THE SPACE OF r-IMMERSIONS OF A UNION OF DISCS IN $\mathbb{R}^n$

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Abstract. For a manifold M and an integer $r > 1$, the space of r-immersions of M in $\mathbb{R}^n$ is defined to be the space of immersions of M in $\mathbb{R}^n$ such that the preimage of every point in $\mathbb{R}^n$ contains fewer than r points. We consider the space of r-immersions when M is a disjoint union of k m-dimensional discs, and prove that it is equivalent to the product of the r-configuration space of k points in $\mathbb{R}^n$ and the k^{th} power of the space of injective linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$. This result is needed in order to apply Michael Weiss's manifold calculus to the study of r-immersions. The analogous statement for spaces of embeddings is "well-known", but a detailed proof is hard to find in the literature, and the existing proofs seem to use the isotopy extension theorem, if only as a matter of convenience. Isotopy extension does not hold for r-immersions, so we spell out the details of a proof that avoids using it, and applies to spaces of r-immersions.

1. Introduction

Embedding calculus (also known as manifold calculus) is a method, invented by M. Weiss [8], for analysing presheaves on manifolds. Suppose F is a contravariant functor defined on a suitable category of m-dimensional manifolds. Let D^m be the open unit disc in $\mathbb{R}^m$. One of the main ideas of embedding calculus is to first focus on the value of F on manifolds of the form $\coprod_{i=1}^{k} D^m$, and then extrapolate from there to get approximations to the value of F on general m-dimensional manifolds. For this approach to be useful, one generally needs a good understanding of the value of F on disjoint unions of copies of D^m.

The original motivating example for embedding calculus is, not coincidentally, the embedding functor $\mathrm{Emb}(-, \mathbb{R}^n)$, were $\mathbb{R}^n$ is a fixed vector space and the domain of the functor is considered to be the category of m-dimensional manifolds and codimension zero embeddings, for some fixed m. One can also replace $\mathbb{R}^n$ with a more general manifold N, but we will restrict ourselves to embeddings into a Euclidean space.

In order to apply Weiss's machinery to the embedding functor, one needs a good understanding of the homotopy type of spaces of the form $\mathrm{Emb}(\coprod_k D^m, \mathbb{R}^n)$. Fortunately, the homotopy type of these spaces is well-understood. To describe it, let $\mathrm{Conf}(k, \mathbb{R}^n) \subset (\mathbb{R}^n)^k$ be the configuration space of ordered k-tuples of pairwise distinct points in $\mathbb{R}^n$. That is, for $\underline{k} = \{1, \ldots, k\}$

$$\mathrm{Conf}(k, \mathbb{R}^n) := \mathrm{Emb}(\underline{k}, \mathbb{R}^n) \cong \{(x_1, \ldots, x_k) \in (\mathbb{R}^n)^k : x_i \neq x_j \text{ for } i \neq j\}.$$

Also let $\mathrm{Linj}(\mathbb{R}^m, \mathbb{R}^n)$ denote the space of injective linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$.

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There is a natural map

$$\mathrm{Emb}\left(\coprod_k D^m, \mathbb{R}^n\right) \xrightarrow{\simeq} \mathrm{Conf}(k, \mathbb{R}^n) \times \mathrm{Linj}(\mathbb{R}^m, \mathbb{R}^n)^k. \tag{1}$$

The map is defined by evaluating an embedding at the centers of the discs, and also differentiating at the centers of the discs. It is well-known that the map (1) is an equivalence.

Recently there has been an emerging interest in applying embedding calculus to the study of r-*immersions* [3, 7]. Given an integer $r > 1$, an r-immersion is an immersion with the property that the preimage of each point consists of fewer than r points. Let $\mathrm{rImm}(M, N)$ denote the space of r-immersions of M into N. For $r = 2$, a 2-immersion is the same thing as an injective immersion. When M is compact it is the same as an embedding. More generally, when M is tame (i.e., is the interior of a compact manifold with boundary), the space of 2-immersions is equivalent to the space of embeddings. Thus for practical purposes, we can identify the space of 2-immersions with the space of embeddings. In general, we have inclusions

$$\mathrm{Emb}(M, N) \subset 3\,\mathrm{Imm}(M, N) \subset \cdots \subset \mathrm{rImm}(M, N) \subset \cdots \subset \mathrm{Imm}(M, N).$$

In order to apply embedding calculus to the study of r-immersions, one would like to have an analogue of the equivalence (1) for r-immersions. Let $\mathrm{rConf}(k, \mathbb{R}^n)$, called the r-*configuration space*, also known as *non r-equal configuration space*, of k points in $\mathbb{R}^n$, be defined to be the space

$$\mathrm{rConf}(k, \mathbb{R}^n) := \mathrm{rImm}(\underline{k}, \mathbb{R}^n) \cong \{(x_1, \ldots, x_k) \in (\mathbb{R}^n)^k : \nexists 1 \leq i_1 < \cdots < i_r \leq k \text{ s.t. } x_{i_1} = \ldots = x_{i_r}\}.$$

There is a natural map

$$\mathrm{rImm}\left(\coprod_k D^m, \mathbb{R}^n\right) \xrightarrow{\simeq} \mathrm{rConf}(k, \mathbb{R}^n) \times \mathrm{Linj}(\mathbb{R}^m, \mathbb{R}^n)^k, \tag{2}$$

defined by evaluation at the centers of the discs and differentiation at the centers of the discs. It is a generalization of (1). Our main result (Theorem 2.1) says that this map is an equivalence. This fact is implicitly assumed in [7], and to some extent also in [3].

As we mentioned above, the case $r = 2$ of our result is well-known. However, we had trouble finding a detailed proof of it in the literature. Furthermore, proofs that we did find tend to use at some point the fact that when M_0 is a closed submanifold of M, the restriction maps $\mathrm{Emb}(M, \mathbb{R}^n) \to \mathrm{Emb}(M_0, \mathbb{R}^n)$ and $\mathrm{Imm}(M, \mathbb{R}^n) \to \mathrm{Imm}(M_0, \mathbb{R}^n)$ are fibrations. This property definitely fails for r-immersions when $2 < r < \infty$. Even when $M = \underline{k}$ is a finite set (a zero-dimensional manifold), and $\underline{k_0} \subset \underline{k}$, the restriction map $\mathrm{rConf}(k, \mathbb{R}^n) \to \mathrm{rConf}(k_0, \mathbb{R}^n)$ is not a fibration.

