Dain's invariant for black hole initial data

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Abstract

Dynamical black holes in the non-perturbative regime are not mathematically well understood. Studying approximate symmetries of spacetimes describing dynamical black holes gives an insight into their structure. Utilising the property that approximate symmetries coincide with actual symmetries when they are present allows one to construct geometric invariants characterising the symmetry. In this paper, we extend Dain's construction of geometric invariants characterising stationarity to the case of initial data sets for the Einstein equations corresponding to black hole spacetimes. We prove the existence and uniqueness of solutions to a boundary value problem showing that one can always find approximate Killing vectors in black hole spacetimes and these coincide with actual Killing vectors when they are present. In the time-symmetric setting we make use of a 2+1 decomposition to construct a geometric invariant on a MOTS that vanishes if and only if the Killing initial data equations are locally satisfied.

1 Introduction

For a generic initial data set of the vacuum Einstein field equations an important question is whether the development of this initial data set possesses a Killing vector or symmetry. This question first arose in the study of linearisation stability [\[16\]](#page-32-0) and is of utmost importance as symmetries greatly simplify problems in all fields within General Relativity. Moreover, some of the most important solutions to the Einstein field equations admit a number of symmetries. In the context of the initial value problem of General Relativity, the existence of continuous symmetries is characterised by solutions of the Killing initial data (KID) equations [\[5\]](#page-32-1) —see also [\[16\]](#page-32-0). The KID equations are a system of overdetermined equations for a scalar and a 3-vector on the initial data set such that, if a solution exists, the development of the data will have a Killing vector with lapse and shift at the initial hypersurface given by the scalar and vector, respectively. In fact, these equations also have a deep connection with the ADM evolution equations and the adjoint linearised constraint map $D\Phi^*$, as described in Section [2.](#page-2-0)

As already mentioned, the condition that a spacetime (M, g) possesses a Killing vector is encoded in the initial data by the Killing initial data (KID) equations. This holds, in particular, for time translations. Given the role that stationary solutions have in the mathematical description of isolated bodies, it is natural to attempt to quantify the deviation of a given initial data set from that of a stationary one. In [\[12\]](#page-32-2), the notion of an approximate Killing vector was introduced

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as a solution to a fourth order, self-adjoint, elliptic partial differential equation whose solution implies the existence of a geometric invariant which allows one to identify static initial data. This idea was developed and extended in [\[17\]](#page-32-3) to the non-time symmetric case —thus allowing for a characterisation of stationary data sets. In these works the approximate Killing vector equation was solved on the whole of an asymptotically Euclidean 3-manifold. That is, the only boundary conditions prescribed in this analysis were those associated to the asymptotic ends. As such, this approach does not allow to encode the presence of a black hole.

The purpose of this article is to extend the results of [\[12\]](#page-32-2) and [\[17\]](#page-32-3) to incorporate not only asymptotic conditions but also an inner boundary that corresponds to the boundary of a black hole. This boundary will be a 2-dimensional surface that encodes the presence of a black hole in the evolution of the initial development. This is accomplished by the condition that the surface is a marginally outer trapped surface (MOTS), sometimes referred to as an apparent horizon.

The main result of this paper is contained in the following theorem.

Main Result. Let (S, h_{ij}, K_{ij}) be a complete, smooth asymptotically Euclidean initial data set for the Einstein vacuum field equations with one asymptotic end. Given smooth fields f, g, f^i and h^i over ∂S , then there exists a solution (X, X^i) to the approximate KID equation boundary value problem

$$
\mathcal{P} \circ \mathcal{P}^* \begin{pmatrix} X \\ X^i \end{pmatrix} = 0 \quad on S,
$$

$$
\begin{cases} X|_{\partial S} = f, \\ \Delta_h X|_{\partial S} = g, \\ X^i|_{\partial S} = f^i, \\ \frac{\partial}{\partial \rho} X^i|_{\partial S} = h^i, \end{cases}
$$

such that on the asymptotic end, one has the asymptotic behaviour

$$
X = \lambda |x| + \vartheta \qquad \vartheta \in H_{1/2}^{\infty}
$$

$$
X^{i} \in H_{1/2}^{\infty}
$$

where λ is Dain's invariant —i.e. if the data is stationary in the sense of Definition [5](#page-7-0) then λ vanishes.

The generality of the boundary values in this theorem is exploited in order to choose physically relevant values of the fields. The construction of the boundary values is clearly non-unique due to the arbitrariness of the fields. We consider the decomposition of the Killing initial data (KID) equations onto ∂S and investigate how much of the KID equations one can solve on this surface. Integral to this construction is the use of the MOTS stability operator [\[2\]](#page-32-4). In this manner, in the time symmetric setting, we incorporate the condition that the MOTS propagates into the evolution of the initial data in the boundary conditions of the approximate KID operator. However, in the non-time symmetric case, how one should incorporate the stability of MOTS is much less obvious.

In addition to the main theorem above, in the time symmetric setting, we also construct a geometric invariant on the surface ∂S that vanishes if and only if the KID equations are solved there. This result is contained in the following theorem and proved in Section [5.](#page-17-0)

Theorem. Given a time symmetric initial data set and a MOTS the KID equations give rise to the elliptic equation on ∂S

$$
\Delta_{\bar{h}}N - \frac{1}{2}(\bar{r} + \bar{K}^{pq}\bar{K}_{pq})N = 0.
$$

For a stable MOTS the unique solution is $N = 0$. Furthermore, using the evolution equations one obtains

$$
\Delta_h N = N = 0, \qquad on \quad \partial S.
$$

Then the KID equations are satisfied on ∂S if and only if $\omega = 0$ where

$$
\omega \equiv \int_{\partial S} |\bar{K}|^2 |D^{\perp}N|^2,
$$

with $|\bar{K}|^2 = \bar{K}_{pq}\bar{K}^{pq}$ and $D^{\perp} \equiv \rho^i D_i$.

Outline of the article

The paper is organised as follows: in Section [2](#page-2-0) the approximate Killing initial data equation (AKE) is constructed as in [\[17\]](#page-32-3) and [\[12\]](#page-32-2). In order to make sure that a suitable boundary value problem is constructed, natural boundary operators are constructed using the Green formula. The compatibility of the boundary operators and the AKE are then verified using the Lopatinskij-Shapiro condition. Section [3](#page-11-0) comprises the main result of this article, namely, the existence and uniqueness of the AKE with inner boundary conditions. In Section [4,](#page-14-0) we investigate and derive the decomposition of the KID equations into components tangential and normal to the inner boundary surface. Finally, in section [5,](#page-17-0) we prove existence of solutions to the tangential KID equations and give a construction of a geometric invariant that vanishes only when one has a Killing vector. In Appendix [A](#page-24-0) we present the Green formula for the approximate KID operator, in Appendix [B](#page-29-0) we show the full derivation of the solution to the ODE arising from the Lopatinski-Shapiro condition and in Appendix [C](#page-30-0) we derive the decomposed components of the KID equations normal and tangential to ∂S .

Notations and Conventions

The indices, a, b, c, \ldots are spacetime indices, i, j, k, \ldots are indices on an initial 3-dimensional hypersurface, S with metric h_{ij} and unit normal n^a . The indices A, B, C, \dots are indices on a 2-dimensional surface Σ with metric \bar{h}_{AB} and unit normal ρ^i embedded in the 3-dimensional surface. The induced covariant derivative on S is D_i and on Σ is \bar{D}_A . We use the positive convention on the extrinsic curvature, that is, $K_{ab} = +h_a{}^c h_b{}^d \nabla_c n_d$ for the extrinsic curvature of S and $\bar{K}_{ij} = +\bar{h}_i{}^k \bar{h}_j{}^l D_k \rho_l$. In the latter sections of the paper, we will make use of Gaussian normal coordinates on 2-dimensional surfaces, that is the acceleration of the foliation vanishes —in other words, $a_i = 0$.

2 The approximate Killing vector equation

In this section, we develop the theory of the approximate Killing vector equation and introduce some compatible boundary operators. The first part of this section was developed in [\[17\]](#page-32-3).

2.1 The approximate Killing operator

Denote an initial data set of the vacuum Einstein field equations by (S, h_{ij}, K_{ij}) where S is a 3-dimensional manifold, h_{ij} is a Riemannian metric on S and K_{ij} is a symmetric rank 2 tensor satisfying the vacuum Einstein constraint equations

$$
r + K^2 - K_{ij} K^{ij} = 0,\t\t(1a)
$$

$$
D^j K_{ij} - D_i K = 0.
$$
 (1b)

We refer to equations [\(1a\)](#page-2-1) and [\(1b\)](#page-2-1) as the *Hamiltonian* and *momentum* constraints, respectively. In the above expressions D_i denotes the Levi-Civita connection of the metric h_{ij} , r is the associated Ricci scalar and $K \equiv K_{ij}h^{ij}$.

In the sequel, we will be particularly interested in initial data sets whose development is endowed with a Killing vector. At the level of the initial data set, this property is encoded through the existence of a solution to the Killing initial data (KID) equations.

Proposition 1. (Killing initial data (KID) equations.) Let (S, h_{ij}, K_{ij}) be an initial data set for the vacuum Einstein field equations. If there exists a pair $(N, Yⁱ)$ such that

$$
NK_{ij} + D_{(i}Y_{j)} = 0,\t\t(2a)
$$

$$
N^{k}D_{k}K_{ij} + D_{i}N^{k}K_{kj} + D_{j}N^{k}K_{ik} + D_{i}D_{j}N = N(r_{ij} + KK_{ij} - 2K_{ik}K_{j}^{k}),
$$
 (2b)

then the development of the initial data is endowed with a Killing vector and $(N, Yⁱ)$ are the lapse and shift of this Killing vector at S.

A proof of this result can be found in [\[6\]](#page-32-5).

Following [\[17\]](#page-32-3), we write the constraint equations [\(1a\)](#page-2-1) and [\(1b\)](#page-2-1) as a map, $\Phi : \mathfrak{M}_2 \times \mathfrak{T}_2 \to \mathfrak{S} \times \mathfrak{X}$, where \mathfrak{M}_2 is the space of 3-dimensional Riemannian metrics, \mathfrak{T}_2 is the space symmetric 2-tensors, $\mathfrak S$ the space of scalars and $\mathfrak X$ is the space of vectors on $\mathcal S$:

$$
\Phi\begin{pmatrix}h_{ij}\\K_{ij}\end{pmatrix}=\begin{pmatrix}r+K^2-K_{ij}K^{ij}\\D^jK_{ij}-D_iK\end{pmatrix}.
$$

Linearising and finding the formal adjoint of this linearisation through integration by parts yields

$$
D\Phi^* \left(\begin{array}{c} X \\ X_i \end{array} \right) = \left(\begin{array}{c} D_i D_j X - X r_{ij} - \Delta_h X h_{ij} + H_{ij} \\ D_{(i} X_{j)} - D^k X_k h_{ij} + F_{ij} \end{array} \right) \tag{3}
$$

where H_{ij} and F_{ij} are as in [\[17\]](#page-32-3).

Remark 1. A calculation shows that having a solution (X, X_i) to $D\Phi^*(X, X_i) = 0$ is equivalent to (X, X_i) satisfying the KID equations —see e.g. Remark 2 in [\[17\]](#page-32-3).

The above remark gives the motivation behind the following definition:

Definition 1. For the operator $\mathcal{P} \circ \mathcal{P}^* : \mathfrak{S} \times \mathfrak{X} \to \mathfrak{S} \times \mathfrak{X}$, the equation

$$
\mathcal{P} \circ \mathcal{P}^* \begin{pmatrix} X \\ X_i \end{pmatrix} = 0 \tag{AKE}
$$

where this operator is given by

$$
\mathscr{P} \circ \mathscr{P}^* \left(\begin{array}{c} X \\ X_i \end{array} \right) \equiv \left(\begin{array}{c} 2\Delta_{\mathbf{h}}\Delta_{\mathbf{h}}X - r^{ij}D_iD_jX + 2r\Delta_{\mathbf{h}}X + \frac{3}{2}D^i rD_iX + (\frac{1}{2}\Delta_{\mathbf{h}}r + r_{ij}r^{ij})X \\ + D^i D^j H_{ij} - \Delta_{\mathbf{h}}H_k^k - r^{ij}H_{ij} + H \end{array} \right)
$$

$$
D^j \Delta_{\mathbf{h}}D_{(i}X_{j)} + D_i \Delta_{\mathbf{h}}D^k X_k + D^j \Delta_{\mathbf{h}}F_{ij} - D_i \Delta_{\mathbf{h}}F_k^k - \bar{F}_i
$$

with

$$
\begin{aligned}\n\bar{H} &\equiv 2(K\bar{Q} - K^{ij}\bar{Q}_{ij}) + 2(K^{ki}K^j{}_k - KK^{ij})\bar{\gamma}_{ij}, \\
\bar{F}_i &\equiv \left(D_i K^{kj} - D^k K^j{}_i\right)\bar{\gamma}_{jk} - \left(K^k{}_i D^j - \frac{1}{2}K^{kj}D_i\right)\bar{\gamma}_{jk} + \frac{1}{2}K^k{}_i D_k \bar{\gamma} \\
\bar{\gamma}_{ij} &\equiv D_i D_j X - Xr_{ij} - \Delta_h X h_{ij} + H_{ij} \\
\bar{Q}_{ij} &\equiv -\Delta_h (D_{(i}X_{j)} - D^k X_k h_{ij} + F_{ij})\n\end{aligned}
$$

$$
H_{ij} \equiv 2X(K^{k}{}_{i}K_{jk} - KK_{ij}) - K_{k(i}D_{j)}X^{k} + \frac{1}{2}K_{ij}D_{k}X^{k} + \frac{1}{2}K_{kl}D^{k}X^{l}h_{ij} - \frac{1}{2}X^{k}D_{k}K_{ij} + \frac{1}{2}X^{k}D_{k}Kh_{ij}
$$

\n
$$
F_{ij} \equiv 2X(Kh_{ij} - K_{ij})
$$

is the approximate Killing vector equation (AKE). A solution (X, X_i) to this equation is called an approximate Killing vector.

Remark 2. Following from Dain, [\[12\]](#page-32-2), every approximate Killing vectors corresponds to one of the ten Killing vectors in flat spacetime. In the following, we focus on the approximate killing vectors that correspond to the Killing vector associated with time translations in flat spacetime. This analysis could be extended to the other Killing vectors corresponding to spatial translations, boosts and rotations.

Remark 3. When the initial data is *time symmetric*, that is $K_{ij} = 0$, the AKE simplifies to

$$
\mathcal{P} \circ \mathcal{P}^* \begin{pmatrix} X \\ X_i \end{pmatrix} = \begin{pmatrix} 2\Delta_h \Delta_h X - r^{ij} D_i D_j X + \frac{1}{2} r_{ij} r^{ij} X \\ D^j \Delta_h D_{(i} X_j) + D_i \Delta_h D^k X_k \end{pmatrix}
$$
(4)

where we have used the constraint equations in this case to set $r = 0$. Note that under the assumption of time symmetry, these equations decouple from one another and can thus be considered as two separate equations: one for the lapse X and one for the shift X_i .

We have the following important properties of the AKE operator:

Lemma 1. The operator $\mathcal{P} \circ \mathcal{P}^*$ as defined above is a self adjoint, fourth order elliptic operator.

A proof of this result can by found in in [\[17\]](#page-32-3). In order to discuss the solvability of the [\(AKE\)](#page-3-0), we will need to introduce weighted Sobolev spaces and the notion of an asymptotically Euclidean manifolds.

