QUANTUM AFFINE VERTEX ALGEBRAS ASSOCIATED TO UNTWISTED QUANTUM AFFINIZATION ALGEBRAS

FEI KONG

Abstract. Let $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ be the untwisted affinization of a symmetrizable quantum Kac-Moody algebra $\mathcal{U}_\hbar(\mathfrak{g})$. For $\ell \in \mathbb{C}$, we construct an $\hbar$-adic quantum vertex algebra $V_{\hat{\mathfrak{g}},\hbar}(\ell,0)$, and establish a one-to-one correspondence between ϕ-coordinated $V_{\hat{\mathfrak{g}},\hbar}(\ell,0)$-modules and restricted $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$-modules of level ℓ. Suppose that ℓ is a positive integer. We construct a quotient $\hbar$-adic quantum vertex algebra $L_{\hat{\mathfrak{g}},\hbar}(\ell,0)$ of $V_{\hat{\mathfrak{g}},\hbar}(\ell,0)$, and establish a one-to-one correspondence between certain ϕ-coordinated $L_{\hat{\mathfrak{g}},\hbar}(\ell,0)$-modules and restricted integrable $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$-modules of level ℓ. Suppose further that $\mathfrak{g}$ is of finite type. We prove that $L_{\hat{\mathfrak{g}},\hbar}(\ell,0)/\hbar L_{\hat{\mathfrak{g}},\hbar}(\ell,0)$ is isomorphic to the simple affine vertex algebra $L_{\hat{\mathfrak{g}}}(\ell,0)$.

1. Introduction

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra, and let $\hat{\mathfrak{g}} = \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus \mathbb{C}c$ be the corresponding affine Kac-Moody Lie algebra. For any complex number $\ell \in \mathbb{C}$, we define the induced module

$$V_{\hat{\mathfrak{g}}}(\ell,0) = \mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\hat{\mathfrak{g}}_+)} \mathbb{C}_\ell, \quad \text{where } \hat{\mathfrak{g}}_+ = \mathfrak{g}\otimes\mathbb{C}[t]\oplus\mathbb{C}c$$

and $\mathbb{C}_\ell = \mathbb{C}$ is a $\hat{\mathfrak{g}}_+$-module with $\mathfrak{g}\otimes\mathbb{C}[t]$ acting trivially and c acting as scalar ℓ. The induced module $V_{\hat{\mathfrak{g}}}(\ell,0)$ carries a vertex algebra structure, and the category of $V_{\hat{\mathfrak{g}}}(\ell,0)$-modules is naturally isomorphic to the category of restricted $\hat{\mathfrak{g}}$-modules of level ℓ [FZ, Li1, LL]. If ℓ is a positive integer, then the unique simple quotient $\hat{\mathfrak{g}}$-module $L_{\hat{\mathfrak{g}}}(\ell,0)$ of $V_{\hat{\mathfrak{g}}}(\ell,0)$ is integrable. Moreover, $L_{\hat{\mathfrak{g}}}(\ell,0)$ carries a simple quotient vertex algebra structure, and the category of $L_{\hat{\mathfrak{g}}}(\ell,0)$-modules is naturally isomorphic to the category of restricted integrable $\hat{\mathfrak{g}}$-modules of level ℓ [FZ, DL, MP1, MP2, DLM1, LL].

Denote by $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ the (untwisted) quantum affine algebra associated to $\hat{\mathfrak{g}}$ ([Dr1] and [Jim]). Like the affinization realization of affine Kac-Moody Lie algebras, Drinfeld provided a quantum affinization realization of $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ in [Dr2]. Based on the Drinfeld presentation, Frenkel and Jing constructed vertex representations for simply-laced untwisted quantum affine algebras in [FJ]. In the very paper, they formulated a fundamental problem of developing certain "quantum vertex algebra theory" associated to quantum

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affine algebras, in parallel with the connection between affine Kac-Moody Lie algebras and vertex algebras.

As one of the fundamental works, Etingof and Kazhdan ([EK]) developed a theory of quantum vertex operator algebras in the sense of formal deformations of vertex algebras. The most visible difference between these and vertex algebras is that the usual locality is replaced by the S-locality. Such S-locality is controlled by a rational quantum Yang-Baxter operator. Partly motivated by the work of Etingof and Kazhdan, H. Li conducted a series of studies. While vertex algebras are analogues of commutative associative algebras, H. Li introduced the notion of nonlocal vertex algebras [Li4], which are analogues of noncommutative associative algebras. A nonlocal vertex algebra is a weak quantum vertex algebra [Li4] if it satisfies the S-locality. In addition, it becomes a quantum vertex algebra [Li4] if the S-locality is controlled by a rational quantum Yang-Baxter operator. Mainly in order to associate quantum vertex algebras to quantum affine algebras, a theory of ϕ-coordinated quasi modules was developed in [Li7, Li8]. The $\hbar$-adic counterparts of these notions were introduced in [Li6]. In this framework, a quantum vertex operator algebra in sense of Etingof-Kazhdan is an $\hbar$-adic quantum vertex algebra whose classical limit is a vertex algebra.

In the pioneer work [EK], Etingof and Kazhdan constructed quantum vertex operator algebras as formal deformations of $V_{\hat{\mathfrak{gl}}_n}(\ell,0)$ and $V_{\hat{\mathfrak{sl}}_n}(\ell,0)$, by using the R-matrix type relations given in [RSTS]. Recently, Butorac, Jing and Kožić ([BJK]) extended Etingof-Kazhdan's construction to type B, C and D rational R-matrices. The modules of these quantum vertex operator algebras are in one-to-one correspondence with restricted modules for the corresponding Yangian doubles (see [Ko1]). Based on the R-matrix presentation of quantum affine algebras (see [DF, JLM1, JLM2]), Kožić constructed the quantum vertex operator algebras associated with trigonometric R-matrices of types A, B, C and D ([Ko2, Ko1]), and established a one-to-one correspondence between ϕ-coordinated modules and restricted modules for quantum affine algebras.

In [JKLT2], N. Jing, H. Li, S. Tan and I developed a method to construct desired quantum vertex algebras. Let V be a nonlocal vertex algebra, and let (H,ρ,τ) be a triple consisting of a cocommutative nonlocal vertex bialgebra H, a right H-comodule nonlocal vertex algebra structure ρ on V, and a "compatible" H-module nonlocal vertex algebra structure τ on V. Then a new nonlocal vertex algebra $\mathfrak{D}_\tau^\rho(V)$ with V as the underlying space was obtained. It was also proved that under certain conditions, $\mathfrak{D}_\tau^\rho(V)$ is a quantum vertex algebra. On the representation side, it was proved that for any ϕ-coordinated V-module W with a "compatible" ϕ-coordinated H-module structure, there exists a ϕ-coordinated $\mathfrak{D}_\tau^\rho(V)$-module structure on W. Later in [JKLT3], we introduced an $\hbar$-adic version of this construction, and constructed a family of $\hbar$-adic quantum vertex algebras $V_L[[\hbar]]^\eta$ as formal deformations of the lattice vertex algebras V_L. When L is a root lattice of A, D, E type, we established a natural connection between twisted quantum affine algebras and equivariant ϕ-coordinated quasi modules for $V_L[[\hbar]]^\eta$ with suitable η.

It is remarkable that Drinfeld's quantum affinization process can be extended to general symmetrizable quantum Kac-Moody algebras ([GKV, Jin, N, CJKT]). We denote by $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ the untwisted quantum affinization of a symmetrizable quantum Kac-Moody algebra $\mathcal{U}_\hbar(\mathfrak{g})$. In this paper, we use the language of Drinfeld presentations to construct the $\hbar$-adic quantum vertex algebra $V_{\hat{\mathfrak{g}},\hbar}(\ell,0)$ and establish a one-to-one correspondence between restricted $\mathcal{U}_\hbar(\mathfrak{g})$-modules of level ℓ and ϕ-coordinated $V_{\hat{\mathfrak{g}},\hbar}(\ell,0)$-modules. Assume further that ℓ is a positive integer. We will study a quotient $\hbar$-adic quantum vertex algebra $L_{\hat{\mathfrak{g}},\hbar}(\ell,0)$ of $V_{\hat{\mathfrak{g}},\hbar}(\ell,0)$, and establish a one-to-one correspondence between restricted integrable $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$-modules of level ℓ and ϕ-coordinated weight modules of $L_{\hat{\mathfrak{g}},\hbar}(\ell,0)$. Finally, when $\mathfrak{g}$ is of finite type, we show that $L_{\hat{\mathfrak{g}},\hbar}(\ell,0)/\hbar L_{\hat{\mathfrak{g}},\hbar}(\ell,0) \cong L_{\hat{\mathfrak{g}}}(\ell,0)$ as vertex algebras.

Now we provide some detailed information. Let $A=(a_{ij})_{i,j\in I}$ be a symmetrizable generalized Cartan matrix (GCM). Then there are unique relatively prime positive integers r_i $(i\in I)$ such that DA is symmetric with $D=\mathrm{diag}\{r_i\}_{i\in I}$. Let $\mathfrak{g}=[\mathfrak{g}(A),\mathfrak{g}(A)]$ be the derived subalgebra of the Kac-Moody Lie algebra $\mathfrak{g}(A)$, and let $\hat{\mathfrak{g}}$ be an "affinization Lie algebra" associated to $\mathfrak{g}$ (see Definition 2.4). Recall that the (untwisted) quantum affinization algebra (see [CJKT]) $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ is the $\mathbb{C}[[\hbar]]$-algebra topologically generated by

$$\left\{h_{i,q}(n),\ x_{i,q}^{\pm}(n),\ c\ \middle|\ i\in I, n\in\mathbb{Z}\right\}, \tag{1.1}$$

and subject to the relations in terms of the generating functions

$$\begin{aligned}
\phi_{i,q}^{\pm}(z) &= q^{\pm h_{i,q}(0)}\exp\left(\pm(q-q^{-1})\sum_{\pm m>0} h_{i,q}(m)z^{-m}\right),\\
x_{i,q}^{\pm}(z) &= \sum_{m\in\mathbb{Z}} x_{i,q}^{\pm}(m)z^{-m},
\end{aligned}$$

The relations are $(i,j\in I)$:

(Q1) $\quad [c,\phi_{i,q}^{\pm}(z)]=0=[\phi_{i,q}^{\pm}(z),\phi_{j,q}^{\pm}(w)]=[c,x_{i,q}^{\pm}(z)]$,

(Q2) $\quad \phi_{i,q}^{+}(z_1)\phi_{j,q}^{-}(z_2)=\phi_{j,q}^{-}(z_2)\phi_{i,q}^{+}(z_1)\iota_{z_1,z_2}g_{ij,q}(q^{rc}z_2/z_1)^{-1}g_{ij,q}(q^{-rc}z_2/z_1)$,

(Q3) $\quad \phi_{i,q}^{+}(z_1)x_{j,q}^{\pm}(z_2)=x_{j,q}^{\pm}(z_2)\phi_{i,q}^{+}(z_1)\iota_{z_1,z_2}g_{ij,q}(q^{\mp\frac{1}{2}rc}z_2/z_1)^{\pm 1}$,

(Q4) $\quad \phi_{i,q}^{-}(z_1)x_{j,q}^{\pm}(z_2)=x_{j,q}^{\pm}(z_2)\phi_{i,q}^{-}(z_1)\iota_{z_2,z_1}g_{ji,q}(q^{\mp\frac{1}{2}rc}z_1/z_2)^{\mp 1}$,

(Q5) $\quad [x_{i,q}^{+}(z_1),x_{j,q}^{-}(z_2)]$

$$=\frac{\delta_{ij}}{q_i-q_i^{-1}}\left(\phi_{i,q}^{+}(z_1q^{-\frac{1}{2}rc})\delta\left(\frac{z_2q^{rc}}{z_1}\right)-\phi_{i,q}^{-}(z_1q^{\frac{1}{2}rc})\delta\left(\frac{z_2q^{-rc}}{z_1}\right)\right),$$

(Q6) $\quad F_{ij,q}^{\pm}(z_1,z_2)x_{i,q}^{\pm}(z_1)x_{j,q}^{\pm}(z_2)=G_{ij,q}^{\pm}(z_1,z_2)x_{j,q}^{\pm}(z_2)x_{i,q}^{\pm}(z_1)$,

$$\text{(Q7)}\quad \sum_{\sigma\in S_{m_{ij}}}\sum_{k=0}^{m_{ij}}(-1)^k\binom{m_{ij}}{k}_{q_i} x_{i,q}^{\pm}(z_{\sigma(1)})\cdots x_{i,q}^{\pm}(z_{\sigma(k)})x_{j,q}^{\pm}(w)$$
$$\times\, x_{i,q}^{\pm}(z_{\sigma(k+1)})\cdots x_{i,q}^{\pm}(z_{\sigma(m_{ij})}) \;=0,\quad \text{if}\ \ a_{ij}<0,$$

where $q=\exp\hbar\in\mathbb{C}[[\hbar]]$, $q_i=q^{r_i}$, r is the least common multiple of $\{r_i\}_{i\in I}$ and

$$F_{ij,q}^{\pm}(z_1,z_2)=z_1-q_i^{\pm a_{ij}}z_2,\quad G_{ij,q}^{\pm}(z_1,z_2)=q_i^{\pm a_{ij}}z_1-z_2,$$
$$g_{ij,q}(z)=\frac{G_{ij,q}^{+}(1,z)}{F_{ij,q}^{+}(1,z)}$$

and the map ι_{z_1,z_2} is defined as in [FHL, §3.1]. The classical limit $\mathcal{U}_\hbar(\hat{\mathfrak{g}})/\hbar U_\hbar(\hat{\mathfrak{g}})$ is isomorphic to the universal enveloping algebra of $\hat{\mathfrak{g}}$.

Let $V_{\hat{\mathfrak{g}}}(\ell,0)$ be the vertex algebra associated to $\hat{\mathfrak{g}}$ (see Definition 2.6). Different from the construction of $V_L[[\hbar]]^\eta$ in [JKLT3], we are not able to find a suitable "deforming triple" for $V_{\hat{\mathfrak{g}}}(\ell,0)$, such that the corresponding $\hbar$-adic quantum vertex algebra can be associated to $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$. Let $\mathcal{U}_\hbar^f(\hat{\mathfrak{g}})$ be the unital associative algebra over $\mathbb{C}[[\hbar]]$ topologically generated by the elements (1.1) subject to relations (Q1)-(Q4), (Q6) and

$$\text{(Q5.5)}\quad (1-q^{r\ell}z_2/z_1)^{\delta_{ij}}(1-q^{-r\ell}z_2/z_1)^{\delta_{ij}}[x_{i,q}^{+}(z_1),x_{j,q}^{-}(z_2)]=0,\quad i,j\in I.$$

It is easy to see that $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ is a quotient algebra of $\mathcal{U}_\hbar^f(\hat{\mathfrak{g}})$. Using the theory of free vertex algebras developed by Roitman ([R]), we construct a vertex algebra $F(A,\ell)$ (see Definition 2.6). Then by using the construction given in [JKLT2, JKLT3], we get a family of $\hbar$-adic quantum vertex algebras $F_\tau(A,\ell)$ as formal deformations of $F(A,\ell)$ (see Theorem 5.13). The category of restricted $\mathcal{U}_\hbar^f(\hat{\mathfrak{g}})$-modules is isomorphic to the category of ϕ-coordinated $F_\tau(A,\ell)$-modules with suitable τ. For this suitable τ, we construct the quotient $\hbar$-adic quantum vertex algebras $V_{\hat{\mathfrak{g}},\hbar}(\ell,0)$ and $L_{\hat{\mathfrak{g}},\hbar}(\ell,0)$ of $F_\tau(A,\ell)$. Finally, based on the "normal ordered" type quantum Serre relations given in [CJKT] and the integrable conditions developed in [DM], we establish a natural connection between restricted (resp. restricted integrable) $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$-modules of level ℓ and ϕ-coordinated modules (resp. ϕ-coordinated weighted modules) for $V_{\hat{\mathfrak{g}},\hbar}(\ell,0)$ (resp. $L_{\hat{\mathfrak{g}},\hbar}(\ell,0)$).

The paper is organized as follows. In Section 2, we recall the basics about vertex algebras and free vertex algebras. Define the vertex algebra $F(A,\ell)$ associated to a symmetrizable generalized Cartan matrix A and a complex number ℓ. Then we realize $V_{\hat{\mathfrak{g}}}(\ell,0)$ and $L_{\hat{\mathfrak{g}}}(\ell,0)$ as quotient vertex algebras of $F(A,\ell)$. In Section 3, we recall the notion of $\hbar$-adic quantum vertex algebras, and the theory of constructing $\hbar$-adic nonlocal vertex algebras introduced in [Li6]. In Section 5, we construct an $\hbar$-adic nonlocal vertex algebra $F_\tau(A,\ell)$. By using the deformation method introduced in [JKLT2, JKLT3] (see Section 4), we prove that $F_\tau(A,\ell)$ is an $\hbar$-adic quantum vertex algebra and $F_\tau(A,\ell)/\hbar F_\tau(A,\ell)\cong F(A,\ell)$ as vertex algebras. In Section

6, we construct two quotient $\hbar$-adic quantum vertex algebras $V_{\hat{\mathfrak{g}},\hbar}(\ell,0)$ and $L_{\hat{\mathfrak{g}},\hbar}(\ell,0)$ of $F_\tau(A,\ell)$. In Section 7, we recall the notion of ϕ-coordinated modules of $\hbar$-adic nonlocal vertex algebras, and study the category of ϕ-coordinated modules of $V_{\hat{\mathfrak{g}},\hbar}(\ell,0)$. In Section 8, we establish the natural connections between certain $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$-modules and ϕ-coordinated modules for the two $\hbar$-adic quantum vertex algebras $V_{\hat{\mathfrak{g}},\hbar}(\ell,0)$ and $L_{\hat{\mathfrak{g}},\hbar}(\ell,0)$.

Throughout this paper, we denote the set of nonnegative integer and positive integers by $\mathbb{N}$ and $\mathbb{Z}_+$, respectively. For any ring R, we denote the set of invertible elements by $R^\times$.

2. Vertex algebras and affine vertex algebras

In this section, we recall the notion of vertex algebras and define some vertex algebras associated to the symmetrizable GCM $A=(a_{ij})_{i,j\in I}$.

A *vertex algebra* is a vector space V equipped with a linear map

$$Y(\cdot,z):V\to\mathcal{E}(V):=\mathrm{Hom}(V,V((z)));\quad v\mapsto Y(v,z)=\sum_{n\in\mathbb{Z}}v_nz^{-n-1} \tag{2.1}$$

and equipped with a distinguished vector $\mathbf{1}$ of V, called the *vacuum vector*, such that

$$Y(\mathbf{1},z)v=v,\quad Y(v,z)\mathbf{1}\in V[[z]],\quad \lim_{z\to0}Y(v,z)\mathbf{1}=v\quad \text{for } v\in V \tag{2.2}$$

and such that for $u,v,w\in V$, the following *Jacobi identity* holds true

$$\begin{aligned}&z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2)w-z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(u,z_1)w\\&\qquad=z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)Y(Y(u,z_0)v,z_2)w.\end{aligned} \tag{2.3}$$

A *conformal Lie algebra* (C,Y^+,T) ([Kac]), also known as a *vertex Lie algebra* ([DLM2, P]), is a vector space C equipped with a linear operator T and a linear map

$$Y^+(\cdot,z):\ C\to\mathrm{Hom}(C,z^{-1}C[z^{-1}]);\quad u\mapsto Y^+(u,z)=\sum_{n\geq0}u_nz^{-n-1} \tag{2.4}$$

such that for any $u,v\in C$,

$$\begin{aligned}&[T,Y^+(u,z)]=Y^+(Tu,z)=\frac{d}{dz}Y^+(u,z),\\&Y^+(u,z)v=\mathrm{Sing}_z\left(e^{zT}Y^+(v,-z)u\right),\\&[Y^+(u,z_1),Y^+(v,z_2)]=\mathrm{Sing}_{z_1}\mathrm{Sing}_{z_2}(Y^+(Y^+(u,z_1-z_2)v,z_2)),\end{aligned}$$

where Sing stands for the singular part. Let (C,Y^+,T), (C_1,Y_1^+,T_1) be conformal Lie algebras. A linear map $f:C\to C_1$ is called conformal Lie algebra homomorphism if $f\big(Y^+(u,z)v\big)=Y_1^+(f(u),z)f(v)$ and $f\big(Tu\big)=T_1f(u)$ for $u,v\in C$.

Let C be an arbitrary conformal Lie algebra. Recall the *coefficient algebra* $\widehat{C}$ ([R]), also known as the *local vertex Lie algebra* ([DLM2]) is the Lie algebra with the underlying vector space $\big(C\otimes\mathbb{C}[t,t^{-1}]\big)/\mathrm{Im}\big(T\otimes 1+1\otimes d/dt\big)$, and the Lie bracket

$$[u(m),v(n)]=\sum_{k\geq 0}\binom{m}{k}(u_k v)(m+n-k), \tag{2.5}$$

where $u,v\in C$, $m,n\in\mathbb{Z}$ and $u(m)$ is the image of $u\otimes t^m$ in $\widehat{C}$. Set $\widehat{C}^-=\mathrm{Span}_{\mathbb{C}}\,\{u(m)\mid u\in C, m<0\}$ and $\widehat{C}^+=\mathrm{Span}_{\mathbb{C}}\,\{u(m)\mid u\in C, m\geq 0\}$, and define

$$V_C=\mathcal{U}(\widehat{C})\otimes_{\mathcal{U}(\widehat{C}^+)}\mathbb{C}, \tag{2.6}$$

where $\mathbb{C}$ is a trivial $\widehat{C}^+$-module. It is proved in [R, Proposition 1.3] (see also [DLM2, Proposition 3.4]) that $\widehat{C}^{\pm}$ are Lie subalgebras of $\widehat{C}$ and $\widehat{C}=\widehat{C}^-\oplus\widehat{C}^+$. Then $V_C\cong\mathcal{U}(\widehat{C}^-)$ as vector spaces. For convenience, we identify u with $u(-1)\otimes 1\in V_C$ for $u\in C$. A $\widehat{C}$-module W is called a *restricted* module if

$$u(z)v=\sum_{m\in\mathbb{Z}}u(m)vz^{-m-1}\in W((z)),\quad \text{for } u\in C,\ v\in W.$$

It is proved in [DLM2] that

Theorem 2.1 ([DLM2, Theorem 4.8]). *There exists a vertex algebra structure on V_C such that $\mathbf{1}=1\otimes 1\in V_C$ is the vacuum vector and*

$$Y(u,z)=\sum_{m\in\mathbb{Z}}u(m)z^{-m-1}\quad \textit{for } u\in C.$$

For any restricted $\widehat{C}$-module W, there exists a V_C-module structure Y_W on W such that

$$Y_W(u,z)=u(z)=\sum_{m\in\mathbb{Z}}u(m)z^{-m-1},\quad u\in C.$$

On the other hand, for any V_C-module W, there exists a restricted $\widehat{C}$-module structure on W such that

$$u(z).w=Y_W(u,z)w$$

where $u\in C$ and $w\in W$.

Let $\mathcal{B}$ be a set, and let $N:\mathcal{B}\times\mathcal{B}\to\mathbb{N}$ be a symmetric function called a *locality function* ([R]). Denote by $\mathfrak{Conf}(\mathcal{B},N)$ the category whose objects are conformal Lie algebras C which contain $\mathcal{B}$ as a generating subset and

$$a_n b=0\quad \text{for all } a,b\in\mathcal{B},\ n\geq N(a,b).$$

The following result is given in [R, Proposition 3.1]

Proposition 2.2. *There exists a conformal Lie algebra $C(\mathcal{B},N)\in\mathfrak{Conf}(\mathcal{B},N)$ such that for any $C\in\mathfrak{Conf}(\mathcal{B},N)$, there is a unique conformal Lie algebra homomorphism f from $C(\mathcal{B},N)$ to C such that*

$$f(a)=a\in\mathcal{B}\subset C\quad \textit{for } a\in\mathcal{B}\subset C(\mathcal{B},N).$$

Moreover, the coefficient algebra $\widehat{C}(\mathcal{B},N)$ of $C(\mathcal{B},N)$ is isomorphic to the Lie algebra generated by

$$\{b(n) \mid b \in \mathcal{B},\ n \in \mathbb{Z}\} \tag{2.7}$$

and subject to the relations

$$\sum_{i\geq 0}(-1)^i \binom{N(a,b)}{i}[a(n-i), b(m+i)] = 0 \quad \textit{for } a,b \in \mathcal{B},\ m,n \in \mathbb{Z}. \tag{2.8}$$

The conformal Lie algebra $C(\mathcal{B},N)$ is called the *free conformal Lie algebra* in [R]. From Theorem 2.1, we get a vertex algebra $V(\mathcal{B},N)$ corresponding to $C(\mathcal{B},N)$, which is called the *free vertex algebra* in [R].

We note that the relation (2.8) is equivalent to the following relation

$$(z_1 - z_2)^{N(a,b)}[a(z_1), b(z_2)] = 0 \quad \text{for } a,b \in \mathcal{B}. \tag{2.9}$$

Proposition 2.3. *Let V be a vertex algebra, and let $f : \mathcal{B} \to V$ such that*

$$(z_1 - z_2)^{N(a,b)}[Y(f(a),z_1), Y(f(b),z_2)] = 0 \quad \textit{for } a,b \in \mathcal{B}.$$

Then there exists a vertex algebra homomorphism $\widehat{f} : V(\mathcal{B},N) \to V$ such that $\widehat{f}|_{\mathcal{B}} = f$.

We introduce the following Lie algebra.

Definition 2.4. Let $\hat{\mathfrak{g}}$ be the Lie algebra generated by

$$\{h_i(m),\ x_i^{\pm}(m) \mid i \in I,\ m \in \mathbb{Z}\}$$

and a central element c, subject to the relations written in terms of generating functions in z:

$$h_i(z) = \sum_{m\in\mathbb{Z}} h_i(m) z^{-m-1}, \quad x_i^{\pm}(z) = \sum_{m\in\mathbb{Z}} x_i^{\pm}(m) z^{-m-1}, \quad i \in I.$$

The relations are $(i,j \in I)$

(L1) $[h_i(z_1), h_j(z_2)] = r_i a_{ij} rc \frac{\partial}{\partial z_2} z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)$,

(L2) $[h_i(z_1), x_j^{\pm}(z_2)] = \pm r_i a_{ij} x_j^{\pm}(z_2) z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)$,

(L3) $[x_i^{+}(z_1), x_j^{-}(z_2)] = \frac{\delta_{ij}}{r_i}\left(h_i(z_2) z_1^{-1}\delta\left(\frac{z_2}{z_1}\right) + rc\frac{\partial}{\partial z_2} z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right)$,

(L4) $(z_1 - z_2)^{n_{ij}}[x_i^{\pm}(z_1), x_j^{\pm}(z_2)] = 0$,

(S) $[x_i^{\pm}(z_1), [x_i^{\pm}(z_2), \ldots, [x_i^{\pm}(z_{m_{ij}}), x_j^{\pm}(z_0)]\cdots]] = 0, \quad \text{if } a_{ij} < 0,$

where $n_{ij} = 1 - \delta_{ij}$ for $i,j \in I$ and $m_{ij} = 1 - a_{ij}$ for $i,j \in I$ with $a_{ij} < 0$.

