Best multi-valued approximants via multi-designs

María José Benac a,b , Noelia Belén Rios c,d , Mariano Ruiz c,d*

^aDto. Académico de Matemática, FCEyT-UNSE, Santiago del Estero, Argentina
 ^bInstituto de Recursos Hídricos - FCEyT- CONICET, Santiago del Estero, Argentina
 ^cCentro de Matemática de La Plata, FCE-UNLP, La Plata, Argentina
 ^dIAM-CONICET, Buenos Aires, Argentina

Abstract

Let $\mathbf{d}=(d_j)_{j\in\mathbb{I}_m}\in\mathbb{N}^m$ be a decreasing finite sequence of positive integers, and let $\alpha=(\alpha_i)_{i\in\mathbb{I}_n}$ be a finite and non-increasing sequence of positive weights. Given a family $\Phi^0=(\mathcal{F}_j^0)_{j\in\mathbb{I}_m}$ of Bessel sequences with $\mathcal{F}_j^0=\{f_{i,j}^0\}_{i\in\mathbb{I}_k}\in(\mathbb{C}^{d_j})^k$ for each $1\leq j\leq m$, our main purpose on this work is to characterize the best approximants of the m-tuple of frame operators of the elements of Φ^0 in the set $D(\alpha,\mathbf{d})$ of the so-called (α,\mathbf{d}) -designs, which are the m-tuples $\Phi=(\mathcal{F}_j)_{j\in\mathbb{I}_m}$ such that each $\mathcal{F}_j=\{f_{i,j}\}_{i\in\mathbb{I}_n}$ is a finite sequence in \mathbb{C}^{d_j} , and $\sum_{j\in\mathbb{I}_m}\|f_{i,j}\|^2=\alpha_i$ for $i\in\mathbb{I}_n$. Specifically, in this work we completely characterize the minimizers of the Joint Frame Operator Distance (JFOD) function: $\Theta:D(\alpha,\mathbf{d})\to\mathbb{R}_{\geq 0}$ given by

$$\Theta(\Phi) = \sum_{j=1}^{m} \|S_{\mathcal{F}_{j}} - S_{\mathcal{F}_{j}^{0}}\|_{2}^{2},$$

where $S_{\mathcal{F}}$ denotes the frame operator of \mathcal{F} and $\|\cdot\|_2$ is the Frobenius norm. Indeed, we show that local minimizers of Θ are also global and we obtain an algorithm to construct the optimal (α, \mathbf{d}) -desings. As an application of the main result, in the particular case that m = 1, we also characterize global minimizers of a G-frames problem recently considered by He, Leng and Xu.

AMS subject classification: 42C15, 15A60.

Keywords: Frames, frames completions, proximity problems, majorization.

1 Introduction

Motivated by many applications in matrix theory, matrix approximation problems (or matrix nearness problems) have been studied for several years. There are many books and papers in the literature that deal with different variants of these problems, see for example [9] and [11] for a more detailed discussion of the subject and references.

^{*}Partially supported by CONICET (PIP 00954CO - 2022), UNLP (11X829), UNSE (23/C190-PIP-2022) e-mail addresses: mjbenac@gmail.com, nbrios@mate.unlp.edu.ar, mruiz@mate.unlp.edu.ar

Let $\mathcal{M}_{n\times d}(\mathbb{C})$ be the space of complex matrices of size $n\times d$. Given a non empty subset \mathcal{X} of $\mathcal{M}_{n\times d}(\mathbb{C})$ and $A\in \mathcal{M}_{n\times d}(\mathbb{C})$, a usual matrix nearness problem is to compute

$$\delta = \min_{X \in \mathcal{X}} N(A - X),$$

where $N(\cdot)$ is a unitary invariant norm, that is, N(UAW) = N(A) for every $A \in \mathcal{M}_{n \times d}$ and every pair of unitary matrices $U \in \mathcal{M}_n(\mathbb{C})$ and $W \in \mathcal{M}_d(\mathbb{C})$. Typically, the matrix norm used is the Frobenius norm: $||A||_2 = \operatorname{tr}(A^*A)$ which has some desirable properties.

If such a distance can be calculated, a natural issue that arises is to characterize the set of best approximants, that is the set

$$\mathcal{X}^{op} = \{ X \in \mathcal{X} : N(A - X) = \delta \}.$$

Finite frame theory provided many of such matrix approximation problems (or Procustes type problems) related to frame designs. Given a finite dimensional complex Hilbert space \mathcal{H} , a frame \mathcal{F} for \mathcal{H} is simply a generating set of vectors of \mathcal{H} . Associated to a frame \mathcal{F} , there is a positive definite bounded linear operator $S_{\mathcal{F}}$ of \mathcal{H} , called *frame operator*, that allows to perform encoding-decoding schemes. For practical reasons sometimes it is useful to find frames with some structure whose frame operators are "close" to some definite positive operator A. These kind of approximation problems were considered by some of the authors in [8] and [14], in which they were attacked with various tools of matrix analysis, such as the Schur-Horn theorem or Lidskii inequalities. These results are also related to optimal designs of frames with specific predetermined characteristics obtained by minimizing some convex potentials on sets of frames (see [16], [17],[18]).

In [13], the authors solved completely a conjecture posed by N. Strawn in [20] related to an approximation problem. Given a $S \in \mathcal{M}_d(\mathbb{C})^+$ and a fixed finite sequence of positive weights $\alpha = (\alpha_i)_{i \in \mathbb{I}_m}$, N. Strawn considered the following setting: let $\mathcal{D}(\alpha, d)$ denote the finite sequences $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n} \in (\mathbb{C}^d)^n$ such that $||f_i||^2 = \alpha_i$, $i \in \mathbb{I}_n$. Consider in $\mathcal{D}(\alpha, d)$ the product metric (i.e. the metric as a subset of $(\mathbb{C}^d)^n$); let $\Theta : \mathcal{D}(\alpha, d) \to \mathbb{R}_{\geq 0}$, be given by $\Theta(\mathcal{F}) = ||S - S_{\mathcal{F}}||_2$, where $||\cdot||_2$ denotes the Frobenius norm. Strawn conjectured that local minimizers of Θ where actually global minimizers. This assertion becomes relevant in applied situations in which numerical methods based on gradient descent or alternating projections methods are used to obtain local minimizers of Θ , [12].

In [13], Strawn's conjeture was settled in the affirmative and the spectral and geometrical structures of the minimizers of the function Θ defined above (called the *frame operator distance*) were explicitly computed.

In this work we consider a natural extension of the previous problem to a simultaneous approximation problem. Now, given a positive integer m, we consider the sets of m-tuples $\Phi = (\mathcal{F}_j)_{j \in \mathbb{I}_m}$, such that each $\mathcal{F}_j = \{f_{ij}\}_{i \in \mathbb{I}_n}$ is a sequence in \mathbb{C}^{d_j} such that

$$\sum_{j \in \mathbb{I}_m} \|f_{ij}\|^2 = \alpha_i.$$

These m- tuples are called (α, \mathbf{d}) -designs. Then, given a fixed sequence of positive operators $\{S_j\}$, we consider the function

$$\Theta(\Phi) = \sum_{j=1}^{m} ||S_j - S_{\mathcal{F}_j}||_2^2,$$

which measure the joint frame operator distance between $\{S_j\}_{j\in\mathbb{I}_m}$ and the frame operators $\{S_{\mathcal{F}_j}\}_{j\in\mathbb{I}_m}$.

The problem we consider in this work is to find the (α, \mathbf{d}) -designs Φ^{op} that minimize Θ , which result in the best simultaneous approximation of S_j , for $1 \leq j \leq m$. Moreover, since the set of (α, \mathbf{d}) -designs can be endowed with a natural (product) metric, we also consider the study of the spectral and geometric structure of the local minimizers of Θ in this set. Notice that the particular case m = 1 represent Strawn's problem described above.

