

Green's Functions and Existence of Solutions of Nonlinear Fractional Implicit Difference Equations with Dirichlet Boundary Conditions

Alberto Cabada^{a,b}, Nikolay D. Dimitrov^c and Jagan Mohan Jonnalagadda^d

^aCITMAga, 15782 Santiago de Compostela, Galicia, Spain.

^b Departamento de Estatística, Análise Matemática e Optimización,
Facultade de Matemáticas, Universidade de Santiago de Compostela,
Galicia, Spain, alberto.cabada@usc.es

^cDepartment of Mathematics, University of Ruse, 7017 Ruse,
Bulgaria; ndimitrov@uni-ruse.bg

^d Department of Mathematics, Birla Institute of Technology
and Science Pilani, Hyderabad - 500078, Telangana, India;
j.jaganmohan@hotmail.com

December 20, 2022

Abstract

This article is devoted to deduce the expression of the Green's function related to a general constant coefficients fractional difference equation coupled to Dirichlet conditions. In this case, due to the points where some of the fractional operators is applied, we are in presence of a implicit fractional difference equation. Such property makes it more complicated to calculate and manage the expression of the Green's function. Such expression, on the contrary to the explicit case where it follows from finite sums, is deduced from series of infinity terms. Such expression will be deduced from the Laplace transform on the time scales of the integers. Finally, we prove two existence results for nonlinear problems, via suitable fixed point theorems.

Key Words: Fractional difference, Dirichlet conditions, Green's function, existence of solutions.

AMS Classification: 39A12, 39A70.

1 Introduction and Preliminaries

Considering integrals and derivatives of arbitrary orders allows modeling many real phenomena in which the value that the solution takes at a given instant depends on the value of the solution in all the previous moments of the process. Thus, fractional calculus became very useful in several fields as viscoelasticity, neurology and control theory [14, 15, 16, 18, 19, 21]. During the last decades, a lot of authors studied fractional difference equations and there has been a progress made in developing the basic theory in this field. We refer to the reader the monographs [12, 17] for more details. We use the standard notation $\mathbb{N}_a = \{a, a + 1, a + 2, \dots\}$ for $a \in \mathbb{R}$, and $[c, c + n_0]_{\mathbb{N}_c} = [c, c + n_0] \cap \mathbb{N}_c$, for $c \in \mathbb{R}$ and $n_0 \in \mathbb{N}_1$.

In [5] Atici and Eloe proved that for all $(t, s) \in [v-2, v+b+1]_{\mathbb{N}_{v-2}} \times [0, b+1]_{\mathbb{N}_0}$, the following function

$$G_0(t, s) = \frac{1}{\Gamma(v)} \begin{cases} \frac{t^{(v-1)}(v+b-s)^{(v-1)}}{(v+b+1)^{(v-1)}} - (t-s-1)^{(v-1)}, & s < t-v+1, \\ \frac{t^{(v-1)}(v+b-s)^{(v-1)}}{(v+b+1)^{(v-1)}}, & t-v+1 \leq s, \end{cases}$$

is the related Green function to the Dirichlet problem

$$\begin{aligned} -\Delta^v y(t) &= h(t+v-1), \quad t \in [0, b+1]_{\mathbb{N}_0}, \\ y(v-2) &= y(v+b+1) = 0, \end{aligned}$$

with $v \in \mathbb{R}$, $1 < v < 2$ and $b \in \mathbb{N}$. Moreover, they proved that $G_0(t, s) > 0$ for all $(t, s) \in [v-1, v+b]_{\mathbb{N}_{v-1}} \times [0, b+1]_{\mathbb{N}_0}$.

Using previous expression and constructing Green's function as a series of functions, in [8], by using the spectral theory is ensured, for a suitable range of values of the nonconstant function $a(t)$, the positiveness of the Green's function related to the following Dirichlet problem

$$\begin{aligned} -\Delta^v y(t) + a(t+v-1)y(t+v-1) &= h(t+v-1), \\ y(v-2) &= y(v+b+1) = 0, \end{aligned}$$

for $t \in [0, b+1]_{\mathbb{N}_0}$, where $v \in \mathbb{R}$ with $1 < v < 2$ and $b \in \mathbb{N}$, $b \geq 5$.

A similar approach has been done in [7] for the following problem with mixed conditions:

$$\begin{aligned} -\Delta^v y(t) + a(t+v-1)y(t+v-1) &= h(t+v-1), \\ y(v-2) &= \Delta^\beta y(v+b+1-\beta) = 0, \end{aligned}$$

with $1 < v \leq 2$ and $0 \leq \beta \leq 1$.

Using another approach in [2, 3, 4] the general expression of several linear n -th order initial value problems is obtained. They use $R_0(f(t))(s)$ the Laplace transform on the time scale of integers [6, 10], which is defined by the following expression:

$$R_{t_0}(f(t))(s) = \sum_{t=t_0}^{\infty} \left(\frac{1}{s+1} \right)^{t+1} f(t).$$

Recently, in [9] the authors considered the problem

$$-\Delta^v y(t) + \alpha \Delta^\mu y(t + v - \mu - 1) = h(t + v - 1), \quad (1)$$

$$y(v - 2) = y(v + b + 1) = 0, \quad (2)$$

for $t \in I \equiv [0, b + 1]_{\mathbb{N}_0}$, where $\mu, v \in \mathbb{R}$ such that $0 < \mu < 1$ and $1 < v < 2$; Δ^v and Δ^μ are the standard v -th and μ -th order Riemann–Liouville fractional difference operators, respectively; α is a real constant and $h : I \rightarrow \mathbb{R}$.

By using the Laplace transform $R_0(f(t))(s)$ they obtained the general expression of equation (1) and deduced the explicit expression of the Green's function related to problem (1)–(2). It was proven that such Green's function has some symmetric properties and is positive on $[v - 1, v + b]_{\mathbb{N}_{v-1}} \times [0, b + 1]_{\mathbb{N}_0}$ for all $\alpha \geq 0$ and $v - \mu - 1 > 0$, which improved the results given in [5]. Moreover, the authors deduced some strong positiveness conditions on the Green's function that allow them to construct suitable cones where to deduce the existence of solutions of related nonlinear problems. We point out that, in such case, the fact that the fractional operator Δ^μ is defined on the points $t + v - \mu - 1$ gives us an explicit equation. Such property gives us the expression of the Green's function as a combination of finite sums.