Remark 2.5. Suppose that the GCM A is of finite or affine type. Recall that $\mathfrak{g} = [\mathfrak{g}(A), \mathfrak{g}(A)]$ is the derived subalgebra of the Kac-Moody Lie algebra associated to the GCM A. If A is not of type $A_1^{(1)}$, then $\hat{\mathfrak{g}}$ is the universal central extension of $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]$ (see [Gar] when A is of finite type and [MERY] when A is of affine type). If A is of type $A_1^{(1)}$, then $\hat{\mathfrak{g}}$ is the quotient algebra of the universal central extension of $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]$ modulo the ideal generated by the following relations

$$(z_1-z_2)[x_i^{\pm}(z_1),x_j^{\pm}(z_2)]=0,\quad \text{for } i\neq j.$$

Introduce a set $\mathcal{B}=\left\{h_i,\, x_i^{\pm}\;\middle|\; i\in I\right\}$, and define a function $N:\mathcal{B}\times\mathcal{B}\to\mathbb{N}$ by

$$N(h_i,h_j)=2,\quad N(h_i,x_j^{\pm})=1,\quad N(x_i^{\pm},x_j^{\pm})=n_{ij},\quad N(x_i^{+},x_j^{-})=\delta_{ij}2.$$

Definition 2.6. For $\ell\in\mathbb{C}$, we let $F(A,\ell)$ be the quotient vertex algebra of $V(\mathcal{B},N)$ modulo the ideal generated by

$$(h_i)_0(h_j),\quad (h_i)_1(h_j)-r_ia_{ij}r\ell\mathbf{1},\quad (h_i)_0(x_j^{\pm})\mp r_ia_{ij}x_j^{\pm}\quad \text{for } i,j\in I.$$

Furthermore, let $V_{\hat{\mathfrak{g}}}(\ell,0)$ be the quotient vertex algebra of $F(A,\ell)$ modulo the ideal generated by

$$(x_i^{+})_0(x_j^{-})-\frac{\delta_{ij}}{r_i}h_i,\quad (x_i^{+})_1(x_j^{-})-\frac{\delta_{ij}}{r_i}r\ell\mathbf{1}\quad \text{for } i,j\in I,$$
$$\left((x_i^{\pm})_0\right)^{m_{ij}}(x_j^{\pm})\quad \text{for } i,j\in I \text{ with } a_{ij}<0.$$

If $\ell\in\mathbb{Z}_+$, we define $L_{\hat{\mathfrak{g}}}(\ell,0)$ to be the quotient vertex algebra of $V_{\hat{\mathfrak{g}}}(\ell,0)$ modulo the ideal generated by

$$\left((x_i^{\pm})_{-1}\right)^{\ell r/r_i}x_i^{\pm}\quad \text{for } i\in I.$$

Remark 2.7. Set $\hat{\mathfrak{g}}_+=\mathfrak{g}\otimes\mathbb{C}[t]\oplus\mathbb{C}c$. For $\ell\in\mathbb{C}$, we let $\mathbb{C}_\ell:=\mathbb{C}$ be a $\hat{\mathfrak{g}}_+$-module, with $\mathfrak{g}\otimes\mathbb{C}[t].\mathbb{C}_\ell=0$ and $c=\ell$. Then there are $\hat{\mathfrak{g}}$-module structures on both $V_{\hat{\mathfrak{g}}}(\ell,0)$ and $L_{\hat{\mathfrak{g}}}(\ell,0)$, such that

$$h_i(z)=Y(h_i,z),\quad x_i^{\pm}(z)=Y(x_i^{\pm},z),\quad i\in I.$$

In addition, as $\hat{\mathfrak{g}}$-modules, we have that

$$V_{\hat{\mathfrak{g}}}(\ell,0)\cong\mathcal{U}(\hat{\mathfrak{g}})\otimes_{\mathcal{U}(\hat{\mathfrak{g}}_+)}\mathbb{C}_\ell$$

Suppose that the GCM A is of finite type. Then $\mathfrak{g}$ is a finite dimensional simple Lie algebra. And $L_{\hat{\mathfrak{g}}}(\ell,0)$ is the unique simple quotient $\hat{\mathfrak{g}}$-module of $V_{\hat{\mathfrak{g}}}(\ell,0)$, when $\ell\in\mathbb{Z}_+$.

3. Quantum vertex algebras

A $\mathbb{C}[[\hbar]]$-module V is said to be *torsion-free* if $\hbar v\neq 0$ for every $0\neq v\in V$, and said to be *separated* if $\cap_{n\geq 1}\hbar^nV=0$. For a $\mathbb{C}[[\hbar]]$-module V, using subsets $v+\hbar^nV$ for $v\in V$, $n\geq 1$ as the basis of open subsets one obtains a topology on V, which is called the *$\hbar$-adic topology.* A $\mathbb{C}[[\hbar]]$-module V is said to be *$\hbar$-adically complete* if every Cauchy sequence in V with respect to this $\hbar$-adic topology has a limit in V. A $\mathbb{C}[[\hbar]]$-module V is *topologically free* if $V=V_0[[\hbar]]$ for some vector space V_0 over $\mathbb{C}$. It is known that a $\mathbb{C}[[\hbar]]$-module is topologically free if and only if it is torsion-free, separated, and $\hbar$-adically complete ([Kas], see [Li6]). For another topologically free $\mathbb{C}[[\hbar]]$-module $U = U_0[[\hbar]]$, we recall the complete tensor

$$U\widehat{\otimes}V = (U_0 \otimes V_0)[[\hbar]].$$

We view a vector space as a $\mathbb{C}[[\hbar]]$-module by letting $\hbar = 0$. Fix a $\mathbb{C}[[\hbar]]$-module W. For $k \in \mathbb{Z}_+$, and some formal variables $z_1, \dots, z_k$, we define

$$\mathcal{E}^{(k)}(W; z_1, \dots, z_k) = \mathrm{Hom}_{\mathbb{C}[[\hbar]]}(W, W((z_1, \dots, z_k))). \tag{3.1}$$

Recall from [Li2, Definition 5.3] that an ordered sequence $(a_1(z), \dots, a_k(z))$ in $\mathcal{E}^{(1)}(W)$ is said to be *compatible* if there exists an $m \in \mathbb{Z}_+$, such that

$$\left(\prod_{1\le i<j\le k} (z_i - z_j)^m\right) a_1(z_1)\cdots a_k(z_k) \in \mathcal{E}^{(k)}(W; z_1, \dots, z_k).$$

Now, we assume $W = W_0[[\hbar]]$ for some vector space W_0. Then W is topologically free. Define

$$\mathcal{E}_\hbar^{(k)}(W; z_1, \dots, z_k) = \mathrm{Hom}_{\mathbb{C}[[\hbar]]}(W, W_0((z_1, \dots, z_k))[[\hbar]]). \tag{3.2}$$

Note that $\mathcal{E}_\hbar^{(k)}(W; z_1, \dots, z_k) = \mathcal{E}^{(k)}(W; z_1, \dots, z_k)[[\hbar]]$ is topologically free. For convenience, we will also write $\mathcal{E}_\hbar^{(k)}(W) = \mathcal{E}_\hbar^{(k)}(W; z_1, \dots, z_k)$ and write $\mathcal{E}_\hbar(W) = \mathcal{E}_\hbar^{(1)}(W)$. For $n, k \in \mathbb{Z}_+$, the quotient map from W to $W/\hbar^n W$ induces the following $\mathbb{C}[[\hbar]]$-module map

$$\widetilde{\pi}_n^{(k)} : (\mathrm{End}_{\mathbb{C}[[\hbar]]}(W))[[z_1^{\pm 1}, \dots, z_k^{\pm 1}]] \to (\mathrm{End}_{\mathbb{C}[[\hbar]]}(W/\hbar^n W))[[z_1^{\pm 1}, \dots, z_k^{\pm 1}]].$$

For $A(z_1, z_2), B(z_1, z_2) \in \mathrm{Hom}_{\mathbb{C}[[\hbar]]}(W, W_0((z_1))((z_2))[[\hbar]])$, we write $A(z_1, z_2) \sim B(z_2, z_1)$ if for each $n \in \mathbb{Z}_+$ there is $k \in \mathbb{N}$ such that

$$(z_1 - z_2)^k \widetilde{\pi}_n^{(2)}(A(z_1, z_2)) = (z_1 - z_2)^k \widetilde{\pi}_n^{(2)}(B(z_2, z_1)).$$

Let $Z(z_1, z_2) : \mathcal{E}_\hbar(W)\widehat{\otimes}\mathcal{E}_\hbar(W)\widehat{\otimes}\mathbb{C}((z))[[\hbar]] \to \mathrm{End}_{\mathbb{C}[[\hbar]]}(W)[[z_1^{\pm 1}, z_2^{\pm 1}]]$ be defined by

$$Z(z_1, z_2)\,(a(z) \otimes b(z) \otimes f(z)) = \iota_{z_1, z_2} f(z_1 - z_2) a(z_1) b(z_2).$$

A subset U of $\mathcal{E}_\hbar(W)$ is said to be *$\hbar$-adically S-local* if for any $a(z), b(z) \in U$, there exists $A(z) \in (\mathbb{C}U \otimes \mathbb{C}U \otimes \mathbb{C}((z)))\,[[\hbar]]$ such that

$$a(z_1) b(z_2) \sim Z(z_2, z_1)\,(A(z))\,,$$

where $\mathbb{C}U$ denotes the subspace spanned by U.

Recall from [Li6, Remark 4.7] that $\widetilde{\pi}_n^{(k)}$ induces a $\mathbb{C}[[\hbar]]$-module map $\pi_n^{(k)} : \mathcal{E}_\hbar^{(k)}(W) \to \mathcal{E}^{(k)}(W/\hbar^n W)$ with kernel $\hbar^n \mathcal{E}_\hbar^{(k)}(W)$. And $\mathcal{E}_\hbar^{(k)}(W)$ is isomorphic to the inverse limit of the following inverse system

$$0 \longleftarrow \mathcal{E}^{(k)}(W/\hbar W) \longleftarrow \mathcal{E}^{(k)}(W/\hbar^2 W) \longleftarrow \mathcal{E}^{(k)}(W/\hbar^3 W) \longleftarrow \cdots$$

If $k = 1$, we will also write $\pi_n = \pi_n^{(1)}$. Then an ordered sequence $(a_1(z) \dots, a_r(z))$ in $\mathcal{E}_\hbar(W)$ is said to be *$\hbar$-adically compatible* if for every $n \in \mathbb{Z}_+$, the sequence $(\pi_n(a_1(z)), \dots, \pi_n(a_r(z)))$ in $\mathcal{E}(W/\hbar^n W)$ is compatible. A subset U of $\mathcal{E}_\hbar(W)$

is said to be $\hbar$-*adically compatible* if every finite sequence in U is $\hbar$-adically compatible.

Let $(a(z), b(z))$ in $\mathcal{E}_\hbar(W)$ be $\hbar$-adically compatible. That is, for any $n \in \mathbb{Z}_+$, we have that

$$(z_1 - z_2)^{k_n} \pi_n(a(z_1)) \pi_n(b(z_2)) \in \mathcal{E}^{(2)}(W/\hbar^n W) \quad \text{for some } k_n \in \mathbb{Z}_+.$$

We recall the following vertex operator introduced in [Li6, Definition 4.11]:

$$Y_{\mathcal{E}}(a(z), z_0)\, b(x) = \sum_{n\in\mathbb{Z}} a(z)_n b(z) z_0^{-n-1} \tag{3.3}$$
$$= \varprojlim_{n>0} z_0^{-k_n} \left((z_1 - z)^{k_n} \pi_n(a(z_1)) \pi_n(b(z)) \right) |_{z_1 = z + z_0}.$$

An $\hbar$-*adic nonlocal vertex algebra* [Li6, Definition 2.9] is a topologically free $\mathbb{C}[[\hbar]]$-module V equipped with a $\mathbb{C}[[\hbar]]$-module map $Y(\cdot, z) : V \to \mathcal{E}_\hbar(V)$ and a distinguished vacuum vector $\mathbf{1}$ such that the vacuum property (2.2) holds, and for $u, v \in V$, $(Y(u,z), Y(v,z))$ is an $\hbar$-adic compatible pair with

$$Y_{\mathcal{E}}(Y(u,z), z_0)\, Y(v,z) = Y(Y(u,z_0)v, z) \tag{3.4}$$

Denote by ∂ the canonical derivation on V:

$$u \mapsto \partial u = \lim_{z\to 0} \frac{d}{dz} Y(u,z)\mathbf{1}. \tag{3.5}$$

An $\hbar$-*adic weak quantum vertex algebra* ([Li6, Definition 2.9]) is an $\hbar$-adic nonlocal vertex algebra V, such that $\{Y(u,z) \mid u \in V\}$ is $\hbar$-adically S-local. And an $\hbar$-*adic quantum vertex algebra* ([Li6, Definition 2.20]) is an $\hbar$-adic weak quantum vertex algebra V equipped with a $\mathbb{C}[[\hbar]]$-module map (called a *quantum Yang-Baxter operator*)

$$S(z) : V \widehat{\otimes} V \to V \widehat{\otimes} V \widehat{\otimes} \mathbb{C}((z))[[\hbar]], \tag{3.6}$$

which satisfies the *shift condition*:

$$[\partial \otimes 1, S(z)] = -\frac{d}{dz} S(z), \tag{3.7}$$

the *quantum Yang-Baxter equation*:

$$S^{12}(z_1) S^{13}(z_1 + z_2) S^{23}(z_2) = S^{23}(z_2) S^{13}(z_1 + z_2) S^{12}(z_1), \tag{3.8}$$

and the *unitarity condition*:

$$S^{21}(z) S(-z) = 1, \tag{3.9}$$

satisfying the following conditions:

(1) The *vacuum property*:

$$S(z)(\mathbf{1} \otimes v) = \mathbf{1} \otimes v, \quad \text{for } v \in V. \tag{3.10}$$

(2) The S-*locality*: For any $u, v \in V$, one has

$$Y(u, z_1) Y(v, z_2) \sim Y(z_2)(1 \otimes Y(z_1)) S(z_2 - z_1)(v \otimes u). \tag{3.11}$$

(3) The *hexagon identity*:

$$S(z_1)(Y(z_2)\otimes 1) = (Y(z_2)\otimes 1)S^{23}(z_1)S^{13}(z_1+z_2). \tag{3.12}$$

We list some technical results that will be used later on.

Lemma 3.1. *Let $(V, S(z))$ be an $\hbar$-adic quantum vertex algebra. Then*

$$S(z)(v\otimes \mathbf{1}) = v\otimes \mathbf{1} \quad \textit{for } v\in V, \tag{3.13}$$

$$[1\otimes\partial, S(z)] = \frac{d}{dz}S(z), \tag{3.14}$$

$$S(z_1)(1\otimes Y(z_2)) = (1\otimes Y(z_2))S^{12}(z_1-z_2)S^{13}(z_1), \tag{3.15}$$

$$S(z)f(\partial\otimes 1) = f\left(\partial\otimes 1+\frac{\partial}{\partial z}\right)S(z) \quad \textit{for } f(z)\in\mathbb{C}[z][[\hbar]], \tag{3.16}$$

$$S(z)f(1\otimes\partial) = f\left(1\otimes\partial-\frac{\partial}{\partial z}\right)S(z) \quad \textit{for } f(z)\in\mathbb{C}[z][[\hbar]]. \tag{3.17}$$

Lemma 3.2. *Let $(V, S(z))$ be an $\hbar$-adic quantum vertex algebra, and let $u, v\in V$, $f(z)\in\mathbb{C}((z))[[\hbar]]$.*

(1) If $S(z)(v\otimes u) = v\otimes u + \mathbf{1}\otimes\mathbf{1}\otimes f(z)$, then

$$\begin{aligned}
&S(z)((v_{-1})^n\mathbf{1}\otimes u)\\
=&(v_{-1})^n\mathbf{1}\otimes u + n(v_{-1})^{n-1}\mathbf{1}\otimes\mathbf{1}\otimes f(z),\\
&S(z)(v\otimes(u_{-1})^n\mathbf{1})\\
=&v\otimes(u_{-1})^n\mathbf{1} + n\mathbf{1}\otimes(u_{-1})^{n-1}\mathbf{1}\otimes f(z),\\
&S(z)\left(\exp(v_{-1})\mathbf{1}\otimes u\right)\\
=&\exp(v_{-1})\mathbf{1}\otimes u + \exp(v_{-1})\mathbf{1}\otimes\mathbf{1}\otimes f(z), \quad \textit{if } v\in\hbar V,\\
&S(z)\left(v\otimes\exp(u_{-1})\mathbf{1}\right)\\
=&v\otimes\exp(u_{-1})\mathbf{1} + \mathbf{1}\otimes\exp(u_{-1})\mathbf{1}\otimes f(z), \quad \textit{if } u\in\hbar V.
\end{aligned}$$

(2) If $S(z)(v\otimes u) = v\otimes u + \mathbf{1}\otimes u\otimes f(z)$, then

$$\begin{aligned}
&S(z)((v_{-1})^n\mathbf{1}\otimes u) = \sum_{i=0}^{n}\binom{n}{i}(v_{-1})^i\mathbf{1}\otimes u\otimes f(z)^{n-i},\\
&S(z)\left(\exp(v_{-1})\mathbf{1}\otimes u\right) = \exp(v_{-1})\mathbf{1}\otimes u\otimes\exp\left(f(z)\right),\\
&\quad \textit{if } v\in\hbar V, f(z)\in\hbar\mathbb{C}((z))[[\hbar]].
\end{aligned}$$

(3) If $S(z)(v\otimes u) = v\otimes u + v\otimes\mathbf{1}\otimes f(z)$, then

$$\begin{aligned}
&S(z)(v\otimes(u_{-1})^n\mathbf{1}) = \sum_{i=0}^{n}\binom{n}{i}v\otimes(u_{-1})^i\mathbf{1}\otimes f(z)^{n-i},\\
&S(z)\left(v\otimes\exp(u_{-1})\mathbf{1}\right) = v\otimes\exp\left(u_{-1}\right)\mathbf{1}\otimes\exp(f(z)),\\
&\quad \textit{if } u\in\hbar V, f(z)\in\hbar\mathbb{C}((z))[[\hbar]].
\end{aligned}$$

For a topologically free $\mathbb{C}[[\hbar]]$-module V and a submodule M, we recall the notation given in [Li6, Definition 3.4]:

$$[M] = \{v \in V \mid \hbar^n V \in M \text{ for some } n \in \mathbb{N}\}.$$

It is straightforward to verify that

Lemma 3.3. *Let $(V, S(z))$ be an $\hbar$-adic quantum vertex algebra, and let $U \subset V$ be a generating subset of V. Let $M \subset V$ be a closed ideal of V such that $[M] = M$ and*

$$S(z)(M \otimes U),\ S(z)(U \otimes M) \subset M \widehat{\otimes} V \widehat{\otimes} \mathbb{C}((z))[[\hbar]] + V \widehat{\otimes} M \widehat{\otimes} \mathbb{C}((z))[[\hbar]].$$

Then V/M is an $\hbar$-adic quantum vertex algebra with quantum Yang-Baxter operator induced from $S(z)$.

In the rest of this section, we recall the construction of $\hbar$-adic nonlocal vertex algebras and their modules introduced in [Li6]. Here, a *module* ([Li6, Definition 2.33]) for an $\hbar$-adic nonlocal vertex algebra V is a topologically free $\mathbb{C}[[\hbar]]$-module W, equipped with a $\mathbb{C}[[\hbar]]$-module map $Y_W(\cdot, z) : V \to \mathcal{E}_\hbar(W)$, satisfying the conditions that $Y_W(\mathbf{1}, z) = 1_W$ and that for $u, v \in V$, $(Y_W(u, z), Y_W(v, z))$ is $\hbar$-adically compatible and

$$Y_{\mathcal{E}}\left(Y_W(u, z), z_0\right) Y_W(v, z) = Y_W\left(Y(u, z_0)v, z\right).$$

The following result is given in [Li6]:

Theorem 3.4. *Let W be a topologically free $\mathbb{C}[[\hbar]]$-module, and let $U \subset \mathcal{E}_\hbar(W)$ be an $\hbar$-adically compatible subset. Then there is a minimal $\hbar$-adically S-local subset $\langle U \rangle \subset \mathcal{E}_\hbar(W)$ containing 1_W and U, such that*

1. *$\langle U \rangle$ is topologically free and $[\langle U \rangle] = \langle U \rangle$.*
2. *$\langle U \rangle$ is $Y_{\mathcal{E}}$ closed in the sense that for any $a(z), b(z) \in U$ and any $n \in \mathbb{Z}$, we have $a(z)_n b(z) \in \langle U \rangle$.*

Then $(\langle U \rangle, Y_{\mathcal{E}}, 1_W)$ carries the structure of an $\hbar$-adic nonlocal vertex algebra and W is a faithful $\langle U \rangle$-module with the module map Y_W defined by $Y_W(a(z), z_0) = a(z_0)$ for $a(z) \in \langle U \rangle$.

Remark 3.5. If U is $\hbar$-adically S-local, then $\langle U \rangle$ is an $\hbar$-adic weak quantum vertex algebra.

We recall the following result from [Li6, Theorem 4.24] for later use.

Theorem 3.6. *Let V be a topologically free $\mathbb{C}[[\hbar]]$-module, $U \subset V$, $\mathbf{1} \in V$, and Y^0 a map*

$$Y^0 : U \to \mathcal{E}_\hbar(V); \quad u \mapsto Y^0(u, z) = u(z) = \sum_{n \in \mathbb{Z}} u_n z^{-n-1}.$$

Assume that all the following conditions hold:

$$Y^0(u, z)\mathbf{1} \in V[[z]] \quad \textit{and} \quad \lim_{z \to 0} Y^0(u, z)\mathbf{1} = u \quad \textit{for } u \in U,$$

$U(z) = \{u(z) \mid u \in U\}$ is $\hbar$-adically S-local, and V is $\hbar$-adically spanned by vectors

$$u^{(1)}_{m_1} \cdots u^{(r)}_{m_r} \mathbf{1} \tag{3.18}$$

for $r \in \mathbb{N}$, $u^{(i)} \in U$, $m_i \in \mathbb{Z}$. In addition we assume that there exists a $\mathbb{C}[[\hbar]]$-module map ψ from V to $\langle U(z) \rangle \subset \mathcal{E}_\hbar(V)$ such that $\psi(\mathbf{1}) = 1_V$ and

$$\psi\left(Y(u, z_0)v\right) = Y_{\mathcal{E}}\left(u(z), z_0\right)\psi(v) \quad \text{for } u \in U,\, v \in V. \tag{3.19}$$

Then the map Y^0 extends uniquely to a $\mathbb{C}[[\hbar]]$-module map Y from V to $\mathcal{E}_\hbar(V)$ such that $(V, Y, \mathbf{1})$ carries the structure of an $\hbar$-adic weak quantum vertex algebra.

4. Deformation by vertex bialgebras

In this section, we recall the deformation of $\hbar$-adic nonlocal vertex algebras by using vertex bialgebras introduced in [JKLT2, JKLT3].

We start by recalling the notion of vertex bialgebras and smash products given in [Li5] (see also [JKLT3]). An *$\hbar$-adic (nonlocal) vertex bialgebra* is an $\hbar$-adic (nonlocal) vertex algebra V equipped with a classical coalgebra structure (Δ, ε) such that (the coproduct) $\Delta : V \to V \widehat{\otimes} V$ and (the counit) $\varepsilon : V \to \mathbb{C}[[\hbar]]$ are homomorphisms of $\hbar$-adic nonlocal vertex algebras.

Remark 4.1. Let (H, Δ, ε) be a bialgebra over $\mathbb{C}[[\hbar]]$ equipped with a derivation ∂. Suppose that H is topologically free. Then H is an $\hbar$-adic nonlocal vertex bialgebra with vacuum 1 and vertex operator defined by

$$Y(a, z)b = \left(e^{z\partial} a\right) b \quad \text{for } a, b \in H.$$

We denote this $\hbar$-adic nonlocal vertex bialgebra by $(H, \partial, \Delta, \varepsilon)$.

Let (H, Δ, ε) be an $\hbar$-adic nonlocal vertex bialgebra. A (left) *H-module (nonlocal) vertex algebra* ([Li5]) is an $\hbar$-adic nonlocal vertex algebra V equipped with a module structure τ on V for H viewed as an $\hbar$-adic nonlocal vertex algebra such that

$$\tau(h, z)v \in V \widehat{\otimes} \mathbb{C}((z))[[\hbar]], \qquad \tau(h, z)\mathbf{1}_V = \varepsilon(h)\mathbf{1}_V, \tag{4.1}$$

$$\tau(h, z_1)Y(u, z_2)v = \sum Y(\tau(h_{(1)}, z_1 - z_2)u, z_2)\tau(h_{(2)}, z_1)v \tag{4.2}$$

for $h \in H$, $u, v \in V$, where $\mathbf{1}_V$ denotes the vacuum vector of V and $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ is the coproduct in the Sweedler notation.

A *(right) H-comodule nonlocal vertex algebra* ([JKLT2]) is a nonlocal vertex algebra V equipped with a homomorphism $\rho : V \to V \widehat{\otimes} H$ of $\hbar$-adic nonlocal vertex algebras such that

$$(\rho \otimes 1)\rho = (1 \otimes \Delta)\rho, \quad (1 \otimes \varepsilon)\rho = \mathrm{Id}_V. \tag{4.3}$$

ρ is *compatible* with an H-module nonlocal vertex algebra structure τ (see [JKLT2]) if

$$\rho(\tau(h, z)v) = (\tau(h, z) \otimes 1)\rho(v) \quad \text{for } h \in H,\, v \in V. \tag{4.4}$$

We introduce the following notion.

Definition 4.2. Let V be an $\hbar$-adic nonlocal vertex algebra. A *deforming triple* for V is a triple (H, ρ, τ), where H is a cocommutative $\hbar$-adic nonlocal vertex bialgebra, (V, ρ) is a right H-comodule nonlocal vertex algebra and (V, τ) is an H-module nonlocal vertex algebra, such that ρ and τ are compatible.

The following result is given in [JKLT2].

