Restricted Log-Exp-Analytic Power Functions

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Abstract. A preparation theorem for compositions of restricted log-exp-analytic functions and power functions of the form

$$h : \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} x^r, & x > 0, \\ 0, & \text{else}, \end{cases}$$

for $r \in \mathbb{R}$ is given. Consequently we obtain a parametric version of Tamm's theorem for this class of functions which is indeed a full generalisation of the parametric version of Tamm's theorem for $\mathbb{R}_{\text{an}}^{\mathbb{R}}$-definable functions.

Introduction

In [8] Opris gave the definition for restricted log-exp-analytic functions. These are $\mathbb{R}_{\text{an,exp}}$-definable functions which are compositions of log-analytic functions and exponentials of functions which are locally bounded where $\mathbb{R}_{\text{an,exp}}$ is the structure generated by all restricted analytic functions and the global exponential function (see [1]). A log-analytic function is piecewise given by compositions from either side of globally subanalytic functions and the global logarithm (see [5], [6] and [9] for the formal definition and elementary properties of log-analytic functions).

Example

The function

$$g :]0,1[^2 \to \mathbb{R}, (t,x) \mapsto \arctan(\log(e^{1/t \cdot \log^2(1/x)} + \log(e^{e^{1/t}} + 2))),$$

is restricted log-exp-analytic.

Since the global exponential function comes only locally bounded into the game one sees that a restricted log-exp-analytic function $f : \mathbb{R} \to \mathbb{R}$ fulfills the following property for all sufficiently small positive y. Either $f(y)$ vanishes identically or there is $c \in \mathbb{R} \setminus \{0\}$, a non-negative integer $r \in \mathbb{N}_0$ and $q_0, \ldots, q_r \in \mathbb{Q}$ such that $f(y) = c \cdot h(y) + o(h(y))$ where $h(y) := y^{q_0} \cdot (-\log(y))^{q_1} \cdot \ldots \cdot \log_{r-1}(-\log(y))^{q_r}$ (see Definition 1.13 and Proposition 3.16 in [9]). A consequence is the following.

Fact

Let $r \in \mathbb{R} \setminus \mathbb{Q}$. The irrational power function

$$h : \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} x^r, & x > 0, \\ 0, & \text{else}, \end{cases}$$

is $\mathbb{R}_{\text{an,exp}}$-definable, but not restricted log-exp-analytic, since one has $h \sim x^r$ as $x \searrow 0$. Because $h(x) = \exp(r \log(x))$ for $x \in \mathbb{R}_{>0}$ we see that $h|_{\mathbb{R}_{>0}}$ is restricted log-exp-analytic.

Consequently Tamm's theorem from [8] is not a generalisation of the version of Tamm's theorem from [3], since in [3] $\mathbb{R}_{\text{an}}^{\mathbb{R}}$-definable functions are considered where $\mathbb{R}_{\text{an}}^{\mathbb{R}}$ is the structure generated by all globally subanalytic functions and irrational power functions. By Miller [7] the structure $\mathbb{R}_{\text{an}}^{\mathbb{R}}$ is o-minimal. (See [2] for the definition and properties of an o-minimal structure.) Even the structure $\mathbb{R}_{\text{an,exp}}$ is o-minimal by Van den Dries [1] which is a proper extension of $\mathbb{R}_{\text{an}}^{\mathbb{R}}$.

This article merges the results from [3] and [8]: We look at compositions of irrational power functions and restricted log-exp-analytic functions. Such compositions form a class of $\mathbb{R}_{\text{an,exp}}$-definable functions which contains all restricted log-exp-analytic functions and all $\mathbb{R}_{\text{an}}^{\mathbb{R}}$-definable functions. We call them *restricted log-exp-analytic power functions*.

As in [8] we give differentiability results of this class of functions in the parametric setting. Thus we introduce variables $(w_1, \ldots, w_l, u_1, \ldots, u_m, z)$, where $(u_1, \ldots, u_m, z)$ is serving as the tuple of independent variables of families of functions parameterized by $w := (w_1, \ldots, w_l)$. (The variable z is needed to describe a preparation theorem for restricted log-exp-analytic power functions with respect to a single variable which is suitable for our purposes.) Then a *restricted log-exp-analytic power function in* (u, z) on X where $X \subset \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}$ is $\mathbb{R}_{\text{an,exp}}$-definable and $X_w := \{(u, z) \in \mathbb{R}^m \times \mathbb{R} \mid (w, u, z) \in X\}$ is open for every $w \in \mathbb{R}^l$ is the composition from either side of restricted log-exp-analytic functions in (u, z) and irrational power functions on X. A *restricted log-exp-analytic function in* (u, z) on X is the composition from either side of log-analytic functions and exponentials of locally bounded functions in (u, z) ($g : X \to \mathbb{R}$ is locally bounded in (u, z) if $g_w : X_w \to \mathbb{R}, (u, z) \mapsto g(w, u, z)$, is locally bounded for every $w \in \mathbb{R}^l$).

One of our main goals for this article is to formulate and prove a preparation theorem for restricted log-exp-analytic power functions in (u, z) (see Theorem C in [9] for a precise preparation theorem for $\mathbb{R}_{\text{an,exp}}$-definable functions, see [4] and [6] for original versions): a restricted log-exp-analytic power function $f : X \to \mathbb{R}, (w, u, z) \mapsto f(w, u, z)$, in (u, z) where $X_w := \{(u, z) \in \mathbb{R}^m \times \mathbb{R} \mid (w, u, z) \in X\}$ is open for $w \in \mathbb{R}^l$ can be cellwise written as $(m + 1, X)$*-power-restricted* (e, r)*-prepared functions* for suitable parameters $e \in \mathbb{N}_0 \cup \{-1\}$ and $r \in \mathbb{N}_0$. Here the parameter e describes the maximal number of iterations of exponentials which occur in such a preparation which have the following form: each of them are exponentials of $(m + 1, X)$-power-restricted (l, r)-prepared functions for $l < e$ which can be extended to a locally bounded function in (u, z) on X. This information about the exponentials is described by the tuple $(m + 1, X)$. The parameter r describes the maximal number of iterations of the logarithm depending on z which occur in every such exponential. These logarithms can be technical described by products of real powers of components of a logarithmic scale (see Definition 1.4 below for the notion of a logarithmic scale). Formally an $(m + 1, X)$-power-restricted (e, r)-prepared function is defined as follows.

Let $n := l + m$, $C \subset \mathbb{R}^n \times \mathbb{R}_{\neq 0}$ be an $\mathbb{R}_{\text{an,exp}}$-definable cell and let $r \in \mathbb{N}_0$. Let $t := (w, u)$ and $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, (t, z) \mapsto t$, be the projection on the first n coordinates. Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be $\mathbb{R}_{\text{an,exp}}$-definable with $C \subset X$ such that $X_w \subset \mathbb{R}^{m+1}$ is open for $w \in \mathbb{R}^l$. We call a function $f : C \to \mathbb{R}, (w, u, z) \mapsto f(w, u, z)$, $(m + 1, X)$-

power-restricted $(-1,r)$*-prepared in* z *with center* $\Theta := (\Theta_0,\ldots,\Theta_r)$ if f is the zero function. For $e \in \mathbb{N}_0$ call a function $f : C \to \mathbb{R}, (w,u,z) \mapsto f(w,u,z)$, $(m+1,X)$-*power-restricted* (e,r)*-prepared in* z *with center* Θ if for $(t,z) \in C$

$$f(t,z) = \sigma a(t)|y_0(t,z)|^{\alpha_0} \cdot \ldots \cdot |y_r(t,z)|^{\alpha_r}\exp(c(t,z)) \cdot \rho(t,z)$$

where $\alpha_0,\ldots,\alpha_r \in \mathbb{R}$, $y_0 = z - \Theta_0(t), y_1 = \log(|y_0|) - \Theta_1(t),\ldots,$ $\sigma \in \{-1,0,1\}$, $a = \prod_{j=1}^{k} h_j^{\lambda_j}$ where $k \in \mathbb{N}$, $\lambda_j \in \mathbb{R}$, c can be extended to a locally bounded function in (u,z) on X and is itself $(m+1,X)$-power-restricted $(e-1,r)$-prepared with center Θ and $\rho(t,z)$ is a unit of a special form which we describe below. Furthermore there is $\delta > 1$ such that $1/\delta < \rho < \delta$ and the functions $\Theta_0,\ldots,\Theta_r : \pi(C) \to \mathbb{R}$ and $h_j : \pi(C) \to \mathbb{R}_{>0}$ are C-nice functions: they are compositions of log-analytic functions and exponentials of the form $\exp(h)$ where h is the component of a center of a logarithmic scale on C. Note that a log-analytic function on $\pi(C)$ is C-nice and that every C-nice function is definable (see [9] for examples and several properties of C-nice functions), but the class of C-nice functions does not necessarily coincide with the class of definable functions: if the cell C is *simple*, i.e. for every $t \in \pi(C)$ there is $d_t \in \mathbb{R}_{>0} \cup \{\infty\}$ such that $C_t =]0,d_t[$ (see for example Definition 2.15 in Kaiser-Opris [5]), the class of C-nice functions coincides with the class of log-analytic ones (in [5] it is shown that the center of a logarithmic scale vanishes on a simple cell).