Our companion paper [1] relies heavily on the map (2) being an equivalence. It thus seems prudent to write out a full proof of this assertion. We hope that the result is interesting enough to stand on its own.

Notation for the derivative of a function. Since the letter D denotes a disc in $\mathbb{R}^m$, we avoid using it to denote the differential. Instead we use the same notation for derivatives as in [2]. Let U, V be open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f\colon U \to V$ be a smooth function. We define f' to be the function $f'\colon U \to \mathrm{L}(\mathbb{R}^m, \mathbb{R}^n)$ which associates to a point $x \in U$ the Fréchet derivative of f at x. Thus for each $x \in U, f'(x)$ is a linear homomorphism from $\mathbb{R}^m$ to $\mathbb{R}^n$. The linear approximation of f at 0 can be written as $Lf(x) = f(0) + f'(0)(x)$.

Similarly $f'' \colon U \to \mathrm{L}\left((\mathbb{R}^m \otimes \mathbb{R}^m)_{\Sigma_2}, \mathbb{R}^n\right)$ denotes the second derivative of f, and $f^{(i)} \colon U \to \mathrm{L}\left((\mathbb{R}^m)^{\otimes i}_{\Sigma_i}, \mathbb{R}^n\right)$ denotes the i-th derivative of f. Thus for each $x \in U$, $f^{(i)}(x)$ is a symmetric multilinear map from $\mathbb{R}^m \times \cdots \times \mathbb{R}^m$ to $\mathbb{R}^n$.

We will equip the space of multilinear maps with the norm metric. Thus

$$||f^{(i)}(x)||_{\mathrm{norm}} = \sup\{||f^{(i)}(x)(u_1, \ldots, u_i)||_{\mathbb{R}^n} : ||u_1||_{\mathbb{R}^m} = \cdots = ||u_i||_{\mathbb{R}^m} = 1\}.$$

Subsequently, we will just write $|| - ||$ to denote the norm of a vector or an operator, trusting that it is clear from the context which norm is meant.

This notion of derivative can be extended to smooth maps $f \colon M \to N$ where M, N are smooth manifolds with boundary. We will only apply it to the case when $N = \mathbb{R}^n$ and M is a finite disjoint union of either open or closed discs in $\mathbb{R}^m$.

Topology on spaces of smooth maps. We endow the space $C^\infty(M, \mathbb{R}^n)$ of smooth maps from M to $\mathbb{R}^n$ with the compact-open C^∞-topology, a.k.a the weak Whitney C^∞-topology. Thus a sequence of smooth functions $f_j \colon M \to \mathbb{R}^n$ converges to f if for every $k \geq 0$ the sequence $f_j^{(k)}$ converges to $f^{(k)}$ uniformly on every compact subspace of M. When M is a subset of $\mathbb{R}^m$, this is equivalent to saying that for every multi-index α, the sequence $\partial^\alpha f_j$ converges to $\partial^\alpha f$ uniformly on every compact subspace of M. All spaces of embeddings, immersions, r-immersions, etc. are endowed with the subspace topology from the space of smooth maps.

2. r-IMMERSIONS OF A UNION OF DISCS

Let D^m be the open unit disc in $\mathbb{R}^m$ and k a natural number. Let $\mathrm{Linj}(\mathbb{R}^m, \mathbb{R}^n)$ denote the space of injective linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$. There is a canonical map

$$\mathrm{ev} \colon \mathrm{rImm}\left(\coprod_k D^m, \mathbb{R}^n\right) \to \mathrm{rConf}(k, \mathbb{R}^n) \times \mathrm{Linj}(\mathbb{R}^m, \mathbb{R}^n)^k.$$

The map ev is defined by evaluating an r-immersion $f \colon \coprod_k D^m \to \mathbb{R}^n$ at the center of each disc D^m and also differentiating f at the center of each disc. The main result of this paper is the following theorem

Theorem 2.1. *The map* ev *is a weak homotopy equivalence.*

The proof of Theorem 2.1 goes through a construction of several intermediate spaces. To begin with, let us introduce notation for some auxiliary spaces of componentwise embeddings.

Definition 2.2. Suppose $\coprod_{i=1}^k M_i$ is a disjoint union of k manifolds, possibly with boundary. Let

$$\mathrm{CEmb}\left(\coprod_{i=1}^k M_i, \mathbb{R}^n\right) \subset C^\infty\left(\coprod_{i=1}^k M_i, \mathbb{R}^n\right)$$

be the space of smooth maps that restrict to an embedding on each component. Similarly, let $\mathrm{rImm}^{\mathrm{ce}}\left(\coprod_{i=1}^k M_i, \mathbb{R}^n\right)$ be the subspace of $\mathrm{rImm}\left(\coprod_{i=1}^k M_i, \mathbb{R}^n\right)$ consisting of those r-immersions that restrict to an embedding on each M_i.

In practice we will use this notation only when each M_i is an open or closed disc in $\mathbb{R}^m$.