Theorem 4.3. *Let V be an $\hbar$-adic nonlocal vertex algebra, and let (H, ρ, τ) be a deforming triple. Set*

$$\mathfrak{D}_\tau^\rho(Y)(a, z) = \sum Y(a_{(1)}, z)\tau(a_{(2)}, z) \quad \text{for } a \in V, \tag{4.5}$$

where $\rho(a) = \sum a_{(1)} \otimes a_{(2)} \in V \otimes H$. Then $(V, \mathfrak{D}_\tau^\rho(Y), \mathbf{1})$ carries the structure of an $\hbar$-adic nonlocal vertex algebra. Denote this $\hbar$-adic nonlocal vertex algebra by $\mathfrak{D}_\tau^\rho(V)$. Moreover, $(\mathfrak{D}_\tau^\rho(V), \rho)$ is also a right H-comodule nonlocal vertex algebra.

Remark 4.4. Let H be a cocommutative $\hbar$-adic nonlocal vertex bialgebra, and let (V, ρ) be an H-comodule nonlocal vertex algebra. We view the counit ε as an H-module nonlocal vertex algebra structure on V as follows

$$\varepsilon(h, z)v = \varepsilon(h)v \quad \text{for } h \in H,\ v \in V. \tag{4.6}$$

Then it is easy to verify that ε and ρ are compatible, and $\mathfrak{D}_\varepsilon^\rho(V) = V$.

Remark 4.5. Let V_1, V_2 be two $\hbar$-adic nonlocal vertex algebras. Suppose that (H, ρ_1, τ_1) and (H, ρ_2, τ_2) are deforming triples of V_1 and V_2 respectively. Suppose that $f : V_1 \to V_2$ is an $\hbar$-adic nonlocal vertex algebra homomorphism such that

$$\rho_2 \circ f = (f \otimes 1) \circ \rho_1, \quad \text{and} \quad f(\tau_1(h, z)v) = \tau_2(h, z)f(v) \quad \text{for } h \in H,\ v \in V_1.$$

Then f is an $\hbar$-adic nonlocal vertex algebra homomorphism from $\mathfrak{D}_{\tau_1}^{\rho_1}(V_1)$ to $\mathfrak{D}_{\tau_2}^{\rho_2}(V_2)$.

Now, we fix a cocommutative $\hbar$-adic nonlocal vertex bialgebra H, and an H-comodule nonlocal vertex algebra (V, ρ). Note that

$$\operatorname{Hom}\left(H, \operatorname{Hom}(V, V \widehat{\otimes} \mathbb{C}((x))[[\hbar]])\right)$$

is a unital associative algebra, where the multiplication is defined by

$$(f * g)(h, z) = \sum f(h_{(1)}, z)g(h_{(2)}, z)$$

for $f, g \in \operatorname{Hom}\left(H, \operatorname{Hom}(V, V \widehat{\otimes} \mathbb{C}((z))[[\hbar]])\right)$, and the unit ε. For $f, g \in \operatorname{Hom}\left(H, \operatorname{Hom}(V, V \widehat{\otimes} \mathbb{C}((z))[[\hbar]])\right)$, we say f and g commute if

$$[f(h, z_1), g(k, z_2)] = 0 \quad \text{for } h, k \in H.$$

Let $\mathfrak{L}_H^\rho(V)$ be the set of H-module nonlocal vertex algebra structures on V that are compatible with ρ. The following is an immediate $\hbar$-adic analogue of [JKLT2, Proposition 3.3 and Proposition 3.4].

Proposition 4.6. *Let τ and τ' are commuting elements in $\mathfrak{L}_H^\rho(V)$. Then $\tau * \tau' \in \mathfrak{L}_H^\rho(V)$ and $\tau * \tau' = \tau' * \tau$. Moreover, $\tau \in \mathfrak{L}_H^\rho\left(\mathfrak{D}_{\tau'}^\rho(V)\right)$ and*

$$\mathfrak{D}_\tau^\rho\left(\mathfrak{D}_{\tau'}^\rho(V)\right) = \mathfrak{D}_{\tau*\tau'}^\rho(V). \tag{4.7}$$

Recall that an element $\tau \in \mathfrak{L}_H^\rho(V)$ is said to be *invertible* if there exists $\tau^{-1} \in \mathfrak{L}_H^\rho(V)$, such that τ and τ^{-1} are commute and $\tau * \tau^{-1} = \varepsilon$. We have the immediate $\hbar$-adic analogue of [JKLT2, Theorem 3.6].

Theorem 4.7. *Let V_0 be a vertex algebra, and let $V = V_0[[\hbar]]$ be the corresponding $\hbar$-adic vertex algebra. Suppose that (H,ρ,τ) is a deforming triple of V, such that τ is invertible in $\mathfrak{L}_H^\rho(V)$. Then $\mathfrak{D}_\tau^\rho(V)$ is an $\hbar$-adic quantum vertex algebra with quantum Yang-Baxter operator $S(z)$ defined by*

$$S(z)(v\otimes u) = \sum \tau(u_{(2)}, -z)v_{(1)} \otimes \tau^{-1}(v_{(2)}, z)u_{(1)} \quad \textit{for } u, v \in V,$$

where $\rho(u) = \sum u_{(1)} \otimes u_{(2)}$ and $\rho(v) = \sum v_{(1)} \otimes v_{(2)}$.

5. Construction of $F_\tau(A,\ell)$

In this section, we fix a GCM $A = (a_{ij})_{i,j\in I}$ and a complex number ℓ. Let $\mathfrak{T}$ be the set of tuples

$$\tau = \left(\tau_{ij}(z), \tau_{ij}^{1,\pm}(z), \tau_{ij}^{2,\pm}(z), \tau_{ij}^{\epsilon_1,\epsilon_2}(z)\right)_{i,j\in I}^{\epsilon_1,\epsilon_2=\pm},$$

where $\tau_{ij}(z)$, $\tau_{ij}^{1,\pm}(z)$, $\tau_{ij}^{2,\pm}(z)$, $\tau_{ij}^{\epsilon_1,\epsilon_2}(z) \in \mathbb{C}((z))[[\hbar]]$, such that

$$\lim_{\hbar\to 0}\tau_{ij}(z) = \lim_{\hbar\to 0}\tau_{ji}(-z), \quad \lim_{\hbar\to 0}\tau_{ij}^{1,\pm}(z) = -\lim_{\hbar\to 0}\tau_{ji}^{2,\pm}(-z),$$
$$\lim_{\hbar\to 0}\tau_{ij}^{\epsilon_1,\epsilon_2}(z) = \lim_{\hbar\to 0}\tau_{ji}^{\epsilon_2,\epsilon_1}(-z) \in \mathbb{C}[[z]]^\times.$$

Definition 5.1. Let $\tau \in \mathfrak{T}$. Define $\mathcal{M}_\tau$ to be the category consisting of topologically free $\mathbb{C}[[\hbar]]$-modules W, equipped with fields $h_{i,\hbar}(z), x_{i,\hbar}^\pm(z) \in \mathcal{E}_\hbar(W)$ $(i \in I)$ satisfying the following conditions

$$\begin{aligned}(5.1)\quad & [h_{i,\hbar}(z_1), h_{j,\hbar}(z_2)] \\ &= r_i a_{ij} r\ell \frac{\partial}{\partial z_2} z_1^{-1}\delta\left(\frac{z_2}{z_1}\right) + \tau_{ij}(z_1-z_2) - \tau_{ji}(z_2-z_1),\end{aligned}$$

$$\begin{aligned}(5.2)\quad & [h_{i,\hbar}(z_1), x_{j,\hbar}^\pm(z_2)] \\ &= \pm x_{j,\hbar}^\pm(z_2)\left(r_i a_{ij} z_1^{-1}\delta\left(\frac{z_2}{z_1}\right) + \tau_{ij}^{1,\pm}(z_1-z_2) + \tau_{ji}^{2,\pm}(z_2-z_1)\right),\end{aligned}$$

$$\begin{aligned}(5.3)\quad & \tau_{ij}^{\pm,\pm}(z_1-z_2)(z_1-z_2)^{n_{ij}} x_{i,\hbar}^\pm(z_1) x_{j,\hbar}^\pm(z_2) \\ &= \tau_{ji}^{\pm,\pm}(z_2-z_1)(z_1-z_2)^{n_{ij}} x_{j,\hbar}^\pm(z_2) x_{i,\hbar}^\pm(z_1),\end{aligned}$$

$$\begin{aligned}(5.4)\quad & \tau_{ij}^{\pm,\mp}(z_1-z_2)(z_1-z_2)^{2\delta_{ij}} x_{i,\hbar}^\pm(z_1) x_{j,\hbar}^\mp(z_2) \\ &= \tau_{ji}^{\mp,\pm}(z_2-z_1)(z_1-z_2)^{2\delta_{ij}} x_{j,\hbar}^\mp(z_2) x_{i,\hbar}^\pm(z_1).\end{aligned}$$

Let $\mathcal{F}_\tau$ be the forgetful functor from $\mathcal{M}_\tau$ to the category of topologically free $\mathbb{C}[[\hbar]]$-modules. Define $\mathrm{End}_{\mathbb{C}[[\hbar]]}(\mathcal{F}_\tau)$ to be the algebra of endomorphisms of the functor $\mathcal{F}_\tau$. For each $W \in \mathcal{M}_\tau$, $\mathrm{End}_{\mathbb{C}[[\hbar]]}(W)$ is a topological algebra over $\mathbb{C}[[\hbar]]$ such that

$$\{(K : \hbar^n W) \mid K \subset W,\, |K| < \infty,\, n \in \mathbb{Z}_+\}$$

forms a local basis at 0, where

$$(K : \hbar^n W) = \left\{\varphi \in \mathrm{End}_{\mathbb{C}[[\hbar]]}(W) \,\middle|\, \varphi(K) \subset \hbar^n W\right\}.$$

Equip $\mathrm{End}_{\mathbb{C}[[\hbar]]}(\mathcal{F}_\tau)$ with the coarsest topology such that for any $W \in \mathcal{M}_\tau$, the canonical $\mathbb{C}[[\hbar]]$-algebra epimorphism from $\mathcal{A}$ to $\mathrm{End}_{\mathbb{C}[[\hbar]]}(W)$ is continuous. It is easy to verify that both $\mathrm{End}_{\mathbb{C}[[\hbar]]}(W)$ and $\mathrm{End}_{\mathbb{C}[[\hbar]]}(\mathcal{F}_\tau)$ are topologically free.

For each $i \in I$, we define endomorphisms $h_{i,\hbar}(n)$ and $x^\pm_{i,\hbar}(n)$ ($n \in \mathbb{Z}$) of $\mathcal{F}_\tau$ as follows:

$$\sum_{n\in\mathbb{Z}} h_{i,\hbar}(n).vz^{-n-1} = h_{i,\hbar}(z)v, \quad \sum_{n\in\mathbb{Z}} x^\pm_{i,\hbar}(n).vz^{-n-1} = x^\pm_{i,\hbar}(z)v.$$

where $v \in W$ and $W \in \mathcal{M}_\tau$. We denote by $\mathcal{A}$ the closed subalgebra of $\mathrm{End}_{\mathbb{C}[[\hbar]]}(\mathcal{F}_\tau)$ generated by $\left\{h_{i,\hbar}(n), x^\pm_{i,\hbar}(n) \,\middle|\, i \in I, n \in \mathbb{Z}\right\}$. Then $\mathcal{A}$ is topologically free. Let $\mathcal{A}_+$ be the minimal closed left ideal of $\mathcal{A}$ containing $h_{i,\hbar}(n)$ and $x^\pm_{i,\hbar}(n)$ ($i \in I$, $n \geq 0$), such that $[\mathcal{A}_+] = \mathcal{A}_+$. Define

$$F_\tau(A, \ell) = \mathcal{A}/\mathcal{A}_+. \tag{5.5}$$

Set $\mathbf{1} = 1 + \mathcal{A}_+ \in F_\tau(A, \ell)$. And for each $i \in I$, we identify $h_{i,\hbar}$ and $x^\pm_{i,\hbar}$ with $h_{i,\hbar}(-1)\mathbf{1}$ and $x^\pm_{i,\hbar}(-1)\mathbf{1}$ in $F_\tau(A, \ell)$, respectively. It is straightforward to verify the following result.

Lemma 5.2. *The topologically free $\mathbb{C}[[\hbar]]$-module $F_\tau(A, \ell)$ equipped with fields $h_{i,\hbar}(z), x^\pm_{i,\hbar}(z)$ ($i \in I$) is an object in $\mathcal{M}_\tau$. Moreover, let W be an object in $\mathcal{M}_\tau$, and let $v_+ \in W$ be such that $h_{i,\hbar}(n).v_+ = 0 = x^\pm_{i,\hbar}(n).v_+$ for $i \in I$ and $n \geq 0$. Then there exists a unique $\mathcal{A}$-module homomorphism $\varphi : F_\tau(A, \ell) \to W$ such that $\varphi(\mathbf{1}) = v_+$.*

Proposition 5.3. *There exists an $\hbar$-adic nonlocal vertex algebra structure on $F_\tau(A, \ell)$ such that $\mathbf{1}$ is the vacuum vector and the vertex operator map Y_τ is determined by the following conditions*

$$Y_\tau(h_{i,\hbar}, z) = h_{i,\hbar}(z), \quad Y_\tau(x^\pm_{i,\hbar}, z) = x^\pm_{i,\hbar}(z), \quad i \in I. \tag{5.6}$$

Moreover, there exists a natural isomorphism between the category of $F_\tau(A, \ell)$-modules and $\mathcal{M}_\tau$.

Proof. Let $(W, h_{i,\hbar}(z), x^\pm_{i,\hbar}(z))$ be an object in the category $\mathcal{M}_\tau$, and let

$$U_W = \left\{h_{i,\hbar}(z),\, x^\pm_{i,\hbar}(z),\, c \,\middle|\, i \in I\right\} \subset \mathcal{E}_\hbar(W).$$

It follows from the relations (5.1), (5.2), (5.3) and (5.4) that U_W is an $\hbar$-adically S-local subset. From Theorem 3.4, we get an $\hbar$-adic weak quantum vertex algebra $\langle U_W\rangle \subset \mathcal{E}_\hbar(W)$ containing $U_W, 1_W$, such that 1_W is the vacuum vector and $Y_\mathcal{E}$ is the vertex operator (see (3.3)). Moreover, W is a faithful $\langle U_W\rangle$-module with $Y_W(a(z), z_0) = a(z_0)$ for $a(z) \in \langle U_W\rangle$.

Using [Li6, Lemma 4.21], we get that $\Big(\langle U_W\rangle, Y_\mathcal{E}(h_{i,\hbar}(z_1), z), Y_\mathcal{E}(x^\pm_{i,\hbar}(z_1), z)\Big)$ is an object in $\mathcal{M}_\tau$. From the vacuum property (2.2), we have that

$$\begin{aligned} &h_{i,\hbar}(z)1_W = Y_\mathcal{E}(h_{i,\hbar}(z_1), z)1_W,\\ &x^\pm_{i,\hbar}(z)1_W = Y_\mathcal{E}(x^\pm_{i,\hbar}(z_1), z)1_W \in \langle U_W\rangle[[z]] \quad \text{for } i \in I. \end{aligned}$$

Combining this with Lemma 5.2, we get an $\mathcal{A}$-module map $\varphi_W : F_\tau(A, \ell) \to \langle U_W\rangle$, such that $\varphi_W(\mathbf{1}) = 1_W$.

Note that $F_\tau(A, \ell)$ is an object in $\mathcal{M}_\tau$ (see Lemma 5.2). Since $\varphi_{F_\tau(A,\ell)}$ is an $\mathcal{A}$-module map, we have that

$$\begin{aligned} &\varphi_{F_\tau(A,\ell)}(a(z_0)v) = a(z_0)\varphi_{F_\tau(A,\ell)}(v) = Y_\mathcal{E}(a(z), z_0)\varphi_{F_\tau(A,\ell)}(v)\\ &\text{for } a = h_{i,\hbar} \text{ or } x^\pm_{i,\hbar},\ v \in F_\tau(A, \ell). \end{aligned}$$

From the definition of $F_\tau(A, \ell)$ we have that $a(z)\mathbf{1} \in F_\tau(A, \ell)[[z]]$ for $a = h_{i,\hbar}, x^\pm_{i,\hbar}$. Then by using Theorem 3.6, we get a unique $\mathbb{C}[[\hbar]]$-module map Y_τ such that

$$Y_\tau(h_{i,\hbar}, z) = h_{i,\hbar}(z), \quad Y_\tau(x^\pm_{i,\hbar}, z) = x^\pm_{i,\hbar}(z), \quad i \in I,$$

and $(F_\tau(A, \ell), Y_\tau, \mathbf{1})$ carries the structure of an $\hbar$-adic weak quantum vertex algebra.

An immediate $\hbar$-adic analogue of [Li4, Proposition 6.7] implies that every $F_\tau(A, \ell)$-module is an object in $\mathcal{M}_\tau$. On the other hand, let $(W, h_{i,\hbar}(z), x^\pm_{i,\hbar}(z))$ be an object in the category $\mathcal{M}_\tau$. Since $\varphi_W : F_\tau(A, \ell) \to \langle U_W\rangle$ is an $\mathcal{A}$-module map, we have that

$$\varphi_W(Y_\tau(a, z_0)v) = \varphi_W(a(z_0)v) = a(z_0)\varphi_W(v) = Y_\mathcal{E}(a(z), z_0)\varphi_W(v),$$

for any $a = h_{i,\hbar}, x^\pm_{i,\hbar}$ and $v \in F_\tau(A, \ell)$. Hence, φ_W is an $\hbar$-adic weak quantum vertex algebra homomorphism. Since W is a faithful $\langle U_W\rangle$-module, we get that W is a $F_\tau(A, \ell)$-module. □

The proof of Proposition 5.3 implies that $F_\tau(A, \ell)$ admits the following universal property.

Proposition 5.4. *Let V be an $\hbar$-adic nonlocal vertex algebra, and let Y be the vertex operator map of V. Suppose there exists $\bar{h}_{i,\hbar}, \bar{x}^\pm_{i,\hbar} \in V$, such that $(V, Y(\bar{h}_{i,\hbar}, z), Y(\bar{x}^\pm_{i,\hbar}, z))$ becomes an object of $\mathcal{M}_\tau$. Then there exists a unique $\hbar$-adic nonlocal vertex algebra homomorphism $\varphi : F_\tau(A, \ell) \to V$, such that $\varphi(h_{i,\hbar}) = \bar{h}_{i,\hbar}$ and $\varphi(x^\pm_{i,\hbar}) = \bar{x}^\pm_{i,\hbar}$ for $i \in I$.*

Corollary 5.5. *Let V be an $\hbar$-adic commutative vertex algebra, and let $\alpha_i, e_i^{\pm} \in V$ $(i \in I)$. Then there exists a unique $\hbar$-adic nonlocal vertex algebra homomorphism $\rho : F_\tau(A, \ell) \to F_\tau(A, \ell)\widehat{\otimes} V$, such that*

$$\rho(h_{i,\hbar}) = h_{i,\hbar} \otimes 1 + \mathbf{1} \otimes \alpha_i, \quad \rho(x_{i,\hbar}^{\pm}) = x_{i,\hbar}^{\pm} \otimes e_i^{\pm}, \quad i \in I.$$

Let H_0 be the symmetric algebra of the following vector space, and let $H = H_0[[\hbar]]$:

$$\bigoplus_{i \in I} \left(\mathbb{C}[\partial] \left(\mathbb{C}\alpha_i \oplus \mathbb{C}e_i^+ \oplus \mathbb{C}e_i^- \right) \right).$$

Then H is a commutative and cocommutative bialgebra with Δ and ε uniquely determined by $(i \in I,\, n \in \mathbb{N})$:

$$\Delta(\partial^n \alpha_i) = \partial^n \alpha_i \otimes 1 + 1 \otimes \partial^n \alpha_i, \quad \Delta(\partial^n e_i^{\pm}) = \sum_{k=0}^{n} \binom{n}{k} \partial^k e_i^{\pm} \otimes \partial^{n-k} e_i^{\pm},$$

$$\varepsilon(\partial^n \alpha_i) = 0, \quad \varepsilon(\partial^n e_i^{\pm}) = \delta_{n,0}.$$

Let ∂ be the derivation on H such that

$$\partial(\partial^n \alpha_i) = \partial^{n+1} \alpha_i, \quad \partial(\partial^n e_i^{\pm}) = \partial^{n+1} e_i^{\pm} \quad \text{for } i \in I,\, n \in \mathbb{N}.$$

It is straightforward to see that $\Delta \circ \partial = (\partial \otimes 1 + 1 \otimes \partial) \circ \Delta$ and $\varepsilon \circ \partial = 0$. From Remark 4.1 we have that $(H, \partial, \Delta, \varepsilon)$ carries an $\hbar$-adic vertex bialgebra structure. It is immediate from Corollary 5.5 that

Lemma 5.6. *There is a unique $\hbar$-adic nonlocal vertex algebra map $\rho : F_\tau(A, \ell) \to F_\tau(A, \ell)\widehat{\otimes} H$, such that*

$$\rho(h_{i,\hbar}) = h_{i,\hbar} \otimes 1 + \mathbf{1} \otimes \alpha_i, \quad \rho(x_{i,\hbar}^{\pm}) = x_{i,\hbar}^{\pm} \otimes e_i^{\pm} \quad \textit{for } i \in I. \tag{5.7}$$

Moreover, $(F_\tau(A, \ell), \rho)$ is an H-comodule nonlocal vertex algebra.

Recall from [Li5] that a *pseudo-endomorphism* of an $\hbar$-adic nonlocal vertex algebra V is a $\mathbb{C}[[\hbar]]$-module map $A(z) : V \to V\widehat{\otimes}\mathbb{C}((z))[[\hbar]]$, such that

$$A(z)\mathbf{1} = \mathbf{1} \otimes 1, \quad A(z_1)Y(u, z_2)v = Y(A(z_1 - z_2)u, z_2)v \quad \text{for } u, v \in V.$$

And a *pseudo-derivation* of V is a $\mathbb{C}[[\hbar]]$-module map $D(z) : V \to V\widehat{\otimes}\mathbb{C}((z))[[\hbar]]$, such that

$$[D(z_1), Y(u, z_2)] = Y(D(z_1 - z_2)u, z_2) \quad \text{for } u \in V.$$

As a straightforward $\hbar$-adic analogue of [Li3, Proposition 2.11], we have that

Proposition 5.7. *Let V be an $\hbar$-adic nonlocal vertex algebra, and view $\mathbb{C}((z))[[\hbar]]$ as an $\hbar$-adic vertex algebra with the vertex operator map*

$$Y(f(z), z_0)g(z) = f(z - z_0)g(z).$$

Suppose that $A(z)$ is an $\hbar$-adic nonlocal vertex algebra homomorphism from V to the tensor product $\hbar$-adic nonlocal vertex algebra $V\widehat{\otimes}\mathbb{C}((z))[[\hbar]]$. Then $A(z)$ is a pseudo-endomorphism of V. Moreover, let

$$D(z) : V \to V\widehat{\otimes}\mathbb{C}((z))[[\hbar]]$$

be a $\mathbb{C}[[\hbar]]$*-module map. We view* $V[\delta]/\delta^2 V[\delta]$ *as an* $\hbar$*-adic nonlocal vertex algebra. If* $1+\delta D(z)$ *is a* $\mathbb{C}[\delta]$*-linear pseudo-endomorphism of* $V[\delta]/\delta^2 V[\delta]$*, then* $D(z)$ *is a pseudo-derivation of* V*.*

Using Corollary 5.5 and Proposition 5.7, we have:

Lemma 5.8. *Let* $\sigma \in \mathfrak{T}$*. Then for each* $i \in I$ *there exists a pseudo-derivation* $\sigma_i(z)$ *on* $F_\tau(A,\ell)$ *such that*

$$\sigma_i(z)h_{j,\hbar} = \mathbf{1}\otimes\sigma_{ij}(z), \quad \sigma_i(z)x^{\pm}_{j,\hbar} = \pm x^{\pm}_{j,\hbar}\otimes\sigma^{1,\pm}_{ij}(z). \tag{5.8}$$

And there exists a pseudo-endomorphism $\sigma_i^{\pm}(x)$ *on* $F_\tau(A,\ell)$ *such that*

$$\sigma_i^{\pm}(z)h_{j,\hbar} = h_{j,\hbar}\otimes 1 \mp \mathbf{1}\otimes\sigma^{2,\pm}_{ij}(z), \quad \sigma_i^{\pm}(z)x^{\epsilon}_{j,\hbar} = x^{\epsilon}_{j,\hbar}\otimes\sigma^{\pm,\epsilon}_{ij}(z)^{-1}, \tag{5.9}$$

where $\epsilon = \pm$ *and* $i,j \in I$*.*

Let V be an $\hbar$-adic nonlocal vertex algebra. A subset U of

$$\mathrm{Hom}(V, V\widehat{\otimes}\mathbb{C}((z))[[\hbar]])$$

is said to be Δ-*closed* (see [Li5]) if for any $a(z)\in U$, there exist two sequences $\{a_{(1n)}(z)\}_{n\geq 1}, \{a_{(2n)}(z)\}_{n\geq 1} \subset U \subset \mathrm{Hom}(V, V\widehat{\otimes}\mathbb{C}((z))[[\hbar]])$ that converge to 0 under the $\hbar$-adic topology and

$$a(z_1)Y(v,z_2) = \sum_{n\geq 1} Y(a_{(1n)}(z_1-z_2)v, z_2)a_{(2n)}(z_1) \quad \text{for } v\in V.$$

Let $B(V)$ be the sum of all Δ-closed subspaces U of $\mathrm{Hom}(V, V\widehat{\otimes}\mathbb{C}((z))[[\hbar]])$ such that

$$a(z)\mathbf{1} \in \mathbb{C}[[\hbar]]\mathbf{1} \quad \text{for } a(z)\in U. \tag{5.10}$$

Note that $\mathrm{Hom}(V, V\widehat{\otimes}\mathbb{C}((z))[[\hbar]]) \cong \mathrm{End}_{\mathbb{C}((z))[[\hbar]]}(V\widehat{\otimes}\mathbb{C}((z))[[\hbar]])$ is an associative algebra.

The following is the immediate $\hbar$-adic analogue of [Li5, Proposition 3.4].

