

A Classification of Modular Functors via Factorization Homology

Adrien Brochier ^a and Lukas Woike ^b

^a *Institut de Mathématiques de Jussieu-Paris Rive Gauche*
Université Paris Cité
Sorbonne Université
UMR 7586 CNRS
Bâtiment Sophie Germain
8 Place Aurélie Nemours
75013 Paris
France

^b *Institut de Mathématiques de Bourgogne*
Université de Bourgogne
UMR 5584 CNRS
Faculté des Sciences Mirande
9 Avenue Alain Savary
21078 Dijon
France

Modular functors are traditionally defined as systems of projective representations of mapping class groups of surfaces that are compatible with gluing. They can formally be described as modular algebras over central extensions of the modular surface operad, with the values of the algebra lying in a suitable symmetric monoidal $(2, 1)$ -category \mathcal{S} of linear categories. In this paper, we prove that modular functors in \mathcal{S} are equivalent to self-dual balanced braided algebras \mathcal{A} in \mathcal{S} (a categorification of the notion of a commutative Frobenius algebra) for which a condition formulated in terms of factorization homology with coefficients in \mathcal{A} is satisfied; we call such \mathcal{A} connected. The equivalence in one direction is afforded by genus zero restriction. Our construction of the inverse equivalence is entirely topological and can be thought of as a far reaching generalization of the construction of modular functors from skein theory. In order to verify the connectedness condition in practice, we prove that it can be reduced to a single condition in genus one. Moreover, we show that cofactorizability of \mathcal{A} , a condition known to be satisfied for modular categories, is sufficient. Therefore, we recover in particular Lyubashenko’s construction of a modular functor from a (not necessarily semisimple) modular category and show that it is determined by its genus zero part. Additionally, we exhibit modular functors that do not come from modular categories and outline applications to the theory of vertex operator algebras.

CONTENTS

1	Introduction and summary	2
1.1	Statement of the main result	3
1.2	Essential uniqueness of extensions to a modular functor	5
1.3	Projective representations and equivalences	6
1.4	Cofactorizability as a sufficient condition	6
1.5	Recovering the Lyubashenko construction	6
1.6	Modular functors that do not come from modular categories	7
1.7	Relation to skein theory	7
1.8	Relation to higher-dimensional fully extended topological field theories	8
2	Preliminaries	8
2.1	Factorization homology of framed E_2 -algebras	8
2.2	The modular extension of cyclic framed little disks algebras	9
2.3	The boundary map	12
2.4	A concrete description of cyclic framed E_2 -algebras with values in linear categories	14
3	The definition of a modular functor and the moduli space of modular functors	15
3.1	The definition of a modular functor	15
3.2	The extension problem for cyclic framed E_2 -algebras and the moduli space of modular functors	18

4	Ansular functors and handlebody skein modules	18
4.1	The handlebody skein modules for a cyclic framed E_2 -algebra	19
4.2	Equivariance properties of handlebody skein modules under the mapping class group action	21
4.3	The connection to skein algebras and skein modules	22
5	Connected cyclic framed E_2 -algebras and their associated extensions of the surface operad	23
5.1	The $\Omega_{\mathcal{A}}$ -groupoids and the map $\text{Surf}_{\mathcal{A}} \rightarrow \text{Surf}$ of modular operads	23
5.2	Connectedness of the $\Omega_{\mathcal{A}}$ -groupoids: General connectedness and reduction to genus one	26
5.3	The notion of connectedness	27
6	The construction and classification of modular functors	29
6.1	The construction of modular functors	29
6.2	Universality of $\text{Surf}_{\mathcal{A}}$ and $\mathfrak{F}_{\mathcal{A}}$	30
6.3	Weak uniqueness of extensions	32
6.4	The classification of modular functors	33
7	Sufficient conditions for connectedness	34
7.1	Triviality of the mapping class group action on factorization homology	34
7.2	Cofactorizability	34
7.3	Simplification of $\text{Surf}_{\mathcal{A}}$	35
8	Applications and examples	36
8.1	The spaces of conformal blocks of a Rex-valued modular functor	36
8.2	Modular categories	37
8.3	Vertex operator algebras	38
8.4	Drinfeld centers of possibly non-spherical pivotal finite tensor categories	39

1 INTRODUCTION AND SUMMARY

A *modular functor*, in its simplest form, is a system of projective representations of mapping class groups of surfaces on vector spaces, the so-called *spaces of conformal blocks* [Seg88, MS89, Tur94, Til98, BK01]. These are subject to compatibility conditions with respect to the gluing of surfaces. The notion of a modular functor grew out of attempts to axiomatically capture (aspects of) the mathematical structure underlying conformal field theory and has, over the last three decades, developed into one of the key concepts at the cross-roads of low-dimensional topology, representation theory and mathematical physics. More specifically, the reasons for the interest in modular functors include the following:

- Modular functors provide a rich source for highly non-trivial and explicitly computable mapping class group representations. A landmark result is the asymptotic faithfulness of the mapping class group representations built from certain quantum groups established by Andersen [And06] and Freedman-Walker-Wang [FWW02] by different methods.
- Modular functors are closely related to three-dimensional topological field theory. In fact, the Reshetikhin-Turaev construction [RT90, RT91] provides in particular a class of modular functors. The input datum for this construction is a *semisimple modular category* (also called *modular fusion category*), a certain type of finitely semisimple braided tensor category introduced by Moore-Seiberg [MS89, MS90] and Turaev [Tur92, Tur94] that can be seen as a categorical version of *modular data* (however, the notion of a modular category is strictly finer [MS21]). Modular categories can for instance be obtained by taking modules over suitable quantum groups [Tur94, BK01, EGNO15, Kas15] or vertex operator algebras [Hua08]. But modular functors exist under far more general assumptions than three-dimensional topological field theories. Most importantly, they can also be built from possibly *non-semisimple* modular categories as demonstrated by Lyubashenko [Lyu95a, Lyu95b] while the construction of the three-dimensional topological field theory requires semisimplicity by [BDSPV15] (see however [DRGG⁺22a] for a partially defined topological field theory using modified traces). Modular functors also provide a topological perspective on various homological invariants associated with these non-semisimple categories [LMSS22, SW21, MW23c].
- From the perspective of mathematical physics, modular functors form one of the key mathematical structures governing two-dimensional conformal field theory. For example, the famous *Verlinde formula* [Ver88, MS88, Car89, Wit89, Tur94], i.e. the diagonalization of the fusion rules through the so-called *S-matrix*, is ultimately a statement about the orbit of the fusion multiplication on the space of conformal blocks of the torus under the action of the mapping class group $\text{SL}(2, \mathbb{Z})$ of the torus. Another reason for the interest in modular functors is the fact that they provide a direct and conceptual access to the so-called *correlators* of a conformal field theory [FRS02, FRS04a, FRS04b, FRS05, FS17],

certain elements in the spaces of conformal blocks invariant under the mapping class group actions and gluing.

The main result of this article is a classification of modular functors. To motivate our result, recall the classical theorem that two-dimensional topological field theories in the sense of Atiyah [Ati88] are classified by commutative Frobenius algebras, i.e. finite-dimensional commutative algebras equipped with a non-degenerate invariant pairing [Abr96, Koc03]. In particular, every ordinary two-dimensional topological field theory yields by evaluation on the circle and genus zero surfaces a commutative Frobenius algebra. In the higher categorical framework, two-dimensional topological field theories and, more generally, modular functors with values in a symmetric monoidal $(2, 1)$ -category \mathcal{S} , yield by evaluation on the circle and genus zero surfaces a *cyclic algebra* over the so-called *framed E_2 -operad*. Those are explicitly characterized in [MW23b]. In the familiar case where \mathcal{S} is some $(2, 1)$ -category of linear categories, cyclic framed E_2 -algebras coincide with balanced braided monoidal categories equipped with an appropriate analog of a non-degenerate pairing. Simply put, this paper is about identifying necessary and sufficient conditions on such a cyclic framed E_2 -algebra to produce a modular functor and the question of whether such an extension to a modular functor is unique.

Most previously known examples of modular functors arise as the restriction to dimension two of an (at least partially defined) three-dimensional topological field theory which, in turn, is constructed from a braided monoidal category satisfying strong finiteness and duality conditions [RT91, BK01, KL01, DRGG⁺22a]. In fact, the modular functor is often the main ingredient in these constructions, and in all of those cases coincides with the one which was defined by Lyubashenko [Lyu95a, Lyu95b, Lyu96, KL01]. It is a common theme in the study of topological field theories that the extension to the top dimension (here this would mean producing numerical invariants of three-dimensional manifolds) is the step that requires the strongest conditions. If one is only interested in constructing modular functors, it is natural to expect that these conditions can be relaxed greatly. Solving the classification problem for modular functors amounts to precisely understanding these conditions.

1.1 Statement of the main result. In order to state precisely the main results of this article, let us discuss the definition of a modular functor in more detail using the language of operads: Recall that an operad, as introduced by Boardman, Vogt and May [BV68, May72, BV73], describes an (algebraic) structure by giving separately the needed nullary, unary, binary — more generally, n -ary operations (operations with n inputs and one output) and a prescription for how to compose them. The strength of this description lies in the fact that the n -ary operations do not necessarily have to form a set, but can form for instance a space, a groupoid or a chain complex. This allows for the description of highly intricate algebraic structures. An important family of operads, that in fact motivated to a large extent the invention of the concept in [BV68, May72, BV73], are the *operads E_r of little disks of dimension $r \geq 1$* . These are topological operads whose space of operations of arity n is given by the space of embeddings of n many r -dimensional disks into one r -dimensional disk that are composed of translations and rescalings. If one additionally allows rotations, one obtains the *framed E_r -operads*.

Cyclic operads, as defined by Getzler and Kapranov [GK95], are operads that come with a prescription how to cyclically permute the inputs of operations with the output, which essentially amounts to consistently erasing the distinction between inputs and the output altogether. Modular operads [GK98] come additionally with a self-composition for operations that can also be thought of as a trace.

The prototypical example of a modular operad is the groupoid-valued modular operad Surf of compact oriented surfaces and their mapping classes or, for short, the modular surface operad. The groupoid of arity n operations has connected compact oriented surfaces with $n + 1$ boundary components as objects (here $n + 1$ has to be read as n inputs plus one output) and mapping classes between these surfaces as morphisms, see Section 2.2 for details. Crucially, the genus zero part of Surf is equivalent to the operad fE_2 of framed little disks. In particular, fE_2 has a natural cyclic structure which is not entirely obvious from its original definition.

Given a modular operad \mathcal{O} , one can consider modular algebras over it. These algebras take values in a suitable (higher) symmetric monoidal category \mathcal{S} . If \mathcal{O} is groupoid-valued, then it is natural to choose \mathcal{S} as a symmetric monoidal $(2, 1)$ -category because it is enriched over groupoids, and this is in fact the only case that will be relevant for the treatment of modular functors in this article. More concretely, one should imagine \mathcal{S} as a suitable choice of symmetric monoidal $(2, 1)$ -category of \mathbf{k} -linear categories, where \mathbf{k} is a fixed algebraically closed field; an example is the symmetric monoidal $(2, 1)$ -category Rex of finitely cocomplete linear categories and finitely cocontinuous functors, equipped with the Deligne-Kelly monoidal product [Fra13]. For an object $X \in \mathcal{S}$, a non-degenerate symmetric pairing $\kappa : X \otimes X \rightarrow I$ (here I is the unit of \mathcal{S}) turns the endomorphism operad of X into a modular operad. This modular operad is denoted by End_{κ}^X . The structure of a modular \mathcal{O} -algebra on (X, κ) is then by definition a map $\mathcal{O} \rightarrow \text{End}_{\kappa}^X$ of modular operads. This is essentially the original definition of Getzler and Kapranov, but we have to take into account

that, since we are considering bicategorical operadic algebras, it is crucial to relax all of the familiar axioms for modular operads and their algebras up to coherent isomorphism. This is done in [MW23b], and we give a concise summary in Section 2.2.

Equipped with the theory of modular operads and their algebras, one might tentatively define a modular functor with values in \mathcal{S} as a modular algebra over Surf . While in principal this seems like a reasonable definition, experience tells us that we should in fact also allow central extensions of mapping class groups. This very important point in the definition of a modular functor takes into account the well-known fact that the naturally occurring mapping class group representations built from representation categories tend to be *projective representations* because of the so-called *framing anomaly* [Ati90], see [GM13, Section 3] for a detailed discussion. In order to incorporate this, we use the following definition: A *modular functor with values in a symmetric monoidal $(2, 1)$ -category \mathcal{S}* is a modular algebra over an extension of Surf as a modular operad (see Section 3.1 for what is meant by extension here). We emphasize that the extension is *part of the data*; it can a priori be arbitrary and will only be subject to the consistency conditions that we give in Definition 3.1 and 3.3. The conditions will actually imply that the extension produces extensions of mapping class groups that are central (Remark 3.2).

As already mentioned, the present article is concerned with an obvious problem in the study of modular functors, namely the *classification of modular functors*. More specifically, the present article offers a classification of modular functors in terms of so-called *self-dual balanced braided algebras* satisfying a condition formulated in terms of factorization homology. Let us first recall the following definition from [MW23b]:

Definition 2.1 ([MW23b, Definition 5.4]). A *self-dual balanced braided algebra* in \mathcal{S} is an object $\mathcal{A} \in \mathcal{S}$ together with the following structure:

- \mathcal{A} is a *balanced braided algebra* in \mathcal{S} — in more detail:
 - \mathcal{A} has a multiplication $\mu : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ that is associative and unital up to coherent isomorphism (we denote the unit 1-morphism by $u : I \rightarrow \mathcal{A}$);
 - \mathcal{A} comes with an isomorphism $c : \mu \rightarrow \mu^{\text{op}} = \mu \circ \tau$ (where τ is the symmetric braiding of \mathcal{S}) called *braiding* which satisfies the hexagon relations;
 - \mathcal{A} comes with an isomorphism $\theta : \text{id}_{\mathcal{A}} \rightarrow \text{id}_{\mathcal{A}}$ referred to as *balancing* satisfying

$$\begin{aligned} \theta \circ \mu &= c^2 \circ \mu(\theta \otimes \theta) , \\ \theta \circ u &= \text{id}_u . \end{aligned}$$

- \mathcal{A} comes with a non-degenerate pairing $\kappa : \mathcal{A} \otimes \mathcal{A} \rightarrow I$ (a 1-morphism exhibiting \mathcal{A} as its own dual in the homotopy category of \mathcal{S}) together with an isomorphism $\gamma : \kappa(u \otimes \mu) \rightarrow \kappa$ subject to the following conditions:
 - The isomorphism $\kappa(u \otimes \mu(u, -)) \rightarrow \kappa(u, -)$ coming from γ coincides with the isomorphism coming from the unit constraint $\mu(u, -) \cong \text{id}_{\mathcal{A}}$.
 - The equality $\kappa(\theta \otimes \text{id}_{\mathcal{A}}) = \kappa(\text{id}_{\mathcal{A}} \otimes \theta)$ holds.

Self-dual balanced braided algebras in \mathcal{S} are equivalent to cyclic framed E_2 -algebras in \mathcal{S} [MW23b]. It is generally not true that such a structure already extends to a modular functor. Rather, a cyclic framed E_2 -algebra uniquely extends to an *ansular functor* [MW23a], a consistent system of representations of mapping class groups of handlebodies (throughout assumed to be compact and oriented) with parametrized disks on their boundaries. The main result of this paper gives, roughly speaking, a necessary and sufficient condition for a given framed cyclic E_2 -algebra to extend to a modular functor, and shows that it does so in an essentially unique way.

The main ingredient for understanding this necessary and sufficient condition is *factorization homology* [Lur, AF15], a type of homology theory with origins in [BD04] that allows us to integrate E_n -algebras in a higher symmetric monoidal category \mathcal{S} over n -dimensional manifolds. We will throughout be interested in oriented manifolds. Therefore, we will need *framed E_n -algebras* as coefficients (the terminology is very unfortunate here). For factorization homology to be well-defined, the symmetric monoidal category \mathcal{S} , in which our framed E_n -algebras take their values, needs to satisfy some rather mild technical assumptions given in Section 2.1, and we restrict throughout to those \mathcal{S} meeting the requirements (these conditions will be satisfied in the cases of interest such as $\mathcal{S} = \text{Rex}$). Let us briefly mention the philosophy, formulated for the two-dimensional situation that is relevant for us: Factorization homology with coefficients in a framed E_2 -algebra \mathcal{A} in \mathcal{S} evaluated on a surface Σ is an object in \mathcal{S} denoted by $\int_{\Sigma} \mathcal{A}$. The functor $\Sigma \mapsto \int_{\Sigma} \mathcal{A}$ is characterized through a local-to-global principle, i.e. an analog of the Eilenberg-Steenrod axioms: If Σ is a disk, then $\int_{\mathbb{D}^2} \mathcal{A}$ is just \mathcal{A} . For a more complicated surface, $\int_{\Sigma} \mathcal{A}$ can be computed via *excision*. When \mathcal{S} is a

suitable symmetric monoidal $(2, 1)$ -category of linear categories, $\int_{\Sigma} \mathcal{A}$ may, in physical terms, be interpreted as a refinement of the category of modules over the algebra of observables of Σ [CG16]. We give a recollection of factorization homology in Section 2.1.

Suppose that \mathcal{A} is a self-dual balanced braided algebra according to Definition 2.1. The first bullet point of the definition tells us exactly that \mathcal{A} is a framed E_2 -algebra in \mathcal{S} by the description of non-cyclic framed E_2 -algebras in [Wah01, SW03]. For this reason, we may consider factorization homology $\int_{\Sigma} \mathcal{A}$ for a compact oriented surface Σ with n boundary components. We observe that the universal property of factorization homology implies that for any three-dimensional handlebody H with boundary Σ the ansular functor of \mathcal{A} gives rise to a 1-morphism

$$\Phi_{\mathcal{A}}(H) : \int_{\Sigma} \mathcal{A} \longrightarrow \mathcal{A}^{\otimes n} \quad (1.1)$$

that can be interpreted as a *generalized skein module for H* , see Section 1.7 of this introduction and Section 4.3 for this viewpoint. A crucial property of factorization homology is that it is *canonically pointed*: There exists a canonical map

$$\mathcal{O}_{\Sigma} : I \longrightarrow \int_{\Sigma} \mathcal{A}$$

induced by the embedding of the empty manifold into Σ , and we show that the value on H of the ansular functor associated with \mathcal{A} is given by the composition

$$I \xrightarrow{\mathcal{O}_{\Sigma}} \int_{\Sigma} \mathcal{A} \xrightarrow{\Phi_{\mathcal{A}}(H)} \mathcal{A}^{\otimes n}.$$

The construction of the map (1.1) is given in Section 4. In fact, a more refined statement can be made: It is a standard fact that the object $\int_{\partial\Sigma \times [0,1]} \mathcal{A}$ is an algebra in \mathcal{S} that acts on both $\int_{\Sigma} \mathcal{A}$ and $\mathcal{A}^{\otimes n}$. With respect to these module structures, $\Phi_{\mathcal{A}}(H)$ comes canonically with the structure of a module map (Proposition 4.3). We may now define $\Omega_{\mathcal{A}}(\Sigma)$ as the replete full subgroupoid (i.e. full and closed under isomorphisms) of the category of module maps $\int_{\Sigma} \mathcal{A} \longrightarrow \mathcal{A}^{\otimes n}$ spanned by the maps $\Phi_{\mathcal{A}}(H)$ with H running over all handlebodies with boundary Σ . We call a cyclic framed E_2 -algebra \mathcal{A} or, equivalently, a self-dual balanced braided algebra in \mathcal{S} *connected* if the groupoid $\Omega_{\mathcal{A}}(\Sigma)$ is connected for any surface Σ (Definition 5.5).

Although this connectedness condition is fairly abstract, we prove in Proposition 5.4 that it can be verified in a concrete way: Let H, H' be two handlebodies with boundary \mathbb{T}_1^2 (the torus with one boundary) whose mapping class groups generate $\text{Map}(\mathbb{T}_1^2)$, i.e. the braid group B_3 on three strands. Then \mathcal{A} is connected if and only if $\Phi_{\mathcal{A}}(H)$ and $\Phi_{\mathcal{A}}(H')$ are isomorphic as \mathcal{A} -module maps. This reduction to genus one is of course a familiar feature of the examples of modular functors constructed from modular categories. There, roughly speaking, one has a natural candidate for such an isomorphism, and the category at hand is modular if and only if this map is indeed invertible. Note however that in those examples, as explained below, this actually follows from the fact that modular categories are in particular cofactorizable, which turns out to be a *genus zero* condition. This seems to be a particular feature that makes modular categories remarkable.

We are now in a position to state our main result, namely the classification of modular functors. Instead of just classifying them up to equivalence, we will in fact describe the *moduli space* $\mathfrak{M}\mathfrak{F}$ of modular functors whose detailed construction is given in Definition 3.10.

Theorem 6.8 (*Classification of modular functors*). *The moduli space $\mathfrak{M}\mathfrak{F}$ of modular functors with values in the symmetric monoidal $(2, 1)$ -category \mathcal{S} is equivalent to the 2-groupoid of connected self-dual balanced braided algebras \mathcal{A} in \mathcal{S} . This equivalence is afforded by restricting modular functors to surfaces of genus zero.*

For a given connected self-dual balanced braided algebra \mathcal{A} , the modular functor corresponding to \mathcal{A} under the above equivalence is denoted by $\mathfrak{F}_{\mathcal{A}}$ and can be explicitly described, see Section 6.1. In Section 8, we focus on the linear case: We give a formula for all spaces of conformal blocks (Corollary 8.1) and conclude that they are always finite-dimensional provided that \mathcal{A} is a finite linear category in the sense of Etingof-Ostrik [EO04].

Having stated the main result, let now us now expand on further results, that are either needed for the proof of the main result or arise as a consequence, and elaborate on the relation to existing modular functor constructions.

1.2 Essential uniqueness of extensions to a modular functor. An intermediate result of independent interest appearing in the proof of the main result is the following uniqueness result for extensions to modular functors:

Theorem 6.6 (*Weak uniqueness of extensions*). *For any modular functor defined in genus zero, i.e. a cyclic framed E_2 -algebra \mathcal{A} in \mathcal{S} , the space of extensions of \mathcal{A} to a modular functor is empty or contractible. In other words, if there is an extension of \mathcal{A} to a modular functor, this extension is unique up to a contractible choice.*

This uniqueness result can be seen as a generalization of a similar result of Andersen-Ueno [AU12] who consider modular functors that are built from finitely semisimple categories equipped with an involution. As a small caveat, we should note that even if we restrict to a symmetric monoidal $(2, 1)$ -category of finitely semisimple linear categories, our definition of a modular functor substantially differs from the one used in [AU12]. Instead, our notion is closer to the one of Tillmann [Til98] and contains as a special case the one of Turaev [Tur94] and Bakalov-Kirillov [BK01].

1.3 Projective representations and equivalences. The construction of the moduli space $\mathfrak{M}\mathfrak{F}$ of modular functors provides in particular a convenient definition of equivalences of modular functors. By definition a modular functor leads to a coherent system of representations of central extensions of mapping class groups of surfaces. We allow arbitrary central extensions, and define $\mathfrak{M}\mathfrak{F}$ in such a way that if, roughly speaking, two modular functors produce systems of representations of possibly different central extensions such that one factors coherently through the other, we regard them to be equivalent. General equivalences are given by zigzags of those kind of maps.

For a fixed connected cyclic framed E_2 -algebra \mathcal{A} , the particular extension $\text{Surf}_{\mathcal{A}}$ of Surf that enters the construction of the modular functor $\mathfrak{F}_{\mathcal{A}}$ associated to \mathcal{A} plays a distinguished role: For any other extension \mathcal{Q} , the modular \mathcal{Q} -algebra structures on \mathcal{A} are in one-to-one correspondence with genuine maps of extensions $\mathcal{Q} \rightarrow \text{Surf}_{\mathcal{A}}$ (Theorem 6.4). Beware that this does not mean that $\text{Surf}_{\mathcal{A}}$ is somehow minimal: For example, if \mathcal{A} can actually be given the structure of a Surf -algebra, leading to genuine representations of mapping class groups, it does not mean that $\text{Surf}_{\mathcal{A}} = \text{Surf}$, but rather that there is a map of extensions $\text{Surf} \rightarrow \text{Surf}_{\mathcal{A}}$, i.e. a section. Different sections could a priori lead to non-equivalent Surf -structures on \mathcal{A} which are still identified in $\mathfrak{M}\mathfrak{F}$. In other words, even if the projective representations at hand can be lifted to genuine representations, the correct notion of equivalence is still that of projective equivalence.

The origin of the projectiveness of the mapping class group representations is perhaps best illustrated in the special case where \mathcal{A} is a modular category and Σ is closed: For a handlebody H with boundary Σ , the connectedness condition guarantees that the functors $\Phi_{\mathcal{A}}(H) : \int_{\Sigma} \mathcal{A} \rightarrow \text{vect}$ for different choices of H are isomorphic, but non-canonically. In that case, we also have

$$\text{Aut}(\Phi_{\mathcal{A}}(H)) \cong \text{Aut}(\text{id}_{\text{vect}}) \cong \mathbf{k}^{\times} .$$

There is a natural action of the mapping class group on $\int_{\Sigma} \mathcal{A}$ which happens to be (again, non-canonically) trivial in that particular situation, so that it can be transported to a trivial action of $\text{Map}(\Sigma)$ on vect . In the categorical setting, the action being trivial means that each group element acts by the identity functor and that the composition is controlled by a \mathbf{k}^{\times} -valued 2-cocycle η on $\text{Map}(\Sigma)$. Having fixed H , changing the trivialization of the $\text{Map}(\Sigma)$ -action leads to an equivalent trivial action, i.e. one defined by a cocycle in the same cohomology class as η . The vector space $\Phi_{\mathcal{A}}(H)\mathcal{O}_{\Sigma}$ then carries a canonical structure of a homotopy fixed point for this action. This is well-known to be the same as a representation of the central extension determined by η on that space, thereby producing a projective representation of $\text{Map}(\Sigma)$ on $\Phi_{\mathcal{A}}(H)\mathcal{O}_{\Sigma}$. Those are precisely the representations that the modular functors $\mathfrak{F}_{\mathcal{A}}$ produces in that special case.

1.4 Cofactorizability as a sufficient condition. The condition for a self-dual balanced braided algebra to be connected is relatively transparent from a topological perspective, but somewhat abstract from an algebraic perspective. Therefore, a large part of the article is devoted to providing instead sufficient conditions that can be verified more easily in concrete situations. Most importantly, we prove that *cofactorizability* in the sense of [BJSS21] is a sufficient condition that can be verified in genus zero:

Theorem 7.6. *For any cofactorizable cyclic framed E_2 -algebra, there is an essentially unique extension to a modular functor. More precisely, we have an embedding of 2-groupoids*

$$\{\text{cofactorizable self-dual balanced braided algebras in } \mathcal{S}\} \hookrightarrow \mathfrak{M}\mathfrak{F} .$$

For general \mathcal{S} , this embedding is not essentially surjective (Remark 7.7).

1.5 Recovering the Lyubashenko construction. We have the following relevant special case of Theorem 7.6: If $\mathcal{S} = \text{Rex}$ and if the self-dual balanced braided algebra \mathcal{A} is actually a finite ribbon category (not every self-dual balanced braided algebra in Rex is of this form, see the next subsection and [MW23a]), then

cofactorizability is equivalent to the non-degeneracy of the braiding of \mathcal{A} , which means that \mathcal{A} is a modular category (this definition of modularity does *not* include semisimplicity). In this case, the modular functor afforded by Theorem 6.8 agrees with Lyubashenko’s modular functor [Lyu95a] (Corollary 8.3). This provides a purely topological construction of this well-known class of modular functors via factorization homology. The new insight about this construction is its universality: For any modular category, Lyubashenko’s construction yields, up to a contractible choice, the *unique* extension to a modular functor. This is a conceptually appealing description of Lyubashenko’s construction that, to the best of our knowledge, is new. As a concrete consequence, this implies that the ribbon automorphisms of a modular category (as 2-group) act on the associated modular functor. The comparison to Lyubashenko’s construction is fairly abstract, but it can be made explicit for ribbon categories using the results of [BZBJ18a], thereby recovering Lyubashenko’s mapping class group representations [Lyu95a] (including the notorious S -transformation) and also their description in [DRGG⁺22b] in a direct way that does not appeal to the uniqueness result. We plan to address this in a forthcoming paper.

1.6 Modular functors that do not come from modular categories. By combining our results with those of Allen-Lentner-Schweigert-Wood [ALSW21], we give in Corollary 8.5 a criterion for a vertex operator algebra V to produce a modular functor. This comes with a prescription to obtain spaces of conformal blocks for V . These spaces of conformal blocks are *universal*: They are up to equivalence the only ones extending the ‘obvious’ hom space genus zero blocks to a modular functor. This construction produces modular functors that do not come from modular categories, for example the Feigin-Fuchs boson (Example 8.8). As another class of not necessarily modular categories that yield modular functors, we discuss Drinfeld centers of (not necessarily spherical) pivotal finite tensor categories (Corollary 8.9) building on the algebraic results of [MW22].

1.7 Relation to skein theory. Our main result is largely inspired by and specializes to the skein-theoretic approach to modular functors [Rob94, MR95]. To illustrate this, let \mathcal{A} be the modular fusion category obtained by taking the semisimplification of the representation category of quantum SL_2 at a root of unity (with a non-standard ribbon structure [Tin17]). Let Σ be a closed surface and H a handlebody with $\partial H = \Sigma$. In that case:

- The vector space $\Phi_{\mathcal{A}}(H)\mathcal{O}_{\Sigma}$ is identified with the \mathcal{A} -skein module $\mathrm{Sk}_{\mathcal{A}}(H)$ of H , a quotient of the famous Kauffman bracket skein module of H evaluated at that root of unity.
- The category $\int_{\Sigma} \mathcal{A}$ is identified with the category of modules over $\mathrm{End}_{\int_{\Sigma} \mathcal{A}}(\mathcal{O}_{\Sigma})$, which is in turn identified with the skein algebra $\mathrm{SkAlg}_{\mathcal{A}}(\Sigma)$ of Σ by [Coo23].
- Since $\Phi_{\mathcal{A}}(H)$ is an equivalence in that case (a result that will be proven in subsequent work; we do not logically depend on it in this article and just mention it for illustration), this recovers the fact proven in [Rob94, MR95] that there is an algebra isomorphism

$$\mathrm{SkAlg}_{\mathcal{A}}(\Sigma) \cong \mathrm{End}(\mathrm{Sk}_{\mathcal{A}}(H)) .$$

- In other words, $\Phi_{\mathcal{A}}(H)$ factors as

$$\int_{\Sigma} \mathcal{A} \simeq \mathrm{End}(\mathrm{Sk}_{\mathcal{A}}(H))\text{-mod} \simeq \mathrm{vect}$$

where the second equivalence is the standard Morita triviality of matrix algebras. Hence, the whole functor $\Phi_{\mathcal{A}}(H)$ can be identified with the skein module of H , this time seen as a $\mathbf{k}\text{-SkAlg}_{\mathcal{A}}(\Sigma)$ bimodule.