Remark 2.3. It is well-known that for a compact M, possibly with boundary, $\mathrm{Emb}(M,N)$ is open in $C^\infty(M,N)$ (since M is compact, the strong and weak Whitney topologies coincide). See [5, Theorem 1.4] or [6, Proposition 9.5.9], where the case with boundary is treated more explicitly. Since there are homeomorphisms $\mathrm{CEmb}\left(\coprod_{i=1}^k M_i, \mathbb{R}^n\right) \xrightarrow{\cong} \prod_i \mathrm{Emb}\left(M_i, \mathbb{R}^n\right)$ and $C^\infty\left(\coprod_{i=1}^k M_i, \mathbb{R}^n\right) \xrightarrow{\cong} \prod_i C^\infty\left(M_i, \mathbb{R}^n\right)$, it follows that when the manifolds M_i are compact, $\mathrm{CEmb}\left(\coprod_{i=1}^k M_i, \mathbb{R}^n\right)$ is an open subset of $C^\infty\left(\coprod_{i=1}^k M_i, \mathbb{R}^n\right)$. Since

$$\mathrm{rImm}^{\mathrm{ce}}\left(\coprod_{i=1}^k M_i, \mathbb{R}^n\right) = \mathrm{rImm}\left(\coprod_{i=1}^k M_i, \mathbb{R}^n\right) \cap \mathrm{CEmb}\left(\coprod_{i=1}^k M_i, \mathbb{R}^n\right),$$

it follows that, again assuming M_i are compact, $\mathrm{rImm}^{\mathrm{ce}}\left(\coprod_{i=1}^k M_i, \mathbb{R}^n\right)$ is an open subset of $\mathrm{rImm}\left(\coprod_{i=1}^k M_i, \mathbb{R}^n\right)$.

We will now define a map that "almost" exhibits $\mathrm{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n)$ as a deformation retract of $\mathrm{rImm}\left(\coprod_k D^m, \mathbb{R}^n\right)$.

Definition 2.4. Let $f \in C^\infty\left(\coprod_k D^m, \mathbb{R}^n\right)$ and $0 \le t \le 1$. Define the map $f_t \in C^\infty\left(\coprod_k D^m, \mathbb{R}^n\right)$ by the formula $f_t(x) = f(tx)$. It is clear that if f is an r-immersion then so is f_t for all $0 < t \le 1$, but not for $t = 0$. Likewise, if f is a componentwise embedding, then so is f_t for all $0 < t \le 1$. Define the map

$$H\colon \mathrm{rImm}\left(\coprod_k D^m, \mathbb{R}^n\right) \times (0,1] \to \mathrm{rImm}\left(\coprod_k D^m, \mathbb{R}^n\right)$$

by the formula

$$H(f,t) = f_t.$$

For all $f \in \mathrm{rImm}\left(\coprod_k D^m, \mathbb{R}^n\right)$, since f is a local embedding, there exists an $\epsilon > 0$ such that for all $0 < t \le \epsilon$, $H(f,t) \in \mathrm{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n)$. The following easy lemma is a strengthening of this obsevation.

Lemma 2.5. *For every $f \in \mathrm{rImm}\left(\coprod_k D^m, \mathbb{R}^n\right)$ there exists an open neighbourhood U of f and an $\epsilon > 0$ such that $H(U \times (0, \epsilon]) \subset \mathrm{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n)$.*

Proof. Let $f\colon \coprod_k D^m \to \mathbb{R}^n$ be an r-immersion. For $0 < \epsilon < 1$, let $B_\epsilon^m \subset D^m$ denote the closed disc of radius ϵ. Since f is a local embedding one can find an $\epsilon > 0$ so that f restricts to an embedding of B_ϵ^m for each copy of D^m. This means that the restriction map

$$\rho\colon \mathrm{rImm}\left(\coprod_k D^m, \mathbb{R}^n\right) \to \mathrm{rImm}\left(\coprod_k B_\epsilon^m, \mathbb{R}^n\right)$$

takes f into the subspace $\mathrm{rImm}^{\mathrm{ce}}(\coprod_k B_\epsilon^m, \mathbb{R}^n) \subset \mathrm{rImm}\left(\coprod_k B_\epsilon^m, \mathbb{R}^n\right)$. By Remark 2.3 it is an open subset. Thus $\rho^{-1}\left(\mathrm{rImm}^{\mathrm{ce}}(\coprod_k B_\epsilon^m, \mathbb{R}^n)\right)$ is the required open neighborhood of f in $\mathrm{rImm}\left(\coprod_k D^m, \mathbb{R}^n\right)$. ☐

Corollary 2.6. *For any compact space K and map $K \to \mathrm{rImm}\left(\coprod_k D^m, \mathbb{R}^n\right)$ there exists an $\epsilon > 0$ such that the composition*

$$K \times (0,1] \to \mathrm{rImm}\left(\coprod_k D^m, \mathbb{R}^n\right) \times (0,1] \xrightarrow{H} \mathrm{rImm}\left(\coprod_k D^m, \mathbb{R}^n\right)$$

takes $K \times (0, \epsilon]$ into $\mathrm{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n)$.

Proof. Suppose we have a map $h : K \to \mathrm{rImm}(\coprod_k D^m, \mathbb{R}^n)$. Let $x \in K$. Then $h(x)$ is an r-immersion. By Lemma 2.5, there is an open neighborhood V of $h(x)$ in $\mathrm{rImm}(\coprod_k D^m, \mathbb{R}^n)$, and a positive number ϵ_x such that all the elements of $H(V \times (0, \epsilon_x])$ are componentwise embeddings. Let $U_x = h^{-1}(V)$. Thus for every $x \in K$ we found an open neighborhood U_x and a positive number ϵ_x such that every r-immersion in $h(U_x)$ restricts to an embedding of discs of radius ϵ_x. By compactness of K, there is a finite collection of points, say $x_1, \ldots, x_l$ such that $U_{x_1}, \ldots, U_{x_l}$ cover K. Let $\epsilon = \min\{\epsilon_{x_1}, \ldots, \epsilon_{x_l}\}$. Then every r-immersion in $h(K)$ restricts to an embedding of discs of radius ϵ. But this means that every element of $H(h(K) \times (0, \epsilon])$ is a componentwise embedding. □

Now we can complete the first important step toward proving Theorem 2.1.

Proposition 2.7. *The inclusion*

$$\mathrm{rImm}^{\mathrm{ce}}\left(\coprod_k D^m, \mathbb{R}^n\right) \hookrightarrow \mathrm{rImm}\left(\coprod_k D^m, \mathbb{R}^n\right) \tag{3}$$

is a weak homotopy equivalence.