The first goal of this paper is to prove that a restricted log-exp-analytic power function in (u,z) can be indeed cellwise prepared as $(m+1,X)$-power-restricted (e,r)-prepared functions in (u,z).

Theorem A

Let $X \subset \mathbb{R}^n \times \mathbb{R}$ *be* $\mathbb{R}_{\text{an,exp}}$*-definable and let* $f : X \to \mathbb{R}$ *be a restricted log-exp-analytic power function in* (u,z)*. Then there are* $e \in \mathbb{N}_0 \cup \{-1\}$*,* $r \in \mathbb{N}_0$ *and an* $\mathbb{R}_{\text{an,exp}}$*-definable cell decomposition* $\mathcal{C}$ *of* $X_{\neq 0}$ *such that for every* $C \in \mathcal{C}$ *there is* $\Theta := (\Theta_0,\ldots,\Theta_r)$ *such that the function* $f|_C$ *is* $(m+1,X)$*-power-restricted* (e,r)*-prepared in* z *with center* Θ*.*

In the case of a restricted log-exp-analytic function we have a similar preparation with the difference that the function a is C-nice and that the logarithms are rational powers of components of logarithmic scales (i.e. $\lambda_1 \ldots \lambda_k \in \mathbb{Q}$ and $\alpha_0 \ldots \alpha_r \in \mathbb{Q}$).

The second goal of this paper is to give some differentiability properties for restricted log-exp-analytic power functions which are versions for Theorem A, Theorem B and Theorem C from [8] for restricted log-exp-analytic power functions. These are also generalizations of the results from [3].

Theorem B

Let $X \subset \mathbb{R}^l \times \mathbb{R}^m$ *be* $\mathbb{R}_{\text{an,exp}}$*-definable such that* X_w *is open for every* $w \in \mathbb{R}^l$ *and let* $f : X \to \mathbb{R}, (w,u) \mapsto f(w,u)$*, be a restricted log-exp-analytic power function in* u*. Then the following holds.*

(1) Closedness under taking derivatives: Let $i \in \{1,\ldots,m\}$ *be such that* f *is*

differentiable with respect to u_i on X. Then $\partial f/\partial u_i$ is a restricted log-exp-analytic power function in u.

(2) *Strong quasianalyticity: There is $N \in \mathbb{N}$ such that if $f(w,-)$ is C^N for $w \in \mathbb{R}^l$ and if there is $a \in X_t$ such that all derivatives up to order N vanish in a then $f(w,-)$ vanishes identically.*

(3) *Parametric version of Tamm's theorem: There is $M \in \mathbb{N}$ such that if $f(w,-)$ in C^M at u for $(w,u) \in X$ then $f(w,-)$ is real analytic at u.*

This paper is organised as follows. In Section 1 we pick up the most important concepts from [8] and [9] like log-analytic functions and their preparation theorem. In Section 2 we give a proof for Theorem A and Section 3 is devoted to the proof of Theorem B divided into three separate propositions.

Notations

By $\mathbb{N} := \{1, 2, \ldots\}$ we denote the set of natural numbers and by $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ the set of nonnegative integers. For $m, n \in \mathbb{N}$ we denote by $\mathcal{M}(m \times n, \mathbb{R})$ respectively $\mathcal{M}(m \times n, \mathbb{Q})$ the set of $m \times n$-matrices with real respectively rational entries.

For $m \in \mathbb{N}$, a set $X \subset \mathbb{R}^m$ and a set E of positive real valued functions on X we set $\log(E) := \{\log(g) \mid g \in E\}$.

For $X \subset \mathbb{R}^n \times \mathbb{R}$ let $X_{\neq 0} := \{(t,z) \in X \mid z \neq 0\}$. For $X \subset \mathbb{R}^l \times \mathbb{R}^m$ and $w \in \mathbb{R}^m$ we set $X_w := \{u \in \mathbb{R}^m \mid (w,u) \in X\}$ and for a function $f : X \to \mathbb{R}, (w,u) \mapsto f(w,u)$, we set $f_w : X_w \to \mathbb{R}, u \mapsto f(w,u)$.

The reader should be familiar with basic facts about o-minimal structures from [2].

Convention

Definable means $\mathbb{R}_{\text{an,exp}}$-definable if not otherwise mentioned.

1 $\mathbb{R}_{\text{an,exp}}$-Definable Functions

1.1 Log-Analytic Functions and the Exponential Number

Compare with [9], Section 1 for a more detailed description of the content in this subsection.

Let $m \in \mathbb{N}$ and $X \subset \mathbb{R}^m$ be definable.

1.1 Definition

Let $f : X \to \mathbb{R}$ be a function.

(a) Let $r \in \mathbb{N}_0$. By induction on r we define that f is **log-analytic of order at most** r.

Base case: The function f is log-analytic of order at most 0 if there is a decomposition $\mathcal{C}$ of X into finitely many definable cells such that for $C \in \mathcal{C}$ there is a globally subanalytic function $F : \mathbb{R}^m \to \mathbb{R}$ such that $f|_C = F|_C$.

Inductive step: The function f is log-analytic of order at most r if the following holds: There is a decomposition $\mathcal{C}$ of X into finitely many definable cells such that for $C \in \mathcal{C}$ there are $k, l \in \mathbb{N}_0$, a globally subanalytic function $F : \mathbb{R}^{k+l} \to \mathbb{R}$, and log-analytic functions $g_1, \ldots, g_k : C \to \mathbb{R}, h_1, \ldots, h_l : C \to \mathbb{R}_{>0}$ of order at most $r - 1$ such that

$$f|_C = F(g_1, \ldots, g_k, \log(h_1), \ldots, \log(h_l)).$$

(b) Let $r \in \mathbb{N}_0$. We call f **log-analytic of order** r if f is log-analytic of order at most r but not of order at most $r - 1$.

(c) We call f **log-analytic** if f is log-analytic of order r for some $r \in \mathbb{N}_0$.

1.2 Definition

Let $f : X \to \mathbb{R}$ be a function. Let E be a set of positive definable functions on X.

(a) By induction on $e \in \mathbb{N}_0$ we define that f has **exponential number at most** e **with respect to** E.

Base Case: The function f has exponential number at most 0 with respect to E if f is log-analytic.

Inductive Step: The function f has exponential number at most e with respect to E if the following holds: There are $k, l \in \mathbb{N}_0$, functions $g_1, \ldots, g_k : X \to \mathbb{R}$ and $h_1, \ldots, h_l : X \to \mathbb{R}$ with exponential number at most $e - 1$ with respect to E and a log-analytic function $F : \mathbb{R}^{k+l} \to \mathbb{R}$ such that

$$f = F(g_1, \ldots, g_k, \exp(h_1), \ldots, \exp(h_l))$$

and $\exp(h_1), \ldots, \exp(h_l) \in E$.

(b) Let $e \in \mathbb{N}_0$. We say that f has **exponential number** e **with respect to** E if f has exponential number at most e with respect to E but not at most $e - 1$ with respect to E.

(c) We say that f **can be constructed from** E if there is $e \in \mathbb{N}_0$ such that f has exponential number e with respect to E.

1.3 Remark

Let $e \in \mathbb{N}_0$. Let E be a set of positive definable functions on X.

(1) Let $f : X \to \mathbb{R}$ be a function with exponential number at most e with respect to E. Then $\exp(f)$ has exponential number at most $e + 1$ with respect to $E \cup \{\exp(f)\}$.

(2) Let $s \in \mathbb{N}_0$. Let $f_1, \ldots, f_s : X \to \mathbb{R}$ be functions with exponential number at most e with respect to E and let $F : \mathbb{R}^s \to \mathbb{R}$ be log-analytic. Then $F(f_1, \ldots, f_s)$ has exponential number at most e with respect to E.

1.2 A Preparation Theorem for Log-Analytic Functions

Compare with [9], Section 2 for a more detailed description of the content in this subsection.

Let $n \in \mathbb{N}$. Let t range over $\mathbb{R}^n$ and z over $\mathbb{R}$. We fix a definable set $C \subset \mathbb{R}^n \times \mathbb{R}$.

1.4 Definition ([9] Section 2.1)

Let $r \in \mathbb{N}_0$. A tuple $\mathcal{Y} := (y_0, \ldots, y_r)$ of functions on C is called an r**-logarithmic scale** on C with **center** $\Theta = (\Theta_0, \ldots, \Theta_r)$ if the following holds:

(a) $y_j > 0$ or $y_j < 0$ for every $j \in \{0, \ldots, r\}$.

(b) Θ_j is a definable function on $\pi(C)$ for every $j \in \{0, \ldots, r\}$.

(c) We have $y_0(t,z) = z - \Theta_0(t)$ and inductively $y_j(t,z) = \log(|y_{j-1}(t,z)|) - \Theta_j(t)$ for every $j \in \{1, \ldots, r\}$ and all $(t,z) \in C$.

(d) Either there is $\epsilon_0 \in]0,1[$ such that $0 < |y_0(t,z)| < \epsilon_0|z|$ for all $(t,z) \in C$ or $\Theta_0 = 0$, and for every $j \in \{1, \ldots, r\}$ either there is $\epsilon_j \in]0,1[$ such that $0 < |y_j(t,z)| < \epsilon_j|\log(|y_{j-1}(t,z)|)|$ for all $(t,z) \in C$ or $\Theta_j = 0$.