- The statement that \mathcal{A} is connected in that case translates into the fact that for any other handlebody H' with $\partial H' = \Sigma$, the skein modules of H and H' are isomorphic not just as vector spaces but as modules over $\mathrm{SkAlg}_{\mathcal{A}}(\Sigma)$.
- The mapping class group action on $\int_{\Sigma} \mathcal{A}$ is induced by the natural action of that group on the skein algebra of Σ . The fact that this categorical action is trivial means that the action on the skein algebra is by inner automorphisms. Of course, since the category at hand is equivalent to vect in that case, the triviality of the action is automatic; this is essentially a reformulation of the Skolem-Noether theorem.

The argument that in the general case of connected cyclic framed E_2 -algebras produces the projective mapping class group representations reduces now in this special case to the argument in [Rob94].

1.8 Relation to higher-dimensional fully extended topological field theories. In some special cases, the structures and conditions featured in this paper have a natural interpretation in the framework of fully extended topological field theories. Let us emphasize that this interpretation relies on various statements which are, as of yet, still conjectural. Our main results thus establish in an independent way (in particular without relying on the cobordism hypothesis) parts of the field theories that are expected to be associated with \mathcal{A} .

Let us explain in more detail the necessary background: By [JFS17, Hau17] E_2 -algebras in a symmetric monoidal $(2,1)$ -category \mathcal{S} form naturally the objects of a certain ‘Morita’ 4-category in which they are automatically 2-dualizable [GS18]. The Baez-Dolan-Lurie cobordism hypothesis [BD95, Lur09b] then implies that any E_2 -algebra \mathcal{A} induces a framed two-dimensional topological field theory valued in that Morita 4-category. If \mathcal{A} is a framed E_2 -algebra, this produces an oriented theory and its value on surfaces agrees with factorization homology with coefficients in \mathcal{A} . If one assumes that \mathcal{A} is in fact 3-dualizable and equipped with the structure of a homotopy $\mathrm{SO}(3)$ -fixed point compatible with its framed structure, this extends to an oriented three-dimensional topological field theory. In particular, the evaluation on a handlebody H with boundary Σ , seen as a three-dimensional bordism from Σ to a disjoint union of n disks yields a map

$$\int_{\Sigma} \mathcal{A} \longrightarrow \mathcal{A}^{\otimes n} . \quad (1.2)$$

We expect that (1.2) agrees with the map $\Phi_{\mathcal{A}}(H)$ in that case. There is then a canonical choice of a certain boundary conditions leading to the construction of a relative topological field theory in the sense of [FT14, JFS17] which roughly speaking attaches an object, rather than a morphism, of \mathcal{S} to a three-dimensional manifold with closed boundary. We expect that the data for this relative theory to itself be oriented includes a cyclic structure on \mathcal{A} , and that the value of such a relative theory on handlebodies should then recover the ansular functor associated with \mathcal{A} .

If \mathcal{A} enjoys the much stronger property of being invertible in the Morita 4-category of E_2 -algebras, then it induces an invertible four-dimensional topological field theory. Invertibility implies that the value of that theory on a handlebody H depends on H only up to four-dimensional bordisms. The relative theory should then produce an anomalous, partially defined, oriented three-dimensional topological field theory. Such a structure induces, by restriction to dimension two, a modular functor, whose value on a surface is given by evaluating the relative theory on any handlebody H for that surface. The anomaly in that case reflects the fact that the theory still depends, albeit very weakly, on the choice of H , which recovers the well-known topological description of the central extensions of mapping class groups in that particular setting [MR95]. In summary, it is widely expected from this perspective that cyclic, invertible framed E_2 -algebras should lead to examples of modular functors. Our results thus provide a direct proof that this is indeed the case, under the generally weaker condition of cofactorizability.

We note that Lyubashenko’s modular functor is expected to fit into this general picture as follows: If \mathcal{A} is a ribbon category with enough projective objects, then the main result of [BJS21] states that \mathcal{A} is 3-dualizable in the Morita category of braided tensor categories. On the one hand, it is expected that the ribbon structure on \mathcal{A} canonically extends to a homotopy $\mathrm{SO}(3)$ -fixed point structure. On the other hand, \mathcal{A} turns out to be canonically cyclic in that case [MW23b]. If \mathcal{A} is finite and non-degenerate (i.e. modular), it is shown in [BJSS21] that it is in fact invertible, hence in particular cofactorizable (in the finite ribbon case, cofactorizability is sufficient for invertibility, although we do not expect this to be the case in general). Finally, if \mathcal{A} is additionally semisimple, the corresponding anomalous field theory is expected [Fre12a, Fre12b, Wal] to extend to dimension three and to recover the Reshetikhin-Turaev theory and its underlying modular functor.

ACKNOWLEDGMENTS. We are grateful to David Jordan, Gwénaél Massuyeau, Lukas Müller, Pavel Safronov, Christoph Schweigert, Noah Snyder, Nathalie Wahl, Simon Wood and Yang Yang for helpful discussions related to this project. LW gratefully acknowledges support by the Danish National Research Foundation through the Copenhagen Centre for Geometry and Topology (DNRF151) at Københavns Universitet during the period in which this project was initiated. Moreover, LW gratefully acknowledges support by the ANR project CPJ n°ANR-22-CPJ1-0001-01 at the Institut de Mathématiques de Bourgogne (IMB). The IMB receives support from the EIPHI Graduate School (ANR-17-EURE-0002).

2 PRELIMINARIES

2.1 Factorization homology of framed E_2 -algebras. In this subsection, we recall the framework of *factorization homology for surfaces*. References include [Lur, Chapter 5.5] and [AF15] for factorization homology in general and [BZBJ18a, BZBJ18b] for category-valued two-dimensional case. It allows us to integrate a framed little 2-disk algebra in a suitable symmetric monoidal $(2,1)$ -category over a surface.

In order to supply a few more details, let us fix some conventions and notation for the symmetric monoidal $(2, 1)$ -category \mathcal{S} that factorization homology and all operadic algebras will take their values in (unless otherwise stated):

For the rest of the article, \mathcal{S} will be a bicomplete symmetric monoidal $(2, 1)$ -category with monoidal product \otimes such that for $M \in \mathcal{S}$ the functor $M \otimes -$ commutes with colimits.

An excellent reference for symmetric monoidal $(2, 1)$ -categories is [SP09, Chapter 2]. However, having in mind the case $\mathcal{S} = \text{Rex}$, the symmetric monoidal $(2, 1)$ -category of finitely cocomplete linear categories, cocomplete (‘right exact’) functors and natural isomorphisms, that we will define in Section 2.4, is fully sufficient.

We denote by E_2 the *little disk operad*, a topological operad whose space of n -ary operations is given by the space of embeddings of n disks into one disk that are obtained as a combination of rescalings and translations. If additionally one allows rotations, one obtains the *framed little disk operad* $\mathfrak{f}E_2$. We refer to [Fre17, Section 5] for a textbook reference on the E_2 -operad and additionally to [SW03] for the framed case. Both E_2 and $\mathfrak{f}E_2$ are aspherical, i.e. their homotopy groups π_ℓ for $\ell \geq 2$ are trivial. Hence, they can be seen as groupoid-valued operads. In fact, in arity n , the spaces $E_2(n)$ and $\mathfrak{f}E_2(n)$ are equivalent to the classifying spaces of the pure braid group on n strands and the framed pure braid group on n strands, respectively. Since \mathcal{S} , by virtue of being a $(2, 1)$ -category, is enriched over Grpd , we can consider E_2 -algebras and $\mathfrak{f}E_2$ -algebras in \mathcal{S} . An algebra in \mathcal{S} over a Grpd -valued operad \mathcal{O} is an object \mathcal{A} in \mathcal{S} together with maps $\mathcal{O}(n) \rightarrow \mathcal{S}(\mathcal{A}^{\otimes n}, \mathcal{A})$ that respect equivariance under the symmetric group actions and operadic composition up to coherent isomorphism; we will explain how to formally define cyclic and modular operadic algebras in Section 2.2.

Factorization homology attaches to a given $\mathfrak{f}E_2$ -algebra \mathcal{A} in \mathcal{S} and a surface Σ an object $\int_\Sigma \mathcal{A} \in \mathcal{S}$. This construction is functorial with respect to oriented embeddings. For us, ‘surface’ means throughout compact oriented two-dimensional manifold with parametrized, possibly empty boundary. A priori, factorization homology with coefficients in a framed E_2 -algebra can only be evaluated on an oriented two-dimensional manifold without boundary. For the extension to the case with boundary, it is understood throughout that we extend the framed E_2 -algebra in a trivial way to a framed Swiss-Cheese algebra, see [BZBJ18a, Remark 2.2] for this convention.

Roughly speaking, factorization homology is defined on disks and their embeddings into each other using the $\mathfrak{f}E_2$ -structure of \mathcal{A} . Then one performs a homotopy left Kan extension along the inclusion of the symmetric monoidal category of disks into the symmetric monoidal category of all surfaces. This entails that $\int_\Sigma \mathcal{A}$ can be described as a homotopy colimit (or simply put, the ‘correct’ $(2, 1)$ -categorical colimit)

$$\int_\Sigma \mathcal{A} = \text{hocolim}_{\substack{\varphi: (\mathbb{D}^2)^{\sqcup n} \rightarrow \Sigma \\ n \geq 0}} \mathcal{A}^{\otimes n} \quad (2.1)$$

over all oriented embeddings of n disks into Σ with n running over all non-negative integers. As usual, $\mathcal{A}^{\otimes 0} = I$.

The object $\int_\Sigma \mathcal{A} \in \mathcal{S}$ is characterized by the fact that $\int_{\mathbb{D}^2} \mathcal{A} \simeq \mathcal{A}$ and by a locality property called *excision*: Suppose that a surface Σ is obtained by gluing surfaces Σ_0 and Σ_1 together along a boundary component, then $\int_{\Sigma_0} \mathcal{A}$ and $\int_{\Sigma_1} \mathcal{A}$ is a left and right module, respectively, over the algebra $\int_{\mathbb{S}^1 \times [0, 1]} \mathcal{A}$, and the embedding $\Sigma_0 \sqcup \Sigma_1 \rightarrow \Sigma$ obtained by choosing a collar of the gluing boundary component induces an equivalence $\int_{\Sigma_0} \mathcal{A} \otimes_{\int_{\mathbb{S}^1 \times [0, 1]} \mathcal{A}} \int_{\Sigma_1} \mathcal{A} \xrightarrow{\simeq} \int_\Sigma \mathcal{A}$ from the relative monoidal product over $\int_{\mathbb{S}^1 \times [0, 1]} \mathcal{A}$ to $\int_\Sigma \mathcal{A}$; we refer to [AF15, Section 3.3] for details.

For any surface Σ , the diffeomorphisms $\Sigma \rightarrow \Sigma$ preserving the orientation and boundary parametrization form a topological group $\text{Diff}(\Sigma)$. The group of path components $\text{Map}(\Sigma) := \pi_0(\text{Diff}(\Sigma))$ is the mapping class group of the surface Σ . If we mention in the sequel diffeomorphisms or mapping classes, it will always be implicit that we are only interested in those that preserve the orientation and boundary parametrization. By construction $\int_\Sigma \mathcal{A}$ comes with a homotopy coherent $\text{Diff}(\Sigma)$ -action.

For a surface Σ , the embedding $\emptyset \rightarrow \Sigma$ yields a map $\mathcal{O}_\Sigma : I \rightarrow \int_\Sigma \mathcal{A}$ from the unit I of \mathcal{S} to $\int_\Sigma \mathcal{A}$. This uses the functoriality of factorization homology with respect to embeddings and the canonical equivalence $\int_\emptyset \mathcal{A} \simeq I$. Hence, $\int_\Sigma \mathcal{A}$ comes with a generalized object that we refer to as the *distinguished object* of $\int_\Sigma \mathcal{A}$. In [BZBJ18a, Section 5.1] it is also referred to as *quantum structure sheaf*. Since the diffeomorphism group of Σ preserves tautologically the embedding $\emptyset \rightarrow \Sigma$, the distinguished object \mathcal{O}_Σ canonically comes with the structure of a homotopy $\text{Diff}(\Sigma)$ -fixed point.

2.2 The modular extension of cyclic framed little disks algebras. The operad $\mathfrak{f}E_2$ is a *cyclic* operad in the sense of Getzler and Kapranov [GK95], i.e. the spaces of operations come with a prescription

to cyclically permute the inputs with the output. Below we will make ample use of this cyclic structure. In order to handle cyclic operads, but also modular operads [GK98] efficiently, we will use the description given in [Cos04] based on certain categories of graphs: First recall that a graph consists of a set of half edges and a set of vertices (for us always finite) plus a map from half edges to vertices telling us to which vertex a half edge is attached and an involution on the set of half edges telling us how the half edges are glued together. A *corolla* is a contractible graph with one vertex and a finite number of legs attached to it. We denote by $\text{Legs}(T)$ the set of legs of a corolla T . Unless otherwise stated, $\text{Legs}(T)$ can be empty. Now we denote by \mathbf{Graphs} the category whose objects are finite disjoint unions of corollas. A morphism $T \rightarrow T'$ between two disjoint unions of corollas is an equivalence class of graphs Γ together with

- an identification φ_1 of T with

$$\nu(\Gamma) := \text{the disjoint union of corollas obtained from } \Gamma \text{ by cutting } \Gamma \text{ at all internal edges}$$

(an identification of graphs is here a bijection between the sets of half edges and vertices compatible in the obvious way with the gluing information),

- and an identification of T' with

$$\pi_0(\Gamma) := \text{the disjoint union of corollas obtained from } \Gamma \text{ by contracting all internal edges.}$$

We illustrate the definition of $\nu(\Gamma)$ and $\pi_0(\Gamma)$ in Figure 1. Such triples $(\Gamma, \varphi_1, \varphi_2)$ and $(\Gamma', \varphi'_1, \varphi'_2)$ are defined to be equivalent if there is an identification $\Gamma \cong \Gamma'$ of graphs compatible in the obvious way with $\varphi_1, \varphi_2, \varphi'_1$ and φ'_2 .

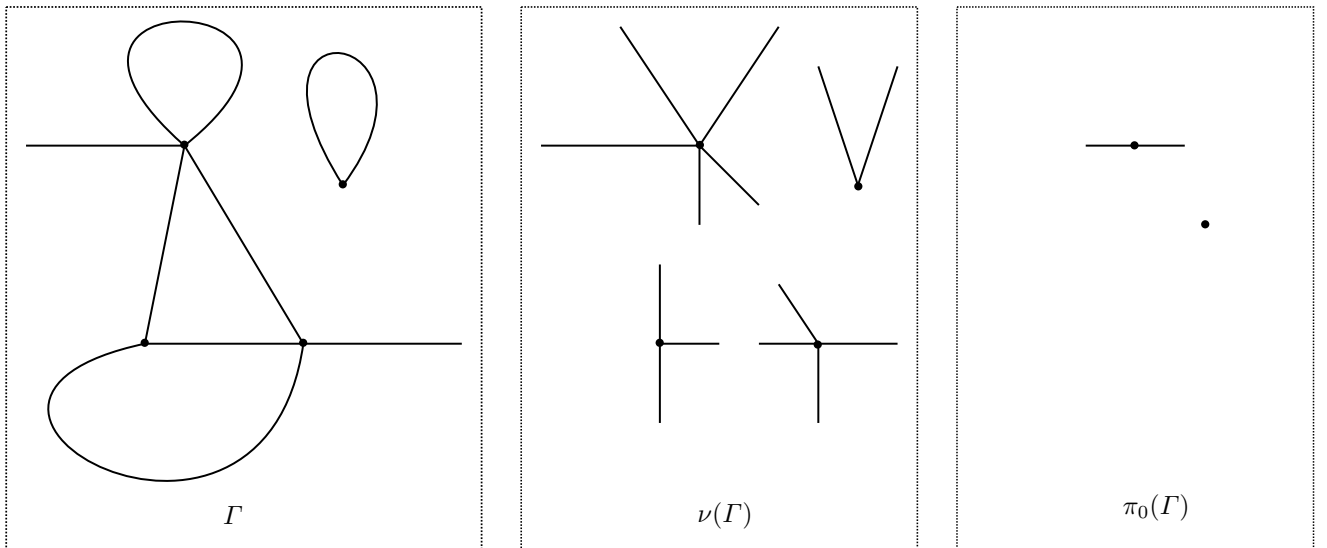


Figure 1: On the definition of $\nu(\Gamma)$ and $\pi_0(\Gamma)$.

In order to form the composition $\Gamma_2 \circ \Gamma_1$, one replaces the vertices of Γ_2 with the graph Γ_1 , we refer to [MW23b, Section 2.1] for more details and explaining pictures. Disjoint union endows \mathbf{Graphs} with a symmetric monoidal structure. We denote by $\mathbf{Forests} \subset \mathbf{Graphs}$ the symmetric monoidal subcategory of \mathbf{Graphs} whose objects are finite disjoint unions of corollas (hence, it has the same objects as \mathbf{Graphs}) and whose morphisms are forests, i.e. disjoint unions of trees (a tree is a contractible graph).

A *cyclic operad* in a symmetric monoidal $(2, 1)$ -category \mathcal{S} (the requirements on \mathcal{S} encountered in the last section on factorization homology are not needed here) can now be defined as a symmetric monoidal functor $\mathcal{O} : \mathbf{Forests} \rightarrow \mathcal{S}$, where the 1-category $\mathbf{Forests}$ is seen as a symmetric monoidal $(2, 1)$ -category without non-identity 2-morphisms and where ‘symmetric monoidal functor’ is understood up to coherent isomorphism [SP09, Chapter 2]; a summary of what that means is given in [MW23b, Section 2.1]. A *modular operad* in \mathcal{S} is a symmetric monoidal functor $\mathcal{O} : \mathbf{Graphs} \rightarrow \mathcal{S}$. (Ordinary non-cyclic operads can be described as symmetric monoidal functors out of $\mathbf{RForests}$, the version of $\mathbf{Forests}$ where the graphs come equipped with a distinguished leg, the so-called *root*.) We have to highlight a minor technical point: Cyclic and modular operads, when defined this way using graph categories, do a priori not have an operadic identity, i.e. a distinguished unary operation that behaves neutrally with respect to composition — of course up to coherent isomorphism. Operadic identities can be included via [MW23b, Definition 2.5]. In this article, the term

‘operad’ (and its cyclic and modular variants) will by default always include operadic identities, and all maps between operads preserve these operadic identities up to coherent isomorphism.

An important example for us is the *modular handlebody operad* $\mathbf{Hbdy} : \mathbf{Graphs} \rightarrow \mathbf{Cat}$. To a corolla T , it assigns the groupoid $\mathbf{Hbdy}(T)$ whose objects are compact oriented three-dimensional handlebodies (hereafter just referred to as handlebodies) that are connected and come with $|\mathbf{Legs}(T)|$ many disks embedded in the boundary. The embedded disks are parametrized, i.e. they come with an identification of the boundary disks of H with $(\mathbb{D}^2)^{\sqcup \mathbf{Legs}(T)}$. Morphisms are isotopy classes of orientation-preserving diffeomorphisms of handlebodies respecting the disk parametrizations (we will just call these ‘mapping classes’). If T has n legs, then in short

$$\mathbf{Hbdy}(T) = \left\{ \begin{array}{l} \text{groupoid of connected handlebodies} \\ \text{with } n \text{ parametrized disks embedded in the boundary} \\ \text{and mapping classes as morphisms} \end{array} \right\}.$$

The assignment is extended monoidally, i.e. $\mathbf{Hbdy}(T \sqcup T') = \mathbf{Hbdy}(T) \times \mathbf{Hbdy}(T')$ for corollas T and T' . We see a pair $(H, H') \in \mathbf{Hbdy}(T \sqcup T') = \mathbf{Hbdy}(T) \times \mathbf{Hbdy}(T')$ as the handlebody $H \sqcup H'$. In other words, non-connected handlebodies are allowed in the operad \mathbf{Hbdy} , but they are associated to an object in \mathbf{Graphs} that is not a corolla. Operadic composition is by gluing. The cyclic action on the parametrizations of embedded disks gives us the cyclic structure; we refer to [Gia11, Section 4.3] for details.

If T has n legs, we might be tempted to denote $\mathbf{Hbdy}(T)$ as $\mathbf{Hbdy}(n)$ with n as the total arity (or, alternatively, $\mathbf{Hbdy}(n-1)$ since we have $n-1$ inputs and one output), but seeing the arity really as a (disjoint union of) corollas has the advantage that no ordering of the legs is implied through the numbering.

If we restrict to genus zero handlebodies, we obtain a cyclic operad that we can identify with the framed E_2 -operad $\mathbf{f}E_2$. In other words, $\mathbf{f}E_2$ is the cyclic operad of genus zero handlebodies or, equivalently, genus zero surfaces.

By replacing in the definition of \mathbf{Hbdy} handlebodies with surfaces (again, let us recall that for us these are always compact and oriented), we obtain the *modular operad of surfaces*, also called the *modular surface operad*. Again, if T has n legs, this definition amounts to

$$\mathbf{Surf}(T) = \left\{ \begin{array}{l} \text{groupoid of connected surfaces} \\ \text{with } n \text{ parametrized boundary components} \\ \text{and mapping classes as morphisms} \end{array} \right\}.$$

For $\Sigma \in \mathbf{Surf}(T)$, $\mathbf{Map}(\Sigma) = \pi_1(\mathbf{Surf}(T), \Sigma)$ is the mapping class group of Σ . Since it consists of *automorphisms* of Σ in $\mathbf{Surf}(T)$, the boundary parametrizations must be preserved. This version of the mapping class group is often referred to as the pure mapping class group. Nonetheless, ‘non-pure’ mapping classes of Σ are still contained in $\mathbf{Surf}(T)$, but as *isomorphisms* instead of *automorphisms*.

Given any cyclic or modular operad \mathcal{O} in \mathbf{Cat} , one can consider cyclic or modular algebras in any symmetric monoidal $(2, 1)$ -category \mathcal{S} . We will give here only a rather dense summary and refer for the details to [GK95] for the 1-categorical case and moreover to [MW23b, Section 2] for the bicategorical generalization that we are interested in: Suppose that X is a self-dual object in the symmetric monoidal $(2, 1)$ -category \mathcal{S} , i.e. X comes with a map $\kappa : X \otimes X \rightarrow I$, called *evaluation*, and a map $\Delta : I \rightarrow X \otimes X$, called *coevaluation*, such that the usual zigzag identities are fulfilled up to isomorphism (this exhibits X as its own dual in the homotopy category of \mathcal{S}). A map $\kappa : X \otimes X \rightarrow I$ that is part of a self-duality of X and that is symmetric up to coherent isomorphism is referred to as *non-degenerate symmetric pairing on X* . For any corolla T , we denote by $X^{\otimes \mathbf{Legs}(T)}$ the unordered monoidal product of X running over the set $\mathbf{Legs}(T)$ of legs of T and set $\mathbf{End}_{\kappa}^X(T) := \mathcal{S}(I, X^{\otimes \mathbf{Legs}(T)})$ for a corolla T (this is dual to the conventions in [MW23b, Section 2], but the difference is insignificant). The definition is extended monoidally to the non-connected case, i.e. $\mathbf{End}_{\kappa}^X(T) = \prod_{c \in \mathbf{C}(T)} \mathcal{S}(I, \mathcal{A}^{\otimes \mathbf{Legs}(c)})$ for a possibly non-connected object T in \mathbf{Graphs} with set $\mathbf{C}(T)$ of path components. This assignment extends to a symmetric monoidal functor $\mathbf{Graphs} \rightarrow \mathbf{Cat}$, i.e. to a modular operad, the *endomorphism operad of (X, κ)* . Its restriction to $\mathbf{Forests}$ is called the *cyclic endomorphism operad of (X, κ)* .

For a given \mathbf{Cat} -valued cyclic operad \mathcal{O} , the structure of a cyclic \mathcal{O} -algebra on $X \in \mathcal{S}$ is defined as the choice of a non-degenerate symmetric pairing $\kappa : X \otimes X \rightarrow I$ on X and a morphism $\mathcal{O} \rightarrow \mathbf{End}_{\kappa}^X$ of cyclic operads. Modular algebras are defined analogously.

For the cyclic framed E_2 -operad $\mathbf{f}E_2$, which can be regarded as being groupoid-valued, one can now consider cyclic $\mathbf{f}E_2$ -algebras in any symmetric monoidal $(2, 1)$ -category \mathcal{S} . Similarly, one can consider modular \mathbf{Hbdy} -algebras in \mathcal{S} . A modular \mathbf{Hbdy} -algebra is called *ansular functor* in [MW23a].

The results in [MW23a] building on [MW23b, MW23c] give an explicit description of cyclic framed E_2 -algebras and ansular functors in $(2, 1)$ -categories. In order to state these results, we need the following notion:

Definition 2.1 ([MW23b, Definition 5.4]). A *self-dual balanced braided algebra* in \mathcal{S} is an object $\mathcal{A} \in \mathcal{S}$ together with the following structure:

- \mathcal{A} is a *balanced braided algebra* in \mathcal{S} — in more detail:
 - \mathcal{A} has a multiplication $\mu : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ that is associative and unital up to coherent isomorphism (we denote the unit 1-morphism by $u : I \rightarrow \mathcal{A}$);
 - \mathcal{A} comes with an isomorphism $c : \mu \rightarrow \mu^{\text{op}} = \mu \circ \tau$ (where τ is the symmetric braiding of \mathcal{S}) called *braiding* which satisfies the hexagon relations;
 - \mathcal{A} comes with an isomorphism $\theta : \text{id}_{\mathcal{A}} \rightarrow \text{id}_{\mathcal{A}}$ referred to as *balancing* satisfying

$$\begin{aligned}\theta \circ \mu &= c^2 \circ \mu(\theta \otimes \theta) , \\ \theta \circ u &= \text{id}_u .\end{aligned}$$

- \mathcal{A} comes with a non-degenerate pairing $\kappa : \mathcal{A} \otimes \mathcal{A} \rightarrow I$ (a 1-morphism exhibiting \mathcal{A} as its own dual in the homotopy category of \mathcal{S}) together with an isomorphism $\gamma : \kappa(u \otimes \mu) \rightarrow \kappa$ subject to the following conditions:
 - The isomorphism $\kappa(u \otimes \mu(u, -)) \rightarrow \kappa(u, -)$ coming from γ coincides with the isomorphism coming from the unit constraint $\mu(u, -) \cong \text{id}_{\mathcal{A}}$.
 - The equality $\kappa(\theta \otimes \text{id}_{\mathcal{A}}) = \kappa(\text{id}_{\mathcal{A}} \otimes \theta)$ holds.

We give examples of self-dual balanced braided algebras in Subsection 2.4. Framed E_2 -algebras in a symmetric monoidal $(2, 1)$ -category are equivalent to balanced braided algebras [Wah01, SW03]. The following is an extension of this result characterizing *cyclic* algebras over that operad:

Theorem 2.2 ([MW23b, MW23a]). *The following structures are equivalent:*

- (i) *A cyclic framed E_2 -algebra in \mathcal{S} .*
- (ii) *A self-dual balanced braided algebra in \mathcal{S} .*
- (iii) *A modular Hbdy-algebra in \mathcal{S} , i.e. an \mathcal{S} -valued ansular functor.*

The algebraic structures described under each of these points form naturally a 2-groupoid, and the equivalence is an equivalence of 2-groupoids.

The equivalence between (i) and (ii) is the main result of [MW23b]. The equivalence between (i) and (iii) is established in [MW23a] and is, roughly, based on the following: Clearly, any modular Hbdy-algebra can be restricted to a cyclic fE_2 -algebra. The inverse is given by the *modular extension* for cyclic algebras [MW23a, Section 4] and uses results of Giansiracusa [Gia11] on the derived modular envelope of fE_2 (the notion of the modular envelope is due to Costello [Cos04]) and the strengthening of Giansiracusa’s result in [MW23c, MW23a]. We denote the modular extension of a framed E_2 -algebra \mathcal{A} , i.e. the unique extension to a Hbdy-algebra (an ansular functor) by $\widehat{\mathcal{A}}$. The evaluation of the modular extension on an arbitrary handlebody is explicitly described in [MW23a, Corollary 6.3] via a ‘coend formula’ that will appear in Corollary 8.1 below.

Remark 2.3. The assumptions on \mathcal{S} from Section 2.1 are not needed in Definition 2.1 and Theorem 2.2. Both are true for any symmetric monoidal $(2, 1)$ -category.

Remark 2.4. Even though we will, for the construction of modular functors in the context of this article, always be interested in mapping classes instead of diffeomorphisms, let us, for some applications in the proof of Theorem 5.2, mention that there is also the modular operad SURF of surfaces and *diffeomorphisms* (as always, it is implicit that surfaces are compact, oriented and with boundary parametrization, and that the diffeomorphisms preserve this). Note that SURF, as opposed to Surf, is not groupoid-valued, but a priori a topological modular operad, or equivalently a modular operad with values in ∞ -groupoids, i.e. a symmetric monoidal functor from Graphs to spaces or ∞ -groupoids up to coherent homotopy. The framework for this ∞ -categorical approach to operads is laid out in [Lur, Chapter 2], see also [HRY20]. The details of the ∞ -categorical approach will not concern us here because we are, at the end of the day, interested in the 1-categorical truncation $\text{Surf} = \text{IISURF}$, where II is the fundamental groupoid.

2.3 The boundary map. For $H \in \text{Hbdy}(T)$ with $T \in \text{Graphs}$, we can form the boundary surface ∂H of H which is defined as the boundary of H minus the interior of the parametrized disks embedded in its boundary. Then we obtain a functor $\partial : \text{Hbdy}(T) \rightarrow \text{Surf}(T)$, and in fact, a morphism of modular operads

$$\partial : \text{Hbdy} \rightarrow \text{Surf} , \tag{2.2}$$

the *boundary map*, see also [MW23b, Section 7.3].

In order to investigate (2.2), let us recall the following notion: We call a functor $F : \Gamma \rightarrow \Omega$ between groupoids a *fibration* if any lifting problem in simplicial sets

$$\begin{array}{ccc} 0 & \longrightarrow & B\Gamma \\ \downarrow & \nearrow \exists & \downarrow BF \\ \Delta^1 & \longrightarrow & B\Omega \end{array}$$

has a solution (here B denotes the nerve of a category). Functors between categories with this lifting property are also called *isofibrations*. For categories, there is the a priori stronger notion of a *Grothendieck fibration*, but for groupoids, both notions agree. Therefore, we will just call this type of functor a fibration. For fibrations, the strict fiber and the homotopy fiber are equivalent. We call a map of (modular) groupoid-valued operads a fibration if it has this property arity-wise.

Proposition 2.5. *The natural map $\partial : \mathbf{Hbdy} \rightarrow \mathbf{Surf}$ taking the boundary of handlebodies is a fibration of modular operads. The fiber and, equivalently, the homotopy fiber over some $\Sigma \in \mathbf{Surf}(T)$ is discrete up to equivalence in the sense that its automorphism groups are trivial. Its isomorphism classes can be identified with the set $\mathbf{Map}(\Sigma)/\mathbf{Map}(H)$ of left cosets of $\mathbf{Map}(H)$ in $\mathbf{Map}(\Sigma)$, where H is any handlebody with boundary Σ .*

Since the fibers of the fibration $\partial : \mathbf{Hbdy} \rightarrow \mathbf{Surf}$ are discrete, we should think of ∂ as a *covering* of modular operads.