The case m > 1 is original and correspond to a natural extension of Strawn's problem, so it is forseeable that similar techniques allow us to find an spectral and geometric characterization of the local and global minimizers of Θ in the set of (α, \mathbf{d}) -designs. Specifically, we solve the multivalue Strawn's problem through a translation of the multi-completion problem given in [3], which means that the minimum in Θ are attained in the (α, \mathbf{d}) -designs that minimize the joint convex potential for a suitable multi-completion problem.

These notes are organized as follows. In Section 2 we include some preliminaries about matrix analysis and (α, \mathbf{d}) -designs. In Section 3 we prove the main result, that local minimizers of Θ are global, and we obtain an spectral characterization of this minimizers. In Section 4 we present an algorithm to find (effectively) the best approximants among (α, \mathbf{d}) -designs. Finally, in Section 5, we apply the m=1 case to an approximation problem for G-frames (see for example [21] and [22]), considered in [19]. This allows us to fully describe the minimizers for the distance problem considered and to suggest an algorithm that will allow us to construct the optimal G-frames.

2 Preliminaries and notation

In this Section we recall the notion of (α, \mathbf{d}) -design, the multi-completions and the main problems considered in [3], that plays a key role in our work. Next, we describe some basic notation and notions used throughout the rest of the paper.

We let $\mathcal{M}_d(\mathbb{C})$ for the algebra of $d \times d$ complex matrices. We denote by $\mathcal{H}(d) \subset \mathcal{M}_d(\mathbb{C})$ the real subspace of selfadjoint matrices and by $\mathcal{M}_d(\mathbb{C})^+ \subset \mathcal{H}(d)$ the cone of positive semidefinite matrices. We let $\mathcal{U}(d) \subset \mathcal{M}_d(\mathbb{C})$ denote the group of unitary matrices. For $d \in \mathbb{N}$, let $\mathbb{I}_d = \{1, \ldots, d\}$ and let $\mathbb{I}_d = \{1\}_{i \in \mathbb{I}_d} \in \mathbb{R}^d$ be the vector with all its entries equal to 1.

Given $x = (x_i)_{i \in \mathbb{I}_d} \in \mathbb{R}^d$ we denote by $x^{\downarrow} = (x_i^{\downarrow})_{i \in \mathbb{I}_d}$ (respectively $x^{\uparrow} = (x_i^{\uparrow})_{i \in \mathbb{I}_d}$) the vector obtained by rearranging the entries of x in non-increasing (respectively non-decreasing) order. We denote by $(\mathbb{R}^d)^{\downarrow} = \{x^{\downarrow} : x \in \mathbb{R}^d\}$, $(\mathbb{R}^d)^{\downarrow} = \{x^{\downarrow} : x \in \mathbb{R}^d\}$ and analogously for $(\mathbb{R}^d)^{\uparrow}$ and $(\mathbb{R}^d_{\geq 0})^{\uparrow}$.

We also denote by $I_d \in \mathcal{M}_d(\mathbb{C})$ the identity matrix. Given $S \in \mathcal{M}_d(\mathbb{C})$ we let $R(S) \subset \mathbb{C}^d$ denote the range (or image) of S and $\mathrm{rk}(S)$ denote the rank of S, i.e. the dimension of R(S). Given a matrix $A \in \mathcal{H}(d)$ we denote by $\lambda(A) = \lambda^{\downarrow}(A) = (\lambda_i(A))_{i \in \mathbb{I}_d} \in (\mathbb{R}^d)^{\downarrow}$ the eigenvalues of A counting multiplicities and arranged in non-increasing order, and by $\lambda^{\uparrow}(A)$ the same vector but arranged in non-decreasing order. On the other hand, we denote by $\sigma(A) \subset \mathbb{R}$ its spectrum, i.e. the set of eigenvalues of A. If $x, y \in \mathbb{C}^d$ we denote by $x \otimes y \in \mathcal{M}_d(\mathbb{C})$ the rank-one matrix given by $(x \otimes y) z = \langle z, y \rangle x$, for $z \in \mathbb{C}^d$.

2.1Finite frames

Given a finite sequence $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ in \mathbb{C}^d , $S_{\mathcal{F}} \in \mathcal{M}_d(\mathbb{C})^+$ (a Bessel sequence using frame terminology) will denote the frame operator of \mathcal{F} , which is given by

$$S_{\mathcal{F}} f = \sum_{i \in \mathbb{I}_n} \langle f, f_i \rangle f_i = \sum_{i \in \mathbb{I}_n} (f_i \otimes f_i) f$$
 for $f \in \mathbb{C}^d$.

If there exists a constant a > 0 such that

$$a \|f\|^2 \le \sum_{i \in \mathbb{I}_n} |\langle f, f_i \rangle|^2 \quad \text{for all} \quad f \in \mathbb{C}^d.$$
 (1)

we say that \mathcal{F} is a frame for \mathbb{C}^d . This condition is equivalent to say that \mathcal{F} spans \mathbb{C}^d or that $S_{\mathcal{F}}$ is a positive invertible operator acting on \mathbb{C}^d .

Recall now the notion of majorization between real vectors, which is a partial pre-order relation in \mathbb{R}^d that arises naturally in matrix analysis, and that will play a central role throughout our work. Let $x, y \in \mathbb{R}^d$. We say that x is submajorized by y, and write $x \prec_w y$, if

$$\sum_{i=1}^{j} x_i^{\downarrow} \le \sum_{i=1}^{j} y_i^{\downarrow} \quad \text{for every} \quad 1 \le j \le d.$$

If $x \prec_w y$ and $\operatorname{tr} x = \sum_{i \in \mathbb{I}_d} x_i = \sum_{i \in \mathbb{I}_d} y_i = \operatorname{tr} y$, then x is majorized by y, and write $x \prec y$. In addition, we say that x is strictly majorized by y if $x \prec y$ and $x^{\downarrow} \neq y^{\downarrow}$.

For convenience, we extend the definition to allow comparing vectors of positive entries and different sizes, if $x \in \mathbb{R}^k_{>0}$ and $y \in \mathbb{R}^d_{>0}$, we note $x \prec_w y$ if

$$\sum_{i=1}^{j} x_i^{\downarrow} \le \sum_{i=1}^{j} y_i^{\downarrow} \quad \text{for every} \quad 1 \le j \le \min\{k, d\}.$$

and x is majorized by y if $x \prec_w y$ and $\sum_{i \in \mathbb{I}_k} x_i = \sum_{i \in \mathbb{I}_d} y_i$. In several applications of finite frame theory, it is important to construct families $\mathcal{F} = \sum_{i \in \mathbb{I}_d} y_i$. $\{f_i\}_{i\in\mathbb{I}_k}\in(\mathbb{C}^{\bar{d}})^k$ in such a way that the frame operator $S_{\mathcal{F}}$ and the squared norms $(\|f_i\|^2)_{i\in\mathbb{I}_k}$ are prescribed in advance. This problem is known as the frame design problem, and its solution can be obtained in terms of the Schur-Horn theorem for majorization.

Theorem 2.1 (See [1]). Let $S \in \mathcal{M}_d(\mathbb{C})^+$ and let $\mathbf{a} = (a_i)_{i \in \mathbb{I}_k} \in (\mathbb{R}_{>0}^k)$. Then, the following statements are equivalent:

1. There exists $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_k} \in (\mathbb{C}^d)^k \text{ such that } S_{\mathcal{F}} = S \text{ and } ||f_i||^2 = a_i \text{ , for } i \in \mathbb{I}_k \text{ ;}$

2.
$$\mathbf{a} \prec \lambda(S)$$
.

Preliminaries on (α, \mathbf{d}) -designs

Given a m-tuple of natural numbers $\mathbf{d} = (d_1, d_2, \dots, d_m)$, arranged in a non-increasing order, a d-design is any family of Bessel sequences:

$$\Phi = \{\mathcal{F}_j\}_{j \in \mathbb{I}_m},$$

such that each $\mathcal{F}_j = \{f_{i,j}\}_{i \in \mathbb{I}_k}$ is a Bessel sequence for \mathbb{C}^{d_j} .