Proof. First of all, the inclusion is surjective on π_0. Indeed, suppose $f \in \mathrm{rImm}\left(\coprod_k D^m, \mathbb{R}^n\right)$. Then $H(\{f\} \times (0,1])$ defines a path from f to a point in $\mathrm{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n)$.

Second, let us show that the inclusion is injective on π_0. Suppose $f, g \in \mathrm{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n)$ and there is a path $\alpha \colon [0,1] \to \mathrm{rImm}\left(\coprod_k D^m, \mathbb{R}^n\right)$ from f to g. Consider the composition

$$[0,1] \times (0,1] \xrightarrow{\alpha \times 1_{(0,1]}} \mathrm{rImm}\left(\coprod_k D^m, \mathbb{R}^n\right) \times (0,1] \xrightarrow{H} \mathrm{rImm}\left(\coprod_k D^m, \mathbb{R}^n\right).$$

It follows from Corollary 2.6 that for some $0 < \epsilon < 1$ this map restricts to a map

$$[0,1] \times [\epsilon, 1] \xrightarrow{H \circ (\alpha \times 1_{[\epsilon,1]})} \mathrm{rImm}\left(\coprod_k D^m, \mathbb{R}^n\right)$$

that sends $[0,1] \times \{\epsilon\}$ into $\mathrm{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n)$. Moreover, since α maps $\partial([0,1])$ into the subspace $\mathrm{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n)$, and H preserves this subspace, it follows that $H \circ (\alpha \times 1_{[\epsilon,1]})$ maps $\partial([0,1]) \times [\epsilon, 1]$ into $\mathrm{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n)$. Altogether it follows that $H \circ (\alpha \times 1_{[\epsilon,1]})$ takes $[0,1] \times \{\epsilon\} \cup \partial([0,1]) \times [\epsilon, 1]$ into $\mathrm{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n)$. Therefore $H \circ (\alpha \times 1_{[\epsilon,1]})$ defines a path homotopy between α, and a path from f to g that lies entirely in $\mathrm{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n)$. We have proved that the inclusion is injective on π_0.

Now let us choose a basepoint in $\mathrm{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n)$ and let it also serve as the basepoint of $\mathrm{rImm}\,(\coprod_k D^m, \mathbb{R}^n)$. We want to show that for all $d \geq 1$ the induced homomorphism

$$\pi_d\left(\mathrm{rImm}^{\mathrm{ce}}\left(\coprod_k D^m, \mathbb{R}^n\right)\right) \to \pi_d\left(\mathrm{rImm}\left(\coprod_k D^m, \mathbb{R}^n\right)\right)$$

is an isomorphism. For surjectivity, let $h\colon S^d \to \mathrm{rImm}\,(\coprod_k D^m, \mathbb{R}^n)$ be a pointed map. We need to show that h is pointed homotopic to a map $S^d \to \mathrm{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n)$. Using Corollary 2.6 once more we conclude that there exists an $0 < \epsilon < 1$ such that the composition

$$S^d \times [\epsilon, 1] \xrightarrow{h\times 1_{[\epsilon,1]}} \mathrm{rImm}\left(\coprod_k D^m, \mathbb{R}^n\right) \times [\epsilon, 1] \xrightarrow{H} \mathrm{rImm}\left(\coprod_k D^m, \mathbb{R}^n\right)$$

takes $S^d \times \{\epsilon\}$ into $\mathrm{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n)$. We obtained an *unpointed* homotopy of h to a map $h_\epsilon\colon S^d \to \mathrm{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n)$. Furthermore, h takes the basepoint of S^d into $\mathrm{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n)$, so it follows that the homotopy $H \circ (h \times 1_{[\epsilon,1]})$, while not constant on the basepoint, keeps the basepoint inside $\mathrm{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n)$. It follows that h is pointed homotopic to a conjugation of h_ϵ by a path in $\mathrm{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n)$, which completes the proof of surjectivity on π_d.

Finally, we need to show that the inclusion is injective on π_d. Suppose $h\colon S^d \to \mathrm{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n)$ represents an element of the kernel. It means that h extends to a map $\tilde{h}\colon D^{d+1} \to \mathrm{rImm}\,(\coprod_k D^m, \mathbb{R}^n)$. Using H and Corollary 2.6 once again, one can show that $\tilde{h}$ can be deformed into a map $D^{d+1} \to \mathrm{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n)$ that defines a null homotopy of h, thus proving that h represents zero in $\pi_d\,(\mathrm{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n))$. The details of this last step are left to the reader. □

Let us say that an immersion of $\coprod_k D^m$ is non-r-overlapping if the intersection of images of every r components is empty. Note that a componentwise embedding is an r-immersion if and only if it is non-r-overlapping.

The following definition is taken from [4].

Definition 2.8. Let $f\colon \coprod_k D^m \to \mathbb{R}^n$ be a map. For each $1 \leq i \leq k$, let 0_i be the center of the i-th copy of D^m in the coproduct $\coprod_k D^m$. For each r-tuple of integers $\vec{i} = (i_1, \ldots, i_r)$, where $1 \leq i_1 < i_2 < \cdots < i_r \leq k$, let

$$f_{\vec{i}} = \frac{f(0_{i_1}) + \cdots + f(0_{i_r})}{r}.$$

Finally define $sd(f)$ by the following formula

$$sd(f) = \frac{1}{\sqrt{r}} \min_{1\leq i_1<\cdots<i_r\leq k} \sqrt{\sum_{j=1}^{r} ||f(0_{i_j}) - f_{\vec{i}}||^2}.$$

The notation sd stands for "safe distance". It is a distance for which it is guaranteed that if the radius of each disc is less than the safe distance, then the immersion is non-r-overlapping (as we will prove shortly). The formula for $sd(f)$ does not give the largest possible safe distance, but what matters is that $sd(f)$ depends continuously on f.

Next, let us make precise the notion of a radius of an immersion of a union of discs.