Proposition 5.9. *For any* $\hbar$*-adic nonlocal vertex algebra* V*,* $B(V)$ *is a* Δ*-closed associative subalgebra of* $\mathrm{Hom}(V, V\widehat{\otimes}\mathbb{C}((z))[[\hbar]])$ *and it is closed under the derivation* $\partial = \frac{d}{dz}$*. Moreover,* $B(V)$ *is an* $\hbar$*-adic nonlocal vertex algebra with the vertex operator defined by*

$$Y(a(z),z_0)b(z) = a(z+z_0)b(z) \quad \textit{for } a(z),\, b(z)\in B(V).$$

Furthermore, V *is a* $B(V)$*-module with the module action* $Y_V(a(z),z_0) = a(z_0)$ *for* $a(z)\in B(V)$*.*

We have that

Proposition 5.10. *For each* $\sigma\in\mathfrak{T}$*, there exists an* H*-module structure* $\sigma(\cdot,z)$ *on* $F_\tau(A,\ell)$ *such that*

$$\sigma(\alpha_i,z) = \sigma_i(z), \quad \sigma(e_i^{\pm},z) = \sigma_i^{\pm}(z), \quad i\in I. \tag{5.11}$$

Moreover, (H,ρ,σ) *is a deforming triple for* $F_\tau(A,\ell)$*.*

Proof. Note that there is a $\mathbb{C}[[\hbar]]$-algebra homomorphism $\varphi : H \to B(F_\tau(A,\ell))$ defined by

$$\varphi(\partial^{n_1}\alpha_{i_1}\partial^{n_2}\alpha_{i_2}\cdots\partial^{n_r}\alpha_{i_r}) = \frac{d^{n_1}\sigma_{i_1}(z)}{dz^{n_1}}\frac{d^{n_2}\sigma_{i_2}(z)}{dz^{n_2}}\cdots\frac{d^{n_r}\sigma_{i_r}(z)}{dz^{n_r}},$$
$$\varphi(\partial^{n_1}e_{i_1}^{\epsilon_1}\partial^{n_2}e_{i_2}^{\epsilon_2}\cdots\partial^{n_r}e_{i_r}^{\epsilon_r}) = \frac{d^{n_1}\sigma_{i_1}^{\epsilon_1}(z)}{dz^{n_1}}\frac{d^{n_2}\sigma_{i_2}^{\epsilon_2}(z)}{dz^{n_2}}\cdots\frac{d^{n_r}\sigma_{i_r}^{\epsilon_r}(z)}{dz^{n_r}},$$

where $i_1, i_2, \dots, i_r \in I$, $\epsilon_1, \epsilon_2, \dots, \epsilon_r = \pm$ and $0 \le n_1, \dots, n_r \in \mathbb{Z}$. It is easy to see that

$$\varphi(\partial h) = \frac{d}{dz}\varphi(h) \quad \text{for } h \in H.$$

Then φ is an $\hbar$-adic vertex algebra homomorphism. Hence, $F_\tau(A,\ell)$ is an H-module, and we denote this module action by $\sigma(\cdot, z)$.

Let H' be the maximal $\mathbb{C}[[\hbar]]$-submodule of H such that for any $h \in H'$, one has $\sigma(h,z)\mathbf{1} = \varepsilon(h)\mathbf{1}$ and

$$\sigma(h,z_1)Y_\tau(u,z_2)v = \sum Y_\tau(\sigma(h_{(1)}, z_1 - z_2)u, z_2)\sigma(h_{(2)}, z_1)v,$$

for all $u, v \in F_\tau(A,\ell)$, where $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$. It is straightforward to check that H' is an $\hbar$-adic vertex subalgebra of H. Since $\{\alpha_i, e_i^\pm \mid i \in I\} \subset H'$ and H is generated by $\{\alpha_i, e_i^\pm \mid i \in I\}$, we get that $H' = H$. Hence, $(F_\tau(A,\ell), \sigma)$ is an H-module nonlocal vertex algebra.

Similarly, one can easily checks that

$$\rho(\sigma(h,z)v) = (\sigma(h,z) \otimes 1) \circ \rho(v) \quad \text{for } h \in H,\ v \in F_\tau(A,\ell). \tag{5.12}$$

This proves that (H, ρ, σ) is a deforming triple for $F_\tau(A,\ell)$. □

Let $\tau, \sigma \in \mathfrak{T}$. Combining Theorem 4.3 and Proposition 5.10, we have the $\hbar$-adic nonlocal vertex algebra

$$\mathfrak{D}_\sigma^\rho(F_\tau(A,\ell)). \tag{5.13}$$

Define $\tau * \sigma$ to be

$$(\tau_{ij}(z) + \sigma_{ij}(z), \tau_{ij}^{1,\pm}(z) + \sigma_{ij}^{1,\pm}(z), \tau_{ij}^{2,\pm}(z) + \sigma_{ij}^{2,\pm}(z), \tau_{ij}^{\epsilon_1,\epsilon_2}(z)\sigma_{ij}^{\epsilon_1,\epsilon_2}(z))_{i,j\in I}^{\epsilon_1,\epsilon_2=\pm}.$$

Then $\mathfrak{T}$ is an abelian group under $*$ with identity ε defined by

$$\varepsilon_{ij}(z) = 0 = \varepsilon_{ij}^{1,\pm}(z) = \varepsilon_{ij}^{2,\pm}(z), \quad \varepsilon_{ij}^{\epsilon_1,\epsilon_2}(z) = 1, \tag{5.14}$$

where $i, j \in I$, $\epsilon_1, \epsilon_2 = \pm$.

Lemma 5.11. *Let $\tau, \tau', \tau'' \in \mathfrak{T}$. Then the following relations hold true on $F_\tau(A,\ell)$*

$$[\tau'(h,z_1), \tau''(h',z_2)] = 0 \quad \textit{for } h, h' \in H.$$

Proof. From (5.11), (5.8) and (5.9), we have that

$$[\tau'(a,z_1), \tau''(b,z_2)]v = 0.$$

for $a,b\in\left\{\alpha_i,e_i^{\pm}\,\middle|\,i\in I\right\}$, $v\in\{h_{i,\hbar},x_{i,\hbar}^{\pm}\mid i\in I\}$. Let V be the $\mathbb{C}[[\hbar]]$-submodule of $F_\tau(A,\ell)$ consisting of elements v such that

$$[\tau'(a,z_1),\tau''(b,z_2)]v=0\quad\text{for } a,b=\alpha_i,e_i^{\pm},\ i\in I.$$

It is easy to verify that V is a subalgebra of $F_\tau(A,\ell)$. Since $F_\tau(A,\ell)$ is generated by $\{h_{i,\hbar},\,x_{i,\hbar}^{\pm}\mid i\in I\}$, we get $V=F_\tau(A,\ell)$. Since H is a commutative $\hbar$-adic vertex algebra generated by $\left\{\alpha_i,e_i^{\pm}\,\middle|\,i\in I\right\}$, we have that

$$[\tau'(h,z_1),\tau''(h',z_2)]v=0\quad\text{for } h,h'\in H,\ v\in F_\tau(A,\ell).$$

Therefore, we complete the proof. □

It follows from Lemma 5.11 and Proposition 4.6 that

$$\mathfrak{D}_{\tau'}^{\rho}\left(\mathfrak{D}_{\tau''}^{\rho}(F_\tau(A,\ell))\right)=\mathfrak{D}_{\tau'*\tau''}^{\rho}(F_\tau(A,\ell))\quad\text{for }\tau,\tau',\tau''\in\mathfrak{T}. \tag{5.15}$$

Moreover, we have that

Proposition 5.12. *Let $\tau,\sigma\in\mathfrak{T}$. Then*

$$\mathfrak{D}_{\sigma}^{\rho}(F_\tau(A,\ell))=F_{\tau*\sigma}(A,\ell).$$

Moreover, $F_\tau(A,\ell)/\hbar F_\tau(A,\ell)\cong F(A,\ell)$.

Proof. From Proposition 4.6, we have that

$$\mathfrak{D}_{\sigma}^{\rho}(Y_\tau)(h_{i,\hbar},z)=Y_\tau(h_{i,\hbar},z)+\sigma_i(z),\quad \mathfrak{D}_{\sigma}^{\rho}(Y_\tau)(x_{i,\hbar}^{\pm},z)=Y_\tau(x_{i,\hbar}^{\pm},z)\sigma_i^{\pm}(z).$$

Using Lemma 5.8, one can easily check that

$$(\mathfrak{D}_{\sigma}^{\rho}(F_\tau(A,\ell)),\mathfrak{D}_{\sigma}^{\rho}(Y_\tau)(h_{i,\hbar},z),\mathfrak{D}_{\sigma}^{\rho}(Y_\tau)(x_{i,\hbar}^{\pm},z))$$

is an object of $\mathcal{M}_{\tau*\sigma}$. Using Proposition 5.4, we get an $\hbar$-adic nonlocal vertex algebra homomorphism $f_{\tau,\sigma}$ from $F_{\tau*\sigma}(A,\ell)$ to $\mathfrak{D}_{\sigma}^{\rho}(F_\tau(A,\ell))$. From Remark 4.5 we see that $f_{\tau,\sigma}$ is also an $\hbar$-adic nonlocal vertex algebra homomorphism from $\mathfrak{D}_{\sigma^{-1}}^{\rho}(F_{\tau*\sigma}(A,\ell))$ to $\mathfrak{D}_{\sigma^{-1}}^{\rho}(\mathfrak{D}_{\sigma}^{\rho}(F_\tau(A,\ell)))$. It follows from Remark 4.4 and (5.15) that $\mathfrak{D}_{\sigma^{-1}}^{\rho}(\mathfrak{D}_{\sigma}^{\rho}(F_\tau(A,\ell)))=F_\tau(A,\ell)$. Then $f_{\tau,\sigma}$ becomes an $\hbar$-adic nonlocal vertex algebra homomorphism from $\mathfrak{D}_{\sigma^{-1}}^{\rho}(F_{\tau*\sigma}(A,\ell))$ to $F_\tau(A,\ell)$. Replacing τ with $\tau*\sigma$ and replacing σ with σ^{-1}, we get an $\hbar$-adic nonlocal vertex algebra homomorphism

$$f_{\tau*\sigma,\sigma^{-1}}:F_\tau(A,\ell)\to\mathfrak{D}_{\sigma^{-1}}^{\rho}(F_{\tau*\sigma}(A,\ell)).$$

Hence, $f_{\tau,\sigma}\circ f_{\tau*\sigma,\sigma^{-1}}$ is an $\hbar$-adic nonlocal vertex algebra endomorphism on $F_\tau(A,\ell)$ which maps a to a for all $a\in\left\{h_{i,\hbar},x_{i,\hbar}^{\pm}\,\middle|\,i\in I\right\}$. From Proposition 5.4, we get that

$$\begin{array}{ccc} F_\tau(A,\ell) & \xrightarrow{f_{\tau*\sigma,\sigma^{-1}}} & \mathfrak{D}_{\sigma^{-1}}^{\rho}(F_{\tau*\sigma}(A,\ell)) \\ & \searrow\!\!\!= & \Big\downarrow f_{\tau,\sigma} \\ & & F_\tau(A,\ell) \end{array} \tag{5.16}$$

By replacing τ and σ with $\tau * \sigma$ and σ^{-1} in (5.16), we get that

$$
\begin{array}{ccc}
F_{\tau*\sigma}(A,\ell) & \xrightarrow{f_{\tau,\sigma}} & \mathfrak{D}^{\rho}_{\sigma}(F_{\tau}(A,\ell)) \\
& \diagdown\!\!\diagdown & \big\downarrow {\scriptstyle f_{\tau*\sigma,\sigma^{-1}}} \\
& & F_{\tau*\sigma}(A,\ell)
\end{array}
\tag{5.17}
$$

Notice that

$$
\begin{aligned}
&\mathfrak{D}^{\rho}_{\sigma^{-1}}(F_{\tau*\sigma}(A,\ell))/\hbar\mathfrak{D}^{\rho}_{\sigma^{-1}}(F_{\tau*\sigma}(A,\ell)) = F_{\tau*\sigma}(A,\ell)/\hbar F_{\tau*\sigma}(A,\ell),\\
&\mathfrak{D}^{\rho}_{\sigma}(F_{\tau}(A,\ell))/\hbar\mathfrak{D}^{\rho}_{\sigma}(F_{\tau}(A,\ell)) = F_{\tau}(A,\ell)/\hbar F_{\tau}(A,\ell).
\end{aligned}
$$

Then $f_{\tau,\sigma}$ and $f_{\tau*\sigma,\sigma^{-1}}$ induce the following $\mathbb{C}$-linear isomorphisms

$$
\begin{array}{ccc}
F_{\tau*\sigma}(A,\ell)/\hbar F_{\tau*\sigma}(A,\ell) & \Longrightarrow & F_{\tau}(A,\ell)/\hbar F_{\tau}(A,\ell) \\
& & \Big\| \\
& & \mathfrak{D}^{\rho}_{\sigma}(F_{\tau}(A,\ell))/\hbar\mathfrak{D}^{\rho}_{\sigma}(F_{\tau}(A,\ell)).
\end{array}
\tag{5.18}
$$

Since both $F_{\tau*\sigma}(A,\ell)$ and $\mathfrak{D}^{\rho}_{\sigma}(F_{\tau}(A,\ell))$ are topologically free, we get that $f_{\tau,\sigma} : F_{\tau*\sigma}(A,\ell) \to \mathfrak{D}^{\rho}_{\sigma}(F_{\tau}(A,\ell))$ is an $\hbar$-adic nonlocal vertex algebra isomorphism.

From (5.14), we see that $F_{\varepsilon}(A,\ell) = F(A,\ell)[[\hbar]]$ as $\hbar$-adic vertex algebras. Recall that ε is the identity of $\mathcal{T}$. Then by replacing τ and σ with ε and τ in (5.18), we get that $F_{\tau}(A,\ell)/\hbar F_{\tau}(A,\ell) \cong F(A,\ell)$. □

For $\tau \in \mathfrak{T}$, we note that

$$
\tau^{-1} = \left(-\tau_{ij}(z), -\tau_{ij}^{1,\pm}(z), -\tau_{ij}^{2,\pm}(z), \tau_{ij}^{\epsilon_1,\epsilon_2}(z)^{-1}\right)_{i,j\in I}^{\epsilon_1,\epsilon_2=\pm}.
$$

It is immediate from Theorem 4.7 that

Theorem 5.13. *For any $\tau \in \mathfrak{T}$, $F_{\tau}(A,\ell)$ is an $\hbar$-adic quantum vertex algebra with the quantum Yang-Baxter operator $S_{\tau}(z)$ defined by*

$$
S_{\tau}(z)(v\otimes u) = \sum \tau(u_{(2)},-z)v_{(1)} \otimes \tau^{-1}(v_{(2)},z)u_{(1)} \quad \text{for } u,v\in F_{\tau}(A,\ell).
$$

Moreover, for any $i,j \in I$ and $\epsilon_1,\epsilon_2 = \pm$, we have that

$$
S_{\tau}(z)(h_{j,\hbar}\otimes h_{i,\hbar}) = h_{j,\hbar}\otimes h_{i,\hbar} + \mathbf{1}\otimes\mathbf{1}\otimes\left(\tau_{ij}(-z)-\tau_{ji}(z)\right),
\tag{5.19}
$$

$$
S_{\tau}(z)(x^{\pm}_{j,\hbar}\otimes h_{i,\hbar}) = x^{\pm}_{j,\hbar}\otimes h_{i,\hbar} \pm x^{\pm}_{j,\hbar}\otimes\mathbf{1}\otimes\left(\tau_{ij}^{1,\pm}(-z)+\tau_{ji}^{2,\pm}(z)\right),
\tag{5.20}
$$

$$
S_{\tau}(z)(h_{j,\hbar}\otimes x^{\pm}_{i,\hbar}) = h_{j,\hbar}\otimes x^{\pm}_{i,\hbar} \mp \mathbf{1}\otimes x^{\pm}_{i,\hbar}\otimes\left(\tau_{ij}^{2,\pm}(-z)+\tau_{ji}^{1,\pm}(z)\right),
\tag{5.21}
$$

$$
S_{\tau}(z)(x^{\epsilon_1}_{j,\hbar}\otimes x^{\epsilon_2}_{i,\hbar}) = x^{\epsilon_1}_{j,\hbar}\otimes x^{\epsilon_2}_{i,\hbar}\otimes \tau_{ji}^{\epsilon_1,\epsilon_2}(z)\tau_{ij}^{\epsilon_2,\epsilon_1}(-z)^{-1}.
\tag{5.22}
$$

6. Construction of quantum affine vertex algebras

Let $\ell \in \mathbb{C}$. In the rest of this paper, we fix a special

$$
\tau = \left(\tau_{ij}(z), \tau_{ij}^{1,\pm}(z), \tau_{ij}^{2,\pm}(z), \tau_{ij}^{\epsilon_1,\epsilon}(z)\right)_{i,j\in I}^{\epsilon_1,\epsilon_2=\pm} \in \mathfrak{T}
$$

defined as follows

$$\tau_{ij}(z) = [r_i a_{ij}]_{q^{\frac{\partial}{\partial z}}} [r\ell]_{q^{\frac{\partial}{\partial z}}} q^{r\ell\frac{\partial}{\partial z}} \frac{e^{-z}}{(1-e^{-z})^2} - r_i a_{ij} r\ell z^{-2}, \tag{6.1}$$

$$\tau_{ij}^{1,\pm}(z) = \tau_{ij}^{2,\pm}(z) = [r_i a_{ij}]_{q^{\frac{\partial}{\partial z}}} q^{r\ell\frac{\partial}{\partial z}} \frac{1+e^{-z}}{2-2e^{-z}} - r_i a_{ij} z^{-1}, \tag{6.2}$$

$$\tau_{ij}^{\pm,\pm}(z) = \begin{cases} \left(e^{z/2}-e^{-z/2}\right)^{-1}\left(q_i^{-1}e^{z/2}-q_ie^{-z/2}\right), & \text{if } a_{ij}>0, \\ z^{-1}\left(q_i^{-a_{ij}/2}e^{z/2}-q_i^{a_{ij}/2}e^{-z/2}\right), & \text{if } a_{ij}\le 0, \end{cases} \tag{6.3}$$

$$\tau_{ij}^{+,-}(z) = z^{-\delta_{ij}}(z+2r\ell\hbar)^{\delta_{ij}}, \tag{6.4}$$

$$\tau_{ij}^{-,+}(z) = z^{-\delta_{ij}}(z-2r\ell\hbar)^{\delta_{ij}} g_{ij,\hbar}(z)^{-1}, \tag{6.5}$$

where $g_{ij,\hbar}(z) = (1-q_i^{a_{ij}}e^{-z})/(q_i^{a_{ij}}-e^{-z})$. Then the equations (5.1), (5.2), (5.3), (5.4) become:

$$\begin{aligned} &[h_{i,\hbar}(z_1), h_{j,\hbar}(z_2)] \\ &\quad= [r_i a_{ij}]_{q^{\frac{\partial}{\partial z_2}}} [r\ell]_{q^{\frac{\partial}{\partial z_2}}} \left(\iota_{z_1,z_2} q^{-r\ell\frac{\partial}{\partial z_2}} - \iota_{z_2,z_1} q^{r\ell\frac{\partial}{\partial z_2}}\right) \frac{e^{-z_1+z_2}}{(1-e^{-z_1+z_2})^2}, \end{aligned} \tag{6.6}$$

$$\begin{aligned} &[h_{i,\hbar}(z_1), x_{j,\hbar}^{\pm}(z_2)] \\ &\quad= \pm x_{j,\hbar}^{\pm}(z_2)[r_i a_{ij}]_{q^{\frac{\partial}{\partial z_2}}} \left(\iota_{z_1,z_2} q^{-r\ell\frac{\partial}{\partial z_2}} - \iota_{z_2,z_1} q^{r\ell\frac{\partial}{\partial z_2}}\right) \frac{1+e^{-z_1+z_2}}{2-2e^{-z_1+z_2}}, \end{aligned} \tag{6.7}$$

$$\begin{aligned} &\iota_{z_1,z_2} \frac{1-q_i^{a_{ij}}e^{-z_1+z_2}}{(1-e^{-z_1+z_2})^{\delta_{ij}}} x_{i,\hbar}^{\pm}(z_1) x_{j,\hbar}^{\pm}(z_2) \\ &\quad= \iota_{z_2,z_1} \frac{q_i^{a_{ij}}-e^{-z_1+z_2}}{(1-e^{-z_1+z_2})^{\delta_{ij}}} x_{j,\hbar}^{\pm}(z_2) x_{i,\hbar}^{\pm}(z_1), \end{aligned} \tag{6.8}$$

$$\begin{aligned} &(z_1-z_2)^{\delta_{ij}}(z_1-z_2+2r\ell\hbar)^{\delta_{ij}} \\ &\quad\times \left(x_{i,\hbar}^{+}(z_1)x_{j,\hbar}^{-}(z_2) - \iota_{z_2,z_1} g_{ij,\hbar}(z_1-z_2) x_{j,\hbar}^{-}(z_2) x_{i,\hbar}^{+}(z_1)\right) = 0. \end{aligned} \tag{6.9}$$

For $i,j\in I$, we set

$$f_{ij,\hbar}(z) = z^{n_{ij}} \tau_{ij}^{+,+}(z) = \frac{(q_i^{-a_{ij}/2}e^{z/2}-q_i^{a_{ij}/2}e^{-z/2})}{(e^{z/2}-e^{-z/2})^{\delta_{ij}}}. \tag{6.10}$$

Then the relation (6.8) is equivalent to the following equation:

$$\begin{aligned} &\iota_{z_1,z_2} f_{ij}(z_1-z_2) x_{i,\hbar}^{\pm}(z_1) x_{j,\hbar}^{\pm}(z_2) \\ &= -(-1)^{\delta_{ij}} \iota_{z_2,z_1} f_{ji}(z_2-z_1) x_{j,\hbar}^{\pm}(z_2) x_{i,\hbar}^{\pm}(z_1). \end{aligned} \tag{6.11}$$

It is straightforward to verify that

Lemma 6.1. *Let $i,j\in I$. Then*

$$\tau_{ij}(z) - \tau_{ji}(-z) = [r_i a_{ij}]_{q^{\frac{\partial}{\partial z}}} [r\ell]_{q^{\frac{\partial}{\partial z}}} \left(q^{r\ell\frac{\partial}{\partial z}} - q^{-r\ell\frac{\partial}{\partial z}}\right) \frac{e^{-z}}{(1-e^{-z})^2}, \tag{6.12}$$

$$\tau_{ij}^{1,+}(z) + \tau_{ji}^{2,+}(-z) = [r_i a_{ij}]_{q^{\frac{\partial}{\partial z}}} \left(q^{r\ell\frac{\partial}{\partial z}} - q^{-r\ell\frac{\partial}{\partial z}}\right) \frac{1+e^{-z}}{2-2e^{-z}} \tag{6.13}$$

$$= \tau_{ij}^{1,-}(z) + \tau_{ji}^{2,-}(-z),$$

$$\tau_{ij}^{\pm,\pm}(z)\tau_{ji}^{\pm,\pm}(-z)^{-1} = g_{ij,\hbar}(z) = \tau_{ij}^{\pm,\mp}(z)^{-1}\tau_{ji}^{\mp,\pm}(-z). \tag{6.14}$$

Moreover, we have that

Lemma 6.2. *Let*

$$F(z) = (q^z - q^{-z})/z \in \hbar\mathbb{C}[z^2][[\hbar]]. \tag{6.15}$$

Then for any $i, j \in I$, we have that

$$\begin{aligned} F\left(\frac{\partial}{\partial z}\right)(\tau_{ij}(z) - \tau_{ji}(-z)) \\ = \left(q^{-r\ell\frac{\partial}{\partial z}} - q^{r\ell\frac{\partial}{\partial z}}\right)\left(\tau_{ij}^{1,+}(z) + \tau_{ji}^{2,+}(-z)\right), \end{aligned} \tag{6.16}$$

$$F\left(\frac{\partial}{\partial z}\right)\left(\tau_{ij}^{1,\pm}(z) + \tau_{ji}^{2,\pm}(-z)\right) = \mp\left(q^{r\ell\frac{\partial}{\partial z}} - q^{-r\ell\frac{\partial}{\partial z}}\right)\log\frac{\tau_{ij}^{+,\pm}(z)}{\tau_{ji}^{\pm,+}(-z)}. \tag{6.17}$$

Proof. From (6.13), we have that

$$\begin{aligned} &\left(q^{-r\ell\frac{\partial}{\partial z}} - q^{r\ell\frac{\partial}{\partial z}}\right)(\tau_{ij}^{1,+}(z) + \tau_{ji}^{2,+}(-z)) \\ =&\left(q^{-\frac{\partial}{\partial z}} - q^{\frac{\partial}{\partial z}}\right)[r\ell]_{q^{\frac{\partial}{\partial z}}}(\tau_{ij}^{1,+}(z) + \tau_{ji}^{2,+}(-z)) \\ =&F\left(\frac{\partial}{\partial z}\right)\frac{\partial}{\partial z}[r\ell]_{q^{\frac{\partial}{\partial z}}}(\tau_{ij}^{1,+}(z) + \tau_{ji}^{2,+}(-z)) \\ =&F\left(\frac{\partial}{\partial z}\right)[r_i a_{ij}]_{q^{\frac{\partial}{\partial z}}}[r\ell]_{q^{\frac{\partial}{\partial z}}}\left(q^{r\ell\frac{\partial}{\partial z}} - q^{-r\ell\frac{\partial}{\partial z}}\right)\frac{e^{-z}}{(1-e^{-z})^2}. \end{aligned}$$

Combining this with (6.12), we prove the equation (6.16). The proof of the equation (6.17) is similar. □

Definition 6.3. For $\ell \in \mathbb{C}$, we let R_1^ℓ be the minimal closed ideal of $F_\tau(A, \ell)$ such that $[R_1^\ell] = R_1^\ell$ and contains the following elements

$$\left(x_{i,\hbar}^+\right)_0 x_{i,\hbar}^- - (q_i - q_i^{-1})^{-1}\left(\mathbf{1} - E(h_{i,\hbar})\right) \quad \text{for } i \in I, \tag{6.18}$$

$$\left(x_{i,\hbar}^+\right)_1 x_{i,\hbar}^- + 2r\ell\hbar(q_i - q_i^{-1})^{-1}E(h_{i,\hbar}) \quad \text{for } i \in I, \tag{6.19}$$

$$\left(x_{i,\hbar}^\pm\right)_0^{m_{ij}} x_{j,\hbar}^\pm \quad \text{for } i, j \in I \text{ with } a_{ij} < 0, \tag{6.20}$$

where

$$E(h_{i,\hbar}) = \left(\frac{F(r_i + r\ell)}{F(r_i - r\ell)}\right)^{\frac{1}{2}}\exp\left(\left(-q^{-r\ell\partial}F(\partial)\,h_{i,\hbar}\right)_{-1}\right)\mathbf{1}. \tag{6.21}$$

Define

$$V_{\hat{\mathfrak{g}},\hbar}(\ell, 0) = F_\tau(A, \ell)/R_1^\ell. \tag{6.22}$$

Definition 6.4. Let $\ell \in \mathbb{Z}_+$. We let R_2^ℓ be the minimal closed ideal of $V_{\hat{\mathfrak{g}},\hbar}(\ell,0)$ such that $[R_2^\ell] = R_2^\ell$ and contains the following elements

$$\left(x_{i,\hbar}^{\pm}\right)_{-1}^{r\ell/r_i} x_{i,\hbar}^{\pm} \quad \text{for } i \in I. \tag{6.23}$$

Define

$$L_{\hat{\mathfrak{g}},\hbar}(\ell,0) = V_{\hat{\mathfrak{g}},\hbar}(\ell,0)/R_2^\ell. \tag{6.24}$$

It is immediate from Propositions 2.3, 5.12 and Definitions 2.6, 6.3, 6.4 that

Proposition 6.5. *For $\ell \in \mathbb{C}$, there is a surjective vertex algebra homomorphism from $V_{\hat{\mathfrak{g}}}(\ell,0)$ to $V_{\hat{\mathfrak{g}},\hbar}(\ell,0)/\hbar V_{\hat{\mathfrak{g}},\hbar}(\ell,0)$ such that*

$$h_i \mapsto h_{i,\hbar}, \quad x_i^{\pm} \mapsto x_{i,\hbar}^{\pm} \quad \textit{for } i \in I. \tag{6.25}$$

If $\ell \in \mathbb{Z}_+$, then (6.25) *defines a surjective vertex algebra homomorphism from $L_{\hat{\mathfrak{g}}}(\ell,0)$ to $L_{\hat{\mathfrak{g}},\hbar}(\ell,0)/\hbar L_{\hat{\mathfrak{g}},\hbar}(\ell,0)$.*

The main purpose of this section is to prove the following results.