For a logarithmic scale $(y_0, \ldots, y_r)$ on a definable set C and $\alpha \in \mathbb{R}^{r+1}$ we often write $|\mathcal{Y}(t,z)|^{\otimes\alpha}$ instead of $\prod_{j=0}^{r} |y_j(t,z)|^{\alpha_j}$ where $(t,z) \in C$.

1.5 Definition ([9] Section 2.3)

We call $g : \pi(C) \to \mathbb{R}$ a C**-heir** if there is $l \in \mathbb{N}_0$, an l-logarithmic scale $\hat{\mathcal{Y}}$ with center $(\hat{\Theta}_0, \ldots, \hat{\Theta}_l)$ on C, and $j \in \{1, \ldots, l\}$ such that $g = \exp(\Theta_j)$.

1.6 Definition ([9] Section 2.3)

We call $g : \pi(C) \to \mathbb{R}$ C**-nice** if there is a set E of C-heirs such that g can be constructed from E.

Note that the class of log-analytic functions on $\pi(C)$ can be a proper subclass of the class of C-nice functions (compare with Example 2.39 in [9]). In the following we give the definition from [9] for log-analytically prepared functions with the difference that we also allow real exponents for the iterations of the logarithms. This is needed to describe preparations of restricted log-exp-analytic power functions on simple cells in an effective way.

1.7 Definition

Let $r \in \mathbb{N}_0$. Let $g : C \to \mathbb{R}$ be a function. We say that g is **r-real-log-analytically prepared in z with center Θ** if

$$g(t,z) = a(t)|\mathcal{Y}(t,z)|^{\otimes\alpha}\rho(t,z)$$

for all $(t,z) \in C$ where a is a definable function on $\pi(C)$ which vanishes identically or has no zero, $\mathcal{Y} = (y_0, \ldots, y_r)$ is an r-logarithmic scale with center Θ on C, $\alpha \in \mathbb{R}^{r+1}$ and the following holds for ρ. There is $s \in \mathbb{N}$ such that $\rho = v \circ \phi$ where v is a power series which converges on an open neighbourhood of $[-1,1]^s$ with $v([-1,1]^s) \subset \mathbb{R}_{>0}$ and $\phi := (\phi_1, \ldots, \phi_s) : C \to [-1,1]^s$ is a function of the form

$$\phi_j(t,z) := b_j(t)|\mathcal{Y}(t,z)|^{\otimes\gamma_j}$$

for $j \in \{1, \ldots, s\}$ and $(t,z) \in C$ where $b_j : \pi(C) \to \mathbb{R}$ is definable for $j \in \{1, \ldots, s\}$ and $\gamma_j := (\gamma_{j0}, \ldots, \gamma_{jr}) \in \mathbb{R}^{r+1}$. We call a **coefficient** and $b := (b_1, \ldots, b_s)$ a tuple of **base functions** for f. An **LA-preparing tuple** for f is then

$$\mathcal{J} := (r, \mathcal{Y}, a, \alpha, s, v, b, \Gamma)$$

where

$$\Gamma := \begin{pmatrix} \gamma_{10} & \cdot & \cdot & \gamma_{1r} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \gamma_{s0} & \cdot & \cdot & \gamma_{sr} \end{pmatrix} \in \mathcal{M}\big(s \times (r+1), \mathbb{R}\big).$$

If $\alpha, \gamma_1, \ldots, \gamma_s \in \mathbb{Q}^{r+1}$ we say that g is **r-log-analytically prepared in z with center Θ**.

The following preparation theorem for log-analytic functions has been established in [9].

1.8 Fact ([9] Theorem A)
Let $m \in \mathbb{N}$, $r \in \mathbb{N}_0$. Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable. Let $f_1, \ldots, f_m : X \to \mathbb{R}$ be log-analytic functions of order at most r. Then there is a definable cell decomposition $\mathcal{C}$ of $X_{\neq 0}$ such that $f_1|_C, \ldots, f_m|_C$ are r-log-analytically prepared in z with C-nice coefficient, C-nice base functions and common C-nice center for $C \in \mathcal{C}$.

2 Restricted Log-Exp-Analytic Power Functions

2.1 Basic Facts and Definitions

The main results of this paper are formulated in the parametric setting. So we set up the concept of restricted log-exp-analytic power functions in single variables.

Let $l, m \in \mathbb{N}_0$. Let w range over $\mathbb{R}^l$ and u over $\mathbb{R}^m$. We fix definable sets $C, X \subset \mathbb{R}^l \times \mathbb{R}^m$ with $C \subset X$. Suppose that X_w is open for every $w \in \mathbb{R}^l$. Let $\pi_l : \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}^l, (w,u) \mapsto w$.

2.1 Definition

We call a function $g : \mathbb{R} \to \mathbb{R}$ **power function** if there is $\chi \in \mathbb{R}$ such that $g(x) = x^\chi$ for $x \in \mathbb{R}_{>0}$ and $g(x) = 0$ otherwise.

Note that power functions are definable since $x^\chi = e^{\chi \log(x)}$ for every $c \in \mathbb{R}$ and $x \in \mathbb{R}_{>0}$. Our next aim is to define restricted log-exp-analytic power functions formally which are compositions of log-analytic functions, exponentials of locally bounded functions and power functions. In the sense of Definition 1.2(c) they are precisely those functions which can be constructed from a set E of positive definable functions such that every $g \in \log(E)$ is locally bounded or $g = \chi \log(h)$ for a constant $\chi \in \mathbb{R}$ and a positive function h which can also be constructed from E. For convenience we call such a set E a *LoPo-set*.

2.2 Definition

Let E be a set of positive definable functions on C. We call E a **LoPo-set on C in u with reference set** X if the following holds: Let $e \in \mathbb{N}_0$ and let $g \in \log(E)$ be with exponential number at most e with respect to E. Then g is locally bounded in u with reference set X (i.e. there is a definable function $\tilde{g} : X \to \mathbb{R}$ with $\tilde{g}|_C = g$ where $\tilde{g}_w$ is locally bounded for $w \in \pi_l(X)$) or there is a function $h : C \to \mathbb{R}_{>0}$ which has exponential number at most e with respect to E and a constant $\chi \in \mathbb{R}$ such that $g = \chi \log(h)$.

2.3 Remark

Let E be a set of positive definable functions on C. Let $Y \subset \mathbb{R}^l \times \mathbb{R}^m$ be definable with $X \subset Y$ such that Y_w is open for every $w \in \mathbb{R}^l$. Let E be a LoPo-set in u with reference set Y. Then E is a LoPo-set in u with reference set X.

Proof

This follows from the following fact. Let $g : C \to \mathbb{R}$ be locally bounded in u with reference set Y. Then $g : C \to \mathbb{R}$ is locally bounded in u with reference set X. ■

2.4 Definition

Let $f : C \to \mathbb{R}$ be a function.

(a) Let $e \in \mathbb{N}_0$. We say that f is a **restricted log-exp-analytic power function (restricted log-exp-analytic function) in u of order (at most) e with reference set** X if f has exponential number (at most) e with respect to a LoPo-set E in u (with respect to a set E of exponentials of locally bounded functions in u) with reference set X on C.

(b) We say that f is a **restricted log-exp-analytic power function (restricted log-exp-analytic function) in u with reference set** X if f can be constructed from a LoPo-set E in u (from a set E of exponentials of locally bounded functions in u) with reference set X on C, i.e. there is $e \in \mathbb{N}_0$ and a LoPo-set E in u (a set E of exponentials of locally bounded functions in u) on C with reference set X such that f has exponential number (at most) e with respect to E.

2.5 Remark

(1) The log-analytic functions are precisely the restricted log-exp-analytic power functions in u of order (at most) 0.

(2) A restricted log-exp-analytic function $f : C \to \mathbb{R}$ in u with reference set X is a restricted log-exp-analytic power function in u with reference set X.

2.6 Example

Let $\chi \in \mathbb{R} \setminus \mathbb{Q}$. The irrational power function

$$f : \mathbb{R} \to \mathbb{R}, u \mapsto \begin{cases} u^\chi, & u > 0, \\ 0, & \text{else}, \end{cases}$$

is a restricted log-exp-analytic power function (of order (at most) 1) in u with reference set $\mathbb{R}$.