In the sequel, we will not only use the lifting property, but also sometimes rather specific lifts. Before giving the full proof of Proposition 2.5, let us record these for later reference: For a diffeomorphism $f : \Sigma \rightarrow \Sigma'$ and a handlebody H with $\partial H = \Sigma$, we define the handlebody $f.H$ as the pushout

$$\begin{array}{ccc} \Sigma \times \{0\} \cong \Sigma = \partial H & \longrightarrow & H \\ f \times 0 \downarrow & & \downarrow \tilde{f} \\ \Sigma' \times [0, 1] & \longrightarrow & f.H \end{array}$$

similar constructions are often described as ‘gluing a mapping cylinder’ to H .

Lemma 2.6. *The diffeomorphism $\tilde{f} : H \rightarrow f.H$ satisfies $\partial \tilde{f} = f$, and the mapping class $\tilde{f} : H \rightarrow f.H$ solves the lifting problem*

$$\begin{array}{ccc} 0 & \xrightarrow{H} & B\mathbf{Hbdy}(T) \\ \downarrow & \nearrow \tilde{f} & \downarrow B\partial \\ \Delta^1 & \xrightarrow{f} & B\mathbf{Surf}(T) \end{array}$$

Proof. This follows immediately from the definition of $f.H$ and \tilde{f} . □

Remark 2.7. In Lemma 2.6 and similar situations below, we will use the same symbol for a diffeomorphism and the associated mapping class in order to avoid clumsy notation. We will always make clear through the context what is meant.

Proof of Proposition 2.5. Lemma 2.6 implies that the boundary map is a fibration. The homotopy fiber ∂/Σ of ∂ over Σ is the groupoid of pairs (H, f) consisting of a handlebody $H \in \mathbf{Hbdy}(T)$ plus a mapping class $f : \partial H \rightarrow \Sigma$. Every object in ∂/Σ , up to isomorphism, is of the form (H_0, f) , where H_0 is one fixed handlebody with $\partial H_0 \cong \Sigma$ and $f : \partial H_0 \rightarrow \Sigma$ is any mapping class (this follows from the fact that two handlebodies with isomorphic boundary are isomorphic, although not isomorphic relative boundary of course). A morphism $(H_0, f) \rightarrow (H_0, f')$ is a mapping class $g : H_0 \rightarrow H_0$ with $f' \partial g = f$ which implies $\partial g = (f')^{-1} f$. Since the mapping class group of a handlebody is a subgroup of the mapping class group of its boundary, ∂ is injective on morphisms. As a result, the automorphism groups of the groupoid ∂/Σ are trivial, and $\pi_0(\partial/\Sigma)$ can be identified with the set $\mathbf{Map}(\Sigma)/\mathbf{Map}(H_0)$ of left cosets of $\mathbf{Map}(H_0)$ in $\mathbf{Map}(\Sigma)$. □

Remark 2.8. Let $f : \Sigma \rightarrow \Sigma'$ be a diffeomorphism. If $f = \partial g$ for a diffeomorphism $g : H \rightarrow H'$ of handlebodies, there is a unique diffeomorphism $\xi_g : f.H \rightarrow H'$ with $\partial \xi_g = \text{id}_{\Sigma'}$ and $\xi_g \circ f = g$. Indeed, we are forced to define $\xi_g := g \circ (\tilde{f})^{-1}$. Now $\partial \xi_g = \text{id}_{\Sigma'}$ follows from Lemma 2.6.

2.4 A concrete description of cyclic framed E_2 -algebras with values in linear categories. There is an inexhaustible supply of examples of cyclic framed E_2 -algebras or, equivalently, self-dual balanced braided algebras coming from representation theory. This section will briefly name the main sources. The reader only interested in the abstract treatment of modular functors may skip this subsection.

If we specialize the symmetric monoidal $(2,1)$ -category \mathcal{S} in Theorem 2.2 to be Lex^f , the symmetric monoidal bicategory of

- finite categories, i.e. \mathbf{k} -linear abelian categories with finite-dimensional morphism spaces, finitely many simple objects up to isomorphism, enough projective objects, finite length for every object (such categories are exactly the ones that are linearly equivalent to finite-dimensional modules over some algebra),
- left exact functors,
- and natural isomorphisms

with the Deligne product as monoidal product, then each of the structures mentioned in Theorem 2.2 can equivalently be described as ribbon Grothendieck-Verdier categories in the sense of Boyarchenko and Drinfeld [BD13] by the results of [MW23b]: A *Grothendieck-Verdier category* in Lex^f is a monoidal category $\mathcal{A} \in \text{Lex}^f$ in which the functor $\mathcal{A}(K, X \otimes -)$ is representable by $DX \in \mathcal{A}$ for each X (that means $\mathcal{A}(K, X \otimes -) \cong \mathcal{A}(DX, -)$) such that the functor $D : \mathcal{A} \rightarrow \mathcal{A}^{\text{op}}$ sending X to DX is an equivalence. One calls D the *duality functor*. It is very important to emphasize that this is not necessarily a rigid duality (we discuss the rigid case in Example 2.9). In fact even if \mathcal{A} is rigid, then D needs not coincide with the rigid duality. A *ribbon Grothendieck-Verdier category* in Lex^f comes additionally with

- a braiding, i.e. isomorphisms $c_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ satisfying the hexagon axioms,
- and a balancing $\theta_X : X \rightarrow X$, i.e. a natural automorphism of the identity functor with
 - $\theta_{X \otimes Y} = c_{Y,X} c_{X,Y} (\theta_X \otimes \theta_Y)$ for $X, Y \in \mathcal{A}$,
 - and $\theta_I = \text{id}_I$ for the monoidal unit I of \mathcal{A} ,

such that additionally the ribbon condition $\theta_{DX} = D\theta_X$ is satisfied for all $X \in \mathcal{A}$.

We should mention that Lex^f , due to lack of (co)completeness, is not suitable as a target category for factorization homology. Instead, one can use Rex , the symmetric monoidal $(2,1)$ -category of

- finitely cocomplete linear categories,
- finitely cocontinuous functors (that one calls right exact functors even though we are not necessarily dealing with abelian categories)
- and natural isomorphisms.

The monoidal product is the Deligne-Kelly product, see [Fra13, Section 4] for details. The symmetric monoidal $(2,1)$ -category Rex meets the technical requirements mentioned in Section 2.1 that allow us to define factorization homology with values in Rex .

There is a symmetric monoidal sub- $(2,1)$ -category Rex^f of Rex spanned by all finite linear categories, as defined above as the objects of Lex^f . We can see Rex^f as dual version of Lex^f . More precisely, the functor taking the opposite category of a finite linear category provides us with a symmetric monoidal equivalence $\text{Lex}^f \simeq \text{Rex}^f$. Therefore, any cyclic framed E_2 -algebra in Lex^f gives rise to a cyclic framed E_2 -algebra in Rex^f and hence in Rex .

Example 2.9 (Finite ribbon categories). A *finite tensor category* in the sense of Etingof, Gelaki, Nikshych and Ostrik [EO04, EGNO15] is a finite linear category with a monoidal product that is rigid, i.e. admits left and right duals, and has a simple monoidal unit. A *finite ribbon category* \mathcal{A} is a finite tensor category that comes with a braiding c , a balancing θ such that $\theta_{X^\vee} = \theta_X^\vee$, where X^\vee is the dual object of $X \in \mathcal{A}$ (in finite ribbon categories, left and right duality coincide). One may obtain a finite ribbon category for example by taking finite-dimensional modules over a finite-dimensional ribbon Hopf algebra, see [Kas15, Section XIV.3] for a textbook reference. Thanks to rigidity, the monoidal product of a finite tensor category is exact. Therefore, a finite ribbon category provides also an example of a cyclic framed E_2 -algebra in Lex^f , Rex^f or Rex . For a braided category \mathcal{A} , one defines the Müger center $Z_2(\mathcal{A}) \subset \mathcal{A}$ as the full subcategory spanned by the *transparent* objects, i.e. those objects that trivially double braid with every other object. A *modular category* [MS88, Tur92, Tur94] is a finite ribbon category whose Müger center contains just the monoidal unit and finite direct sums of it. In this case, we call the braiding *non-degenerate*.

Example 2.10 (*Module categories of vertex operator algebras*). A rich source for ribbon Grothendieck-Verdier categories are vertex operator algebras. This is investigated by Allen, Lentner, Schweigert and Wood in [ALSW21], and this example is essentially a short summary of some of their results. Presenting a detailed definition of vertex operator algebras and the construction of their categories of modules is beyond the scope of this article. Instead, it will suffice for us to use the connection between vertex operator algebras and ribbon Grothendieck-Verdier duality presented in [ALSW21], namely that suitable categories of modules over a C_2 -cofinite vertex operator algebra form a ribbon Grothendieck-Verdier category. Let us discuss an example [ALSW21, Section 3] that can be described in relatively simple terms and that will be relevant for us later: The *Feigin-Fuchs boson* is a ribbon Grothendieck-Verdier category or rather a class of ribbon Grothendieck-Verdier categories. While the underlying monoidal category will still be rigid, the duality functor of the Grothendieck-Verdier structure will not necessarily coincide with the rigid duality. The categories in question are built from *bosonic lattice data*, i.e. a quadruple $\Psi = (\mathfrak{h}, \langle -, - \rangle, \Lambda, \xi)$, where \mathfrak{h} is a finite-dimensional real vector space with non-degenerate symmetric real-valued bilinear form $\langle -, - \rangle$, a lattice $\Lambda \subset \mathfrak{h}$ (i.e. a discrete subgroup) that is even with respect to $\langle -, - \rangle$, and an element $\xi \in \Lambda^*/\Lambda$, where $\Lambda^* := \{x \in \mathfrak{h} \mid \langle x, \Lambda \rangle \subset \mathbb{Z}\}$. We will assume that Λ has full rank. In this case, $G := \Lambda^*/\Lambda$ is a finite abelian group. From bosonic lattice data $\Psi = (\mathfrak{h}, \langle -, - \rangle, \Lambda, \xi)$, one may construct a vertex operator algebra whose category of modules $\text{VM}(\Psi)$ is a ribbon Grothendieck-Verdier category over \mathbb{C} . The details on the vertex operator algebra are given in [ALSW21, Section 3.3] and need not concern us here because there is a group-cohomological description [ALSW21, Theorem 3.12]: A class $\omega \in H_{\text{ab}}^3(G; \mathbb{C}^\times)$ in the third abelian group cohomology of $G = \Lambda^*/\Lambda$ (we refer to [EML53] for a background on abelian group cohomology) determines on the category vect_G of finite-dimensional G -graded vector spaces over \mathbb{C} a braided monoidal structure [EGNO15, Section 8.4] that we denote by vect_G^ω . By [Zet18, Theorem 4.2.2] the choice of a group element $g_0 \in G$ with $g_0 = 2h_0$ for some $h_0 \in G$ gives us in fact a ribbon Grothendieck-Verdier structure whose dualizing object is the ground field \mathbb{C}_{g_0} seen as graded vector space supported in degree g_0 . We denote the resulting ribbon Grothendieck-Verdier category by $\text{vect}_G^{\omega, g_0}$. Now by [ALSW21, Theorem 3.12] we can construct from bosonic lattice data $\Psi = (\mathfrak{h}, \langle -, - \rangle, \Lambda, \xi)$ a class $\omega(\Psi) \in H_{\text{ab}}^3(G; \mathbb{C}^\times)$ such that

$$\text{VM}(\Psi) \simeq \text{vect}_G^{\omega(\Psi), 2\xi} \quad \text{with} \quad G = \Lambda^*/\Lambda$$

as ribbon Grothendieck-Verdier categories. The Feigin-Fuchs boson forms a rigid monoidal category, but its Grothendieck-Verdier duality will not necessarily coincide with the rigid duality. In fact, this is the case if and only if $\xi = 0$. We should mention that the results of [ALSW21] can also produce ribbon Grothendieck-Verdier categories whose monoidal product is not exact, so that the category cannot even be rigid in that case (rigidity would imply exactness).

3 THE DEFINITION OF A MODULAR FUNCTOR AND THE MODULI SPACE OF MODULAR FUNCTORS

The purpose of this section is to give a sufficiently general definition of a modular functor. Having at our disposal the groupoid-valued surface operad Surf and the framework to define its modular $(2, 1)$ -categorical algebras, one might be tempted to define modular functors with values in a symmetric monoidal $(2, 1)$ -category \mathcal{S} as modular \mathcal{S} -valued Surf -algebras. As already briefly explained in the introduction, this definition would be unsatisfactory as it is known to exclude rich classes of examples, namely systems of *projective* mapping class group representations. Those can be included by allowing for central extensions of mapping class groups. The ‘projectiveness’ of the mapping class group actions of a modular functor is sometimes referred to as *anomaly*. We refer to [GM13] for a discussion about how and why this appears in the classical constructions of modular functors.

3.1 The definition of a modular functor. Since we would like to systematically treat all extensions at once, we need to develop a rather general theory of extensions for modular operads, and more specifically the surface operad. Fortunately, a formalism to treat extensions of groupoids is available in [BBF05]. We can easily adapt this to groupoid-valued modular operads.

Definition 3.1. Let \mathcal{O} be a modular groupoid-valued operad. An extension of \mathcal{O} is a map $q : \mathcal{P} \rightarrow \mathcal{O}$ of groupoid-valued modular operads such that for any $o \in \mathcal{O}(T)$ and any $T \in \text{Graphs}$ the homotopy fiber \mathcal{P}_o of $q_T : \mathcal{P}(T) \rightarrow \mathcal{O}(T)$ over $o \in \mathcal{O}(T)$ is connected.

As in this definition, we will throughout denote the homotopy fiber of \mathcal{P} over o by \mathcal{P}_o . For an extension $q : \mathcal{P} \rightarrow \mathcal{Q}$, any $o \in \mathcal{O}(T)$ and any $p \in \mathcal{P}_o$, we obtain a short exact sequence of groups

$$1 \rightarrow \text{Aut}_{\mathcal{P}_o}(p) \rightarrow \text{Aut}_{\mathcal{P}(T)}(p) \rightarrow \text{Aut}_{\mathcal{O}(T)}(o) \rightarrow 1 ; \quad (3.1)$$

here we see p also as object in $\mathcal{P}(T)$ by applying the forgetful map $\mathcal{P}_o \rightarrow \mathcal{P}(T)$, but we use the same symbol by a slight abuse of notation. The short exact sequence (3.1) is the only non-trivial part of the long exact sequence of homotopy groups for the fiber sequence $|BP_o| \rightarrow |B\mathcal{P}(T)| \rightarrow |B\mathcal{O}(T)|$ (B is the nerve of a category; $| - |$ is the geometric realization of a simplicial set). We then say that at $o \in \mathcal{O}$ the extension is by the group $\text{Aut}_{\mathcal{P}_o}(p)$.

Remark 3.2. An extension $\mathcal{P} \rightarrow \mathcal{O}$ of modular operads gives us at every operation $o \in \mathcal{O}(T)$ for some corolla T a group extension, namely (3.1). Generally, a group extension $1 \rightarrow G \rightarrow H \rightarrow J \rightarrow 1$ can equivalently be described by a *Dedecker-Schreier cocycle* following [Sch26, Ded64] which is a J -action up to coherent isomorphism on $\star//G$. In other words, we get for every $j \in J$ an equivalence $\psi_j : \star//G \rightarrow \star//G$ and coherence data consisting of natural isomorphisms $\alpha(j, j') : \psi_j \circ \psi_{j'} \cong \psi_{jj'}$ for $j, j' \in J$. We say that the *action of every element of J on G is by an inner automorphism* if for any $j \in J$ one of the following equivalent conditions are satisfied:

- (I1) The group automorphism of G determined by evaluation of ψ_j on morphisms is inner for every $j \in J$.
- (I2) The functor $\psi_j : \star//G \rightarrow \star//G$ is isomorphic to the identity for every $j \in J$.

Group extensions satisfying (I1) or (I2) will be the type of extensions that we will later on allow in the definition of modular functors as the correct generalization of central extensions (we will discuss the details in Section 3.1). One could ask why we do not concentrate on extensions $1 \rightarrow G \rightarrow H \rightarrow J \rightarrow 1$ that are central in the conventional sense, i.e. extensions for which G maps to the center of H (thereby implying that G is abelian). The reason for this is that the difference of this more familiar description to the conditions (I1) or (I2) is just a matter of perspective: If any of the conditions (I1) or (I2) are satisfied, then after replacing the Dedecker-Schreier cocycle by an equivalent one, we may assume that $\psi_j = \text{id}_{\star//G}$ for every $j \in J$. Nonetheless, the coherence data may be non-trivial. In fact, the coherence data will now amount to a 2-cocycle α on J with coefficients in $Z(G)$, in the conventional abelian sense. This is because the automorphism group of the functor $\text{id}_{\star//G}$ is $Z(G)$. From the abelian cocycle α , we may now build a central extension $1 \rightarrow Z(G) \rightarrow H' \rightarrow J \rightarrow 1$, where the middle group comes with a canonical map $H' \rightarrow H$. In other words, once we are given an extension $1 \rightarrow G \rightarrow H \rightarrow J \rightarrow 1$ such that every element of J acts by an inner automorphism, we may reduce to a central extension because the a priori non-abelian G -valued cocycle on J just takes values in $Z(G)$.

For the definition of a modular functor, we will have to further refine the notion of an extension. This will mean the following:

- We would like to consider extensions of **Surf** that are relative to **Hbdy** (i.e. come with a trivialization over **Hbdy**). Requirements in this direction are already implicit in the discussion of the framing anomaly in [Tur94, IV.3.5], see also the operadic discussion of this point in [MW23b, Lemma 7.15].
- We would like the extension term, i.e. the fiber of the extension, to satisfy some very mild locality properties. Such requirements are standard in the theory of modular functors, see e.g. [FS17, Section 3.2] for a discussion in a slightly different mathematical language.

We will now make such requirements precise in our framework. To this end, we need a few rather technical definitions: Consider the bicategory $\text{ModOp}(\text{Grpd})/\text{Hbdy}$ of groupoid-valued modular operads over **Hbdy**, i.e. the bicategory of groupoid-valued modular operads together with a map of modular operads to **Hbdy**. Clearly, $\text{id}_{\text{Hbdy}} \in \text{ModOp}(\text{Grpd})/\text{Hbdy}$, so we may consider the slice $\text{id}_{\text{Hbdy}}/(\text{ModOp}(\text{Grpd})/\text{Hbdy})$. This is the bicategory of groupoid-valued modular operads \mathcal{E} together with a map $p : \mathcal{E} \rightarrow \text{Hbdy}$ plus a section s of p up to isomorphism. We define the *bicategory Triv/Hbdy of trivial extensions of **Hbdy*** as the full subcategory of $\text{id}_{\text{Hbdy}}/(\text{ModOp}(\text{Grpd})/\text{Hbdy})$ spanned by those (\mathcal{E}, p, s) such that $p : \mathcal{E} \rightarrow \text{Hbdy}$ is an extension and such that for $H \in \text{Hbdy}(T)$ the $\text{Map}(H)$ -action on $\text{Aut}_{\mathcal{E}_H}(s(H))$ determined by the short exact sequence

$$1 \rightarrow \text{Aut}_{\mathcal{E}_H}(s(H)) \rightarrow \text{Aut}_{\mathcal{E}(T)}(s(H)) \rightarrow \text{Aut}_{\text{Hbdy}(T)}(H) = \text{Map}(H) \rightarrow 1$$

is trivial. Here \mathcal{E}_H denotes again the homotopy fiber of $\mathcal{E}(T) \rightarrow \text{Hbdy}(T)$ over H . Clearly, we have a forgetful map $\text{Triv}/\text{Hbdy} \rightarrow \text{ModOp}(\text{Grpd})/\text{Hbdy}$.

Consider now an object in the homotopy pullback

$$\begin{array}{ccc} \text{ModOp}(\text{Grpd})/\text{Surf} \times_{\text{ModOp}(\text{Grpd})/\text{Hbdy}} \text{Triv}/\text{Hbdy} & \longrightarrow & \text{ModOp}(\text{Grpd})/\text{Surf} \\ \downarrow & & \downarrow \partial^* \\ \text{Triv}/\text{Hbdy} & \xrightarrow{\text{forget}} & \text{ModOp}(\text{Grpd})/\text{Hbdy} \end{array} \quad (3.2)$$

in bicategories, where ∂^* is the pullback along $\partial : \mathbf{Hbdy} \rightarrow \mathbf{Surf}$. In plain terms, the homotopy pullback (3.2) is the bicategory whose objects are formed by a groupoid-valued modular operad \mathcal{Q} with a map $q : \mathcal{Q} \rightarrow \mathbf{Surf}$ and an equivalence $\chi : \mathcal{E} \simeq \partial^* \mathcal{Q} := \mathcal{Q} \times_{\mathbf{Surf}} \mathbf{Hbdy}$ over \mathbf{Hbdy} for a trivial extension (\mathcal{E}, p, s) of \mathbf{Hbdy} . This means that χ is a trivialization of \mathcal{Q} over \mathbf{Hbdy} . For brevity, we will often denote an object $(\mathcal{Q}, q, \chi, \mathcal{E}, p, s) \in \mathbf{ModOp}(\mathbf{Grpd})/\mathbf{Surf} \times_{\mathbf{ModOp}(\mathbf{Grpd})/\mathbf{Hbdy}} \mathbf{Triv}/\mathbf{Hbdy}$ by its underlying modular operad \mathcal{Q} . We can make a few observations:

- The map $q : \mathcal{Q} \rightarrow \mathbf{Surf}$ is an extension of \mathbf{Surf} . Indeed, denote by \mathcal{Q}_Σ the homotopy fiber of $q : \mathcal{Q} \rightarrow \mathbf{Surf}$ over a surface Σ and choose a handlebody H with $\partial H = \Sigma$. The equivalence $\chi : \mathcal{E} \simeq \partial^* \mathcal{Q}$ restricts to an equivalence $\mathcal{E}_H \simeq \mathcal{Q}_{\partial H} = \mathcal{Q}_\Sigma$. This tells us that \mathcal{Q}_Σ is connected because \mathcal{E} is assumed to have connected homotopy fibers. This makes $q : \mathcal{Q} \rightarrow \mathbf{Surf}$ an extension of \mathbf{Surf} in the sense of Definition 3.1. Every element of the mapping class group $\mathbf{Map}(\Sigma)$ of any surface Σ acts by an inner automorphism on the extension term $\pi_1(\mathcal{Q}_\Sigma)$. Equivalently, any mapping class of Σ acts by a non-canonically trivializable functor $\mathcal{Q}_\Sigma \rightarrow \mathcal{Q}_\Sigma$ (Remark 3.2). Indeed, this is true for all mapping classes of all handlebodies since \mathcal{Q} is a trivial extension when restricted to \mathbf{Hbdy} . It then also holds for mapping classes of surfaces.
- The tuple $(\mathcal{Q}, p, \chi, G)$ canonically determines a map $\mathbf{Hbdy} \rightarrow \mathcal{Q}$ of modular operads over \mathbf{Surf} . Indeed, the needed map $\mathbf{Hbdy} \rightarrow \mathcal{Q}$ can be obtained as follows:

$$\mathbf{Hbdy} \xrightarrow{s} \mathcal{E} \xrightarrow{\chi} \partial^* \mathcal{Q} = \mathcal{Q} \times_{\mathbf{Surf}} \mathbf{Hbdy} \xrightarrow{\text{projection}} \mathcal{Q}. \quad (3.3)$$

By abuse of notation we will denote this map again by s (there will be no confusion as long as we specify source and target).

- After the choice of a handlebody H with $\partial H = \Sigma$, the homotopy fiber \mathcal{Q}_Σ comes with a specific object, i.e. a pointing, namely $\star \xrightarrow{s} \mathcal{E}_H \xrightarrow{\chi} \mathcal{Q}_\Sigma$. As a consequence, $\mathcal{Q}_{\mathbb{D}^2}$ also comes with a pointing because there is up to unique isomorphism only one handlebody with boundary \mathbb{D}^2 ; in fact, this is true for all genus zero surfaces. This implies that for any oriented embedding $\varphi : (\mathbb{D}^2)^{\sqcup J} \rightarrow \Sigma$ we have a functor

$$\lambda_\varphi : \mathcal{Q}_{\Sigma \setminus \mathring{\text{im}} \varphi} \xrightarrow{\text{pointing} \times \text{id}} \mathcal{Q}_{(\mathbb{D}^2)^{\sqcup J}} \times \mathcal{Q}_{\Sigma \setminus \mathring{\text{im}} \varphi} \xrightarrow{\text{operadic composition}} \mathcal{Q}_\Sigma, \quad (3.4)$$

where $\mathring{\text{im}} \varphi$ is the interior of the image of φ . We say that an object \mathcal{Q} in the homotopy pullback (3.2) admits insertions of vacua if for any surface and any oriented embedding the functor (3.4) is an equivalence.

Definition 3.3. We define the bicategory $\mathbf{Ext}^\circ(\mathbf{Surf})$ of extensions of \mathbf{Surf} relative \mathbf{Hbdy} that admit insertions of vacua as the full subcategory of the homotopy pullback (3.2) spanned by those objects that admit insertions of vacua.

This finally gives us the class of ‘reasonable’ extensions of \mathbf{Surf} that we would like to consider: They are extensions relative genus zero or, equivalently, relative to the handlebody operad, and they satisfy a rather mild locality property in the sense that they are compatible with the insertion of disks.

Lemma 3.4. The 2-morphisms in the bicategory $\mathbf{Ext}^\circ(\mathbf{Surf})$ are invertible, thereby making $\mathbf{Ext}^\circ(\mathbf{Surf})$ a $(2, 1)$ -category. In other words, this means that the morphism categories $\mathbf{Map}_{\mathbf{Ext}^\circ(\mathbf{Surf})}(-, -)$ in $\mathbf{Ext}^\circ(\mathbf{Surf})$ are groupoids.

Proof. Let $\mathcal{Q}, \mathcal{Q}' \in \mathbf{Ext}^\circ(\mathbf{Surf})$ and let $\alpha_0, \alpha_1 : \mathcal{Q} \rightarrow \mathcal{Q}'$ be 1-morphisms. We have seen in (3.3) that \mathcal{Q} and \mathcal{Q}' come with their canonical maps $s : \mathbf{Hbdy} \rightarrow \mathcal{Q}$ and $s' : \mathbf{Hbdy} \rightarrow \mathcal{Q}'$. After unpacking the definitions, it follows from $\alpha_i : \mathcal{Q} \rightarrow \mathcal{Q}'$ being a 1-morphism in $\mathbf{Ext}^\circ(\mathbf{Surf})$ that $\alpha_i \circ s \xrightarrow{\beta_i} s'$ by a canonical isomorphism for $i = 0, 1$. Now if $\gamma : \alpha_0 \rightarrow \alpha_1$ is a 2-morphism in $\mathbf{Ext}^\circ(\mathbf{Surf})$, then we have in particular $\beta_{1,H} \gamma_s(H) = \beta_{0,H}$ for any handlebody. This tells us that all components of γ that lie in the image of $s : \mathbf{Hbdy} \rightarrow \mathcal{Q}$ are invertible. But s is arity-wise essentially surjective (because the homotopy fibers of \mathcal{Q} over \mathbf{Surf} are connected). Therefore, it follows that all components of γ are invertible. \square

We have now achieved the main goal of this section and are finally in a position to define modular functors:

Definition 3.5. A modular functor is a pair $(\mathcal{Q}, \mathcal{B})$, where $\mathcal{Q} \in \mathbf{Ext}^\circ(\mathbf{Surf})$ is an extension of \mathbf{Surf} in the sense of Definition 3.3 (i.e. it is implied that the extension is relative \mathbf{Hbdy} and admits insertions of vacua), and \mathcal{B} is a modular \mathcal{Q} -algebra.

3.2 The extension problem for cyclic framed E_2 -algebras and the moduli space of modular functors. In the next step, we will, using Definition 3.5 as a starting point, define a *moduli space of modular functors*. To this end, we will define some auxiliary notions that will be convenient later.

Given an extension \mathcal{Q} of Surf relative Hbdy and a cyclic framed E_2 -algebra, it is natural to ask about a description of the space of modular \mathcal{Q} -algebras that extend \mathcal{A} .

Definition 3.6 (*Extensions of a cyclic framed E_2 -algebra over a fixed extension of Surf*). For a cyclic framed E_2 -algebra \mathcal{A} in \mathcal{S} and $\mathcal{Q} \in \text{Ext}^\circ(\text{Surf})$, we define the groupoid $\text{Ext}^\circ(\mathcal{A}; \mathcal{Q})$ of extensions of \mathcal{A} on \mathcal{Q} as the homotopy fiber of

$$\text{ModAlg}(\mathcal{Q}) \xrightarrow{\text{restriction along the section Hbdy} \rightarrow \mathcal{Q}} \text{ModAlg}(\text{Hbdy}) \xrightarrow{\text{genus zero restriction}} \text{CycAlg}(fE_2)$$

over \mathcal{A} .

Remark 3.7. Cyclic and modular algebras with values in a symmetric monoidal bicategory form 2-groupoids [MW23b, Proposition 2.18]. Therefore, $\text{Ext}^\circ(\mathcal{A}; \mathcal{Q})$, as a homotopy fiber of a map between 2-groupoids, is a priori a 2-groupoid itself. However, from the definition of 1-morphisms and 2-morphisms of cyclic and modular algebras [MW23b, Section 2.4] and the definition of the homotopy fiber, it follows that all 2-automorphisms of $\text{Ext}^\circ(\mathcal{A}; \mathcal{Q})$ are actually trivial. For this reason, we will see $\text{Ext}^\circ(\mathcal{A}; \mathcal{Q})$ as groupoid.

Definition 3.8 (*All extensions of a cyclic framed E_2 -algebra*). For a cyclic framed E_2 -algebra \mathcal{A} in \mathcal{S} , we define the bicategory $\text{Ext}^\circ(\mathcal{A})$ of all extensions of \mathcal{A} as the bicategory of pairs $(\mathcal{Q}, \mathcal{B})$ formed by $\mathcal{Q} \in \text{Ext}^\circ(\text{Surf})$ and $\mathcal{B} \in \text{Ext}^\circ(\mathcal{A}; \mathcal{Q})$. A 1-morphism $(\mathcal{Q}, \mathcal{B}) \rightarrow (\mathcal{Q}', \mathcal{B}')$ is a 1-morphism $\psi : \mathcal{Q} \rightarrow \mathcal{Q}'$ in $\text{Ext}^\circ(\text{Surf})$ together with a map $\alpha : \mathcal{B} \rightarrow \psi^* \mathcal{B}'$ in $\text{Ext}^\circ(\mathcal{A}; \mathcal{Q})$. The 2-morphisms are defined in a similar way.