Definition 2.9. Let $f\colon \coprod_k D^m \to \mathbb{R}^n$ be a map. Let D^m_i denote the i-th copy of D^m in the coproduct. Define the radius of f to be the following

$$R(f) = \sup_{1\le i\le k,\, x\in D^m_i} ||f(x) - f(0_i)||.$$

Note that $R(f)$ can be ∞. But if, for example, f is the restriction of a map defined on a union of closed unit discs then $R(f)$ is finite.

The point of the last two definitions is that they give a condition for an immersion to be non-r-overlapping. The following lemma is present implicitly in [4].

Lemma 2.10. *Let $f\colon \coprod_k D^m \to \mathbb{R}^n$ be a map. If $R(f) < sd(f)$ then f is non-r-overlapping.*

Proof. Suppose by contradiction that f is r-overlapping. Then there exists an r-tuple $\vec{i} = (i_1, \ldots, i_r)$ and points $x_{i_j} \in D^m_{i_j}$ such that $f(x_{i_1}) = \cdots = f(x_{i_r})$. Let z denote this common value. Recall that $f_{\vec{i}}$ is the centroid of $f(0_{i_1}), \ldots, f(0_{i_r})$. We have the following inequalities

$$r\cdot sd(f)^2 \le \sum_{j=1}^{r} ||f(0_{i_j}) - f_{\vec{i}}||^2 \le \sum_{j=1}^{r} ||f(0_{i_j}) - z||^2 \le r\cdot R(f)^2$$

which contradicts the assumption $R(f) < sd(f)$. □

Next, let us introduce the homotopy between a smooth map $f\colon \coprod_k D^m \to \mathbb{R}^n$ and its linearization. It is a standard tool in the study of embeddings of a disc or a union of discs.

Definition 2.11. Define the map $\Phi\colon C^\infty(\coprod_k D^m, \mathbb{R}^n) \times [0,1] \to C^\infty(\coprod_k D^m, \mathbb{R}^n)$ as follows. Let D^m_i denote the i-th copy of D^m in $\coprod_k D^m$, let 0_i be the center of D^m_i and let $x \in D^m_i$. Then

$$\Phi(f,t)(x) = \begin{cases} f(0_i) + \frac{f(xt) - f(0_i)}{t} & t > 0 \\ f(0_i) + f'(0_i)(x) & t = 0 \end{cases}$$

We need to know that Φ is a continuous function. This is a standard result, but we did not find a detailed proof of it, so for the reader's convenience we include one.

Lemma 2.12. *The function Φ of Definition 2.11 is continuous.*

Proof. Continuity at points where $t > 0$ really is obvious and is left to the reader. We shall address continuity at points where $t = 0$. The weak topology on $C^\infty(\coprod_k D^m, \mathbb{R}^n)$ is first countable, so it is enough to prove that Φ is sequentially continuous. Suppose we have a sequence (f_j, t_j) in $C^\infty(\coprod_k D^m, \mathbb{R}^n) \times [0,1]$ converging to $(f, 0) \in C^\infty(\coprod_k D^m, \mathbb{R}^n) \times [0,1]$. We have to show that $\Phi(f_j, t_j)$ converges to the linearization of f. This means that we have to show that $\Phi(f_j, t_j)$ converges uniformly to f on any compact subset of $\coprod_k D^m$, and the same holds for all derivatives of these functions.

Let us first prove convergence on the level of functions themselves. We can use linear Taylor approximation to write, for each j and $x \in D^m_i$

$$f_j(x) = f_j(0_i) + f_j'(0_i)(x) + E_j(x)$$

where E_j is the error term. It follows that

$$\Phi(f_j, t_j)(x) = f_j(0_i) + f_j'(0_i)(x) + \frac{E_j(xt_j)}{t_j} \tag{4}$$

where by convention $\frac{E_j(xt)}{t} = 0$ when $t = 0$.

Let $K \subset \coprod_k D^m$ be a compact subset. We need to show that $\Phi(f_j, t_j)(x)$ converges to $\Phi(f, 0)(x)$ uniformly in x, when x is restricted to K. Let us define the constants M_j, M as follows

$$M_j = \sup_{x \in K}(||f_j''(x)||), \quad M = \sup_{x \in K}(||f''(x)||)$$

where $||-||$ denotes the operator norm. Since K is compact, M_j, M are finite. Since f_j converges to f in the C^∞ topology, the sequence M_j converges to M, and in particular it is bounded.

By Taylor's theorem for vector-valued functions [2, Theorem 5.6.2] we have the estimate

$$||E_j(xt)|| \leq \frac{M_j}{2}||x||^2 t^2 \leq \frac{M_j}{2} t^2.$$

Note that $E_j(xt) \in \mathbb{R}^n$ and $x \in \mathbb{R}^m$, so the two occurrences of $|| - ||$ denote the Euclidean norm in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. It follows that the error estimate $\frac{E_j(xt_j)}{t_j}$ in (4) satisfies the following estimate for all j and x:

$$\left\| \frac{E_j(xt_j)}{t_j} \right\| \leq \frac{M_j}{2} t_j. \tag{5}$$

Therefore the following holds, where as usual $x \in D_i^m$:

$$\begin{aligned} \|\Phi(f,0)(x) - \Phi(f_j, t_j)(x)\| &= \left\| f(0_i) + f'(0_i)(x) - \left(f_j(0_i) + f_j'(0_i)(x) + \tfrac{E_j(xt_j)}{t_j} \right) \right\| \\ &\leq \|f(0_i) - f_j(0_i)\| + \left\| f'(0_i) - f_j'(0_i) \right\| + \tfrac{M_j}{2} t_j. \end{aligned}$$

Here the $|| - ||$ sign refers to the euclidean norm in $\mathbb{R}^n$ in $\|f(0_i) - f_j(0_i)\|$, and to the operator norm in $\left\| f'(0_i) - f_j'(0_i) \right\|$. Since f_j converges to f in the weak Whitney C^∞-topology, and $t_j \xrightarrow{n \to \infty} 0$, and M_j is bounded, it is clear that the right hand side of the inequality converges to zero as $j \to \infty$, independently of x. We have proved the convergence on the level of functions.