2.2 The Approximate Killing vector equation on asymptotically Euclidean manifolds

In this section, we summarise the results of [\[17\]](#page-32-3) for the solvability of the AKE on asymptotically Euclidean manifolds. We make use of weighted Sobolev spaces to discuss the decay of various tensor fields on the 3-dimensional manifold S .

2.2.1 Weighted Sobolev spaces and asymptotically Euclidean manifolds

We begin with the definition of a weighted Sobolev space, H^s_δ :

Definition 2. Given points $p, x \in \mathcal{S}$, let s be a non-negative integer and $\delta \in \mathbb{R}$. The weighted Sobolev space *denoted by* H^s_δ consists of of all functions, u, such that the weighted Sobolev norm is finite

$$
||u||_{s,\delta}\equiv\sum_{0\leq|\alpha|\leq s}||D^{\alpha}u||_{\delta-|\alpha|}<\infty
$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multiindex and the norm in the summand is the weighted L^2 -norm

$$
||\phi||_{\delta} \equiv \left(\int_{\mathcal{S}} |\phi|^2 \sigma^{-2\delta - 3} d^3 x\right)^{1/2}
$$

with $\sigma(x) \equiv (1+d(p,x)^2)^{1/2}$ and d denotes the Riemannian distance on S. One says that $u \in H^{\infty}_{\delta}$ if $u \in H^s_\delta$ for all s.

Remark 4. We follow Bartnik's conventions [\[3\]](#page-32-6) for the weighted Sobolev spaces and norms. Note also that we are slightly abusing notation since these norms seem to be dependent on p . However, different choices of p give rise to equivalent weighted Sobolev norms, see e.g. [\[3,](#page-32-6) [9\]](#page-32-7). Thus, we denote these norms with the same symbol.

Using these weighted Sobolev spaces, we are in a position to discuss the necessary fall-off conditions in order to form an asymptotically Euclidean manifold. Consider an initial data set (S, h_{ij}, K_{ij}) for the Einstein vacuum field equations that has n asymptotically Euclidean ends. That is, there exists a compact set β such that

$$
\mathcal{S}\setminus\mathcal{B}=\sum_{k=1}^n\mathcal{S}_{(k)}
$$

where $\mathcal{S}_{(k)}$ are open sets diffeomorphic to the complement of a closed ball in \mathbb{R}^3 . Each $\mathcal{S}_{(k)}$ is called an asymptotic end. On each of these ends one can introduce asymptotically Cartesian coordinates $x = (x^{\alpha})$. The definition of an asymptotically Euclidean manifold is then defined on these ends.

Definition 3. The 3-dimensional manifold S is called asymptotically Euclidean if on each asymptotic end one has that

$$
h_{\alpha\beta} - \delta_{\alpha\beta} \in H^{\infty}_{-\frac{1}{2}},
$$

$$
K_{\alpha\beta} \in H^{\infty}_{-\frac{3}{2}}.
$$

2.2.2 Green's Formula and the Fredholm alternative

Green's formula will be fundamental to the choice of boundary operators in constructing the boundary value problem for the [\(AKE\)](#page-3-0). In this section, we outline the basic theory of Green's formula and use this to motivate the Fredholm alternative. The latter will be necessary in proving the main theorem.

Green's formula of an elliptic differential operator $\mathscr A$ arises when considering the formal adjoint \mathscr{A}^* of \mathscr{A} [\[18,](#page-32-8) [13\]](#page-32-9). Let Ω be a bounded smooth domain in \mathbb{R}^n . For all \mathbf{u}, \mathbf{v} compactly supported in Ω , the *formal adjoint* is defined through

$$
\int_{\Omega} \mathbf{v} \cdot \mathscr{A} \mathbf{u} \mathrm{d} \mu = \int_{\Omega} \mathbf{u} \cdot \mathscr{A}^* \mathbf{v} \mathrm{d} \mu.
$$

The adjoint is calculated by performing integration by parts. If we now consider \mathbf{u}, \mathbf{v} to be not compactly supported on Ω , performing integration by parts yields boundary terms. The resulting relation is known as *Green's formula* for $\mathscr A$

$$
\int_{\Omega} (\mathbf{v} \cdot \mathscr{A} \mathbf{u} - \mathbf{u} \cdot \mathscr{A}^* \mathbf{v}) d\mu = \oint_{\partial \Omega} \mathbf{v} \cdot \mathscr{B} \mathbf{u} - \mathbf{u} \cdot \mathscr{T} \mathbf{v} dS
$$

where $\mathscr T$ and $\mathscr B$ are differential boundary operators. In order to formalise this discussion, we first define a Dirichlet system.

Definition 4. Let Γ be a subset of $\partial\Omega$. The boundary value operators $b_j^{\alpha}(x, D)$, $j = 1, ..., n$ and $\alpha = 1, ..., N$ is the number of equations, form a Dirichlet system on Γ if and only if

- *i.* The order, m_j^{α} , of b_j^{α} is such that $m_i^{\alpha} \neq m_j^{\alpha}$ for $i \neq j = 1, ..., n$,
- ii. The symbol of the operator $\sigma_i(x, \vec{\xi}) \neq 0 \ \forall x \in \Gamma$ and $\vec{\xi} \neq 0$ and is normal to $\partial \Omega$ at x,
- iii. For each α , the orders m_j^{α} run through all numbers $0, 1, ..., n-1$ (without loss of generality $m_j^{\alpha} = j - 1$). The number n is called the order of the Dirichlet system.

A set of boundary value operators satisfying only points 1 and 2 above is said to be normal. Then Green's formula can be expressed as the following:

Proposition 2. Let $\mathscr{A}(x, D)$ be an elliptic operator on $\overline{\Omega}$ and $b^{\alpha}_{j}(x, D)$, $j = 1, ..., m$, $\alpha = 1, ..., N$, be a normal boundary value system on $\partial\Omega$. Then on $\partial\Omega$ one can find another boundary value system $S_j^{\alpha}, j = 1, ..., m, \alpha = 1, ..., N$, with orders $\mu_j^{\alpha} < 2m - 1$ so that $\{b_1^1, ..., b_m^N, S_1^1, ..., S_m^N\}$ is a Dirichlet system of order $2mN$ on $\partial \Omega$.^{[1](#page-6-0)} Additionally, one can construct a further $2mN$ boundary value operators $B_j^{\prime\alpha}, T_j^{\alpha}, j = 1, ..., m$ with the properties:

- *i.* The orders of $B_j'^{\alpha}$ and T_j^{α} are given by $2m 1 \mu_j^{\alpha}$ and $2m 1 m_j^{\alpha}$, respectively.
- ii. The set $\{B_1^{\prime 1},...,B_m^{\prime N},T_1^1,...,T_m^N\}$ is also a Dirichlet system of order $2m$ on $\partial\Omega$.
- iii. We have Green's formula:

$$
\int_{\Omega} (\mathbf{v} \cdot \mathscr{A} \mathbf{u} - \mathbf{u} \cdot \mathscr{A}^* \mathbf{v}) d\mu = \sum_{j=1}^{m} \sum_{\alpha=1}^{N} \oint_{\partial \Omega} \left(S_j^{\alpha} \mathbf{u} \cdot B_j^{\prime \alpha} \mathbf{v} - b_j^{\alpha} \mathbf{u} \cdot T_j^{\alpha} \mathbf{v} \right) dS. \tag{5}
$$

Thus, the operators arising in Green's formula are the natural boundary operators to consider. The above discussion generalises to operators over a manifold.

In the sequel, we will need to make use of the Fredholm alternative for elliptic operators acting between weighted Sobolev spaces which relies on the asymptotic homogeneity of the approximate Killing operator. We outline the notion of asymptotic homogeneity here. In local coordinates on \mathcal{S} , the [\(AKE\)](#page-3-0) can be written as

$$
\mathscr{L} \boldsymbol{u} \equiv (A^{\alpha \beta \gamma \delta} + a^{\alpha \beta \gamma \delta}) \cdot \partial_{\alpha} \partial_{\beta} \partial_{\gamma} \partial_{\delta} \boldsymbol{u} + a^{\alpha \beta \gamma} \cdot \partial_{\alpha} \partial_{\beta} \partial_{\gamma} \boldsymbol{u} + a^{\alpha \beta} \cdot \partial_{\alpha} \partial_{\beta} \boldsymbol{u} + a^{\alpha} \cdot \partial_{\alpha} \boldsymbol{u} + \boldsymbol{a} \cdot \boldsymbol{u} = 0,
$$

where $u : \mathcal{S} \to \mathbb{R}^4$ is a vector valued function over $\mathcal{S}, A^{\alpha\beta\gamma\delta}$ are a constant matrices, while $a^{\alpha\beta\gamma\delta}, a^{\alpha\beta\gamma}, a^{\alpha\beta}, a^{\alpha}$ and a are smooth matrix-valued functions of the coordinates (x^{α}) . Then $\mathscr L$ is asymptotically homogeneous if the matrix-valued functions belong to the following weighted Sobolev spaces

$$
\boldsymbol{a}^{\alpha\beta\gamma\delta}\in H^{\infty}_{\tau},\quad \boldsymbol{a}^{\alpha\beta\gamma}\in H^{\infty}_{\tau-1},\quad \boldsymbol{a}^{\alpha\beta}\in H^{\infty}_{\tau-2},\quad \boldsymbol{a}^{\alpha}\in H^{\infty}_{\tau-3},\quad \boldsymbol{a}\in H^{\infty}_{\tau-4},
$$

for $\tau < 0$, see [\[7,](#page-32-10) [14\]](#page-32-11). With this definition, we can classify the asymptotic homogeneity of the approximate Killing operator in local coordinates.

Lemma 2. If the [3](#page-5-0)-dimensional manifold S is asymptotically Euclidean as in Definition 3 then $\mathscr L$ is asymptotically homogeneous with $\tau = -1/2$.

We will make use of the following form of the Fredholm alternative, as proved in [\[18\]](#page-32-8) (see also $[7]$:

Proposition 3. Let $\mathscr L$ be a fourth order asymptotically homogeneous elliptic operator over a smooth domain Ω with smooth coefficients and let b_i^{α} , $i = 1, 2, \alpha = 1, ..., N$, be smooth boundary operators on ∂Ω. Given some non-negative integer δ, the boundary value problem

$$
\begin{cases}\n\mathscr{L}\boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega \\
b_1^1 \boldsymbol{u} = \boldsymbol{g}_1^1 \\
\vdots & \text{on } \partial \Omega \\
b_2^N \boldsymbol{u} = \boldsymbol{g}_2^N\n\end{cases}
$$

where $f, g_1^1, ..., g_2^N \in H^0_{\delta-4}$ possesses at least one solution $u \in H^4_{\delta}$ if and only if

$$
\int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v} \mathrm{d} \mu + \sum_{j=1}^{2} \sum_{\alpha=1}^{N} \int_{\partial \Omega} \boldsymbol{g}_{j}^{\alpha} \cdot T_{j}^{\alpha} \boldsymbol{v} \mathrm{d} \sigma = 0 \qquad \forall \boldsymbol{v} \in N^{*}
$$

¹The choice of S_j^{α} is not unique.

where T_i are the boundary operators coming from Green's formula and N^* is the space of solutions to the homogeneous adjoint boundary value problem

$$
\begin{cases}\n\mathcal{L}^* \mathbf{v} = 0 & \text{in } \Omega \\
B_1'^1 \mathbf{v} = 0 & \text{on } \partial \Omega \\
\vdots & \text{on } \partial \Omega \\
B_2'^N \mathbf{v} = 0\n\end{cases}
$$

for $v \in H_{1-\delta}^0$. \mathscr{L}^* denotes the formal adjoint of \mathscr{L} .

2.2.3 Existence of solutions to the AKE on asymptotically Euclidean manifolds

The notion of approximate Killing vectors and approximate symmetries was introduced by Dain [\[12\]](#page-32-2). For time symmetric initial data it was shown that:

- a. every Killing vector is also an approximate Killing vector;
- b. for generic initial data that admits no Killing vector (i.e. no solution to the KID equations) there always exists an approximate Killing vector;
- c. every approximate Killing vector can be uniquely associated with a Killing vector in flat spacetime.

An invariant denoted by $\lambda_{(k)}$ was also constructed on each asymptotic end such that $\lambda_{(k)} = 0$ if and only if the initial data is stationary in the sense of the following definition:

Definition 5. An asymptotically Euclidean initial data set (S, h_{ij}, K_{ij}) is called stationary if there exists non-trivial $(Y, Y_i) \in H^2_{\frac{1}{2}}$ such that

$$
\mathscr{P}^*\begin{pmatrix} Y \\ Y_i \end{pmatrix} = 0.
$$

Moreover, if the initial data is also time symmetric, i.e. $K_{ij} = 0$ then, if the above condition holds, the initial data is called static.^{[2](#page-7-1)}

The above results were extended in [\[17\]](#page-32-3) to the non-time symmetric setting. We state the main theorem (Theorem 1) of this work here, without proof:

Theorem 1. Let (S, h_{ij}, K_{ij}) be a complete, smooth asymptotically Euclidean initial data set for the Einstein vacuum field equations with n asymptotic ends. Then there exists a solution $(X, Xⁱ)$ to the AKE,

$$
\mathscr{P}\circ\mathscr{P}^*\left(\begin{array}{c} X \\ X^i \end{array}\right)=0,
$$

such that at each asymptotic end one has the asymptotic behaviour

$$
\begin{split} X_{(k)}&=\lambda_{(k)}|x|+\vartheta_{(k)},\qquad \vartheta_{(k)}\in H^{\infty}_{\frac{1}{2}},\\ X^i_{(k)}&\in H^{\infty}_{\frac{1}{2}}, \end{split}
$$

where $\lambda_{(k)}$, $k = 1, \ldots, n$, are constants and $\lambda_{(k)} = 0$ for some k if and only if (S, h_{ij}, K_{ij}) is stationary in the sense of Definition [5.](#page-7-0) Moreover, the solution is unique up to constant rescaling.

²Note that this condition is equivalent to the KID equations due to the fall off on (Y, Y_i) . See Remark (6) in [\[17\]](#page-32-3).

Remark 5. In the following work, we restrict to one asymptotic end with one inner boundary. This can be extended to one asymptotic end with multiple inner boundaries corresponding to spacetimes with multiple black holes.

The goal of the present work is to extend this theorem to include inner boundary conditions on S. To do this, it is important that the constructed boundary value problem is solvable and that the boundary 2-dimensional surface represents that of a black hole boundary. In this context the natural surface to consider is a marginally outer trapped surface (MOTS), sometimes referred to as an apparent horizon. The remainder of this section will tackle constructing a solvable boundary value problem. In particular, we will compute Green's formula in order to obtain natural boundary operators and verify that the Lopatinskij-Shapiro condition holds for the AKE with these natural boundary operators. Thereby showing that this boundary value problem is Fredholm.

Remark 6. As a general principle, for an elliptic boundary value problem, the derivative order of the boundary condition has to be lower than the operator and the number of boundary conditions must be half the order of the equation. The AKE is comprised of 4 fourth-order equations. Thus, we must have 8 boundary conditions.

2.3 Green's formula for the AKE

In this section, we derive Green's formula for the AKE using equation [\(5\)](#page-6-1). The derivation is essentially integration by parts and thus, due to the size of some of the terms, the calculations can be found in full in Appendix [A.](#page-24-0)

By inspecting the calculation in Appendix A, we can construct Green's formula for the (AKE). We work here using the components of the equation instead of vectorial quantities as in equation [\(5\)](#page-6-1). Thus, the obtained Dirichlet systems will have in total 16 elements since the [\(AKE\)](#page-3-0) has four components.