We can now define the moduli space of modular functors:

Definition 3.9. We define the *bicategory of modular functors* MF as the bicategory formed by pairs $(\mathcal{Q}, \mathcal{B})$, where $\mathcal{Q} \in \text{Ext}^\circ(\text{Surf})$ and $\mathcal{B} \in \text{ModAlg}(\mathcal{Q})$. A 1-morphism $(\mathcal{Q}, \mathcal{B}) \rightarrow (\mathcal{Q}', \mathcal{B}')$ is a 1-morphism $\psi : \mathcal{Q} \rightarrow \mathcal{Q}'$ in $\text{Ext}^\circ(\text{Surf})$ together with a map $\alpha : \mathcal{B} \rightarrow \psi^* \mathcal{B}'$ in $\text{Ext}^\circ(\mathcal{A}; \mathcal{Q})$, where \mathcal{A} is the cyclic framed E_2 -algebra obtained from \mathcal{B} via genus zero restriction. The 2-morphisms are defined in a similar way.

By taking nerves of morphism categories we may see a bicategory as a simplicial category to which we can assign its homotopy coherent nerve, see e.g. [Rie14, Section 16.3]. This allows us to speak about the nerve BC of a bicategory \mathcal{C} . This version of the nerve is also called the *Duskin nerve* [Dus02].

Definition 3.10. We define the *moduli space of modular functors* as the geometric realization $\mathfrak{MF} := |BMF|$ of the Duskin nerve of MF.

Remark 3.11. One may ask why one should consider the realization $|BMF|$ of the nerve of MF as the moduli space of modular functors and not MF, as plain bicategory. The reason is the following: Suppose we are given $(\mathcal{Q}, \mathcal{B}), (\mathcal{Q}', \mathcal{B}')$ in MF. A 1-morphism $(\mathcal{Q}, \mathcal{B}) \rightarrow (\mathcal{Q}', \mathcal{B}')$ in MF is a map $\alpha : \mathcal{Q} \rightarrow \mathcal{Q}'$ of extensions of Surf relative to Hbdy plus an equivalence $\phi : \alpha^* \mathcal{B}' \simeq \mathcal{B}$ of modular \mathcal{Q} -algebras. In other words, \mathcal{B} factors through α and produces \mathcal{B}' . By definition the map α is over Surf; it just changes the fibers of the extensions. Spelled out in more detail, this means that for a surface Σ the modular algebra \mathcal{B} produces an action of an extension $\pi : G_{\mathcal{Q}, \Sigma} \rightarrow \text{Map}(\Sigma)$ on $\mathcal{B}(\Sigma)$ while \mathcal{B}' produces an action of an extension $\pi' : G_{\mathcal{Q}', \Sigma} \rightarrow \text{Map}(\Sigma)$ on $\mathcal{B}'(\Sigma)$. The map $\alpha : \mathcal{Q} \rightarrow \mathcal{Q}'$, when evaluated at Σ , gives us a group morphism $\alpha_\Sigma : G_{\mathcal{Q}, \Sigma} \rightarrow G_{\mathcal{Q}', \Sigma}$ with $\pi' \alpha_\Sigma = \pi$, and $\phi : \alpha^* \mathcal{B}' \simeq \mathcal{B}$ provides an isomorphism $\phi_\Sigma : \alpha_\Sigma^* \mathcal{B}'(\Sigma) \cong \mathcal{B}(\Sigma)$ of $G_{\mathcal{Q}, \Sigma}$ -representations. Then we want to regard $(\mathcal{Q}, \mathcal{B})$ and $(\mathcal{Q}', \mathcal{B}')$ as the same modular functor because \mathcal{B} and \mathcal{B}' both factor through \mathcal{Q}' and agree as \mathcal{Q}' -algebras. But these pairs are only identified through a 1-isomorphism in $\mathfrak{MF} = |BMF|$, and not MF. In general, two objects in \mathfrak{MF} are isomorphic if they are connected by a zigzag of such maps.

4 ANSULAR FUNCTORS AND HANDLEBODY SKEIN MODULES

This section is devoted to our first result, a connection between the ansular functor associated to a framed cyclic E_2 -algebra and factorization homology. The connection will be through handlebody skein modules that we define abstractly in Section 4.1. The connection to more classical skein-theoretic methods is summarized in Section 4.3, but this is not the focus of this article and also logically not relevant for our main results.

boundary components of Σ / embedded disks of H

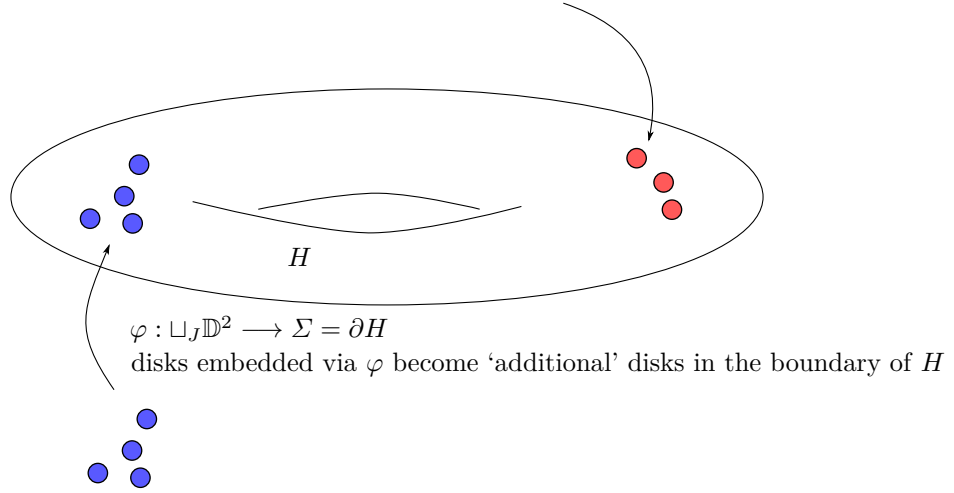


Figure 2: On the definition of H^φ .

4.1 The handlebody skein modules for a cyclic framed E_2 -algebra. Let \mathcal{A} be a cyclic framed E_2 -algebra in \mathcal{S} , $\Sigma \in \text{Surf}(T)$ for $T \in \text{Graphs}$ (we assume for the moment that T is a corolla) and $\varphi : \sqcup_J \mathbb{D}^2 \rightarrow \Sigma$ an oriented embedding (J is a finite set). If $H \in \text{Hbdy}(T)$ is a handlebody with $\partial H = \Sigma$, then the boundary components of Σ induce embedded disks in ∂H . Additionally, we obtain embedded disks in ∂H from φ . In total, we obtain a handlebody with $|J| + |\text{Legs}(T)|$ embedded disks. We denote this handlebody by H^φ , see Figure 2.

The evaluation of the modular extension $\widehat{\mathcal{A}}$ (see Theorem 2.2 and the remarks afterwards) on H^φ yields a map $I \rightarrow \mathcal{A}^{\otimes \text{Legs}(T)} \otimes \mathcal{A}^{\otimes J}$. Since \mathcal{A} is a cyclic algebra, we can use the cyclic structure to obtain a map

$$\widehat{\mathcal{A}}(H^\varphi) : \mathcal{A}^{\otimes J} \rightarrow \mathcal{A}^{\otimes \text{Legs}(T)}. \quad (4.1)$$

Proposition 4.1. Let \mathcal{A} be a cyclic framed E_2 -algebra in \mathcal{S} , $T \in \text{Graphs}$ a corolla, $H \in \text{Hbdy}(T)$ and $\Sigma = \partial H$. The maps $\widehat{\mathcal{A}}(H^\varphi) : \mathcal{A}^{\otimes J} \rightarrow \mathcal{A}^{\otimes \text{Legs}(T)}$ from (4.1) are natural in the embedding $\varphi : \sqcup_J \mathbb{D}^2 \rightarrow \Sigma$ and descend to the factorization homology $\int_\Sigma \mathcal{A}$, inducing a map

$$\Phi_{\mathcal{A}}(H) : \int_\Sigma \mathcal{A} \rightarrow \mathcal{A}^{\otimes \text{Legs}(T)}.$$

We call $\Phi_{\mathcal{A}}(H)$ the *handlebody skein module* for the cyclic framed E_2 -algebra \mathcal{A} and the handlebody H , or just the Φ -map for \mathcal{A} and H .

Proof. In order for the maps (4.1) to descend to $\int_\Sigma \mathcal{A}$, we need to show that they provide a coherent co-cone; this is because of the homotopy colimit description of factorization homology recalled in (2.1). To this end, let $J = \{1, \dots, n\}$ and $J' = \{1, \dots, m\}$ (here n or m can be zero in which case $J = \emptyset$ or $J' = \emptyset$, respectively); moreover, let $\varphi : \sqcup_J \mathbb{D}^2 = (\mathbb{D}^2)^{\sqcup n} \rightarrow \Sigma$ be an oriented embedding and $\varphi' : \sqcup_{J'} \mathbb{D}^2 = (\mathbb{D}^2)^{\sqcup m} \rightarrow \mathbb{D}^2$ an fE_2 -operation. The operation φ' gives rise to a morphism $\mathcal{A}(\varphi') : \mathcal{A}^{\otimes m} \rightarrow \mathcal{A}$. It suffices now to prove that for the i -th partial composition $\varphi \circ_i \varphi'$ the triangle

$$\begin{array}{ccc} \mathcal{A}^{\otimes(j-1)} \otimes \mathcal{A}^{\otimes m} \otimes \mathcal{A}^{\otimes(n-j)} & & \\ \downarrow & \searrow^{\widehat{\mathcal{A}}(H^{\varphi \circ_i \varphi'})} & \\ \mathcal{A}^{\otimes(j-1)} \otimes \mathcal{A}(\varphi') \otimes \mathcal{A}^{\otimes(n-j)} & & \mathcal{A}^{\otimes \text{Legs}(T)}, \\ \downarrow & \nearrow_{\widehat{\mathcal{A}}(H^\varphi)} & \\ \mathcal{A}^{\otimes n} & & \end{array}$$

commutes up to coherent isomorphism. This isomorphism is obtained as follows: Observe that $\mathcal{A}^{\otimes(j-1)} \otimes \mathcal{A}(\varphi') \otimes \mathcal{A}^{\otimes(n-j)}$ can be written up to canonical isomorphism as

$$\mathcal{A}^{\otimes(j-1)} \otimes \mathcal{A}(\varphi') \otimes \mathcal{A}^{\otimes(n-j)} = \widehat{\mathcal{A}}\left(\left([0, 1] \times \mathbb{D}^2\right)^{\sqcup(j-1)} \sqcup \mathbb{B}_{n,1}^3 \sqcup \left([0, 1] \times \mathbb{D}^2\right)^{\sqcup(n-j)}\right),$$

where $\mathbb{B}_{n,1}^3$ is a closed three-dimensional ball with n incoming and one outgoing embedded disks in its boundary; this is because the modular algebra $\widehat{\mathcal{A}}$ extends the cyclic algebra \mathcal{A} . The statement follows now from the canonical isomorphism

$$H^{\varphi \circ \iota \varphi'} \cong H^\varphi \cup_i \left(([0,1] \times \mathbb{D}^2)^{\sqcup(j-1)} \sqcup \mathbb{B}_{n,1}^3 \sqcup ([0,1] \times \mathbb{D}^2)^{\sqcup(n-j)} \right)$$

and the fact that $\widehat{\mathcal{A}}$ is a modular algebra and hence respects gluing. \square

We now obtain a factorization homology description of the ansular functor. The key ingredient, namely the Φ -maps, have already been established. The rest is relatively formal: Since $\int_\Sigma \mathcal{A}$ is the homotopy colimit of $\mathcal{A}^{\otimes J}$ running over all embeddings $\varphi : \sqcup_J \mathbb{D}^2 \rightarrow \Sigma$, there is for any oriented embedding $\varphi : \sqcup_J \mathbb{D}^2 \rightarrow \Sigma$ a canonical map $\mathcal{A}^{\otimes J} \rightarrow \int_\Sigma \mathcal{A}$ that by abuse of notation we will again denote by φ . In the case $J = \emptyset$, we have the unique embedding $\varphi_0 : \emptyset \rightarrow \Sigma$. We consider the following diagram:

$$\begin{array}{ccccc}
I & \xrightarrow{\mathcal{O}_\Sigma} & \int_\Sigma \mathcal{A} & \xrightarrow{\Phi_{\mathcal{A}}(H)} & \mathcal{A}^{\otimes \text{Legs}(T)} \\
& \searrow \text{id}_I & \uparrow \varphi_0 & \nearrow \widehat{\mathcal{A}}(H^{\varphi_0}) & \\
& & \mathcal{A}^{\otimes 0} = I & & \\
& & \widehat{\mathcal{A}}(H) & &
\end{array}$$

First we investigate the two upper triangles: The left upper triangle commutes by definition of \mathcal{O}_Σ while the right upper triangle commutes by construction of $\Phi_{\mathcal{A}}(H)$ in Proposition 4.1. The lower triangle commutes up to a canonical isomorphism, again by construction. Let us summarize this:

Theorem 4.2 (Factorization homology description of the ansular functor). *Let \mathcal{A} be a cyclic framed E_2 -algebra in \mathcal{S} . Then for any corolla $T \in \text{Graphs}$ and $H \in \text{Hbdy}(T)$ with boundary $\Sigma = \partial H$, the diagram*

$$\begin{array}{ccccc}
I & \xrightarrow{\mathcal{O}_\Sigma} & \int_\Sigma \mathcal{A} & \xrightarrow{\Phi_{\mathcal{A}}(H)} & \mathcal{A}^{\otimes \text{Legs}(T)} \\
& \searrow & & \nearrow & \\
& & \widehat{\mathcal{A}}(H) & &
\end{array}$$

commutes up to canonical isomorphism.

For a surface Σ (which, for simplicity, we assume to be connected), it is standard that $\int_\Sigma \mathcal{A}$ is a module over the algebra $\int_{\partial\Sigma \times [0,1]} \mathcal{A}$, see e.g. [Gin15, Lemma 5.2]. The algebra structure on $\int_{\partial\Sigma \times [0,1]} \mathcal{A}$ comes from stacking cylinders over $\partial\Sigma$. One can obtain the action $\int_{\partial\Sigma \times [0,1]} \mathcal{A} \otimes \int_\Sigma \mathcal{A} \rightarrow \int_\Sigma \mathcal{A}$ by choosing a collar for $\partial\Sigma$, i.e. an embedding

$$j : (\partial\Sigma \times [0,1]) \sqcup \Sigma \rightarrow \Sigma \quad (4.2)$$

which by functoriality of factorization homology with respect to embeddings induces the action. Up to isomorphism, the action does not depend on the collar. By applying this to the disk, we find that \mathcal{A} is a $\int_{\mathbb{S}^1 \times [0,1]} \mathcal{A}$ -module. This also implies that $\mathcal{A}^{\otimes \text{Legs}(T)}$ is a module over $\int_{\partial\Sigma \times [0,1]} \mathcal{A}$ for any corolla T and $\Sigma \in \text{Surf}(T)$.

Proposition 4.3. *Let \mathcal{A} be a cyclic framed E_2 -algebra in \mathcal{S} , $T \in \text{Graphs}$ a corolla and $H \in \text{Hbdy}(T)$ with boundary $\Sigma = \partial H$. Then the map $\Phi_{\mathcal{A}}(H) : \int_\Sigma \mathcal{A} \rightarrow \mathcal{A}^{\otimes \text{Legs}(T)}$ comes naturally with the structure of a $\int_{\partial\Sigma \times [0,1]} \mathcal{A}$ -module map.*

Proof. Let $\psi : \sqcup_K \mathbb{D}^2 \rightarrow \partial\Sigma \times [0,1]$ and $\varphi : \sqcup_J \mathbb{D}^2 \rightarrow \Sigma$ be oriented embeddings. Postcomposition with (4.2) gives us an embedding $j(\psi, \varphi) : \sqcup_{K \sqcup J} \mathbb{D}^2 \rightarrow \Sigma$. Since $\widehat{\mathcal{A}}$ is a modular algebra, the map

$$\mathcal{A}^{\otimes (K \sqcup J)} \xrightarrow{\widehat{\mathcal{A}}(H^{j(\psi, \varphi)})} \mathcal{A}^{\otimes \text{Legs}(T)} \quad (4.3)$$

is canonically isomorphic to the partial composition of

$$\widehat{\mathcal{A}}(H^\varphi) : \mathcal{A}^{\otimes J} \longrightarrow \mathcal{A}^{\otimes \text{Legs}(T)} \quad (4.4)$$

with

$$\widehat{\mathcal{A}}((\partial\Sigma \times [0,1])^\psi) : \mathcal{A}^{\otimes K} \otimes \mathcal{A}^{\otimes \text{Legs}(T)} \longrightarrow \mathcal{A}^{\otimes \text{Legs}(T)} . \quad (4.5)$$

(A priori, $\widehat{\mathcal{A}}((\partial\Sigma \times [0,1])^\psi)$ is a map $\mathcal{A}^{\otimes K} \longrightarrow \mathcal{A}^{\otimes \text{Legs}(T)} \otimes \mathcal{A}^{\otimes \text{Legs}(T)}$, but we move one of the $\mathcal{A}^{\otimes \text{Legs}(T)}$ to the left via the pairing.) But the maps (4.3) induce the map

$$\int_{\partial\Sigma \times [0,1]} \mathcal{A} \otimes \int_{\Sigma} \mathcal{A} \xrightarrow{\text{action}} \int_{\Sigma} \mathcal{A} \xrightarrow{\Phi_{\mathcal{A}}(H)} \mathcal{A}^{\otimes \text{Legs}(T)} \quad (4.6)$$

while the (partial) composition of (4.4) and (4.5)

$$\mathcal{A}^{\otimes (K \sqcup J)} \xrightarrow{\text{id} \otimes \widehat{\mathcal{A}}(H^\varphi)} \mathcal{A}^{\otimes K} \otimes \mathcal{A}^{\otimes \text{Legs}(T)} \xrightarrow{\widehat{\mathcal{A}}((\partial\Sigma \times [0,1])^\psi)} \mathcal{A}^{\otimes \text{Legs}(T)}$$

induces the map

$$\left(\int_{\partial\Sigma \times [0,1]} \mathcal{A} \right) \otimes \int_{\Sigma} \mathcal{A} \xrightarrow{\text{id} \otimes \Phi_{\mathcal{A}}(H)} \left(\int_{\partial\Sigma \times [0,1]} \mathcal{A} \right) \otimes \mathcal{A}^{\otimes \text{Legs}(T)} \xrightarrow{\text{action}} \mathcal{A}^{\otimes \text{Legs}(T)} . \quad (4.7)$$

Therefore, we obtain a canonical isomorphism between (4.6) and (4.7). It is a straightforward observation that this equips $\Phi_{\mathcal{A}}(H)$ with the structure of a module map. \square

4.2 Equivariance properties of handlebody skein modules under the mapping class group action. In this subsection, we discuss the transformation behavior of handlebody skein modules under the action of the mapping class group on factorization homology.

Definition 4.4. Let \mathcal{A} be a cyclic framed E_2 -algebra in \mathcal{S} and T a corolla. For $\Sigma \in \text{Surf}(T)$, we denote by $\mathcal{S}_{\partial}(\int_{\Sigma} \mathcal{A}, \mathcal{A}^{\otimes \text{Legs}(T)})$ the groupoid of $\int_{\partial\Sigma \times [0,1]} \mathcal{A}$ -module maps $\int_{\Sigma} \mathcal{A} \longrightarrow \mathcal{A}^{\otimes \text{Legs}(T)}$ with module map isomorphisms as morphisms.

Remark 4.5. Any diffeomorphism $f : \Sigma \longrightarrow \Sigma'$ induces a 1-morphism $f_* : \int_{\Sigma} \mathcal{A} \longrightarrow \int_{\Sigma'} \mathcal{A}$ and an algebra map $\alpha_f : \int_{\partial\Sigma \times [0,1]} \mathcal{A} \longrightarrow \int_{\partial\Sigma' \times [0,1]} \mathcal{A}$ (if $\Sigma = \Sigma'$, this is the identify) such that f_* comes naturally with the structure of a $\int_{\partial\Sigma \times [0,1]} \mathcal{A}$ -module map $\int_{\Sigma} \mathcal{A} \longrightarrow \alpha_f^* \int_{\Sigma'} \mathcal{A}$. Here $\alpha_f^* \int_{\Sigma'} \mathcal{A}$ is the restriction of $\int_{\Sigma'} \mathcal{A}$ along α_f . The isomorphism class of the module map that f gives rise to depends only on the mapping class of f .

Proposition 4.6. Let \mathcal{A} be a cyclic framed E_2 -algebra in \mathcal{S} , $T \in \text{Graphs}$ a corolla and $f : \Sigma \longrightarrow \Sigma'$ a diffeomorphism. Then for any $H \in \text{Hbdy}(T)$ with $\partial H = \Sigma$, the diagram

$$\begin{array}{ccc} \int_{\Sigma} \mathcal{A} & \xrightarrow{\Phi_{\mathcal{A}}(H)} & \mathcal{A}^{\otimes \text{Legs}(T)} \\ \downarrow f_* & & \uparrow \Phi_{\mathcal{A}}(f.H) \\ \int_{\Sigma'} \mathcal{A} & & \end{array} \quad (4.8)$$

commutes up to canonical isomorphism of $\int_{\partial\Sigma \times [0,1]} \mathcal{A}$ -module maps, i.e. up to an isomorphism in the category $\mathcal{S}_{\partial}(\int_{\Sigma} \mathcal{A}, \mathcal{A}^{\otimes \text{Legs}(T)})$. This isomorphism, when combined with the isomorphism from Theorem 4.2 and the homotopy fixed point structure of \mathcal{O}_{Σ} , yield the isomorphism $\widehat{\mathcal{A}}(\vec{f}) : \widehat{\mathcal{A}}(H) \longrightarrow \widehat{\mathcal{A}}(f.H)$:

$$\begin{array}{ccc} & \widehat{\mathcal{A}}(H) & \\ & \curvearrowright & \\ \begin{array}{ccc} \int_{\Sigma} \mathcal{A} & \xrightarrow{\Phi_{\mathcal{A}}(H)} & \mathcal{A}^{\otimes \text{Legs}(T)} \\ \downarrow f_* & & \uparrow \Phi_{\mathcal{A}}(f.H) \\ \int_{\Sigma'} \mathcal{A} & & \end{array} & = & \widehat{\mathcal{A}}(\vec{f}) \\ & \curvearrowleft & \\ & \widehat{\mathcal{A}}(f.H) & \end{array} \quad (4.9)$$

Proof. Let $\varphi : \sqcup_J \mathbb{D}^2 \rightarrow \Sigma$ be an oriented embedding, then $f \circ \varphi : \sqcup_J \mathbb{D}^2 \rightarrow \Sigma'$ is also an oriented embedding. The diffeomorphism $\tilde{f} : H \rightarrow f.H$ from Lemma 2.6 induces a diffeomorphism $H^\varphi \rightarrow (f.H)^{f \circ \varphi}$ giving us, after evaluation with the modular extension $\widehat{\mathcal{A}}$, an isomorphism $\widehat{\mathcal{A}}(H^\varphi) \rightarrow \widehat{\mathcal{A}}((f.H)^{f \circ \varphi})$. By the construction of $\Phi_{\mathcal{A}}$ these isomorphisms descend to factorization homology to yield the isomorphism (4.8). It is an isomorphism of $\int_{\partial \Sigma \times [0,1]} \mathcal{A}$ -module maps because f respects the boundary parametrizations. Now (4.9) can be verified directly using the definitions. \square

A relevant special case of Proposition 4.6 is the following:

Corollary 4.7. *Let \mathcal{A} be a cyclic framed E_2 -algebra in \mathcal{S} , $T \in \text{Graphs}$ a corolla and $g : H \rightarrow H'$ a diffeomorphism between $H, H' \in \text{Hbdy}(T)$. Set $f := \partial g$. Then after the identification $\Phi_{\mathcal{A}}(f.H) \cong \Phi_{\mathcal{A}}(H')$ induced by ξ_g from Remark 2.8, we obtain an isomorphism of $\int_{\partial \Sigma \times [0,1]} \mathcal{A}$ -module maps filling the triangle*

$$\begin{array}{ccc} \int_{\Sigma} \mathcal{A} & \xrightarrow{\Phi_{\mathcal{A}}(H)} & \mathcal{A}^{\otimes \text{Legs}(T)} \\ f_* = \partial g_* \downarrow & & \uparrow \Phi_{\mathcal{A}}(H') \\ \int_{\Sigma'} \mathcal{A} & & \end{array}$$

such that

$$\begin{array}{ccc} & \widehat{\mathcal{A}}(H) & \\ & \uparrow \mathcal{O}_{\Sigma} & \\ I & \int_{\Sigma} \mathcal{A} & \xrightarrow{\Phi_{\mathcal{A}}(H)} \mathcal{A}^{\otimes \text{Legs}(T)} \\ & \downarrow f_* = \partial g_* & \\ & \int_{\Sigma'} \mathcal{A} & \xrightarrow{\Phi_{\mathcal{A}}(H')} \mathcal{A}^{\otimes \text{Legs}(T)} \\ & \downarrow \mathcal{O}_{\Sigma'} & \\ & \widehat{\mathcal{A}}(H') & \end{array} = \widehat{\mathcal{A}}(g)$$

In particular, if a diffeomorphism g of Σ lies in the diffeomorphism group of some handlebody H , there is a canonical isomorphism

$$\Phi_{\mathcal{A}}(H)g_* \cong \Phi_{\mathcal{A}}(H) .$$

This endows $\Phi_{\mathcal{A}}(H)$ with the structure of a homotopy fixed point for the $\text{Diff}(H)$ -action on the category $\mathcal{S}_{\partial}(\int_{\Sigma} \mathcal{A}, \mathcal{A}^{\otimes \text{Legs}(T)})$.

4.3 The connection to skein algebras and skein modules. It was briefly mentioned in the introduction that the $\Phi_{\mathcal{A}}(H)$ -maps can be understood as generalized skein modules for handlebodies, formulated via the modular envelope and factorization homology. We will explain now why this is indeed just a different perspective on Theorem 4.2. These considerations are not vital for the rest of the article, but hopefully insightful to anyone interested in skein theory: For a framed E_2 -algebra \mathcal{A} in \mathcal{S} and a surface Σ , one defines the *skein algebra of \mathcal{A} and Σ* by

$$\text{SkAlg}_{\mathcal{A}}(\Sigma) := \text{End}(\mathcal{O}_{\Sigma}) ,$$

where the endomorphisms on the right hand side are taken in the hom category $\mathcal{S}(I, \int_{\Sigma} \mathcal{A})$. The crucial fact that this definition of the skein algebra coincides with the ‘traditional’ definition in the case where \mathcal{A} is a semisimple ribbon category follows from [Coo23].

We will now see that this implies that the value $\widehat{\mathcal{A}}(H)$ of the modular extension $\widehat{\mathcal{A}}$ of \mathcal{A} on H becomes a module over the skein algebra of $\Sigma = \partial H$. Indeed, the action is the algebra map

$$\text{SkAlg}_{\mathcal{A}}(\Sigma) \rightarrow \text{End}(\widehat{\mathcal{A}}(H))$$

(the endomorphisms on the right hand side are taken in the hom category $\mathcal{S}(I, \int_{\Sigma} \mathcal{A}^{\otimes \text{Legs}(T)})$) defined as the composition

$$\text{SkAlg}_{\mathcal{A}}(\Sigma) = \text{End}(\mathcal{O}_{\Sigma}) \xrightarrow{\Phi_{\mathcal{A}}(H) \circ -} \text{End}(\Phi_{\mathcal{A}}(H) \circ \mathcal{O}_{\Sigma}) \stackrel{\text{Theorem 4.2}}{\cong} \text{End}(\widehat{\mathcal{A}}(H)) .$$

In case \mathcal{A} is ribbon and semisimple, following [Wal, GJS23], any handlebody H with n embedded disks in its boundary $\partial H = \Sigma$ also leads to a functor

$$\mathrm{Sk}_{\mathcal{A}}(H) : \int_{\Sigma} \mathcal{A} \longrightarrow \mathcal{A}^{\otimes n}$$

whose components are given by relative skein modules. This functor tautologically also satisfies the universal property of Proposition 4.2 so that there is a canonical isomorphism $\mathrm{Sk}_{\mathcal{A}}(H) \cong \Phi_{\mathcal{A}}(H)$ in that case.

5 CONNECTED CYCLIC FRAMED E_2 -ALGEBRAS AND THEIR ASSOCIATED EXTENSIONS OF THE SURFACE OPERAD

One of the main goals of this article is to identify general necessary and sufficient conditions for a cyclic framed E_2 -algebra to extend to a modular functor. In this subsection, we encounter the main technical condition that will be relevant in this context, namely the notion of *connectedness* that will be defined in terms of handlebody skein modules. In this section, we will be occupied with studying this notion in detail; we will see why the notion is relevant in Section 6, and we will discuss how to produce examples in Section 7.

5.1 The $\Omega_{\mathcal{A}}$ -groupoids and the map $\mathrm{Surf}_{\mathcal{A}} \longrightarrow \mathrm{Surf}$ of modular operads. This subsection contains a technical construction that lies at the heart of the paper: For any cyclic framed E_2 -algebra \mathcal{A} , we will construct a modular operad $\mathrm{Surf}_{\mathcal{A}}$ (depending on \mathcal{A}) and a canonical map $\mathrm{Surf}_{\mathcal{A}} \longrightarrow \mathrm{Surf}$. This map will later on produce for us the extensions of Surf over which the modular functors will live.

Definition 5.1. Let \mathcal{A} be a cyclic framed E_2 -algebra in \mathcal{S} and T a corolla. For $\Sigma \in \mathrm{Surf}(T)$, we define

$$\Omega_{\mathcal{A}}(\Sigma) := \left\{ \begin{array}{l} \text{replete full subgroupoid of } \mathcal{S}_{\partial} \left(\int_{\Sigma} \mathcal{A}, \mathcal{A}^{\otimes \mathrm{Legs}(T)} \right) \\ \text{spanned by all module maps } \Phi_{\mathcal{A}}(H) : \int_{\Sigma} \mathcal{A} \longrightarrow \mathcal{A}^{\otimes \mathrm{Legs}(T)} \\ \text{given in Proposition 4.1 for any handlebody } H \text{ with boundary } \Sigma \\ \text{(with the module map structure from Proposition 4.3)} \end{array} \right\}. \quad (5.1)$$

Here *replete full* (as opposed to just full) means that $\Omega_{\mathcal{A}}(\Sigma)$ is closed under isomorphisms: It does not only contain all the $\Phi_{\mathcal{A}}(H)$, but also all maps $\int_{\Sigma} \mathcal{A} \longrightarrow \mathcal{A}^{\otimes \mathrm{Legs}(T)}$ isomorphic, as module map, to some $\Phi_{\mathcal{A}}(H)$. (Of course, passing from a full subcategory to its repletion does not change the subcategory up to equivalence.) The definition (5.1) is extended to the case of non-connected $T = \sqcup_{i=1}^n T^{(i)}$ by $\Omega_{\mathcal{A}}(\sqcup_{i=1}^n \Sigma_i) = \prod_{i=1}^n \Omega_{\mathcal{A}}(\Sigma_i)$ for $\Sigma_i \in \mathrm{Surf}(T^{(i)})$ for $i = 1, \dots, n$.