The aim of this paper is to continue our work in this direction as we consider the following equation

$$-\Delta^v y(t) + \alpha \Delta^\mu y(t + v - \mu) = h(t + v - 1), \quad t \in I \equiv \{0, 1, \dots, b + 1\}, \quad (3)$$

coupled to the boundary conditions (2).

Here $\mu, v \in \mathbb{R}$ such that $0 < \mu < 1$ and $1 < v < 2$; Δ^v and Δ^μ are the standard v -th and μ -th order Riemann–Liouville fractional difference operators, respectively; α is a constant and $h : I \rightarrow \mathbb{R}$. We point out that even if we use Laplace transform $R_0(f(t))(s)$ to equation (3), we deduce that the sums are not finite as ones given in [9]. As a result, we study the convergence of the series and we apply some fixed point to deduce some existence results for a related nonlinear problem. We remark that problem (3) coupled to the Dirichlet conditions (2) has been studied in [13] for the particular case of $\alpha = 0$.

The paper is organized as follows: After an introduction where we compile the main concepts and properties that we will use along the paper, we obtain, in Section 2, the expression of the Green's function related to problem (1)–(2) for $|\alpha| < 1$ (which is the condition that characterizes the convergence of the used series). In Section 3 we deduce two existence results for nonlinear problems. Such existence results follow from the expression of the Green's function by constructing an operator whose fixed points coincide with the solutions of the problems that we are looking for. We finalize the paper with two examples that point out the applicability of the obtained existence results.

First we recall some basic definitions and lemmas, which will be used till the end of this work.

Definition 1 We define $t^{(v)} = \frac{\Gamma(t+1)}{\Gamma(t+1-v)}$, for any t and v for which the right-hand side is well defined. We also appeal to the convention that if $t + 1 - v$ is a pole of the Gamma function and $t + 1$ is not a pole, then $t^{(v)} = 0$.

Definition 2 The v -th fractional sum of a function f , for $v > 0$ and $t \in \mathbb{N}_{a+v}$, is defined as

$$\Delta^{-v} f(t) = \Delta^{-v} f(t; a) := \frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} (t-s-1)^{(v-1)} f(s).$$

We also define the v -th fractional difference for $v > 0$ by $\Delta^v f(t) := \Delta^N \Delta^{v-N} f(t)$, where $t \in \mathbb{N}_{a+v}$ and $N \in \mathbb{N}$ is chosen so that $0 \leq N-1 < v \leq N$.

Lemma 3 ([11, Lemma 2.3]) Let t and v be any numbers for which $t^{(v)}$ and $t^{(v-1)}$ be defined. Then $\Delta t^{(v)} = vt^{(v-1)}$.

Lemma 4 ([4, Lemma 2.1])

$$R_{v-1} \left(t^{(v-1)} \right) (s) = \frac{\Gamma(v)}{s^v}. \quad (4)$$

Lemma 5 ([4, Lemma 2.2]) If $\mu > 0$ and $m-1 < \mu < m$, where m denotes a positive integer and f is defined on $\mathbb{N}_{\mu-m}$, then

$$R_0 \left(\Delta_{\mu-m}^{\mu} f \right) (s) = s^{\mu} R_{\mu-m} (f) (s) - \sum_{k=0}^{m-1} s^{m-k-1} \left(\Delta^k \Delta_{\mu-m}^{-(\mu-m)} f \right) \Big|_{t=0}. \quad (5)$$

Lemma 6 ([4, Lemma 2.4])

$$R_{v-2} (f *_{v-2} g) (s) = (s+1)^{v-1} R_{v-2} (f) (s) R_{v-2} (g) (s), \quad (6)$$

where

$$f *_{v-2} g (t) = \sum_{s=v-2}^t f(t-s+v-2) g(s) \quad (7)$$

is the convolution product of two functions defined on \mathbb{N}_{v-2} .

Definition 7 The two parameter delta discrete Mittag-Leffler function is defined by

$$e_{\alpha, \beta}(\lambda, t-a) = \sum_{k=0}^{\infty} \lambda^k \frac{(t-a+k\alpha+\beta-1)^{(k\alpha+\beta-1)}}{\Gamma(k\alpha+\beta)},$$

for $\alpha > 0$, $\beta \in \mathbb{R}$ and $t \in \mathbb{N}_a$.

Observe that,

$$e_{\alpha, \beta}(\lambda, -1) = \sum_{k=0}^{\infty} \lambda^k \frac{(k\alpha+\beta-2)^{(k\alpha+\beta-1)}}{\Gamma(k\alpha+\beta)} = 0.$$

Clearly, if $|\lambda| < 1$, then $e_{\alpha, \beta}(\lambda, 0) = \frac{1}{1-\lambda}$. Also,

$$e_{\alpha, \beta}(\lambda, 1) = \sum_{k=0}^{\infty} \lambda^k \frac{(k\alpha+\beta)^{(k\alpha+\beta-1)}}{\Gamma(k\alpha+\beta)} = \sum_{k=0}^{\infty} \lambda^k (k\alpha+\beta) = \frac{\alpha\lambda}{(1-\lambda)^2} + \frac{\beta}{(1-\lambda)}.$$

Remark 8 Using D'Alembert's Ratio test, one can easily check that the above function converges for all $|\lambda| < 1$ and diverges for $|\lambda| > 1$. [20, Theorem 6].

2 Construction of the Green's Function

In this section we will construct the Green's function related to Problem (3) - (2), following the approach given in [9]. To this end we use the main properties of the Laplace transform. Since the delta discrete Mittag-Leffler function will be used, along all the section we assume that $|\alpha| < 1$, in order to ensure its convergence by the characterization given in Remark 8.