Lemma 3. The Green formula for the AKE can be written as

$$
\int_{\mathcal{S}} \mathcal{P} \circ \mathcal{P}^* \begin{pmatrix} X \\ X_i \end{pmatrix} \cdot \begin{pmatrix} Z \\ Z_i \end{pmatrix} - \int_{\mathcal{S}} \mathcal{P} \circ \mathcal{P}^* \begin{pmatrix} Z \\ Z_i \end{pmatrix} \cdot \begin{pmatrix} X \\ X_i \end{pmatrix}
$$

=
$$
\sum_{j=1}^2 \sum_{\alpha=1}^4 \left(\oint_{\partial \mathcal{S}} S_j^{\alpha}(X, X_i) \cdot B_j^{'\alpha}(Z, Z_i) - \oint_{\partial \mathcal{S}} b_j^{\alpha}(X, X_i) \cdot T_j^{\alpha}(Z, Z_i) \right).
$$

Thus, one has the Dirichlet systems $\{b_1^1, ..., b_2^4, S_1^1, ..., S_2^4\}$ and $\{B_1^{'1}, ..., B_2^{'4}, T_1^1, ..., T_2^4\}$.

A useful property that elliptic boundary value problems can have is self-adjointness. That is, if the b_j appearing in the formula in Lemma [3](#page-8-0) are the operators appearing in a boundary value problem then the B'_{i} *i* are the *adjoint boundary operators*. If $b_j^{\alpha} = B_j^{'\alpha}$ then the boundary value problem $(L, b_1^1, ..., b_2^4)$ is self-adjoint for $L = L^*$. Since we have that the AKE is self adjoint, it will prove incredibly useful to consider a self-adjoint boundary value problem —particularly, when employing the Fredholm alternative.

Corollary 1. One can choose the boundary operators b_j^{α} to be

$$
\left\{I, \Delta_h X, \frac{\partial}{\partial \rho} X_i\right\}.
$$
\n(6)

where I is the identity operator acting on (X, X_i) . This choice yields a self-adjoint boundary value problem for the AKE.

Remark 7. The final operator in this set is found by decomposing terms of the form $D_i X_j$ into tangential and normal components i.e.

$$
D_i X_j = h_i^k D_k X_j = (\bar{h}_i^k + \rho_i \rho^k) D_k X_j = \bar{h}_i^k D_k X_j + \rho_i \rho^k D_k X_j.
$$

The tangential term is determined by X_i , leaving only the normal derivative to be determined. We also note the change in direction of the unit normal. The reason for this choice is so that the normal vector is normal to what will become a MOTS.

Next, we check that these boundary operators are compatible with the approximate KID equation. To do this, we make use of the Lopatinskij-Shapiro condition.

2.4 Verifying the Lopatinskij-Shapiro condition

The Lopatinskij-Shapiro (LS) condition allows us to establish the compatibility of an elliptic operator \mathscr{L} , with some boundary operators \mathscr{B} . That is, if $(\mathscr{L}, \mathscr{B})$ satisfy the LS condition then $(\mathscr{L}, \mathscr{B})$ is Fredholm i.e. its kernel and cokernel are finite dimensional. We will give a brief overview of the LS condition and then prove that it holds for the AKE with the boundary operators derived in corollary [1.](#page-8-1) For more details see [\[18\]](#page-32-8).

Let u^A , $A = 1, ..., N$ be a collection of fields on a subset $\Omega \subseteq \mathbb{R}^n$ with coordinates $x = (x^{\alpha})$ and suppose we have N equations of at most order l . We consider operators of the form

$$
(\mathscr{L} u)_i = \sum_{0 \leq |\gamma| \leq l} L^\gamma_{iB}(x^\alpha,u) \partial_\gamma u^B
$$

where $i = 1, ..., n$ are equation indices, γ is a multiindex. We complement $\mathscr L$ with boundary operators on $\partial\Omega$ of the form

$$
(\mathcal{B}u)_j = \sum_{0 \le |\gamma| \le k} B_{jB}^\gamma(x^\alpha, u) \partial_\gamma u^B
$$

with $j = 1, ..., m$ so that there are m boundary conditions.

In order to state the LS condition, we need the definition of the principal part of a differential operator. Recall that to obtain the *principal part* of an operator $\mathscr L$ we consider only the highest order derivative terms in the operator —namely

$$
(\mathscr{L}^H u)_i = \sum_{|\gamma|=l} L_{iB}^\gamma(x^\alpha,u) \mathcal{D}_\gamma u^B \equiv A_{iB}^{\gamma_1...\gamma_l} \partial_{\gamma_1}...\partial_{\gamma_l} u^B,
$$

where the Einstein summation convention is used in the second equality. In particular, the principal part of the AKE is

$$
(\mathscr{P}\circ\mathscr{P}^*)^H\begin{pmatrix}X\\X_i\end{pmatrix}=\begin{pmatrix}2\Delta_{\delta}\Delta_{\delta}X\\ \partial^j\Delta_{\delta}\partial_{(i}X_{j)}+\partial_i\Delta_{\delta}\partial^kX_k\end{pmatrix}.
$$

An important detail which we will exploit in the following lemma is that the components of the principal part of the AKE operator decouple from one another and can thus be considered separately. We now are in a position to state the LS condition

Condition. (Lopatinskij-Shapiro (LS)). Focus only on the principal parts of $\mathscr L$ and $\mathscr B$. Let $x_* \in \partial \Omega$ and pick $x^1 = \rho$ such that ρ^i is the outward pointing normal to $\partial \Omega$. Consider the ODE problem

$$
\begin{cases} \mathscr{L}_{iB}(x_*; \frac{\mathrm{d}}{\mathrm{d}\rho}, \vec{\xi}) u^B = 0, \\ \mathscr{B}_{jB}(x_*; \frac{\mathrm{d}}{\mathrm{d}\rho}, \vec{\xi}) u^B = 0, \end{cases}
$$

where $A_{iB}(x_*, \frac{d}{dp}, \vec{\xi})$ is obtained by the replacement

$$
\partial_1 \to \frac{\mathrm{d}}{\mathrm{d}\rho}
$$
, $\partial_j \to \mathrm{i}\xi_j$ $j = 2,...,n$, $\vec{\xi} \neq 0$

in the principal part: $A_{iB}^{\gamma_1...\gamma_l}\partial_{\gamma_1}...\partial_{\gamma_l}$ and similarly for $B_{jB}(x_*,\frac{d}{dp},\vec{\xi})$. Then, the pair $(\mathscr{L},\mathscr{B})$ is said to satisfy the LS condition of the only stable solution^{[3](#page-10-0)} to the above ODE is the trivial one.

It is important to note that the LS condition is verified about an arbitrary point on the boundary. Thus, we can generalise to S and ∂S . We now verify the LS condition for the AKE and boundary conditions given in Lemma [3.](#page-8-0)

Lemma 4. For an initial data set (S, h_{ij}, K_{ij}) of the vacuum Einstein field equations, with inner boundary ∂S , the boundary value problem consisting of the [\(AKE\)](#page-3-0)

$$
\mathscr{P}\circ\mathscr{P}^*\begin{pmatrix} X \\ X_i \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}
$$

along with the boundary operators $(I, \Delta_h, \frac{\partial}{\partial \rho})$ given on ∂S satisfy the Lopatinskij-Shapiro condition.

Proof. Due to the decoupling of the principal parts of the AKE one can consider the lapse and shift components independently. Thus, begin by considering the principal part of the lapse component of the AKE operator

$$
\mathscr{P} \circ \mathscr{P}^*(X) = 2\Delta_{\delta} \Delta_{\delta} X.
$$

with the the boundary operators on ∂S given by (I, Δ_{δ}) . Focusing on the principal part of the lapse component of the AKE and the principal part of the boundary operators, consider the ordinary differential equation problem given by

$$
\begin{cases}\n\left(\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - |\xi|^2\right) \left(\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - |\xi|^2\right) X = 0, \\
\Delta_\delta X = 0, \\
X = 0,\n\end{cases}
$$

where this has been obtained by choosing a point on ∂S and performing the replacement

$$
\partial_1 \to \frac{\mathrm{d}}{\mathrm{d}\rho}, \qquad \partial_j \to \mathrm{i}\xi_j.
$$

Since one is performing the replacement about a point, one also has that $h_{ij} \to \delta_{ij}$. The stable solution is given by

$$
X = c_1 e^{-|\xi|\rho} + c_2 \rho e^{-|\xi|\rho}.
$$

Then, applying the boundary conditions above at $\rho = 0$, one sees that $c_1 = c_2 = 0$ and thus the solution is trivial and the Lopatinskij-Shapiro condition is satisfied for the lapse component of the AKE with the above boundary operators.

The shift component of the AKE has principal part given by

$$
\partial^j \Delta_\delta \partial_{(i} X_{j)} + \partial_i \Delta_\delta \partial^k X_k.
$$

Unlike the case of the lapse component of the AKE there are three equations in this case corresponding to $i = 1, 2, 3$. Thus, one requires six boundary conditions. Due to the derivatives either

³That is, $u^B \to 0$ as $\rho \to \infty$.

side of the Laplacian, carrying out the transformation in order to verify the LS condition yields a system of ODEs with terms containing derivatives of fourth order and lower.

Using the commutativity of partial derivatives one finds that, multiplying through by 2, the principal part is

$$
\Delta_{\delta}(\Delta_{\delta}X_i + 3\partial_i\partial^j X_j).
$$

Making use of this expression and performing the replacement $X_i \to (X^{\perp}, X_A)$ with $A = 1, 2$ to this expression directly yields the system of ordinary differential equations

$$
\begin{cases}\n\left(\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - |\xi|^2\right) \left(4\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}X^\perp - |\xi|^2X + 3i\xi^A\frac{\mathrm{d}}{\mathrm{d}\rho}X_A\right) = 0, \\
\left(\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - |\xi|^2\right) \left(\left(\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - |\xi|^2\right)X_A + 3i\xi_A\left(\frac{\mathrm{d}}{\mathrm{d}\rho}X^\perp + i\xi^BX_B\right)\right) = 0.\n\end{cases}
$$

Analysing the fundamental matrix of this system one finds that the solution to the above system of ODEs is of the form

$$
X_i = \sum_{k=0}^{2} X_{*i} \rho^k e^{\pm |\xi| \rho},
$$

Following a computation (see Appendix [B\)](#page-29-0) one finds that the stable solution is given by

$$
\begin{cases} X = ae^{-|\xi|\rho} + \frac{\mathrm{i}}{|\xi|} b_A \xi^A \rho \mathrm{e}^{-|\xi|\rho} + c(\frac{3}{10}|\xi| \rho^2 + \rho) \mathrm{e}^{-|\xi|\rho}, \\ X_A = a_A \mathrm{e}^{-|\xi|\rho} + b_A \rho \mathrm{e}^{-|\xi|\rho} - c \frac{3}{10} \mathrm{i} \xi_A \rho^2 \mathrm{e}^{-|\xi|\rho}. \end{cases}
$$

One then requires 6 boundary conditions to fix the 6 constants corresponding to a, a_A, b_A, c . Consider the principal parts of the boundary operators X_i and $\frac{\partial}{\partial \rho}X_i$. Performing the Lopatinskij-Shapiro replacements yields the initial conditions

$$
\begin{cases}\nX(0) = 0, \\
X_A(0) = 0, \\
\frac{d}{d\rho}X(0) = 0, \\
\frac{d}{d\rho}X_A(0) = 0.\n\end{cases}
$$

Substituting the stable solution into these conditions yields $a = a_1 = a_2 = b_1 = b_2 = c = 0$ and thus this boundary value problem has only the trivial solution and the Lopatinskij-Shapiro condition is satisfied by the shift component of the AKE along with the boundary operators $(I, \frac{\partial}{\partial \rho}).$

Thus, we have shown that the Lopintskij-Shapiro condition is satisfied for the 8 boundary operators $(I, \Delta_h N, \frac{\partial}{\partial \rho} N_i)$. \Box

Since the LS condition is satisfied we have the following corollary:

Corollary 2. The approximate Killing vector operator $\mathscr{P} \circ \mathscr{P}^*$ $\begin{pmatrix} X \\ Y \end{pmatrix}$ X_i ã along with the 8 boundary operators $\{I, \Delta_h X, \frac{\partial}{\partial \rho}\}\$ is elliptic.

3 Main existence theorem

We are now in a position to prove a series of existence results to the (AKE) . We begin with an auxiliary existence result which will be essential to prove our main existence result.

Lemma 5. Let (S, h_{ij}, K_{ij}) be a complete, smooth asymptotically Euclidean initial data set for the Einstein vacuum field equations with one asymptotic end. On ∂S , suppose one has the conditions

$$
\begin{cases} N|_{\partial S} = 0, \\ \Delta N|_{\partial S} = 0, \\ N^i|_{\partial S} = 0, \\ \frac{\partial}{\partial \rho} N^i|_{\partial S} = 0. \end{cases}
$$

For $0 < \beta < \frac{1}{2}$, $\mathscr{P} \circ \mathscr{P}^* : H_\beta^\infty \to H_{\beta-4}^\infty$, the equation

$$
\mathscr{P} \circ \mathscr{P}^* \begin{pmatrix} N \\ N^i \end{pmatrix} = 0,
$$

admits a non-zero solution $N, N^i \in H_\beta^\infty$ if and only if we have that

$$
\mathscr{P}^*\begin{pmatrix}N\\N^i\end{pmatrix}=0,
$$

i.e. the data is stationary in the sense of Definition [5.](#page-7-0)

Proof. Assume that $\mathscr{P} \circ \mathscr{P}^*(N, N^i) = 0$. One has the following identity from [\[17\]](#page-32-3)

$$
\int_{\mathcal{S}} \mathcal{P}^* \left(\begin{array}{c} X \\ X_i \end{array} \right) \cdot \mathcal{P}^* \left(\begin{array}{c} X \\ X_i \end{array} \right) = \oint_{\partial \mathcal{S}} \rho^k (\mathcal{A}_k + \mathcal{B}_k + \mathcal{C}_k + \mathcal{D}_k) dS - \oint_{\partial \mathcal{S}_{\infty}} s^k (\mathcal{A}_k + \mathcal{B}_k + \mathcal{C}_k + \mathcal{D}_k) dS
$$

where S_{∞} is a sphere at infinity, ρ^k is the *inward* pointing normal to ∂S and s^k is the outward pointing normal to the sphere at infinity. The second boundary integral vanishes by virtue of the decay of $N, Nⁱ$ as shown in [\[17\]](#page-32-3). The boundary integrands are defined as

$$
\mathcal{A}_k = ND^i \gamma_{ik} - D^i N \gamma_{ik} + D_k N \gamma - N D_k \gamma,
$$

\n
$$
\mathcal{B}_k = 2(K^{ij} q_{kij} - K q_{kj}^j) N,
$$

\n
$$
\mathcal{C}_k = N^i D^l q_{lik} - D^j N^i q_{kij} + D_i N^i q_{kj}^j - N_i D^l q_{lj}^j,
$$

\n
$$
\mathcal{D}_k = \frac{1}{2} N_k K^{lj} \gamma_{jl} + \frac{1}{2} N^i K_{ik} \gamma - N^i K_i^l \gamma_{kl},
$$

\n
$$
\gamma_{ij} = D_i D_j N - N r_{ij} - \Delta_h N h_{ij} + H_{ij},
$$

\n
$$
q_{kij} = D_k (D_{(i} N_{j)} - D^l N_l h_{ij} - F_{ij}),
$$

\n
$$
F_{ij} = 2N (K h_{ij} - K_{ij}).
$$

Directly applying the boundary conditions, one observes immediately that

$$
\mathcal{B}_k = \mathcal{D}_k = F_{ij} = 0 \quad \text{on} \quad \partial \mathcal{S}.
$$

Using that $N^i = 0$, \mathcal{C}_k reduces to

$$
\mathcal{C}_k = -D^j N^i q_{kij} + D_i N^i q_{kj}^j.
$$

Using the decomposition of the metric onto the boundary we have that

$$
D_i X_j = h_i^k D_k X_j = (\bar{h}_i^k + \rho_i \rho^k) D_k X_j = \bar{h}_i^k D_k X_j + \rho_i \rho^k D_k X_j.
$$

The first term on the right hand side vanishes since X_i is constant and thus does not change along the boundary, and the second term vanishes by the boundary condition $\rho^k D_k X_j = 0$. Thus, $\mathcal{C}_k = 0$. One also readily sees that the trace of γ_{ij} vanishes. Accordingly, the only term

left is the second one in \mathcal{A}_k : $\rho^k D^i N \gamma_{ik}$. This reduces to $\rho^k D^i N D_i D_k N$ as all other terms in γ_{ij} vanish. We have that, using the condition on the Laplacian of N , it follows that

$$
0 = h^{ij} D_i D_j N = \bar{h}^{ij} D_i D_j N + \rho^i \rho^j D_i D_j N.
$$

The first term on the right hand side vanishes as the derivatives tangential to ∂S of N vanish. Thus, one obtains the identity

$$
\rho^j \frac{\partial}{\partial \rho} D_j N = 0.
$$

Then, since the 3-dimensional covariant derivative is assumed to be torsion free, we can write

$$
\rho^k D^i N D_i D_k N = \rho^k D^i N D_k D_i N = D^i N \frac{\partial}{\partial \rho} D_i N = \rho^i \rho_k D^k N \frac{\partial}{\partial \rho} D_i N = 0
$$

applying the previous identity.