For the definition of $\mathrm{Surf}_{\mathcal{A}}$, recall that for a functor $F : \mathcal{C} \longrightarrow \mathrm{Cat}$, i.e. a category-valued presheaf, the *Grothendieck construction* of F is the category $\mathrm{Gr}F$ formed by pairs (c, x) , where $c \in \mathcal{C}$ and $x \in F(c)$, see [MLM92, Section I.5] for a textbook reference. A morphism $(c, x) \longrightarrow (c', x')$ is a morphism $f : c \longrightarrow c'$ in \mathcal{C} together with a morphism $\alpha : F(f)x \longrightarrow x'$ in $F(c')$. The Grothendieck construction comes with a canonical projection functor $\mathrm{Gr}F \longrightarrow \mathcal{C}$. (Often, the Grothendieck construction of F is denoted by $\int F$. In this article, we will not use the integral sign for the Grothendieck construction because of the notational conflict with factorization homology.) There is also a version of the Grothendieck construction for functors $F : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Cat}$, i.e. for category-valued presheaves. It is defined as the category $\mathrm{Gr}F$ formed by pairs (c, x) , where $c \in \mathcal{C}$ and $x \in F(c)$. A morphism $(c, x) \longrightarrow (c', x')$ is a morphism $f : c \longrightarrow c'$ in \mathcal{C} together with a morphism $\alpha : x \longrightarrow F(f)x'$ in $F(c)$. It comes again with a projection functor $\mathrm{Gr}F \longrightarrow \mathcal{C}$ (not to $\mathcal{C}^{\mathrm{op}}$).

Theorem 5.2. For any cyclic framed E_2 -algebra \mathcal{A} in \mathcal{S} and $T \in \mathrm{Graphs}$, the assignment

$$\mathrm{Surf}_{\mathcal{A}} : \mathrm{Graphs} \longrightarrow \mathrm{Cat}, \quad T \longmapsto \mathrm{Surf}_{\mathcal{A}}(T) := \mathrm{Gr}(\mathrm{Surf}(T)^{\mathrm{op}} \longrightarrow \mathrm{Cat}), \quad \Sigma \longmapsto \Omega_{\mathcal{A}}(\Sigma) \quad (5.2)$$

yields a groupoid-valued modular operad $\mathrm{Surf}_{\mathcal{A}}$ that comes with a map $p_{\mathcal{A}} : \mathrm{Surf}_{\mathcal{A}} \longrightarrow \mathrm{Surf}$ and a map $s_{\mathcal{A}} : \mathrm{Hbdy} \longrightarrow \mathrm{Surf}_{\mathcal{A}}$ of modular operads over Surf .

From the statement of the result, it is not supposed to be obvious how $\Sigma \longmapsto \Omega_{\mathcal{A}}(\Sigma)$ is a functor (which is needed to form the Grothendieck construction). It is certainly clear how to define it on surfaces and diffeomorphisms, but it is non-trivial that only the mapping classes are detected. This will be covered in the proof.

Proof. (i) We need some preliminary considerations: Let us recall from Remark 2.4 the topological modular operad SURF of surfaces and *diffeomorphisms* as opposed to mapping classes. Clearly, we have for every corolla T a functor

$$\mathcal{S}_{\partial} \left(\int_{-} \mathcal{A}, \mathcal{A}^{\otimes \mathrm{Legs}(T)} \right) : \mathrm{SURF}^{\mathrm{op}}(T) \longrightarrow \mathrm{Grpd}, \quad \Sigma \longmapsto \mathcal{S}_{\partial} \left(\int_{\Sigma} \mathcal{A}, \mathcal{A}^{\otimes \mathrm{Legs}(T)} \right). \quad (5.3)$$

This gives us also a functor

$$\Omega_{\mathcal{A}} : \text{SURF}^{\text{op}}(T) \longrightarrow \text{Grpd} , \quad \Sigma \longmapsto \Omega_{\mathcal{A}}(\Sigma) ,$$

by restriction in range. To see that this is true, we only need to verify that the $\Omega_{\mathcal{A}}$ -subgroupoids $\Omega_{\mathcal{A}}(\Sigma) \subset \mathcal{S}_{\partial}(\int_{\Sigma} \mathcal{A}, \mathcal{A}^{\otimes \text{Legs}(T)})$ are stable under the precomposition with diffeomorphisms, but this follows from their definition and Proposition 4.6.

- (ii) The key step in the proof of the result will be the following auxiliary statement: There is a topological modular operad $\mathcal{G}_{\mathcal{A}}$ sending a corolla T to the Grothendieck construction

$$(\mathcal{G}_{\mathcal{A}})(T) := \text{Gr} \left(\text{SURF}^{\text{op}}(T) \xrightarrow{\mathcal{S}_{\partial} \left(\int_{-} \mathcal{A}, \mathcal{A}^{\otimes \text{Legs}(T)} \right)} \text{Grpd} \right) . \quad (5.4)$$

of the functor (5.3). For this definition, we need a Grothendieck construction which is more general than the 1-categorical one recalled above, but this is available; the case needed here is covered e.g. by the treatment in [Lur09a, Chapter 3.2].

The construction of this modular operad is rather involved: First of all, we extend the prescription (5.4) to non-connected objects in **Graphs** by sending disjoint unions of corollas to products of groupoids. Next we need to extend the definition of $\mathcal{G}_{\mathcal{A}}$ to morphisms in **Graphs**. Let $\Gamma : T \longrightarrow T'$ be a morphism in **Graphs**. Without loss of generality, we can assume that T' is connected, i.e. a corolla. We write the decomposition of T into corollas as $T = \sqcup_{i=1}^n T^{(i)}$. The image $(\mathcal{G}_{\mathcal{A}})(\Gamma)$ of Γ under $\mathcal{G}_{\mathcal{A}}$, that we now want to construct, is by definition a functor

$$(\mathcal{G}_{\mathcal{A}})(\Gamma) : \prod_{i=1}^n \text{Gr} \left(\text{Surf}^{\text{op}}(T) \xrightarrow{\mathcal{S}_{\partial} \left(\int_{-} \mathcal{A}, \mathcal{A}^{\otimes \text{Legs}(T^{(i)})} \right)} \text{Grpd} \right) \longrightarrow \text{Gr} \left(\text{Surf}^{\text{op}}(T) \xrightarrow{\mathcal{S}_{\partial} \left(\int_{-} \mathcal{A}, \mathcal{A}^{\otimes \text{Legs}(T')} \right)} \text{Grpd} \right) .$$

We define this functor as follows: An object on the left consists of a surface $\Sigma = \sqcup_{i=1}^n \Sigma_i \in \text{Surf}(T)$ and a family $\Psi = (\Psi_i)_{1 \leq i \leq n}$ of maps $\Psi_i \in \mathcal{S}_{\partial} \left(\int_{\Sigma_i} \mathcal{A}, \mathcal{A}^{\otimes \text{Legs}(T^{(i)})} \right)$. Since **Surf** is already a modular operad, we may evaluate **Surf** on Γ and obtain a functor $\Gamma_* : \text{Surf}(T) \longrightarrow \text{Surf}(T')$ sending Σ to $\Gamma_*\Sigma$. We will accomplish the definition of $(\mathcal{G}_{\mathcal{A}})(\Gamma)$ by assigning to Ψ an object $\Gamma_*\Psi \in \mathcal{S}_{\partial} \left(\int_{\Gamma_*\Sigma} \mathcal{A}, \mathcal{A}^{\otimes \text{Legs}(T')} \right)$. Then we can define

$$(\mathcal{G}_{\mathcal{A}})(\Gamma)(\Sigma, \Psi) := (\Gamma_*\Sigma, \Gamma_*\Psi) . \quad (5.5)$$

The definition of $\Gamma_*\Psi$ is the hard part and done as follows: First we observe that Γ induces a map

$$\bigotimes_{i=1}^n \mathcal{A}^{\otimes \text{Legs}(T^{(i)})} \longrightarrow \mathcal{A}^{\otimes \text{Legs}(T')} ; \quad (5.6)$$

this is *dual* to constructions used for the definition of the modular endomorphism operad in [MW23b, Section 2.3]. Essentially, this map performs cyclic permutations and contracts copies of \mathcal{A} via the pairing $\kappa : \mathcal{A} \otimes \mathcal{A} \longrightarrow I$ — just as prescribed by Γ . As a consequence, we obtain the following map:

$$\prod_{i=1}^n \mathcal{S}_{\partial} \left(\int_{\Sigma_i} \mathcal{A}, \mathcal{A}^{\otimes \text{Legs}(T^{(i)})} \right) \xrightarrow{\otimes} \mathcal{S}_{\partial} \left(\bigotimes_{i=1}^n \int_{\Sigma_i} \mathcal{A}, \bigotimes_{i=1}^n \mathcal{A}^{\otimes \text{Legs}(T^{(i)})} \right) \xrightarrow{(5.6)} \mathcal{S}_{\partial} \left(\bigotimes_{i=1}^n \int_{\Sigma_i} \mathcal{A}, \mathcal{A}^{\otimes \text{Legs}(T')} \right) \quad (5.7)$$

The ‘ ∂ ’ reminds us of the restriction to module maps with respect to the boundary, which means precisely:

- For the first step, these are maps of $\int_{\partial \Sigma_i \times [0,1]} \mathcal{A}$ -modules in each factor.
- For the second step, these are maps of $\int_{\partial \Sigma \times [0,1]} \mathcal{A}$ -modules.
- For the third step, these are maps of $\int_{\partial \Gamma_* \Sigma \times [0,1]} \mathcal{A}$ -modules, where $\bigotimes_{i=1}^n \int_{\Sigma_i} \mathcal{A}$ is seen as $\int_{\partial \Gamma_* \Sigma \times [0,1]} \mathcal{A}$ -module by restriction along the algebra map

$$\int_{\partial \Gamma_* \Sigma \times [0,1]} \mathcal{A} \longrightarrow \int_{\partial \Sigma \times [0,1]} \mathcal{A}$$

induced by $\partial \Gamma_* \Sigma \subset \partial \Sigma$.

In order to obtain the desired map $\Gamma_*\Psi : \int_{\Gamma_*\Sigma} \mathcal{A} \longrightarrow \mathcal{A}^{\otimes \text{Legs}(T')}$, we need to show that the image of Ψ under (5.7) that we will denote by Ψ^κ actually factors naturally through the map $\bigotimes_{i=1}^n \int_{\Sigma_i} \mathcal{A} \longrightarrow \int_{\Gamma_*\Sigma} \mathcal{A}$ induced by the gluing prescribed by Γ . To this end, we will use the excision property of factorization homology which, in the case at hand, tells us the following: If Γ prescribes a gluing of r pairs of boundary components, then $\bigotimes_{i=1}^n \int_{\Sigma_i} \mathcal{A}$ is a module over $\left(\int_{\mathbb{S}^1 \times [0,1]} \mathcal{A}\right)^{\otimes r}$ in two ways; we denote the actions by $\alpha_\ell : \left(\int_{\mathbb{S}^1 \times [0,1]} \mathcal{A}\right)^{\otimes r} \otimes \bigotimes_{i=1}^n \int_{\Sigma_i} \mathcal{A} \longrightarrow \bigotimes_{i=1}^n \int_{\Sigma_i} \mathcal{A}$, $\ell = 1, 2$. By the excision property of factorization homology, $\int_{\Gamma_*\Sigma} \mathcal{A}$ is the homotopy coequalizer of α_1 and α_2 . Hence, we need to show that $\Psi^\kappa : \bigotimes_{i=1}^n \int_{\Sigma_i} \mathcal{A} \longrightarrow \mathcal{A}^{\otimes \text{Legs}(T')}$ coequalizes α_1 and α_2 up to isomorphism (this is structure and not just a property). To see this, we first observe that $\bigotimes_{i=1}^n \mathcal{A}^{\otimes \text{Legs}(T^{(i)})}$ also comes with two actions of $\left(\int_{\mathbb{S}^1 \times [0,1]} \mathcal{A}\right)^{\otimes r}$, namely by action on the first and the second copy of \mathcal{A} associated to a gluing pair. We denote these two actions β_1 and β_2 . Now consider the diagram:

$$\begin{array}{ccccc}
\left(\int_{\mathbb{S}^1 \times [0,1]} \mathcal{A}\right)^{\otimes r} \otimes \bigotimes_{i=1}^n \int_{\Sigma_i} \mathcal{A} & \xrightarrow{\alpha_\ell} & \bigotimes_{i=1}^n \int_{\Sigma_i} \mathcal{A} & \xrightarrow{\Psi^\kappa} & \mathcal{A}^{\otimes \text{Legs}(T')} \\
\text{id} \otimes \bigotimes_{i=1}^n \Psi_i \downarrow & & \downarrow \bigotimes_{i=1}^n \Psi_i & \nearrow & \\
\left(\int_{\mathbb{S}^1 \times [0,1]} \mathcal{A}\right)^{\otimes r} \otimes \bigotimes_{i=1}^n \mathcal{A}^{\otimes \text{Legs}(T^{(i)})} & \xrightarrow{\beta_\ell} & \bigotimes_{i=1}^n \mathcal{A}^{\otimes \text{Legs}(T^{(i)})} & &
\end{array} \quad (5.6)$$

The left square commutes by a canonical isomorphism (this is part of data that makes the Ψ_i module maps). The triangle on the right commutes by definition. The needed isomorphism $\Psi^\kappa \alpha_1 \cong \Psi^\kappa \alpha_2$ can now be obtained by evaluating the above diagram at $\ell = 1$ and $\ell = 2$ and by combining these two cases. This uses additionally the fact that (5.6) homotopy coequalizes β_1 and β_2 thanks to the cyclicity of the pairing. This completes the construction of $\Gamma_*\Psi$.

Having completed the definition of $\Gamma_*\Psi$, we can define $(\mathcal{G}_\mathcal{A})(\Gamma)(\Sigma, \Psi)$ via (5.5). All of the constructions so far are functorial in module maps. Therefore, the definition extends to morphisms, thereby giving us the functor $(\mathcal{G}_\mathcal{A})(\Gamma)$. This concludes the construction of $\mathcal{G}_\mathcal{A}$ as modular operad. The canonical projection to the base of the Grothendieck construction yields a morphism $\mathcal{G}_\mathcal{A} \longrightarrow \text{SURF}$ of modular operads.

- (iii) Next we observe that we can perform the Grothendieck construction (5.4) also with the $\Omega_\mathcal{A}$ -groupoids instead of the $\mathcal{S}_\partial \left(\int_- \mathcal{A}, \mathcal{A}^{\otimes \text{Legs}(T)}\right)$ -groupoids (these are full subgroupoids). Then all of the constructions in (ii) restrict in range and give us a new topological modular operad $\mathcal{G}_\mathcal{A}^\Omega$ with a map $\mathcal{G}_\mathcal{A}^\Omega \longrightarrow \mathcal{G}_\mathcal{A}$. This is because the $\Omega_\mathcal{A}$ -subgroupoids are stable under compositions and cyclic permutations. The latter is obvious, the former follows from the fact that the gluing of module maps $\int_\Sigma \mathcal{A} \longrightarrow \mathcal{A}^{\otimes \text{Legs}(T)}$, when applied to maps of the form $\Phi_\mathcal{A}(H)$, translates to the gluing of handlebodies. More precisely, the functor $(\mathcal{G}_\mathcal{A})(\Gamma)$ associated to a morphism $\Gamma : T \longrightarrow T'$ in **Graphs** sends $\Phi_\mathcal{A}(H)$ for $H \in \text{Hbdy}(T)$ to $\Phi_\mathcal{A}(\Gamma_*H)$. This is a consequence of the construction of the $\Phi_\mathcal{A}$ -maps in Proposition 4.1 and properties of the modular extension $\widehat{\mathcal{A}}$.
- (iv) So far we have maps $\mathcal{G}_\mathcal{A}^\Omega \longrightarrow \mathcal{G}_\mathcal{A} \longrightarrow \text{SURF}$ of modular operads. After applying the symmetric monoidal fundamental groupoid functor, we obtain a map $\Pi \mathcal{G}_\mathcal{A}^\Omega \longrightarrow \text{Surf}$ of groupoid-valued operads. Now we define $\text{Surf}_\mathcal{A}$ via

$$\text{Surf}_\mathcal{A} := \Pi \mathcal{G}_\mathcal{A}^\Omega, \quad (5.8)$$

thereby giving us a map $p_\mathcal{A} : \text{Surf}_\mathcal{A} \longrightarrow \text{Surf}$ of modular operads. It is however not obvious that this definition of $\text{Surf}_\mathcal{A}$ has arity-wise the claimed description (5.2). Clearly, we can write $\text{Surf}_\mathcal{A}(T)$ again as a Grothendieck construction over the homotopy fibers of $p_\mathcal{A} : \text{Surf}_\mathcal{A} \longrightarrow \text{Surf}$, see [Hol08], but it is unclear that the homotopy fiber over $\Sigma \in \text{Surf}(T)$ is actually the groupoid $\Omega_\mathcal{A}(\Sigma)$. By construction the homotopy fiber of $\mathcal{G}_\mathcal{A}^\Omega \longrightarrow \text{SURF}$ over Σ is $\Omega_\mathcal{A}(\Sigma)$, but the homotopy fiber will generally not commute with taking the fundamental groupoid. To see that $\Omega_\mathcal{A}(\Sigma)$ is indeed the fiber of $p_\mathcal{A}$ over Σ , we look at the long exact sequence of the homotopy groups for the fibration $\mathcal{G}_\mathcal{A}^\Omega(T) \longrightarrow \text{SURF}(T)$. To this end, choose a base point in the fiber over Σ , i.e. some $\Phi_\mathcal{A}(H)$ with $\partial H = \Sigma$. With the definition (5.8), we find the exact sequence

$$\begin{array}{ccccccc}
1 & \longrightarrow & \pi_2(\mathcal{G}_{\mathcal{A}}^{\Omega}(T), \Phi_{\mathcal{A}}(T)) & \longrightarrow & \pi_1(\text{Diff}_0(\Sigma)) & \longrightarrow & \\
\longleftarrow & & \longleftarrow & & \longleftarrow & & \\
\pi_1(\Omega_{\mathcal{A}}(\Sigma), \Phi_{\mathcal{A}}(H)) & \longrightarrow & \pi_1(\text{Surf}_{\mathcal{A}}(T), \Phi_{\mathcal{A}}(H)) & \longrightarrow & \text{Map}(\Sigma) & \longrightarrow & \\
\longleftarrow & & \longleftarrow & & \longleftarrow & & \\
\pi_0(\Omega_{\mathcal{A}}(\Sigma)) & \longrightarrow & \pi_0(\text{Surf}_{\mathcal{A}}(T)) & \longrightarrow & \pi_0(\text{Surf}(T)) & \longrightarrow &
\end{array}$$

where $\text{Diff}_0(\Sigma)$ is the unit component of the topological group $\text{Diff}(\Sigma)$. For $\Omega_{\mathcal{A}}(\Sigma)$ to be the homotopy fiber of $p_{\mathcal{A}}$ over Σ , we need $\pi_1(\Omega_{\mathcal{A}}(\Sigma), \Phi_{\mathcal{A}}(H)) \rightarrow \pi_1(\text{Surf}_{\mathcal{A}}(T), \Phi_{\mathcal{A}}(H))$ to be injective. By exactness this is the case if and only if $\pi_1(\text{Diff}_0(\Sigma)) = \pi_2(\text{SURF}(T), \Sigma) \rightarrow \pi_1(\Omega_{\mathcal{A}}(\Sigma), \Phi_{\mathcal{A}}(H))$ is trivial. This map is indeed trivial because the map $\text{Diff}_0(H) \rightarrow \text{Diff}_0(\Sigma)$ is an equivalence (this is a consequence of [Hat76, Theorem 2], see also the comments in [MW23c, Section 2]) and because $\Phi_{\mathcal{A}}(H)$ is a homotopy fixed point under the $\text{Diff}(H)$ -action by Corollary 4.7. This tells us that $p_{\mathcal{A}} : \text{Surf}_{\mathcal{A}} \rightarrow \text{Surf}$ is indeed a map with homotopy fiber $\Omega_{\mathcal{A}}(\Sigma)$ over Σ , thereby justifying (5.2). \square

5.2 Connectedness of the $\Omega_{\mathcal{A}}$ -groupoids: General connectedness and reduction to genus one.

We will now investigate the connectedness of the $\Omega_{\mathcal{A}}$ -groupoids. This will inform the definition of a connected cyclic framed E_2 -algebra in the next subsection.

To this end, we need to establish some terminology: With the presentation of $\text{Map}(\Sigma)$ given in [FM12, Section 4.4.4] we can find two handlebodies H and H' such that the subgroups $\text{Map}(H)$ and $\text{Map}(H')$ together generate $\text{Map}(\Sigma)$. We call such a pair (H, H') of handlebodies a *generating pair of handlebodies with boundary Σ* .

Lemma 5.3. *For a cyclic framed E_2 -algebra \mathcal{A} in \mathcal{S} , the groupoid $\Omega_{\mathcal{A}}(\Sigma)$ is connected if and only if for some generating pair (H, H') of handlebodies with boundary Σ the $\int_{\partial\Sigma \times [0,1]}$ \mathcal{A} -module maps $\Phi_{\mathcal{A}}(H)$ and $\Phi_{\mathcal{A}}(H')$ are isomorphic as module maps.*

Proof. If $\Omega_{\mathcal{A}}(\Sigma)$ is connected, then clearly $\Phi_{\mathcal{A}}(H) \cong \Phi_{\mathcal{A}}(H')$ for any two handlebodies with boundary Σ (simply by definition). Let us prove the converse: After the choice of an isomorphism $u : H \rightarrow H'$, we have a group isomorphism

$$\text{Map}(H) \longrightarrow \text{Map}(H'), \quad a \longmapsto uau^{-1}.$$

With $s := \partial u \in \text{Map}(\Sigma)$, we may now without loss of generality assume that $H' = s.H$, see Remark 2.8. For arbitrary handlebody \tilde{H} with boundary Σ , we now need to show $\Phi_{\mathcal{A}}(\tilde{H}) \cong \Phi_{\mathcal{A}}(H)$ as module maps. First note that \tilde{H} can be written as $\tilde{H} = f.H$ for some $f \in \text{Map}(\Sigma)$. Since $\text{Map}(H)$ and $\text{Map}(H')$ generate $\text{Map}(\Sigma)$ and since $\text{Map}(H) \cong \text{Map}(H')$ via conjugation with u , we can write $f = a_1ub_1u^{-1} \dots a_nub_nu^{-1}$ with $a_1, b_1, \dots, a_n, b_n \in \text{Map}(H)$. With Proposition 4.6, we find

$$\Phi_{\mathcal{A}}(\tilde{H}) = \Phi_{\mathcal{A}}(f.H) \cong \Phi_{\mathcal{A}}(H)f_*^{-1} \cong \Phi_{\mathcal{A}}(H)s_*\partial b_n^{-1} s_*^{-1}\partial a_n^{-1} \dots s_*\partial b_1^{-1} s_*^{-1}\partial a_1^{-1},$$

where \cong is always an isomorphism of module maps. For this reason, it suffices to prove:

$$\Phi_{\mathcal{A}}(H)s_* \cong \Phi_{\mathcal{A}}(H), \quad (5.9)$$

$$\Phi_{\mathcal{A}}(H)s_*^{-1} \cong \Phi_{\mathcal{A}}(H), \quad (5.10)$$

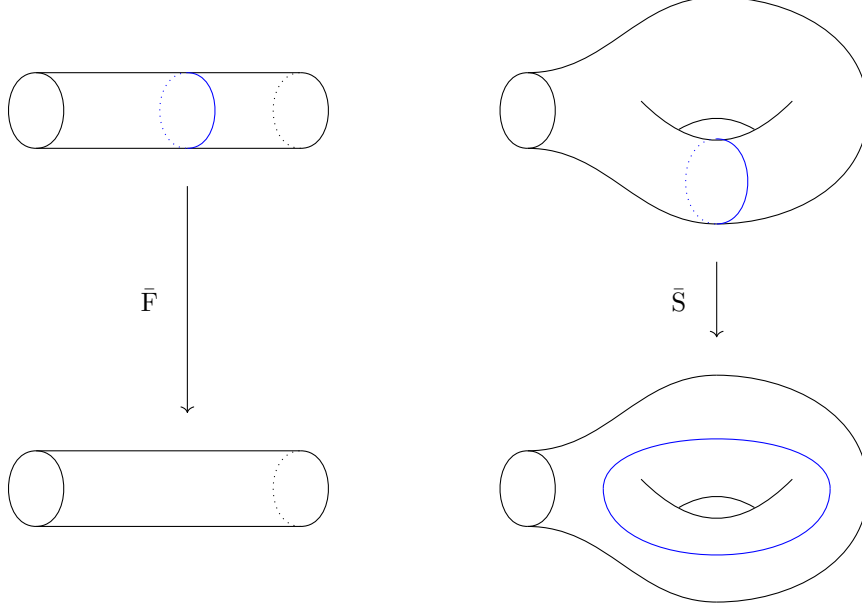
$$\Phi_{\mathcal{A}}(H)\partial a_* \cong \Phi_{\mathcal{A}}(H) \quad \text{for all } a \in \text{Map}(H). \quad (5.11)$$

First we use Proposition 4.6 to conclude $\Phi_{\mathcal{A}}(H') = \Phi_{\mathcal{A}}(s.H) = \Phi_{\mathcal{A}}(H)s_*^{-1}$. With $\Phi_{\mathcal{A}}(H') \cong \Phi_{\mathcal{A}}(H)$ — which holds by assumption — this yields $\Phi_{\mathcal{A}}(H) \cong \Phi_{\mathcal{A}}(H)s_*^{-1}$ and proves (5.10). After applying s_* from the right to both sides, we obtain (5.9). Finally, (5.11) is a consequence of the homotopy $\text{Diff}(H)$ -fixed point structure of $\Phi_{\mathcal{A}}(H)$, see Corollary 4.7. \square

Proposition 5.4. *For a cyclic framed E_2 -algebra \mathcal{A} in \mathcal{S} , the following conditions are equivalent:*

- (i) All groupoids $\Omega_{\mathcal{A}}(\Sigma)$, where Σ runs over all surfaces, are connected.
- (ii) For some generating pair (H, H') of handlebodies whose boundary is the torus \mathbb{T}_1^2 with one boundary component, the handlebody skein modules $\Phi_{\mathcal{A}}(H)$ and $\Phi_{\mathcal{A}}(H')$ are isomorphic as $\int_{\mathbb{S}^1 \times [0,1]}$ \mathcal{A} -module maps.

Proof. Clearly, (i) \Rightarrow (ii). Let us prove (ii) \Rightarrow (i): By Lemma 5.3 the condition (ii) implies that $\Omega_{\mathcal{A}}(\mathbb{T}_1^2)$ is connected. Therefore, it remains to prove that connectedness of $\Omega_{\mathcal{A}}(\mathbb{T}_1^2)$ implies connectedness of $\Omega_{\mathcal{A}}(\Sigma)$ for all surfaces Σ . For this, we will make use of the Lego Teichmüller game from [BK00]. For each surface Σ , one can define a groupoid $\mathcal{C}(\Sigma)$ of cut systems of Σ . Roughly, these are pair of pants decompositions; for details, we refer to [BK00, Section 7.1-7.3]. The morphisms are generated by the following two moves that can be applied to subsurfaces (the cuts are displayed in blue):



The \bar{F} -move deletes a cut in a cylindrical region, assuming of course that we are still left with a cut system afterwards. The \bar{S} -move replaces a cut on \mathbb{T}_1^2 with a transversal one. If we freely generate a groupoid with these moves, we obtain a groupoid $\tilde{\mathcal{C}}(\Sigma)$. The actual groupoid $\mathcal{C}(\Sigma)$ of cut system is then obtained by imposing further relations. These will however not be relevant for us. For us, it suffices to know that $\mathcal{C}(\Sigma)$ is connected and simply connected [BK00, Theorem 7.9]. This implies that $\tilde{\mathcal{C}}(\Sigma)$ is connected. We will now construct a functor

$$T : \tilde{\mathcal{C}}(\Sigma) \longrightarrow \Omega_{\mathcal{A}}(\Sigma) . \quad (5.12)$$

For a cut system $C \in \tilde{\mathcal{C}}(\Sigma)$, we take the genus zero surface Σ_C obtained by cutting Σ at C . We fill each component of Σ_C with a handlebody; this choice is essentially unique because $\mathbf{Hbdy} \rightarrow \mathbf{Surf}$ is an equivalence in genus zero. This gives us a handlebody H_C with $\partial H_C = \Sigma_C$. The cut system describes a gluing operation from Σ_C to Σ , i.e. $\Sigma = \cup \Sigma_C$, or more formally, $\Sigma = \Gamma_* \Sigma_C$ for a morphism Γ in \mathbf{Graphs} described by C . The operadic composition of $\mathbf{Surf}_{\mathcal{A}}$ sends $\Phi_{\mathcal{A}}(H_C)$ to an object that we define to be $T(C)$. Thanks to step (iii) in the proof of Theorem 5.2, we know

$$T(C) \cong \Phi_{\mathcal{A}}(\cup H_C) . \quad (5.13)$$

Every handlebody with boundary Σ can be obtained as $\cup H_C$ for some cut system C for Σ . Thanks to 5.13, this makes T , so far just as an assignment on object level, essentially surjective. To an \bar{F} -move $C \rightarrow C'$, we associate an isomorphism $T(C) \cong T(C')$ that we obtain from (5.13) and the fact that $\cup H_C = \cup H_{C'}$. In order to define T an \bar{S} -move $C \rightarrow C'$, we can, thanks to the compatibility of the $\Phi_{\mathcal{A}}$ -maps with gluing (again, this is step (iii) in the proof of Theorem 5.2 and (5.13)), assume that $\Sigma = \mathbb{T}_1^2$. In that case, we just assign to $C \rightarrow C'$ some morphism $T(C) \rightarrow T(C')$ existing by the assumption that $\Omega_{\mathcal{A}}(\mathbb{T}_1^2)$ is connected. In $\tilde{\mathcal{C}}(\Sigma)$ we do not impose further relations on the moves. Therefore, the assignments provide us with the functor (5.12). Since $\tilde{\mathcal{C}}(\Sigma)$ is connected and T , as already remarked, is essentially surjective, $\Omega_{\mathcal{A}}(\Sigma)$ must be connected. \square

5.3 The notion of connectedness. Having established Proposition 5.4, we can now finally define:

Definition 5.5. We call a cyclic framed E_2 -algebra \mathcal{A} or, equivalently, a self-dual balanced braided algebra in \mathcal{S} *connected* if any of the two equivalent conditions from Proposition 5.4 hold.

With the results and constructions of the previous two subsection, we can now see easily that, for any connected cyclic framed E_2 -algebra \mathcal{A} , the map $\mathbf{Surf}_{\mathcal{A}} \rightarrow \mathbf{Surf}$ is an extension. In fact, we will then see later in Theorem 6.4 that these extensions are universal in a precise sense.

