mathbb{N}$, a contradiction since $\langle f_n, g_n \rangle = 1$. We have now concluded that $f_n = h_n$ and clearly this holds for all $n \in \mathbb{N}$. Therefore, $\{h_n\}_{n=1}^\infty = \{f_n\}_{n=1}^\infty$ is a complete Riesz sequence in H , hence a Riesz basis for H . The same of course holds for its biorthogonal sequence $\{g_n\}_{n=1}^\infty$. \square

Corollary 3.1. *Suppose that an exponential system $E = \{e^{i\lambda_n t}\}$, with real λ_n , is a Riesz-Fischer sequence in $L^2(-a, a)$, and E has a complete biorthogonal sequence. Then E is a Riesz basis for $L^2(-a, a)$.*

Proof. It follows from Lindner [4] that E is a Bessel sequence in $L^2(-a, a)$ as well. Hence E is a Riesz sequence in $L^2(-a, a)$ and from Lemma 3.1 we obtain the result. \square

Consider now the assumptions of Theorem 1.1 and without loss of generality, suppose that $\{f_n\}_{n=1}^\infty$ is complete in H , hence it is exact since it is also a minimal sequence. Thus it has a unique biorthogonal sequence and clearly this is $\{g_n\}_{n=1}^\infty$. Since $\{f_n\}_{n=1}^\infty$ is a Riesz-Fischer sequence, then by Proposition **A** (Part b) it has a biorthogonal Bessel sequence, and by uniqueness, this is $\{g_n\}_{n=1}^\infty$. Therefore $\{g_n\}_{n=1}^\infty$ is a Bessel sequence and a Riesz-Fischer sequence simultaneously, hence a Riesz sequence. It then follows from Lemma 3.1 that $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ are Riesz bases for H . The proof of Theorem 1.1 is now complete.

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