Putting this all together yields

$$
\int_{\mathcal{U}} \mathscr{P}^* \left(\begin{array}{c} X \\ X_i \end{array} \right) \cdot \mathscr{P}^* \left(\begin{array}{c} X \\ X_i \end{array} \right) = 0,
$$

so that we must have \mathscr{P}^* $\begin{pmatrix} X \\ Y \end{pmatrix}$ $= 0$, i.e. the data is stationary. Uniqueness then follows X_i directly from Lemma 4 of [\[17\]](#page-32-3) since we have the same decay on $(N, Nⁱ)$. \Box

We are now able to prove that a solution to the [\(AKE\)](#page-3-0) exists for an arbitrary choice of boundary data.

Theorem 2. Let (S, h_{ij}, K_{ij}) be a complete, smooth asymptotically Euclidean initial data set for the Einstein vacuum field equations with one asymptotic end. Given smooth fields $f, g, fⁱ$ and hⁱ over ∂S , then there exists a solution $(X, Xⁱ)$ to the approximate KID equation boundary value problem

$$
\mathcal{P} \circ \mathcal{P}^* \begin{pmatrix} X \\ X^i \end{pmatrix} = 0 \quad on S,
$$

$$
\begin{cases} X|_{\partial S} = f, \\ \Delta_h X|_{\partial S} = g, \\ X^i|_{\partial S} = f^i, \\ \frac{\partial}{\partial \rho} X^i|_{\partial S} = h^i, \end{cases}
$$

such that on the asymptotic end, one has the asymptotic behaviour

$$
X_{(k)} = \lambda |x| + \vartheta, \qquad \vartheta \in H_{\frac{1}{2}}^{\infty}
$$

$$
X^{i} \in H_{\frac{1}{2}}^{\infty}
$$

where λ is Dain's invariant —i.e. if the data is stationary in the sense of Definition [5—](#page-7-0) then λ vanishes.

Proof. We make use of the Fredholm alternative, Proposition [3.](#page-6-2) Note first that the boundary value problem is self adjoint by virtue of Corollary [1.](#page-8-1) From Lemma [5,](#page-12-0) in the non-stationary case we can only have the trivial solution to the adjoint homogeneous problem. Thus, a solution to the AKE boundary value problem always exists in this case.

In the stationary case, we have a solution to the AKE boundary value problem for specific boundary data coming from the Killing vector associated to stationarity. The decay of $(X, Xⁱ)$ allows one to prove existence and uniqueness exactly as in [\[17\]](#page-32-3). \Box

In summary, we have shown that there always exists an approximate Killing vector for an arbitrary choice of the functions specifying the value of the lapse, its Laplacian, the shift and its normal derivative on the inner boundary. Ideally, we would like to restrict the choice of data so that it has physical relevance and connects with the mathematical description of a black hole. In order to do this we analyse the tangential parts of the KID equations on a MOTS. In the sequel, we will show that one can always solve these equations on the boundary. Thus, we derive a natural prescription of boundary data.

4 The KID equations on an apparent horizon

In this section we decompose the KID equations into parts tangential and normal to a MOTS using a $2+1$ projector formalism. This works in the same way as the $3+1$ decomposition. We first outline a condition on the extrinsic curvature of S and ∂S for the presence of an apparent horizon in [4.1.](#page-14-1) In section [4.2](#page-14-2) we decompose the KID equations using a $2+1$ decomposition' making use of the MOTS condition.

4.1 MOTS condition

In this section we outline the mathematical condition for the existence of a MOTS or apparent horizon, see [\[8\]](#page-32-12). The spacelike 2-surface Σ is embedded into a 4-dimensional spacetime. In this setting, the orthogonal space to Σ is 2-dimensional and is Lorentzian. Thus, one can choose two future directed null vectors l^+ and l^- and define the null mean curvatures of Σ by

$$
\chi^+ = \bar{h}^{ab} \nabla_a l_b^+, \qquad \chi^- = \bar{h}^{ab} \nabla_a l_b^-.
$$

The MOTS condition is then defined through the vanishing of one of these null mean curvatures —by convention this is chosen to be $\chi^+ = 0$. One can translate the above formalism into a condition on the extrinsic curvature of S and Σ as in [\[8\]](#page-32-12) to obtain

$$
\chi^+ = -K + \rho^i \rho^j K_{ij} + \bar{K}.
$$

Thus, the MOTS condition can be expressed as

$$
0 = -K + \rho^i \rho^j K_{ij} + \bar{K}.\tag{7}
$$

Using this equation, we will be able to encode into the presence of a black hole into the boundary data of the [\(AKE\)](#page-3-0). The notion of MOTS stability [\[1,](#page-32-13) [2\]](#page-32-4) will be utilised in order to guarantee that the MOTS propagates into the development of the initial data. In other words, so that the MOTS forms a so-called trapping horizon. Fundamental to the study of MOTS stability, is the MOTS stability operator —see [\[2\]](#page-32-4) for details. In the case of a spacelike, time-symmetric initial hypersurface the MOTS stability operator takes the form

$$
\mathcal{L} = -\Delta_{\bar{h}} - (r_{ij}\rho^i \rho^j + \bar{K}_{pq}\bar{K}^{pq}).
$$
\n(8)

4.2 2+1 Decomposition

In this subsection we discuss the $2+1$ decomposition of the KID equations. We first make a brief comment about notation.

Notation. Throughout this section all barred quantities correspond to 2-dimensional quantities. For example, h_{ij} is 3-dimensional and \bar{h}_{AB} is 2-dimensional. Following the convention in [\[10\]](#page-32-14), we denote the decomposition of a vector field at a point p by $u = u^{\parallel} + u^{\perp}$ where u^{\parallel} is its component along $T_p\Sigma$ for a 2-surface Σ and u^{\perp} its normal component. This extends to higher rank tensor fields.

In the following consider a 3-dimensional hypersurface S of the spacetime. Following the conventions outlined above let h denote the metric induced on S. Moroever, let Σ denote a 2-dimensional surface within S. Let ς be a scalar function such that the covector $\alpha_i = D_i \varsigma$ is normal to this foliation. In order to define the unit normal, let us set

$$
\alpha^i\alpha_i\equiv\frac{1}{X^2}.
$$

Thus, the *unit normal to the foliation* is $\rho_i \equiv X\alpha_i$. By applying the above contraction, one sees $\rho_i \rho^i = 1$. In this way we obtain a foliation of 2-surfaces, one 2-surface for each value of ς .

The Riemannian 3-metric h_{ij} induces a Riemannian 2-metric \bar{h}_{AB} on Σ_r , where indices $A, B, ...$ indicate the intrinsic 2-dimensional nature of \bar{h}_{AB} . The metrics h_{ij} and \bar{h}_{AB} are related through the projector

$$
\bar{h}_{ij} = h_{ij} - \rho_i \rho_j
$$

One associates a 2-covariant derivative with the metric \bar{h}_{AB} . For a scalar this is defined as

$$
\bar{D}_j \phi \equiv \bar{h}_j{}^i D_i \phi.
$$

The associated Riemann curvature tensor is defined through

$$
\bar{D}_i \bar{D}_j v^k - \bar{D}_j \bar{D}_i v^k = \bar{r}^k_{\ iij} v^l
$$

for an intrinsic vector v^A . Note that since this curvature tensor is 2-dimensional, it only has one independent component. Similarly, one obtains the Ricci tensor and Ricci scalar

$$
\bar{r}_{ij} = \bar{r}_{ikj}^k \qquad \bar{r} = \bar{h}^{ij} \bar{r}_{ij}.
$$

The extrinsic curvature of the 2-surface embedded in the 3-dimensional surface is defined by

$$
\bar{K}_{pq} = \bar{h}_p^{\ \ i} \bar{h}_q^{\ \ j} D_i \rho_j
$$

matching the positive sign convention on K_{ij} . This tensor is symmetric and lies purely on the 2-dimensional surfaces. Another important quantity is the acceleration of the foliation

$$
a_i = \rho^j D_j \rho_i.
$$

Using $\rho^i D_j \rho_i = 0$ one can write

$$
\bar{K}_{ij} = D_i \rho_j - \rho_i a_j.
$$

Remark 8. In the following, we will often make use of Gaussian normal coordinate so that this acceleration vanishes, $a_i = 0$.

Now, one can project the KID equations [\(2a\)](#page-3-1)-[\(2b\)](#page-3-1) onto the 2-surfaces of constant ς . Let N^{\parallel}^i denote the projection of the shift vector N^i onto the 2-surfaces —i.e. $N^{\parallel i} \equiv \bar{h}^i_{\ j} N^j$. We proceed now to decompose objects into parts perpendicular and parallel to the normal ρ^i . Projecting with

$$
\bar{h}_p^{i}\bar{h}_q^{j}
$$

we obtain the projection of the the first KID equation [\(2a\)](#page-3-1)

$$
NK_{pq}^{\parallel} + \bar{D}_{(p}N_{q)}^{\parallel} + N_k \rho^k \bar{K}_{pq} = 0,
$$
\n(9)

where $K_{pq}^{\parallel} \equiv \bar{h}_p^{i} \bar{h}_q^{j} K_{ij}$.

For the second KID equation [\(2b\)](#page-3-1), we project term by term making use of the Gauss-Codazzi equation and Gaussian normal coordinates to obtain the projection of the second KID equation onto Σ:

$$
N^{k} \left(D_{k} K_{pq}^{\parallel} + K_{iq} \left(\rho^{i} (\bar{K}_{kp}) + (\bar{K}_{k}^{i}) \rho_{p} \right) + K_{pj} \left(\rho^{j} (\bar{K}_{kq}) + (\bar{K}_{k}^{j}) \rho_{q} \right) \right.- K_{ij} \left(\rho^{j} (\bar{K}_{k}^{i}) \rho_{p} \rho_{q} + \rho^{i} (\bar{K}_{k}^{j}) \rho_{p} \rho_{q} + \rho^{i} \rho^{j} (\bar{K}_{kq}) \rho_{p} + \rho^{i} \rho^{j} (\bar{K}_{kp}) \rho_{q} \right) \right)+ \bar{h}_{p}^{i} \bar{h}_{q}^{j} D_{i} N^{k} K_{kj} + \bar{h}_{p}^{i} \bar{h}_{q}^{j} D_{j} N^{k} K_{ik} + \bar{D}_{p} \bar{D}_{q} N + \rho^{k} D_{k} N \bar{K}_{pq}= N \left(\bar{h}_{p}^{i} \bar{h}_{q}^{j} r_{ij} + K K_{pq}^{\parallel} + \bar{h}_{p}^{i} \bar{h}_{q}^{j} K_{ik} K^{k}{}_{j} \right).
$$
\n
$$
(10)
$$

Together, equations [\(9\)](#page-15-0) and [\(10\)](#page-16-0) give conditions on Σ that Killing vectors of the the spacetime must satisfy on this 2-surface. However, the converse is not true. If $(N, Nⁱ)$ a solution to these two equations that does not necessarily mean that one has a solution to the KID equations. This is because there are two other components of the KID equations in this decomposition which we call the normal-normal and normal-tangential components. For the first KID equation [\(2a\)](#page-3-1), one has

normal – normal :
$$
N K_{pq}^{\perp} + \rho_{(p|} \rho^j D_{j|} N_{q}^{\perp} - N_i a^i \rho_p \rho_q - N_{(p}^{\perp} a_q) = 0,
$$

normal – tangential : $N K_{pq}^{\perp \parallel} + \frac{1}{2} (\rho_p \rho^i D_i N_q^{\parallel} + N_j a^j \rho_p \rho_q + N_p^{\perp} a_q)$
 $+ \frac{1}{2} (\bar{h}_q{}^j D_j N_p^{\perp} - N_i \rho_p \bar{K}_q{}^i - \rho^i N_i \bar{K}_{pq}) = 0,$

where $K_{pq}^{\perp} \equiv \rho_p \rho^i \rho_q \rho^j K_{ij}$ and $K_{pq}^{\perp} \equiv \rho_p \rho^i \bar{h}_j{}^q K_{ij}$. For the second KID equation [\(2b\)](#page-3-1), one can obtain the projection under the assumption of Gaussian coordinates. The normal-normal and normal-tangential components of the second KID equations are

normal – normal:
$$
\rho_p \rho_q N^k D_k (K - \bar{K}) - \rho_p \rho_q N^k K_{ij} (\rho^i \bar{K}_k{}^j + \rho^j \bar{K}_k{}^i)
$$

$$
+ \rho_p \rho_q \rho^i \rho^j (D_i N^k K_{kj} + D_j N^k K_{ik}) + \rho_p \rho^i D_i (\rho_q \rho^j D_j N)
$$

$$
= N(-\bar{h}_p{}^i \bar{h}_q{}^j r_{ij} + K K_{pq}^{\perp} + \rho_p \rho_q \rho^i \rho^j K_{ik} K_j^k),
$$

normal – tangential:
$$
h_p{}^i \rho_q \rho^j (N^k D_k K_{ij} + D_i N^k K_{kj} + D_j N^k K_{ik})
$$

$$
+ \rho_p \rho^i D_i \bar{D}_q N
$$

 $=N(h_p{}^i \rho_q \rho^j r_{ij} + KK^{\perp \parallel} + h_p{}^i \rho_q \rho^j K_i{}^k K_{kj}).$

4.2.1 Time symmetric hypersurfaces

In order to gain some intuition into the structure of the decomposition of the KID equations we consider, in first instance, the case of time symmetric hypersurfaces so that the extrinsic curvature K_{ij} vanishes. Under this assumption, the trapping condition simplifies to $\bar{K}=0$ i.e the surface Σ is minimal. Moroever, the decomposition of the first KID equation implies that

$$
\bar{D}_p N^{\parallel p} = 0. \tag{11}
$$

after taking the trace.