Proof

This is immediately seen with the fact that $f(u) = \exp(\chi \log(u))$ for $u \in \mathbb{R}_{>0}$ and $f(u) = 0$ otherwise: let

$$g : \mathbb{R} \to \mathbb{R}, u \mapsto \begin{cases} \exp(\chi \log(u)), & u > 0, \\ 1, & \text{else}, \end{cases}$$

and let $E := \{g\}$. Then E is a LoPo-set in u with reference set $\mathbb{R}$, since $\log(g) = \chi \log(h)$ for the log-analytic function $h : \mathbb{R} \to \mathbb{R}_{>0}$ with $h(u) = u$ for $u > 0$ and $h = 1$ otherwise. Let

$$G : \mathbb{R}^2 \to \mathbb{R}, (x_1, x_2) \mapsto \begin{cases} x_1, & x_2 > 0, \\ 0, & \text{else}. \end{cases}$$

Then G is log-analytic (and even globally subanalytic). Since $f(u) = G(g(u), u)$ for $u \in \mathbb{R}$ we see that f is a restricted log-exp-analytic power function of order 1 in u with reference set $\mathbb{R}$ (since f is not log-analytic). ■

2.7 Remark

Let $e \in \mathbb{N}_0$. Let $Y \subset \mathbb{R}^l \times \mathbb{R}^m$ be definable with $X \subset Y$ such that Y_w is open for every $w \in \mathbb{R}^l$. Let $f : C \to \mathbb{R}$ be a restricted log-exp-analytic power function in u of order at most e with reference set Y. Then f is a restricted log-exp-analytic power function in u of order at most e with reference set X.

Proof

This is directly seen with Remark 2.3. ■

2.8 Remark

Let $k \in \mathbb{N}$. For $j \in \{1, \ldots, k\}$ let $f_j : C \to \mathbb{R}$ be a restricted log-exp-analytic power function in u with reference set X. Let $F : \mathbb{R}^k \to \mathbb{R}$ be log-analytic. Then

$$C \to \mathbb{R}, u \mapsto F(f_1(u), \ldots, f_k(u)),$$

is a restricted log-exp-analytic power function in u with reference set X.

Proof

Note that f_j can be constructed from a set E_j of positive definable functions which is a LoPo-set in u with reference set X for $j \in \{1, \ldots, k\}$. Then $E := E_1 \cup \ldots \cup E_k$ is a LoPo-set in u with reference set X and for $j \in \{1, \ldots, k\}$ the function f_j can be constructed from E. With Remark 1.3(2) we are done. ■

2.9 Remark

Let $C_1, C_2 \subset \mathbb{R}^m$ be disjoint and definable with $C_1 \cup C_2 = C$. For $j \in \{1, 2\}$ let $f_j : C_j \to \mathbb{R}$ be a restricted log-exp-analytic power function in u with reference set X. Then

$$f : C \to \mathbb{R}, (w, u) \mapsto \begin{cases} f_1(w, u), & (w, u) \in C_1, \\ f_2(w, u), & (w, u) \in C_2, \end{cases}$$

is a restricted log-exp-analytic power function in u with reference set X.

Proof

For $j \in \{1, 2\}$ let E_j be a LoPo-set on C_j in x with reference set X such that f_j can be constructed from E_j. For $j \in \{1, 2\}$ let

$$\tilde{E}_j := \{\delta : C \to \mathbb{R} \mid \delta \text{ is a function with } \delta|_{C_j} \in E_j \text{ and } 1 \text{ otherwise}\}.$$

Then $\tilde{E}_j$ is a LoPo-set on C in u with reference set X: let $g \in \log(\tilde{E}_j)$ be with exponential number at most e with respect to $\tilde{E}_j$. Then $g|_{C_j} \in \log(E_j)$ has exponential number at most e with respect to $\tilde{E}_j|_{C_j} = E_j$.
If $g|_{C_j}$ is of the form $\chi \log(h)$ then g is of the form $\chi \log(\tilde{h})$ with $\tilde{h}(w, u) = h(w, u)$ for $(w, u) \in C_j$ and $\tilde{h}(w, u) = 1$ otherwise. Note that $\tilde{h}$ has exponential number at most e with respect to $\tilde{E}_j$.
If g is locally bounded in u with reference set X then $\tilde{g} : C \to \mathbb{R}$ with $\tilde{g}(w, u) = g(w, u)$ for $(w, u) \in C_j$ and 0 otherwise is also locally bounded in u with reference set X. Therefore $E := \tilde{E}_1 \cup \tilde{E}_2$ is a LoPo-set in u with reference set X from which f can be constructed. ■

2.10 Definition

A function $f : X \to \mathbb{R}$ is called a **restricted log-exp-analytic power function in** u if f is a restricted log-exp-analytic power function in u with reference set X.

2.11 Remark

Let $k \in \mathbb{N}_0$. Let $v := (v_1, \ldots, v_k)$ range over $\mathbb{R}^k$. Let $g : \mathbb{R}^k \to \mathbb{R}^m$ be log-analytic and continuous. Let

$$V := \{(w, u, v) \in X \times \mathbb{R}^k \mid (w, u + g(v)) \in X\}.$$

Let $f : X \to \mathbb{R}, (w, u) \mapsto f(w, u)$, be a restricted log-exp-analytic power function in u. Then $F : V \to \mathbb{R}, (w, u, v) \mapsto f(w, u + g(v))$, is a restricted log-exp-analytic power function in (u, v).

Proof

Note that V_w is open in $\mathbb{R}^m \times \mathbb{R}^k$ for every $w \in \mathbb{R}^l$. Let E be a LoPo-set in u with reference set X such that f can be constructed from E. Consider

$$\tilde{E} := \{V \to \mathbb{R}_{>0}, (w,u,v) \mapsto h(w, u + g(v)) \mid h \in E\}.$$

Note that F can be constructed from $\tilde{E}$. We show that $\tilde{E}$ is a LoPo-set in (u,v) with reference set V and we are done. Let $\beta \in \tilde{E}$. Then there is $h \in E$ with $\beta(w,u,v) = h(w, u+g(v))$ for $(w,u,v) \in V$. Let $e \in \mathbb{N}$ be such that h has exponential number at most e with respect to E.

Case 1: Let h be locally bounded in u with reference set X. Then β is locally bounded in (u,v) with reference set V by the claim in the proof of Remark 2.10 in [8].

Case 2: Let $\chi \in \mathbb{R}$ be a constant and $\eta : X \to \mathbb{R}_{>0}$ be with exponential number at most $e-1$ with respect to E such that $h = \exp(\chi \log(\eta))$. We obtain

$$\beta(w,u,v) = h(w, u + g(v)) = \exp(\chi \log(\eta(w, u + g(v))))$$

for $(w,u,v) \in V$. Note that $V \to \mathbb{R}_{>0}, (w,u,v) \mapsto \eta(w, u + g(v))$, has exponential number at most $(e-1)$ with respect to $\tilde{E}$. This finishes the proof. ■

2.2 A Preparation Theorem for Restricted Log-Exp-Analytic Power Functions

In this section we give a preparation theorem for restricted log-exp-analytic power functions. Our considerations start with Theorem B from [9].

Let $m, l \in \mathbb{N}_0$. Let w range over $\mathbb{R}^l$ and u over $\mathbb{R}^m$. Here $(u_1, \ldots, u_m, z)$ is serving as the tuple of independent variables of families of functions parameterized by $w := (w_1, \ldots, w_l)$. Furthermore we fix definable sets $C, X \subset \mathbb{R}^l \times \mathbb{R}^m$ with $C \subset X$ such that X_w is open for $w \in \mathbb{R}^n$.

2.12 Definition

Let $f : C \to \mathbb{R}$ be definable. Suppose that $f(x) > 0$ for every $x \in C$, $f(x) < 0$ for every $x \in C$ or $f = 0$. Then f is **a finite product of powers of definable functions** $g_1, \ldots, g_k : C \to \mathbb{R}_{>0}$ for $k \in \mathbb{N}$ if there are $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ and $\sigma \in \{-1, 0, 1\}$ such that $f = \sigma \prod_{j=1}^k g_j^{\lambda_j}$.

2.13 Definition

Let $f : C \to \mathbb{R}, (w,u) \mapsto f(w,u)$, be a function. By induction on $e \in \mathbb{N}_0 \cup \{-1\}$ we define that f is **(m,X)-power-restricted e-prepared**. To this preparation we associate a **finite set of log-analytic functions L on C which "occur" in this preparation**.

$e = -1$: The function f is (m,X)-power-restricted (-1)-prepared if f is the zero function. Then $L := \{0\}$.

$e-1 \to e$: The function f is (m,X)-power-restricted e-prepared if the following holds. There is $s \in \mathbb{N}$ such that

$$f(w,u) = a(w,u)\exp(c(w,u))v(b_1(w,u)\exp(d_1(w,u)),\ldots,b_s(w,u)\exp(d_s(w,u)))$$

for $(w,u) \in C$ where $a, b_1, \ldots, b_s$ are finite products of powers of log-analytic functions, $c, d_1, \ldots, d_s$ are locally bounded in u with reference set X and are (m,X)-power-restricted $(e-1)$-prepared. Additionally we have $b_j(w,u)\exp(d_j(w,u) \in [-1,1]$ for $(w,u) \in C$ and v is a power series which converges on an open neighbourhood of $[-1,1]^s$ with $v([-1,1]^s) \subset \mathbb{R}_{>0}$. Suppose that for c and $d_1, \ldots, d_s$ corresponding sets of log-analytic functions $L_c, L_{d_1}, \ldots, L_{d_s}$ have already been defined. Let $b_0 := a$. For $j \in \{0, \ldots, s\}$ let $\sigma_j \in \{-1,0,1\}$ and $\lambda_{j0}, \ldots, \lambda_{jk} \in \mathbb{R}$, $h_{j0}, \ldots, h_{jk} : C \to \mathbb{R}_{>0}$ be log-analytic with

$$b_j = \sigma_j \prod_{i=1}^{k} h_{ji}^{\lambda_{ji}}$$

where $k \in \mathbb{N}$. We set

$$L := L_c \cup L_{d_1} \cup \ldots \cup L_{d_s} \cup \{h_{ji} \mid j \in \{0, \ldots, s\}, i \in \{1, \ldots, k\}\}.$$

Convention

For a set E of positive definable functions on X we say that $f : X \to \mathbb{R}$ has exponential number at most -1 with respect to E if f is the zero function.