First, from (5), we have

$$R_0 [\Delta^v y(t)](s) = s^v R_{v-2} [y(t)](s) - sA - B, \quad (8)$$

where $A = [\Delta^{v-2} y(t)]_{t=0}$ and $B = [\Delta^{v-1} y(t)]_{t=0}$. One can check that

$$A = \frac{1}{\Gamma(2-v)} (1-v)^{(1-v)} y(v-2) = y(v-2) \quad (9)$$

and

$$B = (1-v)y(v-2) + y(v-1).$$

Denote by $Y_1(t) = y(t+v-\mu)$. Then,

$$\begin{aligned} R_{\mu-1} [Y_1(t)](s) &= \sum_{t=\mu-1}^{\infty} \left(\frac{1}{s+1} \right)^{t+1} Y_1(t) \\ &= \sum_{t=v-1}^{\infty} \left(\frac{1}{s+1} \right)^{t-v+\mu+1} y(t) \\ &= (s+1)^{v-\mu} \sum_{t=v-1}^{\infty} \left(\frac{1}{s+1} \right)^{t+1} y(t) \\ &= (s+1)^{v-\mu} \left[\sum_{t=v-2}^{\infty} \left(\frac{1}{s+1} \right)^{t+1} y(t) - \left(\frac{1}{s+1} \right)^{v-2+1} y(v-2) \right] \\ &= (s+1)^{v-\mu} R_{v-2} [y(t)](s) - (s+1)^{1-\mu} y(v-2). \end{aligned} \quad (10)$$

Next, from (5), we have

$$R_0 [\Delta^\mu y(t)](s) = s^\mu R_{\mu-1} [y(t)](s) - [\Delta^{\mu-1} y(t)]_{t=0}. \quad (11)$$

Using (10) and (11), we obtain

$$\begin{aligned} R_0 [\Delta^\mu Y_1(t)](s) &= s^\mu R_{\mu-1} [Y_1(t)](s) - [\Delta^{\mu-1} Y_1(t)]_{t=0} \\ &= s^\mu [(s+1)^{v-\mu} R_{v-2} [y(t)](s) - (s+1)^{1-\mu} y(v-2)] - [\Delta^{\mu-1} Y_1(t)]_{t=0}. \end{aligned} \quad (12)$$

Now, consider

$$\begin{aligned}
[\Delta^{\mu-1}Y_1(t)]_{t=0} &= [\Delta^{-(1-\mu)}Y_1(t)]_{t=0} \\
&= \left[\frac{1}{\Gamma(1-\mu)} \sum_{s=\mu-1}^{t-(1-\mu)} (t-s-1)^{(1-\mu-1)} Y_1(s) \right]_{t=0} \\
&= \left[\frac{1}{\Gamma(1-\mu)} \sum_{s=\mu-1}^{t-(1-\mu)} (t-s-1)^{(1-\mu-1)} y(s+v-\mu) \right]_{t=0} \\
&= \frac{1}{\Gamma(1-\mu)} (-\mu)^{(-\mu)} y(v-1) \\
&= y(v-1) = B - (1-v)A.
\end{aligned} \tag{13}$$

Using (9) and (13) in (12), we deduce

$$\begin{aligned}
R_0 [\Delta^\mu Y_1(t)](s) &= s^\mu (s+1)^{v-\mu} R_{v-2} [y(t)](s) - s^\mu (s+1)^{1-\mu} A - B + (1-v)A \\
&= s^\mu (s+1)^{v-\mu} R_{v-2} [y(t)](s) + [(1-v) - s^\mu (s+1)^{1-\mu}] A - B.
\end{aligned} \tag{14}$$

Denote $H_1(t) = h(t+v-1)$. Then,

$$\begin{aligned}
R_0 [H_1(t)](s) &= \sum_{t=0}^{\infty} \left(\frac{1}{s+1} \right)^{t+1} H_1(t) \\
&= \sum_{t=v-1}^{\infty} \left(\frac{1}{s+1} \right)^{t-v+1+1} h(t) \\
&= (s+1)^{v-1} \sum_{t=v-1}^{\infty} \left(\frac{1}{s+1} \right)^{t+1} h(t) \\
&= (s+1)^{v-1} \sum_{t=v-2}^{\infty} \left(\frac{1}{s+1} \right)^{t+1} h(t) - h(v-2) \\
&= (s+1)^{v-1} R_{v-2} [h(t)](s) - h(v-2).
\end{aligned} \tag{15}$$

By applying R_0 to each side of (3) and employing (8), (14) and (15), we obtain

$$\begin{aligned}
&- [s^v R_{v-2} [y(t)](s) - sA - B] \\
&+ \alpha [s^\mu (s+1)^{v-\mu} R_{v-2} [y(t)](s) + [(1-v) - s^\mu (s+1)^{1-\mu}] A - B] \\
&= (s+1)^{v-1} R_{v-2} [h(t)](s) - h(v-2).
\end{aligned}$$

Rearranging the terms gives us

$$\begin{aligned}
&(s^v - \alpha s^\mu (s+1)^{v-\mu}) R_{v-2} [y(t)](s) \\
&= (s + \alpha(1-v) - \alpha s^\mu (s+1)^{1-\mu}) A + (1-\alpha)B - (s+1)^{v-1} R_{v-2} [h(t)](s) + h(v-2).
\end{aligned}$$

This implies that

$$R_{v-2}[y(t)](s) = \frac{(s + \alpha(1-v) - \alpha s^\mu(s+1)^{1-\mu})}{(s^v - \alpha s^\mu(s+1)^{v-\mu})} A + \frac{(1-\alpha)}{(s^v - \alpha s^\mu(s+1)^{v-\mu})} B$$

$$- \frac{(s+1)^{v-1}}{(s^v - \alpha s^\mu(s+1)^{v-\mu})} R_{v-2}[h(t)](s) + \frac{1}{(s^v - \alpha s^\mu(s+1)^{v-\mu})} h(v-2). \quad (16)$$

Denote $Z(t) = y(t + n(v - \mu))$. Then,

$$R_{v-2}[Z(t)](s) = \sum_{t=v-2}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} Z(t)$$

$$= \sum_{t=(n+1)v-n\mu-2}^{\infty} \left(\frac{1}{s+1}\right)^{t-nv+n\mu+1} y(t)$$

$$= (s+1)^{nv-n\mu} \sum_{t=(n+1)v-n\mu-2}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} y(t)$$

$$= (s+1)^{nv-n\mu} R_{(n+1)v-n\mu-2}[y(t)](s). \quad (17)$$

Note that using (4) and (17), we obtain

$$\frac{s}{s^v - \alpha s^\mu(s+1)^{v-\mu}} = \frac{1}{s^{v-1}} \frac{1}{\left[1 - \alpha \left(\frac{s+1}{s}\right)^{v-\mu}\right]}$$

$$= \frac{1}{s^{v-1}} \sum_{k=0}^{\infty} \alpha^k \left(\frac{s+1}{s}\right)^{kv-k\mu}$$

$$= \sum_{k=0}^{\infty} \alpha^k (s+1)^{kv-k\mu} \frac{1}{s^{(k+1)v-k\mu-1}}$$

$$= \sum_{k=0}^{\infty} \alpha^k (s+1)^{kv-k\mu} \frac{R_{(k+1)v-k\mu-2}[t^{((k+1)v-k\mu-2)}](s)}{\Gamma((k+1)v-k\mu-1)}$$