Remark 9. Due to the decay condition placed on $Nⁱ$ in the previous section, in the time symmetric setting one has that $N^i = 0$ [\[11,](#page-32-15) [12\]](#page-32-2). By performing integration by parts on the time symmetric [\(AKE\)](#page-3-0), one finds that $N^i = 0$ on ∂S . Direct inspection of the KID equations also shows that N^i must be a Killing vector of S in the time symmetric case.

In the current setting the decomposition of the second KID reduces to

$$
\bar{D}_p \bar{D}_q N = N \bar{h}_p{}^i \bar{h}_q{}^j r_{ij}.
$$

Taking the trace of the above expression one has that

$$
\Delta_{\bar{h}} N = N \bar{h}^{ij} r_{ij} = \frac{1}{2} N(\bar{r} + \bar{K}^{pq} \bar{K}_{pq}),
$$

where in the second equality it has been used that

$$
\bar{r} + \bar{K}^{ij}\bar{K}_{ij} - \bar{K}^2 = 2\bar{h}^{ij}r_{ij},
$$

which is a consequence of the Gauss-Codazzi equation. Accordingly, the decomposition of the second KID equation implies that

$$
\Delta_{\bar{h}} N - \frac{1}{2} (\bar{r} + \bar{K}^{pq} \bar{K}_{pq}) N = 0.
$$
 (12)

Remark 10. This equation has a very similar form to the MOTS stability operator given in [\[2\]](#page-32-4).

One can perform the same simplification on the normal-normal and normal-tangential components of the equations. Additionally, using Gaussian normal coordinates one obtains

normal – normal :
$$
\rho_p \rho^i D_i(\rho_q \rho^j D_j N) = N(h_p^i h_q^j - \bar{h}_p^i \bar{h}_q^j) r_{ij}
$$
 (13a)

$$
normal - tangential : \qquad \rho_p \rho^i D_i \bar{D}_q N = N \rho_p \rho^i \bar{h}_q{}^j r_{ij} \tag{13b}
$$

5 The KID equations projected onto ∂S

In this section, we study the existence of solutions to the decomposed KID equations on an apparent horizon. We first study the simpler time symmetric case, $K_{ij} = 0$ in Subsection [5.1](#page-17-1) and move on to the full non-time symmetric case in Subsection [5.2.](#page-20-0)

5.1 The time symmetric case

Throughout this section we assume that $K_{ij} = 0$. We begin with some general observations.

5.1.1 What do Killing vector quantities look like on MOTS in a static spacetime?

Before proceeding to the analysis, we briefly explore the behaviour of the quantities $X, \Delta_h X, X_i$ and $\frac{\partial}{\partial n}X_i$ on the boundary of a static black hole. The prototypical example of a static black hole is the Schwarzschild black hole. In this case the outermost MOTS is the event horizon [\[4\]](#page-32-16). Thus, we only need to consider what happens at $r = 2m$ in Schwarzschild coordinates. Now, we need to choose an appropriate slicing. Choosing the standard Schwarzschild slicing, the shift vector vanishes everywhere by definition so that $X_i = \frac{\partial}{\partial n} X_i = 0$. The lapse is $X = (1 - \frac{2m}{r})^{1/2}$ so that on the event horizon $X = 0$. Finally, the Laplacian of the lapse vanishes by virtue of one of the the static Einstein equations —namely, $D_i D_j X = X r_{ij}$. Taking the trace and using that $X = 0$ on the event horizon, we obtain $\Delta_h X = 0$. Accordingly, the type of boundary conditions given in Corollary [1](#page-8-1) are physically relevant.

5.1.2 Existence and uniqueness of solutions

We now want to discuss the existence of solutions to equation [\(12\)](#page-17-2). Recall that this equation arises from the 2+1 decomposition of the second KID equation, equation along the 2-surface ∂S . Letting —namely

$$
\mathcal{K}N \equiv \Delta_{\bar{h}}N - \frac{1}{2}(\bar{r} + \bar{K}_{pq}\bar{K}^{pq})N = 0.
$$
\n(14)

Starting from the quantity NKN , integrating over ∂S and integrating by parts, one finds that

$$
\oint_{\partial S} \frac{1}{2} (\bar{r} + \bar{K}_{pq} \bar{K}^{pq}) N^2 \, \mathrm{d}S = \oint_{\partial S} N \Delta_{\bar{h}} N \, \mathrm{d}S = -\oint_{\partial S} \bar{D}_i N \bar{D}^i N \, \mathrm{d}S. \tag{15}
$$

Thus, if $\bar{r} > -\bar{K}_{pq}\bar{K}^{pq}$, in particular if $\bar{r} > 0$ then $N = 0$ is the only solution to equation [\(14\)](#page-17-3). To further this analysis, assume that the MOTS is stable. In this case, since we have the MOTS is a surface immersed in a time symmetric slice, the stability operator takes exactly the form of [\(8\)](#page-14-3). Using the Gauss-Codazzi equation the expression [\(8\)](#page-14-3) can be rewritten as

$$
\mathcal{L} = -\Delta_{\bar{h}} + \frac{1}{2}(\bar{r} - \bar{K}_{pq}\bar{K}^{pq}).
$$

Remark 11. Note the remarkable similarity with the equation for the tangential component of the second KID equation above [\(14\)](#page-17-3). In fact, notice that

$$
\mathcal{K} = \mathcal{L} + |\bar{K}|^2.
$$

Making use of the above observations one obtains the following lemma:

Lemma 6. Assuming the MOTS is stable, the only solution to [\(14\)](#page-17-3) is the trivial one, i.e. $N = 0$.

Proof. In the following let λ and μ denote, respectively, the lowest positive eigenvalues of \mathcal{L} and K. We proceed to compare λ and μ . For this, we make use of the Rayleigh-Ritz characterisation of these eigenvalues. Namely, one has that

$$
\lambda = \inf_{u} \oint_{\partial S} \left(|\bar{D}u|^2 + \frac{1}{2} (\bar{r} - |\bar{K}|^2) u^2 \right) dS,
$$

$$
\mu = \inf_{u} \oint_{\partial S} \left(|\bar{D}u|^2 + \frac{1}{2} (\bar{r} + |\bar{K}|^2) u^2 \right) dS,
$$

where the infimum is take over functions u on ∂S with $||u||_{L^2}^2 = 1$. It then follows that $\lambda \leq \mu$. Thus, if ∂S is a stable MOTS, then $\lambda > 0$ and accordingly $\mu > 0$. Now, using Lemma 4.2, (iii) in [\[2\]](#page-32-4) it follows that the Kernel of K is trivial. Thus, necessarily $N = 0$. П

The existence and uniqueness of the trivial solution in the time symmetric setting implies that these are natural boundary values to place on the [\(AKE\)](#page-3-0) here. Interestingly, this coincides with the boundary values of lemma [5](#page-12-0) so that in the time symmetric case, with the boundary values we have just prescribed, it turns out that necessarily one has a solution to the KID equations and therefore the spacetime evolving from this initial data will have a killing vector.

5.1.3 Measuring the deviation from staticity on ∂S

In the previous subsection it has been shown that, in the time symmetric setting, a natural prescription of the lapse N on the stable MOTS ∂S is $N = 0$. Moreover, the Einstein evolution equations under time symmetry imply that $\Delta_h N = 0$.

Given the choice

$$
N = \Delta_h N = 0, \qquad \text{on} \quad \partial S,\tag{16}
$$

it is natural to ask how much of the (time symmetric) KID equations are satisfied on ∂S . Setting $N = \Delta_h N = 0$ into equations [\(13a\)](#page-17-4) and [\(13b\)](#page-17-5) yields

normal – normal :
$$
\rho_p \rho^i D_i(\rho_q \rho^j D_j N) = 0
$$
,
normal – surface : $\rho_p \rho^i D_i \bar{D}_q N = 0$.

For the 'normal-normal' component observe that, using Gaussian coordinates, it follows from the conditions [\(16\)](#page-18-0) that

$$
0 = \rho_p \rho_q \rho^i \rho^j D_i D_j N
$$

= $\rho_p \rho_q (h^{ij} - \bar{h}^{ij}) D_i D_j N$
= $\rho_p \rho_q \Delta_{\bar{h}} N$,

so that $\Delta_{\bar{h}}N = 0$. This is consistent with equation [\(12\)](#page-17-2). Thus, solving the intrinsic equation [\(12\)](#page-17-2) also solves the normal-normal equation. On the other hand, the normal-surface component does not fully vanish as a consequence of the conditions (16) . Accordingly, one obtains an extra condition that needs to be imposed to satisfy the time symmetric KID equations on ∂S . The observation is contained in the following lemma in which we derive a *Dain-like* invariant characterising staticity of of the initial data set on the MOTS ∂S.

Lemma 7. Given time symmetric initial data set, let

$$
N = \Delta_h N = 0, \qquad on \quad \partial S.
$$

Then the time symmetric KID equations are satisfied on ∂S if and only if $\omega = 0$ where

$$
\omega \equiv \int_{\partial S} |\bar{K}|^2 |D^{\perp} N|^2,\tag{17}
$$

with $|\bar{K}|^2 = \bar{K}_{pq}\bar{K}^{pq}$ and $D^{\perp} \equiv \rho^i D_i$.

Proof. For the normal-surface component of the second KID equation one has that

$$
0 = \bar{h}_p{}^i \rho_q \rho^j D_i D_j N
$$

= $\bar{h}_p{}^i \rho_q (D_i (\rho^j D_j N) - D_j N D_i \rho^j)$
= $\bar{h}_p{}^i \rho_q (D_i (\rho^j D_j N) + D_j N \bar{K}_i{}^j)$
= $\rho_q \bar{D}_p (\rho^j D_j N)$.

Thus, this condition is equivalent to the statement that $\rho^i D_i N$ is constant along ∂S . We can use this observation to construct a quantity that measures the non-staticity of the boundary ∂S . Taking the L^2 norm of the quantity $\bar{D}_i D^{\perp} N$ on Σ , the above condition becomes, by integrating by parts

$$
0=\int_{\partial\mathcal{S}}\bar{D}_iD^\perp N\bar{D}^iD^\perp N=-\int_{\partial\mathcal{S}}D^\perp N\Delta_{\bar{h}}D^\perp N.
$$

In order to simplify this further, recall equation [\(12\)](#page-17-2). Taking the normal derivative of this equation yields

$$
D^{\perp} \Delta_{\bar{h}} N - \frac{1}{2} (\bar{r} + |\bar{K}|^2) D^{\perp} N = 0
$$

where we have used the Leibniz rule on the second term and that $N = 0$ on ∂S . To use this expression in the integral above, one commutes derivatives in the first term. Using the assumption of Gaussian coordinates

$$
D^{\perp} \Delta_{\bar{h}} N = \bar{h}^{ij} \rho^k D_k D_i D_j N
$$

=
$$
\bar{h}^{ij} \rho^k (r^l{}_{jki} D_l N + D_i D_k D_j N).
$$

In the second term, we now use the Leibniz rule to obtain

$$
\bar{h}^{ij}\rho^k D_i D_k D_j N = \Delta_{\bar{h}} D^\perp N - \bar{h}^{ij} D_i (D_j \rho^k D_k N) - \bar{K}^{jk} D_k D_j N
$$

$$
= \Delta_{\bar{h}} D^{\perp} N - \bar{h}^{ij} D_i (\bar{K}_j{}^k D_k N) - \bar{K}^{pq} (\bar{D}_p \bar{D}_q N + \bar{K}_{pq} D^{\perp} N)
$$

$$
= \Delta_{\bar{h}} D^{\perp} N - \bar{h}^{ij} \bar{K}_j{}^k D_i (\bar{D}_k N) - |\bar{K}|^2 D^{\perp} N
$$

$$
= \Delta_{\bar{h}} D^{\perp} N - |\bar{K}|^2 D^{\perp} N
$$

where we have, again, used that $N = 0$ on ∂S and changed the 3-covariant derivative to the 2-covariant derivative by contraction with an intrinsic quantity. To get to the fourth line one uses that $N = 0$ on ∂S . To take care of the Riemann tensor, we observe that

$$
\begin{aligned}\n\bar{h}^{ij}\rho^k r^l{}_{jki}D_l N &= \bar{h}^{ij}\rho^k r^l{}_{jki}h_l{}^m D_m N \\
&= \bar{h}^{ij}\rho^k r^l{}_{jki}(\bar{h}_l{}^m + \rho_l \rho^m)D_m N \\
&= \bar{h}^{ij}\rho^k r^l{}_{jki}\rho_l \rho^m D_m N \\
&= \bar{h}^{ij}\rho^k \rho^l r_{ljki}D^\perp N.\n\end{aligned}
$$

Now, we can apply the Gauss-Codazzi identity to obtain

$$
\bar{h}^{ij}\rho^k\rho^l r_{ljki} = \frac{1}{2}(\bar{r}+|\bar{K}|^2).
$$

Thus,

$$
D^{\perp} \Delta_{\bar{h}} N = \frac{1}{2} (\bar{r} - |\bar{K}|^2) D^{\perp} N + \Delta_{\bar{h}} D^{\perp} N,
$$

so that we finally obtain

$$
\Delta_{\bar{h}} D^{\perp} N = \frac{1}{2} (\bar{r} + |\bar{K}|^2) D^{\perp} N - \frac{1}{2} (\bar{r} - |\bar{K}|^2) D^{\perp} N = |\bar{K}|^2 D^{\perp} N.
$$

Thus, we can rewrite the condition that the second KID equation is satisfied in terms of the vanishing of

$$
\omega \equiv \int_{\Sigma} |\bar{K}|^2 |D^\perp N|^2.
$$

In other words, in time symmetric initial data, if $N = \Delta_h N = 0$ on the ∂S then the KID equations will be satisfied at the boundary if and only if $\omega = 0$. \Box

5.2 The non-time symmetric case

Having shown in the previous section that the KID equations can be used to choose suitable boundary values for the approximate Killing equation as well as constructing an invariant characterising the stationarity at the apparent horizon, we now move on to the non-time symmetric case, $K_{ab} \neq 0$. In this setting, the decomposition of the KID equations is much more complicated. We can no longer consider $N^i = 0$ and thus have to study a system of equations. Using the time symmetric case as a blueprint, we begin by manipulating the equations in order to obtain a system of equations that has a similar form to the operator K above so that we can investigate how much of the full KID equations can be satisfied on a boundary ∂S which is assume to be a MOTS.