Theorem 5.6. *For a cyclic framed E_2 -algebra \mathcal{A} in \mathcal{S} , the map $p_{\mathcal{A}} : \text{Surf}_{\mathcal{A}} \rightarrow \text{Surf}$ is an extension if and only if \mathcal{A} is connected. In that case, the extension of $\text{Map}(\Sigma)$ provided by $\text{Surf}_{\mathcal{A}}$ takes the form of a short exact sequence of groups*

$$1 \rightarrow \text{Aut}_{\partial}(\Phi_{\mathcal{A}}(H)) \rightarrow \text{Map}_{\mathcal{A}}(\Sigma) \rightarrow \text{Map}(\Sigma) \rightarrow 1 \quad \text{with} \quad \text{Map}_{\mathcal{A}}(\Sigma) = \pi_1(\text{Surf}_{\mathcal{A}}(T), \Sigma),$$

where $\text{Aut}_{\partial}(\Phi_{\mathcal{A}}(H))$ denotes the group of automorphisms of $\Phi_{\mathcal{A}}(H) : \int_{\Sigma} \mathcal{A} \rightarrow \mathcal{A}^{\otimes \text{Legs}(T)}$ as maps of $\int_{\partial \Sigma \times [0,1]} \mathcal{A}$ -modules. The action of every element of $\text{Map}(\Sigma)$ on $\text{Aut}_{\partial}(\Phi_{\mathcal{A}}(H))$ determined by this short exact sequence is by an inner automorphism.

Proof. It follows from (5.2) immediately that $p_{\mathcal{A}}$ is an extension if and only if are groupoids $\Omega_{\mathcal{A}}(\Sigma)$ are connected. In that case, the resulting extension of mapping class groups takes the form $1 \rightarrow \text{Aut}_{\partial}(\Phi_{\mathcal{A}}(H)) \rightarrow \text{Map}_{\mathcal{A}}(\Sigma) \rightarrow \text{Map}(\Sigma) \rightarrow 1$ (this just amounts to specializing the sequence (3.1) to the case at hand).

It remains to prove that every element in $\text{Map}(\Sigma)$ acts on $\text{Aut}_{\partial}(\Phi_{\mathcal{A}}(H)) = \pi_1(\Omega_{\mathcal{A}}(\Sigma), \Phi_{\mathcal{A}}(H))$ by an inner automorphism. To this end, it suffices by Remark 3.2 to prove that the $\text{Map}(\Sigma)$ -action on $\Omega_{\mathcal{A}}(\Sigma)$ is (the one that it obtains by being the homotopy fiber of $p_{\mathcal{A}} : \text{Surf}_{\mathcal{A}} \rightarrow \text{Surf}$), for each mapping class separately, trivializable. For this, in turn, we just need to prove that the $\text{Map}(H)$ -action on $\Omega_{\mathcal{A}}(\Sigma)$ is trivializable for every handlebody H with $\partial H = \Sigma$ (of course, this relies on the fact that the handlebody groups generate the mapping class groups of surfaces). This can be seen as follows: Consider the inclusion functor

$$\chi_H : \star // \text{Aut}_{\partial}(\Phi_{\mathcal{A}}(H)) \rightarrow \Omega_{\mathcal{A}}(\partial H) = \Omega_{\mathcal{A}}(\Sigma) \quad (5.14)$$

that sends \star to $\Phi_{\mathcal{A}}(H)$ and is defined tautologically on morphisms. If we equip $\star // \text{Aut}_{\partial}(\Phi_{\mathcal{A}}(H))$ with the trivial $\text{Map}(H)$ -action, then χ_H is $\text{Map}(H)$ -equivariant up to coherent isomorphism as follows from the homotopy $\text{Diff}(H)$ -fixed point structure of $\Phi_{\mathcal{A}}(H)$ from Corollary 4.7. By construction χ_H is fully faithful. Since $\Omega_{\mathcal{A}}(\Sigma)$ is assumed to be connected, it is also essentially surjective and hence an equivalence. The existence of a $\text{Map}(H)$ -equivariant equivalence from a category with trivial $\text{Map}(H)$ -action to $\Omega_{\mathcal{A}}(\Sigma)$ proves that the $\text{Map}(H)$ -action on $\Omega_{\mathcal{A}}(\Sigma)$ is trivial up to coherent isomorphism (again, let us emphasize that the $\text{Map}(\Sigma)$ -action on $\Omega_{\mathcal{A}}(\Sigma)$ will be trivial in a possibly non-coherent way). \square

In Definition 3.3 we single out the admissible extensions of Surf that we will be interested in. This leads to the obvious question whether the extension $\text{Surf}_{\mathcal{A}}$ from Theorem 5.2 fits into this framework. This is answered affirmatively in the next result.

Theorem 5.7. *Let \mathcal{A} be a connected cyclic framed E_2 -algebra. Then $p_{\mathcal{A}} : \text{Surf}_{\mathcal{A}} \rightarrow \text{Surf}$ is in a canonical way an extension of Surf relative Hbdy that admits insertions of vacua in the sense of Definition 3.3, i.e. we have $\text{Surf}_{\mathcal{A}} \in \text{Ext}^{\circ}(\text{Surf})$.*

Proof. By Theorem 5.2 the connectedness assumption ensures that $\text{Surf}_{\mathcal{A}}$ is an extension. Moreover, the pullback $\text{Surf}_{\mathcal{A}} \times_{\text{Surf}} \text{Hbdy} \rightarrow \text{Hbdy}$ comes with a canonical trivialization, i.e. with a canonical equivalence to a trivial extension of Hbdy : On the fiber over a handlebody H , the trivialization is given by the $\text{Map}(H)$ -equivariant equivalence

$$\chi : \star // \text{Aut}_{\partial}(\Phi_{\mathcal{A}}(H)) \rightarrow \Omega_{\mathcal{A}}(\partial H) \quad (5.15)$$

from (5.14). It is straightforward to see that the maps (5.15) combine into an equivalence of modular operads over Hbdy .

It remains to establish the insertion of vacua property. This will be implied by Proposition 5.9 that follows below. We formulate it as a separate statement because we would like to prove a slightly stronger statement that does not depend on the connectedness of the $\Omega_{\mathcal{A}}$ -groupoids. \square

Remark 5.8. The map $\text{Hbdy} \rightarrow \text{Surf}_{\mathcal{A}}$ that the trivialization over Hbdy gives rise to thanks to (3.3) agrees with the map $s_{\mathcal{A}} : \text{Hbdy} \rightarrow \text{Surf}_{\mathcal{A}}$ already given in Theorem 5.2.

Proposition 5.9 (Insertion of vacua). *Let \mathcal{A} be a cyclic framed E_2 -algebra in \mathcal{S} , $\Sigma \in \text{Surf}(T)$ for $T \in \text{Graphs}$ and $\varphi : \sqcup_J \mathbb{D}^2 \rightarrow \Sigma$ an oriented embedding. Then*

$$\Omega_{\mathcal{A}}(\Sigma \setminus \text{im } \varphi) \xrightarrow{\text{id} \times \Phi_{\mathcal{A}}(\mathbb{B}_1^3)} \Omega_{\mathcal{A}}(\Sigma \setminus \text{im } \varphi) \times \Omega_{\mathcal{A}}(\mathbb{D}^2)^{\times J} \xrightarrow{\text{gluing}} \Omega_{\mathcal{A}}(\Sigma) \quad (5.16)$$

is an equivalence.

Proof. Without loss of generality, we can assume that T is a corolla. For any handlebody H with boundary Σ , we can form the handlebody H^φ which has additional embedded disks in its boundary coming from φ ; we have used this notation on page 19 already. On the object level, (5.16) sends $\Phi_{\mathcal{A}}(H^\varphi) \mapsto \Phi_{\mathcal{A}}(H)$, thereby making the functor essentially surjective. In order to see that (5.16) is also fully faithful, we observe that it is given by the restriction in domain and range of the equivalence

$$\begin{aligned} \mathcal{S}_\partial \left(\int_{\Sigma \setminus \text{im } \varphi} \mathcal{A}, \mathcal{A}^{\otimes \text{Legs}(T)} \otimes \mathcal{A}^{\otimes J} \right) &\simeq \mathcal{S}_\partial \left(\int_{\Sigma \setminus \text{im } \varphi} \mathcal{A} \otimes \left(\int_{\mathbb{S}^1 \times [0,1]} \mathcal{A} \right)^{\otimes J} \mathcal{A}^{\otimes J}, \mathcal{A}^{\otimes \text{Legs}(T)} \right) \\ &\simeq \mathcal{S}_\partial \left(\int_{\Sigma} \mathcal{A}, \mathcal{A}^{\otimes \text{Legs}(T)} \right) \end{aligned}$$

which uses the pairing and excision. \square

6 THE CONSTRUCTION AND CLASSIFICATION OF MODULAR FUNCTORS

The goal of this section is to give a general construction procedure for modular functors and to ultimately use this construction to classify modular functors.

6.1 The construction of modular functors. As a first result, we will construct a modular $\text{Surf}_{\mathcal{A}}$ -algebra for any cyclic framed E_2 -algebra \mathcal{A} . This modular $\text{Surf}_{\mathcal{A}}$ -algebra will be a modular functor if and only if \mathcal{A} is connected.

Definition 6.1. For any cyclic framed E_2 -algebra \mathcal{A} , a corolla $T \in \text{Graphs}$ and $\Sigma \in \text{Surf}(T)$, the distinguished object $\mathcal{O}_\Sigma : I \rightarrow \int_{\Sigma} \mathcal{A}$ gives us by precomposition a functor

$$\mathfrak{F}_{\mathcal{A}}^{T, \Sigma} : \Omega_{\mathcal{A}}(\Sigma) \xrightarrow{\text{inclusion}} \mathcal{S}_\partial \left(\int_{\Sigma} \mathcal{A}, \mathcal{A}^{\otimes \text{Legs}(T)} \right) \xrightarrow{-\circ \mathcal{O}_\Sigma} \mathcal{S} \left(I, \mathcal{A}^{\otimes \text{Legs}(T)} \right).$$

This is extended to the case of non-connected T .

Theorem 6.2 (Construction of modular functors). For any cyclic framed E_2 -algebra \mathcal{A} , the functors from Definition 6.1 yield the structure of a modular $\text{Surf}_{\mathcal{A}}$ -algebra $\mathfrak{F}_{\mathcal{A}}$ whose restriction to Hbdy can be canonically identified with the modular extension $\widehat{\mathcal{A}}$ of \mathcal{A} . This modular $\text{Surf}_{\mathcal{A}}$ -algebra \mathcal{A} is a modular functor if and only if \mathcal{A} is connected.

Proof. Let $T \in \text{Graphs}$ be a corolla. Then for any diffeomorphism $f : \Sigma \rightarrow \Sigma'$, the triangle

$$\begin{array}{ccc} \Omega_{\mathcal{A}}(\Sigma') & & \\ \downarrow -\circ f_* & \searrow \mathfrak{F}_{\mathcal{A}}^{T, \Sigma'} & \\ \Omega_{\mathcal{A}}(\Sigma) & & \mathcal{S}(I, \mathcal{A}^{\otimes \text{Legs}(T)}) \\ & \nearrow \mathfrak{F}_{\mathcal{A}}^{T, \Sigma} & \end{array}$$

commutes by a canonical natural isomorphism coming from the homotopy fixed point structure $f_* \mathcal{O}_{\Sigma'} \cong \mathcal{O}_\Sigma$ of the distinguished object \mathcal{O}_Σ under diffeomorphisms (this homotopy fixed point structure was recalled in Section 2.1). This entails that the functors $\mathfrak{F}_{\mathcal{A}}^{T, \Sigma}$ descend to the Grothendieck construction $\mathcal{G}_{\mathcal{A}}^\Omega(T) = \text{Gr} \left(\text{SURF}(T)^{\text{op}} \xrightarrow{\Omega_{\mathcal{A}}} \text{Grpd} \right)$ and yield functors $\mathfrak{F}_{\mathcal{A}} : \text{Surf}_{\mathcal{A}}(T) \rightarrow \mathcal{S}(I, \mathcal{A}^{\otimes \text{Legs}(T)})$ because $\text{Surf}_{\mathcal{A}}(T)$ is the fundamental groupoid of $\mathcal{G}_{\mathcal{A}}^\Omega(T)$, see (5.8), and because $\mathcal{S}(I, \mathcal{A}^{\otimes \text{Legs}(T)})$ is just a 1-category. This extends in a straightforward way to the case of non-connected T . Now the functors $\mathfrak{F}_{\mathcal{A}}$ endow \mathcal{A} with the structure of a modular $\text{Surf}_{\mathcal{A}}$ -algebra. The restriction of $\mathfrak{F}_{\mathcal{A}}$ along the map $\text{Hbdy} \rightarrow \text{Surf}_{\mathcal{A}}$ from Theorem 5.2 sends a handlebody H to $\Phi_{\mathcal{A}}(H) \mathcal{O}_{\partial H}$, which can be canonically identified with $\widehat{\mathcal{A}}(H)$ by Theorem 4.2. In other words, it agrees with the ansular functor associated to \mathcal{A} . By Theorem 5.2 and 5.7 $\text{Surf}_{\mathcal{A}}$ is an extension if and only if \mathcal{A} is connected. Therefore, $(\text{Surf}_{\mathcal{A}}, \mathfrak{F}_{\mathcal{A}})$ is a modular functor if and only if \mathcal{A} is connected. \square

Remark 6.3. The construction of Theorem 6.2 is suitably functorial in maps of cyclic framed E_2 -algebras that by [MW23b, Proposition 2.18] are always invertible. More precisely, it is straightforward to verify the following: Let $\varphi : \mathcal{A} \xrightarrow{\cong} \mathcal{B}$ be an equivalence of cyclic framed E_2 -algebras. Then φ gives rise to a modular operad $\text{Surf}_{\mathcal{A}, \mathcal{B}}$ that comes with equivalences $\varepsilon_{\mathcal{A}} : \text{Surf}_{\mathcal{A}, \mathcal{B}} \xrightarrow{\cong} \text{Surf}_{\mathcal{A}}$ and $\varepsilon_{\mathcal{B}} : \text{Surf}_{\mathcal{A}, \mathcal{B}} \xrightarrow{\cong} \text{Surf}_{\mathcal{B}}$ such that the map φ induces an equivalence $\varepsilon_{\mathcal{A}}^* \mathfrak{F}_{\mathcal{A}} \xrightarrow{\cong} \varepsilon_{\mathcal{B}}^* \mathfrak{F}_{\mathcal{B}}$ of modular $\text{Surf}_{\mathcal{A}, \mathcal{B}}$ -algebras.

6.2 Universality of $\text{Surf}_{\mathcal{A}}$ and $\mathfrak{F}_{\mathcal{A}}$. We will ultimately use the construction $\mathcal{A} \mapsto \mathfrak{F}_{\mathcal{A}}$ from the previous subsection to classify modular functors. To this end, the next result will be crucial: We will prove that, if \mathcal{A} is connected, then an extension of \mathcal{A} to a modular functor living over an extension \mathcal{Q} of Surf amounts exactly to a map $\mathcal{Q} \rightarrow \text{Surf}_{\mathcal{A}}$ of extensions. In other words, $\text{Surf}_{\mathcal{A}}$ is a ‘classifying space’ for modular functors extending \mathcal{A} .

Theorem 6.4 (*Universality of $\text{Surf}_{\mathcal{A}}$ and $\mathfrak{F}_{\mathcal{A}}$*). *For any connected cyclic framed E_2 -algebra \mathcal{A} , the $\text{Surf}_{\mathcal{A}}$ -algebra $\mathfrak{F}_{\mathcal{A}}$ from Theorem 6.2 is universal in the sense that, for $\mathcal{Q} \in \text{Ext}^\circ(\text{Surf})$, the map*

$$\text{Map}_{\text{Ext}^\circ(\text{Surf})}(\mathcal{Q}, \text{Surf}_{\mathcal{A}}) \xrightarrow{\simeq} \text{Ext}^\circ(\mathcal{A}; \mathcal{Q}) , \quad \psi \mapsto \psi^* \mathfrak{F}_{\mathcal{A}} \quad (6.1)$$

is an equivalence of groupoids.

The fact that the morphism category in $\text{Ext}^\circ(\text{Surf})$ is indeed a groupoid is a consequence of Lemma 3.4.

Proof. The proof is based on a thorough investigation of $\text{Ext}^\circ(\mathcal{A}; \mathcal{Q})$ that we will present first. Afterwards, we will be able to describe a weak inverse to (6.1) directly.

- (i) Let $\mathcal{B} \in \text{Ext}^\circ(\mathcal{A}; \mathcal{Q})$, moreover $T \in \text{Graphs}$ a corolla and $\Sigma \in \text{Surf}(T)$. In this first step, it is our goal to construct natural maps

$$A_{\mathcal{B}}^- : \mathcal{Q}_{\Sigma} \rightarrow \mathcal{S}_{\partial} \left(\int_{\Sigma} \mathcal{A}, \mathcal{A}^{\otimes \text{Legs}(T)} \right) , \quad q \mapsto A_{\mathcal{B}}^q .$$

As usual \mathcal{Q}_{Σ} is the homotopy fiber of the extension $\mathcal{Q} \rightarrow \text{Surf}$ over Σ . For any oriented embedding $\varphi : (\mathbb{D}^2)^{\sqcup J} \rightarrow \Sigma$, we have the equivalence

$$\lambda_{\varphi} : \mathcal{Q}_{\Sigma \setminus \text{im } \varphi} \xrightarrow{\simeq} \mathcal{Q}_{\Sigma} \quad (6.2)$$

from Definition 3.3 since \mathcal{Q} admits insertions of vacua by assumption. Now for $q \in \mathcal{Q}_{\Sigma}$ and $r = (o, \alpha) \in \lambda_{\varphi}/q$, i.e. an object $o \in \mathcal{Q}_{\Sigma \setminus \text{im } \varphi}$ and a morphism $\lambda_{\varphi} o \xrightarrow{\alpha} q$, the algebra \mathcal{B} gives us a map

$$\mathcal{B}^{\otimes J} \xrightarrow{\mathcal{B}_o} \mathcal{B}^{\otimes \text{Legs}(T)} .$$

These maps combine into a map

$$\mathcal{B}_q^{\varphi} : \text{hocolim}_{q' \in \lambda_{\varphi}/q} \mathcal{B}^{\otimes J} \rightarrow \mathcal{B}^{\otimes \text{Legs}(T)} .$$

We will now show that \mathcal{B}_q^{φ} is natural in φ and hence gives us a map

$$A_{\mathcal{B}}^q : \text{hocolim}_{\varphi} \text{hocolim}_{r \in \lambda_{\varphi}/q} \mathcal{B}^{\otimes J} \rightarrow \mathcal{B}^{\otimes \text{Legs}(T)} . \quad (6.3)$$

To see this, we write $J = \{1, \dots, n\}$ and $J' = \{1, \dots, m\}$ and pick in addition to φ a framed E_2 -operation $\varphi' : \sqcup_{J'} \mathbb{D}^2 = (\mathbb{D}^2)^{\sqcup m} \rightarrow \mathbb{D}^2$. We will see φ' not only as a $\text{f}E_2$ -operation, but also as genus zero surface. The functor

$$c_i(\varphi, \varphi') : \mathcal{Q}_{\Sigma \setminus \text{im } \varphi} \xrightarrow{\text{pointing of } \mathcal{Q}_{\varphi'}} \mathcal{Q}_{\Sigma \setminus \text{im } \varphi} \times \mathcal{Q}_{\varphi'} \xrightarrow{\text{gluing}} \mathcal{Q}_{\Sigma \setminus \text{im } (\varphi \circ_i \varphi')}$$

makes the triangle

$$\begin{array}{ccc} \mathcal{Q}_{\Sigma \setminus \text{im } \varphi} & & \\ \downarrow c_i(\varphi, \varphi') & \searrow \lambda_{\varphi} & \\ \mathcal{Q}_{\Sigma \setminus \text{im } (\varphi \circ_i \varphi')} & & \mathcal{Q}_{\Sigma} , \\ & \nearrow \lambda_{\varphi \circ_i \varphi'} & \end{array}$$

commute up to a canonical natural isomorphism. Since λ_{φ} and $\lambda_{\varphi \circ_i \varphi'}$ are equivalences by assumption, so is $c_i(\varphi, \varphi')$. As a result, we may identify

$$\lambda_{\varphi \circ_i \varphi'} / q \simeq \lambda_{\varphi} / q . \quad (6.4)$$

Consider now the following triangle:

$$\begin{array}{ccc}
\text{hocolim}_{r' \in \lambda_{\varphi \circ_i \varphi'} / q} \mathcal{B}^{\otimes(j-1)} \otimes \mathcal{B}^{\otimes m} \otimes \mathcal{B}^{\otimes(n-j)} & & \\
\downarrow (*) & \searrow \mathcal{B}_q^{\varphi \circ_i \varphi'} & \\
\text{hocolim}_{r \in \lambda_{\varphi} / q} \mathcal{B}^{\otimes n} & \xrightarrow{\mathcal{B}_q^{\varphi}} & \mathcal{B}^{\otimes \text{Legs}(T)} ,
\end{array}$$

The map $(*)$ is induced by $\mathcal{B}_{s(\varphi')} : \mathcal{B}^{\otimes m} \rightarrow \mathcal{B}$ (we are using here the section $s : \text{Hbdy} \rightarrow \mathcal{Q}$ associated to \mathcal{Q}) and (6.4). Since \mathcal{B} is a modular algebra, this triangle commutes up to a canonical isomorphism. This tells us that we indeed obtain the map (6.3).

The key point is that the slices $\lambda_{\varphi} / -$ are contractible (see (6.2)). For this reason, the double homotopy colimit on the left hand side of (6.3) models the factorization homology of the restriction of \mathcal{B} to $\text{f}E_2$. But as part the data, we have an equivalence of cyclic framed E_2 -algebras between this restriction of \mathcal{B} and \mathcal{A} . Therefore, we may identify the left hand side of (6.3) canonically with $\int_{\Sigma} \mathcal{A}$. In other words, we may see $\Lambda_{\mathcal{B}}^q$ as a map

$$\Lambda_{\mathcal{B}}^q : \int_{\Sigma} \mathcal{A} \rightarrow \mathcal{A}^{\otimes \text{Legs}(T)} .$$

By construction this comes with the structure of an $\int_{\partial \Sigma \times [0,1]} \mathcal{A}$ -module map (this is essentially the same argument as in the proof of Proposition 4.3). Moreover, the assignment $q \mapsto \Lambda_{\mathcal{B}}^q$ naturally extends to a functor out of \mathcal{Q}_{Σ} , and it sends the morphisms in \mathcal{Q}_{Σ} to morphisms of $\int_{\partial \Sigma \times [0,1]} \mathcal{A}$ -module maps. Therefore, we obtain natural functors

$$\Lambda_{\mathcal{B}}^- : \mathcal{Q}_{\Sigma} \rightarrow \mathcal{S}_{\partial} \left(\int_{\Sigma} \mathcal{A}, \mathcal{A}^{\otimes \text{Legs}(T)} \right) ,$$

where $\mathcal{S}_{\partial} \left(\int_{\Sigma} \mathcal{A}, \mathcal{A}^{\otimes \text{Legs}(T)} \right)$ is the groupoid of $\int_{\partial \Sigma \times [0,1]} \mathcal{A}$ -module maps $\int_{\Sigma} \mathcal{A} \rightarrow \mathcal{A}^{\otimes \text{Legs}(T)}$ with module isomorphisms as morphisms. Then

$$\mathcal{Q}_{\Sigma} \xrightarrow{\Lambda_{\mathcal{B}}^-} \mathcal{S}_{\partial} \left(\int_{\Sigma} \mathcal{A}, \mathcal{A}^{\otimes \text{Legs}(T)} \right) \xrightarrow{-\circ \mathcal{O}_{\Sigma}} \mathcal{S}(I, \mathcal{A}^{\otimes \text{Legs}(T)}) \quad (6.5)$$

can be canonically identified with $\mathcal{Q}_{\Sigma} \xrightarrow{\mathcal{B}} \mathcal{S}(I, \mathcal{A}^{\otimes \text{Legs}(T)})$ under the identification of \mathcal{A} and \mathcal{B} as cyclic algebras (this is the same argument as in the proof of Theorem 4.2).

(ii) Next consider the following diagram (the dashed arrow is to be ignored for the moment):

$$\begin{array}{ccccc}
\partial^{-1}(\Sigma) & \xrightarrow{s_{\mathcal{A}}} & \Omega_{\mathcal{A}}(\Sigma) & & \\
\downarrow s & \nearrow \psi_{\Sigma}^{\mathcal{B}} & \downarrow \iota & \searrow \tilde{\mathfrak{F}}_{\mathcal{A}}^{T, \Sigma} & \\
\mathcal{Q}_{\Sigma} & \xrightarrow{\Lambda_{\mathcal{B}}^-} & \mathcal{S}_{\partial} \left(\int_{\Sigma} \mathcal{A}, \mathcal{A}^{\otimes \text{Legs}(T)} \right) & \xrightarrow{-\circ \mathcal{O}_{\Sigma}} & \mathcal{S}(I, \mathcal{A}^{\otimes \text{Legs}(T)}) \\
& \searrow \mathcal{B} & & & \downarrow \simeq \\
& & & & \mathcal{S}(I, \mathcal{B}^{\otimes \text{Legs}(T)})
\end{array} \quad (6.6)$$

Here ι is the full subgroupoid inclusion. The lower surface commutes by a canonical isomorphism because the composition (6.5), as just discussed, can be identified with \mathcal{B} . The right triangle commutes strictly by Definition 6.1. By construction the square on the left commutes up to a canonical isomorphism (this is the identification of \mathcal{B} with $\widehat{\mathcal{A}}$ over Hbdy that is part of data). The fact that by connectedness of \mathcal{Q}_{Σ} the functor $s : \partial^{-1}(\Sigma) \rightarrow \mathcal{Q}_{\Sigma}$ is essentially surjective implies that $\Lambda_{\mathcal{B}}^-$ must factor through $\Omega_{\mathcal{A}}(\Sigma)$. In other words, there is a unique functor $\psi_{\Sigma}^{\mathcal{B}} : \mathcal{Q}_{\Sigma} \rightarrow \Omega_{\mathcal{A}}(\Sigma)$ with $\iota \psi_{\Sigma}^{\mathcal{B}} = \Lambda_{\mathcal{B}}^-$. Now $\iota \psi_{\Sigma}^{\mathcal{B}} s = \Lambda_{\mathcal{B}}^- s \cong \iota s_{\mathcal{A}}$ by a canonical isomorphism. Since ι is a full subgroupoid inclusion, this gives us a canonical isomorphism $\psi_{\Sigma}^{\mathcal{B}} s \cong s_{\mathcal{A}}$. We have now established that the entire diagram (6.6) commutes up to a canonical isomorphism.

(iii) The map $\psi_{\Sigma}^{\mathcal{B}}$, being natural in Σ , gives us a map $\psi^{\mathcal{B}} : \mathcal{Q} \rightarrow \text{Surf}_{\mathcal{A}}$ of modular operads, in fact, of modular operads over Surf (because it is constructed fiberwise anyway). Over Hbdy , the map is trivialized thanks to $\psi^{\mathcal{B}} s \cong s_{\mathcal{A}}$.

With all these preparations, we can describe the desired inverse $\text{Ext}^{\circ}(\mathcal{A}; \mathcal{Q}) \rightarrow \text{Map}_{\text{Ext}^{\circ}(\text{Surf})}(\mathcal{Q}, \text{Surf}_{\mathcal{A}})$ to (6.1). It sends $\mathcal{B} \in \text{Ext}^{\circ}(\mathcal{A}; \mathcal{Q})$ to the map $\psi^{\mathcal{B}} : \mathcal{Q} \rightarrow \text{Surf}_{\mathcal{A}}$ from step (iii). This gives us indeed an inverse: For any map $\theta : \mathcal{Q} \rightarrow \text{Surf}_{\mathcal{A}}$ of extensions, we find $\Lambda_{\theta^* \mathfrak{F}_{\mathcal{A}}}^- \simeq \Lambda_{\mathfrak{F}_{\mathcal{A}}}^{\theta(-)}$, and hence $\psi^{\theta^* \mathfrak{F}_{\mathcal{A}}} \simeq \psi^{\mathfrak{F}_{\mathcal{A}}} \circ \theta = \theta$ (this uses that $\psi^{\mathfrak{F}_{\mathcal{A}}}$ is the identity). Similarly, for $\mathcal{B} \in \text{Ext}^{\circ}(\mathcal{A}; \mathcal{Q})$, we find $(\psi^{\mathcal{B}})^* \mathfrak{F}_{\mathcal{A}} \simeq \mathcal{B}$ by the commutativity of the lower surface in the above diagram. \square

Lemma 6.5. *Let \mathcal{A} be a cyclic framed E_2 -algebra. If the bicategory $\text{Ext}^{\circ}(\mathcal{A})$ of extensions of \mathcal{A} is not empty, then \mathcal{A} is connected.*

Proof. Let us assume that we have some $\mathcal{B} \in \text{Ext}^{\circ}(\mathcal{A}; \mathcal{Q})$ for some $\mathcal{Q} \in \text{Ext}^{\circ}(\text{Surf})$. Even though \mathcal{A} is not connected, the proof of Theorem 6.4 still gives us a map $\psi^{\mathcal{B}} : \mathcal{Q} \rightarrow \text{Surf}_{\mathcal{A}}$ of modular operads over Surf (the construction of this map in the proof did not need anywhere the $\Omega_{\mathcal{A}}$ -groupoids are connected; without the assumption, we only have the problem that the map $\psi^{\mathcal{B}} : \mathcal{Q} \rightarrow \text{Surf}_{\mathcal{A}}$ is not a map in $\text{Ext}^{\circ}(\text{Surf})$, simply because $\text{Surf}_{\mathcal{A}}$ might have non-connected fibers and hence might not qualify as an extension). We can verify $\psi^{\mathcal{B}} s \simeq s_{\mathcal{A}}$ for the canonical maps $s : \text{Hbdy} \rightarrow \mathcal{Q}$ and $s_{\mathcal{A}} : \text{Hbdy} \rightarrow \text{Surf}$. This entails that the essentially surjective functor $\partial^{-1}(\Sigma) \xrightarrow{s_{\mathcal{A}}} \Omega_{\mathcal{A}}(\Sigma)$ factors as $\partial^{-1}(\Sigma) \xrightarrow{s} \mathcal{Q}_{\Sigma} \xrightarrow{\psi^{\mathcal{B}}} \Omega_{\mathcal{A}}(\Sigma)$. But this is only possible if $\psi^{\mathcal{B}} : \mathcal{Q}_{\Sigma} \rightarrow \Omega_{\mathcal{A}}(\Sigma)$ is essentially surjective. Since \mathcal{Q}_{Σ} is connected by assumption, so is $\Omega_{\mathcal{A}}(\Sigma)$. \square

6.3 Weak uniqueness of extensions. For a cyclic framed E_2 -algebra \mathcal{A} , we defined the bicategory of extensions of \mathcal{A} to a modular functor in Definition 3.8. The following result is mostly an application of the universality of $\text{Surf}_{\mathcal{A}}$:

Theorem 6.6 (Weak uniqueness of extensions). *For any cyclic framed E_2 -algebra \mathcal{A} , the realization $|\text{BExt}^{\circ}(\mathcal{A})|$ of the Duskin nerve of the bicategory $\text{Ext}^{\circ}(\mathcal{A})$ is empty or contractible. In other words, if there is an extension of \mathcal{A} to a modular functor, this extension is unique up to a contractible choice.*

In the case of cyclic framed E_2 -algebras whose underlying category is finitely semisimple and equipped with an involution, such a uniqueness result has been established by Andersen-Ueno [AU12] (with a different definition of modular functor). The above result applies to cyclic framed E_2 -algebras regardless of finiteness and semisimplicity assumptions and makes a statement about the *space of extensions* rather than extensions up to equivalence.