2.14 Proposition

Let $e \in \mathbb{N}_0$. Let $f : X \to \mathbb{R}$ be a restricted log-exp-analytic power function in u of order at most e. Then there is a decomposition $\mathcal{C}$ of X into finitely many definable cells such that for every $C \in \mathcal{C}$ the function $f|_C$ is (m,X)-power-restricted e-prepared.

Proof

Let $e \in \mathbb{N}_0 \cup \{-1\}$ and E be a LoPo-set in u with reference set X such that f has exponential number at most e with respect to E. We proceed by induction on e. For $e = -1$ the assertion is clear.

$e-1 \to e$: There is a decomposition $\mathcal{D}$ of X into finitely many definable cells such that for every $D \in \mathcal{D}$ there is $s \in \mathbb{N}$ such that

$$f(w,u) = a(w,u)\exp(d_0(w,u))v(b_1(w,u)\exp(d_1(w,u)),\ldots,b_s(w,u)\exp(d_s(w,u)))$$

for $(w,u) \in D$ where $a, b_1, \ldots, b_s : D \to \mathbb{R}$ are log-analytic and $d_0, d_1, \ldots, d_s : D \to \mathbb{R}$ are finite $\mathbb{Q}$-linear combinations of functions from $\log(E)$ which have exponential number at most $(e-1)$ with respect to E. Additionally $b_j(w,u)\exp(d_j(w,u)) \in [-1,1]$ for $(w,u) \in D$ and v is a power series which converges absolutely on an open neighbourhood of $[-1,1]^s$ with $v([-1,1]^s) \subset \mathbb{R}_{>0}$ (see Theorem B in [9]). Fix $D \in \mathcal{D}$ with the corresponding preparation for $f|_D$. Note that there are locally bounded $\delta_0, \dots, \delta_s : D \to \mathbb{R}$ in u with reference set X which have exponential number at most $(e-1)$ with respect to E and functions $\zeta_0, \dots, \zeta_s : D \to \mathbb{R}$ such that $d_j = \delta_j + \zeta_j$ for $j \in \{0, \dots, s\}$ and the following holds. There are $k \in \mathbb{N}$ and constants $\chi_{j1}, \dots, \chi_{jk} \in \mathbb{R}$ and positive functions $\eta_{j1}, \dots, \eta_{jk} : D \to \mathbb{R}_{>0}$ which have exponential number at most $e-1$ with respect to E such that $\zeta_j = \sum_{i=1}^k \chi_{ji} \log(\eta_{ji})$. So we obtain

$$f|_D = a(\prod_{i=1}^k \eta_{0i}^{\chi_{0i}})\exp(\delta_0)v(b_1(\prod_{i=1}^k \eta_{1i}^{\chi_{1i}})\exp(\delta_1), \dots, b_s(\prod_{i=1}^k \eta_{si}^{\chi_{si}})\exp(\delta_s)).$$

Now we use the inductive hypothesis on η_{ji} and find a decomposition $\mathcal{A}$ of D into finitely many definable cells such that for $j \in \{0, \dots, s\}$ and $i \in \{1, \dots, k\}$ we have that η_{ji} is (m, X)-power-restricted $(e-1)$-prepared, i.e. for $(w, u) \in A$

$$\eta_{ji}(w, u) = \hat{a}_{ji}(w, u)\exp(\nu_{j,i,0}(w, u))\cdot$$

$$\hat{v}_{ji}(\hat{b}_{j,i,1}(w, u)\exp(\nu_{j,i,1}(w, u)), \dots, \hat{b}_{j,i,\hat{s}}(w, u)\exp(\nu_{j,i,\hat{s}}(w, u)))$$

where $\nu_{j,i,0}, \dots, \nu_{j,i,\hat{s}} : A \to \mathbb{R}$ are locally bounded in u with reference set X, the functions $\hat{a}_{ji}, \hat{b}_{j,i,1}, \dots, \hat{b}_{j,i,\hat{s}} : A \to \mathbb{R}$ are finite products of powers of log-analytic functions and $\hat{v}_{ji}$ is a power series which converges absolutely on an open neighbourhood of $[-1, 1]^{\hat{s}}$ with $\hat{v}_{ji}([-1, 1]^{\hat{s}}) \subset \mathbb{R}_{>0}$. (By redefining the single v_{ij} we may assume that $\hat{s}$ does not depend on j.) Note also that $\hat{a}_{ji}$ is positive.

Fix $A \in \mathcal{A}$ and the corresponding preparation for $\eta_{ji}|_A$. Let

$$\beta_{ji} := \hat{v}_{ji}(\hat{b}_{j,i,1}\exp(\nu_{j,i,1}), \dots, \hat{b}_{j,i,\hat{s}}\exp(\nu_{j,i,\hat{s}})).$$

For $j \in \{0, \dots, s\}$ let

$$\omega_j := \prod_{i=1}^k \beta_{ji}^{\chi_{ji}}, \; \kappa_j := \sum_{i=1}^k \chi_{ji}\nu_{j,i,0}, \; \mu_j := \prod_{i=1}^k \hat{a}_{ji}^{\chi_{ji}}.$$

Note that $\hat{v}_{ji}^{\chi_{ji}}$ is a power series which converges absolutely on an open neighbourhood of $[-1, 1]^s$ with $\hat{v}_{ji}^{\chi_{ji}}([-1, 1]^s) \subset \mathbb{R}_{>0}$ (by using the exponential series, the logarithmic series and the fact that $\hat{v}_{ji}^{\chi_{ji}} = \exp(\chi_{ji} \log(\hat{v}_{ji}))$). We obtain

$$f|_A = a\mu_0 e^{\delta_0+\kappa_0}\omega_0 v(b_1\mu_1 e^{\delta_1+\kappa_1}\omega_1, \dots, b_s\mu_s e^{\delta_s+\kappa_s}\omega_s).$$

Note that $a\mu_0$ and $b_j\mu_j$ for $j \in \{1, \dots, s\}$ are finite product of powers of log-analytic functions. Additionally $\delta_j + \kappa_j$ is locally bounded in u with reference set X and has exponential number at most $e-1$ with respect to E for $j \in \{0, \dots, s\}$. So by the inductive hypothesis on $\delta_j + \kappa_j$ for $j \in \{0, \dots, \hat{s}\}$ we find a decomposition $\mathcal{B}$ of A into finitely many definable cells such that for every $B \in \mathcal{B}$ we have that $(\delta_j + \kappa_j)|_B$ is (m, X)-power-restricted $(e-1)$-prepared for $j \in \{0, \dots, s\}$. We are done by composition of power series. ∎

For the rest of Section 2.2 let $n \in \mathbb{N}_0$ be with $n = l + m$. Let z range over $\mathbb{R}$. Now $(u_1, \ldots, u_m, z)$ is serving as the tuple of independent variables of families of functions parameterized by $w := (w_1, \ldots, w_l)$. Let $t := (w, u)$ and let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n, (t, z) \mapsto t$. Now let $C, X \subset \mathbb{R}^n \times \mathbb{R}$ be definable sets with $C \subset X$ such that X_w is open for every $w \in \mathbb{R}^l$.

2.15 Definition

Let $e \in \mathbb{N}_0 \cup \{-1\}$ and $r \in \mathbb{N}_0$. By induction on $e \in \mathbb{N}_0 \cup \{-1\}$ we define that $f : C \to \mathbb{R}, (w, u, z) \mapsto f(w, u, z)$, is $(m + 1, X)$**-power-restricted** (e, r)**-prepared in** z and associate a **preparing tuple** to this preparation.

$e = -1$: We call f $(m + 1, X)$-power-restricted $(-1, r)$-prepared in z if f is the zero function. A preparing tuple for f is then (0).