5.2.1 Intrinsic equations over ∂S

We begin with the second decomposed KID equation [\(10\)](#page-16-0). Taking the trace and using that $\bar{h}^{pq}\rho_p = 0$ yields

$$
N^k \left(\bar{h}^{pq} D_k K_{pq}^{\parallel} - K_{iq} \left(\rho^i \bar{K}_k{}^q \right) - K_{pj} \left(\rho^j \bar{K}_k{}^p \right) \right) + \bar{h}^{ij} D_i N^k K_{kj} + \bar{h}^{ij} D_j N^k K_{ik} + \Delta_{\bar{h}} N - \rho^k D_k N \bar{K}
$$

= $N \left(\bar{h}^{ij} r_{ij} + \bar{h}^{ij} K_{ij} K + \bar{h}^{ij} K_{ik} K^k{}_j \right).$

One can simplify the first term as follows

$$
\bar{h}^{pq} D_k K_{pq}^{\parallel} = D_k(\bar{h}^{ij} K_{ij}) - K_{ij} \bar{h}_p^{i} \bar{h}_q^{j} D_k(\rho^p \rho^q)
$$

=
$$
D_k \bar{K},
$$

where, to get from the first to the second line, one uses $\bar{h}^{pq}\rho_p = 0$ and the MOTS condition [\(7\)](#page-14-4). Thus, one obtains

$$
N^k D_k \overline{K} - N_{\parallel}^A K_{ij} \rho^i \overline{K}_A{}^j - N^A K_{ij} \rho^j \overline{K}_A{}^i + \overline{h}^{ij} D_i N^k K_{kj} + \overline{h}^{ij} D_j N^k K_{ik} + \Delta_{\overline{h}} N - \rho^k D_k N \overline{K}
$$

= $N (\overline{h}^{ij} r_{ij} + \overline{K} K + \overline{h}^{ij} K_{ik} K^k{}_j).$

Now, we assume that the quantities $\rho_k N^k$ and $\frac{\partial}{\partial \rho} N^i$ are and prescribed on ∂S . We separate terms into their normal and tangential components as follows:

$$
N^k D_k \overline{K} = (\overline{h}_i^k + \rho_i \rho^k) N^i D_k \overline{K}
$$

= $N_{\parallel}^A \overline{D}_A \overline{K} + \rho_i N^i \rho^k D_k \overline{K},$

and

$$
h^{ij} D_i N^k K_{ij} = h^{ij} D_i N^k_{\parallel} K_{kj} + h^{ij} D_i (\rho^k \rho_l N^l) K_k j
$$

= $\bar{D}^A N^B_{\parallel} K^{\parallel}_{AB} - \rho^l K_{li} \bar{K}^i_A N^A_{\parallel} + h^{ij} D_i (\rho^k \rho_l N^l) K_{kj}.$

Thus, the second KID equation on ∂S implies that

$$
\Delta_{\bar{h}}N - \rho^k D_k N \bar{K} + 2\bar{D}^A N_{\parallel}^B K_{AB}^{\parallel} + N_{\parallel}^A (\bar{D}_A \bar{K} - 4K_{ij}\rho^i \bar{K}_A{}^j) - N (\bar{h}^{ij}r_{ij} + \bar{K}K + \bar{h}^{ij}K_{ik}K_{\ j}^k) = \tilde{F}
$$

where

$$
\tilde{F} \equiv -2h^{ij}D_i(\rho^k \rho_l N^l)K_{kj} - \rho_i N^i \rho^k D_k \bar{K}.
$$

In order to remove the normal derivate of N in the above equation, consider the trace of the first decomposed KID equation [\(9\)](#page-15-0)

$$
N\bar{K} = N_k \rho^k - \bar{D}_A N_{\parallel}^A.
$$

Taking the normal derivative of this quantity, one obtains an expression for the normal derivative of N in terms of prescribed quantities and thus the second KID equation on ∂S can be written

$$
\Delta_{\bar{h}} N + (2K_{\parallel}^{AB} - \bar{K}^{AB}) \bar{D}_A N_B^{\parallel} + N_{\parallel}^A (\bar{D}_A \bar{K} - 4K_{ij} \rho^i \bar{K}_A{}^j - \rho^k r_{kA}) - N (\bar{h}^{ij} r_{ij} + \bar{K} K + \bar{h}^{ij} K_{ik} K^k{}_j - \rho^k \bar{D}_k \bar{K}) = F
$$

where F now includes terms involving the normal derivative of N^i .

For the first KID equation, we cannot just take a derivative tangential to ∂S and take the trace as we did in the time symmetric case as the resulting equation is not elliptic. Instead, we consider the trace free part of [\(9\)](#page-15-0), namely

$$
NK_{AB}^{\parallel} + \bar{D}_{(A}N_{B)}^{\parallel} + N_k \rho^k \bar{K}_{AB} - \frac{1}{2} \bar{h}_{AB} (\bar{D}_C N_{\parallel}^C + \bar{K}(N + N_k \rho^k)) = 0.
$$

Taking the divergence of this equations and using the special form of the Riemann tensor in 2-dimensions we can write

$$
\Delta_{\bar{h}} N_B^{\parallel} + (2K_{AB}^{\parallel} - \bar{K} \bar{h}_{AB}) \bar{D}^A N + R N_B^{\parallel} + N (2 \bar{D}^A K_{AB}^{\parallel} - \bar{D}_B \bar{K}) = F_B,
$$

where

$$
F_B \equiv \bar{D}_B(\bar{K}N_k\rho^k) - 2\bar{D}^A(N_k\rho^k\bar{K}_{AB}).
$$

Thus, we have the following system of equations on ∂S :

$$
\Delta_{\bar{h}} N_B^{\parallel} + (2K_{AB}^{\parallel} - \bar{K} \bar{h}_{AB}) \bar{D}^A N + R N_B^{\parallel} + N (2\bar{D}^A K_{AB}^{\parallel} - \bar{D}_B \bar{K}) = F_B \tag{18a}
$$

$$
\Delta_{\bar{h}} N + (2K_{\parallel}^{AB} - \bar{K}^{AB}) \bar{D}_A N_B^{\parallel} + N_{\parallel}^A (\bar{D}_A \bar{K} - 4K_{ij}\rho^i \bar{K}_A{}^j - \rho^k r_{kA}) - N (\bar{h}^{ij} r_{ij} + \bar{K}K + \bar{h}^{ij} K_{ik} K^k{}_j - \rho^k \bar{D}_k \bar{K}) = F
$$
(18b)

where F and F_B are source terms. These source terms are completely determined in terms of the intrinsic geometry of ∂S , the extrinsic curvatures K_{ij} and \bar{K}_{AB} , the normal component of N^i , and the normal derivative of N^i , $\frac{\partial}{\partial \rho}N^i$.

Remark 12. As in the time symmetric case, the system [\(18a\)](#page-22-0)-[\(18b\)](#page-22-1) does not incorporate all parts of the KID equation on ∂S . Equation [\(18a\)](#page-22-0) is a formulation of the tracefree part of [\(9\)](#page-15-0) while [\(18b\)](#page-22-1) is the trace of [\(10\)](#page-16-0).

The system [\(18a\)](#page-22-0)-[\(18b\)](#page-22-1) is manifestly elliptic for (N, N_B^{\parallel}) . It can be succinctly written in the matricial form

$$
\Delta_{\bar h} \vec N + T^A \bar D_A \vec N + C \cdot \vec N = \vec F
$$

where T_A and C are 3×3 matrices and $\vec{N} \equiv (N, N_A^{\parallel})$. We note the formal similarity of this equation with the time symmetric equation [\(14\)](#page-17-3). However, in contrast to the time symmetric case, it is not clear how to connect the solvability of the system [\(18a\)](#page-22-0)-[\(18b\)](#page-22-1) to, for example, the stability of the MOTS ∂S. This is an interesting question which falls beyond the scope of the present article.

In order to provide some intuition into the consequences of the system [\(18a\)](#page-22-0)-[\(18b\)](#page-22-1), for the rest of this section, we make the following assumption:

Assumption 1. The elliptic operator associated to the system [\(18a\)](#page-22-0)-[\(18b\)](#page-22-1) as well as its adjoint have trivial Kernel.

As mentioned earlier, the above assumption ensures the existence of a unique solution (N, N_A^{\parallel}) to the system [\(18a\)](#page-22-0)-[\(18b\)](#page-22-1). Observe that this is independently of whether the 3-manifold S admits a solution to the KID equations. However, it is important to note that this system will always be solved by a solution to the KID equations. The values obtained as solutions to the system [\(18a\)](#page-22-0)- [\(18b\)](#page-22-1) then provide boundary values for the AKE boundary value problem in the following way: N and the tangential components of N^i are obtained through solving the system. The value of Δ_hN can be obtained by using Einstein's equations and the other quantities were already prescribed. One could obtain values for the quantities that we prescribed here through analysing the normalnormal and normal-tangential components of the decomposed KID equations independently of the above analysis. These components are derived in appendix [C.](#page-30-0)

5.2.2 Constructing an invariant on ∂S

Constructing an invariant on a MOTS ∂S immersed on a non-time symmetric hypersurface S is more involved than in the time symmetric case. It is important to note that the way one derives an invariant is not unique. We outline below one possible way is to consider the parts of the decomposed KID equations on ∂S that are not included in the system [\(18a\)](#page-22-0)-[\(18b\)](#page-22-1).

For example, under Assumption [1,](#page-22-2) the solution to the system [\(18a\)](#page-22-0)-[\(18b\)](#page-22-1) may not solve the trace of the first decomposed KID equation [\(9\)](#page-15-0):

$$
Q \equiv N\bar{K} - N_k \rho^k + \bar{D}_A N_{\parallel}^A = 0.
$$

The non-zero value of Q, in conjunction with the tracefree part of the second decomposed KID equation and the normal-normal and normal-tangential components of the decomposed KID equa-tions derived in appendix [C,](#page-30-0) will characterise the non-stationarity of $\mathcal S$ on the boundary 2-surface

 ∂S . In particular, the sum of the L^2 norms of these quantities will provide a geometric invariant that incorporates all of the above quantities.

6 Conclusion

We have shown that there exists solutions to the (AKE) , approximate Killing vectors, on asymptotically Euclidean initial data along with boundary conditions on an inner boundary 2-dimensional surface. We associated this boundary with a boundary of a black hole characterised by a MOTS. In the time symmetric case, we have then constructed invariants on this MOTS that classify the staticity of the initial data set. In particular, the invariant ω vanishes when there exists a Killing vector.

Combining the analysis of the latter sections with the main theorem allows us to write down the following theorem in the time symmetric case:

Theorem 3. Let λ be the Dain invariant associated to the boundary value problem

$$
\mathcal{P} \circ \mathcal{P}^* \begin{pmatrix} X \\ X^i \end{pmatrix} = 0 \quad on \ \mathcal{S},
$$

$$
\begin{cases} X|_{\partial \mathcal{S}} = 0, \\ \Delta_h X|_{\partial \mathcal{S}} = 0, \\ X^i|_{\partial \mathcal{S}} = 0, \\ \frac{\partial}{\partial \rho} X^i|_{\partial \mathcal{S}} = 0, \end{cases}
$$

in a time symmetric complete, smooth asymptotically Euclidean initial data set for the Einstein vacuum field equations with one asymptotic end and an inner boundary ∂S . Then, if on the one hand $\lambda = 0$ then the initial data is static. On the other hand, if the 2-surface invariant ω is non-zero on ∂S then the initial data cannot be static —and thus $\lambda \neq 0$.

Note that, by Lemma [5,](#page-12-0) since we have vanishing boundary conditions, we have a solution to the KID equations and thus Dain's invariant vanishes.

In the non-time symmetric case one could write down an analogue of the above theorem with the explicit solutions coming from the system [\(18a\)](#page-22-0)-[\(18b\)](#page-22-1) as well as prescribed values for the other boundary values. In this case Dain's invariant λ would be non-zero. Further work would entail removing Assumption 1 and the precise construction of invariants for specific given initial data. It would also be of interest to explore the conditions one would have to impose on the initial data in order to guarantee existence of unique solutions to the projected KID equations on ∂S.

Acknowledgements

We have benefited from discussions with J.L. Jaramillo on the MOTS stability operator. Further enriching discussions with J.L. Williams about various aspects of this project are acknowledged.The calculations in this article have been carried out in the suit xAct for the Wolfram programming language [\[15\]](#page-32-17). We have made used of routined for integration by parts in xAct originally written by T. Bäckdahl. We thank him for his help in adapting these to the setting considered in this article.

A Green's Formula for the full AKE

We will show that the Green formula indeed gives the form found in Lemma [3.](#page-8-0) Throughout this appendix we will suppress the volume and surface forms d μ and dS as well as ∂S on the surface integrals for readability. Start by considering the expression

$$
\int_{S} \mathscr{P} \circ \mathscr{P}^* \begin{pmatrix} X \\ X_i \end{pmatrix} \cdot \begin{pmatrix} Z \\ Z^i \end{pmatrix} - \int_{S} \mathscr{P} \circ \mathscr{P}^* \begin{pmatrix} Z \\ Z^i \end{pmatrix} \cdot \begin{pmatrix} X \\ X_i \end{pmatrix}
$$

=
$$
\int_{S} \mathscr{P} \circ \mathscr{P}^* (X)Z + \mathscr{P} \circ \mathscr{P}^* (X_i)Z^i - \int_{S} \mathscr{P} \circ \mathscr{P}^* (Z)X + \mathscr{P} \circ \mathscr{P}^* (Z^i)X_i.
$$

Note the slight abuse of notation in writing $\mathscr{P} \circ \mathscr{P}^*(X)$ to mean the lapse component of the AKE and $\mathscr{P} \circ \mathscr{P}^{*}(X_i)$ to mean the shift component. Since the AKE operator is self-adjoint, we only need to perform integration by parts on one of the integrals and the form of the bulk integrals will not matter. The AKE takes the form

$$
\mathcal{P} \circ \mathcal{P}^* \left(\begin{array}{c} X \\ X_i \end{array} \right) \equiv \left(\begin{array}{c} 2\Delta_h \Delta_h X - r^{ij} D_i D_j X + 2r \Delta_h X + \frac{3}{2} D^i r D_i X + (\frac{1}{2} \Delta_h r + r_{ij} r^{ij}) X \\ + D^i D^j H_{ij} - \Delta_h H_k^k - r^{ij} H_{ij} + \tilde{H} \\ D^j \Delta_h D_{(i} X_{j)} + D_i \Delta_h D^k X_k + D^j \Delta_h F_{ij} - D_i \Delta_h F_k^k - \bar{F}_i \end{array} \right)
$$

where

$$
\bar{H} \equiv 2(K\bar{Q} - K^{ij}\bar{Q}_{ij}) + 2(K^{ki}K^{j}{}_{k} - KK^{ij})\bar{\gamma}_{ij},
$$
\n
$$
\bar{F}_{i} \equiv (D_{i}K^{kj} - D^{k}K^{j})\bar{\gamma}_{jk} - (K^{k}{}_{i}D^{j} - \frac{1}{2}K^{kj}D_{i})\bar{\gamma}_{jk} + \frac{1}{2}K^{k}{}_{i}D_{k}\bar{\gamma}
$$
\n
$$
\bar{\gamma}_{ij} \equiv D_{i}D_{j}X - Xr_{ij} - \Delta_{h}Xh_{ij} + H_{ij}
$$
\n
$$
\bar{Q}_{ij} \equiv -\Delta_{h}(D_{(i}X_{j)} - D^{k}X_{k}h_{ij} + F_{ij})
$$
\n
$$
H_{ij} \equiv 2X(K^{k}{}_{i}K_{jk} - KK_{ij}) - K_{k(i}D_{j)}X^{k} + \frac{1}{2}K_{ij}D_{k}X^{k} + \frac{1}{2}K_{kl}D^{k}X^{l}h_{ij} - \frac{1}{2}X^{k}D_{k}K_{ij} + \frac{1}{2}X^{k}D_{k}Kh_{ij}
$$
\n
$$
F_{ij} \equiv 2X(Kh_{ij} - K_{ij}).
$$

We consider each component separately, beginning with the *lapse component*.