Theorem 6.6. *Let $\ell \in \mathbb{C}$. Then $V_{\hat{\mathfrak{g}},\hbar}(\ell,0)$ is an $\hbar$-adic quantum vertex algebra. Moreover, if $\ell \in \mathbb{Z}_+$, then $L_{\hat{\mathfrak{g}},\hbar}(\ell,0)$ is also an $\hbar$-adic quantum vertex algebra. Furthermore, the quantum Yang-Baxter operators of both $V_{\hat{\mathfrak{g}},\hbar}(\ell,0)$ and $L_{\hat{\mathfrak{g}},\hbar}(\ell,0)$ satisfy the relations* (5.19), (5.20), (5.21) *and* (5.22).

We start with some technical results.

Lemma 6.7. *Let W be a topologically free $\mathbb{C}[[\hbar]]$-module,*

$$\beta(z) = \sum_{m\in\mathbb{Z}} \beta_m z^{-m-1} \in W((z))$$

and $f(z) = \sum_{i=0}^n a_i z^i \in \mathbb{C}[z]$, such that $a_n = 1$. Suppose that

$$\operatorname{Sing}_z z^k \hbar^n f(z/\hbar)\beta(z) = 0 \quad \textit{for some } k \in \mathbb{Z}.$$

Then $\beta_m \in \operatorname{Span}_{\mathbb{C}[[\hbar]]} \{\beta_i \mid k \le i < k+n\}$ for all $m \ge k$. Moreover, if $\beta_i = 0$ for all $k \le i < k+n$, then $\operatorname{Sing}_z z^k \beta(z) = 0$.

Proof. Notice that

$$0 = \operatorname{Sing}_z z^k \hbar^n f(z/\hbar)\beta(z) = \sum_{m\ge 0} z^{-m-1} \sum_{i=0}^n a_i \hbar^{n-i} \beta_{m+k+i}.$$

It follows that

$$\beta_{m+n} = -\sum_{i=0}^{n-1} a_i \hbar^{n-i} \beta_{m+i} \quad \text{for } m \ge k.$$

By using induction on m, we complete the proof. □

Lemma 6.8. *Let V be an $\hbar$-adic nonlocal vertex algebra, and let $u, v \in V$, $f(z) = (\sum_{i=0}^{n} a_i z^i)/(\sum_{j=0}^{m} b_j z^j) \in \mathbb{C}(z)$. Suppose*

$$\iota_{z_1,z_2,\hbar}\hbar^{n-m} f\big((z_1-z_2)/\hbar\big) Y(u,z_1)Y(v,z_2) \in \mathcal{E}_\hbar^{(2)}(V). \tag{6.26}$$

Then we have that

$$\mathrm{Sing}_z\, \iota_{z,\hbar}\hbar^{n-m} f(z/\hbar) Y(u,z)v = 0.$$

Proof. Let $U = \{Y(u,z) \mid u \in V\} \subset \mathcal{E}_\hbar(V)$. From (3.4), we get an $\hbar$-adic nonlocal vertex algebra homomorphism $V \to \langle U \rangle$ defined by $u \mapsto Y(u,z)$. Then from (6.26), we have that

$$\begin{aligned}
&\hbar^{n-m} f(z/\hbar) Y(Y(u,z)v, z_2)\\
=&\hbar^{n-m} f\big((z_1-z_2)/\hbar\big) Y_{\mathcal{E}}\left(Y(u,z_2), z\right) Y(v,z_2)\\
=&\big(\hbar^{n-m} f\big((z_1-z_2)/\hbar\big) Y(u,z_1)Y(v,z_2)\big)\,|_{z_1=z_2+z}.
\end{aligned}$$

Notice that

$$\begin{aligned}
&\big(\hbar^{n-m} f\big((z_1-z_2)/\hbar\big) Y(u,z_1)Y(v,z_2)\big)\,|_{z_1=z_2+z}\\
&\quad \in \mathrm{Hom}(V, V((z_2))[[z]]) + \hbar^n \operatorname{End} V[[z_2^{\pm 1}, z]] \quad \text{for all } n \geq 0,
\end{aligned}$$

and that $f(z,\hbar) Y(Y(u,z)v, z_2)\mathbf{1} \in V[[z, z^{-1}, z_2]]$. It follows that

$$\begin{aligned}
&\hbar^{n-m} f(z/\hbar) Y(Y(u,z)v, z_2)\mathbf{1}\\
=&\big(\hbar^{n-m} f\big((z_1-z_2)/\hbar\big) Y(u,z_1)Y(v,z_2)\mathbf{1}\big)\,|_{z_1=z_2+z} \in V[[z_2, z]].
\end{aligned}$$

Setting $z_2 = 0$, we get that

$$\hbar^{n-m} f(z/\hbar) Y(u,z)v = \big(\hbar^{n-m} f(z/\hbar) Y(Y(u,z)v, z_2)\mathbf{1}\big)\,|_{z_2=0} \in V[[z]].$$

Therefore, $\mathrm{Sing}_z\, \iota_{z,\hbar}\hbar^{n-m} f(z/\hbar) Y(u,z)v = 0$. □

Combining Lemmas 6.8 and equations (6.11), (6.9), we have that

Lemma 6.9. *In the $\hbar$-adic quantum vertex algebra $F_\tau(A,\ell)$, we have that*

$$\mathrm{Sing}_z (z - r_i a_{ij}\hbar) Y_\tau(x_{i,\hbar}^{\pm}, z) x_{j,\hbar}^{\pm} = 0, \quad \text{if } a_{ij} < 0, \tag{6.27}$$

$$\mathrm{Sing}_z\, z^{-1}(z - r_i a_{ii}\hbar) Y_\tau(x_{i,\hbar}^{\pm}, z) x_{i,\hbar}^{\pm} = 0, \tag{6.28}$$

$$\mathrm{Sing}_z\, Y_\tau(x_{i,\hbar}^{+}, z) x_{j,\hbar}^{-} = 0, \quad \text{if } i \neq j, \tag{6.29}$$

$$\mathrm{Sing}_z\, z(z + 2r\ell\hbar) Y_\tau(x_{i,\hbar}^{+}, z) x_{i,\hbar}^{-} = 0. \tag{6.30}$$

We have that

Proposition 6.10. *For $i \in I$, we define $A_i(z)$ to be*

$$\mathrm{Sing}_z\, Y_\tau(x_{i,\hbar}^{+}, z) x_{i,\hbar}^{-} - (q_i - q_i^{-1})^{-1}\left(\mathbf{1}z^{-1} - E(h_{i,\hbar})(z + 2r\ell\hbar)^{-1}\right).$$

Then $A_i(z) = 0$ in $V_{\hat{\mathfrak{g}},\hbar}(\ell, 0)$. Moreover, we have that

$$\begin{aligned}
&Y_\tau(x_{i,\hbar}^{+}, z_1) Y_\tau(x_{j,\hbar}^{-}, z_2) - \iota_{z_2,z_1} g_{ij,\hbar}(z_1 - z_2) Y_\tau(x_{j,\hbar}^{-}, z_2) Y_\tau(x_{i,\hbar}^{+}, z_1)\\
=&\frac{\delta_{ij}}{q_i - q_i^{-1}}\left(z_1^{-1}\delta\left(\frac{z_2}{z_1}\right) - Y_\tau\left(E(h_{i,\hbar}), z_2\right) z_1^{-1}\delta\left(\frac{z_2 - 2r\ell\hbar}{z_1}\right)\right).
\end{aligned} \tag{6.31}$$

Proof. Let

$$\bar{A}_i(z) = Y_\tau(x^+_{i,\hbar}, z)x^-_{i,\hbar} - (q_i - q_i^{-1})^{-1} \left(\mathbf{1}z^{-1} - E(h_{i,\hbar})(z + 2r\ell\hbar)^{-1}\right).$$

It is easy to see that $A_i(z) = \mathrm{Sing}_z \bar{A}_i(z)$. From (6.30), we have that $\mathrm{Sing}_z z(z + 2r\ell\hbar)\bar{A}_i(z) = 0$. Then it follows from Lemma 6.7 that

$$A_i(z) = \mathrm{Sing}_z \bar{A}_i(z) \in \bigoplus_{n=0}^{1} \mathbb{C}[z^{-1}][[\hbar]] \left(\mathrm{Res}_z z^n \bar{A}_i(z)\right).$$

Recall from (6.18) and (6.19) that $\mathrm{Res}_z z^n \bar{A}_i(z) = 0$ for $n = 0, 1$. Therefore, $A_i(z) = 0$ in $V_{\hat{\mathfrak{g}},\hbar}(\ell, 0)$.

From [Li6, (2.25)], we have that

$$\begin{aligned}
&z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_\tau(x^+_{i,\hbar}, z_1)Y_\tau(x^-_{j,\hbar}, z_2)\\
&- z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) \iota_{z_2,z_1} g_{ij,\hbar}(z_1 - z_2)Y_\tau(x^-_{j,\hbar}, z_2)Y_\tau(x^+_{i,\hbar}, z_1)\\
=&z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) Y_\tau\left(Y_\tau(x^+_{i,\hbar}, z_0)x^-_{j,\hbar}, z_2\right).
\end{aligned}$$

Taking Res_{z_0}, we get that

$$\begin{aligned}
&Y_\tau(x^+_{i,\hbar}, z_1)Y_\tau(x^-_{j,\hbar}, z_2) - \iota_{z_2,z_1} g_{ij,\hbar}(z_1 - z_2)Y_\tau(x^-_{j,\hbar}, z_2)Y_\tau(x^+_{i,\hbar}, z_1)\\
=&\mathrm{Res}_{z_0} z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) Y_\tau\left(Y_\tau(x^+_{i,\hbar}, z_0)x^-_{j,\hbar}, z_2\right)\\
=&\mathrm{Res}_{z_0} z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) Y_\tau\left(\mathrm{Sing}_{z_0} Y_\tau(x^+_{i,\hbar}, z_0)x^-_{j,\hbar}, z_2\right)\\
=&\mathrm{Res}_{z_0} z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) (q_i - q_i^{-1})^{-1} \left(z_0^{-1} - Y_\tau\left(E(h_{i,\hbar}), z_2\right)(z_0 + 2r\ell\hbar)^{-1}\right)\\
=&\delta_{ij}(q_i - q_i^{-1})^{-1} \left(z_1^{-1}\delta\left(\frac{z_2}{z_1}\right) - Y_\tau\left(E(h_{i,\hbar}), z_2\right) z_1^{-1}\delta\left(\frac{z_2 - 2r\ell\hbar}{z_1}\right)\right),
\end{aligned}$$

where the third equation follows from the first statement. ☐

For any positive integer k and $i_1, \dots, i_k \in I$, we set

$$\begin{aligned}
(6.32)\quad &Y^\pm_{i_1,\dots,i_k}(z_1, \dots, z_k)\\
&= \left(\prod_{1\le a<b\le k} f_{i_a,i_b,\hbar}(z_a - z_b)\right) Y_\tau(x^\pm_{i_1,\hbar}, z_1)Y_\tau(x^\pm_{i_2,\hbar}, z_2)\cdots Y_\tau(x^\pm_{i_k,\hbar}, z_k).
\end{aligned}$$

It is immediate from equations (6.11) and (6.9) that

Lemma 6.11. *For any positive integer k, $i_1, \dots, i_k \in I$ and $\sigma \in S_k$, we have that*

$$Y^{\pm}_{i_{\sigma(1)},\dots,i_{\sigma(k)}}(z_{\sigma(1)}, \dots, z_{\sigma(k)}) = \left(\prod_{\substack{1\le a<b\le k\\ \sigma(a)>\sigma(b)}} C_{i_a,i_b} \right) Y^{\pm}_{i_1,\dots,i_k}(z_1, \dots, z_k), \tag{6.33}$$

where

$$C_{ij} = -(-1)^{\delta_{ij}}. \tag{6.34}$$

Moreover, $Y^{\pm}_{i_1,\dots,i_k}(z_1, \dots, z_k) \in \mathcal{E}^{(k)}_{\hbar}(F_\tau(A, \ell))$.

Lemma 6.12. *Let $i, j \in I$ with $a_{ij} < 0$ and $k \in \mathbb{N}$. Then in $F_\tau(A, \ell)$, we have that*

$$\begin{aligned} &\operatorname{Sing}_{z_1,\dots,z_k} Y_\tau(x^{\pm}_{i,\hbar}, z_1) \cdots Y_\tau(x^{\pm}_{i,\hbar}, z_k) x^{\pm}_{j,\hbar} \\ &\quad \in \mathbb{C}[z_1^{-1}, \dots, z_k^{-1}][[\hbar]] \left(\left(x^{\pm}_{i,\hbar} \right)^k_0 x^{\pm}_{j,\hbar} \right). \end{aligned} \tag{6.35}$$

Moreover, we define $\bar{Y}^{\pm}_{ij,k}(z)$ to be

$$Y^{\pm}_{i,\dots,i,j}\left(z + r_i\big((k-1)a_{ii} + a_{ij}\big)\hbar, z + r_i\big((k-2)a_{ii} + a_{ij}\big)\hbar, \dots, z + r_i a_{ij}\hbar, z\right).$$

Then we have that

$$Y_\tau\left(\left(x^{\pm}_{i,\hbar} \right)^k_0 x^{\pm}_{j,\hbar}, z \right) = c_k \bar{Y}^{\pm}_{ij,k}(z) \quad \textit{for some } c_k \in \mathbb{C}[[\hbar]]^\times. \tag{6.36}$$

Proof. We prove the lemma by using induction on k. From equations (6.11) and (6.9), we have that

$$\begin{aligned} &\left(\prod_{1\le a\le k} f_{ii,\hbar}\left(z_1 - z_2 - r_i\big((a-1)a_{ii} + a_{ij}\big)\hbar\right) \right) f_{ij,\hbar}(z_1 - z_2) Y_\tau\left(x^{\pm}_{i,\hbar}, z_1\right) \bar{Y}^{\pm}_{ij,k}(z_2) \\ =& Y^{\pm}_{i,\dots,i,j}\left(z_1, z_2 + r_i((k-1)a_{ii} + a_{ij})\hbar, z_2 + r_i((k-2)a_{ii} + a_{ij})\hbar, \dots, z_2 + r_i a_{ij}\hbar, z_2\right) \end{aligned}$$

lies in $\mathcal{E}^{(2)}_{\hbar}(F_\tau(A, \ell))$. Notice that

$$\begin{aligned} &f_{ij,\hbar}(z_1 - z_2) \prod_{1\le a\le k} f_{ii,\hbar}\left(z_1 - z_2 - r_i\big((a-1)a_{ii} + a_{ij}\big)\hbar\right) \\ &\quad = q_i^{-k-a_{ij}/2} e^{(z_1-z_2)/2} - q_i^{k+a_{ij}/2} e^{-(z_1-z_2)/2}. \end{aligned}$$

and that $\left(q_i^{-k-a_{ij}/2} e^{(z_1-z_2)/2} - q_i^{k+a_{ij}/2} e^{-(z_1-z_2)/2}\right) / (z_1 - z_2 - r_i(ka_{ii} + a_{ij})\hbar) \in \mathbb{C}[[z_1, z_2]]^\times$. Then

$$\begin{aligned} &(z_1 - z_2 - r_i(ka_{ii} + a_{ij}\hbar))\, Y_\tau\left(x^{\pm}_{i,\hbar}, z_1\right) \bar{Y}^{\pm}_{ij,k}(z_2) \\ =& d_k(z_1 - z_2) Y^{\pm}_{i,\dots,i,j}\big(z_1, z_2 + r_i((k-1)a_{ii} + a_{ij})\hbar, \\ &\qquad z_2 + r_i((k-2)a_{ii} + a_{ij})\hbar, \dots, z_2 + r_i a_{ij}\hbar, z_2\big) \end{aligned}$$

lies in $\mathcal{E}_\hbar^{(2)}(F_\tau(A,\ell))$ for some $d_k(z)\in\mathbb{C}[[z,\hbar]]^\times$. From induction assumptions (6.35), (6.36) and Lemmas 6.7, 6.8, we complete the proof of (6.35) for $k+1$.

From induction assumption (6.36) and (3.4), we have that

$$\begin{aligned}
&(z-r_i(ka_{ii}+a_{ij})\hbar)Y\left(Y\left(x_{i,\hbar}^\pm,z\right)\left(\left(x_{i,\hbar}^\pm\right)_0^k x_{j,\hbar}^\pm\right),z_2\right)\\
=&c_k(z-r_i(ka_{ii}+a_{ij})\hbar)Y_{\mathcal{E}}\left(Y(x_{i,\hbar}^\pm,z_2),z\right)\bar{Y}_{ij,k}^\pm(z_2)\\
=&c_k\left(\left(z_1-z-r_i(ka_{ii}+a_{ij}\hbar)\right)Y_\tau\left(x_{i,\hbar}^\pm,z_1\right)\bar{Y}_{ij,k}^\pm(z_2)\right)\Big|_{z_1=z_2+z}\\
=&c_kd_k(z)Y_{i,\dots,i,j}^\pm\big(z_2+z,z_2+r_i((k-1)a_{ii}+a_{ij})\hbar,\\
&\qquad z_2+r_i((k-2)a_{ii}+a_{ij})\hbar,\dots,z_2+r_ia_{ij}\hbar,z_2\big).
\end{aligned}$$

By multiplying $(z-r_i(ka_{ii}+a_{ij})\hbar)^{-1}$ and taking Sing_z on both hand sides, we get that

$$\begin{aligned}
&Y\left(\mathrm{Sing}_z\,Y\left(x_{i,\hbar}^\pm,z\right)\left(x_{i,\hbar}^\pm\right)_0^k x_{j,\hbar}^\pm,z_2\right)\\
=&(z-r_i(ka_{ii}+a_{ij})\hbar)^{-1}c_{k+1}\bar{Y}_{ij,k+1}^\pm(z_2),
\end{aligned}\tag{6.37}$$

where $c_{k+1}=c_kd_k(r_i(ka_{ii}+a_{ij})\hbar)$. Then from the induction assumption (6.35), we complete the proof of (6.35) for $k+1$. Notice that both c_k and $d_k(r_i(ka_{ii}+a_{ij})\hbar)$ are invertible in $\mathbb{C}[[\hbar]]$. We have that c_{k+1} is invertible. By taking Res_z on both hand sides of (6.37), we complete the proof of (6.36) for $k+1$. Therefore, we complete the proof of the lemma. $\square$

Similarly, we have that

Lemma 6.13. *Let $i\in I$ and $k\in\mathbb{N}$. Then in $F_\tau(A,\ell)$, we have that*

$$\begin{aligned}
&\mathrm{Sing}_{z_1,\dots,z_k}\,z_1^{-1}\cdots z_k^{-1}Y_\tau(x_{i,\hbar}^\pm,z_1)\cdots Y_\tau(x_{i,\hbar}^\pm,z_k)x_{i,\hbar}^\pm\\
&\qquad\in\mathbb{C}[z_1^{-1},\dots,z_k^{-1}][[\hbar]]\left(\left(x_{i,\hbar}^\pm\right)_{-1}^k x_{i,\hbar}^\pm\right).
\end{aligned}\tag{6.38}$$

Moreover, there is $c_k'\in\mathbb{C}[[\hbar]]^\times$, such that

$$Y_\tau\left(\left(x_{i,\hbar}^\pm\right)_{-1}^k x_{i,\hbar}^\pm,z\right)=c_k'Y_{i,\dots,i}^\pm\left(z+kr_ia_{ii}\hbar,z+(k-1)r_ia_{ii}\hbar,\dots,z\right).$$

It is immediate from Lemma 6.12 that

Proposition 6.14. *For $i,j\in I$ with $a_{ij}<0$, we let*

$$\begin{aligned}
&Q_{ij}^\pm(z_1,\dots,z_{m_{ij}})\\
=&\,\mathrm{Sing}_{z_1,z_2,\dots,z_{m_{ij}}}\,Y_\tau(x_{i,\hbar}^\pm,z_1)Y_\tau(x_{i,\hbar}^\pm,z_2)\cdots Y_\tau(x_{i,\hbar}^\pm,z_{m_{ij}})x_{j,\hbar}^\pm.
\end{aligned}$$

Then $Q_{ij}^\pm(z_1,\dots,z_{m_{ij}})=0$ in $V_{\hat{\mathfrak{g}},\hbar}(\ell,0)$.

And it is immediate from Lemma 6.13 that

Proposition 6.15. *For $\ell \in \mathbb{Z}_+$, we let*

$$\begin{aligned}&M_i^{\pm}(z_1, \dots, z_{r\ell/r_i})\\ =&\operatorname{Sing}_{z_1,z_2,\dots,z_{r\ell/r_i}} z_1^{-1}\cdots z_{r\ell/r_i}^{-1} Y_\tau(x_{i,\hbar}^{\pm}, z_1)Y_\tau(x_{i,\hbar}^{\pm}, z_2)\cdots Y_\tau(x_{i,\hbar}^{\pm}, z_{r\ell/r_i})x_{i,\hbar}^{\pm}.\end{aligned}$$

Then $M_i^{\pm}(z_1, \dots, z_{r\ell/r_i}) = 0$ in $L_{\hat{\mathfrak{g}},\hbar}(\ell, 0)$.

Combining Propositions 6.10, 6.14 and 6.15, we have that

Proposition 6.16. *R_1^ℓ is the minimal closed ideal of $F_\tau(A, \ell)$ such that $[R_1^\ell] = R_1^\ell$, and contains all coefficients of $A_i(z)$ for $i \in I$ and all coefficients of $Q_{ij}^{\pm}(z_1, \dots, z_{m_{ij}})$ for $i, j \in I$ with $a_{ij} < 0$. Moreover, if $\ell \in \mathbb{Z}_+$, then R_2^ℓ is the minimal closed ideal of $V_{\hat{\mathfrak{g}},\hbar}(\ell, 0)$ such that $[R_2^\ell] = R_2^\ell$ and contains all coefficients of $M_i^{\pm}(z_1, \dots, z_{r\ell/r_i})$ for $i \in I$.*

Theorem 6.6 is immediate from Lemma 3.3, Proposition 6.16 and the following three technical results.