Our interest is to consider **d**-designs with some restriction on the sizes of the vectors on the Bessel sequences.

Namely, let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in (\mathbb{R}^n_{>0})$ be a sequence of weights. Then, an (α, \mathbf{d}) -design $\Phi = \{\mathcal{F}_j\}_{j \in \mathbb{I}_m}$ is a \mathbf{d} -design such that

$$\sum_{j \in \mathbb{I}_m} ||f_{ij}||^2 = \alpha_i, \text{ for } i \in \mathbb{I}_n.$$

The set of all (α, \mathbf{d}) -designs shall be denoted by $\mathcal{D}(\alpha, \mathbf{d})$. Also, with the aim to simplify some calculations, we assume that the weights are arranged in a non-increasing order.

Notice that, if m=1 and $\mathbf{d}=d$, (α, \mathbf{d}) -designs generalize the notion of the structured Bessel sequences for \mathbb{C}^d with prescribed norms given by α . That is, those Bessel sequences $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ whose vectors lie in the α -torus

$$\mathcal{B}_{\alpha,d} = \{ \mathcal{F} = \{ f_i \}_{i \in \mathbb{I}_n} \in (\mathbb{C}^d)^n : \| f_i \|^2 = \alpha_i, \ i \in \mathbb{I}_n \}.$$
 (2)

In [3], the authors studied the problem of finding (α, \mathbf{d}) -designs $\Phi^{op} = \{\mathcal{F}_j^{op}\}_{j \in \mathbb{I}_m}$ that complete an initial \mathbf{d} -design $\Phi^0 = \{\mathcal{F}_j^0\}_{j \in \mathbb{I}_m}$ in an optimal sense.

In what follows, we will detail this multi-completion problem and the results obtained in [3], which will be useful in the next Section.

Consider an *n*-tuple $\alpha = (\alpha_i)_{i \in \mathbb{I}_n} \in \mathbb{R}^n_{>0}$, arranged in a non-increasing order and let $\mathbf{d} = (d_j)_{j \in \mathbb{I}_m} \in (\mathbb{N}^m)^{\downarrow}$ be such that $d_1 \leq n$ (this last condition is to assure that the optimal completions are frames for their respective spaces).

We shall consider the set of (α, \mathbf{d}) -designs, $\mathcal{D}(\alpha, \mathbf{d})$, endowed with the metric

$$m(\Phi, \Phi') = \sum_{j \in \mathbb{T}_m} \left(\sum_{i \in \mathbb{T}_m} \|f_{i,j} - f'_{i,j}\|^2 \right)^{1/2},$$

where $\Phi, \Phi' \in \mathcal{D}(\alpha, \mathbf{d})$.

We set a fixed **d**-design $\Phi^0 = (\mathcal{F}_j^0)_{j \in \mathbb{I}_m}$. The goal is to find and to characterize optimal (multi) completions of Φ^0 among the (α, \mathbf{d}) -designs. That completion is obtained by appending to each \mathcal{F}_j^0 the vectors of the respective Bessel sequence in the (α, \mathbf{d}) -design.

Here, the optimality is measured in terms of (joint) Benedetto-Fickus potential P of the multicompletions. That is, the goal is to find the local minimizers of the function $\Psi : \mathcal{D}(\alpha, \mathbf{d}) \to \mathbb{R}_{\geq 0}$, given by

$$\Psi(\Phi) = P(\Phi^0, \Phi) = \sum_{j \in \mathbb{I}_m} \operatorname{tr}(S^2_{(\mathcal{F}^0_j, \mathcal{F}_j)}) = \sum_{j \in \mathbb{I}_m} \sum_{i \in \mathbb{I}_{d_j}} \lambda_i^2(S_{(\mathcal{F}^0_j, \mathcal{F}_j)}),$$
(3)

where $S_{(\mathcal{F}_{j}^{0},\mathcal{F}_{j})} = S_{\mathcal{F}_{j}^{0}} + S_{\mathcal{F}_{j}}$ denotes the frame operator of the sequence $(\mathcal{F}_{j}^{0},\mathcal{F}_{j}) \in (\mathbb{C}^{d_{j}})^{k+n}$, for $j \in \mathbb{I}_{m}$ and the metric in $\mathcal{D}(\alpha,\mathbf{d})$ is induced by the distance defined above.

Now we are able to present a summarized version of the main result of [3] that shall be useful in the sequel.

First, let $\lambda_j = (\lambda_{i,j})_{i \in \mathbb{I}_{d_j}} = \lambda^{\uparrow}(S_{\mathcal{F}_j^0}) \in (\mathbb{R}_{\geq 0}^{d_j})^{\uparrow}$, for $j \in \mathbb{I}_m$ be the vectors of eigenvalues (arranged in a non-decreasing order) of each frame operator $S_{\mathcal{F}_j^0}$.

Theorem 2.2 ([3]). There exist vectors $\nu_j \in \mathbb{R}^{d_j}$ such that, for $\tilde{\Phi} \in \mathcal{D}(\alpha, \mathbf{d})$, we have

$$\tilde{\Phi}$$
 is a local minimizer of Ψ on $\mathcal{D}(\alpha, \mathbf{d})$ \iff $\lambda(S_{(\mathcal{F}_i^0, \tilde{\mathcal{F}}_i)}) = \nu_j^{\downarrow}$.

Moreover,

- 1. $\tilde{\Phi}$ is a global minimizer of Ψ .
- 2. For $j \in \mathbb{I}_m$, $S_{\tilde{\mathcal{F}}_j}$ commutes with $S_{\mathcal{F}_j^0}$ and $S_{\tilde{\mathcal{F}}_j} + S_{\mathcal{F}_j^0}$ is invertible. In particular, $(\mathcal{F}_j^0, \tilde{\mathcal{F}}_j)$ is a frame for \mathbb{C}^{d_j} .

The results proved in [3] state a stronger feature fulfilling local minima: the frame operators $S_{\tilde{\mathcal{F}}_j}$ not only commute with $S_{\mathcal{F}_j^0} + S_{\tilde{\mathcal{F}}_j}$ but also the vectors $\tilde{\mathcal{F}}_j = \{\tilde{f}_{ij}\}$ are eigenvectors of $S_{\mathcal{F}_j^0} + S_{\tilde{\mathcal{F}}_j}$. As a consequence, $\tilde{\mathcal{F}}_j$ decomposes into mutually orthogonal sets of vectors for each $j \in \mathbb{I}_m$.

These results allow to describe the spectra ν_j as $\nu_j = \max\{\mathbf{c}, \lambda_j\}$ (entry-wise maximum) where $\mathbf{c} \in (\mathbb{R}^{d_1})^{\downarrow}$ is a vector constructed from the data α and λ_j , $j \in \mathbb{I}_m$.

3 Local minimizers for the joint frame operator distance (JFOD)

In this section we will present a simultaneous approximation problem for Bessel sequences that generalizes previous results shown in [13]. Taking as a starting point a **d**-design $\Phi^0 = \{\mathcal{F}_j^0\}_{j\in\mathbb{I}_m}$ the goal is to characterize (α,\mathbf{d}) -designs that are local minimizers for some distance function defined on the frame operators. The approach is similar to the one developed in [13]: it reduces to finding the local (global) minima of a suitable joint convex potential for the (α,\mathbf{d}) -design problem described in the previous section for a particular case of initial data.

Given two **d**-designs $\Phi^1 = \{\mathcal{F}_j^1\}_{j \in \mathbb{I}_m}$ and $\Phi^2 = \{\mathcal{F}_j^2\}_{j \in \mathbb{I}_m}$, whose sequences of frame operators are $\Sigma_{\Phi^1} = \{S_{\mathcal{F}_j^1}\}_{j \in \mathbb{I}_m}$ and $\Sigma_{\Phi^2} = \{S_{\mathcal{F}_j^2}\}_{j \in \mathbb{I}_m}$, respectively, we define their joint frame operator distance (JFOD) as follows:

$$\operatorname{dist}_{JFOD}(\Sigma_{\Phi^1}, \Sigma_{\Phi^2}) = \left(\sum_{j \in \mathbb{I}_m} \|S_{\mathcal{F}_j^1} - S_{\mathcal{F}_j^2}\|_2^2\right)^{\frac{1}{2}}$$

where the norm $\|\cdot\|_2$ is the Frobenius norm.