A.1 The lapse component

The lapse component of the AKE is given by

$$
\mathscr{P} \circ \mathscr{P}^*(X) = 2\Delta_h \Delta_h X - r^{ij} D_i D_j X + 2r \Delta_h X + \frac{3}{2} D^i r D_i X + (\frac{1}{2} \Delta_h r + r_{ij} r^{ij}) X + D^i D^j H_{ij} - \Delta_h H_k^k - r^{ij} H_{ij} + \bar{H}.
$$

Therefore, we perform integration by parts term by term on the expression

$$
\int_{\mathcal{S}} \mathscr{P} \circ \mathscr{P}^*(X) Z.
$$

Since we know that the operator is self-adjoint, we know the bulk integral obtained through integration by parts. accordingly, we concentrate our attention on the boundary terms of the associated bulk integral in each term of $\mathscr{P} \circ \mathscr{P}^*$.

$$
\int_{S} 2\Delta_h \Delta_h X : \oint \frac{\partial}{\partial \rho} X \Delta_h Z - \oint \frac{\partial}{\partial \rho} (\Delta_h Z) X + \oint \frac{\partial}{\partial \rho} (\Delta_h X) Z - \oint \frac{\partial}{\partial \rho} Z \Delta_h X,
$$

$$
\int_{S} r^{ij} D_{i} D_{j} XZ : \oint r^{ij} \rho_{i} D_{j} (X) Z - \oint X \rho_{j} D_{i} (r^{ij} Z) ,
$$

$$
\int_{S} r \Delta_{h} XZ : \oint rZ \frac{\partial}{\partial \rho} X - \oint X \frac{\partial}{\partial \rho} (rZ) ,
$$

$$
\int_{S} \frac{3}{2} D^{i} r D_{i} XZ : \frac{3}{2} \oint \frac{\partial}{\partial \rho} (rZ) X,
$$

$$
\int_{S} D^{i}D^{j}H_{ij}Z : \oint 2\rho^{i}D^{j}(XB_{ij})Z - \oint 2\rho^{j}XB_{ij}D^{i}Z \n- \frac{1}{2} \left(\oint \rho^{i}D^{j}(K_{ki}D_{j}X^{k})Z - \oint \rho^{j}K_{ki}D_{j}X^{k}D^{i}Z + \oint K_{ki}\rho_{j}X^{k}D^{j}D^{i}Z \right) \n- \frac{1}{2} \left(\oint \rho^{i}D^{j}(K_{kj}D_{i}X^{k})Z - \oint \rho^{j}K_{kj}D_{i}X^{k}D^{i}Z + \oint K_{kj}\rho_{i}X^{k}D^{j}D^{i}Z \right) \n+ \frac{1}{2} \left(\oint \rho^{i}D^{j}(K_{ij}D_{k}X^{k})Z - \oint \rho^{j}K_{ij}D_{k}X^{k}D^{i}Z + \oint \rho_{k}X^{k}K_{ij}D^{j}D^{i}Z \right) \n+ \frac{1}{2} \left(\oint \frac{\partial}{\partial \rho} (K_{kl}D^{k}X^{l})Z - \oint K_{kl}D^{k}X^{l} \frac{\partial}{\partial \rho}Z + \oint \rho^{k}K_{kl}X^{l}\Delta_{h}Z \right) \n- \frac{1}{2} \left(\oint \rho^{i}D^{j}X^{k}D_{k}(K_{ij})Z - \oint X^{k}\rho^{j}D_{k}(K_{ij})D^{i}Z \right) \n+ \frac{1}{2} \left(\oint \frac{\partial}{\partial \rho} (X^{k}D_{k}K)Z - \oint X^{k}D_{k}K \frac{\partial}{\partial \rho}Z \right),
$$

$$
\int_{S} \Delta_{h} H_{k}{}^{k} Z : \oint \frac{\partial}{\partial \rho} \left(X B_{k}{}^{k} \right) Z - \oint 2 X B_{k}{}^{k} \frac{\partial}{\partial \rho} Z \n+ \frac{1}{2} \left(\oint \rho_{j} D^{j} (K_{kl} D^{k} X^{l}) Z - \oint \rho^{j} K_{kl} D^{k} X^{l} D_{j} Z + \oint K_{kl} \rho^{k} X^{l} D^{j} D_{j} Z \right) \n+ \frac{1}{2} \left(\oint \rho_{j} D^{j} (K D_{k} X^{k}) Z - \oint \rho^{j} K D_{k} X^{k} D_{j} Z + \oint \rho_{k} X^{k} D^{j} D_{j} Z \right) \n+ \oint \frac{\partial}{\partial \rho} \left(X^{k} D_{k} K \right) Z - \oint X^{k} D_{k} K \frac{\partial}{\partial \rho} Z,
$$

$$
\int_{S} r^{ij} H_{ij} Z \cdot \frac{1}{2} \left(- \oint r^{ij} K_{ki} \rho_j X^k Z + \oint r^{ij} K_{ij} \rho_k X^k Z + \oint r K_{kl} \rho^k X^l Z \right),
$$

$$
\int_{S} \bar{H}Z : 4\left(\oint K\rho_{i}D^{i}D_{k}X^{k}Z - \oint \rho^{i}D_{k}X^{k}D_{i}(KZ) + \oint \rho_{k}X^{k}\Delta_{h}(KZ)\right) \n- 8\left(\oint \frac{\partial}{\partial \rho}XZ - \oint X\frac{\partial}{\partial \rho}Z\right) \n+ 2\left(\oint \rho_{k}K^{ij}D^{k}D_{i}X_{j}Z - \oint \rho^{k}D_{i}X_{j}D_{k}(K^{ij}Z) + \oint \rho_{i}X_{j}\Delta_{h}(K^{ij}Z)\right) \n- 2\left(\oint K\rho_{i}D^{i}D_{k}X^{k}Z - \oint n^{i}D_{k}X^{k}D_{i}(KZ) + \oint \rho_{k}X^{k}\Delta_{h}(KZ)\right) \n+ 4\left(\oint K^{ij}\rho_{k}D^{k}(A_{ij}X)Z - \oint n^{k}A_{ij}XD_{k}(K^{ij}Z)\right)
$$

+2
$$
\left(\oint B^{ij}\rho_i D_j(X)Z - \oint X\rho_j D_i(B^{ij}Z)\right)
$$

\n-2 $\left(\oint BZ \frac{\partial}{\partial \rho}X - \oint X \frac{\partial}{\partial \rho}(BZ)\right)$
\n+ $\left(-\oint r^{ij}K_{ki}\rho_j X^k Z + \oint r^{ij}K_{ij}\rho_k X^k Z + \oint rK_{kl}n^k X^l Z\right)$,

where $A_{ij} \equiv Kh_{ij} - K_{ij}$ and $B_{ij} \equiv K_{ik}K^{k}{}_{j} - KK_{ij}$. By inspection one sees that $X = 0, \Delta_h X =$ $0, X_i = 0, \frac{\partial}{\partial \rho} X^k = 0$ and the same for Z is enough to for all the boundary terms to vanish.

A.2 The shift component

The shift component of the AKE is

$$
\mathscr{P} \circ \mathscr{P}^*(X_i) = D^j \Delta_{\mathbf{h}} D_{(i} X_{j)} + D_i \Delta_{\mathbf{h}} D^k X_k + D^j \Delta_{\mathbf{h}} F_{ij} - D_i \Delta_{\mathbf{h}} F_k^k - \bar{F}_i.
$$

In analogy to the lapse component, we want to perform integration by parts on

$$
\int_{\mathcal{S}} \mathcal{P} \circ \mathcal{P}^*(X_i) Z^i.
$$

As above, we compute term by term ignoring the final bulk term. The first two terms (i.e. the ones with highest order derivatives) have the following boundary terms:

$$
\oint \frac{1}{2} n^j Z^i \Delta_h D_i X_j - \oint \frac{1}{2} \rho_i \Delta_h D^j Z^i X_j
$$
\n
$$
+ \oint \frac{1}{2} n^j Z^i \Delta_h D_j X_i - \oint \frac{1}{2} \rho_j \Delta_h D^j Z^i X_i
$$
\n
$$
+ \oint \rho_i Z^i \Delta_h D^k X_k - \oint n^k \Delta_h D_i Z^i X_k
$$
\n
$$
+ \oint \frac{1}{2} \frac{\partial}{\partial \rho} (D^j Z^i) D_j X_i - \oint \frac{1}{2} D^j Z^i \frac{\partial}{\partial \rho} (D_i X_j)
$$
\n
$$
+ \oint \frac{1}{2} \frac{\partial}{\partial \rho} (D^j Z^i) D_i X_j - \oint \frac{1}{2} D^j Z^i \frac{\partial}{\partial \rho} (D_j X_i)
$$
\n
$$
+ \oint \frac{\partial}{\partial \rho} (D_i Z^i) D^k X_k - \oint D_i Z^i \frac{\partial}{\partial \rho} (D^j X_j),
$$

$$
\int_{\mathcal{S}} D^j \Delta_h F_{ij} Z^i : \oint n^j \Delta_h (2A_{ij}X) Z^i - \oint \rho_k D^k (2A_{ij}X) D^j Z^i + \oint 2A_{ij}X n^k D_k D^j Z^i,
$$

$$
\int_{\mathcal{S}} D_i \Delta_{\pmb{h}} F_k^{\ k}: \oint \rho_i \Delta_h(4KX) Z^i - \oint \rho_k D^k(4KX) D_i Z^i + \oint 4n^k KX D_k D_i Z^i.
$$

Now, let $C_i^{jk} \equiv D_i K^{kj} - D^k K_i^j$ so that \bar{F}_i can be written as

$$
\bar{F}_i = C_i{}^{jk}\bar{\gamma}_{jk} - \left(K^k{}_i D^j - \frac{1}{2}K^{kj}D_i\right)\bar{\gamma}_{jk} + \frac{1}{2}K^k{}_i D_k\bar{\gamma}.
$$

Then

$$
\int_{S} \bar{F}_{i} Z^{i} : \left(\oint C_{i}{}^{jk} \rho_{j} D_{k} \left(X \right) Z^{i} - \oint X \rho_{k} D_{j} \left(C_{i}{}^{jk} Z^{i} \right) \right) - \left(\oint C_{i} Z^{i} \frac{\partial}{\partial \rho} X - \oint X \frac{\partial}{\partial \rho} \left(C_{i} Z^{i} \right) \right)
$$

$$
+\frac{1}{2}\left(-\oint r^{ij}K_{ki}\rho_{j}X^{k}Z+\oint r^{ij}K_{ij}\rho_{k}X^{k}Z+\oint rK_{kl}n^{k}X^{l}Z\right) -\left[\oint K^{k}{}_{i}\rho_{l}D^{l}D_{k}XZ^{i}-\oint n^{l}D_{k}XD_{l}(K^{k}{}_{i}Z^{i})+\oint \rho_{k}X\Delta_{h}(K^{k}{}_{i}Z^{i})\right. -\int \oint n^{j}K_{i}{}^{k}Xr_{jk}Z^{i} -\left(\oint K_{i}{}^{k}\rho_{k}\Delta_{h}XZ^{i}-\oint \rho_{l}D^{l}XD_{k}(K_{i}{}^{k}Z^{i})+\oint n^{l}XD_{l}D_{k}(K_{i}{}^{k}Z^{i})\right) +\frac{1}{2}\left(\oint K_{i}{}^{k}n^{j}K_{ij}D_{k}X^{l}Z^{i}-\oint K_{lj}\rho_{k}X^{l}D^{j}(K_{i}{}^{k}Z^{i})\right) -\frac{1}{2}\left(\oint K_{i}{}^{k}n^{j}K_{ij}D_{k}X^{l}Z^{i}-\oint K_{li}\rho_{j}X^{l}D^{j}(K_{i}{}^{k}Z^{i})\right) +\frac{1}{2}\left(\oint K_{i}{}^{k}n^{j}K_{jk}D_{l}X^{l}Z^{i}-\oint K_{ik}\rho_{j}X^{l}D^{j}(K_{i}{}^{k}Z^{i})\right) +\frac{1}{2}\left(\oint K_{i}{}^{k}n^{j}K_{jk}D_{l}X^{l}Z^{i}-\oint K_{j}k\rho_{l}X^{l}D^{j}(K_{i}{}^{k}Z^{i})\right) +\frac{1}{2}\left(\oint K_{i}{}^{k}n_{j}X^{l}D_{i}K_{j}Z^{i}-\oint K_{lm}n^{l}X^{m}D_{k}(K_{i}{}^{k}Z^{i})\right) -\frac{1}{2}\oint K_{i}{}^{k}n_{j}X^{l}D_{l}K_{j}Z^{i} \right) +\frac{1}{2}\left[\oint K_{i}{}^{k}n_{j}X^{l}D_{l}K_{j}Z^{i} - \oint \rho_{j}D_{k}XD_{l}(K^{k}{}^{k}Z^{i})\right) +\frac
$$

i)

$$
+\frac{1}{2}\left[-\left(\oint K_i^k \rho_k \Delta_h X Z^i - \oint \rho_l D^l X D_k (K_i^k Z^i) + \oint n^l X D_l D_k (K_i^k Z^i)\right)\right]
$$

$$
-\frac{1}{2}\oint K_i^k \rho_k X r Z^i
$$

$$
+\frac{1}{4}\left(\oint K_i^k \rho_k K l_m D^l X^m Z^i - \oint n^l X^m K_{lm} D_k (K_i^k Z^i)\right)
$$

$$
+\frac{1}{4}\left(\oint K_i^k \rho_k K D_l X^l Z^i - \oint \rho_l X^l K D_k (K_i^k Z^i)\right)
$$

$$
+\frac{1}{2}\oint K_i^k \rho_k X^l D_l K Z^i\right].
$$

We see that setting $X = 0$, $\Delta_h X = 0$, $X^i = 0$ and $\frac{\partial}{\partial \rho} X^i = 0$ and, equivalently, for $Z: Z = 0$, $\Delta_h Z = 0$, $Z^i = 0$ and $\frac{\partial}{\partial \rho} Z^i = 0$ that all boundary integrals in the above expressions vanish. This assertion can be verified by using the fact that we can separate the derivative terms like $D^i X^j$ into an tangential part and normal part to the boundary ∂S. Integrating by parts on the intrinsic derivatives yields $Xⁱ$ and the normal derivative part. Both of these vanish using the vanishing of X^i and its normal derivative.

In summary, we can write Green's formula as

$$
\int_{S} \mathcal{P} \circ \mathcal{P}^* \begin{pmatrix} X \\ X_i \end{pmatrix} \cdot \begin{pmatrix} Z \\ Z^i \end{pmatrix} - \int_{S} \mathcal{P} \circ \mathcal{P}^* \begin{pmatrix} Z \\ Z^i \end{pmatrix} \cdot \begin{pmatrix} X \\ X_i \end{pmatrix}
$$

$$
= \sum_{j=1}^2 \sum_{\alpha=1}^4 \left(\oint S_j^{\alpha}(X, X^i) B_j^{\prime \alpha}(Z, Z^i) - \oint b_j^{\alpha}(X, X^i) T_j^{\alpha}(Z, Z^i) \right)
$$

where

$$
b_1^1 = X
$$

\n
$$
b_2^1 = \Delta_h X
$$

\n
$$
b_1^{2,3,4} = X^i
$$

\n
$$
b_2^{2,3,4} = \frac{\partial}{\partial \rho} X^i
$$

 $B_1^{\prime 1} = Z$ $B_2^{'1} = \Delta_h Z$ $B_1^{'2,3,4} = Z^i$

and

Thus, we verify that the boundary operators satisfy
$$
b_j^{\alpha} = B_j^{;\alpha}
$$
 and, accordingly, the associated boundary value problems is self-adjoint.