Now let us look at derivatives. Fix a compact set K as above. It is easy to check that for all j and $x \in D_i^m$, $\Phi(f_j, t_j)'(x) = f_j'(xt_j)$, while $\Phi(f, 0)'(x) = f'(0_i)$. We have an estimate

$$\left\| f_j'(xt_j) - f_j'(0_i) \right\| \leq M_j t_j.$$

Since t_j converges to 0, it follows that by taking j large enough we can make $\left\| f_j'(xt_j) - f_j'(0_i) \right\|$ arbitrarily small, for all $x \in K \cap D_i^m$. Since f_j converges to f in the C^∞-topology, we can also make $||f_j'(0_i) - f'(0_i)||$ arbitrarily small. It follows that by taking j large enough we can make $||f_j'(xt_j) - f'(0_j)|| = \|\Phi(f_j, t_j)'(x) - \Phi(f, 0)'(x)\|$ arbitrarily small, which means that $\Phi(f_j, t_j)'$ converges to $\Phi(f, 0)'$ uniformly on K.

Finally, suppose $i > 1$. It is not hard to check that $\Phi(f_j, t_j)^{(i)}(x) = f_j^{(i)}(xt_j)t_j^{i-1}$ and $\Phi(f, 0)^{(i)}(x) = 0$. Since $f_j^{(i)}$ converges to $f^{(i)}$ uniformly on K, it follows that $||f_j^{(i)}(x)||$ is uniformly bounded on K, and thus $||f_j^{(i)}(xt_j)t_j^{i-1}||$ can be made arbitrarily small by taking j large enough. This proves convergence for higher derivatives. □

We need to define one more invariant of a map of a union of discs. It measures the largest possible radius attained by f during the homotopy Φ.

Definition 2.13. Let $f\colon \coprod_k D^m \to \mathbb{R}^n$ be a map. Let D^m_i denote the i-th copy of D^m in the coproduct. Define $LR(f)$ by the following formula

$$LR(f) = \sup_{1\le i\le k,\, 0\le t\le 1,\, x\in D^m_i} ||\Phi(f,t)(x) - f(0_i)|| = \sup_{1\le i\le k,\, 0<t\le 1,\, x\in D^m_i} \frac{||f(tx) - f(0_i)||}{t}.$$

Just as with $R(f)$, $LR(f)$ can be infinite. Indeed, it always holds that $R(f) \le LR(f)$. But if, for example, f is the restriction of a differentiable function defined on a union of closed discs then $LR(f) < \infty$. Recall that for $0 \le s \le 1$, f_s is defined by the formula $f_s(x) = f(sx)$. It follows that if f is differentiable and $0 \le s < 1$ then $LR(f_s) < \infty$. Let us record this simple fact in a lemma.

Lemma 2.14. *Suppose f is differentiable*

1. *Whenever $s < 1$, $LR(f_s) < \infty$.*
2. *$LR(f_s) \le s \cdot LR(f)$.*

Proof. (1) The case $s = 0$ is trivial, because f_0 is constant on each disc, and $LR(f_0) = 0$. For $x \in D^m_i$ and $0 < s < 1$, $sx \in sD^m_i$, where sD^m_i is the open disc of radius s contained in D^m_i whose closure $\overline{sD^m_i}$ is also contained in D^m_i, so it is easy to define an extension of f_s to a union of closed discs.

(2) The case $s = 0$ is trivial again. Let $s > 0$. For $x \in D^m_i$, $sx \in sD^m_i$, and if 0_i is the center of the disc D^m_i, then 0_i is also the center of the disc sD^m_i. By definition,

$$\begin{aligned} LR(f_s) = \sup_{1\le i\le k,\, 0<t\le 1,\, sx\in sD^m_i} \frac{||f(stx) - f(0_i)||}{t} &= s \cdot \sup_{1\le i\le k,\, 0<t\le 1,\, sx\in sD^m_i} \frac{||f(stx) - f(0_i)||}{st} \\ &\le s \cdot \sup_{1\le i\le k,\, 0<t\le 1,\, x\in D^m_i} \frac{||f(tx) - f(0_i)||}{t} \\ &= s \cdot LR(f) \end{aligned}$$

□

We have the following simple but important observation.

Lemma 2.15. *Fix an $0 < s < 1$. Then $LR(f_s)$ depends continuously on f.*

Proof. For a fixed $f \in C^\infty(\coprod_k D^m, \mathbb{R}^n)$, let us define the function $\Psi_f\colon (\coprod_k D^m) \times [0,1] \to \mathbb{R}^n$ as follows: for $x \in D^m_i$ and $0 \le t \le 1$

$$\Psi_f(x,t) = \Phi(f,t)(x) - f(0_i) = \begin{cases} \frac{f(xt)-f(0_i)}{t} & t > 0 \\ f'(0_i)(x) & t = 0 \end{cases}$$

Suppose we have a sequence $f_j \in C^\infty(\coprod_k D^m, \mathbb{R}^n)$, converging to f in the usual C^∞-topology, and we fix an $s < 1$. We claim that in this case $\Psi_{(f_j)_s}$ converges uniformly to Ψ_{f_s}. Since $LR(f) = \sup(\Psi_f)$, it follows that the sequence $LR((f_j)_s)$ converges to $LR(f_s)$, which is what we want to prove.

It remains to prove the claim. Since the sequence f_j converges to f, and $s < 1$, it follows that the sequence $(f_j)_s$ converges uniformly to f_s on the entire space $\coprod_k D^m$, and same holds for all derivatives. It follows easily that for any $\delta > 0$, $\Psi_{(f_j)_s}$ converges uniformly to Ψ_{f_s} on the space

$(\coprod_k D^m) \times [\delta, 1]$. To establish convergence near $t = 0$, we use the estimates in the proof of Lemma 2.12. Applying formula (4), we can write the following:

$$\Psi_{(f_j)_s}(x,t) - \Psi_{f_s}(x,t) = (f_j)'_s(0_i)(x) + \frac{E_j(xt)}{t} - \left(f'_s(0_i)(x) + \frac{E(xt)}{t} \right).$$

Here E is the error term for the linear approximation of f. Defining M_j and M as in the proof of Lemma 2.12, and using inequality (5), we get the following inequalities

$$\begin{aligned} \left\| \Psi_{(f_j)_s}(x,t) - \Psi_{f_s}(x,t) \right\| &\leq \|(f_j)'_s(0_i)(x) - f'_s(0_i)(x)\| + \left\| \tfrac{E_j(xt)}{t} \right\| + \left\| \tfrac{E(xt)}{t} \right\| \\ &\leq \|(f_j)'_s(0_i) - f'_s(0_i)\| + \tfrac{M_j}{2}t + \tfrac{M}{2}t. \end{aligned}$$

Note that in the last line $\|(f_j)'_s(0_i) - f'_s(0_i)\|$ denotes the operator norm of $(f_j)'_s(0_i) - f'_s(0_i)$.