Lemma 6.17. *For $i, j \in I$, we have that*

$$\begin{aligned}&S_\tau(z_1)(A_i(z_2) \otimes h_{j,\hbar})\\ =&A_i(z_2) \otimes h_{j,\hbar} + \operatorname{Sing}_{z_2}\left(A_i(z_2) \otimes \mathbf{1} \otimes \left(e^{z_2\frac{\partial}{\partial z_1}} - 1\right)\left(\tau_{ij}^{1,+}(-z_1) + \tau_{ji}^{2,+}(z_1)\right)\right),\\ &S_\tau(z_1)(h_{j,\hbar} \otimes A_i(z_2))\\ =&h_{j,\hbar} \otimes A_i(z_2) + \operatorname{Sing}_{z_2}\left(\mathbf{1} \otimes A_i(z_2) \otimes \left(1 - e^{-z_2\frac{\partial}{\partial z_1}}\right)\left(\tau_{ji}^{2,+}(-z_1) + \tau_{ij}^{1,+}(z_1)\right)\right),\\ &S_\tau(z_1)(A_i(z_2) \otimes x_{j,\hbar}^{\pm})\\ =&\operatorname{Sing}_{z_2}\left(A_i(z_2) \otimes x_{j,\hbar}^{\pm} \otimes \exp\left(\left(e^{z_2\frac{\partial}{\partial z_1}} - 1\right)\log\frac{\tau_{ij}^{+,\pm}(z_1)}{\tau_{ji}^{\pm,+}(-z_1)}\right)\right),\\ &S_\tau(z_1)(x_{j,\hbar}^{\pm} \otimes A_i(z_2))\\ =&\operatorname{Sing}_{z_2}\left(x_{j,\hbar}^{\pm} \otimes A_i(z_2) \otimes \exp\left(\left(1 - e^{-z_2\frac{\partial}{\partial z_1}}\right)\log\frac{\tau_{ij}^{+,\pm}(-z_1)}{\tau_{ji}^{\pm,+}(z_1)}\right)\right).\end{aligned}$$

Proof. From the relations (3.12), (5.20) and (6.13), we have that

$$\begin{aligned}&S_\tau(z_1)\left(\operatorname{Sing}_{z_2} Y_\tau(x_{i,\hbar}^+, z_2)x_{i,\hbar}^- \otimes h_{j,\hbar}\right)\\ =&\operatorname{Sing}_{z_2} Y_\tau^{12}(z_2)S_\tau^{23}(z_1)S_\tau^{13}(z_1 + z_2)(x_{i,\hbar}^+ \otimes x_{i,\hbar}^- \otimes h_{j,\hbar})\\ =&\operatorname{Sing}_{z_2} Y_\tau^{12}(z_2)S_\tau^{23}(z_1)\Big(x_{i,\hbar}^+ \otimes x_{i,\hbar}^- \otimes h_{j,\hbar}\\ &\quad + x_{i,\hbar}^+ \otimes x_{i,\hbar}^- \otimes \mathbf{1} \otimes \otimes \left(\tau_{ji}^{1,+}(-z_1 - z_2) + \tau_{ij}^{2,+}(z_1 + z_2)\right)\Big)\\ =&\operatorname{Sing}_{z_2} Y_\tau^{12}(z_2)\Big(x_{i,\hbar}^+ \otimes x_{i,\hbar}^- \otimes h_{j,\hbar}\\ &\quad + x_{i,\hbar}^+ \otimes x_{i,\hbar}^- \otimes \mathbf{1} \otimes \left(e^{z_2\frac{\partial}{\partial z_1}} - 1\right)\left(\tau_{ji}^{1,+}(-z_1) + \tau_{ij}^{2,+}(z_1)\right)\Big)\end{aligned}$$

$$=\operatorname{Sing}_{z_2} Y_\tau(x^+_{i,\hbar}, z_2)x^-_{i,\hbar}\otimes h_{j,\hbar} + \operatorname{Sing}_{z_2,z}\left(Y_\tau(x^+_{i,\hbar}, z_2)x^-_{i,\hbar}\otimes \mathbf{1}\otimes\left(e^{z_2\frac{\partial}{\partial z_1}}-1\right)\left(\tau^{1,+}_{ji}(-z_1)+\tau^{2,+}_{ij}(z_1)\right)\right).$$

From the relations (3.16), (5.19) and (6.16), we have that

$$\begin{aligned}
&S_\tau(z)\left(q^{-r\ell\partial}F(\partial)h_{i,\hbar}\otimes h_{j,\hbar}\right)\\
=&q^{-r\ell\partial\otimes 1-r\ell\frac{\partial}{\partial z}}F\left(\partial\otimes 1+\frac{\partial}{\partial z}\right)S(z)(h_{i,\hbar}\otimes h_{j,\hbar})\\
=&q^{-r\ell\partial\otimes 1-r\ell\frac{\partial}{\partial z}}F\left(\partial\otimes 1+\frac{\partial}{\partial z}\right)(h_{i,\hbar}\otimes h_{j,\hbar}+\mathbf{1}\otimes\mathbf{1}\otimes(\tau_{ji}(-z)-\tau_{ij}(z)))\\
=&q^{-r\ell\partial}F(\partial)h_{i,\hbar}\otimes h_{j,\hbar}+\mathbf{1}\otimes\mathbf{1}\otimes q^{-r\ell\frac{\partial}{\partial z}}F\left(\frac{\partial}{\partial z}\right)(\tau_{ji}(-z)-\tau_{ij}(z))\\
=&q^{-r\ell\partial}F(\partial)h_{i,\hbar}\otimes h_{j,\hbar}+\mathbf{1}\otimes\mathbf{1}\otimes\left(1-q^{-2r\ell\frac{\partial}{\partial z}}\right)\left(\tau^{1,+}_{ji}(-z)+\tau^{2,+}_{ij}(z)\right).
\end{aligned}$$

Then by using Lemma 3.2, we get that

$$\begin{aligned}
&S_\tau(z)\left(\exp\left(\left(-q^{-r\ell\partial}F(\partial)h_{i,\hbar}\right)_{-1}\right)\mathbf{1}\otimes h_{j,\hbar}\right)\\
=&\exp\left(\left(-q^{-r\ell\partial}F(\partial)h_{i,\hbar}\right)_{-1}\right)\mathbf{1}\otimes h_{j,\hbar}\\
&+\exp\left(\left(-q^{-r\ell\partial}F(\partial)h_{i,\hbar}\right)_{-1}\right)\mathbf{1}\otimes\mathbf{1}\otimes\left(q^{-2r\ell\frac{\partial}{\partial z}}-1\right)\left(\tau^{1,+}_{ji}(-z)+\tau^{2,+}_{ij}(z)\right).
\end{aligned}$$

Hence, we have that

$$\begin{aligned}
&S_\tau(z_1)\left(A_i(z_2)\otimes h_{j,\hbar}\right)=\operatorname{Sing}_{z_2} Y_\tau(x^+_{i,\hbar}, z_2)x^-_{i,\hbar}\otimes h_{j,\hbar}+\operatorname{Sing}_{z_2}\\
&\quad\left(\operatorname{Sing}_{z_2} Y_\tau(x^+_{i,\hbar}, z_2)x^-_{i,\hbar}\otimes\mathbf{1}\otimes\left(e^{z_2\frac{\partial}{\partial z_1}}-1\right)\left(\tau^{1,+}_{ji}(-z_1)+\tau^{2,+}_{ij}(z_1)\right)\right)\\
&\quad-\frac{1}{q_i-q_i^{-1}}\left(\mathbf{1}\otimes h_{j,\hbar}z_2^{-1}-E(h_{i,\hbar})\otimes h_{j,\hbar}(z_2+2r\ell\hbar)^{-1}\right)\\
&\quad+\frac{(z_2+2r\ell\hbar)^{-1}}{q_i-q_i^{-1}}E(h_{i,\hbar})\otimes\mathbf{1}\otimes\left(q^{-2r\ell\frac{\partial}{\partial z_1}}-1\right)\left(\tau^{1,+}_{ji}(-z_1)+\tau^{2,+}_{ij}(z_1)\right)
\end{aligned}$$

(6.39)

$$\begin{aligned}
=&A_i(z_2)\otimes h_{j,\hbar}+\operatorname{Sing}_{z_2} A_i(z_2)\otimes\mathbf{1}\otimes\left(e^{z_2\frac{\partial}{\partial z_1}}-1\right)\left(\tau^{1,+}_{ji}(-z_1)+\tau^{2,+}_{ij}(z_1)\right)\\
&+\frac{1}{q_i-q_i^{-1}}\operatorname{Sing}_{z_2}\left(\mathbf{1}\otimes\mathbf{1}\otimes z_2^{-1}\left(e^{z_2\frac{\partial}{\partial z_1}}-1\right)\left(\tau^{1,+}_{ji}(-z_1)+\tau^{2,+}_{ij}(z_1)\right)\right)\\
&-\frac{1}{q_i-q_i^{-1}}\operatorname{Sing}_{z_2}\left(E(h_{i,\hbar})\otimes\mathbf{1}\otimes\frac{e^{z_2\frac{\partial}{\partial z_1}}-1}{z_2+2r\ell\hbar}\left(\tau^{1,+}_{ji}(-z_1)+\tau^{2,+}_{ij}(z_1)\right)\right)\\
&+\frac{(z_2+2r\ell\hbar)^{-1}}{q_i-q_i^{-1}}E(h_{i,\hbar})\otimes\mathbf{1}\otimes\left(q^{-2r\ell\frac{\partial}{\partial z_1}}-1\right)\left(\tau^{1,+}_{ji}(-z_1)+\tau^{2,+}_{ij}(z_1)\right).
\end{aligned}$$

Notice that

$$\operatorname{Sing}_z z^{-1}\left(e^{z\frac{\partial}{\partial x}}-1\right)=0,$$
$$\operatorname{Sing}_z(z+2r\ell\hbar)^{-1}\left(e^{z\frac{\partial}{\partial x}}-1\right)=\left(q^{-2r\ell\frac{\partial}{\partial x}}-1\right)(z+2r\ell\hbar)^{-1}.$$

Combining these equations and (6.39), we get that

$$\begin{aligned}&S_\tau(z_1)\left(A_i(z_2)\otimes h_{j,\hbar}\right)\\=&A_i(z_2)\otimes h_{j,\hbar}+\operatorname{Sing}_{z_2}A_i(z_2)\otimes\mathbf{1}\otimes\left(e^{z_2\frac{\partial}{\partial z_1}}-1\right)\left(\tau_{ji}^{1,+}(-z_1)+\tau_{ij}^{2,+}(z_1)\right).\end{aligned}$$

Therefore, we complete the proof of the first equation. The proof of the rest equations are similar. □

Similar to the proof of Lemma 6.17, we have the following two results.

Lemma 6.18. *For $i,j,k\in I$ such that $a_{ij}<0$, we have that*

$$\begin{aligned}&S_\tau(z)\left(Q_{ij}^{\pm}(z_1,\dots,z_{m_{ij}})\otimes h_{k,\hbar}\right)\\=&Q_{ij}^{\pm}(z_1,\dots,z_{m_{ij}})\otimes h_{k,\hbar}\pm\operatorname{Sing}_{z_1,\dots,z_{m_{ij}}}\Bigg(Q_{ij}^{\pm}(z_1,\dots,z_{m_{ij}})\otimes\mathbf{1}\\&\quad\otimes\left(\tau_{kj}^{1,\pm}(-z)+\tau_{jk}^{2,\pm}(z)+\sum_{a=1}^{m_{ij}}\left(\tau_{ki}^{1,\pm}(-z-z_a)+\tau_{ik}^{2,\pm}(z+z_a)\right)\right)\Bigg),\\&S_\tau(z)\left(h_{k,\hbar}\otimes Q_{ij}^{\pm}(z_1,\dots,z_{m_{ij}})\right)\\=&h_{k,\hbar}\otimes Q_{ij}^{\pm}(z_1,\dots,z_{m_{ij}})\mp\operatorname{Sing}_{z_1,\dots,z_{m_{ij}}}\Bigg(\mathbf{1}\otimes Q_{ij}^{\pm}(z_1,\dots,z_{m_{ij}})\\&\quad\otimes\left(\tau_{jk}^{2,\pm}(-z)+\tau_{kj}^{1,\pm}(z)+\sum_{a=1}^{m_{ij}}\left(\tau_{ik}^{2,\pm}(-z+z_a)+\tau_{ki}^{1,\pm}(z-z_a)\right)\right)\Bigg),\\&S_\tau(z)\left(Q_{ij}^{\pm}(z_1,\dots,z_{m_{ij}})\otimes x_{k,\hbar}^{\epsilon}\right)=\operatorname{Sing}_{z_1,\dots,z_{m_{ij}}}\Bigg(Q_{ij}^{\pm}(z_1,\dots,z_{m_{ij}})\otimes x_{k,\hbar}^{\epsilon}\\&\quad\otimes\tau_{kj}^{\epsilon,\pm}(-z)^{-1}\tau_{jk}^{\pm,\epsilon}(z)\prod_{a=1}^{m_{ij}}\tau_{ki}^{\epsilon,\pm}(-z-z_a)^{-1}\tau_{ik}^{\pm,\epsilon}(z+z_a)\Bigg),\\&S_\tau(z)\left(x_{k,\hbar}^{\epsilon}\otimes Q_{ij}^{\pm}(z_1,\dots,z_{m_{ij}})\right)=\operatorname{Sing}_{z_1,\dots,z_{m_{ij}}}\Bigg(x_{k,\hbar}^{\epsilon}\otimes Q_{ij}^{\pm}(z_1,\dots,z_{m_{ij}})\\&\quad\otimes\tau_{jk}^{\epsilon,\pm}(-z)^{-1}\tau_{kj}^{\pm,\epsilon}(z)\prod_{a=1}^{m_{ij}}\tau_{ik}^{\epsilon,\pm}(-z+z_a)^{-1}\tau_{ki}^{\pm,\epsilon}(z-z_a)\Bigg).\end{aligned}$$

Lemma 6.19. *For any $\ell\in\mathbb{Z}_+$ and $i,j\in I$, we have that*

$$S_\tau(z)\left(M_i^{\pm}(z_1,\dots,z_{r\ell/r_i})\otimes h_{j,\hbar}\right)=M_i^{\pm}(z_1,\dots,z_{r\ell/r_i})\otimes h_{j,\hbar}$$

$$\pm \operatorname{Sing}_{z_1,\dots,z_{r\ell/r_i}} z_1^{-1}\cdots z_{r\ell/r_i}^{-1}$$
$$\left(M_i^{\pm}(z_1,\dots,z_{r\ell/r_i})\otimes \mathbf{1}\otimes \sum_{a=1}^{r\ell/r_i+1}\Big(\tau_{ji}^{1,\pm}(-z-z_a)+\tau_{ij}^{2,\pm}(z+z_a)\Big)\right),$$
$$S_\tau(z)\left(h_{j,\hbar}\otimes M_i^{\pm}(z_1,\dots,z_{r\ell/r_i})\right)=h_{j,\hbar}\otimes M_i^{\pm}(z_1,\dots,z_{r\ell/r_i})$$
$$\mp \operatorname{Sing}_{z_1,\dots,z_{r\ell/r_i}} z_1^{-1}\cdots z_{r\ell/r_i}^{-1}$$
$$\left(\mathbf{1}\otimes M_i^{\pm}(z_1,\dots,z_{r\ell/r_i})\otimes \sum_{a=1}^{r\ell/r_i+1}\Big(\tau_{ij}^{1,\pm}(-z+z_a)+\tau_{ji}^{2,\pm}(z-z_a)\Big)\right),$$
$$S_\tau(z)\left(M_i^{\pm}(z_1,\dots,z_{r\ell/r_i})\otimes x_{j,\hbar}^{\epsilon}\right)=\operatorname{Sing}_{z_1,\dots,z_{r\ell/r_i}} z_1^{-1}\cdots z_{r\ell/r_i}^{-1}$$
$$\left(M_i^{\pm}(z_1,\dots,z_{r\ell/r_i})\otimes x_{j,\hbar}^{\epsilon}\otimes \prod_{a=1}^{r\ell/r_i+1}\tau_{ji}^{\epsilon,\pm}(-z-z_a)^{-1}\tau_{ij}^{\pm,\epsilon}(z+z_a)\right),$$
$$S_\tau(z)\left(x_{j,\hbar}^{\epsilon}\otimes M_i^{\pm}(z_1,\dots,z_{r\ell/r_i})\right)=\operatorname{Sing}_{z_1,\dots,z_{r\ell/r_i}} z_1^{-1}\cdots z_{r\ell/r_i}^{-1}$$
$$\left(x_{j,\hbar}^{\epsilon}\otimes M_i^{\pm}(z_1,\dots,z_{r\ell/r_i})\otimes \prod_{a=1}^{r\ell/r_i+1}\tau_{ij}^{\epsilon,\pm}(-z+z_a)^{-1}\tau_{ji}^{\pm,\epsilon}(z-z_a)\right),$$

where $z_{r\ell/r_i+1}=0$.

7. ϕ-coordinated modules

In this section, we recall the construction of $\hbar$-adic nonlocal vertex algebras and their ϕ-coordinated modules introduced in [Li7].

First, we fix an associate $\phi(z_1,z)=z_1e^z$, which is a particular associate of the additive formal group $F_a(z_1,z_2)=z_1+z_2$. Let V be a nonlocal vertex algebra. Recall from [Li7] that a *ϕ-coordinated V-module* is a vector space W equipped with a linear map $Y_W^{\phi}(\cdot,z):V\to\mathcal{E}(W)$, satisfying the condition that $Y_W^{\phi}(\mathbf{1},z)=1_W$ and that for $u,v\in V$, there exists positive integer k such that

$$(1-z_2/z_1)^kY_W^{\phi}(u,z_1)Y_W^{\phi}(v,z_2)\in\mathcal{E}^{(2)}(W), \tag{7.1}$$

$$\Big((1-z_2/z_1)^kY_W^{\phi}(u,z_1)Y_W^{\phi}(v,z_2)\Big)\Big|_{z_1=z_2e^{z_0}} = (1-e^{-z_0})^kY_W^{\phi}(Y(u,z_0)v,z_2). \tag{7.2}$$

Then we have that

Lemma 7.1. *Let V be a nonlocal vertex algebra, and let (W,Y_W^{ϕ}) be a ϕ-coordinated V-module. Suppose*

$$\sum_{i\geq 1}a_i\otimes b_i\otimes f_i(z),\quad \sum_{j\geq 1}\alpha_j\otimes\beta_j\otimes g_j(z)\in V\otimes V\otimes\mathbb{C}(z),$$

such that

$$(7.3)\qquad \sum_{i\geq 1}\iota_{z_1,z_2}f_i(e^{z_1-z_2})Y(a_i,z_1)Y(b_i,z_2) = \sum_{j\geq 1}\iota_{z_2,z_1}g_j(e^{z_1-z_2})Y(\alpha_j,z_2)Y(\beta_j,z_1).$$

Then we have that

$$\sum_{i\geq 1}\iota_{z_1,z_2}f_i(z_1/z_2)Y_W^{\phi}(a_i,z_1)Y_W^{\phi}(b_i,z_2) = \sum_{j\geq 1}\iota_{z_2,z_1}g_j(z_1/z_2)Y_W^{\phi}(\alpha_j,z_2)Y_W^{\phi}(\beta_j,z_1).$$

Proof. By letting the both hand sides of (7.3) act on $\mathbf{1}$, we get that

$$(7.4)\qquad \sum_{i\geq 1}\iota_{z_1,z_2}f_i(e^{z_1-z_2})Y(a_i,z_1-z_2)b_i = \sum_{j\geq 1}\iota_{z_2,z_1}g_j(e^{z_1-z_2})e^{(z_1-z_2)\partial}Y(\alpha_j,z_2-z_1)\beta_j.$$

Let k be a positive integer such that $(i,j\geq 1)$

$$(7.5)\qquad z^kf_i(e^z),\quad z^kg_j(e^z)\in\mathbb{C}[[z]],\quad z^kY(a_i,z)b_i,\quad z^kY(\alpha_j,z)\beta_j\in V[[z]].$$

By multiplying $(z_1-z_2)^{2k}$ on both hand sides of (7.4), we get that

$$\sum_{i\geq 1}\left((z_1-z_2)^kf_i(e^{z_1-z_2})\right)\left((z_1-z_2)^kY(a_i,z_1-z_2)b_i\right)$$
$$(7.6)\qquad =\sum_{j\geq 1}\left((z_1-z_2)^kg_j(e^{z_1-z_2})\right)e^{(z_1-z_2)\partial}\left((z_1-z_2)^kY(\alpha_j,z_2-z_1)\beta_j\right).$$

From (7.5), we can take $z_2=0$ in (7.6):

$$z^{2k}\sum_{i\geq 1}f_i(e^z)Y(a_i,z)b_i=\sum_{j\geq 1}z^kg_j(e^z)e^{z\partial}z^kY(\alpha_j,-z)\beta_j.$$

Since z^k is invertible, we get that

$$(7.7)\qquad \sum_{i\geq 1}f_i(e^z)Y(a_i,z)b_i=\sum_{j\geq 1}g_j(e^z)e^{z\partial}Y(\alpha_j,-z)\beta_j.$$

From (7.5), we get that

$$\left.\left((z_1-z_2)^{2k}\sum_{i\geq 1}f_i(e^{z_1-z_2})Y(a_i,z_1)Y(b_i,z_2)\right)\right|_{z_1=z_2+z_0}$$
$$=\sum_{i\geq 1}z_0^kf_i(e^{z_0})z_0^kY_{\mathcal{E}}\left(Y(a_i,z_2),z_0\right)Y(b_i,z_2)$$
$$=\sum_{i\geq 1}z_0^kf_i(e^{z_0})z_0^kY\left(Y(a_i,z_0)b_i,z_2\right),$$

where the last equation follows from (3.4). We note that the condition (7.3) shows that

$$\sum_{i\geq 1} f_i(e^{z_1-z_2})Y(a_i,z_1)Y(b_i,z_2)\in \mathcal{E}^{(2)}(V).$$

Combining these relations, we get that

$$\begin{aligned} &z_0^{2k}\left(\sum_{i\geq 1} f_i(e^{z_1-z_2})Y(a_i,z_1)Y(b_i,z_2)\right)\Big|_{z_1=z_2+z_0}\\ =&\sum_{i\geq 1} z_0^k f_i(e^{z_0}) z_0^k Y\left(Y(a_i,z_0)b_i,z_2\right). \end{aligned}$$

Then we have that

$$\begin{aligned} &\sum_{i\geq 1} f_i(e^{z_0})Y\left(Y(a_i,z_0)b_i,z_2\right)\\ =&\left(\sum_{i\geq 1} f_i(e^{z_1-z_2})Y(a_i,z_1)Y(b_i,z_2)\right)\Bigg|_{z_1=z_2+z_0}\in \mathcal{E}(V)[[z_0]]. \end{aligned}$$

Acting on $\mathbf{1}$ and taking $z_2\to 0$, we get that

$$\sum_{i\geq 1} f_i(e^{z_0})Y(a_i,z_0)b_i\in V[[z_0]].$$

Viewing $\log(1+z_0)$ as an element in $\mathbb{C}[[z_0]]$, we get that

$$\sum_{i\geq 1} f_i(1+z_0)Y(a_i,\log(1+z_0))b_i\in V[[z_0]]. \tag{7.8}$$

We replace k with a larger one if necessary so that $(i,j\geq 1)$:

$$(1-z_2/z_1)^k Y_W^{\phi}(a_i,z_1)Y_W^{\phi}(b_i,z_2),\ (1-z_2/z_1)^k Y_W^{\phi}(\alpha_j,z_2)Y_W^{\phi}(\beta_j,z_1)\in \mathcal{E}^{(2)}(W).$$

From the weak associativity of ϕ-coordinated modules (7.2), we get that

$$\begin{aligned} &\left((1-z/z_1)^k Y_W^{\phi}(a_i,z_1)Y_W^{\phi}(b_i,z)\right)\Big|_{z_1=\phi(z,z_0)} \qquad (7.9)\\ =&(1-e^{-z_0})^k Y_W^{\phi}(Y(a_i,z_0)b_i,z),\\ &\left((1-z/z_1)^k Y_W^{\phi}(\alpha_j,z)Y_W^{\phi}(\beta_j,z_1)\right)\Big|_{z=\phi(z_1,-z_0)} \qquad (7.10)\\ =&(1-e^{-z_0})^k Y_W^{\phi}(Y(\alpha_j,-z_0)\beta_j,z_1). \end{aligned}$$

From the equation (7.10) and [Li7, Remark 2.8], we have that

$$\begin{aligned} &\left((1-z/z_1)^k Y_W^{\phi}(\alpha_j,z)Y_W^{\phi}(\beta_j,z_1)\right)\Big|_{z_1=\phi(z,z_0)}\\ =&\left(\left((1-z/z_1)^k Y_W^{\phi}(\alpha_j,z)Y_W^{\phi}(\beta_j,z_1)\right)\Big|_{z=\phi(z_1,-z_0)}\right)\Bigg|_{z_1=\phi(z,z_0)} \end{aligned}$$

$$= \left((1-e^{-z_0})^k Y_W^{\phi}(Y(\alpha_j,-z_0)\beta_j,z_1)\right)\Big|_{z_1=\phi(z,z_0)}$$
$$=(1-e^{-z_0})^k Y_W^{\phi}(Y(\alpha_j,-z_0)\beta_j,\phi(z,z_0))$$
$$=(1-e^{-z_0})^k Y_W^{\phi}(e^{z_0\partial}Y(\alpha_j,-z_0)\beta_j,z), \tag{7.11}$$

where the (7.11) follows from [Li7, Lemma 3.7]. Combining this with equations (7.7) and (7.9), we get

$$\left((1-z/z_1)^{2k}\sum_{i\geq 1} f_i(z_1/z)Y_W^{\phi}(a_i,z_1)Y_W^{\phi}(b_i,z)\right)\Bigg|_{z_1=\phi(z,z_0)}$$
$$=\sum_{i\geq 1}\left(\left((1-z/z_1)^k f_i(z_1/z)\right)\left((1-z/z_1)^k Y_W^{\phi}(a_i,z_1)Y_W^{\phi}(b_i,z)\right)\right)\Big|_{z_1=\phi(z,z_0)}$$
(7.12)
$$=(1-e^{-z_0})^{2k}\sum_{i\geq 1} f_i(e^{z_0})Y_W^{\phi}(Y(a_i,z_0)b_i,z)$$
$$=(1-e^{-z_0})^{2k}\sum_{j\geq 1} g_j(e^{z_0})Y_W^{\phi}(e^{z_0\partial}Y(\alpha_j,-z_0)\beta_j,z)$$
$$=\sum_{j\geq 1}\left(\left((1-z/z_1)^k g_j(z_1/z)\right)\left((1-z/z_1)^k Y_W^{\phi}(\alpha_j,z)Y_W^{\phi}(\beta_j,z_1)\right)\right)\Big|_{z_1=\phi(z,z_0)}$$
$$=\left((1-z/z_1)^{2k}\sum_{j\geq 1} g_j(z_1/z)Y_W^{\phi}(\alpha_j,z)Y_W^{\phi}(\beta_j,z_1)\right)\Bigg|_{z_1=\phi(z,z_0)}.$$

Notice that

$$(1-z_2/z_1)^{2k}\sum_{i\geq 1} f_i(z_1/z_2)Y_W^{\phi}(a_i,z_1)Y_W^{\phi}(b_i,z_2)\in\mathcal{E}^{(2)}(W),$$
$$(1-z_2/z_1)^{2k}\sum_{j\geq 1} g_j(z_1/z_2)Y_W^{\phi}(\alpha_j,z_2)Y_W^{\phi}(\beta_j,z_1)\in\mathcal{E}^{(2)}(W).$$

Then we get from [Li7, Remark 2.8] that

$$(1-z_2/z_1)^{2k}\sum_{i\geq 1} f_i(z_1/z_2)Y_W^{\phi}(a_i,z_1)Y_W^{\phi}(b_i,z_2)$$
$$=(1-z_2/z_1)^{2k}\sum_{j\geq 1} g_j(z_1/z_2)Y_W^{\phi}(\alpha_j,z_2)Y_W^{\phi}(\beta_j,z_1).$$

Combining this with equation (7.12) and [Li7, Lemma 5.8], we get that

$$(z_2z_0)^{-1}\delta\left(\frac{z_1-z_2}{z_2z_0}\right)\sum_{i\geq 1} f_i(z_1/z_2)Y_W^{\phi}(a_i,z_1)Y_W^{\phi}(b_i,z_2)$$
$$-(z_2z_0)^{-1}\delta\left(\frac{z_2-z_1}{-z_2z_0}\right)\sum_{j\geq 1} g_j(z_1/z_2)Y_W^{\phi}(\alpha_j,z_2)Y_W^{\phi}(\beta_j,z_1)$$

$$=z_1^{-1}\delta\left(\frac{z_2(1+z_0)}{z_1}\right)\sum_{i\geq 1} f_i(1+z_0)Y_W^{\phi}(Y(a_i,\log(1+z_0))b_i,z_2).$$