 $B_2^{'2,3,4} = \frac{\partial}{\partial \rho} Z^i$

B Deriving the solution to the ODE arising from the LS condition

In this appendix we derive the solution to the system of equations

$$
\begin{cases}\n\left(\frac{d^2}{d\rho^2} - |\xi|^2\right) \left(4\frac{d^2}{d\rho^2}X - |\xi|^2X + 3i\xi^A\frac{d}{d\rho}X_A\right) = 0, \\
\left(\frac{d^2}{d\rho^2} - |\xi|^2\right) \left(\left(\frac{d^2}{d\rho^2} - |\xi|^2\right)X_A + 3i\xi_A\left(\frac{d}{d\rho}X + i\xi^BX_B\right)\right) = 0.\n\end{cases}
$$

This system is used in the analysis of the Lopatinskij-Shapiro conditions in Section [2.4.](#page-9-0)

Using the Ansatz

$$
X_i = \sum_{n=0}^{k} X_{*i} \rho^n e^{\pm |\xi| \rho}
$$

where $k = 0, 1, 2$, we obtain the general solution in vector notation

$$
\vec{X} = \vec{a}e^{|\xi|\rho} + \vec{b}e^{-|\xi|\rho} + \alpha \left(\frac{c_1}{\frac{-c_1\xi_1 + ic|\xi|}{\xi_2}}\right)\rho e^{|\xi|\rho} + \beta \left(\frac{d_1}{\frac{-d_1\xi_1 - id|\xi|}{\xi_2}}\right)\rho e^{-|\xi|\rho} \n+ \gamma \left(\rho^2 \left(\frac{\frac{3r|\xi|}{10}\left(-ir + \frac{r_1(\xi_1 + \xi_2)}{|\xi|}\right)}{\frac{3r_1(\xi_1 + \xi_2)}{10}\xi_2\left(-ir + \frac{r_1(\xi_1 + \xi_2)}{|\xi|}\right)}\right) + \rho \left(\frac{r}{r_1}\right)\right)e^{|\xi|\rho} \n+ \delta \left(\rho^2 \left(\frac{\frac{3s|\xi|}{10}\left(-is - \frac{3is_1(\xi_1 + \xi_2)}{|\xi|}\right)}{\frac{3r_1(\xi_1 + \xi_2)}{10}\xi_2\left(-is + \frac{s_1(\xi_1 + \xi_2)}{|\xi|}\right)}\right) + \rho \left(\frac{s}{s_1}\right)\right)e^{-|\xi|\rho},
$$
\n(19)

where \vec{a} and \vec{b} are constant vectors, $\{c, c_1, d, d_1, r, r_1, r_2, s, s_1, s_2\}$ are constants and $\alpha, \beta, \gamma, \delta$ are constants of multiplicity. We write the vector \vec{a} as

$$
\vec{a} = \begin{pmatrix} a \\ a_1 \\ a_2 \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + a_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
$$

where these vectors are linearly independent. This is the same as setting the values of a, a_1, a_2 and using the multiplicity of this solution of the system of ODEs to then multiply each vector by a constant. We can do the same thing with the vector $\vec{c} = (c, c_1, c_2)$ where $c_2 = \frac{-c_1\xi_1 + ic|\xi|}{\xi_2}$ $\frac{1+ic|\xi|}{\xi_2}$. Since there are two constants and c_2 is a linear combination of c and c_1 there are two linearly independent vectors we can construct from this. Rearranging the expression for c_2 so that c_1 and c_2 are free, one can then make the choice of $c_1 = 1, c_2 = 0$ and $c_1 = 0, c_2 = 1$ to get two solutions

$$
\gamma \begin{pmatrix} \frac{-\mathrm{i}\xi_1}{|\xi|} \\ 1 \\ 0 \end{pmatrix} \rho \mathrm{e}^{|\xi|\rho} + \delta \begin{pmatrix} \frac{-\mathrm{i}\xi_2}{|\xi|} \\ 0 \\ 1 \end{pmatrix} \rho \mathrm{e}^{|\xi|\rho}.
$$
 (20)

Finally, one notes that in the final two terms of the full solution above, equation [\(19\)](#page-29-1), that the vector $\vec{r} = (r, r_1, r_2)$ (and $\vec{s} = (s, s_1, s_2)$) has to be linearly independent to the two vectors in [\(20\)](#page-29-2). Choosing $r = 1, r_1 = r_2 = 0$, the term becomes

$$
\gamma \left(\rho^2 \begin{pmatrix} -\frac{3|\xi|}{10} \\ -\frac{3}{10} \mathbf{i} \xi_1 \\ -\frac{3}{10} \mathbf{i} \xi_2 \end{pmatrix} + \rho \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) e^{|\xi| \rho}.
$$

One can then perform these manipulations to all the terms with negative exponential solution to obtain the full general solution the ODE system

$$
\vec{X}(\rho) = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-|\xi|\rho} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-|\xi|\rho} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-|\xi|\rho} + c_4 \begin{pmatrix} \frac{i\xi_1}{|\xi|} \\ 1 \\ 0 \end{pmatrix} \rho e^{-|\xi|\rho} + c_5 \begin{pmatrix} \frac{i\xi_2}{|\xi|} \\ 0 \\ 1 \end{pmatrix} \rho e^{-|\xi|\rho}
$$

$$
c_6 \left(\frac{3}{10} \begin{pmatrix} |\xi| \\ -i\xi_1 \\ -i\xi_2 \end{pmatrix} \rho^2 + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} p \right) e^{-|\xi|\rho} + c_7 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{|\xi|\rho} + c_8 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{|\xi|\rho} + c_9 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{|\xi|\rho}
$$

$$
c_{10} \begin{pmatrix} \frac{-i\xi_1}{|\xi|} \\ 1 \\ 0 \end{pmatrix} \rho e^{|\xi|\rho} + c_{11} \begin{pmatrix} \frac{-i\xi_2}{|\xi|} \\ 0 \\ 1 \end{pmatrix} \rho e^{|\xi|\rho} + c_{12} \begin{pmatrix} -\frac{3}{10} \begin{pmatrix} |\xi| \\ i\xi_1 \\ i\xi_2 \end{pmatrix} \rho^2 + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rho \end{pmatrix} e^{|\xi|\rho}.
$$

with 12 constants. We can then write the stable solution as the first six terms of the this solution:

$$
\vec{X}_s(\rho) = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-|\xi|\rho} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-|\xi|\rho} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-|\xi|\rho} + c_4 \begin{pmatrix} \frac{|\xi_1|}{|\xi|} \\ 1 \\ 0 \end{pmatrix} \rho e^{-|\xi|\rho} + c_5 \begin{pmatrix} \frac{|\xi_2|}{|\xi|} \\ 0 \\ 1 \end{pmatrix} \rho e^{-|\xi|\rho}
$$

$$
c_6 \left(\frac{3}{10} \begin{pmatrix} |\xi| \\ -i\xi_1 \\ -i\xi_2 \end{pmatrix} \rho^2 + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rho \right) e^{-|\xi|\rho}.
$$

By relabelling the constants, one obtains the form of solution found in the proof of Lemma [4.](#page-10-1)

C Deriving the decomposed KID equations

One can project the KID equations [\(2a\)](#page-3-1)-[\(2b\)](#page-3-1) onto the 2-surfaces of constant ς . Let N^{\parallel} ^{*i*} denote the projection of the shift vector N^i onto the 2-surfaces —i.e. $N^{\parallel i} \equiv \bar{h}^i{}_j N^j$. We proceed now to decompose objects into parts perpendicular and parallel to the normal ρ^i . Observing that

$$
D_i N_j = D_i \left(\delta_j^k N_k \right)
$$

=
$$
D_i \left((\bar{h}_j^k + \rho_j \rho^k) N_k \right),
$$

one concludes that

$$
D_i N_j = D_i N_j^{\parallel} + \rho_j \rho^k D_i N_k - N_k \rho^k (\bar{K}_{ij} - \rho_i a_j) - N_k \rho_j (\bar{K}_i^k - \rho_i a^k).
$$

Thus, the projection of D_iN_j onto the 2-surfaces is given by

$$
\bar{h}_p^{\ \ i} \bar{h}_q^{\ \ j} D_i N_j = \bar{D}_p N_q^{\parallel} + N_k \rho^k \bar{K}_{pq}.
$$

Making use of this expression on the projection of the first KID equation [\(2a\)](#page-3-1) with $\bar{h}_p^{i} \bar{h}_q^{j}$ yields

$$
NK_{pq}^{\parallel} + \bar{D}_{(p}N_{q)}^{\parallel} + N_k \rho^k \bar{K}_{pq} = 0,
$$

where $K_{pq}^{\parallel} \equiv \bar{h}_p^{i} \bar{h}_q^{j} K_{ij}$.

For the second KID equation [\(2b\)](#page-3-1), we project term by term to derive the following using Gaussian normal coordinates:

$$
\bar{h}_p^{\ \ i}\bar{h}_q^{\ \ j}D_iD_jN=\bar{D}_p\bar{D}_qN+\rho^k\bar{K}_{pq}D_kN,
$$

$$
\bar{h}_{p}{}^{i}\bar{h}_{q}{}^{j}N^{k}D_{k}K_{ij} = N^{k}\left(D_{k}K_{pq}^{||} + K_{iq}(\rho^{i}(\bar{K}_{kp} + a_{p}\rho_{k}) + (\bar{K}_{k}{}^{i} + a^{i}\rho_{k})\rho_{p}\right) \n+ K_{pj}(\rho^{j}(\bar{K}_{kq} + a_{q}\rho_{k}) + (\bar{K}_{k}{}^{j} + a^{j}\rho_{k})\rho_{q}) - K_{ij}(\rho^{j}(\bar{K}_{k}{}^{i} + a^{i}\rho_{k})\rho_{p}\rho_{q} \n+ \rho^{i}(\bar{K}_{k}{}^{j} + a^{j}\rho_{k})\rho_{p}\rho_{q} + \rho^{i}\rho^{j}(\bar{K}_{kq} + a_{q}\rho_{k})\rho_{p} + \rho^{i}\rho^{j}(\bar{K}_{kp} + a_{p}\rho_{k})\rho_{q})\right),
$$

 $2\bar{h}^{ij}r_{ij} = \bar{r} - \bar{K}^2 + \bar{K}_{pr}\bar{K}^{pr},$

where we have used the Gauss-Codazzi equation to derive the final identity. Putting these expression together, using Gaussian normal coordinates, we obtain the projection of the second KID equation onto ∂S :

$$
N^{k} \Big(D_{k} K_{pq}^{\parallel} + K_{iq} \Big(\rho^{i} (\bar{K}_{kp}) + (\bar{K}_{k}^{i}) \rho_{p} \Big) + K_{pj} \Big(\rho^{j} (\bar{K}_{kq}) + (\bar{K}_{k}^{j}) \rho_{q} \Big) - K_{ij} \Big(\rho^{j} (\bar{K}_{k}^{i}) \rho_{p} \rho_{q} + \rho^{i} (\bar{K}_{k}^{j}) \rho_{p} \rho_{q} + \rho^{i} \rho^{j} (\bar{K}_{kq}) \rho_{p} + \rho^{i} \rho^{j} (\bar{K}_{kp}) \rho_{q} \Big) \Big) + \bar{h}_{p} {^{i} \bar{h}_{q} {^{j} D}_{i} N^{k} K_{kj} + \bar{h}_{p} {^{i} \bar{h}_{q} {^{j} D}_{j} N^{k} K_{ik} + \bar{D}_{p} \bar{D}_{q} N + \rho^{k} D_{k} N \bar{K}_{pq} = N \Big(\bar{h}_{p} {^{i} \bar{h}_{q} {^{j} r_{ij}} + K K_{pq}^{\parallel} + \bar{h}_{p} {^{i} \bar{h}_{q} {^{j} K_{ik} K^{k} }_{j} \Big) .
$$

Together, equations [\(9\)](#page-15-0) and [\(10\)](#page-16-0) give conditions on ∂S that Killing vectors of the the spacetime must satisfy on this 2-surface. However, the converse is not true. If $(N, Nⁱ)$ a solution to these two equations that does not necessarily mean that one has a solution to the KID equations. This is because there are two other components of the KID equations in this decomposition which we call the normal-normal and normal-tangential components. For the first KID equation [\(2a\)](#page-3-1), one has

normal – normal :
$$
N K_{pq}^{\perp} + \rho_{(p|} \rho^j D_{j|} N_q^{\perp} - N_i a^i \rho_p \rho_q - N_{(p}^{\perp} a_q) = 0
$$

normal – tangential : $N K_{pq}^{\perp \parallel} + \frac{1}{2} (\rho_p \rho^i D_i N_q^{\parallel} + N_j a^j \rho_p \rho_q + N_p^{\perp} a_q)$
 $+ \frac{1}{2} (\bar{h}_q{}^j D_j N_p^{\perp} - N_i \rho_p \bar{K}_q{}^i - \rho^i N_i \bar{K}_{pq}) = 0,$

where $K_{pq}^{\perp} \equiv \rho_p \rho^i \rho_q \rho^j K_{ij}$ and $K_{pq}^{\perp} \equiv \rho_p \rho^i \bar{h}_j{}^q K_{ij}$. For the second KID equation [\(2b\)](#page-3-1), one can obtain the projection under the assumption of Gaussian coordinates. The normal-normal component of the first term of [\(2b\)](#page-3-1) can be written as

$$
\rho_p \rho_q \rho^i \rho^j N^k D_k K_{ij} = \rho_p \rho_q N^k D_k (\rho^i \rho^j K_{ij}) - \rho_p \rho_q N^k K_{ij} (\rho^i \bar{K}_k{}^j + \rho^j \bar{K}_k{}^i)
$$

=
$$
\rho_p \rho_q N^k D_k (K - \bar{K}) - \rho_p \rho_q N^k K_{ij} (\rho^i \bar{K}_k{}^j + \rho^j \bar{K}_k{}^i),
$$

where we have used the MOTS condition in the final line. Then the normal-normal component of the second KID equations is

normal – normal
$$
\rho_p \rho_q N^k D_k (K - \bar{K}) - \rho_p \rho_q N^k K_{ij} (\rho^i \bar{K}_k{}^j + \rho^j \bar{K}_k{}^i)
$$

$$
+ \rho_p \rho_q \rho^i \rho^j (D_i N^k K_{kj} + D_j N^k K_{ik}) + \rho_p \rho^i D_i (\rho_q \rho^j D_j N)
$$

$$
= N (-\bar{h}_p{}^i \bar{h}_q{}^j r_{ij} + K K_{pq}^{\perp} + \rho_p \rho_q \rho^i \rho^j K_{ik} K_j^k).
$$

For the normal-tangential component, we obtain

normal – tangential
$$
h_p{}^i \rho_q \rho^j (N^k D_k K_{ij} + D_i N^k K_{kj} + D_j N^k K_{ik})
$$

$$
+ \rho_p \rho^i D_i \bar{D}_q N
$$

$$
= N (h_p{}^i \rho_q \rho^j r_{ij} + K K^{\perp \parallel} + h_p{}^i \rho_q \rho^j K_i{}^k K_{kj}).
$$

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