Let $\Phi^0 = \{\mathcal{F}_j^0\}_{j \in \mathbb{I}_m}$ be a (fixed) **d**-design and consider a set of weights α as in previous section. As it was announced, our objective is to characterize those (α, \mathbf{d}) -designs that best approximate Φ^0 in terms of the JFOD.

In order to properly pose the problem to study, we define the function to minimize:

Definition 3.1. Let $\Phi^0 = \{\mathcal{F}_j^0\}_{j \in \mathbb{I}_m}$ be a **d**-design for $m \in \mathbb{N}$ and $\alpha = (\alpha_i)_{i \in \mathbb{I}_n} \in (\mathbb{R}_{>0}^n)^{\downarrow}$. Consider the function

$$\Theta: \mathcal{D}(\alpha, \mathbf{d}) \to \mathbb{R}_{\geq 0} \quad \text{given by} \quad \Theta(\Phi) = \operatorname{dist}_{JFOD}^{2}(\Sigma_{\Phi^{0}}, \Sigma_{\Phi}) = \sum_{j \in \mathbb{I}_{m}} \|S_{\mathcal{F}_{j}^{0}} - S_{\mathcal{F}_{j}}\|_{2}^{2},$$

for
$$\Phi = \{\mathcal{F}_j\}_{j \in \mathbb{I}_m} \in \mathcal{D}(\alpha, \mathbf{d}).$$

Next proposition shows that we can restate the JFOD problem as a multi-completion problem, as in the case m = 1 studied in [13].

Remark 3.2. Given an initial (fixed) **d**-design $\Phi^0 = \{\mathcal{F}_j^0\}_{j \in \mathbb{I}_m}$, with frame operators $\Sigma_{\Phi^0} = \{S_{\mathcal{F}_j^0}\}$, take $M = \max_{j \in \mathbb{I}_m} \|S_{\mathcal{F}_j^0}\|$ and choose any **d**-design $\tilde{\Phi}^0 = \{\tilde{\mathcal{F}}_j^0\}_{j \in \mathbb{I}_m}$ such as $\Sigma_{\tilde{\Phi}^0} = \{M \cdot I_{d_j} - S_{\mathcal{F}_j^0}\}_{j \in \mathbb{I}_m}$. It is clear that such a **d**-design exists since $M \cdot I_{d_j} - S_{\mathcal{F}_j^0} \in \mathcal{M}_{d_j}^+(\mathbb{C})$ for every $j \in \mathbb{I}_m$.

Proposition 3.3. Consider Φ^0 , $\tilde{\Phi}^0$ as in Remark 3.2. Then, $\Phi = \{\mathcal{F}_j\}_{j \in \mathbb{I}_m} \in \mathcal{D}(\alpha, \mathbf{d})$ is a local minimizer of Θ if and only if it is a local minimizer of Ψ in $\mathcal{D}(\alpha, \mathbf{d})$, given the initial data $\tilde{\Phi}$, α and the strictly convex function $\varphi(t) = t^2$.

Proof. Given $\Phi = \{\mathcal{F}_j\}_{j \in \mathbb{I}_m} \in \mathcal{D}(\alpha, \mathbf{d})$ and M as in Remark 3.2,

$$S_{\mathcal{F}_{j}^{0}} - S_{\mathcal{F}_{j}} = M \cdot I_{d_{j}} + (S_{\mathcal{F}_{j}^{0}} - M \cdot I_{d_{j}}) - S_{\mathcal{F}_{j}} = M \cdot I_{d_{j}} - (S_{\tilde{\mathcal{F}}_{j}^{0}} + S_{\mathcal{F}_{j}}),$$

thus,

$$\Theta(\Phi) = \sum_{j \in \mathbb{I}_m} \|S_{\mathcal{F}_j^0} - S_{\mathcal{F}_j}\|_2^2 = \sum_{j \in \mathbb{I}_m} \operatorname{tr}([M \cdot I_{d_j} - (S_{\tilde{\mathcal{F}}_j^0} + S_{\mathcal{F}_j})]^2)
= \sum_{j \in \mathbb{I}_m} [M^2 d_j - 2M \operatorname{tr}(S_{\tilde{\mathcal{F}}_j^0} + S_{\mathcal{F}_j})] + \sum_{j \in \mathbb{I}_m} \operatorname{tr}([S_{\tilde{\mathcal{F}}_j^0} + S_{\mathcal{F}_j}]^2)
= k + \sum_{j \in \mathbb{I}_m} \operatorname{tr}([S_{\tilde{\mathcal{F}}_j^0} + S_{\mathcal{F}_j}]^2) = k + \Psi(\Phi).$$

Where $k = \sum_{j \in \mathbb{I}_m} M^2 d_j - 2M \operatorname{tr}(S_{\tilde{\mathcal{F}}_j^0} + S_{\mathcal{F}_j})$ denotes a constant. Hence $\Theta(\Phi) = \Psi(\Phi) + k$ for every $\Phi \in \mathcal{D}(\alpha, \mathbf{d})$. In particular, local minimizers of Θ and Ψ (with their respective initial data) must coincide.

As a consequence, the complete characterization of local (global) minimizers for Θ can be carried out by the results summarized in Theorem 2.2:

Let $\tilde{\nu}_j$, $j \in \mathbb{I}_m$, denote the spectra of $S_{(\tilde{\mathcal{F}}_j^0, \mathcal{F}_j)}$, where $\Phi = \{\mathcal{F}_j\}_{j \in \mathbb{I}_m}$ is a local (global) minimizer of Ψ , for the initial data given by the **d**-design $\tilde{\Phi}^0$ constructed from Φ^0 as it was described in Remark 3.2. Notice that, if we denote by $\lambda_j = \lambda^{\downarrow}(S_{\mathcal{F}_j^0})$ the vector of eigenvalues of $S_{\mathcal{F}_j^0}$ arranged in non-increasing order, then the eigenvalues $\tilde{\lambda}_j = \lambda^{\uparrow}(S_{\tilde{\mathcal{F}}_j^0}) = M\mathbb{1}_{d_j} - \lambda_j \in (\mathbb{R}_{\geq 0}^{d_j})^{\uparrow}$.

Theorem 3.4. Let $\delta_j = (\delta_{i,j})_{i \in \mathbb{I}_{d_i}} = M \mathbb{1}_{d_j} - \tilde{\nu}_j$. Then,

1.
$$\min \left\{ \Theta(\Phi) = \sum_{j \in \mathbb{I}_m} \left\| S_{\mathcal{F}_j^0} - S_{\mathcal{F}_j} \right\|_2^2 : \Phi \in \mathcal{D}(\alpha, \mathbf{d}) \right\} = \sum_{j \in \mathbb{I}_m} \|\delta_j\|^2.$$

2. If $\Phi \in \mathcal{D}(\alpha, \mathbf{d})$, then

$$\Theta(\Phi) = \sum_{j \in \mathbb{I}_m} \left\| S_{\mathcal{F}_j^0} - S_{\mathcal{F}_j} \right\|_2^2 = \sum_{j=1}^m \|\delta_j\|^2 \quad \text{if and only if,} \quad \lambda_j (S_{\mathcal{F}_j^0} - S_{\mathcal{F}_j}) = \delta_j^{\downarrow}.$$

In this case, there exists an onb $\{v_{i,j}\}_{i\in\mathbb{I}_{d_j}}$ of \mathbb{C}^{d_j} (for each $j\in\mathbb{I}_m$) such that

$$S_{\mathcal{F}_{j}^{0}} = \sum_{i \in \mathbb{I}_{d_{j}}} \lambda_{i,j} \ v_{i,j} \otimes v_{i,j} \quad and \quad S_{\mathcal{F}_{j}} = \sum_{i \in \mathbb{I}_{d_{j}}} (\lambda_{i,j} - \delta_{i,j}) \ v_{i,j} \otimes v_{i,j}.$$
 (4)

3. Local minimizers of Θ in $\mathcal{D}(\alpha, \mathbf{d})$ are also global minimizers.

Proof. The assertions of the statement are consequences of the Proposition 3.3 and Theorem 2.2. In fact, by Proposition 3.3 local minima for Θ in $\mathcal{D}(\alpha, \mathbf{d})$ coincide with local (and hence global) minimizers for Ψ , where the initial data is given by $\tilde{\Phi}^0$.