$$= \sum_{k=0}^{\infty} \alpha^k \frac{R_{v-2}[(t+k(v-\mu))^{((k+1)v-k\mu-2)}](s)}{\Gamma((k+1)v-k\mu-1)}$$

$$= R_{v-2} \left[\sum_{k=0}^{\infty} \alpha^k \frac{(t+k(v-\mu))^{((k+1)v-k\mu-2)}}{\Gamma((k+1)v-k\mu-1)} \right] (s)$$

$$= R_{v-2}[e_{v-\mu, v-1}(\alpha, t-v+2)](s). \quad (18)$$

Similar to (17), we have

$$R_{v-1}[Z(t)](s) = (s+1)^{nv-n\mu} R_{(n+1)v-n\mu-1}[y(t)](s). \quad (19)$$

Moreover, using (4) and (19) we obtain

$$\begin{aligned}
\frac{1}{s^v - \alpha s^\mu (s+1)^{v-\mu}} &= \frac{1}{s^v} \sum_{k=0}^{\infty} \alpha^k \left(\frac{s+1}{s} \right)^{kv-k\mu} \\
&= \sum_{k=0}^{\infty} \alpha^k (s+1)^{kv-k\mu} \frac{1}{s^{(k+1)v-k\mu}} \\
&= \sum_{k=0}^{\infty} \alpha^k (s+1)^{kv-k\mu} \frac{R_{(k+1)v-k\mu-1} [t^{((k+1)v-k\mu-1)}] (s)}{\Gamma((k+1)v-k\mu)} \\
&= \sum_{k=0}^{\infty} \alpha^k \frac{R_{v-1} [(t+k(v-\mu))^{((k+1)v-k\mu-1)}] (s)}{\Gamma((k+1)v-k\mu)} \\
&= R_{v-1} \left[\sum_{k=0}^{\infty} \alpha^k \frac{(t+k(v-\mu))^{((k+1)v-k\mu-1)}}{\Gamma((k+1)v-k\mu)} \right] (s) \\
&= R_{v-2} \left[\sum_{k=0}^{\infty} \alpha^k \frac{(t+k(v-\mu))^{((k+1)v-k\mu-1)}}{\Gamma((k+1)v-k\mu)} \right] (s) \\
&\quad - (s+1)^{1-v} \left[\sum_{k=0}^{\infty} \alpha^k \frac{(t+k(v-\mu))^{((k+1)v-k\mu-1)}}{\Gamma((k+1)v-k\mu)} \right]_{t=v-2} \\
&= R_{v-2} \left[\sum_{k=0}^{\infty} \alpha^k \frac{(t+k(v-\mu))^{((k+1)v-k\mu-1)}}{\Gamma((k+1)v-k\mu)} \right] (s) \\
&= R_{v-2} [e_{v-\mu, v}(\alpha, t-v+1)] (s). \tag{20}
\end{aligned}$$

Denote $Z_1(t) = y(t+n(v-\mu)-\mu+1)$. Then,

$$\begin{aligned}
R_{v-2} [Z_1(t)] (s) &= \sum_{t=v-2}^{\infty} \left(\frac{1}{s+1} \right)^{t+1} Z_1(t) \\
&= \sum_{t=(n+1)(v-\mu)-1}^{\infty} \left(\frac{1}{s+1} \right)^{t-n(v-\mu)+\mu-1+1} y(t) \\
&= (s+1)^{nv-(n+1)\mu+1} \sum_{t=(n+1)(v-\mu)-1}^{\infty} \left(\frac{1}{s+1} \right)^{t+1} y(t) \\
&= (s+1)^{nv-(n+1)\mu+1} R_{(n+1)(v-\mu)-1} [y(t)] (s). \tag{21}
\end{aligned}$$

Note that using (4) and (21), we obtain

$$\begin{aligned}
& \frac{s^\mu(s+1)^{1-\mu}}{s^v - \alpha s^\mu(s+1)^{v-\mu}} \\
&= \frac{s^\mu(s+1)^{1-\mu}}{s^v} \frac{1}{1 - \alpha s^{\mu-v}(s+1)^{v-\mu}} \\
&= \frac{s^\mu(s+1)^{1-\mu}}{s^v} \frac{1}{\left[1 - \alpha \left(\frac{s+1}{s}\right)^{v-\mu}\right]} \\
&= \frac{s^\mu(s+1)^{1-\mu}}{s^v} \sum_{k=0}^{\infty} \alpha^k \left(\frac{s+1}{s}\right)^{kv-k\mu} \\
&= \sum_{k=0}^{\infty} \alpha^k (s+1)^{kv-(k+1)\mu+1} \frac{1}{s^{(k+1)v-(k+1)\mu}} \\
&= \sum_{k=0}^{\infty} \alpha^k (s+1)^{kv-(k+1)\mu+1} \frac{R_{(k+1)v-(k+1)\mu-1} \left[t^{((k+1)v-(k+1)\mu-1)} \right] (s)}{\Gamma((k+1)v - (k+1)\mu)} \\
&= \sum_{k=0}^{\infty} \alpha^k \frac{R_{v-2} \left[(t+k(v-\mu) - \mu + 1)^{(k+1)v-k\mu-1} \right] (s)}{\Gamma((k+1)v - k\mu)} \\
&= R_{v-2} \left[\sum_{k=0}^{\infty} \alpha^k \frac{(t+k(v-\mu) - \mu + 1)^{(k+1)v-k\mu-1}}{\Gamma((k+1)v - k\mu)} \right] (s) \\
&= R_{v-2} [e_{v-\mu, v-\mu}(\alpha, t-v+2)] (s). \tag{22}
\end{aligned}$$