$e - 1 \to e$: We call f $(m+1, X)$-power-restricted (e, r)-prepared in z if for $(t, z) \in C$

$$f(t, z) = a(t)|\mathcal{Y}(t, z)|^{\otimes\alpha}\exp(d_0(t, z))\rho(t, z)$$

where $a : \pi(C) \to \mathbb{R}$ is a finite product of powers of C-nice functions, $\alpha \in \mathbb{R}^{r+1}$, $d_0 : C \to \mathbb{R}$ is locally bounded in (u, z) with reference set X and is $(m + 1, X)$-power-restricted $(e - 1, r)$-prepared in z. Additionally $\rho : C \to \mathbb{R}$ is of the following form. There is $s \in \mathbb{N}$ such that $\rho = v \circ \phi$ where $\phi := (\phi_1, \ldots, \phi_s) : C \to [-1, 1]^s$ with

$$\phi_j(t, z) = b_j(t)|\mathcal{Y}(t, z)|^{\otimes\gamma_j}\exp(d_j(t, z))$$

for $j \in \{1, \ldots, s\}$ where $\gamma_j \in \mathbb{R}^{r+1}$, $b_j : \pi(C) \to \mathbb{R}$ is a finite product of powers of C-nice functions, $d_j : C \to \mathbb{R}$ is $(m + 1, X)$-power-restricted $(e - 1, r)$-prepared in z and locally bounded in (u, z) with reference set X and v is a power series which converges absolutely on an open neighbourhood of $[-1, 1]^s$ with $v([-1, 1]^s) \subset \mathbb{R}_{>0}$. A preparing tuple for f is then

$$(r, \mathcal{Y}, a, \exp(d_0), \alpha, s, v, b, \exp(d), \Gamma)$$

with $b := (b_1, \ldots, b_s)$, $\exp(d) := (\exp(d_1), \ldots, \exp(d_s))$ and

$$\Gamma := \begin{pmatrix} \gamma_{10} & \cdot & \cdot & \gamma_{1r} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \gamma_{s0} & \cdot & \cdot & \gamma_{sr} \end{pmatrix} \in \mathcal{M}\big(s \times (r + 1), \mathbb{R}\big).$$

A full preparation theorem for restricted log-exp-analytic power functions in (u, z) is the following.

2.16 Proposition

Let $e \in \mathbb{N}_0$. Let $f : X \to \mathbb{R}, (w, u, z) \mapsto f(w, u, z)$, be a restricted log-exp-analytic power function in (u, z) of order at most e. Then there is $r \in \mathbb{N}_0$ and a definable cell decomposition $\mathcal{C}$ of $X_{\neq 0}$ such that for every $C \in \mathcal{C}$ the function $f|_C$ is $(m + 1, X)$-power-restricted (e, r)-prepared in z.

Proof

By Proposition 2.14 there is a decomposition $\mathcal{D}$ of X into finitely many definable cells such that for every $D \in \mathcal{D}$ we have that $f|_D$ is $(m+1, X)$-power-restricted e-prepared. Fix $D \in \mathcal{D}$ and a corresponding finite set L of log-analytic functions on D from Definition 2.13. Let $L := \{l_1, \ldots, l_\kappa\}$ for $\kappa \in \mathbb{N}$. By Fact 1.8 there is a decomposition $\mathcal{C}$ of $D_{\neq 0}$ into finitely many definable cells such that for every $C \in \mathcal{C}$ the functions $l_1, \ldots, l_\kappa$ are r-log-analytically prepared in z with C-nice coefficent, C-nice base functions and common C-nice center. Fix $C \in \mathcal{C}$ and the corresponding center $\Theta := (\Theta_0, \ldots, \Theta_r)$ for this preparation. We proceed by induction on e. For $e = -1$ there is nothing to show.

$e-1 \to e$: We have

$$f|_C = \sigma_{b_0} b_0 \exp(d_0) v(\sigma_{b_1} b_1 \exp(d_1), \ldots, \sigma_{b_s} b_s \exp(d_s))$$

where $\sigma_{b_0}, \ldots, \sigma_{b_s} \in \{-1, 0, 1\}$, $b_j = \prod_{i=1}^k h_{ji}^{\lambda_{ji}}$ with $k \in \mathbb{N}$, $\lambda_{ji} \in \mathbb{R}$ and h_{ji} is a positive log-analytic function on C with $h_{ji} \in L$ for $i \in \{1, \ldots, k\}$ and $j \in \{0, \ldots, s\}$. Additionally $d_0, \ldots, d_s : C \to \mathbb{R}$ are locally bounded in z with reference set X and are (m, X)-power-restricted $(e-1)$-prepared. We have that $\sigma_{b_j} b_j(t,z)\exp(d_j(t,z)) \in [-1, 1]$ for $(t, z) \in C$ and the function $v : [-1, 1]^s \to \mathbb{R}$ is a power series which converges absolutely on an open neighbourhood of $[-1, 1]^s$ with $v([-1, 1]^s) \subset \mathbb{R}_{>0}$. By the inductive hypothesis we have that $d_0, \ldots, d_s$ are $(m+1, X)$-power-restricted $(e-1, r)$-prepared in z with center Θ. Let $j \in \{0, .., s\}$. Since h_{ji} is r-log-analytically prepared in z with C-nice coefficient, base functions and center Θ for $i \in \{1, \ldots, k\}$ one sees immediately that

$$b_j(t, z) = \hat{a}(t) |\mathcal{Y}(t, z)|^{\otimes \alpha} \hat{v}(\hat{b}_1(t) |\mathcal{Y}(t, z)|^{\otimes p_1}, \ldots, \hat{b}_{\hat{s}}(t) |\mathcal{Y}(t, z)|^{\otimes p_{\hat{s}}})$$

for $\alpha \in \mathbb{R}^{r+1}$, $p_i \in \mathbb{Q}^{r+1}$ and $\hat{a} : \pi(C) \to \mathbb{R}$ is a finite product of powers of C-nice functions, $\hat{b}_1, \ldots, \hat{b}_{\hat{s}} : \pi(C) \to \mathbb{R}$ are C-nice functions and $\hat{v} : [-1, 1]^{\hat{s}} \to \mathbb{R}$ is a power series which converges absolutely on an open neighbourhood of $[-1, 1]^{\hat{s}}$ with $\hat{v}([-1, 1]^{\hat{s}}) \subset \mathbb{R}_{>0}$. We are done with composition of power series. ■

With this preparation theorem we are able to prove differentiability results for restricted log-exp-analytic power functions similar as in the restricted log-exp-analytic case in [8]: we have to consider preparations of restricted log-exp-analytic power functions on simple cells (see Definition 2.18 below or Definition 3.7 in [8]). In [8] it is shown that every C-nice function is log-analytic if C is simple, i.e. the preparation simplifies. As in [8] for the restricted log-exp-analytic case and [9] for the log-analytic case we call a such a preparation *pure*.

2.17 Definition

Let $e \in \mathbb{N}_0 \cup \{-1\}$ and $r \in \mathbb{N}_0$. By induction on $e \in \mathbb{N}_0 \cup \{-1\}$ we define that $f : C \to \mathbb{R}, (w, u, z) \mapsto f(w, u, z)$, is **purely** $(m+1, X)$**-power-restricted** (e, r)**-prepared in** z and associate a **purely preparing tuple** to this preparation.

$e = -1$: We call f purely $(m+1, X)$-power-restricted $(-1, r)$-prepared in z if f is the zero function. A preparing tuple for f is then (0).

$e-1 \to e$: We call f purely $(m+1,X)$-power-restricted (e,r)-prepared in z if for $(t,z) \in C$

$$f(t,z) = a(t)|\mathcal{Y}(t,z)|^{\otimes\alpha}\exp(d_0(t,z)) \cdot \rho(t,z)$$

where $a : \pi(C) \to \mathbb{R}$ is a finite product of powers of log-analytic functions, $\alpha \in \mathbb{R}^{r+1}$, $d_0 : C \to \mathbb{R}$ is locally bounded in (u,z) with reference set X and is purely $(m+1,X)$-power-restricted $(e-1,r)$-prepared in z and ρ is a function on C of the following form. There is $s \in \mathbb{N}$ such that $\rho = v \circ \phi$ where $\phi := (\phi_1,\ldots,\phi_s) : C \to [-1,1]^s$ with

$$\phi_j(t,z) = b_j(t)|\mathcal{Y}(t,z)|^{\otimes\gamma_j}\exp(d_j(t,z))$$

for $j \in \{1,\ldots,s\}$ where $b_1,\ldots,b_s : \pi(C) \to \mathbb{R}$ are finite products of powers of log-analytic functions, $d_1,\ldots,d_s : C \to \mathbb{R}$ are locally bounded in (u,z) with reference set X and are purely $(m+1,X)$-power-restricted $(e-1,r)$-prepared in z, $\gamma_j := (\gamma_{j0},\ldots,\gamma_{jr}) \in \mathbb{R}^{r+1}$ and v is a power series on $[-1,1]^s$ which converges absolutely on an open neighbourhood of $[-1,1]^s$ and fulfills $v([-1,1]^s) \subset \mathbb{R}_{>0}$. A purely preparing tuple for f is then

$$(r,\mathcal{Y},a,\exp(d_0),\alpha,s,v,b,\exp(d),\Gamma)$$

with $b := (b_1,\ldots,b_s)$, $\exp(d) := (\exp(d_1),\ldots,\exp(d_s))$ and

$$\Gamma := \begin{pmatrix} \gamma_{10} & \cdot & \cdot & \gamma_{1r} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \gamma_{s0} & \cdot & \cdot & \gamma_{sr} \end{pmatrix} \in \mathcal{M}\big(s \times (r+1), \mathbb{R}\big).$$

2.18 Definition ([8] Definition 3.4)

Let $C \subset \mathbb{R}^n \times \mathbb{R}_{\neq 0}$ be a definable cell. We call C **simple** if for every $t \in \pi(C)$ we have $C_t =]0, d_t[$.

2.19 Proposition

Let $f : X \to \mathbb{R}, (w,u,z) \mapsto f(w,u,z)$, be a restricted log-exp-analytic power function in (u,z). Then there are $r \in \mathbb{N}_0$, $e \in \mathbb{N}_0 \cup \{-1\}$ and a definable cell decomposition $\mathcal{C}$ of X such that for every simple $C \in \mathcal{C}$ the restriction $f|_C$ is purely $(m+1,X)$-power-restricted (e,r)-prepared in z with center (0).