In particular, $\Phi = \{\mathcal{F}_j\}_{j \in \mathbb{I}_m}$ is a local minimizer for Θ in $\mathcal{D}(\alpha, \mathbf{d})$ if and only if, for each $j \in \mathbb{I}_m$, the spectrum of

$$S_{(\tilde{\mathcal{F}}_j^0,\mathcal{F}_j)} = S_{\tilde{\mathcal{F}}_j^0} + S_{\mathcal{F}_j} = M \cdot I_{d_j} - S_{\mathcal{F}_j^0} + S_{\mathcal{F}_j}$$

is given by the vector $\tilde{\nu}_j$, that characterize the minimizers of Ψ , according Theorem 2.2. In particular, the spectra of $S_{\mathcal{F}_j} - S_{\mathcal{F}_j^0}$ are given by the vectors δ_j , $\forall j \in \mathbb{I}_m$, which implies

$$||S_{\mathcal{F}_j} - S_{\mathcal{F}_j^0}||_2^2 = ||\delta_j||^2.$$

Moreover, by the same result, $S_{\mathcal{F}_j}$ commutes with $M \cdot I_{d_j} - S_{\mathcal{F}_i^0}$, so

$$S_{\mathcal{F}_j} \cdot S_{\mathcal{F}_j^0} = S_{\mathcal{F}_j^0} \cdot S_{\mathcal{F}_j}.$$

Therefore, $S_{\mathcal{F}_j}$ and $S_{\mathcal{F}_j^0}$ can be simultaneously diagonalized as in (4).

4 Explicit computation of $\delta = (\delta_j)_{j \in \mathbb{I}_m}$

Once we established in Prop. 3.3 the link between the approximation problem with optimal (α, \mathbf{d}) - multi-completions, we can compute the vectors δ_j , $j \in \mathbb{I}_m$, by reinterpreting the description of the optimal spectra ν_j done in [3] using the "translated" initial data.

Consider the notation introduced in the previous section, so that $\lambda_j = (\lambda_{i,j})_{i \in \mathbb{I}_{d_j}} \in (\mathbb{R}^{d_j}_{\geq 0})^{\downarrow}$ denote the spectrum of each $S_{\mathcal{F}_j^0}$ and $M = \max_{j \in \mathbb{I}_m} \|S_{\mathcal{F}_j^0}\|$. Then, for the construction of the vectors $\tilde{\nu}_j$ we shall use

$$\tilde{\lambda}_j = M \cdot \mathbb{1}_{d_j} - \lambda_j = (M - \lambda_{i,j})_{i \in \mathbb{I}_{d_j}} \in (\mathbb{R}^{d_j}_{\geq 0})^{\uparrow}, \tag{5}$$

i.e. the vector of eigenvalues of $S_{\tilde{\mathcal{F}}_{j}^{0}} = M \cdot I_{d_{j}} - S_{\mathcal{F}_{j}^{0}}$, counted with multiplicities and arranged in non decreasing order, along with the weights $\alpha = (\alpha_{i})_{i \in \mathbb{I}_{n}}$.

According the results shown in [3], there is a unique vector $\mathbf{c} \in (\mathbb{R}^{d_1})^{\downarrow}$, computable from $\{\tilde{\lambda}_j\}_{j\in\mathbb{I}_m}$ and α , such that each spectrum $\tilde{\nu}_j$ of the optimal completion $S_{\tilde{\mathcal{F}}_j^0} + S_{\mathcal{F}_j}$ is described as

$$\tilde{\nu}_i = \max((\mathbf{c})_{d_i}, \tilde{\lambda}_i) \,, \tag{6}$$

where the maximum is taken entry-wise and $(\mathbf{c})_{d_j}$ is the truncation of \mathbf{c} on its d_j first entries. The construction of the vector \mathbf{c} is done with some detail in [3]. Mainly, it can be characterized as the unique (up to rearrangements) vector in \mathbb{R}^{d_1} such that, if

$$\mathbf{c} = (c_1 \mathbb{1}_{s_1}, c_2 \mathbb{1}_{s_2}, \cdots, c_p \mathbb{1}_{s_p}), \tag{7}$$

where $c_1 > c_2 > \cdots > c_p > 0$ and $\sum_{k=1}^p s_k = d_1$, then each c_k and s_k satisfy

$$(\alpha_i)_{i=i_{k-1}+1}^{i_k} \prec \left(\sum_{j:i \leq d_j} (c_k - \tilde{\lambda}_{i,j})^+\right)_{i=i_{k-1}+1}^{i_k} = \left(\sum_{j:i \leq d_j} (\lambda_{i,j} - (M - c_k))^+\right)_{i=i_{k-1}+1}^{i_k} \tag{8}$$

and

$$(\alpha_i)_{i=i_{p-1}+1}^n \prec \left(\sum_{j:i \leq d_j} (c_k - \tilde{\lambda}_{i,j})^+\right)_{i=i_{p-1}+1}^{d_1} = \left(\sum_{j:i \leq d_j} (\lambda_{i,j} - (M - c_k))^+\right)_{i=i_{p-1}+1}^{d_1} \tag{9}$$

for $i_0 = 0$ and $i_j = \sum_{i=1}^{j} s_i$.

Since for each $j \in \mathbb{I}_m$, $\delta_j = M \mathbb{1}_{d_j} - \tilde{\nu}_j$, where $\tilde{\nu}_j$ can be constructed as before, we are able to propose an algorithm that computes δ_j from the previous assertions.

First, let **b** be the vector in $(\mathbb{R}^{d_1})^{\uparrow}$ defined as $\mathbf{b} = M \cdot \mathbb{1}_{d_1} - \mathbf{c}$. Then, by equations (5), (6) and the characterization of δ_i :

$$\delta_i = \min((\mathbf{b})_{d_i}, \lambda_i). \tag{10}$$

From the equations (8) and (9) that define the vector \mathbf{c} we construct

$$\mathbf{b} = (b_1 \mathbb{1}_{s_1}, b_2 \mathbb{1}_{s_2}, \cdots, b_p \mathbb{1}_{s_p}) \in (\mathbb{R}^{d_1})^{\uparrow},$$

inductively as follows (as before, we let $i_0 = 0$ and $i_j = \sum_{k=1}^{j} s_k$):

Definition 4.1. Let us suppose that we have found the indices $i_0 = 0 < i_1 < \cdots < i_k$ (therefore we have $s_1 = i_1, s_j = i_j - i_{j-1}$, for $j = 1, \ldots, k$) and the constants $b_1 < b_2 < \cdots < b_k$.