Proof of Theorem 6.6. We need to show that the bicategory $\text{Ext}^{\circ}(\mathcal{A})$ is either empty or has a contractible nerve. We will do this by proving that $\text{Ext}^{\circ}(\mathcal{A})$ is contractible under the assumption that it is not empty. Under this assumption, it is automatic that \mathcal{A} is connected by Lemma 6.5.

Let us now prove the contractibility of $\text{Ext}^{\circ}(\mathcal{A})$ in the case that \mathcal{A} is connected: By Definition 3.8 $\text{Ext}^{\circ}(\mathcal{A})$ is the Grothendieck construction of the bicategory-valued functor $\text{Ext}^{\circ}(\mathcal{A}; -) : \text{Ext}^{\circ}(\text{Surf})^{\text{op}} \rightarrow \text{BiCat}$ (the bicategorical version of the Grothendieck construction can be defined in complete analogy to the categorical one). By Theorem 6.4 we have equivalences $\text{Ext}^{\circ}(\mathcal{A}; \mathcal{Q}) \simeq \text{Map}_{\text{Ext}^{\circ}(\text{Surf})}(\mathcal{Q}, \text{Surf}_{\mathcal{A}})$ natural in \mathcal{Q} , which implies the equivalence

$$\text{Ext}^{\circ}(\mathcal{A}) \simeq \text{Gr} \left(\mathcal{Q} \mapsto \text{Map}_{\text{Ext}^{\circ}(\text{Surf})}(\mathcal{Q}, \text{Surf}_{\mathcal{A}}) \right) \quad (6.7)$$

of bicategories. But the Grothendieck construction $\text{Gr} \left(\mathcal{Q} \mapsto \text{Map}_{\text{Ext}^{\circ}(\text{Surf})}(\mathcal{Q}, \text{Surf}_{\mathcal{A}}) \right)$ is, after spelling out the definition, the slice bicategory $\text{Ext}^{\circ}(\text{Surf})/\text{Surf}_{\mathcal{A}}$. The identity of $\text{Surf}_{\mathcal{A}}$ is a terminal object in this slice bicategory. This proves that the nerve of $\text{Gr} \left(\mathcal{Q} \mapsto \text{Map}_{\text{Ext}^{\circ}(\text{Surf})}(\mathcal{Q}, \text{Surf}_{\mathcal{A}}) \right)$ is contractible. By (6.7) and homotopy invariance of the nerve under equivalences of bicategories we find that $\text{Ext}^{\circ}(\mathcal{A})$ is contractible as well. \square

Theorem 6.6 proves under very general conditions that a modular functor is determined up to a contractible choice by its genus zero data. But for this it is important to understand ‘genus zero data’ precisely as the data of a cyclic framed E_2 -algebra. Prescribing just the non-cyclic genus zero data, i.e. a balanced braided monoidal structure, leads already to a different situation about which a different statement can be made: Let \mathcal{A} be a framed E_2 -algebra in Rex with underlying finite category (that is a balanced braided category). We denote by $Z_2^{\text{bal}}(\mathcal{A}) \subset Z_2(\mathcal{A})$ the subcategory of the Müger center spanned by the objects with trivial balancing and denote by $\text{PIC}(Z_2^{\text{bal}}(\mathcal{A}))$ the Picard groupoid of $Z_2^{\text{bal}}(\mathcal{A})$, i.e. the groupoid of invertible objects in $Z_2^{\text{bal}}(\mathcal{A})$. Then the following holds:

Corollary 6.7. *For any non-cyclic framed E_2 -algebra \mathcal{A} in Rex with underlying finite category, the space $|B\text{Ext}^\circ(\mathcal{A})|$ of extensions of \mathcal{A} to a modular functor (with maps between such extensions being relative to \mathcal{A}) is either empty or*

$$|B\text{Ext}^\circ(\mathcal{A})| \simeq |BPIC(Z_2^{\text{bal}}(\mathcal{A}))| .$$

If the braiding of \mathcal{A} is non-degenerate and the unit of \mathcal{A} is simple, then the space on the right is the classifying space $B\mathbf{k}^\times$ of the group of units of the ground field \mathbf{k} .

Proof. If $|B\text{Ext}^\circ(\mathcal{A})|$ is not empty, then a cyclic structure exists on \mathcal{A} and an extension to a modular functor is unique up to a contractible choice once a cyclic structure is chosen; this follows from Theorem 6.6. Therefore, the remaining choice is exactly the ambiguity in choosing a cyclic structure. In Lex^f and, equivalently, Rex^f this ambiguity is described in [MW22, Theorem 4.2] through $PIC(Z_2^{\text{bal}}(\mathcal{A}))$ as groupoid or $|BPIC(Z_2^{\text{bal}}(\mathcal{A}))|$ as space. \square

6.4 The classification of modular functors. In this subsection, we state and prove the main result of this paper, namely an explicit description of the moduli space of modular functors. This will however be relatively straightforward because most of the work was already done in the previous subsections.

Theorem 6.8 (*Classification of modular functors*). *The moduli space $\mathfrak{M}\mathfrak{F}$ of modular functors with values in \mathcal{S} is equivalent to the 2-groupoid of connected cyclic framed E_2 -algebras \mathcal{A} in \mathcal{S} . The equivalence is afforded by restriction to genus zero. An inverse sends \mathcal{A} to $\mathfrak{F}_{\mathcal{A}}$.*

If we take into account that cyclic framed E_2 -algebras are self-dual balanced braided algebras by the main result of [MW23a] (this was recalled as Theorem 2.2 in this article), $\mathfrak{M}\mathfrak{F}$ is also equivalent to the 2-groupoid of connected self-dual balanced braided algebras in \mathcal{S} .

The proof of Theorem 6.8 will use a bicategorical version of Thomason's Theorem [Tho79, Theorem 1.2] that can be obtained by a straightforward modification of the version of this result given in [CCG10, Ceg11]: Recall that for a functor $F : \mathcal{C} \rightarrow \text{BiCat}$ from a 2-groupoid \mathcal{C} , seen as tricategory, to the tricategory of bicategories (by default we understand this to be a functor in the weak sense), the Grothendieck construction $\text{Gr}F$ is the bicategory formed by pairs (c, x) , where $c \in \mathcal{C}$ and $x \in F(c)$, in analogy to the usual Grothendieck construction. Thomason's Theorem gives us a canonical homotopy equivalence

$$B\text{Gr}F \simeq \text{hocolim}_{c \in \mathcal{C}} BF(c) \tag{6.8}$$

of simplicial sets (but we can of course apply the geometric realization $|-|$ to get a homotopy equivalence of spaces).

Proof of Theorem 6.8. With Thomason's Theorem (6.8), we can prove the desired statement as follows: We consider the genus zero restriction map $R : \text{MF} \rightarrow \text{CycAlg}(fE_2)$ and first observe that by Lemma 6.5 it actually factors through the full sub-2-groupoid $\text{CycAlg}^{\Omega}(fE_2) \subset \text{CycAlg}(fE_2)$ of connected cyclic framed E_2 -algebras. Therefore, the restriction map $R : \text{MF} \rightarrow \text{CycAlg}(fE_2)$ restricts in range to a map

$$\tilde{R} : \text{MF} \rightarrow \text{CycAlg}^{\Omega}(fE_2) . \tag{6.9}$$

The homotopy fiber of (6.9) is $\text{Ext}^\circ(\mathcal{A})$ by definition. In other words, we can describe MF as a Grothendieck construction $\text{MF} = \text{Gr}(\mathcal{A} \mapsto \text{Ext}^\circ(\mathcal{A}))$; in fact, that is just a different way to express the definition of MF . Now Thomason's Theorem (6.8) gives us

$$B\text{MF} \simeq \text{hocolim}_{\mathcal{A} \in \text{CycAlg}^{\Omega}(fE_2)} B\text{Ext}^\circ(\mathcal{A}) .$$

But $B\text{Ext}^\circ(\mathcal{A})$, for an arbitrary cyclic framed E_2 -algebra is always empty or contractible by Theorem 6.6, and for those algebras in $\text{CycAlg}^{\Omega}(fE_2)$, the latter is the case by Lemma 6.5. This implies that (6.9) induces an equivalence after taking the nerve. The statement about the inverse equivalence follows from the description (6.7) of $\text{Ext}^\circ(\mathcal{A})$. \square

Since by Theorem 6.8 $\mathfrak{M}\mathfrak{F}$ forms a full sub-2-groupoid of cyclic framed E_2 -algebras, we immediately obtain:

Corollary 6.9. *Let \mathcal{A} be a cyclic framed E_2 -algebra. Then the 2-group of automorphisms of \mathcal{A} as cyclic framed E_2 -algebra is equivalent to the 2-group of automorphisms of any modular functor extending \mathcal{A} .*

The connectedness condition that is needed for a cyclic framed E_2 -algebra to uniquely extend to a modular functor is relatively abstract. Even though one of the different equivalent characterizations of connectedness just boils down to a genus one condition, it will still be involved to check the condition in practice. In this section, we provide concrete algebraic conditions that are sufficient for connectedness. These will be used in the next section to discuss classes of examples of modular functors.

7.1 Triviality of the mapping class group action on factorization homology. We start with the following observation that will still not produce a concrete criterion, but instead a valuable tool:

Lemma 7.1. *If for a cyclic framed E_2 -algebra \mathcal{A} and every surface Σ , the 1-morphism $f_* : \int_{\Sigma} \mathcal{A} \rightarrow \int_{\Sigma} \mathcal{A}$ is trivializable for all $f \in \text{Map}(\Sigma)$, i.e. isomorphic as a $\int_{\partial\Sigma \times [0,1]} \mathcal{A}$ -module map to the identity, then \mathcal{A} is connected.*

The Lemma assumes that we can trivialize each f_* separately. It is not implied that we can trivialize coherently.

Proof. For handlebodies H and H' with $\partial H = \partial H' = \Sigma$, pick an isomorphism $g : H \rightarrow H'$. Then $\Phi_{\mathcal{A}}(H')f_* \cong \Phi_{\mathcal{A}}(H)$ with $f := \partial g$ by Proposition 4.6. This implies the assertion. \square

The triviality of the mapping class group action on factorization homology can be checked locally:

Lemma 7.2. *Let \mathcal{A} be a cyclic framed E_2 -algebra in \mathcal{S} . The action $f_* : \int_{\Sigma} \mathcal{A} \rightarrow \int_{\Sigma} \mathcal{A}$ of every diffeomorphism $f : \Sigma \rightarrow \Sigma$ of every surface Σ is isomorphic, as $\int_{\partial\Sigma \times [0,1]} \mathcal{A}$ -module map, to the identity if and only if this is the case for the action $d_* : \int_{\mathbb{S}^1 \times [0,1]} \mathcal{A} \rightarrow \int_{\mathbb{S}^1 \times [0,1]} \mathcal{A}$ of the Dehn twist on the factorization homology of the cylinder.*

Proof. First recall from Remark 4.5 that the isomorphism class of $f_* : \int_{\Sigma} \mathcal{A} \rightarrow \int_{\Sigma} \mathcal{A}$, as $\int_{\partial\Sigma \times [0,1]} \mathcal{A}$ -module map, depends only on the mapping class of f . Now the statement follows from the fact that the pure mapping class group of any surface is generated by Dehn twists (see [FM12, Theorem 4.11] for a textbook reference) and the excision property of factorization homology. \square

7.2 Cofactorizability. Using the tools from the previous subsection, we will now prove that cofactorizability of E_2 -algebras [BJSS21, Section 3.3] is sufficient for connectedness. Let us review the notion: If \mathcal{S} has an internal hom $[-, -]$, then for any algebra \mathcal{A} in \mathcal{S} the multiplication $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ induces a map $\mathcal{A} \rightarrow [\mathcal{A}, \mathcal{A}]$. If \mathcal{A} is an E_2 -algebra, this map descends to the cocenter $\int_{\mathbb{S}^1 \times [0,1]} \mathcal{A}$ and yields an algebra map

$$\int_{\mathbb{S}^1 \times [0,1]} \mathcal{A} \rightarrow [\mathcal{A}, \mathcal{A}]. \quad (7.1)$$

Definition 7.3. Let \mathcal{S} be a symmetric monoidal $(2, 1)$ -category with internal hom. An E_2 -algebra \mathcal{A} is called cofactorizable if (7.1) is an equivalence.

Remark 7.4. Let \mathcal{A} be a cyclic framed E_2 -algebra in a symmetric monoidal $(2, 1)$ -category with internal hom. Then by direct inspection the map

$$\Phi_{\mathcal{A}}(\mathbb{D}^2 \times [0, 1]) : \int_{\mathbb{S}^1 \times [0,1]} \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A} \quad (7.2)$$

agrees with (7.1) after the identification $[\mathcal{A}, \mathcal{A}] \simeq \mathcal{A} \otimes \mathcal{A}$ via the pairing. This implies that \mathcal{A} is cofactorizable if and only if (7.2) is an equivalence. Consequently, we will by definition refer to a cyclic framed E_2 -algebra \mathcal{A} as cofactorizable if (7.2) is an equivalence regardless of whether \mathcal{S} has internal homs. Since cofactorizability is just the invertibility of $\Phi_{\mathcal{A}}(\mathbb{D}^2 \times [0, 1])$, it is a genus zero condition.

Proposition 7.5. *For any cofactorizable cyclic framed E_2 -algebra \mathcal{A} in \mathcal{S} , any diffeomorphism $f : \Sigma \rightarrow \Sigma$ of any surface Σ acts on $\int_{\Sigma} \mathcal{A}$ by a 1-morphism that is, as $\int_{\partial\Sigma \times [0,1]} \mathcal{A}$ -module map, isomorphic to the identity. In particular, cofactorizability implies connectedness.*

Proof. For the Dehn twist d of the cylinder, Corollary 4.7 gives us an isomorphism

$$\Phi_{\mathcal{A}}(\mathbb{D}^2 \times [0, 1]) \circ d_* \cong \Phi_{\mathcal{A}}(\mathbb{D}^2 \times [0, 1])$$

of maps of modules over $\left(\int_{\mathbb{S}^1 \times [0,1]} \mathcal{A}\right)^{\otimes 2}$. Since $\Phi_{\mathcal{A}}(\mathbb{D}^2 \times [0,1])$ is assumed to be invertible, this implies that d_* is isomorphic, as $\left(\int_{\mathbb{S}^1 \times [0,1]} \mathcal{A}\right)^{\otimes 2}$ -module map, to the identity. By Lemma 7.2 this implies that every mapping class acts trivially on factorization homology. Thanks to Lemma 7.1, this implies connectedness. \square

Theorem 7.6. *For any cofactorizable cyclic framed E_2 -algebra, there is an essentially unique extension to a modular functor. More precisely, we have an embedding of 2-groupoids*

$$\{\text{cofactorizable self-dual balanced braided algebras in } \mathcal{S}\} \hookrightarrow \mathfrak{MF}, \quad \mathcal{A} \longmapsto \mathfrak{F}_{\mathcal{A}}.$$

Proof. Proposition 7.5 says that cofactorizable cyclic framed E_2 -algebras are connected. The rest is a consequence of Theorem 6.8. \square

Remark 7.7. This embedding is not an equivalence in general: If we take \mathcal{S} to be a symmetric monoidal 1-category, seen as a symmetric monoidal $(2,1)$ -category with only identity 2-morphisms, then cyclic framed E_2 -algebras in \mathcal{S} are just commutative Frobenius algebras, so that any of those gives rise to a two-dimensional topological field theory (hence to a modular functor). This is a consequence of $\pi_0(\text{Hbdy}) \cong \pi_0(\text{Surf})$ and recovers the well-known classification of ordinary two-dimensional topological field theories [Abr96, Koc03]. But non-trivial Frobenius algebras are in general not cofactorizable.

7.3 Simplification of $\text{Surf}_{\mathcal{A}}$. So far, we have said relatively little about the extension $\text{Surf}_{\mathcal{A}} \longrightarrow \text{Surf}$ provided by a connected cyclic framed E_2 -algebra. In particular, we have not been very explicit about the fibers of the extension (we only gave a relatively abstract description in Theorem 5.6). In this subsection we will calculate the fibers in relevant special cases.

The following notion is a natural generalization of cofactorizability:

Definition 7.8. A cyclic framed E_2 -algebra \mathcal{A} in \mathcal{S} is called Φ -invertible on a connected surface $\Sigma \in \text{Surf}(T)$ if for some handlebody H with boundary $\partial H = \Sigma$ the 1-morphism $\Phi_{\mathcal{A}}(H) : \int_{\partial H} \mathcal{A} \longrightarrow \mathcal{A}^{\otimes \text{Legs}(T)}$ is an equivalence. A cyclic framed E_2 -algebra is called Φ -invertible if it is Φ -invertible on all connected surfaces.

Remark 7.9. If $\Phi_{\mathcal{A}}(H)$ is an equivalence for *some* handlebody with boundary Σ , then this is true for *all* handlebodies with boundary Σ . This follows from Proposition 4.6.

Remark 7.10. In detecting whether $\Omega_{\mathcal{A}}(\Sigma)$ is connected, the following implications are helpful:

$$\mathcal{A} \text{ is } \Phi\text{-invertible at } \Sigma \implies f_* \cong \text{id for all } f \in \text{Map}(\Sigma) \text{ as module map} \implies \Omega_{\mathcal{A}}(\Sigma) \text{ connected}.$$

The proof of the first implication repeats essentially the argument of Proposition 7.5 while the second one is Lemma 7.1.

Proposition 7.11. *Let \mathcal{A} be a cyclic framed E_2 -algebra. If \mathcal{A} is Φ -invertible on a connected surface $\Sigma \in \text{Surf}(T)$ (as defined in Definition 7.8), then the extension of $\text{Map}(\Sigma)$ defined by $p_{\mathcal{A}} : \text{Surf}_{\mathcal{A}} \longrightarrow \text{Surf}$ is of the form*

$$1 \longrightarrow \text{Aut}_{\partial}(\text{id}_{\int_{\Sigma} \mathcal{A}}) \longrightarrow \text{Map}_{\mathcal{A}}(\Sigma) \longrightarrow \text{Map}(\Sigma) \longrightarrow 1,$$

i.e. it is an extension of $\text{Map}(\Sigma)$ by the abelian group $\text{Aut}_{\partial}(\text{id}_{\int_{\Sigma} \mathcal{A}})$ of automorphisms of $\text{id}_{\int_{\Sigma} \mathcal{A}}$ as $\int_{\partial \Sigma \times [0,1]} \mathcal{A}$ -module map. In other words, the extension can be described by a 2-cocycle $H^2(\text{Map}(\Sigma); \text{Aut}_{\partial}(\text{id}_{\int_{\Sigma} \mathcal{A}}))$.

Proof. We first note that Theorem 5.2 combined with Remark 7.10 tells us that we get extensions of the form

$$1 \longrightarrow \text{Aut}_{\partial}(\Phi_{\mathcal{A}}(H)) \longrightarrow \text{Map}_{\mathcal{A}}(\Sigma) \longrightarrow \text{Map}(\Sigma) \longrightarrow 1$$

for H with $\partial H = \Sigma$. Additionally, the module equivalence $\Phi_{\mathcal{A}}(H) : \int_{\Sigma} \mathcal{A} \longrightarrow \mathcal{A}^{\otimes \text{Legs}(T)}$ provides us with an isomorphism $\text{Aut}_{\partial}(\text{id}_{\int_{\Sigma} \mathcal{A}}) \cong \text{Aut}_{\partial}(\Phi_{\mathcal{A}}(H))$ of groups. Finally, we observe that $\text{Aut}_{\partial}(\text{id}_{\int_{\Sigma} \mathcal{A}})$ is abelian because it is a subgroup of $\text{Aut}(\text{id}_{\int_{\Sigma} \mathcal{A}})$ which by the bicategorical Eckmann-Hilton argument is abelian. \square

In the particular case that \mathcal{A} is cofactorizable, one obtains a modular functor featuring extensions of mapping class groups that are of a particularly nice form.

Proposition 7.12. *Let \mathcal{A} be a cofactorizable cyclic framed E_2 -algebra. Then the resulting extensions of mapping class groups of connected surfaces are of the form*

$$1 \longrightarrow \mathrm{Aut}(\Phi_{\mathcal{A}}(\mathbb{B}^3)) \longrightarrow \mathrm{Map}_{\mathcal{A}}(\Sigma) \longrightarrow \mathrm{Map}(\Sigma) \longrightarrow 1. \quad (7.3)$$

Proof. By Proposition 7.5 $\mathrm{Surf}_{\mathcal{A}}$ is connected. It remains to prove that the resulting extensions of mapping class groups are of the form (7.3): Let Σ be a connected surface with $n + 2$ boundary components and Σ' the surface obtained by gluing two of these boundary components together. With the map $\Phi_{\mathcal{A}}(\mathbb{D}^2 \times [0, 1]) : \int_{\mathbb{S}^1 \times [0, 1]} \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$ that is in fact a map of $\left(\int_{\mathbb{S}^1 \times [0, 1]} \mathcal{A}\right)^{\otimes 2}$ -modules by Proposition 4.3, we obtain a square

$$\begin{array}{ccc} \mathcal{S}_{\partial} \left(\int_{\Sigma} \mathcal{A}, \mathcal{A}^{\otimes(n+2)} \right) & \xrightarrow{\text{gluing, see proof of Theorem 5.2}} & \mathcal{S}_{\partial} \left(\int_{\Sigma'} \mathcal{A}, \mathcal{A}^{\otimes n} \right) \\ \simeq \text{via pairing } \kappa \downarrow & & \downarrow \simeq \text{via excision} \\ \mathcal{S}_{\partial} \left(\int_{\Sigma} \mathcal{A} \otimes \left(\int_{\mathbb{S}^1 \times [0, 1]} \mathcal{A} \right)^{\otimes 2}, \mathcal{A}^{\otimes n} \right) & \xrightarrow{- \circ \Phi_{\mathcal{A}}(\mathbb{D}^2 \times [0, 1])} & \mathcal{S}_{\partial} \left(\int_{\Sigma} \mathcal{A} \otimes \left(\int_{\mathbb{S}^1 \times [0, 1]} \mathcal{A} \right)^{\otimes 2}, \int_{\mathbb{S}^1 \times [0, 1]} \mathcal{A}, \mathcal{A}^{\otimes n} \right). \end{array} \quad (7.4)$$

It is straightforward to observe that it commutes up to natural isomorphism. Since \mathcal{A} is assumed to be cofactorizable, $\Phi_{\mathcal{A}}(\mathbb{D}^2 \times [0, 1])$ is by definition an equivalence. This implies that the lower horizontal map in the square is an equivalence. But then the upper horizontal map must be an equivalence as well. The gluing functor $\Omega_{\mathcal{A}}(\Sigma) \longrightarrow \Omega_{\mathcal{A}}(\Sigma')$ is by the construction in the proof of Theorem 5.2 the restriction the upper horizontal map in (7.4) in domain and range. Therefore, it must be fully faithful. Moreover, it is essentially surjective because both groupoids are connected. This makes $\Omega_{\mathcal{A}}(\Sigma) \longrightarrow \Omega_{\mathcal{A}}(\Sigma')$ an equivalence and proves after iterated application $\Omega_{\mathcal{A}}(\Sigma_{0, n+2g}) \simeq \Omega_{\mathcal{A}}(\Sigma_{g, n})$, where $\Sigma_{g, n}$ denotes the surface with genus g and n boundary components. Thanks to the insertion of vacua (Proposition 5.9), we have additionally $\Omega_{\mathcal{A}}(\Sigma_{g, n}) \simeq \Omega_{\mathcal{A}}(\Sigma_{g, 0})$. This gives us $\Omega_{\mathcal{A}}(\Sigma) \simeq \Omega_{\mathcal{A}}(\mathbb{S}^2)$ for every connected surface Σ . Thanks to $\mathrm{Aut}(\Phi_{\mathcal{A}}(\mathbb{B}^3)) = \pi_1(\Omega_{\mathcal{A}}(\mathbb{S}^2))$, this proves that the extensions resulting from \mathcal{A} are of the form (7.3). \square

Let \mathcal{A} be any cyclic framed E_2 -algebra. The factorization homology for the sphere \mathbb{S}^2 is equivalent to $\mathcal{A} \otimes \int_{\mathbb{S}^1 \times [0, 1]} \mathcal{A}$ thanks to excision. Under this identification, the map $\Phi_{\mathcal{A}}(\mathbb{B}^3) : \int_{\mathbb{S}^2} \mathcal{A} \longrightarrow I$ is the map

$$\mathcal{A} \otimes \int_{\mathbb{S}^1 \times [0, 1]} \mathcal{A} \longrightarrow I \quad (7.5)$$

induced by the pairing $\kappa : \mathcal{A} \otimes \mathcal{A} \longrightarrow I$. This leads to the following notion that is analogous to cofactorizability:

Definition 7.13. We call a cyclic framed E_2 -algebra \mathcal{A} *co-non-degenerate* if it is Φ -invertible on the sphere, i.e. if the map $\mathcal{A} \otimes \int_{\mathbb{S}^1 \times [0, 1]} \mathcal{A} \longrightarrow I$ induced by the pairing of \mathcal{A} is an equivalence.

Remark 7.14. If \mathcal{A} is co-non-degenerate, then $\mathrm{Aut}(\Phi_{\mathcal{A}}(\mathbb{B}^3)) \cong \mathrm{Aut}(\mathrm{id}_I)$, which for $\mathcal{S} = \mathrm{Rex}$ is just the group \mathbf{k}^{\times} of units in the ground field \mathbf{k} . If the ambient symmetric monoidal $(2, 1)$ -category \mathcal{S} has internal homs, then application of $[-, I]$ to (7.5) yields the canonical map $I \longrightarrow Z_2(\mathcal{A})$ from the monoidal unit of \mathcal{S} to the Müger center of \mathcal{A} (we recalled the definition on page 14). This follows from the description of the Müger center in [BJSS21, Definition 2.6] as the internal endomorphisms of \mathcal{A} as $\int_{\mathbb{S}^1 \times [0, 1]} \mathcal{A}$ -module. This implies: If \mathcal{A} is co-non-degenerate, then \mathcal{A} is non-degenerate (i.e. $Z_2(\mathcal{A})$ is trivial). The converse holds if $\int_{\mathbb{S}^2} \mathcal{A}$ is dualizable in \mathcal{S} , which applies for instance to a finite ribbon category. This has an important consequence: A finite ribbon category, seen as cyclic framed E_2 -algebra in Rex , is co-non-degenerate if and only if its braiding is non-degenerate, i.e. if its Müger center $Z_2(\mathcal{A})$ is trivial. In this case, we find $\mathrm{Aut}(\Phi_{\mathcal{A}}(\mathbb{B}^3)) \cong \mathbf{k}^{\times}$. Hence, the extensions appearing in (7.3) will really be central extensions by \mathbf{k}^{\times} .

8 APPLICATIONS AND EXAMPLES

This section is devoted to applications of the main result in the case that our ambient symmetric monoidal $(2, 1)$ -category \mathcal{S} is given by Rex ; this special case was covered in Section 2.4.

8.1 The spaces of conformal blocks of a Rex -valued modular functor. Let \mathcal{A} be a cyclic framed E_2 -algebra in Rex . In fact, we will assume that the underlying linear category of \mathcal{A} is a finite linear category, as defined in Section 2.4. This certainly restricts the generality, but for all of the examples that we want to discuss, it will be sufficient. Under these assumptions, \mathcal{A} is in fact a Grothendieck-Verdier category in $\mathrm{Rex}^{\mathrm{f}}$.

Corollary 8.1. *Let \mathcal{A} be a ribbon Grothendieck-Verdier in Rex^f . Then \mathcal{A} extends in at most one way to a modular functor. In that case, the space of conformal blocks for the surface of genus g and n boundary components labeled with $X_1, \dots, X_n \in \mathcal{A}$ is isomorphic to*

$$\mathcal{A}(X_1 \otimes \cdots \otimes X_n \otimes \mathbb{A}^{\otimes g}, K)^* , \quad (8.1)$$

where $K \in \mathcal{A}$ is the dualizing object and $\mathbb{A} \in \mathcal{A}$ is the result of applying the monoidal product to the end $\int_{X \in \mathcal{A}} X \boxtimes DX \in \mathcal{A} \boxtimes \mathcal{A}$.

- (i) A sufficient condition for the existence of the modular functor extension is cofactorizability of \mathcal{A} .
- (ii) The spaces of conformal blocks are always finite-dimensional.

Proof. The uniqueness statement is Theorem 6.6, and the existence in the cofactorizable case, i.e. statement (i), is Theorem 7.6. The formula for the vector space (8.1) follows immediately from the description of the underlying ansular functor, i.e. the Hbdy -algebra, see [MW23a, Corollary 6.3] (note that those results refer to Lex^f ; we need to dualize them to Rex^f). Statement (ii) follows from (8.1). \square

Remark 8.2. We should warn the reader that the identification of the space of conformal blocks with the vector space (8.1), while being very explicit, uses non-canonical choices. This problem is often ignored in the classical literature on modular functors, see [MW23b, Remark 7.10] for a more detailed comment.

8.2 Modular categories. Let us now treat the case of modular categories, as defined in Example 2.9:

Corollary 8.3. *For any modular category \mathcal{A} , seen as non-cyclic framed E_2 -algebra in Rex , there is an essentially unique extension to a modular functor, and this modular functor is equivalent to Lyubashenko's modular functor. The 2-group of ribbon autoequivalences of \mathcal{A} acts, up to coherent isomorphism, on this modular functor.*

This proves that for a given modular category the Lyubashenko construction is the essentially unique possible modular functor construction in a precise sense. To the best of our knowledge, this was previously not known.

Proof. Since \mathcal{A} is a finite ribbon category, it is a cyclic framed E_2 -algebra in Rex by Example 2.9. Moreover, this is the unique cyclic structure by [MW22, Corollary 4.4] because the braiding of \mathcal{A} is non-degenerate.

Thanks to [Shi19, Theorem 1.1], \mathcal{A} is factorizable and then also cofactorizable by [BJSS21, Theorem 1.6]. Now Theorem 7.6 gives us a unique extension to a modular functor. The extensions of mapping class groups are central extensions by the abelian group \mathbf{k}^\times by Remark 7.14. By Theorem 6.6 this modular functor must agree, up to equivalence with Lyubashenko's modular functor, provided that Lyubashenko's modular functor in genus zero actually agrees with \mathcal{A} as cyclic framed E_2 -algebra. This is indeed the case by [MW23b, Proposition 7.14].