Using (18), (20) and (22) in (16), we deduce

$$\begin{aligned}
R_{v-2} [y(t)] (s) &= \left[R_{v-2} [e_{v-\mu, v-1}(\alpha, t-v+2)] (s) \right. \\
&\quad + \alpha(1-v)R_{v-2} [e_{v-\mu, v}(\alpha, t-v+1)] (s) \\
&\quad \left. - \alpha R_{v-2} [e_{v-\mu, v-\mu}(\alpha, t-v+2)] (s) \right] A \\
&\quad + (1-\alpha)R_{v-2} [e_{v-\mu, v}(\alpha, t-v+1)] (s) B \\
&\quad - (s+1)^{v-1} R_{v-2} [e_{v-\mu, v}(\alpha, t-v+1)] (s) R_{v-2} [h(t)] (s) \\
&\quad \quad + R_{v-2} [e_{v-\mu, v}(\alpha, t-v+1)] (s) h(v-2),
\end{aligned}$$

which, by (6) can be written as

$$\begin{aligned}
R_{v-2} [y(t)] (s) &= \left[R_{v-2} [e_{v-\mu, v-1}(\alpha, t-v+2)] (s) \right. \\
&\quad + \alpha(1-v)R_{v-2} [e_{v-\mu, v}(\alpha, t-v+1)] (s) \\
&\quad \left. - \alpha R_{v-2} [e_{v-\mu, v-\mu}(\alpha, t-v+2)] (s) \right] A \\
&\quad + (1-\alpha)R_{v-2} [e_{v-\mu, v}(\alpha, t-v+1)] (s) B \\
&\quad - R_{v-2} [e_{v-\mu, v}(\alpha, t-v+1) *_{v-2} h] (s) \\
&\quad \quad + R_{v-2} [e_{v-\mu, v}(\alpha, t-v+1)] (s) h(v-2).
\end{aligned}$$

Apply to each side the inverse of R_{v-2} , we obtain

$$\begin{aligned} y(t) = & \left[e_{v-\mu, v-1}(\alpha, t-v+2) + \alpha(1-v)e_{v-\mu, v}(\alpha, t-v+1) \right. \\ & \left. - \alpha e_{v-\mu, v-\mu}(\alpha, t-v+2) \right] A + (1-\alpha)e_{v-\mu, v}(\alpha, t-v+1)B \\ & - e_{v-\mu, v}(\alpha, t-v+1) *_{v-2} h + e_{v-\mu, v}(\alpha, t-v+1)h(v-2). \end{aligned}$$

Thus, using (7), we have

$$\begin{aligned} y(t) = & \left[e_{v-\mu, v-1}(\alpha, t-v+2) + \alpha(1-v)e_{v-\mu, v}(\alpha, t-v+1) \right. \\ & \left. - \alpha e_{v-\mu, v-\mu}(\alpha, t-v+2) \right] A + (1-\alpha)e_{v-\mu, v}(\alpha, t-v+1)B \\ & - \sum_{s=v-2}^t e_{v-\mu, v}(\alpha, t-s+v-2-v+1)h(s) + e_{v-\mu, v}(\alpha, t-v+1)h(v-2). \end{aligned}$$

That is,

$$\begin{aligned} y(t) = & \left[e_{v-\mu, v-1}(\alpha, t-v+2) + \alpha(1-v)e_{v-\mu, v}(\alpha, t-v+1) \right. \\ & \left. - \alpha e_{v-\mu, v-\mu}(\alpha, t-v+2) \right] A + (1-\alpha)e_{v-\mu, v}(\alpha, t-v+1)B \\ & - \sum_{s=v-1}^t e_{v-\mu, v}(\alpha, t-s-1)h(s). \quad (23) \end{aligned}$$

Using $y(v-2) = 0$ in (23), we have

$$\begin{aligned} 0 = & \left[e_{v-\mu, v-1}(\alpha, 0) + \alpha(1-v)e_{v-\mu, v}(\alpha, -1) \right. \\ & \left. - \alpha e_{v-\mu, v-\mu}(\alpha, 0) \right] A + (1-\alpha)e_{v-\mu, v}(\alpha, -1)B \\ & - \sum_{s=v-1}^{v-2} e_{v-\mu, v}(\alpha, t-s-1)h(s). \end{aligned}$$

That is,

$$0 = \left[\frac{1}{(1-\alpha)} - \frac{\alpha}{(1-\alpha)} \right] A.$$

Using $y(v+b+1) = 0$ in (23) and taking $A = 0$, we have

$$0 = (1-\alpha)e_{v-\mu, v}(\alpha, b+2)B - \sum_{s=v-1}^{v+b+1} e_{v-\mu, v}(\alpha, v+b-s)h(s),$$

or

$$\begin{aligned} B &= \frac{1}{(1-\alpha)e_{v-\mu,v}(\alpha, b+2)} \sum_{s=v-1}^{v+b+1} e_{v-\mu,v}(\alpha, v+b-s)h(s) \\ &= \frac{1}{(1-\alpha)e_{v-\mu,v}(\alpha, b+2)} \sum_{s=v-1}^{v+b} e_{v-\mu,v}(\alpha, v+b-s)h(s). \end{aligned} \quad (24)$$

Using (24) and $A = 0$ in (23), we obtain

$$\begin{aligned} y(t) &= (1-\alpha)e_{v-\mu,v}(\alpha, t-v+1) \left[\frac{1}{(1-\alpha)e_{v-\mu,v}(\alpha, b+2)} \sum_{s=v-1}^{v+b} e_{v-\mu,v}(\alpha, v+b-s)h(s) \right] \\ &\quad - \sum_{s=v-1}^t e_{v-\mu,v}(\alpha, t-s-1)h(s). \end{aligned}$$

Rearranging the terms, we obtain

$$\begin{aligned} y(t) &= \frac{e_{v-\mu,v}(\alpha, t-v+1)}{e_{v-\mu,v}(\alpha, b+2)} \sum_{s=v-1}^{v+b} e_{v-\mu,v}(\alpha, v+b-s)h(s) \\ &\quad - \sum_{s=v-1}^t e_{v-\mu,v}(\alpha, t-s-1)h(s). \end{aligned}$$

That is,

$$\begin{aligned} y(t) &= \sum_{s=v-1}^t \left[\frac{e_{v-\mu,v}(\alpha, t-v+1)}{e_{v-\mu,v}(\alpha, b+2)} e_{v-\mu,v}(\alpha, v+b-s) - e_{v-\mu,v}(\alpha, t-s-1) \right] h(s) \\ &\quad - \sum_{s=t+1}^{v+b} \left[\frac{e_{v-\mu,v}(\alpha, t-v+1)}{e_{v-\mu,v}(\alpha, b+2)} e_{v-\mu,v}(\alpha, v+b-s) \right] h(s). \end{aligned}$$