Then, we define i_{k+1} and b_{k+1} as:

$$i_{k+1} := \max \left\{ i : i_k + 1 \le i \le d_1 : (\alpha_i)_{i=i_k+1}^{i^*} \prec (\beta_{i,k+1})_{i=i_k+1}^{i} \right\} \quad \text{for } 0 \le k \le p-1, \quad (11)$$

where i^* and $\beta_{i,k+1}$ are determined from the following cases:

Case 1) If $i \in \mathbb{I}_{d_1-1}$ then $i^* = i$, and

$$(\beta_{i,k+1})_{i=i_k+1}^i = \left(\sum_{j:i \le d_j} (\lambda_{i,j} - b_{k+1,i}^*)^+\right)_{i=i_k+1}^i$$

and $b_{k+1,i}^*$, for $i \geq i_k + 1$, is the unique solution of the equation

$$\sum_{i=i_k+1}^{i} \alpha_i = \sum_{i=i_k+1}^{i} \sum_{j: i < d_i} (\lambda_{i,j} - x)^+.$$

Case 2) If $i = d_1$, then $i^* = n$, and

$$(\beta_{i,k+1})_{i=i_{p-1}+1}^{d_1} = \left(\sum_{j:i \le d_j} (\lambda_{i,j} - b_{k+1,d_1}^*)^+\right)_{i=i_{p-1}+1}^{d_1}$$

and b_{k+1,d_1}^* , is the unique solution of the equation

$$\sum_{i=i_{p-1}+1}^{n} \alpha_i = \sum_{i=i_{p-1}+1}^{d_1} \sum_{j: i \le d_j} (\lambda_{i,j} - x)^+.$$

We denote $b_{k+1} = b_{k+1,i_{k+1}}^*$ for $k \in \mathbb{I}_{p-1}$.

 \triangle

This algorithm that produces the vector **b**, which generates $\delta = \{\delta_j\}_{j \in \mathbb{I}_m}$ is deduced from the characterization of **c** given in [3]. We omit the proof of the correct ordering in the entries of **b** (that is, that the b_{k+1} produced in this way is such that $b_k < b_{k+1}$) and of the uniqueness of **b** since they follow directly from the results proved in [3].

Remark 4.2. Note that the construction of δ proposed in Definition 4.1 does not depend on the translation parameter M. This means that the algorithm that produces δ only requires as initial data the spectra λ_j of the frame operators $S_{\mathcal{F}_j^0}$ (with multiplicities and arranged in non decreasing order) and the set of weights $\alpha = (\alpha_i)_{i \in \mathbb{I}_n}$, as expected.

Example 4.3. In the following example, we implement the described algorithm for the same initial data considered in [3, Example 5.5].

That is: for d=(7,5,3) consider $\Phi^0=\{\mathcal{F}_j^0\}_{j\in\mathbb{I}_3}$ be given in matrix form by

$$\mathcal{F}_{1}^{0} = \begin{bmatrix} 0.3066 & 1.6919 & -1.14 & 0.0488 \\ 0.9339 & -0.4353 & -0.2197 & 0.2354 \\ -1.8151 & 0.8134 & 0.3742 & 0.2428 \\ 1.7690 & 1.0168 & 0.8745 & -0.045 \\ -0.4706 & 0.7223 & 0.8595 & 0.0609 \\ 1.1678 & -0.0164 & 0.0839 & 0.2206 \\ -0.1574 & 0.48 & 0.042 & -0.3589 \end{bmatrix}$$

$$\begin{bmatrix} -2.723 & -0.068 & -0.5242 \end{bmatrix}$$

$$\mathcal{F}_2^0 = \begin{bmatrix} -2.723 & -0.068 & -0.5242 \\ -2.2341 & -0.5975 & 0.2401 \\ -1.5660 & 0.7992 & 0.0219 \\ 2.2048 & -0.1835 & -0.4038 \\ 0.5298 & 0.2569 & 0.0631 \end{bmatrix}$$

and

$$\mathcal{F}_3^0 = \begin{bmatrix} -0.8048 & -0.9958 & -0.1026 \\ 1.0153 & -0.5127 & -0.4653 \\ 0.5669 & -0.4955 & 0.6877 \end{bmatrix}$$

In this case, the spectra of $S_{\mathcal{F}_{i}^{0}}$ are:

$$\lambda_1 = (9, 5.5, 3, 0.3, 0, 0, 0), \qquad \lambda_2 = (20, 1.1, 0.5, 0, 0) \quad \text{and} \quad \lambda_3 = (2, 1.5, 0.7).$$

Consider the set of weights $\alpha = (40, 35, 9, 5, 4.5, 3, 2.4, 2)$. An implementation of the previously discussed algorithm produces

$$\mathbf{b} = 20 \cdot \mathbb{1}_7 - \mathbf{c} = (-5.9833, -5.9833, -2.3778, -2.3778, -2.3778, -2.3778, -2.3778),$$

so $\delta_1 = \mathbf{b}$, $\delta_2 = (\mathbf{b})_5$ and $\delta_3 = (\mathbf{b})_3$. Thus, the minimal value for the multi-approximation is

$$\|\delta_1\|^2 + \|\delta_2\|^2 + \|\delta_3\|^2 = 265.685$$
.

Moreover, by applying well-known algorithms that allow the construction of matrices with prescribed spectra and column norms (see for example [10]), we obtain the following solution to the multi-approximation problem:

$$\tilde{\mathcal{F}}_1 = \begin{bmatrix} -1.8371 & 1.6608 & 1.2662 & 0.5185 & 0.3026 & 0.3396 & 0.1226 & 0.5687 \\ -0.5473 & -1.2442 & 0.7725 & 0.2088 & -0.5880 & -1.0094 & -0.4046 & -0.0201 \\ 1.0932 & 2.3814 & -0.0088 & -0.7383 & -0.8395 & -0.1102 & -0.3142 & -0.2439 \\ -2.7000 & -0.2801 & -0.9373 & -0.8292 & -0.2106 & -0.4169 & 0.2434 & -0.1652 \\ -0.1785 & 1.1942 & -1.1371 & 0.9572 & -0.5076 & -0.4684 & 0.0565 & -0.3832 \\ -1.1612 & -0.9604 & -0.0092 & 0.1824 & -0.6634 & 0.9483 & -0.6628 & -0.5109 \\ -0.2751 & 0.6683 & -0.0512 & -0.0647 & 1.0101 & -0.3055 & -0.8896 & -0.7003 \end{bmatrix}$$

$$\tilde{\mathcal{F}}_2 = \begin{bmatrix} -2.3145 & -2.0750 & -1.2187 & 0.0102 & -0.2369 & -0.4077 & -0.1708 & -0.1559 \\ -1.0396 & -2.7753 & 0.4457 & 0.3734 & 0.5631 & 0.3362 & 0.4060 & 0.3706 \\ -2.6610 & 0.4670 & 0.04 & -0.2241 & 0.5553 & 0.2283 & 0.4003 & 0.3654 \\ 2.2525 & 1.2076 & -0.9958 & -0.1239 & 0.6724 & 0.0069 & 0.4848 & 0.4425 \\ 0.0638 & 0.8863 & -0.1217 & 1.4798 & -0.0001 & -0.0314 & -0.0001 & 0 \end{bmatrix}$$

and

$$\tilde{\mathcal{F}}_3 = \begin{bmatrix}
-0.1356 & 2.7412 & -0.1706 & -0.04 & -0.0834 & -0.0451 & -0.0601 & -0.0549 \\
-2.2991 & -0.3745 & -0.7734 & -0.1812 & -0.3781 & -0.2047 & -0.2726 & -0.2488 \\
-1.5759 & 0.1557 & 1.1430 & 0.2679 & 0.5587 & 0.3025 & 0.4028 & 0.3677
\end{bmatrix}$$

5 An application to a distance problem for G-frames

In the previous section we generalized the approximation problem studied in [13], to the setting of (α, \mathbf{d}) -designs, that is, as a simultaneous approximation to a family of semi-definite positive matrices with some structured matrices, that come from these (α, \mathbf{d}) -designs.

In this section we study another natural generalization, to the set of G-frames, that was posed and studied in [19]. So, we briefly recall the concept of G-frames, introduced by W. Sun in [21].

A family $\mathcal{F} = \{T_i\}_{i \in I}$ of linear bounded operators T_i from \mathbb{C}^d to an analysis space \mathbb{C}^n is a G-frame for \mathbb{C}^d if there exist constants a, b > 0 such that

$$a||x||^2 \le \sum_{i \in I} ||T_i x||^2 \le b||x||^2,$$

for every $x \in \mathbb{C}^d$. If only the upper inequality holds, we say that \mathcal{F} is a G-Bessel sequence for \mathbb{C}^d .