Taking Res_{z_0} on both hand sides, we get that

$$\sum_{i\geq 1} f_i(z_1/z_2)Y_W^{\phi}(a_i,z_1)Y_W^{\phi}(b_i,z_2)-\sum_{j\geq 1} g_j(z_1/z_2)Y_W^{\phi}(\alpha_j,z_2)Y_W^{\phi}(\beta_j,z_1)$$
$$=\mathrm{Res}_{z_0}\, z_1^{-1}\delta\left(\frac{z_2(1+z_0)}{z_1}\right)\sum_{i\geq 1} f_i(1+z_0)Y_W^{\phi}(Y(a_i,\log(1+z_0))b_i,z_2)=0,$$

where the last equation follows from (7.8). We complete the proof. □

Proposition 7.2. *Let* (W,Y_W^{ϕ}) *be a* ϕ*-coordinated module of a nonlocal vertex algebra* V*. Let*

$$\sum_{i\geq 1} a_i\otimes b_i\otimes f_i(z),\quad \sum_{j\geq 1}\alpha_j\otimes\beta_j\otimes g_j(z)\in V\otimes V\otimes V\otimes\mathbb{C}(x),$$
$$\text{and}\quad \sum_{k\geq 0}\gamma_k\in V$$

be finite sums, such that

$$\sum_{i\geq 1}\iota_{z_1,z_2}f_i(e^{z_1-z_2})Y(a_i,z_1)Y(b_i,z_2)$$
$$-\sum_{j\geq 1}\iota_{z_2,z_1}g_j(e^{z_1-z_2})Y(\alpha_j,z_2)Y(\beta_j,z_1)$$
$$=\sum_{k\geq 0}Y(\gamma_k,z_2)\frac{1}{k!}\frac{\partial^k}{\partial z_2^k}z_1^{-1}\delta\left(\frac{z_2}{z_1}\right). \tag{7.13}$$

Then we have that

$$\sum_{i\geq 1}\iota_{z_1,z_2}f_i(z_1/z_2)Y_W^{\phi}(a_i,z_1)Y_W^{\phi}(b_i,z_2)$$
$$-\sum_{j\geq 1}\iota_{z_2,z_1}g_j(z_1/z_2)Y_W^{\phi}(\alpha_j,z_2)Y_W^{\phi}(\beta_j,z_1)$$
$$=\sum_{k\geq 0}Y_W^{\phi}(\gamma_k,z_2)\frac{1}{k!}\left(z_2\frac{\partial}{\partial z_2}\right)^k\delta\left(\frac{z_2}{z_1}\right). \tag{7.14}$$

Proof. For any integer $i\geq 0$, we set $b_{-i}=\alpha_{-i}=\gamma_i$, $a_{-i}=\beta_{-i}=\mathbf{1}$ and

$$f_{-i}(z)=g_{-i}(z)=\frac{1}{2i!}\left(-z\frac{\partial}{\partial z}\right)^i\frac{z+1}{z-1}\in\mathbb{C}(z).$$

It is straightforward to verify that

$$\iota_{z_1,z_2}f_{-i}(z_1/z_2)-\iota_{z_2,z_1}g_{-i}(z_1/z_2)=\frac{1}{i!}\left(z_2\frac{\partial}{\partial z_2}\right)^i\delta\left(\frac{z_2}{z_1}\right),$$

$$\iota_{z_1,z_2}f_{-i}(e^{z_1-z_2})-\iota_{z_2,z_1}g_{-i}(e^{z_1-z_2})=\frac{1}{i!}\frac{\partial^i}{\partial z_2^i}z_1^{-1}\delta\left(\frac{z_2}{z_1}\right).$$

Then the equation (7.13) is equivalent to

$$\begin{aligned}&\sum_{i\in\mathbb{Z}}\iota_{z_1,z_2}f_i(e^{z_1-z_2})Y(a_i,z_1)Y(b_i,z_2)\\=&\sum_{j\in\mathbb{Z}}\iota_{z_2,z_1}g_j(e^{z_1-z_2})Y(\alpha_j,z_2)Y(\beta_j,z_1),\end{aligned}$$

and the equation (7.14) is equivalent to

$$\begin{aligned}&\sum_{i\in\mathbb{Z}}\iota_{z_1,z_2}f_i(z_1/z_2)Y_W^\phi(a_i,z_1)Y_W^\phi(b_i,z_2)\\=&\sum_{j\in\mathbb{Z}}\iota_{z_2,z_1}g_j(z_1/z_2)Y_W^\phi(\alpha_j,z_2)Y_W^\phi(\beta_j,z_1).\end{aligned}$$

Therefore, this proposition follows immediate from Lemma 7.1. □

Let V be an $\hbar$-adic nonlocal vertex algebra. A *ϕ-coordinated V-module* is a topologically free $\mathbb{C}[[\hbar]]$-module W equipped with a $\mathbb{C}[[\hbar]]$-linear map $Y_W^\phi(\cdot,x):V\to\mathcal{E}_\hbar(W)$, such that $(W/\hbar^nW,Y_{W,n}^\phi)$ is a ϕ-coordinated $V/\hbar^nV$-module, where $Y_{W,n}^\phi:V/\hbar^nV\to\mathcal{E}(W/\hbar^nW)$ is the $\mathbb{C}[[\hbar]]$-linear map induced from Y_W^ϕ. As an immediate $\hbar$-adic analogue of Proposition 7.2, we have that

Proposition 7.3. *Let V be an $\hbar$-adic nonlocal vertex algebra, and let (W,Y_W^ϕ) be a ϕ-coordinated V-module. Suppose that*

$$\sum_{i\ge1}a_i\otimes b_i\otimes f_i(z),\quad\sum_{j\ge1}\alpha_j\otimes\beta_j\otimes g_j(z)\in V\widehat{\otimes}V\widehat{\otimes}V\widehat{\otimes}\mathbb{C}(z)[[\hbar]],$$
$$\text{and}\quad\sum_{k\ge0}\gamma_k\in V,$$

such that

$$\begin{aligned}&\sum_{i\ge1}\iota_{z_1,z_2}f_i(e^{z_1-z_2})Y(a_i,z_1)Y(b_i,z_2)\\&-\sum_{j\ge1}\iota_{z_2,z_1}g_j(e^{z_1-z_2})Y(\alpha_j,z_2)Y(\beta_j,z_1)\\&\qquad=\sum_{k\ge0}Y(\gamma_k,z_2)\frac{1}{k!}\frac{\partial^k}{\partial z_2^k}z_1^{-1}\delta\left(\frac{z_2}{z_1}\right).\end{aligned}$$

Then we have that

$$\begin{aligned}&\sum_{i\ge1}\iota_{z_1,z_2}f_i(z_1/z_2)Y_W^\phi(a_i,z_1)Y_W^\phi(b_i,z_2)\\&-\sum_{j\ge1}\iota_{z_2,z_1}g_j(z_1/z_2)Y_W^\phi(\alpha_j,z_2)Y_W^\phi(\beta_j,z_1)\end{aligned}$$

$$= \sum_{k\geq 0} Y_W^{\phi}(\gamma_k, z_2)\frac{1}{k!}\left(z_2\frac{\partial}{\partial z_2}\right)^k \delta\left(\frac{z_2}{z_1}\right).$$

Let $Z^{\phi}(z_1, z_2) : \mathcal{E}_\hbar(W)\widehat{\otimes}\mathcal{E}_\hbar(W)\widehat{\otimes}\mathbb{C}(z)[[\hbar]] \to \mathrm{End}(W)[[z_1^{\pm 1}, z_2^{\pm 1}]]$ be the $\mathbb{C}[[\hbar]]$-module map defined by

$$Z^{\phi}(z_1, z_2)(a(z) \otimes b(z) \otimes f(z)) = f(z_1/z_2)a(z_1)b(z_2).$$

Recall from [Li7] that a subset U of $\mathcal{E}_\hbar(W)$ is said to be *$\hbar$-adically S_{trig}-local* if for any $a(z), b(z) \in U$, there is $A(z) \in (\mathbb{C}U \otimes \mathbb{C}U \otimes \mathbb{C}(z))\,[[\hbar]]$, such that

$$a(z_1)b(z_2) \sim Z^{\phi}(z_2, z_1)\,(A(z))\,.$$

Let U be an $\hbar$-adically S_{trig}-local subset of $\mathcal{E}_\hbar(W)$. For $a(z), b(z) \in U$, the locality implies that for any positive integer n, there is positive integer k_n such that

$$(1 - z_1/z_2)^{k_n}\pi_n(a(z_1))\pi_n(b(z_2)) \in \mathcal{E}^{(2)}(W/\hbar^n W).$$

The following is a partial $\hbar$-adic analogue of [Li7, Definition 4.4]:

$$Y_{\mathcal{E}}^{\phi}(a(z), z_0)b(z) = \sum_{n\in\mathbb{Z}} a(z)_n^{\phi} b(z) z_0^{-n-1}$$
$$= \varprojlim_{n>0} (1 - e_0^{z})^{-k_n}\left((1 - z_1/z)^{k_n} a(z_1)b(z)\right)\Big|_{z_1 = ze^{z_0}}.$$

We have the partial $\hbar$-adic analogue of [Li7, Theorem 4.8]:

Theorem 7.4. *Let U be an $\hbar$-adic S_{trig}-local subset of $\mathcal{E}_\hbar(W)$. Then there is a minimal $\hbar$-adically S_{trig}-local subset $\langle U\rangle_\phi \subset \mathcal{E}_\hbar(W)$ containing U and 1_W, such that*

(1) $\langle U\rangle_\phi$ is topologically free and $[\langle U\rangle_\phi] = \langle U\rangle_\phi$.

(2) $\langle U\rangle_\phi$ is $Y_{\mathcal{E}}^{\phi}$ closed, that is, for $a(x), b(x) \in \langle U\rangle_\phi$, $a(x)_n b(x) \in \langle U\rangle_\phi$.

Then $(\langle U\rangle_\phi, Y_{\mathcal{E}}^{\phi}, 1_W)$ carries the structure of an $\hbar$-adic weak quantum vertex algebra and W is a faithful ϕ-coordinated $\langle U\rangle_\phi$-module with $Y_W(a(z), z_0) = a(z_0)$ for $a(z) \in \langle U\rangle_\phi$.

The following result is a partial $\hbar$-adic analogue of [JKLT1, Theorem 2.21]:

Proposition 7.5. *Let W be a topologically free $\mathbb{C}[[\hbar]]$-module, and let $V \subset \mathcal{E}_\hbar(W)$ be an $\hbar$-adic S_{trig}-local subset such that $V = \langle V\rangle_\phi$. Suppose that*

$$\sum_{i\geq 1} a_i(z) \otimes b_i(z) \otimes f_i(z), \quad \sum_{j\geq 1} \alpha_j(z) \otimes \beta_j(z) \otimes g_j(z) \in V\widehat{\otimes}V\widehat{\otimes}\mathbb{C}(z)[[\hbar]],$$

and $\sum_{k\geq 1} \gamma_k(z) \in V,$

such that the following relation holds on W:

$$\sum_{i\geq 1}\iota_{z_1,z_2}f_i(z_1/z_2)a_i(z_1)b_i(z_2)-\sum_{j\geq 1}\iota_{z_2,z_1}g_j(z_1/z_2)\alpha_j(z_2)\beta_j(z_1)$$
$$=\sum_{k\geq 1}\gamma_k(z_2)\frac{1}{k!}\left(z_2\frac{\partial}{\partial z_2}\right)^k\delta\left(\frac{z_2}{z_1}\right).$$

Then we have that

$$\sum_{i\geq 1}\iota_{z_1,z_2}f_i(e^{z_1-z_2})Y_{\mathcal{E}}^{\phi}(a_i(z),z_1)Y_{\mathcal{E}}^{\phi}(b_i(z),z_2)$$
$$-\sum_{j\geq 1}\iota_{z_2,z_1}g_j(e^{z_1-z_2})Y_{\mathcal{E}}^{\phi}(\alpha_j(z),z_2)Y_{\mathcal{E}}^{\phi}(\beta_j(z),z_1)$$
$$=\sum_{k\geq 1}Y_{\mathcal{E}}^{\phi}(\gamma_k(z),z_2)\frac{1}{k!}\frac{\partial^k}{\partial z_2^k}z_1^{-1}\delta\left(\frac{z_2}{z_1}\right).$$

Now we start to determine the category of ϕ-coordinated $V_{\hat{\mathfrak{g}},\hbar}(\ell,0)$-modules.

Definition 7.6. Let $\ell\in\mathbb{C}$. Define $\mathcal{M}_\ell^\phi$ to be the category consisting of topologically free $\mathbb{C}[[\hbar]]$-modules W, equipped with fields $\psi_{i,q}(z),y_{i,q}^\pm(z)\in\mathcal{E}_\hbar(W)$ that satisfy the following relations:

(7.15) $[\psi_{i,q}(z_1),\psi_{j,q}(z_2)]$
$$=[r_ia_{ij}]_{q^{z_2\frac{\partial}{\partial z_2}}}[r\ell]_{q^{z_2\frac{\partial}{\partial z_2}}}\left(\iota_{z_1,z_2}q^{-r\ell z_2\frac{\partial}{\partial z_2}}-\iota_{z_2,z_1}q^{r\ell z_2\frac{\partial}{\partial z_2}}\right)\frac{z_2/z_1}{(1-z_2/z_1)^2},$$

(7.16) $[\psi_{i,q}(z_1),y_{j,q}^\pm(z_2)]$
$$=\pm\, y_{j,q}^\pm(z_2)[r_ia_{ij}]_{q^{z_2\frac{\partial}{\partial z_2}}}\left(\iota_{z_1,z_2}q^{-r\ell z_2\frac{\partial}{\partial z_2}}-\iota_{z_2,z_1}q^{r\ell z_2\frac{\partial}{\partial z_2}}\right)\frac{1+z_2/z_1}{2-2z_2/z_1},$$

(7.17) $(1-z_2/z_1)^{\delta_{ij}}(1-q^{-2r\ell}z_2/z_1)^{\delta_{ij}}$
$$\times\left(y_{i,q}^+(z_1)y_{j,q}^-(z_2)-\iota_{z_2,z_1}g_{ij,q}(z_1/z_2)y_{j,q}^-(z_2)y_{i,q}^+(z_1)\right)=0,$$

(7.18) $\iota_{z_1,z_2}f_{ij,q}(z_1,z_2)y_{i,q}^\pm(z_1)y_{j,q}^\pm(z_2)=C_{ij}\iota_{z_2,z_1}f_{ji,q}(z_2,z_1)y_{j,q}^\pm(z_2)y_{i,q}^\pm(z_1),$

where

$$f_{ij,q}(z_1,z_2)=\iota_{z_1,z_2}(z_1-q_i^{a_{ij}}z_2)(z_1-z_2)^{-\delta_{ij}},\quad g_{ij,q}(z)=\frac{q_i^{a_{ij}}-z}{1-q_i^{a_{ij}}z}.\tag{7.19}$$

For an object $(W,\psi_{i,q}(z),y_{i,q}^\pm(z))$ of $\mathcal{M}_\ell^\phi$, we set

$$U_W=\left\{\psi_{i,q}(z),y_{i,q}^\pm(z)\,\middle|\,i\in I\right\}.\tag{7.20}$$

Then it is immediate from equations (7.15), (7.16), (7.17) and (7.18) that U_W is an $\hbar$-adically S_{trig}-local subset of $\mathcal{E}_\hbar(W)$.

Definition 7.7. Let $\ell \in \mathbb{C}$. Define $\mathcal{R}_\ell^\phi$ to be the full subcategory of $\mathcal{M}_\ell^\phi$ consisting of objects $(W, \psi_{i,q}(z), y_{i,q}^\pm(z))$ such that

$$y_{i,q}^+(z_1)y_{j,q}^-(z_2) - \iota_{z_2,z_1} g_{ij}(z_1/z_2) y_{j,q}^-(z_2) y_{i,q}^+(z_1)$$

$$= \frac{\delta_{ij}}{q_i - q_i^{-1}} \left(\delta\left(\frac{z_1}{z_2}\right) - E(\psi_{i,q}(z))\delta\left(\frac{q^{-2r\ell} z_2}{z_1}\right)\right), \tag{7.21}$$

$$\left(\left(y_{i,q}^\pm(z)\right)_0^\phi\right)^{m_{ij}} y_{j,q}^\pm(z) = 0, \quad \text{if } a_{ij} < 0. \tag{7.22}$$

Proposition 7.8. *Let $(W, \psi_{i,q}(z), y_{i,q}^\pm(z))$ be an object of $\mathcal{R}_\ell^\phi$. Then W becomes a ϕ-coordinated $V_{\hat{\mathfrak{g}},\hbar}(\ell, 0)$-module such that*

$$Y_W^\phi(h_{i,\hbar}, z) = \psi_{i,q}(z), \quad Y_W^\phi(x_{i,\hbar}^\pm, z) = y_{i,q}^\pm(z), \quad i \in I.$$

On the other hand, let (W, Y_W^ϕ) be a ϕ-coordinated $V_{\hat{\mathfrak{g}},\hbar}(\ell, 0)$-module. Then

$$(W, Y_W^\phi(h_{i,\hbar}, z), Y_W^\phi(x_{i,\hbar}^\pm, z))$$

is an object of $\mathcal{R}_\ell^\phi$.

Proof. Let $(W, \psi_{i,q}(z), y_{i,q}^\pm(z))$ be an object of $\mathcal{R}_\ell^\phi$. Then it is an object of $\mathcal{M}_\ell^\phi$. Recall the $\hbar$-adically S_{trig}-local subset U_W (see (7.20) for details). From Theorem 7.4, we get an $\hbar$-adic nonlocal vertex algebra $\langle U_W \rangle_\phi$, and W becomes a ϕ-coordinated $\langle U_W \rangle_\phi$-module with module action $Y_W^\phi(a(z), z_0) = a(z_0)$ for $a(z) \in \langle U_W \rangle_\phi$. From Proposition 7.5 and equations (7.15), (7.16), (7.17), (7.18), we get that $(W, Y_\mathcal{E}^\phi(\psi_{i,q}(z_1), z), Y_\mathcal{E}^\phi(y_{i,q}^\pm(z_1), z))$ is an object of $\mathcal{M}_\tau$ (see Definition 5.1). By using Proposition 5.4, we get an $\hbar$-adic nonlocal vertex algebra homomorphism $\varphi : F_\tau(A, \ell) \to \langle U_W \rangle_\phi$ such that

$$\varphi(h_{i,\hbar}) = \psi_{i,q}(z), \quad \varphi(x_{i,\hbar}^\pm) = y_{i,q}^\pm(z), \quad i \in I.$$

Since $(W, \psi_{i,q}(z), y_{i,q}^\pm(z))$ be an object of $\mathcal{R}_\ell^\phi$. We apply Proposition 7.5 to (7.21) and get that

$$Y_\mathcal{E}^\phi(y_{i,q}^+(z), z_1) Y_\mathcal{E}^\phi(y_{j,q}^-(z), z_2) - \iota_{z_2,z_1} g_{ij,\hbar}(z_1 - z_2) Y_\mathcal{E}^\phi(y_{j,q}^-(z), z_2) Y_\mathcal{E}^\phi(y_{i,q}^+(z), z_1)$$
$$= \delta_{ij}(q_i - q_i^{-1})^{-1} \left(z_1^{-1}\delta\left(\frac{z_2}{z_1}\right) - Y_\mathcal{E}^\phi\left(E(\psi_{i,q}(z)), z_2\right) z_1^{-1}\delta\left(\frac{z_2 - 2r\ell\hbar}{z_1}\right)\right).$$

Let the both hand sides act on 1_W, and take $\mathrm{Sing}_{z_1} \mathrm{Res}_{z_2} z_2^{-1}$, we get that

$$\mathrm{Sing}_{z_1} Y_\mathcal{E}^\phi(y_{i,q}^+(z), z_1) y_{j,q}^- = \delta_{ij}(q_i - q_i^{-1})^{-1}\left(1_W z_1^{-1} - \theta_{i,q}(z)(z_1 + 2r\ell\hbar)^{-1}\right).$$

Combining this with (7.22) and Definition 6.3, we get that φ factor through $V_{\hat{\mathfrak{g}},\hbar}(\ell, 0)$. Therefore, W becomes a ϕ-coordinated $V_{\hat{\mathfrak{g}},\hbar}(\ell, 0)$-module such that $(i \in I)$:

$$Y_W^\phi(h_{i,\hbar}, z_0) = Y_W^\phi(\varphi(h_{i,\hbar}), z_0) = Y_W^\phi(\psi_{i,q}(z), z_0) = \psi_{i,q}(z_0),$$
$$Y_W^\phi(x_{i,\hbar}^\pm, z_0) = Y_W^\phi(\varphi(x_{i,\hbar}^\pm), z_0) = Y_W^\phi(y_{i,q}(z)^\pm, z_0) = y_{i,q}^\pm(z_0).$$

On the other hand, let (W, Y_W^ϕ) be a ϕ-coordinated $V_{\hat{\mathfrak{g}},\hbar}(\ell,0)$-module. From Proposition 7.3 and equations (6.6), (6.7), (6.11), (6.9), (6.31), we have that

$$\begin{aligned}
&[Y_W^\phi(h_{i,\hbar},z_1), Y_W^\phi(h_{j,\hbar},z_2)]\\
&\quad= [r_ia_{ij}]_{q^{z_2\frac{\partial}{\partial z_2}}}[r\ell]_{q^{z_2\frac{\partial}{\partial z_2}}}\left(\iota_{z_1,z_2}q^{-r\ell z_2\frac{\partial}{\partial z_2}} - \iota_{z_2,z_1}q^{r\ell z_2\frac{\partial}{\partial z_2}}\right)\frac{z_2/z_1}{(1-z_2/z_1)^2},\\
&[Y_W^\phi(h_{i,\hbar},z_1), Y_W^\phi(x_{j,\hbar}^\pm,z_2)]\\
&\quad= \pm Y_W^\phi(x_{j,\hbar}^\pm,z_2)[r_ia_{ij}]_{q^{z_2\frac{\partial}{\partial z_2}}}\left(\iota_{z_1,z_2}q^{-r\ell z_2\frac{\partial}{\partial z_2}} - \iota_{z_2,z_1}q^{r\ell z_2\frac{\partial}{\partial z_2}}\right)\frac{1+z_2/z_1}{2-2z_2/z_1},\\
&(1-z_2/z_1)^{\delta_{ij}}(1-q^{-2r\ell}z_2/z_1)^{\delta_{ij}}\\
&\quad\times\left(y_{i,q}^+(z_1)y_{j,q}^-(z_2) - \iota_{z_2,z_1}g_{ij,q}(z_1/z_2)y_{j,q}^-(z_2)y_{i,q}^+(z_1)\right) = 0,\\
&\iota_{z_1,z_2}f_{ij,q}(z_1,z_2)Y_W^\phi(x_{i,\hbar}^\pm,z_1)Y_W^\phi(x_{j,\hbar}^\pm,z_2)\\
&\quad= C_{ij}\iota_{z_2,z_1}f_{ji,q}(z_2,z_1)Y_W^\phi(x_{j,\hbar}^\pm,z_2)Y_W^\phi(x_{i,\hbar}^\pm,z_1),
\end{aligned}$$

and

$$\begin{aligned}
&Y_W^\phi(x_{i,\hbar}^+,z_1)Y_W^\phi(x_{j,\hbar}^-,z_2) - \iota_{z_2,z_1}g_{ij,q}(z_1/z_2)Y_W^\phi(x_{j,\hbar}^-,z_2)Y_W^\phi(x_{i,\hbar}^+,z_1)\\
=&\frac{\delta_{ij}}{q_i-q_i^{-1}}\left(\delta\left(\frac{z_1}{z_2}\right) - Y_W^\phi(E(h_{i,\hbar}),z)\delta\left(\frac{q^{-2r\ell}z_2}{z_1}\right)\right)\\
=&\frac{\delta_{ij}}{q_i-q_i^{-1}}\Bigg(\delta\left(\frac{z_1}{z_2}\right) - \left(\frac{F(r_i+r\ell)}{F(r_i-r\ell)}\right)^{\frac{1}{2}}\\
&\quad\times Y_W^\phi\left(\exp\left(\left(-q^{-r\ell\partial}F(\partial)h_{i,\hbar}\right)_{-1}\right)\mathbf{1},z\right)\delta\left(\frac{q^{-2r\ell}z_2}{z_1}\right)\Bigg)\\
=&\frac{\delta_{ij}}{q_i-q_i^{-1}}\left(\delta\left(\frac{z_1}{z_2}\right) - E\left(Y_W^\phi(h_{i,\hbar},z)\right)\delta\left(\frac{q^{-2r\ell}z_2}{z_1}\right)\right).
\end{aligned}$$

From Definition 6.3, we also have that

$$0 = Y_W^\phi\left(\left(x_{i,\hbar}^\pm\right)_0^{m_{ij}}x_{j,\hbar}^\pm,z\right) = \left(\left(Y_W^\phi(x_{i,\hbar}^\pm,z)\right)_0^\phi\right)^{m_{ij}}Y_W^\phi(x_{j,\hbar}^\pm,z)$$

for $i,j\in I$ with $a_{ij}<0$. Therefore, $(W, Y_W^\phi(h_{i,\hbar},z), Y_W^\phi(x_{i,\hbar}^\pm,z))$ is an object of $\mathcal{R}_\ell^\phi$. □

8. Quantum affinization algebras

Let $\mathcal{U}_\hbar^l(\hat{\mathfrak{g}})$ be the unital associative algebra over $\mathbb{C}[[\hbar]]$ topologically generated by the elements (1.1) subject to relations (Q1)-(Q6). Then $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ is naturally a quotient algebra of $\mathcal{U}_\hbar^l(\hat{\mathfrak{g}})$, and $\mathcal{U}_\hbar^l(\hat{\mathfrak{g}})$ is naturally a quotient algebra of $\mathcal{U}_\hbar^f(\hat{\mathfrak{g}})$. We call a $\mathcal{U}_\hbar^f(\hat{\mathfrak{g}})$-module W a *restricted module* if W is topologically free and

$$\phi^{\pm}_{i,q}(z),\quad x^{\pm}_{i,q}(z)\in\mathcal{E}_\hbar(W).$$

In addition, a $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$-module (resp. $\mathcal{U}^l_\hbar(\hat{\mathfrak{g}})$-module) is said to be *restricted*, if it is restricted as a $\mathcal{U}^f_\hbar(\hat{\mathfrak{g}})$-module. For $i_1,\dots,i_m\in I$, we set

$$\begin{aligned}(8.1)\qquad & x^{\pm}_{i_1,\dots,i_m,q}(z_1,\dots,z_m)\\ &=\left(\prod_{1\le r<s\le m} f^{\pm}_{i_r,i_s,q}(z_r,z_s)\right)x^{\pm}_{i_1,q}(z_1)\cdots x^{\pm}_{i_m,q}(z_m),\end{aligned}$$

where $f^{\pm}_{ij,q}(z_1,z_2)=(z_1-q_i^{\pm a_{ij}}z_2)(z_1-z_2)^{-\delta_{ij}}$ is defined in (7.19) and C_{ij} is defined in (6.34). The following result is proved in [CJKT, Proposition 5.8]:

Lemma 8.1. *Let W be a restricted $\mathcal{U}^l_\hbar(\hat{\mathfrak{g}})$-module. Then for positive integer m, and $i_1,\dots,i_m\in I$,*

$$x^{\pm}_{i_1,\dots,i_m,q}(z_1,\dots,z_m)\in\mathcal{E}^{(m)}_\hbar(W). \tag{8.2}$$