For the additional statement on the action of the 2-group $\text{AUT}_{fE_2}(\mathcal{A})$ of ribbon autoequivalences of \mathcal{A} on the modular functor, we observe that ribbon autoequivalences are automatically compatible with the duality, so that they come automatically with the structure of a cyclic automorphism; in other words, the forgetful map $\text{cAUT}_{fE_2}(\mathcal{A}) \rightarrow \text{AUT}_{fE_2}(\mathcal{A})$ from cyclic fE_2 -automorphisms to non-cyclic ones has a section $s : \text{AUT}_{fE_2}(\mathcal{A}) \rightarrow \text{cAUT}_{fE_2}(\mathcal{A})$, see also [MW22, Corollary 4.6]. Now the action of $\text{AUT}_{fE_2}(\mathcal{A})$ on the modular functor $\mathfrak{F}_{\mathcal{A}}$ is given by

$$\text{AUT}_{fE_2}(\mathcal{A}) \xrightarrow{s} \text{cAUT}_{fE_2}(\mathcal{A}) \stackrel{\text{Corollary 6.9}}{\simeq} \text{AUT}(\mathfrak{F}_{\mathcal{A}}) .$$

\square

Since, in the semisimple case, the spaces of conformal blocks of this modular functor are exactly the state spaces of the Reshetikhin-Turaev construction, Corollary 8.3 is in line with [AKZ17], where the Reshetikhin-Turaev state spaces of an anomaly-free modular fusion category (i.e. a semisimple modular category) — as vector spaces, not as mapping class group representations — are produced from factorization homology.

Example 8.4 (The Hopf algebra case). If a modular category \mathcal{A} arises as finite-dimensional modules over a ribbon factorizable Hopf algebra H , the spaces of conformal blocks $\mathcal{A}(X_1 \otimes \cdots \otimes X_n \otimes \mathbb{A}^{\otimes g}, K)^*$ reduce to the familiar ones $\text{Hom}_H(X_1 \otimes \cdots \otimes X_n \otimes H_{\text{ad}}^{\otimes g}, k)^*$, where H_{ad} is the adjoint representation of H and k is the ground field with H -action via the counit. This is dual to the statement in [MW23b, Corollary 7.13].

8.3 Vertex operator algebras. Using the classification of modular functors we can give a universal construction for spaces of conformal blocks from a suitable vertex operator algebra. We refer to [Fre00] for one possible introduction to the topic of vertex operator algebras. Let V be a C_2 -cofinite vertex operator algebra and fix a notion of V -module that meets the requirements listed in [ALSW21, Theorem 2.12], so that the category of V -modules becomes a ribbon Grothendieck-Verdier category in Rex^f that we denote by $V\text{-mod}$. We will not list these conditions here and refer to [ALSW21] for the details, but these conditions are roughly speaking the weakest known conditions that make $V\text{-mod}$ braided monoidal and closed under taking the contragredient representation $X \mapsto X^*$.

Corollary 8.5 (*Universal construction of spaces of conformal blocks for a vertex operator algebra*). *Let V be a C_2 -cofinite vertex operator together with a notion of V -modules, subject to the conditions just mentioned. Then we have the following universal construction procedure for spaces of conformal blocks: We define spaces of conformal blocks in genus zero by sending an n -holed sphere whose boundary components are labeled by V -modules X_1, \dots, X_n to the dual hom space*

$$\text{Hom}_V(X_1 \otimes \cdots \otimes X_n, V^*)^* \quad (8.2)$$

while the mapping class groups in genus zero (ribbon braid groups) act through the ribbon Grothendieck-Verdier structure present on $V\text{-mod}$. Then the following statements hold:

- (i) This definition of spaces of genus zero conformal blocks (understood as cyclic Surf_0 -structure on $V\text{-mod}$) is the unique possible one that extends the balanced braided structure of $V\text{-mod}$ if the Müger center of $V\text{-mod}$ is trivial (for example if $V\text{-mod}$ is modular).
- (ii) There is, up to equivalence, at most one extension of the genus zero blocks (8.2) to a modular functor. The higher genus blocks are then given by the formula (8.1).
- (iii) The unique extension to a modular functor mentioned under (ii) exists if and only if $V\text{-mod}$ is connected. A sufficient condition is cofactorizability (which is satisfied for instance if $V\text{-mod}$ is modular).
- (iv) If $V\text{-mod}$ fails to be connected, then the higher genus blocks still exist and are compatible with gluing (sometimes called ‘factorization’). They will still carry a representation of handlebody groups instead of mapping class groups of surfaces.

Proof. The hypotheses guarantee that $V\text{-mod}$ is a ribbon Grothendieck-Verdier category in Rex^f by [ALSW21, Theorem 2.12] and hence a cyclic framed E_2 -algebra in Rex^f and also in Rex by the results from [MW23b] recalled in Section 2.4. The genus zero blocks are then given by (8.2), as follows from Corollary 8.1.

If the Müger center of $V\text{-mod}$ is trivial, then so is the balanced Müger center. Therefore, this cyclic structure is the only one by [MW22, Corollary 4.4]. This proves (i).

Moreover, Theorem 6.6 and Corollary 8.1 give us (ii) while the Theorems 6.8 and 7.6 prove (iii). Finally, [MW23a, Theorem 5.9] implies (iv). \square

Corollary 8.6. *The ansular functor from Corollary 8.5 (iv) built from the vertex operator algebra V extends always, regardless of connectedness of the category of V -modules, to a modular $\text{Surf}_{V\text{-mod}}$ -algebra that is the best possible approximation to an extension of the cyclic framed E_2 -algebra $V\text{-mod}$ to a modular functor in the sense that any possibly existing extension of $V\text{-mod}$ to a modular functor factors, uniquely up to a contractible choice, through this modular $\text{Surf}_{V\text{-mod}}$ -algebra structure.*

Proof. This is a consequence of Theorems 6.2 and 6.4. \square

Remark 8.7. For a given vertex operator algebra V , one can attempt to build spaces of conformal blocks/a modular functor in at least two ways:

- (i) By forming a suitable category of modules over V (this usually involves making some choices) to obtain a nice representation category (which in good cases will be a modular category) that can be inserted into the available modular functor constructions [Tur94, BK01, Lyu95a].
- (ii) By performing constructions directly using the vertex operator algebra, see the monograph of Ben-Zvi-Frenkel [BZF04] or, as a summary, [Fre00], and additionally the construction of Damiani-Gibney-Tarasca [DGT21].

The comparison between these approaches is a major open problem in conformal field theory, see e.g. the review article [FRS10, Section 3.2 & 3.3]. We will not attempt here to describe the construction (ii) within the framework of this article (it is not at all clear that this is in fact possible), but Corollary 8.5 provides a potentially powerful comparison tool by introducing in a sense a third construction which is doubtlessly closer to (i), but is ultimately just characterized by a universal property and not tied to certain specific construction or even a certain ‘ansatz’ of what the spaces of conformal blocks at a certain genus should look like. If one then exhibits arbitrary constructions of a modular functor, along the lines of (i), (ii) or even something else,

such that they all agree in genus zero — as cyclic framed E_2 -algebras — then all these constructions agree with the universal one up to equivalence. However, let us emphasize again that it is by no means obvious that a construction following (ii) actually produces a modular functor in the sense of this article. In fact, even for nice classes of rational vertex operator algebras V , the gluing property (‘factorization’) for the blocks constructed directly from V is extremely non-trivial and a major and recent result [DGT19]. In contrast to that, the blocks appearing in Corollary 8.5 always glue correctly.

The cofactorizability of $V\text{-mod}$ is an algebraic condition that one could hope to verify in relevant examples, but an in-depth investigation of this point is beyond the scope of this article. Nonetheless, we want to give at least one example that goes beyond the situation of Corollary 8.3:

Example 8.8 (*Feigin-Fuchs boson*). Let $\Psi = (\mathfrak{h}, \langle -, - \rangle, A, \xi)$ be bosonic lattice data in the sense of [ALSW21, Definition 3.1] as recalled in Example 2.10. Denote by $\text{VM}(\Psi)$ the associated ribbon Grothendieck-Verdier category, i.e. the category associated to the *Feigin-Fuchs boson*. The braiding of this category is non-degenerate. This is a consequence of the fact that the bilinear form $\langle -, - \rangle$ is assumed to be non-degenerate. The underlying braided monoidal category of $\text{VM}(\Psi)$ is rigid (but careful: the rigid duality is generally not the ribbon Grothendieck-Verdier duality of $\text{VM}(\Psi)$). By the [Shi19, Theorem 1.1] and [BJSS21, Theorem 1.6] this implies that $\text{VM}(\Psi)$ is cofactorizable (recall that this is a property of the underlying E_2 -algebra). Now Corollary 8.5 tells us that the ribbon Grothendieck-Verdier category describing the Feigin-Fuchs boson admits an essentially unique extension to a modular functor. We should remark that this category is not modular unless $\xi = 0$. The ribbon Grothendieck-Verdier category $\text{VM}(\Psi)$ has a group-cohomological description by [ALSW21, Theorem 3.12] (we have recalled this in Example 2.10): $\text{VM}(\Psi) \simeq \text{vect}_{A^*/A}^{\omega(\Psi), 2\xi}$. The underlying Hbdy-algebra for group-cohomological ribbon Grothendieck-Verdier categories is given in detail [MW23b, Example 7.12]. In particular, this allows us to conclude that the space of conformal blocks associated to a closed surface of genus g has dimension $|A^*/A|^g$ if $2(g-1)\xi = 0$. Otherwise, it is zero. In [ALSW21] a suggestion of a space of conformal blocks for the Feigin-Fuchs boson is already made in the case of the torus, and the question is raised whether a full modular functor can be built. The considerations of this example prove that this is indeed the case.

8.4 Drinfeld centers of possibly non-spherical pivotal finite tensor categories. We conclude with an example built from categories which are neither modular nor necessarily semisimple:

Corollary 8.9. *Let \mathcal{A} be a pivotal finite tensor category. Then the Drinfeld center $Z(\mathcal{A})$ comes naturally with a cyclic framed E_2 -structure that admits an essentially unique extension to a modular functor. This is, up to equivalence, the only modular functor whose underlying balanced braided monoidal category is $Z(\mathcal{A})$ with its usual balanced braided structure.*

Proof. By [MW22, Theorem 2.12] we have a cyclic framed E_2 -structure (equivalently, ribbon Grothendieck-Verdier structure) on $Z(\mathcal{A})$, namely the one with the distinguished invertible object of \mathcal{A} [ENO04] as dualizing object of $Z(\mathcal{A})$. This cyclic framed E_2 -structure is the only one whose underlying non-cyclic framed E_2 -structure is the usual balanced braided structure of $Z(\mathcal{A})$.

Now once again, we use that $Z(\mathcal{A})$ is not only factorizable by [ENO04, Proposition 4.4], but in fact cofactorizable [BJSS21, Theorem 1.6] and apply Theorem 7.6. \square

For the cyclic framed E_2 -structure on $Z(\mathcal{A})$ from Corollary 8.9, we have

$$Z(\mathcal{A}) \text{ is modular} \iff \mathcal{A} \text{ is spherical},$$

by [MW22, Corollary 2.13]. If $Z(\mathcal{A})$ is modular, then the associated modular functor is a special case of the modular functor associated to a modular category, but Corollary 8.9 does *not* assume that \mathcal{A} is spherical. We refer to [MW22, Section 3] for a description of the underlying ansular functor and the discussion of non-spherical examples.

The modular functor for $Z(\mathcal{A})$, with \mathcal{A} being a pivotal finite tensor category, should be thought of as a generalization of a modular functor of Turaev-Viro type. Indeed if \mathcal{A} is a spherical fusion category, it coincides with the Reshetikhin-Turaev modular functor of the modular category $Z(\mathcal{A})$ by Corollary 8.3, and therefore with the Turaev-Viro type modular functor of \mathcal{A} [TV10, Bal10]. In that case, there is a string-net description [Kir11, FSY23]. For non-spherical fusion categories, one would expect a relation to the non-spherical string-nets in [Run20]. For a pivotal finite tensor category \mathcal{A} that is not necessarily fusion, it is conjectured in [DSPS20, Conjecture 3.5.11] and the comments following afterwards that \mathcal{A} gives rise to a three-dimensional combed local field theory (a combing on a three-dimensional manifold is a choice of a non-vanishing vector field). Corollary 8.9 provides the underlying modular functor. If one drops the pivotal structure, then at least a framed modular functor is guaranteed through [DSPS20, Theorem 3.2.2

and Corollary 3.2.3] via the cobordism hypothesis. A state sum construction of this framed modular functor is given in [FSS22]. The modular functor from Corollary 8.9 should admit a description that is an oriented extension of this state sum construction, see also [FSS22, Remark 5.28].

REFERENCES

- [Abr96] L. Abrams. Two-dimensional topological quantum field theories and Frobenius algebras. *J. Knot Theory Ramif.*, 5(5):569–587, 1996.
- [AF15] D. Ayala and J. Francis. Factorization homology of topological manifolds. *J. Top.*, 8(4):1045–1084, 2015.
- [AKZ17] Y. Ai, L. Kong, and H. Zheng. Topological orders and factorization homology. *Adv. Theor. Math. Phys.*, 21(8):1845–1894, 2017.
- [ALSW21] R. Allen, S. Lentner, C. Schweigert, and S. Wood. Duality structures for module categories of vertex operator algebras and the Feigin Fuchs boson. arXiv:2107.05718 [math.QA], 2021.
- [And06] J. E. Andersen. Asymptotic faithfulness of the quantum $SU(n)$ representations of the mapping class groups. *Ann. Math.*, 163(1):347–368, 2006.
- [Ati88] M. Atiyah. Topological quantum field theory. *Publ. math. IHÉS*, 68:175–186, 1988.
- [Ati90] M. Atiyah. On Framings of 3-Manifolds. *Topology*, 29(1):1–7, 1990.
- [AU12] J. E. Andersen and K. Ueno. Modular functors are determined by their genus zero data. *Quantum Topol.*, 3(4):255–291, 2012.
- [Bal10] B. Balsam. Turaev-Viro invariants as an extended TQFT III. arXiv:1012.0560 [math.QA], 2010.
- [BBF05] V. Blanco, M. Bulejos, and E. Faro. Categorical non abelian cohomology, and the Schreier theory of groupoids. *Math. Z.*, 251:41–59, 2005.
- [BD95] J. Baez and J. Dolan. Higher-Dimensional Algebra and Topological Quantum Field Theory. *J. Math. Phys.*, 36(11):6073–6105, 1995.
- [BD04] A. Beilinson and V. Drinfeld. *Chiral Algebras*, volume 51 of *Colloquium Publications*. Amer. Math. Soc., 2004.
- [BD13] M. Boyarchenko and V. Drinfeld. A duality formalism in the spirit of Grothendieck and Verdier. *Quantum Top.*, 4(4):447–489, 2013.
- [BDSPV15] B. Bartlett, C. L. Douglas, C. Schommer-Pries, and J. Vicary. Modular categories as representations of the 3-dimensional bordism category. arXiv:1509.06811 [math.AT], 2015.
- [BJS21] A. Brochier, D. Jordan, and N. Snyder. On dualizability of braided tensor categories. *Compositio Math.*, 157(3):435–483, 2021.
- [BJSS21] A. Brochier, D. Jordan, P. Safronov, and N. Snyder. Invertible braided tensor categories. *Alg. Geom. Top.*, 21(4):2107–2140, 2021.
- [BK00] B. Bakalov and A. Kirillov. On the Lego-Teichmüller game. *Transf. Groups*, 6:207–244, 2000.
- [BK01] B. Bakalov and A. Kirillov. *Lectures on tensor categories and modular functors*, volume 21 of *University Lecture Series*. Amer. Math. Soc., 2001.
- [BV68] J. M. Boardman and R. M. Vogt. Homotopy-everything H -spaces. *Bull. Amer. Math. Soc.*, 74:1117–1122, 1968.
- [BV73] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces*, volume 347 of *Lecture Notes in Math*. Springer, 1973.
- [BZBJ18a] D. Ben-Zvi, A. Brochier, and D. Jordan. Integrating quantum groups over surfaces. *J. Top.*, 11(4):874–917, 2018.
- [BZBJ18b] D. Ben-Zvi, A. Brochier, and D. Jordan. Quantum character varieties and braided module categories. *Selecta Math. New Series*, 24(35):4711–4748, 2018.
- [BZF04] D. Ben-Zvi and E. Frenkel. *Vertex Algebras and Algebraic Curves*, volume 88. Amer. Math. Soc., 2004.
- [Car89] J. Cardy. Boundary conditions, fusion rules and the Verlinde formula. *Nucl. Phys. B.*, 324(3):581–596, 1989.
- [CCG10] P. Carrasco, A. M. Cegarra, and A. R. Garzón. Nerves and classifying spaces for bicategories. *Alg. Geom. Top.*, 10:219–274, 2010.
- [Ceg11] A. M. Cegarra. Homotopy fiber sequences induced by 2-functors. *J. Pure Appl. Alg.*, 215(4):310–334, 2011.
- [CG16] K. Costello and O. Gwilliam. *Factorization Algebras in Quantum Field Theory*, volume 31 of *New Mathematical Monographs*. Cambridge University Press, 2016.
- [Coo23] J. Cooke. Excision of skein categories and factorisation homology. *Adv. Math.*, 414:108848, 2023.

- [Cos04] K. Costello. The A-infinity operad and the moduli space of curves. arXiv:math/0402015 [math.AG], 2004.
- [Ded64] P. Dedecker. Les foncteurs Ext II, H² II et H² II non abéliens. *C. R. Acad. Sc. Paris*, 258:4891–4895, 1964.
- [DGT19] C. Damiolini, A. Gibney, and N. Tarasca. On factorization and vector bundles of conformal blocks from vertex algebras. To appear in the *Ann. Sci. ENS*, arXiv:1909.04683 [math.AG], 2019.
- [DGT21] C. Damiolini, A. Gibney, and N. Tarasca. Conformal blocks from vertex algebras and their connections on Mgn-bar. *Geom. Top.*, 25(5):22235–2286, 2021.
- [DRGG⁺22a] M. De Renzi, A. M. Gainutdinov, N. Geer, B. Patureau-Mirand, and I. Runkel. 3-Dimensional TQFTs from Non-Semisimple Modular Categories. *Selecta Math. New Ser.*, 28(42), 2022.
- [DRGG⁺22b] M. De Renzi, A. M. Gainutdinov, N. Geer, B. Patureau-Mirand, and I. Runkel. Mapping class group representations from non-semisimple TQFTs. *Comm. Contemp. Math.*, 2150091, 2022.
- [DSPS20] C. L. Douglas, C. Schommer-Pries, and N. Snyder. Dualizable Tensor Categories. *Mem. Amer. Math. Soc.*, 2020.
- [Dus02] J. Duskin. Simplicial matrices and the nerves of weak n-categories I: nerves of bicategories. *Theory App. Cat.*, 9(10):198–308, 2002.
- [EGNO15] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik. *Tensor categories*, volume 205 of *Math. Surveys Monogr.* Amer. Math. Soc., 2015.
- [EML53] S. Eilenberg and S. Mac Lane. On the groups $H(\pi, n)$, I. *Ann. Math.*, 58(1):55–106, 1953.
- [ENO04] P. Etingof, D. Nikshych, and V. Ostrik. An analogue of Radford’s S^4 formula for finite tensor categories. *Int. Math. Res. Not.*, 54:2915–2933, 2004.
- [EO04] P. Etingof and V. Ostrik. Finite tensor categories. *Mosc. Math. J.*, 4(3):627–654, 2004.
- [FM12] B. Farb and D. Margalit. *A Primer on Mapping Class Groups*, volume 49 of *Princeton Math. Series.* Princeton University Press, 2012.
- [Fra13] Ignacio López Franco. Tensor products of finitely cocomplete and abelian categories. *J. Alg.*, 396:207–219, 2013.
- [Fre00] E. Frenkel. Vertex Algebras and Algebraic Curves. Séminaire Bourbaki, Exp. 875 arXiv:math/0007054 [math.QA], 2000.
- [Fre12a] D. S. Freed. 3-dimensional TQFTs through the lens of the cobordism hypothesis. Slides available at <https://www.ma.utexas.edu/users/dafr/StanfordLecture.pdf>, 2012.
- [Fre12b] D. S. Freed. 4-3-2-8-7-6. Slides available at <https://www.ma.utexas.edu/users/dafr/Aspects.pdf>, 2012.
- [Fre17] B. Fresse. *Homotopy of operads and Grothendieck-Teichmüller groups. Part 1: The Algebraic Theory and its Topological Background*, volume 217 of *Math. Surveys and Monogr.* Amer. Math. Soc., 2017.
- [FRS02] J. Fuchs, I. Runkel, and C. Schweigert. TFT construction of RCFT correlators. I: Partition functions. *Nucl. Phys. B*, 646:353–497, 2002.
- [FRS04a] J. Fuchs, I. Runkel, and C. Schweigert. TFT construction of RCFT correlators. II: Unoriented world sheets. *Nucl. Phys. B*, 678:511–637, 2004.
- [FRS04b] J. Fuchs, I. Runkel, and C. Schweigert. TFT construction of RCFT correlators. III: Simple currents. *Nucl. Phys. B*, 694:277–353, 2004.
- [FRS05] J. Fuchs, I. Runkel, and C. Schweigert. TFT construction of RCFT correlators. IV: Structure constants and correlation functions. *Nucl. Phys. B*, 715:539–638, 2005.
- [FRS10] J. Fuchs, I. Runkel, and C. Schweigert. Twenty-five years of two-dimensional rational conformal field theory. *J. Math. Phys.*, 51(015210), 2010.
- [FS17] J. Fuchs and C. Schweigert. Consistent systems of correlators in non-semisimple conformal field theory. *Adv. Math.*, 307:598–639, 2017.
- [FSS22] J. Fuchs, G. Schaumann, and C. Schweigert. A modular functor from state sums for finite tensor categories and their bimodules. *Theory App. Cat.*, 38:436–594, 2022.
- [FSY23] J. Fuchs, C. Schweigert, and Y. Yang. *String-net construction of RCFT correlators*, volume 45 of *SpringerBriefs Math. Phys.* Springer, 2023.
- [FT14] D. S. Freed and C. Teleman. Relative quantum field theory. *Commun. Math. Physics*, 326:459–476, 2014.
- [FWW02] M. Freedman, K. Walker, and Z. Wang. Quantum SU(2) faithfully detects mapping class groups modulo center. *Geom. Top.*, 6:523–539, 2002.
- [Gia11] J. Giansiracusa. The framed little 2-discs operad and diffeomorphisms of handlebodies. *J. Top.*, 4(4):919–941, 2011.

- [Gin15] G. Ginot. Notes on factorization algebras, factorization homology and applications. In D. Calaque and T. Strobl, editors, *Mathematical Physics Studies*, pages 429–552. Springer, 2015.
- [GJS23] S. Gunningham, D. Jordan, and P. Safronov. The finiteness conjecture for skein modules. *Invent. Math.*, 232:301–363, 2023.
- [GK95] E. Getzler and M. Kapranov. Cyclic operads and cyclic homology. In R. Bott and S.-T. Yau, editors, *Conference proceedings and lecture notes in geometry and topology*, pages 167–201. Int. Press, 1995.
- [GK98] E. Getzler and M. Kapranov. Modular operads. *Compositio Math.*, 110:65–126, 1998.
- [GM13] P. M. Gilmer and G. Masbaum. Maslov index, Lagrangians, Mapping Class Groups and TQFT. *Forum Math.*, 25(5):1067–1106, 2013.
- [GS18] O. Gwilliam and C. Scheimbauer. Duals and adjoints in higher morita categories. arXiv:1804.10924 [math.CT], 2018.
- [Hat76] A. Hatcher. Spaces of incompressible surfaces. Updated version of the paper ‘Homeomorphisms of sufficiently large p^2 -irreducible 3-manifolds’ in *Topology* 15:343–347, 1976, available from the author’s web page at pi.math.cornell.edu/~hatcher/Papers/emb.pdf. 1976.
- [Hau17] R. Haugseng. The higher Morita category of E_n -algebras. *Geom. Top.*, 21(3):1631–1730, 2017.
- [Hol08] S. Hollander. A homotopy theory for stacks. *Israel J. Math.*, 163:193–124, 2008.
- [HRY20] P. Hackney, M. Robertson, and D. Yau. A graphical category for higher modular operads. *Adv. Math.*, 365:107044, 2020.
- [Hua08] Y.-Z. Huang. Rigidity and modularity of vertex tensor categories. *Comm. Contemp. Math.*, 10(1):871–911, 2008.
- [JFS17] Theo Johnson-Freyd and Claudia Scheimbauer. (Op) lax natural transformations, twisted quantum field theories, and “even higher” Morita categories. *Adv. Math.*, 307:147–223, 2017.
- [Kas15] C. Kassel. *Quantum Groups*, volume 155 of *Graduate Texts in Math*. Springer, 2015.
- [Kir11] A. Kirillov. String-net model of Turaev-Viro invariants. arXiv:1106.6033 [math.AT], 2011.
- [KL01] T. Kerler and V. V. Lyubashenko. *Non-Semisimple Topological Quantum Field Theories for 3-Manifolds with Corners*, volume 1765 of *Lecture Notes in Math*. Springer, 2001.
- [Koc03] J. Kock. *Frobenius Algebras and 2D Topological Quantum Field Theories*, volume 2003. London Math. Soc. Student Texts, 2003.
- [LMSS22] S. Lentner, S. N. Mierach, C. Schweigert, and Y. Sommerhäuser. Hochschild Cohomology, Modular Tensor Categories, and Mapping Class Groups. *Springer Briefs Math. Phys.*, 2022.
- [Lur] J. Lurie. Higher algebra. Available at <https://www.math.ias.edu/~lurie/papers/HA.pdf>.
- [Lur09a] J. Lurie. *Higher Topos Theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, 2009.
- [Lur09b] J. Lurie. On the classification of topological field theories. *Current development in math*, 2008:129–280, 2009.
- [Lyu95a] V. V. Lyubashenko. Invariants of 3-manifolds and projective representations of mapping class groups via quantum groups at roots of unity. *Comm. Math. Phys.*, 172:467–516, 1995.
- [Lyu95b] V. V. Lyubashenko. Modular transformations for tensor categories. *J. Pure Appl. Alg.*, 98:279–327, 1995.
- [Lyu96] V. V. Lyubashenko. Ribbon abelian categories as modular categories. *J. Knot Theory and its Ramif.*, 5:311–403, 1996.
- [May72] P. May. *The Geometry of Iterated Loop Spaces*, volume 217 of *Lecture Notes in Math*. Springer, 1972.
- [MLM92] S. Mac Lane and I. Moerdijk. *Sheaves in Geometry and Logic*. Springer Universitext. Springer, 1992.
- [MR95] G. Masbaum and J. Roberts. On central extensions of mapping class groups. *Math. Ann.*, 302:131–150, 1995.
- [MS88] G. Moore and N. Seiberg. Polynomial equations for rational conformal field theories. *Phys. Letters B*, 212(4):451–460, 1988.
- [MS89] G. Moore and N. Seiberg. Classical and Quantum Conformal Field Theory. *Comm. Math. Phys.*, 123:177–254, 1989.
- [MS90] G. Moore and N. Seiberg. Lectures on RCFT. In H. C. Lee, editor, *NATO ASI Series*, pages 263–361. Springer, 1990.
- [MS21] M. Mignard and P. Schauenburg. Modular categories are not determined by their modular data. *Lett. Math. Phys.*, 111(60), 2021.
- [MW22] L. Müller and L. Woike. The distinguished invertible object as ribbon dualizing object in the Drinfeld center. arXiv:2212.07910 [math.QA], 2022.

- [MW23a] L. Müller and L. Woike. Classification of Consistent Systems of Handlebody Group Representations. *Int. Math. Res. Not.*, rnad178, 2023.
- [MW23b] L. Müller and L. Woike. Cyclic framed little disks algebras, Grothendieck-Verdier duality and handlebody group representations. *Quart. J. Math.*, 74(1):163–245, 2023.
- [MW23c] L. Müller and L. Woike. The Diffeomorphism Group of the Genus One Handlebody and Hochschild homology. *Proc. Amer. Math. Soc.*, 151(6):2311–2324, 2023.
- [Rie14] E. Riehl. *Categorical Homotopy Theory*, volume 24 of *New Math. Monogr.* Cambridge University Press, 2014.
- [Rob94] J. Roberts. Skeins and mapping class groups. *Math. Proc. Camb. Phil. Soc.*, 115:53–77, 1994.
- [RT90] N. Reshetikhin and V. G. Turaev. Ribbon graphs and their invariants derived from quantum groups. *Comm. Math. Phys.*, 127:1–26, 1990.
- [RT91] N. Reshetikhin and V. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.*, 103:547–597, 1991.
- [Run20] I. Runkel. String-net models for non-spherical pivotal fusion categories. *J. Knot Theory Ramif.*, 29(6):2050035, 2020.
- [Sch26] O. Schreier. Über die Erweiterung von Gruppen I. *Monatsh. Math. Phys.*, 34(1):165–180, 1926.
- [Seg88] G. Segal. Two-dimensional conformal field theories and modular functors. In *IX International Conference on Mathematical Physics (IAMP)*, 1988.
- [Shi19] K. Shimizu. Non-degeneracy conditions for braided finite tensor categories. *Adv. Math.*, 355:106778, 2019.
- [SP09] C. J. Schommer-Pries. *The classification of two-dimensional extended topological field theories*. PhD thesis, Berkeley, 2009.
- [SW03] P. Salvatore and N. Wahl. Framed discs operads and Batalin-Vilkovisky algebras. *Quart. J. Math.*, 54:213–231, 2003.
- [SW21] C. Schweigert and L. Woike. Homotopy Coherent Mapping Class Group Actions and Excision for Hochschild Complexes of Modular Categories. *Adv. Math.*, 386:107814, 2021.
- [Tho79] R. W. Thomason. Homotopy colimits in the category of small categories. *Math. Proc. Cambridge Philos. Soc.*, 85(1):91–109, 1979.
- [Til98] U. Tillmann. \mathcal{S} -Structures for k -Linear Categories and the Definition of a Modular Functor. *J. London Math. Soc.*, 58(1):208–228, 1998.
- [Tin17] P. Tingley. A minus sign that used to annoy me but now I know why it is there (two constructions of the Jones polynomial). In S. Morrison and D. Penneys, editors, *Proc. Centre Math. Appl.*, pages 415–427. Mathematical Sciences Institute, The Australian National University, 2017.
- [Tur92] V. Turaev. Modular categories and 3-manifold invariants. *Int. J. Modern Physics B*, 06(11n12):1807–1824, 1992.
- [Tur94] V. G. Turaev. *Quantum Invariants of Knots and 3-Manifolds*, volume 18 of *Studies in Math.* De Gruyter, 1994.
- [TV10] V. Turaev and A. Virelizier. On two approaches to 3-dimensional TQFTs. arXiv:1006.3501 [math.GT], 2010.
- [Ver88] E. Verlinde. Fusion rules and modular transformations in 2D conformal field theory. *Nucl. Phys. B*, 300:360–376, 1988.
- [Wah01] N. Wahl. *Ribbon braids and related operads*. PhD thesis, Oxford, 2001.
- [Wal] K. Walker. TQFTs. Notes available at <http://canyon23.net/math/tc.pdf>.
- [Wit89] E. Witten. Quantum field theory and the Jones polynomial. *Comm. Math. Phys.*, 121(3):351–399, 1989.
- [Zet18] S. J. Zetsche. Generalised duality theory for monoidal categories and applications. Master thesis, Hamburg, 2018.