Given a G-Bessel sequence $\mathcal{F} = \{T_i\}_{i \in I}$, its frame operator $S_{\mathcal{F}}$ is defined as

$$S_{\mathcal{F}} = \sum_{i \in I} T_i^* T_i.$$

Let $\alpha = (\alpha_i)_{i \in I_m}$ be a non increasing finite sequence of positive weights. Consider the set

$$\Lambda_{\alpha} = \{ \mathcal{F} = \{ T_i \}_{i \in \mathbb{I}_m} : \mathcal{F} \text{ is a G-Bessel sequence for } \mathcal{H} \text{ with } ||T_i||_2^2 = \alpha_i \}.$$

Let A be a positive semi definite operator of \mathcal{H} . Our goal is to study the following approximation problem:

Compute

$$\min_{\mathcal{F} \in \Lambda_{\alpha}} \|A - S_{\mathcal{F}}\|_{2},\tag{12}$$

and characterize the G-Bessel sequences that attain the minimum distance.

We shall see that this problem can be treated as a particular case of the problem considered in [13, 14].

First, we need the following characterization of the frame operators of elements in Λ_{α} . Recall that the dimension of the analysis space that we are considering is n.

Proposition 5.1. Let $S \in \mathcal{M}_d(\mathbb{C})^+$ with eigenvalues given by $\lambda \in \mathbb{R}^d_{\geq 0}$ and let $\alpha = (\alpha_i)_{i \in \mathbb{I}_m} \in (\mathbb{R}^m_{>0})^{\downarrow}$. Then, there exists $\mathcal{F} \in \Lambda_{\alpha}$ with $S_{\mathcal{F}} = S$ if and only if

$$\left(\frac{\alpha_1}{n}\mathbb{1}_n, \frac{\alpha_2}{n}\mathbb{1}_n, \cdots, \frac{\alpha_m}{n}\mathbb{1}_n\right) \prec \lambda. \tag{13}$$

Proof. On one direction, if (13) holds, the Schur-Horn theorem implies the existence of a (vector) frame $\mathcal{F}_{vec} = \{f_j\}_{j \in \mathbb{I}_{nm}}$ such that $||f_j||^2 = \frac{\alpha_i}{n}$, for $(i-1)n+1 \leq j \leq in$, and whose frame operator $S_{\mathcal{F}_{vec}}$ is S. Let $T^* \in L(\mathbb{C}^{nm}, \mathcal{H})$ be the bounded linear operator such that $T^*e_i = f_i$, where $\{e_j\}_{j \in \mathbb{I}_{nm}}$ is the standard orthonormal basis in \mathbb{C}^{nm} .

Consider a (fixed) orthonormal basis $\{b_j\}_{j\in\mathbb{I}_n}$ for \mathcal{K} . Let $W_i\in L(\mathbb{C}^{nm},\mathcal{K})$ be the partial isometry defined such that

$$W_i e_{(i-1)n+j} = b_j$$
, for $j = 1, ..., n$ and $W_i e_k = 0$ otherwise.

That is, it implies that $W_iW_i^* = I_n$ and $W_i^*W_i = P_i$, where I_n is the identity in \mathcal{K} and P_i is the diagonal projection of \mathbb{C}^{mn} onto the subspace generated by

$$J_i = \{e_{(i-1)n+j} : j = 1, \dots, n\}.$$

Thus, $\sum_{i\in\mathbb{I}_m} P_i = I_{mn}$, the identity in \mathbb{C}^{mn} .

Notice that, if we set $\mathcal{F} = \{T_i\}_{i \in \mathbb{I}_m}$ where $T_i \in L(\mathcal{H}, \mathcal{K})$ is defined such that $T_i = W_i T$, for $i \in \mathbb{I}_m$, then $\mathcal{F} \in \Lambda_{\alpha}$ and $S_{\mathcal{F}} = S_{\mathcal{F}_{vec}} = S$.

Indeed,

$$S_{\mathcal{F}} = \sum_{i \in \mathbb{I}_m} T_i^* T_i = \sum_{i \in \mathbb{I}_m} T^* W_i^* W_i T = T^* \left(\sum_{i \in \mathbb{I}_m} W_i^* W_i \right) T = T^* \left(\sum_{i \in \mathbb{I}_m} P_i \right) T = T^* T = S_{\mathcal{F}_{vec}}$$

and

$$||T_i||_2^2 = \operatorname{tr}(T_i^*T_i) = \operatorname{tr}(TT^*P_i) = \sum_{j \in \mathbb{I}_{mn}} ||T^*P_ie_j||^2 = \sum_{j \in \mathbb{I}_n} ||f_{(i-1)n+j}||^2 = \alpha_i.$$

On the other side, suppose that $\mathcal{F} = \{T_i\}_{i \in \mathbb{I}_m}$ is a G-Bessel sequence in Λ_{α} such that $S_{\mathcal{F}} = S$. Then,

$$\alpha_i = ||T_i||_2^2 = \operatorname{tr}(T_i^*T_i)$$
 for all $i \in \mathbb{I}_m$,

in particular, if we denote $f_{ij} = T_i^* b_j$, we have

$$\frac{\alpha_i}{n} \mathbb{1}_n \prec (\|f_{i1}\|^2, \|f_{i2}\|^2, \cdots, \|f_{in}\|^2) \prec \lambda(T_i^*T_i).$$

Here, the majorization on the left holds since it is easy to see that for every $x \in \mathbb{R}^d_{\geq 0}$, $(\frac{\operatorname{tr} x}{d})\mathbb{1}_d \prec x$, while the comparison on the right is due to Theorem 2.1.

Again by use of Schur-Horn theorem we deduce from the previous majorization relationship that there is, for each $i \in \mathbb{I}_m$, a Bessel sequence $G_i = \{g_{ij}\}_{j \in \mathbb{I}_d}$ for \mathcal{H} such that $\|g_{ij}\|^2 = \frac{\alpha_i}{d}$ and such that $S_{G_i} = T_i^* T_i$.

Define the linear operator $T^* \in L(\mathbb{C}^{mn}, \mathcal{H})$ by $T^*b_{(i-1)n+j} = g_{ij}$, for $i \in \mathbb{I}_m$ and $j \in \mathbb{I}_n$.

In particular, $T^*P_iT = T_i^*T_i$, using the previous definition for the orthogonal projections P_i . Then,

$$T^*T = T^* \left(\sum_{i \in \mathbb{I}_m} P_i\right) T = \sum_{i \in \mathbb{I}_m} T_i^* T_i = S.$$

Therefore, the sequence $\mathcal{G} = \{G_i\}_{i \in \mathbb{I}_m}$, constructed by juxtaposition is a Bessel sequence for \mathcal{H} , with synthesis operator $T^* \in L(\mathbb{C}^{mn}, \mathcal{H})$ and frame operator $S_{\mathcal{G}} = T^*T = S$.

Finally, since the squared norms of the elements in \mathcal{G} are given by the vector $(\frac{\alpha_1}{n}\mathbb{1}_n, \frac{\alpha_2}{n}\mathbb{1}_n, \cdots, \frac{\alpha_m}{n}\mathbb{1}_n)$, by Schur-Horn theorem we conclude that

$$\left(\frac{\alpha_1}{n}\mathbb{1}_n, \frac{\alpha_2}{n}\mathbb{1}_n, \cdots, \frac{\alpha_m}{n}\mathbb{1}_n\right) \prec \lambda(S_{\mathcal{G}}) = \lambda.$$

We are now in a position to prove the existence and characterization of approximants of the problem posed in Eq. (12).