Since $(f_j)'_s(0_i)$ converges to $f'_s(0_i)$, and the sequence M_j is bounded, it is clear that for all $\epsilon > 0$ we can find a $\delta > 0$ and an integer j_1, such that for all $0 \leq t \leq \delta$ and $j > j_1$ the terms $||(f_j)'_s(0_i) - f'_s(0_i)||$, $\frac{M_j}{2}t$ and $\frac{M}{2}t$ are each smaller than $\frac{\epsilon}{3}$. It follows that for all $j > j_1$ and $(x,t) \in \coprod_k D^m \times [0, \delta]$, $\left\| \Psi_{(f_j)_s}(x,t) - \Psi_{f_s}(x,t) \right\| < \epsilon$.

We also can find an j_2 such that for all $j > j_2$ and $(x,t) \in \coprod_k D^m \times [\delta, 1]$ it holds $||\Psi_{(f_j)_s}(x,t) - \Psi_{f_s}(x,t)|| < \epsilon$. Thus for all $j > \max(j_1, j_2)$ the inequality $||\Psi_{(f_j)_s}(x,t) - \Psi_{f_s}(x,t)|| < \epsilon$ holds for all x and t. This means that $\Psi_{(f_j)_s}$ converges uniformly to Ψ_{f_s}, and we have proved the claim. ☐

Definition 2.16. We say that a function $f\colon \coprod_k D^m \to \mathbb{R}^n$ is small, if $LR(f) < sd(f)$. Let

$$\mathrm{rImm}^{\mathrm{sm}}\left(\coprod_k D^m, \mathbb{R}^n \right) \subset \mathrm{rImm}^{\mathrm{ce}}\left(\coprod_k D^m, \mathbb{R}^n \right)$$

denote the subspace consisting of r-immersions that are componentwise embeddings and are small.

Note that if f is small then $\Phi(f,t)$ satisfies the hypothesis of Lemma 2.10 for all t, and thus $\Phi(f,t)$ is non-r-overlapping for all t.

Proposition 2.17. *The inclusion* $\mathrm{rImm}^{\mathrm{sm}}(\coprod_k D^m, \mathbb{R}^n) \hookrightarrow \mathrm{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n)$ *is a homotopy equivalence.*

Proof. Let us define the function $\alpha\colon \mathrm{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n) \to (0,1)$ by the formula

$$\alpha(f) = \min\left(\frac{1}{2}, \frac{sd(f)}{4LR(f_{\frac{1}{2}})} \right).$$

Notice that $LR(f_{\frac{1}{2}}) < \infty$ by Lemma 2.14 (1), and therefore $\alpha(f)$ is a well-defined positive number smaller than 1. Also notice that $sd(f)$ is obviously continuous in f, and $LR(f_{\frac{1}{2}})$ is continuous by Lemma 2.15. Therefore α is a continuous function.

Next, let $j_f\colon [0,1] \to [\alpha(f), 1]$ be the canonical linear homeomorphism. Let j be the function

$$j\colon \mathrm{rImm}^{\mathrm{ce}}\left(\coprod_k D^m, \mathbb{R}^n \right) \times [0,1] \to [0,1]$$

defined by the formula $j(f,t) = j_f(t)$, then j is continuous in both f and t.

Now let us define a homotopy

$$H\colon \operatorname{rImm}^{\mathrm{ce}}\left(\coprod_k D^m, \mathbb{R}^n\right) \times [0,1] \to \operatorname{rImm}^{\mathrm{ce}}\left(\coprod_k D^m, \mathbb{R}^n\right)$$

by the formula $H(f,t) = f_{j_f(t)}$. Since j_f is continuous in f and t, H is continuous. Since $j_f(1) = 1$ and $j_f(0) = \alpha(f)$, H is a homotopy between the identity function on $\operatorname{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n)$ and the function that sends f to $f_{\alpha(f)} = f_{\min(\frac{1}{2}, \frac{sd(f)}{4LR(f_{\frac{1}{2}})})}$.

Let us check that $f_{\min(\frac{1}{2}, \frac{sd(f)}{4LR(f_{\frac{1}{2}})})}$ is small. This means to check that

$$LR(f_{\min(\frac{1}{2}, \frac{sd(f)}{4LR(f_{\frac{1}{2}})})}) \leq sd(f_{\min(\frac{1}{2}, \frac{sd(f)}{4LR(f_{\frac{1}{2}})})}).$$

Note that $sd(f)$ only depends on the images of the centers of D^ms under f, and therefore $sd(f_s) = sd(f)$ for any s. So we need to prove that $LR(f_{\min(\frac{1}{2}, \frac{sd(f)}{4LR(f_{\frac{1}{2}})})}) \leq sd(f)$.