Proof

By Proposition 2.16 there are $r \in \mathbb{N}_0$, $e \in \mathbb{N}_0 \cup \{-1\}$ and a definable cell decomposition $\mathcal{C}$ of X such that for every $C \in \mathcal{C}$ the function $f|_C$ is $(m+1,X)$-power-restricted (e,r)-prepared in z. Fix a simple $C \in \mathcal{C}$. We show by induction on $l \in \{-1,\ldots,e\}$ that f is purely $(m+1,X)$-power-restricted (e,r)-prepared in z with center (0). For $l = -1$ there is nothing to show.

$l-1 \to l$: Let

$$(r,\mathcal{Y},a,\exp(c),\alpha,s,v,b,\exp(d),\Gamma)$$

be a preparing tuple for f with $b := (b_1,\ldots,b_s)$ and $\exp(d) := (\exp(d_1),\ldots,\exp(d_s))$. By Proposition 2.15 in [5] we have that $\hat{\Theta} = 0$ for every center $\hat{\Theta}$ of a k-logarithmic scale on C (where $k \in \mathbb{N}_0$). Consequently every C-nice function on $\pi(C)$ is log-analytic and the center Θ of $\mathcal{Y}$ vanishes. So we have that a and $b_1, \ldots, b_s$ are finite products of powers of log-analytic functions on C. So we see that f is purely $(m+1, X)$-power-restricted (e, r)-prepared in z with center (0) by the inductive hypothesis and we are done. ■

A consequence of this preparation theorem is Theorem A, a version of Proposition 3.16 from [8] for restricted log-exp-analytic power functions: a restricted log-exp-analytic power function $f : X \to \mathbb{R}, (w, u, z) \mapsto f(w, u, z)$, in (u, z) can be real log-analytically prepared in z on simple cells with coefficient and base functions which are also restricted log-exp-analytic power functions in u. This result is crucial for proving differentiability properties for restricted log-exp-analytic power functions.

2.20 Proposition

Suppose that 0 *is interior point of* X_t *for every* $t \in \pi(X)$. *Let* $f : X \to \mathbb{R}$ *be a restricted log-exp-analytic power function in* (u, z). *Then there is* $r \in \mathbb{N}_0$ *and a definable cell decomposition* $\mathcal{C}$ *of* X *such that for every simple* $C \in \mathcal{C}$ *the following holds. The restriction* $f|_C$ *is* r*-real log-analytically prepared with LA-preparing tuple*

$$(r, \mathcal{Y}, a, \alpha, s, v, b, \Gamma)$$

where a *and* $b_1, \ldots, b_s$ *are restricted log-exp-analytic power functions in* u *with reference set* $\pi(X)$ *and* $\mathcal{Y}$ *is an* r*-logarithmic scale with center* 0 *on* C.

Proof

The proof of this theorem is very similar to the proof of Proposition 3.16 in [8]. For the readers convenience we give some details.

By Proposition 2.19 there are $r \in \mathbb{N}_0$, $e \in \mathbb{N}_0 \cup \{-1\}$ and a definable cell decomposition $\mathcal{Q}$ of X such that for every simple $Q \in \mathcal{Q}$ the restriction $f|_Q$ is purely $(m+1, X)$-power-restricted (e, r)-prepared in z. Fix such a simple $Q \in \mathcal{Q}$. The following claim is the analogue of the corresponding claim from the proof of Proposition 3.16 from [8] to our situation. We omit its proof since it is the same as in [8] by replacing "log-analytically prepared" with "real log-analytically prepared" and "restricted-log-exp-analytic" with "restricted log-exp-analytic power function".

Claim

Let h be locally bounded in (u, z) with reference set X and r-real log-analytically prepared in z with coefficient and base functions which are restricted log-exp-analytic power functions in u with reference set $\pi(X)$. Then there is a definable simple set $D \subset Q$ with $\pi(D) = \pi(Q)$ such that $h = h_1 + h_2$ where

(1) $h_1 : \pi(D) \to \mathbb{R}$ is a function such that $\exp(h_1) : \pi(D) \to \mathbb{R}$ is a restricted log-exp-analytic power function in u with reference set $\pi(X)$ and

(2) $h_2 : D \to \mathbb{R}$ is a bounded function such that $\exp(h_2)$ is r-real log-analytically prepared in z with coefficient 1 and base functions which are restricted log-exp-analytic power functions in u with reference set $\pi(X)$.

We show by induction on e that there is a simple definable $A \subset Q$ with $\pi(A) = \pi(Q)$ such that $f|_A$ is r-real log-analytically prepared in z with coefficient and base functions which are restricted log-exp-analytic power functions in u with reference set $\pi(X)$. For $l = -1$ it is clear by choosing $A := Q$.

$l - 1 \to l$: Let

$$(r, \mathcal{Y}, a, e^d, \alpha, s, v, b, e^c, \Gamma)$$

be a purely preparing tuple for f where $b := (b_1, \ldots, b_s)$, $e^c := (e^{c_1}, \ldots, e^{c_s})$, and $\Gamma := (\gamma_1, \ldots, \gamma_s)^t$. Note that $a, b_1, \ldots, b_s$ are finite products of powers of log-analytic functions and that $d, c_1, \ldots, c_s$ are purely $(m+1, X)$-power-restricted $(l-1, e)$-prepared in z. We have

$$f(t,z) = a(t)|\mathcal{Y}(z)|^{\otimes\alpha} e^{d(t,z)} v(b_1(t)|\mathcal{Y}(z)|^{\otimes\gamma_1} e^{c_1(t,z)}, \ldots, b_s(t)|\mathcal{Y}(x)|^{\otimes\gamma_s} e^{c_s(t,z)})$$

for every $(t,z) \in Q$. By the inductive hypothesis and the claim we find a simple definable set $A \subset Q$ with $\pi(A) = \pi(Q)$ and functions $d_1, c_{11}, \ldots, c_{1s} : \pi(A) \to \mathbb{R}$ and $d_2, c_{21}, \ldots, c_{2s} : A \to \mathbb{R}$ with the following properties:

(1) The functions $\exp(d_1)$ and $\exp(c_{11}), \ldots, \exp(c_{1s})$ are restricted log-exp-analytic power functions in u with reference set $\pi(X)$,

(2) the functions $\exp(d_2)$ and $\exp(c_{21}), \ldots, \exp(c_{2s})$ are r-real log-analytically prepared in x with coefficient 1 and base functions which are restricted log-exp-analytic power functions in u with reference set $\pi(X)$,

(3) we have $d|_A = d_1 + d_2$ and $c_j|_A = c_{1j} + c_{2j}$ for $j \in \{1, \ldots, s\}$.

Since a and $b_1, \ldots, b_s$ are products of powers of log-analytic functions we see that the functions

$$\hat{a} : \pi(A) \to \mathbb{R}, (w,u) \mapsto a(w,u)\exp(d_1(w,u)),$$

and

$$\hat{b}_j : \pi(A) \to \mathbb{R}, (w,u) \mapsto b_j(w,u)\exp(c_{1j}(w,u)),$$

for $j \in \{1, \ldots, s\}$ are restricted log-exp-analytic power functions in u with reference set $\pi(X)$. For $(w,u,z) \in A$ we have

$$f(w,u,z) = \hat{a}(w,u)|\mathcal{Y}(z)|^{\otimes\alpha} e^{d_2(w,u,z)} v(\hat{\phi}_1(w,u,z), \ldots, \hat{\phi}_s(w,u,z))$$

where $\hat{\phi}_j(w,u,z) := \hat{b}_j(w,u)|\mathcal{Y}(z)|^{\otimes\gamma_j} e^{c_{2j}(w,u,z)}$ for $(w,u,z) \in A$ and $j \in \{1, \ldots, s\}$. By composition of power series we obtain the desired r-real log-analytical preparation for h in z.

So we find a simple definable set $\hat{C} \subset Q$ with $\pi(\hat{C}) = \pi(Q)$ such that $f|_{\hat{C}}$ is r-real log-analytically prepared in x with coefficient and base functions which are restricted log-exp-analytic in u with reference set $\pi(X)$. With the cell decomposition theorem applied to every such $\hat{C}$ we are done (compare with [2], Chapter 3). ■

3 Differentiability Properties of Restricted Log-Exp-Analytic Power Functions

Outgoing from the preparation theorem in Proposition 2.20 we give some differentiability properties of restricted log-exp-analytic power functions. We will not give all the details in the proofs, since everything from Proposition 3.17 in [8] can be formulated and proven for this class of functions in a very similar way. However we will point out the relevance of our new preparation result and explain the main differences to [8]. For an example we prove the first proposition in this section completely.

For this section we fix $l, m \in \mathbb{N}_0$. Let w range over $\mathbb{R}^l$, u over $\mathbb{R}^m$ and z over $\mathbb{R}$. Let $n := l+m$, $t := (w,u)$, and let $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, (t,z) \mapsto t$, be the projection on the first n coordinates.