Theorem 9 Assuming that $|\alpha| < 1$, we have that Problem (3) - (2) has a unique solution if and only if

$$e_{v-\mu,v}(\alpha, b+2) \neq 0.$$

Denote by

$$I_1 = \{(t, s) : v-1 \leq s \leq t \leq v+b+1\},$$

and

$$I_2 = \{(t, s) : v-1 \leq t+1 \leq s \leq v+b\}.$$

Then, the expression for the related Green's function, when $|\alpha| < 1$, is given by

$$G(t, s) = \begin{cases} \frac{e_{v-\mu,v}(\alpha, t-v+1)}{e_{v-\mu,v}(\alpha, b+2)} e_{v-\mu,v}(\alpha, v+b-s) - e_{v-\mu,v}(\alpha, t-s-1), & (t, s) \in I_1, \\ \frac{e_{v-\mu,v}(\alpha, t-v+1)}{e_{v-\mu,v}(\alpha, b+2)} e_{v-\mu,v}(\alpha, v+b-s), & (t, s) \in I_2. \end{cases}$$

3 Existence of Solutions of Nonlinear Problems

In this section we will apply the following Krasnosel'skii–Zabreiko fixed point theorem to obtain nontrivial solutions of

$$-\Delta^v y(t) + \alpha \Delta^\mu y(t + v - \mu) = f(t + v - 1, y(t + v - 1)), \quad t \in I, \quad (25)$$

coupled to the boundary conditions (2).

Here we assume that $f : [v - 1, v + b]_{\mathbb{N}_{v-1}} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

Theorem 10 *Let X be a Banach space and $F : X \rightarrow X$ be a completely continuous operator. If there exists a bounded linear operator $A : X \rightarrow X$ such that 1 is not an eigenvalue and*

$$\lim_{\|y\| \rightarrow \infty} \frac{\|F(y) - A(y)\|}{\|y\|} = 0,$$

then F has a fixed point in X .

We will apply Theorem 10 to a nonlinear summation operator whose kernel is $G(t, s)$. The arguments are in the line to the ones used in [13].

In this context, let the Banach space $(X, \|\cdot\|)$ be defined by

$$X := \{h : [v - 2, v + b + 1]_{\mathbb{N}_{v-2}} \rightarrow \mathbb{R}\},$$

with norm

$$\|h\| := \max_{t \in [v-2, v+b+1]_{\mathbb{N}_{v-2}}} |h(t)|.$$

Clearly, $y \in X$ is a fixed point of the completely continuous operator $F : X \rightarrow X$ defined by

$$(Fy)(t) := \sum_{s=v-1}^{v+b} G(t, s)f(s, y(s)), \quad t \in [v - 2, v + b + 1]_{\mathbb{N}_{v-2}}.$$

We now apply Theorem 10 to the operator F defined in (3) and to an associated linear operator in establishing solutions of (25)–(2).

Theorem 11 *Assume that $|\alpha| < 1$ and $f : [v - 1, v + b]_{\mathbb{N}_{v-1}} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and*

$$\lim_{|r| \rightarrow \infty} \frac{f(t + v - 1, r)}{r} = m$$

for all $t \in I$.

If

$$|m| < d := \frac{1}{\max_{t \in [v-2, v+b+1]_{\mathbb{N}_{v-2}}} \sum_{s=v-1}^{v+b} |G(t, s)|},$$

then the boundary value problem (25)–(2) has a solution y , and moreover, $y \neq 0$ on $[v - 2, v + b + 1]_{\mathbb{N}_{v-2}}$, when $f(t, 0) \neq 0$ for at least one $t \in I$.

Proof.

Corresponding to (25)–(2), we consider the following linear equation

$$-\Delta^v y(t) + \alpha \Delta^\mu y(t + v - \mu) = m y(t + v - 1), \quad t \in I, \quad (26)$$

coupled to the boundary conditions (2). We define a completely continuous linear operator $A : X \rightarrow X$ by

$$(Ay)(t) := m \sum_{s=v-1}^{v+b} G(t, s)y(s), \quad t \in [v-2, v+b+1]_{\mathbb{N}_{v-2}}.$$

Clearly, solutions of (26)–(2) are fixed points of A , and conversely.

First, we show that 1 is not an eigenvalue of A . To see this, we consider two cases: (a) $m = 0$ and (b) $m \neq 0$.

For (a), if $m = 0$, since the boundary value problem (26)–(2) has only the trivial solution, it is immediately that 1 is not an eigenvalue of A .

For (b), if $m \neq 0$ and (26)–(2) has a nontrivial solution, then $\|y\| > 0$. And so, we have

$$\begin{aligned} \|y\| &= \|(Ay)\| \\ &= \max_{t \in [v-2, v+b+1]_{\mathbb{N}_{v-2}}} \left| m \sum_{s=v-1}^{v+b} G(t, s)y(s) \right| \\ &= |m| \max_{t \in [v-2, v+b+1]_{\mathbb{N}_{v-2}}} \left| \sum_{s=v-1}^{v+b} G(t, s)y(s) \right| \\ &\leq |m| \|y\| \max_{t \in [v-2, v+b+1]_{\mathbb{N}_{v-2}}} \sum_{s=v-1}^{v+b} |G(t, s)| \\ &< d \|y\| \frac{1}{d} \\ &= \|y\|, \end{aligned}$$

a contradiction. Again, 1 is not an eigenvalue of A .

Our next claim is that

$$\lim_{\|y\| \rightarrow \infty} \frac{\|F(y) - A(y)\|}{\|y\|} = 0.$$

In this direction, let $\varepsilon > 0$ be given. Now, let

$$\lim_{|r| \rightarrow \infty} \frac{f(t + v - 1, r)}{r} = m$$

for all $t \in I$, implies that there exists an $N_1 > 0$ such that, for $|r| > N_1$,

$$|f(t + v - 1, r) - mr| < \varepsilon |r|, \quad t \in I. \quad (27)$$

Let

$$N = \sup_{|r| \leq N_1, t \in I} |f(t + v - 1, r)|,$$

and let $L \geq N_1$ be such that

$$\frac{N + |m|N_1}{L} < \varepsilon.$$

Next, choose $y \in X$ with $\|y\| > L$. Now, for $s \in [v - 2, v + b + 1]_{\mathbb{N}_{v-2}}$, if $|y(s)| \leq N_1$, we have

$$|f(s, y(s)) - my(s)| \leq |f(s, y(s))| + |m| |y(s)| \leq N + |m| N_1 < \varepsilon L < \varepsilon \|y\|.$$