Suppose first that $\frac{sd(f)}{4LR(f_{\frac{1}{2}})} \leq \frac{1}{2}$. Then we have the inequalities (here we use Lemma 2.14 (2))

$$LR(f_{\min(\frac{1}{2}, \frac{sd(f)}{4LR(f_{\frac{1}{2}})})}) = LR(f_{\frac{sd(f)}{4LR(f_{\frac{1}{2}})}}) \leq \frac{sd(f)}{2LR(f_{\frac{1}{2}})} LR(f_{\frac{1}{2}}) = \frac{sd(f)}{2} < sd(f).$$

Now suppose that $\frac{1}{2} \leq \frac{sd(f)}{4LR(f_{\frac{1}{2}})}$. Then we have the inequality $LR(f_{\frac{1}{2}}) \leq \frac{sd(f)}{2}$ and

$$LR(f_{\min(\frac{1}{2}, \frac{sd(f)}{4LR(f_{\frac{1}{2}})})}) = LR(f_{\frac{1}{2}}) \leq \frac{sd(f)}{2} < sd(f).$$

We have shown that $H(f,0)$ is small for every f. It is clear that if f is small, then $H(f,t)$ is small for all t. We have shown that H induces a homotopy between the identity map on $\operatorname{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n)$ and a map $\operatorname{rImm}^{\mathrm{ce}}(\coprod_k D^m, \mathbb{R}^n) \to \operatorname{rImm}^{\mathrm{sm}}(\coprod_k D^m, \mathbb{R}^n)$, which serves as a homotopy inverse to the inclusion. $\square$

The next step is to show that the space of small r-immersions that are componentwise embeddings is equivalent to the space of small r-immersions that are componentwise affine.

Definition 2.18. Let $\operatorname{rImm}^{\mathrm{aff}}(\coprod_k D^m, \mathbb{R}^n) \subset \operatorname{rImm}^{\mathrm{sm}}(\coprod_k D^m, \mathbb{R}^n)$ be the subspace consisting of r-immersions that are affine on each component (and are small).

Proposition 2.19. *The space* $\operatorname{rImm}^{\mathrm{aff}}(\coprod_k D^m, \mathbb{R}^n)$ *is a deformation retract of* $\operatorname{rImm}^{\mathrm{sm}}(\coprod_k D^m, \mathbb{R}^n)$.

Proof. Recall the map $\Phi\colon C^\infty(\coprod_k D^m, \mathbb{R}^n) \times [0,1] \to C^\infty(\coprod_k D^m, \mathbb{R}^n)$ from Definition 2.11. It is easy to check from the definitions that

1. Φ restricts to a map $\operatorname{rImm}^{\mathrm{sm}}(\coprod_k D^m, \mathbb{R}^n) \times [0,1] \to \operatorname{rImm}^{\mathrm{sm}}(\coprod_k D^m, \mathbb{R}^n)$
2. If f is affine on each component, meaning that $f(x) = f(0_i) + f'(0_i)(x)$ for all $x \in D^m_i$, then it's easily seen that $\Phi(f,t)(x) = f(x)$ for all t. So the homotopy Φ is constant on $\operatorname{rImm}^{\mathrm{aff}}(\coprod_k D^m, \mathbb{R}^n)$.
3. For every f the function $x \mapsto \Phi(f,0)(x)$ is affine on each component.

It follows that Φ defines a deformation retraction of $\operatorname{rImm}^{\mathrm{sm}}(\coprod_k D^m, \mathbb{R}^n)$ onto $\operatorname{rImm}^{\mathrm{aff}}(\coprod_k D^m, \mathbb{R}^n)$. $\square$

Next, we can prove that the evaluation map restricted to $\mathrm{rImm}^{\mathrm{aff}}(\coprod_k D^m, \mathbb{R}^n)$ is an equivalence.

Lemma 2.20. *The map*

$$\mathrm{ev}\colon\ \mathrm{rImm}^{\mathrm{aff}}\left(\coprod_k D^m, \mathbb{R}^n\right) \to \mathrm{rConf}(k, \mathbb{R}^n) \times \mathrm{Linj}(\mathbb{R}^m, \mathbb{R}^n)^k$$

defined by evaluating at the center of each disc D^m and also differentiating at the center of each disc, is a homotopy equivalence.

Proof. For affine maps, the smallness condition amounts to the following inequality, that has to hold for each i between 1 and k:

$$||f'(0_i)|| \leq sd(f).$$

Here $||f'(0_i)||$ denotes the operator norm of $f'(0_i)$. It is easy to show that the space $\mathrm{rConf}(k, \mathbb{R}^n) \times \mathrm{Linj}(\mathbb{R}^m, \mathbb{R}^n)^k$ deformation retracts onto the image of small affine r-immersions. One has to multiply the linear transformations from $\mathbb{R}^m$ to $\mathbb{R}^n$ by a factor that will make their norms smaller than $sd(f)$. □

Finally we can prove the main result.

Proof of Theorem 2.1. We have constructed the following composition of maps

$$\mathrm{rImm}^{\mathrm{aff}}\left(\coprod_k D^m, \mathbb{R}^n\right) \hookrightarrow \mathrm{rImm}^{\mathrm{sm}}\left(\coprod_k D^m, \mathbb{R}^n\right) \hookrightarrow \mathrm{rImm}^{\mathrm{ce}}\left(\coprod_k D^m, \mathbb{R}^n\right) \hookrightarrow$$
$$\hookrightarrow \mathrm{rImm}\left(\coprod_k D^m, \mathbb{R}^n\right) \xrightarrow{\mathrm{ev}} \mathrm{rConf}(k, \mathbb{R}^n) \times \mathrm{Linj}(\mathbb{R}^m, \mathbb{R}^n)^k.$$

We have shown that each one of the inclusions is an equivalence, and that the composition is an equivalence. It follows that the map marked ev is an equivalence. □

Corollary 2.21. *Choose a basepoint in $\mathrm{Imm}\left(\coprod_k D^m, \mathbb{R}^n\right)$, and let $\overline{\mathrm{rImm}}\left(\coprod_k D^m, \mathbb{R}^n\right)$ be the homotopy fiber of the map $\mathrm{rImm}\left(\coprod_k D^m, \mathbb{R}^n\right) \to \mathrm{Imm}\left(\coprod_k D^m, \mathbb{R}^n\right)$. Then there exists an equivalence*

$$\overline{\mathrm{rImm}}\left(\coprod_k D^m, \mathbb{R}^n\right) \simeq \mathrm{rConf}(k, \mathbb{R}^n).$$

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