3.1 Proposition

Let $X \subset \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}$ be definable such that X_w is open for every $w \in \mathbb{R}^l$. Let $f : X \to \mathbb{R}, (w,u,z) \mapsto f(w,u,z)$, be a restricted log-exp-analytic power function in (u,z). Assume that $\lim_{z \searrow 0} f(t,z) \in \mathbb{R}$ for every $t \in \pi(X)$. Then

$$h : \pi(X) \to \mathbb{R}, (w,u) \mapsto \lim_{z \searrow 0} f(w,u,z),$$

is a restricted log-exp-analytic power function in u.

Proof

By Proposition 2.20 there is $r \in \mathbb{N}_0$ and a definable cell decomposition $\mathcal{C}$ of X such that for every simple $C \in \mathcal{C}$ the restriction $f|_C$ is r-real log-analytically prepared in z with coefficient and base functions which are restricted log-exp-analytic power functions in u with reference set $\pi(X)$. Let $C \in \mathcal{C}$ be such a simple cell. Set $g := f|_C$ and let

$$(r, \mathcal{Y}, a, \alpha, s, v, b, \Gamma)$$

be a corresponding LA-preparing tuple for g. Note that $\mathcal{Y}$ has center 0. Then

$$g(w,u,z) = a(w,u)|\mathcal{Y}(z)|^{\otimes \alpha} v(b_1(w,u)|\mathcal{Y}(z)|^{\otimes \gamma_1}, \ldots, b_s(w,u)|\mathcal{Y}(z)|^{\otimes \gamma_s})$$

for $(w,u,z) \in C$. For $j \in \{1,\ldots,s\}$ we have $\lim_{z \searrow 0} |\mathcal{Y}(z)|^{\otimes \gamma_j} \in \mathbb{R}$ since $b_j \neq 0$ and $b_j(u,w)|\mathcal{Y}(z)|^{\otimes \gamma_j} \in [-1,1]$ for $(u,w,z) \in C$. Additionally we have $\lim_{z \searrow 0} |\mathcal{Y}(z)|^{\otimes \alpha} \in \mathbb{R}$ if $a \neq 0$ since $u([-1,1]^s) \subset [1/c, c]$ for a constant $c > 1$ and $\lim_{z \searrow 0} f(u,w,z) \in \mathbb{R}$ for $(u,w) \in \pi(X)$. Therefore we see that

$$A : \pi(C) \to \mathbb{R}, (w,u) \mapsto \lim_{z \searrow 0} a(w,u)|\mathcal{Y}(z)|^{\otimes \alpha},$$

and, for $j \in \{1,\ldots,s\}$, that

$$B_j : \pi(C) \to [-1,1], (w,u) \mapsto \lim_{z \searrow 0} b_j(w,u)|\mathcal{Y}(z)|^{\otimes \gamma_j},$$

are well-defined restricted log-exp-analytic power functions in u with reference set $\pi(X)$. We obtain for $(w,u) \in \pi(C)$

$$h(w,u) = A(w,u)v(B_1(w,u),\ldots,B_s(w,u)).$$

Hence $h|_{\pi(C)}$ is a restricted log-exp-analytic power function in u with reference set $\pi(X)$ by Remark 2.8. By Remark 2.9 we obtain that h is a restricted log-exp-analytic power function in u with reference set $\pi(X)$. ■

Now we obtain part (1) of Theorem B.

3.2 Proposition (Closedness under taking derivatives)
Let $X \subset \mathbb{R}^l \times \mathbb{R}^m$ be definable such that X_w is open for every $w \in \mathbb{R}^l$. Let $f : X \to \mathbb{R}, (w,u) \mapsto f(w,u)$, be a restricted log-exp-analytic power function in u. Let $i \in \{1,\ldots,m\}$ be such that f is differentiable with respect to u_i on X. Then $\partial f/\partial u_i$ is a restricted log-exp-analytic power function in u.

Proof
The proof follows exactly the proof of Theorem A in [8] with the difference that we use Remark 2.11, Proposition 2.20 and Proposition 3.1 instead of the corresponding results from [8]. ■

The fact that a restricted log-exp-analytic power function $g : Y \to \mathbb{R}, (t,z) \mapsto g(t,z)$, in z, where $Y \subset \mathbb{R}^n \times \mathbb{R}$ is definable and Y_t is open for every $t \in \mathbb{R}^n$, can be real log-analytically prepared in z on simple definable cells gives a univariate result concerning strong quasianalyticity of g in z at 0: there is $N \in \mathbb{N}$ such that if $g(t,-)$ is C^N at 0 and all derivatives of g of order at most N with respect to z vanish at 0 then $g(t,-)$ vanishes identically at a small interval around zero (compare with the proof of Proposition 3.19 in [8] with real exponents in the log-analytical preparation instead of rational ones). With this consideration we get part (2) of Theorem B.

3.3 Proposition (Strong quasianalyticity)
Let $X \subset \mathbb{R}^l \times \mathbb{R}^m$ be definable such that X_w is open and connected for every $w \in \mathbb{R}^n$. Let $f : X \to \mathbb{R}, (w,u) \mapsto f(w,u)$, be a restricted log-exp-analytic power function in u. Then there is $N \in \mathbb{N}$ with the following property. If for $w \in \mathbb{R}^l$ the function f_w is C^N and if there is $a \in X_w$ such that all derivatives up to order N vanish in a then f_w vanishes identically.

Proof
The proof follows exactly the proof of Theorem B in [8] with the difference that we use Remark 2.11 and strong quasianalyticity of restricted log-exp-analytic power functions $g : Y \to \mathbb{R}, (t,z) \mapsto g(t,z)$, in z at 0 instead of the corresponding results from [8]. ■

Another consequence of Proposition 2.20 which can be immediately shown as in [8] is a univariate result concerning real analyticity of a restricted log-exp-analytic power function $f : Y \to \mathbb{R}, (t,z) \mapsto f(t,z)$, in z at 0: there is $M \in \mathbb{N}$ such that if $f(t,-)$ is C^M at 0 then $f(t,-)$ is real analytic at 0 (compare with the proof of Proposition

3.21 in [8] with real exponents in the preparation instead of rational ones). With this consideration we get part (3) of Theorem B.

3.4 Theorem (Tamm's theorem)
Let $X \subset \mathbb{R}^l \times \mathbb{R}^m$ be definable such that X_w is open for every $w \in \mathbb{R}^l$. Let $f : X \to \mathbb{R}, (w, u) \mapsto f(w, u)$, be a restricted log-exp-analytic power function in u. Then there is $M \in \mathbb{N}$ such that for all $w \in \mathbb{R}^l$ if $f(w, -)$ is C^M at u then $f(w, -)$ is real analytic at u.

Proof
The proof is the same as the proof of Proposition 3.25 in [8] with the minor difference that we use Remark 2.11 and the property about real analyticity of restricted log-exp-analytic power functions at 0 instead of the corresponding results from [8]. ■

3.5 Corollary
Let $X \subset \mathbb{R}^l \times \mathbb{R}^m$ be definable such that X_w is open for every $w \in \mathbb{R}^l$ and let $f : X \to \mathbb{R}, (w, u) \mapsto f(w, u)$, be a restricted log-exp-analytic power function in u. Then the set of all $(w, u) \in X$ such that $f(w, -)$ is real analytic at u is definable.

3.6 Remark
The function

$$f : \mathbb{R} \to \mathbb{R}, u \mapsto \begin{cases} e^{-\frac{1}{u}}, & u > 0, \\ 0, & u \leq 0, \end{cases}$$

is not a restricted log-exp-analytic power function in u.

Proof
Note that f is flat at 0, but not the zero function. So f is not strong quasianalytic. Furthermore f is C^∞ at 0, but not real analytic. So we see with Theorem B that f is not a restricted log-exp-analytic power function in u. ■

References

[1] L. van den Dries: Geometric categories in o-minimal structures. *Duke Math. J.*, **84** no. 2 (1996), 497-540.

[2] L. van den Dries: Tame Topology and O-Minimal Structures. *London Math. Soc. Lecture Note Series*, **248**, Cambridge University Press, 1998.

[3] L. van den Dries and C. Miller: Extending Tamm's theorem. *Annales de l'Institut Fourier*, **44** no. 5 (1994), 1367-1395.

[4] L. van den Dries and P. Speissegger: O-minimal preparation theorems. *Model Theory and Applications (Quad. Mat.)*, Aracne Rome, **11** (2002), 87-116.

[5] T. Kaiser, A. Opris: Differentiability Properties of Log-Analytic Functions. *Rocky Mountain J. Math,* **52** no. 4 (2022), 1423-1443.

[6] J.-M. Lion and J.-P. Rolin: Théorème de préparation pour les fonctions logarithmico-exponentielles. *Ann. Ins. Fourier*, **47** no. 3 (1997), 859-884.

[7] C. Miller: Expansions of the real field with power functions, *Ann. Pure Appl. Logic*, **68** no. 1 (1994), 79-94.

[8] A. Opris: Restricted log-exp-analytic functions and some differentiability properties. *llinois J. Math,* **66** no. 4 (2022), 509-543.

[9] A. Opris: On preparation theorems for $\mathbb{R}_{\text{an,exp}}$-definable functions. arXiv:2112.08161, 2021.

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