On the other hand, if $|y(s)| > N_1$, we have from (27) that

$$|f(s, y(s)) - my(s)| < \varepsilon |y(s)| \leq \varepsilon \|y\|.$$

Thus, for $s \in [v - 2, v + b + 1]_{\mathbb{N}_{v-2}}$,

$$|f(s, y(s)) - my(s)| \leq \varepsilon \|y\|. \quad (28)$$

It follows from (28) that, for $y \in X$ with $\|y\| > L$,

$$\begin{aligned} \|F(y) - A(y)\| &= \max_{t \in [v-2, v+b+1]_{\mathbb{N}_{v-2}}} \left| \sum_{s=v-1}^{v+b} G(t, s) [f(s, y(s)) - my(s)] \right| \\ &\leq \max_{t \in [v-2, v+b+1]_{\mathbb{N}_{v-2}}} \sum_{s=v-1}^{v+b} |G(t, s)| |f(s, y(s)) - my(s)| \\ &\leq \varepsilon \|y\| \max_{t \in [v-2, v+b+1]_{\mathbb{N}_{v-2}}} \sum_{s=v-1}^{v+b} |G(t, s)| \\ &= \varepsilon \|y\| \frac{1}{d}. \end{aligned}$$

Therefore,

$$\lim_{\|y\| \rightarrow \infty} \frac{\|F(y) - A(y)\|}{\|y\|} = 0.$$

By Theorem 10, F has a fixed point $y \in X$, and y is a desired solution of (25), (2). Moreover $y \not\equiv 0$ on $[v - 2, v + b + 1]_{\mathbb{N}_{v-2}}$, when $f(t, 0) \neq 0$ for at least one $t \in I$ and the proof is complete. ■

Let us recall the following theorem

Theorem 12 [1] (*Leray–Schauder Nonlinear Alternative*) *Let $(E, \|\cdot\|)$ be a Banach space, K be a closed and convex subset of E , U be a relatively open subset of K such that $0 \in U$, and $T : \bar{U} \rightarrow K$ be completely continuous. Then, either*

(i) $u = Tu$ has a solution in \bar{U} ; or

(ii) There exist $u \in \partial U$ and $\lambda \in (0, 1)$ such that $u = \lambda Tu$.

Our second main result in this paper is as follows.

Theorem 13 Assume that $|\alpha| < 1$ and that the following properties are fulfilled:

(A1) There exist a nondecreasing function $\psi : [0, \infty) \rightarrow [0, \infty)$ and $g : [v - 2, v + b + 1]_{\mathbb{N}_{v-2}} \rightarrow [0, \infty)$ such that

$$|f(t + v - 1, r)| \leq g(t + v - 1)\psi(|r|), \quad t \in I, \quad r \in \mathbb{R}.$$

(A2) There exists $L > 0$ such that

$$\frac{L}{\psi(L) \max_{t \in [v-2, v+b+1]_{\mathbb{N}_{v-2}}} \sum_{s=v-1}^{v+b} g(s) |G(t, s)|} > 1.$$

Then, the boundary value problem (25)–(2) has a solution defined on $[v - 2, v + b + 1]_{\mathbb{N}_{v-2}}$. Moreover, $y \not\equiv 0$ on $[v - 2, v + b + 1]_{\mathbb{N}_{v-2}}$, when $f(t, 0) \not\equiv 0$ for at least one $t \in I$.

Proof. We first show that F maps bounded sets into bounded sets. For this purpose, for $r > 0$, let

$$B_r = \{y \in X : \|y\| \leq r\},$$

be a bounded subset of X . Then, by (A1), for $y \in B_r$,

$$|(Fy)(t)| \leq \sum_{s=v-1}^{v+b} |G(t, s)| |f(s, y(s))| \leq \psi(\|y\|) \sum_{s=v-1}^{v+b} g(s) |G(t, s)|,$$

implying that

$$|(Fy)(t)| \leq \psi(r) \sum_{s=v-1}^{v+b} g(s) |G(t, s)|.$$

Thus, F maps B_r into a bounded set. Since $[v - 2, v + b + 1]_{\mathbb{N}_{v-2}}$ is a discrete set, it follows immediately that F maps B_r into an equicontinuous set. Therefore, by the Arzela–Ascoli theorem, F is completely continuous. Next, suppose that $y \in X$ and $y = \lambda Fy$ for some $0 < \lambda < 1$. Then, from (A1), for $t \in [v - 2, v + b + 1]_{\mathbb{N}_{v-2}}$, we have

$$|y(t)| = |\lambda(Fy)(t)| \leq \sum_{s=v-1}^{v+b} |G(t, s)| |f(s, y(s))| \leq \psi(\|y\|) \sum_{s=v-1}^{v+b} g(s) |G(t, s)|,$$

implying that

$$\frac{\|y\|}{\psi(\|y\|) \sum_{s=v-1}^{v+b} g(s) |G(t, s)|} \leq 1.$$

It follows from (A2) that $\|y\| \neq L$. If we set

$$U = \left\{ y \in X : \|y\| < L \right\},$$

then the operator $F : \bar{U} \rightarrow X$ is completely continuous. From the choice of U , then there is no $y \in \partial U$ such that $y = \lambda Fy$ for some $0 < \lambda < 1$. It follows from Theorem 12 that F has a fixed point $y_0 \in \bar{U}$, which is a desired solution of (25)–(2).

Obviously, if $f(t, 0) \not\equiv 0$ on $[v - 2, v + b + 1]_{\mathbb{N}_{v-2}}$, we have that $y \not\equiv 0$ on $[v - 2, v + b + 1]_{\mathbb{N}_{v-2}}$. ■

As a direct consequence of the previous result, we deduce the following corollary:

Corollary 14 *Assume that $|\alpha| < 1$ and condition (A1) in Theorem 13 holds. Then, if*

$$\lim_{L \rightarrow +\infty} \frac{L}{\psi(L)} = +\infty$$

the boundary value problem (25)–(2) has a solution defined on $[v - 2, v + b + 1]_{\mathbb{N}_{v-2}}$. Moreover, $y \not\equiv 0$ on $[v - 2, v + b + 1]_{\mathbb{N}_{v-2}}$, when $f(t, 0) \neq 0$ for at least one $t \in I$.

4 Examples

In this section, we provide two example to demonstrate the applicability of Theorems 11 and 13.