Theorem 5.2. Let $\alpha = (\alpha_j)_{j \in \mathbb{I}_m} (\mathbb{R}^m_{\geq 0})^{\downarrow}$, and consider the set of G-frames Λ_{α} as before. Let $A \in \mathcal{M}_d(\mathbb{C})^+$. Then, there exists $\mathcal{F}^{\text{op}} = \{T_j^{\text{op}}\}_{j \in \mathbb{I}_m} \in \Lambda_{\alpha} \text{ such that }$

$$||A - S_{\mathcal{F}^{\text{op}}}||_2 \le ||A - S_{\mathcal{F}}||_2 \quad for \ all \quad \mathcal{F} \in \Lambda_{\alpha}.$$
 (14)

Moreover, the minimal distance (and the approximants) can be computed using the spectrum of A and the weights $(\frac{\alpha_1}{n}\mathbb{1}_n, \frac{\alpha_2}{n}\mathbb{1}_n, \cdots, \frac{\alpha_m}{n}\mathbb{1}_n)$ as the initial data for the classical approximation problem.

Proof. Given a $\mathcal{F} \in \Lambda_{\alpha}$, by Proposition 5.1 there is a Bessel sequence \mathcal{F}_{vec} for \mathcal{H} such that $S_{\mathcal{F}} = S_{\mathcal{F}_{vec}}$ and such that the norms of the vectors in \mathcal{F}_{vec} are given by the weights:

$$\left(\frac{\alpha_1}{n}\mathbb{1}_n, \frac{\alpha_2}{n}\mathbb{1}_n, \cdots, \frac{\alpha_m}{n}\mathbb{1}_n\right).$$

In particular, the computation of the distance can be done using the results in [13] (or the results in previous section, for the particular case of $\mathbf{d} = d$). Notice that, as it was done in the proof of the Proposition 5.1, optimal G-frames can be constructed from optimal vector frames.

Remark 5.3. The problem considered in [19], is actually solved in terms of unitarily invariant norms (briefly uin). Recall that a norm $N(\cdot)$ in $\mathcal{M}_d(\mathbb{C})$ is unitarily invariant if

$$N(UAV) = N(A)$$
 for every $A \in \mathcal{M}_d(\mathbb{C})$ and $U, V \in \mathcal{U}(d)$,

and $N(\cdot)$ is *strictly convex* if its restriction to diagonal matrices is a strictly convex norm in \mathbb{C}^d . Examples of uin are the spectral norm and the *p*-norms, for $p \geq 1$ (strictly convex if

p > 1). Note that, in particular, when p = 2, we get the Frobenius norm. Then, the problem posed in [19] is:

Given $N(\cdot)$ an strictly convex uin in $\mathcal{M}_d(\mathbb{C})$, $\mathcal{H} = \mathbb{C}^d$, $\mathcal{K} = \mathbb{C}^n$, $\alpha = (\alpha_i)_{i \in \mathbb{I}_m}$ and $A \in \mathcal{M}_d(\mathbb{C})^+$, compute

$$\min_{\mathcal{F}\in\Lambda_{\alpha}}N\left(A-S_{\mathcal{F}}\right),$$

and characterize the G-Bessel sequences that reach the minimum distance. Following the same steps as for the Frobenius norm, and applying Theorem 4.1 in [14], we get the following generalization of Theorem 5.2, since the minimizers do not depend on the unitary invariant norm chosen. \triangle

Theorem 5.4. Let $\alpha = (\alpha_j)_{j \in \mathbb{I}_m}(\mathbb{R}^m_{\geq 0})^{\downarrow}$, $N(\cdot)$ an strictly convex uin in $\mathcal{M}_d(\mathbb{C})$ and consider the set of G-frames Λ_{α} as before. Let $A \in \mathcal{M}_d(\mathbb{C})^+$. Then, there exists $\mathcal{F}^{op} = \{T_j^{op}\}_{j \in \mathbb{I}_m} \in \Lambda_{\alpha}$ such that

$$N(A - S_{\mathcal{F}^{\text{op}}}) \le N(A - S_{\mathcal{F}}) \quad \text{for all} \quad \mathcal{F} \in \Lambda_{\alpha} \,.$$
 (15)

Moreover, the minimal distance (and the approximants) can be computed using the spectrum of A and the weights $(\frac{\alpha_1}{n}\mathbb{1}_n, \frac{\alpha_2}{n}\mathbb{1}_n, \cdots, \frac{\alpha_m}{n}\mathbb{1}_n)$ as the initial data for the classical approximation problem. Even more, the best approximants do not depend on the choice of the strictly convex uin.

References

- [1] J. Antezana, P. Massey, M. Ruiz and D. Stojanoff; The Schur-Horn theorem for operators and frames with prescribed norms and frame operator. Illinois J. Math., 51 (2007) 537-560.
- [2] M. J. Benac, P. Massey, M. Ruiz, D. Stojanoff; Optimal frame designs for multitasking devices with weight restrictions. Adv. Comput. Math. 46 (2020), no. 2, Paper No. 22, 23 pp.
- [3] M. J. Benac, P. Massey, M. Ruiz, D. Stojanoff; Optimal (α, \mathbf{d}) -multi-completion of designs, 2020.
- [4] R. Bhatia; Matrix Analysis, Berlin-Heildelberg-New York, Springer 1997.
- [5] J. Cahill, M. Fickus, D.G. Mixon, M.J. Poteet, N. Strawn, Constructing finite frames of a given spectrum and set of lengths, Appl. Comput. Harmon. Anal. 35 (2013), 52-73.
- [6] P. G. Casazza, G. Kutyniok; Finite Frames: Theory and Applications. Birkhauser, 2012. xii + 483 pp.
- [7] O. Christensen; An introduction to frames and Riesz bases. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, 2003. xxii+440 pp.
- [8] G. Corach, P. Massey, M. Ruiz; Procrustes problems and Parseval quasi-dual frames. Acta Appl. Math. 131 (2014) 179-195.
- [9] J.C. Gower, G. B. Dijksterhuis; Procrustes problems. Vol. 30. OUP Oxford, 2004.

- [10] I.S. Dhillon, R.W. Heath, M.A. Sustik, J.A. Tropp, Generalized finite algorithms for constructing Hermitian matrices with prescribed diagonal and spectrum, SIAM J. Matrix Anal. Appl. 27 (2005) 61-71.
- [11] N.J. Higham; Matrix nearness problems and applications. In: Applications of Matrix Theory. Inst. Math. Appl. Conf. Ser. New Ser., vol. 22, pp. 1-27. Oxford University Press, Oxford (1989)
- [12] H-F. Liu; Frame completion with prescribed norms via alternating projection method. Applied Numerical Mathematics 164 (2021) 161-174.
- [13] P. Massey, N. Rios, D. Stojanoff; Frame completions with prescribed norms: local minimizers and applications. Adv. Comput. Math. 44 (2018), no. 1, 51-86.
- [14] P. Massey, N. Rios, D. Stojanoff; Generalized frame operator distance problems. J. Math. Anal. App., 479, (2019), pp 1738-1763
- [15] P. Massey, M. Ruiz; Minimization of convex functionals over frame operators. Adv. Comput. Math. 32 (2010), no. 2, 131-153.
- [16] P. Massey, M. Ruiz, D. Stojanoff; Optimal dual frames and frame completions for majorization. Appl. Comput. Harmon. Anal. 34 (2013), no. 2, 201-223.
- [17] P. Massey, M. Ruiz, D. Stojanoff; Optimal frame completions. Advances in Computational Mathematics 40 (2014), 1011-1042.
- [18] P. Massey, M. Ruiz, D. Stojanoff; Optimal frame completions with prescribed norms for majorization. J. Fourier Anal. Appl. 20 (2014), no. 5, 1111-1140.
- [19] M. He, J. Leng, Y. Xu; G-frame operator distance problems. Ann. Funct. Anal. 12 (2021), no. 3, 40.
- [20] N. Strawn; Optimization over finite frame varieties and structured dictionary design. Appl. Comput. Harmon. Anal. 32 (2012) 413-434.
- [21] W. Sun; G-frames and G-Riesz bases. J. Math. Anal. Appl. 322 (2006) no. 1, 437-452.
- [22] W. Sun; Stability of G-frames. Journal of Mathematical Analysis and Applications 326.2 (2007) 858-868.