Moreover, for $\sigma\in S_m$, we have

$$\begin{aligned}& x^{\pm}_{i_{\sigma(1)},\dots,i_{\sigma(m)},q}(z_{\sigma(1)},\dots,z_{\sigma(m)})\\ &=\left(\prod_{\substack{1\le r<s\le m\\ \sigma(r)>\sigma(s)}} C_{i_r,i_s}\right)x^{\pm}_{i_1,\dots,i_m,q}(z_1,\dots,z_m).\end{aligned}$$

We need the following special case of [CJKT, Theorem 5.17]:

Proposition 8.2. *A restricted $\mathcal{U}^l_\hbar(\hat{\mathfrak{g}})$-module is a restricted $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$-module if and only if*

$$x^{\pm}_{i,\dots,i,j,q}(q_i^{\pm a_{ij}}z,q_i^{\pm a_{ij}\pm 2}z,\dots,q_i^{\mp a_{ij}}z,z)=0\quad\textit{for } i,j\in I \textit{ with } a_{ij}<0.$$

From the definition of $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$, we immediately get that

Lemma 8.3. *For each $i\in I$, there is a $\mathbb{C}$-algebra homomorphism from $\jmath_i:\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)\to\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ given by $\hbar\mapsto r_i\hbar$, $c\mapsto rc/r_i$, and*

$$h_{1,q}(0)\mapsto\frac{h_{i,q}(0)}{r_i},\quad h_{1,q}(m)\mapsto\frac{h_{i,q}(m)}{[r_i]_q},\quad x^{\pm}_{i,q}(n)\mapsto x^{\pm}_{i,q}(n). \tag{8.3}$$

Let d be the derivation of $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ defined by

$$[d,\phi^{\pm}_{i,q}(z)]=-z\frac{\partial}{\partial z}\phi^{\pm}_{i,q}(z),\quad [d,x^{\pm}_{i,q}(z)]=-z\frac{\partial}{\partial z}x^{\pm}_{i,q}(z)\quad\text{for } i\in I, \tag{8.4}$$

and let $\mathfrak{h}:=\oplus_{i\in I}\mathbb{C}h_{i,q}(0)\oplus\mathbb{C}c\oplus\mathbb{C}d$. For a $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$-module W and $\lambda\in\mathfrak{h}^*$, we denote

$$W_\lambda=\{v\in W\mid h.v=\lambda(h)v,\ h\in\mathfrak{h}\}.$$

A $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$-module W is called a *weight module* if $W=\oplus_{\lambda\in\mathfrak{h}^*}W_\lambda$. And a weight module W for $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ is said to be *integrable*, if $x_{i,q}^\pm(m)$ acts locally nilpotently for any $i\in I$, $m\in\mathbb{Z}$. It is immediate to see that W is integrable if and only if W is integrable as a $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)$-module through $\jmath_i$ for all $i\in I$ (see Lemma 8.3). Then we get from [DM, Theorem 7, Theorem 8] that

Proposition 8.4. *A nontrivial restricted weight $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$-module W of level ℓ is integrable if and only if*

$$\ell\in\mathbb{Z}_+,\quad \textit{and}\quad x^\pm_{i,\dots,i,q}(q^{\pm 2r\ell}z,q^{\pm 2r\ell\mp 2r_i}z,\dots,q^{\pm 2r_i}z,z)=0\quad \textit{for } i\in I.$$

We need another presentation of restricted $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$-modules. A straightforward calculation shows that

Lemma 8.5. *Let $\ell\in\mathbb{C}$ and let W be a restricted $\mathcal{U}_\hbar^f(\hat{\mathfrak{g}})$-module of level ℓ. Define $\hat{x}^+_{i,q}(z)=x^+_{i,q}(z)$ and*

$$\hat\phi_{i,q}(z)=F\left(z\frac{\partial}{\partial z}\right)^{-1}\log\frac{\phi^+_{i,q}(zq^{\frac12 r\ell})}{\phi^-_{i,q}(zq^{-\frac12 r\ell})},\quad \hat{x}^-_{i,q}(z)=x^-_{i,q}(zq^{-r\ell})\phi^+_{i,q}(zq^{-\frac12 r\ell})^{-1},$$

where $F(z)$ is defined in (6.15). Then $(W,\hat\phi_{i,q}(z),\hat{x}^\pm_{i,q}(z))$ is an object of $\mathcal{M}_\ell^\phi$.

On the other hand, let $(W,\psi_{i,q}(z),y^\pm_{i,q}(z))$ be an object of $\mathcal{M}_\ell^\phi$. Define $\check{y}^+_{i,q}(z)=y^+_{i,q}(z)$ and

$$\psi^\pm_{i,q}(z)=\sum_{\pm n>0}\psi_{i,q}(n)z^{-n}+\frac12\psi_{i,q}(0),\tag{8.5}$$

$$\textit{where } \psi_{i,q}(z)=\sum_{n\in\mathbb{Z}}\psi_{i,q}(n)z^{-n},$$

$$\check\psi^\pm_{i,q}(z)=\exp\left(\pm F\left(z\frac{\partial}{\partial z}\right)\psi^\pm_{i,q}(zq^{\mp\frac12 r\ell})\right),\tag{8.6}$$

$$\check{y}^-_{i,q}(z)=y^-_{i,q}(zq^{r\ell})\check\psi^+_{i,q}(zq^{\frac12 r\ell}).\tag{8.7}$$

Then W becomes a restricted $\mathcal{U}_\hbar^f(\hat{\mathfrak{g}})$-module of level ℓ with module action

$$\phi^\pm_{i,q}(z)=\check\psi^\pm_{i,q}(z),\quad x^\pm_{i,q}(z)=\check{y}^\pm_{i,q}(z),\quad i\in I.$$

We need the following technical result given in [JKLT3].

Lemma 8.6. *Let W be a topologically free $\mathbb{C}[[\hbar]]$-module, and*

$$\alpha(z)\in\mathrm{Hom}(W,W\widehat\otimes\mathbb{C}[z,z^{-1}][[\hbar]]),\quad \beta(z)\in\mathcal{E}_\hbar(W),$$
$$\gamma(z)\in\mathrm{Hom}(W,W\widehat\otimes\mathbb{C}(z)[[\hbar]]),$$

be such that

$$[\alpha(z_1),\alpha(z_2)]=0=[\beta(z_1),\beta(z_2)]=[\alpha(z_1),\gamma(z_2)]=[\beta(z_1),\gamma(z_2)],$$
$$[\alpha(z_1),\beta(z_2)]=\iota_{z_1,z_2}\gamma(z_2/z_1).$$

Suppose that

$$\exp\left(A(z)\right)=\sum_{n\geq 0}\frac{A(z)^n}{n!}$$

is well defined in $\mathrm{Hom}(W,W[[z,z^{-1}]])$ *for* $A(z)=\alpha(z),\beta(z),\gamma(z)$ *or* $\alpha(z)+\beta(z)$. *Then we have that*

$$\begin{aligned}&\exp\left((\alpha(z)+\beta(z))_{-1}^{\phi}\right)1_W\\=&\exp\left(\beta(z)\right)\exp\left(\alpha(z)\right)\exp\left(\mathrm{Res}_z\, z^{-1}\gamma(e^{-z})/2\right).\end{aligned}$$

Let $U_W=\left\{\hat{\phi}_{i,q}(z),\hat{x}_{i,q}^{\pm}(z)\,\middle|\, i\in I\right\}$. From Lemma 8.5 , we have that U_W is a $\hbar$-adically S_{trig}-subset. Then by using Theorem 7.4, we get an $\hbar$-adic nonlocal vertex algebra $\langle U_W\rangle_\phi$.

Lemma 8.7. *For* $i\in I$, *we have that*

$$E\left(\hat{\phi}_{i,q}(z)\right)=\phi_{i,q}^{-}(zq^{-\frac{3}{2}r\ell})\phi_{i,q}^{+}(zq^{-\frac{1}{2}r\ell})^{-1}.$$

Proof. Recall the definition of $\hat{\phi}_{i,q}^{\pm}(z)$ in (8.5). From the equation (7.15), we get that

$$[\hat{\phi}_{i,q}^{+}(z_1),\hat{\phi}_{i,q}^{-}(z_2)]=[r_ia_{ii}]_{q^{z_2\frac{\partial}{\partial z_2}}}[r\ell]_{q^{z_2\frac{\partial}{\partial z_2}}}\iota_{z_1,z_2}q^{-r\ell z_2\frac{\partial}{\partial z_2}}\frac{z_2/z_1}{(1-z_2/z_1)^2}. \tag{8.8}$$

Let

$$\widetilde{\phi}_{i,q}^{\pm}(z)=-q^{-r\ell z\frac{\partial}{\partial z}}F\left(z\frac{\partial}{\partial z}\right)\hat{\phi}_{i,q}^{\pm}(z).$$

We note from (8.6) and (8.7) that

$$\exp\left(\widetilde{\phi}_{i,q}^{+}(z)\right)=\phi_{i,q}^{\pm}(zq^{-r\ell\pm\frac{1}{2}r\ell})^{\mp 1}. \tag{8.9}$$

By using the relation (8.8), we have that

$$\begin{aligned}&-z_1\frac{\partial}{\partial z_1}z_2\frac{\partial}{\partial z_2}[\widetilde{\phi}_{i,q}^{+}(z_1),\widetilde{\phi}_{i,q}^{-}(z_2)]\\
=&\left(z_2\frac{\partial}{\partial z_2}\right)^2F\left(z_2\frac{\partial}{\partial z_2}\right)^2[r_ia_{ii}]_{q^{z_2\frac{\partial}{\partial z_2}}}[r\ell]_{q^{z_2\frac{\partial}{\partial z_2}}}\iota_{z_1,z_2}q^{-r\ell z_2\frac{\partial}{\partial z_2}}\frac{z_2/z_1}{(1-z_2/z_1)^2}\\
=&\left(q^{z_2\frac{\partial}{\partial z_2}}-q^{-z_2\frac{\partial}{\partial z_2}}\right)^2[r_ia_{ii}]_{q^{z_2\frac{\partial}{\partial z_2}}}[r\ell]_{q^{z_2\frac{\partial}{\partial z_2}}}\iota_{z_1,z_2}q^{-r\ell z_2\frac{\partial}{\partial z_2}}\frac{z_2/z_1}{(1-z_2/z_1)^2}\\
=&\left(q^{r_ia_{ii}z_2\frac{\partial}{\partial z_2}}-q^{-r_ia_{ii}z_2\frac{\partial}{\partial z_2}}\right)\left(1-q^{-2r\ell z_2\frac{\partial}{\partial z_2}}\right)\iota_{z_1,z_2}\frac{z_2/z_1}{(1-z_2/z_1)^2}\\
=&\left(q^{r_ia_{ii}z_2\frac{\partial}{\partial z_2}}-q^{-r_ia_{ii}z_2\frac{\partial}{\partial z_2}}\right)\left(1-q^{-2r\ell z_2\frac{\partial}{\partial z_2}}\right)\iota_{z_1,z_2}z_1\frac{\partial}{\partial z_1}z_2\frac{\partial}{\partial z_2}\log\left(1-z_2/z_1\right).\end{aligned}$$

Notice that $\mathrm{Res}_{z_2}\, z_2^{-1}\circ\left(z_2\frac{\partial}{\partial z_2}\right)=0$. Then we have that

$$\mathrm{Res}_{z_2}\, z_2^{-1}[\widetilde{\phi}_{i,q}^{+}(z_1),\widetilde{\phi}_{i,q}^{-}(z_2)]$$

$$=q^{-r\ell z_1\frac{\partial}{\partial z_1}}F\left(z_1\frac{\partial}{\partial z_1}\right)\mathrm{Res}_{z_2}\,z_2^{-1}[\hat{\phi}_{i,q}^{+}(z_1),\hat{\phi}_{i,q}^{-}(z_2)]$$
$$=q^{-r\ell z_1\frac{\partial}{\partial z_1}}F\left(z_1\frac{\partial}{\partial z_1}\right)\mathrm{Res}_{z_2}\,z_2^{-1}[r_ia_{ii}]_{q^{z_2\frac{\partial}{\partial z_2}}}[r\ell]_{q^{z_2\frac{\partial}{\partial z_2}}}\iota_{z_1,z_2}q^{-r\ell z_2\frac{\partial}{\partial z_2}}\frac{z_2/z_1}{(1-z_2/z_1)^2}$$
$$=q^{-r\ell z_1\frac{\partial}{\partial z_1}}F\left(z_1\frac{\partial}{\partial z_1}\right)\mathrm{Res}_{z_2}\,z_2^{-1}r_ia_{ii}r\ell\iota_{z_1,z_2}\frac{z_2/z_1}{(1-z_2/z_1)^2}=0.$$

Hence, we get that

$$[\widetilde{\phi}_{i,q}^{+}(z_1),\widetilde{\phi}_{i,q}^{-}(z_2)]$$
$$=\left(q^{r_ia_{ii}z_2\frac{\partial}{\partial z_2}}-q^{-r_ia_{ii}z_2\frac{\partial}{\partial z_2}}\right)\left(q^{-2r\ell z_2\frac{\partial}{\partial z_2}}-1\right)\iota_{z_1,z_2}\log\left(1-z_2/z_1\right)$$
$$=\left(q^{r_ia_{ii}z_2/z_1\frac{\partial}{\partial z_2/z_1}}-q^{-r_ia_{ii}z_2/z_1\frac{\partial}{\partial z_2/z_1}}\right)\left(q^{-2r\ell z_2/z_1\frac{\partial}{\partial z_2/z_1}}-1\right)\iota_{z_1,z_2}\log\left(1-z_2/z_1\right)$$
$$=\iota_{z_1,z_2}\gamma(z_2/z_1),$$

where

$$\gamma(z)=\left(q^{r_ia_{ii}z\frac{\partial}{\partial z}}-q^{-r_ia_{ii}z\frac{\partial}{\partial z}}\right)\left(q^{-2r\ell z\frac{\partial}{\partial z}}-1\right)\log\left(1-z\right).$$

Since $\mathrm{Res}_z\,z^{-1}\frac{\partial^2}{\partial z^2}\log z=0=\mathrm{Res}_z\,z^{-1}\frac{\partial^2}{\partial z^2}z=\mathrm{Res}_z\,z^{-1}\frac{\partial^2}{\partial z^2}1$, we get that

$$\mathrm{Res}_z\,z^{-1}\gamma(e^{-z})=\mathrm{Res}_z\,z^{-1}\left(q^{-2r_i\frac{\partial}{\partial z}}-q^{2r_i\frac{\partial}{\partial z}}\right)\left(q^{2r\ell\frac{\partial}{\partial z}}-1\right)\log\left(1-e^{-z}\right)$$
$$=\mathrm{Res}_z\,z^{-1}\left(q^{-2r_i\frac{\partial}{\partial z}}-q^{2r_i\frac{\partial}{\partial z}}\right)\left(q^{2r\ell\frac{\partial}{\partial z}}-1\right)\log\frac{e^{\frac12 z}-e^{-\frac12 z}}{z/2}$$
$$=\mathrm{Res}_z\log\frac{e^{\frac12 z}-e^{-\frac12 z}}{z/2}\left(q^{2r_i\frac{\partial}{\partial z}}-q^{-2r_i\frac{\partial}{\partial z}}\right)\left(q^{-2r\ell\frac{\partial}{\partial z}}-1\right)z^{-1}$$
$$=\mathrm{Res}_z\left(\frac{1}{z+2r_i\hbar-2r\ell\hbar}-\frac{1}{z-2r_i\hbar-2r\ell\hbar}-\frac{1}{z+2r_i\hbar}+\frac{1}{z-2r_i\hbar}\right)\log\frac{e^{\frac12 z}-e^{-\frac12 z}}{z/2}$$
$$=\log\frac{F(r_i)F(-r_i+r\ell)}{F(-r_i)F(r_i+r\ell)}=\log\frac{F(r_i-r\ell)}{F(r_i+r\ell)}.$$

By using Lemma 8.6, we get that

$$E\left(\hat{\phi}_{i,q}(z)\right)=\left(\frac{F(r_i+r\ell)}{F(r_i-r\ell)}\right)^{\frac12}\exp\left(\left(\widetilde{\phi}_{i,q}^{+}(z)+\widetilde{\phi}_{i,q}^{-}(z)\right)_{-1}^{\phi}\right)1_W$$
$$=\exp\left(\widetilde{\phi}_{i,q}^{-}(z)\right)\exp\left(\widetilde{\phi}_{i,q}^{+}(z)\right)=\phi_{i,q}^{-}(zq^{-\frac32 r\ell})\phi_{i,q}^{+}(zq^{-\frac12 r\ell})^{-1},$$

where the last equation follows from (8.9). We complete the proof. □

For any positive integer m and $i_1,\dots,i_m\in I$, we define

$$\hat{x}_{i_1,\dots,i_m,q}^{\pm}(z_1,\dots,z_m)\tag{8.10}$$
$$=\left(\prod_{1\le a<b\le m}f_{i_a,i_b,q}^{+}(z_a,z_b)\right)\hat{x}_{i_1,q}^{\pm}(z_1)\cdots\hat{x}_{i_m,q}^{\pm}(z_m).\tag{8.11}$$

From the relation (7.18), we have that

$$\hat{x}^{\pm}_{i_1,\dots,i_m,q}(z_1,\dots,z_m)\in\mathcal{E}^{(m)}_{\hbar}(W).$$

Then from a similar argument to the proof of Lemma 6.12, we get the following two results.

Lemma 8.8. *For $i,j\in I$ with $a_{ij}<0$, we have that*

$$\left(\left(\hat{x}^{\pm}_{i,q}(z)\right)^{\phi}_{0}\right)^{m_{ij}}\hat{x}^{\pm}_{j,q}(z)=c\hat{x}^{\pm}_{i,\dots,i,j,q}(q_i^{a_{ij}}z,q_i^{a_{ij}+2}z,\dots,q_i^{-a_{ij}}z,z)$$

for some invertible element $c\in\mathbb{C}[[\hbar]]$.

Lemma 8.9. *Suppose that $\ell\in\mathbb{Z}_+$. For any $i\in I$, we have that*

$$\left(\left(\hat{x}^{\pm}_{i,q}(z)\right)^{\phi}_{-1}\right)^{r\ell/r_i}\hat{x}^{\pm}_{i,q}(z)=c\hat{x}^{\pm}_{i,\dots,i,q}(q^{2r\ell}z,q^{2r\ell-2r_i}z,\dots,q^{2r_i}z,z)$$

for some invertible element $c\in\mathbb{C}[[\hbar]]$.

Then we have that

Proposition 8.10. *Let W be a restricted $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$-module of level ℓ. Then $(W,\hat{\phi}_{i,q}(z),\hat{x}^{\pm}_{i,q}(z))$ is an object of $\mathcal{R}^{\phi}_{\ell}$. On the other hand, let $(W,\psi_{i,q}(z),y^{\pm}_{i,q}(z))$ be an object of $\mathcal{R}^{\phi}_{\ell}$. Then W is a restricted $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$-module of level ℓ with the module action defined by*

$$\phi^{\pm}_{i,q}(z)=\check{\psi}^{\pm}_{i,q}(z),\quad x^{\pm}_{i,q}(z)=\check{y}^{\pm}_{i,q}(z)\quad \textit{for } i\in I.$$

Proof. From Lemma 8.5, we get that W is a restricted $\mathcal{U}^{f}_{\hbar}(\hat{\mathfrak{g}})$-module of level ℓ if and only if $(W,\hat{\phi}_{i,q}(z),\hat{x}^{\pm}_{i,q}(z))$ is an object of $\mathcal{M}^{\phi}_{\ell}$. Then by a straightforward calculation, we get that W is a restricted $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$-module of level ℓ if and only if

$$\begin{aligned}&\hat{x}^{+}_{i,q}(z_1)\hat{x}^{-}_{j,q}(z_2)-g_{ji,q}(z_1/z_2)\hat{x}^{-}_{j,q}(z_2)\hat{x}^{+}_{i,q}(z_1)\\&\quad=\frac{\delta_{ij}}{q_i-q_i^{-1}}\left(\delta\left(\frac{z_2}{z_1}\right)-\phi^{-}_{i,q}(zq^{-\frac{3}{2}r\ell})\phi^{+}_{i,q}(zq^{-\frac{1}{2}r\ell})^{-1}\delta\left(\frac{z_2q^{-2r\ell}}{z_1}\right)\right),\\&\hat{x}^{\pm}_{i,\dots,i,j,q}(q_i^{a_{ij}}z,q_i^{a_{ij}+2}z,\dots,q_i^{-a_{ij}}z,z)=0,\quad \text{if } a_{ij}<0.\end{aligned}$$

Therefore, the proposition follows immediate from Lemmas 8.7 and 8.8. □

Proposition 8.11. *Let $\ell\in\mathbb{Z}_+$, and let W be a restricted weight $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$-module of level ℓ. Then W is integrable if and only if*

$$\left(\left(\hat{x}^{\pm}_{i,q}(z)\right)^{\phi}_{-1}\right)^{r\ell/r_i}\hat{x}^{\pm}_{i,q}(z)=0,\quad i\in I. \tag{8.12}$$

Proof. From Proposition 8.4 and the definition of $\hat{x}^{\pm}_{i,q}(z)$ (see Lemma 8.5), we have that W is integrable if and only if

$$\hat{x}^{\pm}_{i,\dots,i,q}(q^{2r\ell}z,q^{2r\ell-2r_i}z,\dots,q^{2r_i}z,z)=0\quad\text{for } i\in I.$$

Then the proposition follows from Lemma 8.9 and Proposition 8.10. □

Combining Propositions 7.8 and 8.10, we get that

Theorem 8.12. *Let $\ell \in \mathbb{C}$. Then the category of restricted $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$-modules of level ℓ is isomorphic to the category of ϕ-coordinated $V_{\hat{\mathfrak{g}},\hbar}(\ell,0)$-modules. To be more precise, let W be a restricted $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$-module of level ℓ. Then W becomes a ϕ-coordinated $V_{\hat{\mathfrak{g}},\hbar}(\ell,0)$-module such that*

$$Y_W^\phi(h_{i,\hbar},z)=\hat{\phi}_{i,q}(z),\quad Y_W^\phi(x_{i,\hbar}^\pm,z)=\hat{x}_{i,q}^\pm(z),\quad i\in I.$$

On the other hand, let (W,Y_W^ϕ) be a ϕ-coordinated $V_{\hat{\mathfrak{g}},\hbar}(\ell,0)$-module. Then W becomes a restricted $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$-module of level ℓ such that

$$\hat{\phi}_{i,q}(z)=Y_W^\phi(h_{i,\hbar},z),\quad \hat{x}_{i,q}^\pm(z)=Y_W^\phi(x_{i,\hbar}^\pm,z),\quad i\in I.$$

A ϕ-coordinated $V_{\hat{\mathfrak{g}},\hbar}(\ell,0)$-module (W,Y_W^ϕ) is called a *weight module* if $(h_{i,\hbar})_0^\phi$ acts semi-simply for all $i\in I$, and there exists a $\mathbb{C}[[\hbar]]$-module map $\partial: W\to W$, such that

$$[\partial,Y_W^\phi(u,z)]=z\frac{\partial}{\partial z}Y_W^\phi(u,z)\quad\text{for } u\in V_{\hat{\mathfrak{g}},\hbar}(\ell,0). \tag{8.13}$$

A ϕ-coordinated $L_{\hat{\mathfrak{g}},\hbar}(\ell,0)$ is called a *weight module* if it is a weight module viewed as a ϕ-coordinated $V_{\hat{\mathfrak{g}},\hbar}(\ell,0)$-module. Combining Proposition 8.11 with Theorem 8.12, we get that

Theorem 8.13. *Let $\ell\in\mathbb{Z}_+$. Then the category isomorphism obtained in Theorem 8.12 induces a category isomorphism between the category of restricted integrable $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$-modules of level ℓ and the category of ϕ-coordinated weight modules of $L_{\hat{\mathfrak{g}},\hbar}(\ell,0)$.*

Theorem 8.14. *Suppose that the GCM A is of finite type. Let $\ell\in\mathbb{Z}_+$. Then there is a vertex algebra isomorphism $L_{\hat{\mathfrak{g}}}(\ell,0)\to L_{\hat{\mathfrak{g}},\hbar}(\ell,0)/\hbar L_{\hat{\mathfrak{g}},\hbar}(\ell,0)$ defined by*

$$h_i\mapsto h_{i,\hbar},\quad x_i^\pm\mapsto x_{i,\hbar}^\pm,\quad i\in I.$$

Proof. Recall from Proposition 6.5 that there is a surjective vertex algebra homomorphism $\varphi: L_{\hat{\mathfrak{g}}}(\ell,0)\to L_{\hat{\mathfrak{g}},\hbar}(\ell,0)/\hbar L_{\hat{\mathfrak{g}},\hbar}(\ell,0)$ determined by $h_i\mapsto h_{i,\hbar}$ and $x_i^\pm\mapsto x_{i,\hbar}^\pm$ for $i\in I$. It is straightforward to verify that both $L_{\hat{\mathfrak{g}}}(\ell,0)$ and $L_{\hat{\mathfrak{g}},\hbar}(\ell,0)/\hbar L_{\hat{\mathfrak{g}},\hbar}(\ell,0)$ are $\hat{\mathfrak{g}}$-modules such that

$$\begin{aligned}&a_i(z).u=Y(a_i,z)u \text{ for } u\in L_{\hat{\mathfrak{g}}}(\ell,0)\quad\text{and}\\&a_i(z).v=Y_\tau(a_{i,\hbar},z)v \text{ for } v\in L_{\hat{\mathfrak{g}},\hbar}(\ell,0)/\hbar L_{\hat{\mathfrak{g}},\hbar}(\ell,0),\end{aligned}$$

where $a=h$ or $x^\pm$. Then φ is also a surjective $\hat{\mathfrak{g}}$-module homomorphism. Since the GCM A is of finite type, we get from Remark 2.7 that $L_{\hat{\mathfrak{g}}}(\ell,0)$ is a simple $\hat{\mathfrak{g}}$-module. Then φ is either an isomorphism or 0.

Suppose that $\varphi=0$. Since $L_{\hat{\mathfrak{g}},\hbar}(\ell,0)$ is topologically free, we get that $L_{\hat{\mathfrak{g}},\hbar}(\ell,0)=0$. Then $L_{\hat{\mathfrak{g}},\hbar}(\ell,0)$ has only trivial ϕ-coordinated modules. From Theorem 8.13, we get that the $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ has only trivial restricted integrable modules of level ℓ, which contradict to [Lu, Section 4.13]. Therefore, φ must be an isomorphism. □

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Key Laboratory of Computing and Stochastic Mathematics (Ministry of Education), School of Mathematics and Statistics, Hunan Normal University, Changsha, China 410081

Email address: `kongmath@hunnu.edu.cn`