Example 15 *Consider the boundary value problem (25), (2) with $a = 0$, $b = 5$, $v = 1.5$, $\mu = 0.5$, $\alpha = 0.5$ and*

$$f(t + v - 1, r) = \frac{r}{3\pi} \left| \tan^{-1} \left((t + v - 1)^2 (r + 1)^3 \right) \right| + e^{(t+v-1)^2} \sqrt{|r + 1|}.$$

Clearly,

$$m = \lim_{|r| \rightarrow \infty} \frac{f(t + v - 1, r)}{r} = \frac{1}{6} \quad \text{for all } t \in I.$$

The Green's function associated with the boundary value problem is given by

$$G(t, s) = \begin{cases} \frac{e_{1,1.5}(0.5, t-0.5)}{e_{1,1.5}(0.5, 7)} e_{1,1.5}(0.5, 6.5 - s) - e_{1,1.5}(0.5, t - s - 1), & (t, s) \in I_1, \\ \frac{e_{1,1.5}(0.5, t-0.5)}{e_{1,1.5}(0.5, 7)} e_{1,1.5}(0.5, 6.5 - s), & (t, s) \in I_2, \end{cases} \quad (29)$$

where

$$I_1 = \{(t, s) : 0.5 \leq s \leq t \leq 7.5\},$$

and

$$I_2 = \{(t, s) : 0.5 \leq t + 1 \leq s \leq 6.5\}.$$

Since

$$d = \frac{1}{\max_{t \in [-0.5, 7.5]_{\mathbb{N}_{-0.5}}} \sum_{s=0.5}^{6.5} |G(t, s)|} = 0.241342 > |m|,$$

by Theorem 11, the boundary value problem has a non trivial solution defined on $[-0.5, 7.5]_{\mathbb{N}_{-0.5}}$.

Example 16 Consider the boundary value problem (25), (2) with $a = 0$, $b = 5$, $v = 1.5$, $\mu = 0.5$, $\alpha = 0.5$ and $f(t + v - 1, r) = (t + v - 1) \sqrt[4]{|r|^3 + t + v - 1}$. Clearly,

$$|f(t + v - 1, r)| \leq g(t + v - 1)\psi(|r|), \quad t \in I, \quad r \in \mathbb{R},$$

where

$$g(t + v - 1) = t + v - 1, \quad t \in I,$$

and

$$\psi(|r|) = \sqrt[4]{|r|^3 + b + v}, \quad r \in \mathbb{R}.$$

Also, $g : [v-2, v+b+1]_{\mathbb{N}_{v-2}} \rightarrow [0, \infty)$ and $\psi : [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing function. Thus, the assumption (A1) of Theorem 13 holds.

Now, since

$$\lim_{L \rightarrow +\infty} \frac{L}{\psi(L)} = +\infty,$$

from Corollary 14, we have that the considered problem has at least a nontrivial solution y .

Notice that, in this case, we can estimate the minimum value of the parameter L as 74,395.4. We point out that, from the proof of Theorem 13, this value of L is the better a priori bound that we have for $\|y\|$.

Acknowledgments First author is partially supported by Xunta de Galicia (Spain), project EM2014/032 and AIE, Spain and FEDER, grant PID2020-113275GB-I00. The second author is supported by Project FNSE-03.

References

- [1] Agarwal, R. P.; Meehan, M.; O'Regan, D. *Fixed point theory and applications*. Cambridge Tracts in Mathematics, 141. Cambridge University Press, Cambridge, 2001.
- [2] Atici, F M, Eloe, P W, *A transform method in discrete fractional calculus*. Int. J. Difference Equ. 2 (2007), 2, 165-176.
- [3] Atici, F M, Eloe, P W, *Initial value problems in discrete fractional calculus*. Proc. Amer. Math. Soc. 137 (2009), no. 3, 981-989.
- [4] Atici, F M, Eloe, P W, *Linear Forward Fractional Difference Equations*, Com. Appl. Anal. 19 (2015), 31-42.

- [5] Atici, F M, Eloe, P W, *Two-point boundary value problems for finite fractional difference equations*. J. Difference Equ. Appl. 17 (2011), 4, 445-456.
- [6] Bohner, M, Peterson, A, *Dynamic equations on time scales. An introduction with applications.*, Birkhäuser Boston Inc., Boston, MA, 2001.
- [7] Bourguiba, R. Cabada, A. Wanassi, O K. *Existence of solutions of discrete fractional problem coupled to mixed fractional boundary conditions*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM 116 (2022), 4, 175, 15 pp.
- [8] Cabada, A, Dimitrov, N, *Nontrivial solutions of non-autonomous Dirichlet fractional discrete problems.*, Fract. Calc. Appl. Anal. 23 (2020), no. 4, 980-995.
- [9] Cabada, A., Dimitrov, N., Jagan Mohan, J., *Green's functions for fractional difference equations with Dirichlet boundary conditions*, Chaos Solitons Fractals 153 (2021), part 1, 111455, 14 pp.
- [10] Donahue, R, *The development of a transform method for use in solving difference equations*, Master's thesis, University of Dayton, 1987.
- [11] Goodrich, C.S. *On positive solutions to nonlocal fractional and integer-order difference equations*. Appl. Anal. Discrete Math 5 (2011), 122-132.
- [12] Goodrich, C, Peterson, A C, *Discrete fractional calculus*. Springer, Cham, 2015.
- [13] Henderson, J, *Nontrivial solutions for a nonlinear ν th order Atici-Eloe fractional difference equation satisfying Dirichlet boundary conditions*. *Differ. Equ. Appl.* 14 (2022), 2, 137-143.
- [14] Heymans, N, Podlubny, I, *Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives*. Rheol. Acta 45, 765-771 (2006).
- [15] Kilbas, A. A., Srivastava, H. H., Trujillo, J. J., *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam, 2006.
- [16] Magin, R. L., *Fractional Calculus in Bioengineering*. Begell House Publisher, Connecticut, 2006.
- [17] Miller, K. S., Ross, B., *Fractional difference calculus*. In Univalent functions, fractional calculus, and their applications (Koriyama, 1988), Ellis Horwood Ser. Math. Appl., Horwood, Chichester, (1989), 139-152.
- [18] Podlubny, I., *Fractional Differential Equations*. Academic Press, San Diego (1999).
- [19] Samko, S. G., Kilbas, A. A., Marichev, O. I., *Fractional Integrals and Derivatives - Theory and Applications*. Gordon & Breach, Linghorne (1993).

- [20] Shobanadevi, N., Jagan Mohan, J., Analysis of Discrete Mittag - Leffler Functions, *Int. J. Anal. Appl.*, 7 (2) (2015), 129-144.
- [21] West, B. J., Bologna, M., Grigolini, P., *Physics of Fractal Operators*. Springer, New York (2003).