

Scaling limits of nonlinear functions of random grain model, with application to Burgers' equation

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Abstract

We study scaling limits of nonlinear functions G of random grain model X on \mathbb{R}^d with long-range dependence and marginal Poisson distribution. Following [15] we assume that the intensity M of the underlying Poisson process of grains increases together with the scaling parameter λ as $M = \lambda^\gamma$, for some $\gamma > 0$. The results are applicable to the Boolean model and exponential G and rely on an expansion of G in Charlier polynomials and a generalization of Mehler's formula. Application to solution of Burgers' equation with initial aggregated random grain data is discussed.

Keywords: Random grain model, nonlinear function, Boolean model, long-range dependence, scaling limit, aggregation, Poisson distribution, Charlier polynomials, Mehler's formula, Gaussian random field, stable random field, Burgers' equation, initial random grain data

1 Introduction

Limit theorems for random fields (RFs) with continuous or discrete d -dimensional argument have been extensively studied in the literature. Quite often, such work refer to the limit distribution of integrals

$$X_\lambda(\phi) := \int_{\mathbb{R}^d} X(\mathbf{t})\phi(\mathbf{t}/\lambda)d\mathbf{t}, \quad \text{as } \lambda \rightarrow \infty, \quad (1.1)$$

(or respective sums in the discrete argument case), where $X = \{X(\mathbf{t}); \mathbf{t} \in \mathbb{R}^d\}$ is a given stationary RF, for each ϕ from a class of (test) functions $\Phi = \{\phi : \mathbb{R}^d \rightarrow \mathbb{R}\}$. A suitably normalized limit of (1.1) is a RF indexed by $\phi \in \Phi$, called the (isotropic) scaling limit of X in this paper. The above approach is common in the theory of generalized RFs, and is discussed in

[5] together with various classes of Φ . For $\phi(\mathbf{t}) = \mathbb{I}(\mathbf{t} \in A)$, where A is a bounded Borel subset of \mathbb{R}^d , (1.1) is the integral (sum) of $X(\mathbf{t})$'s over $\mathbf{t} \in \lambda A$ whose limit distribution was studied in [18] for linear RF X on \mathbb{Z}^d and A having irregular boundary. For $\Phi = \Phi_{\text{rec},d} := \{\mathbb{I}(\mathbf{t} \in]\mathbf{0}, \mathbf{x}]); \mathbf{x} \in \mathbb{R}_+^d\}$, $]0, \mathbf{x}] := \prod_{i=1}^d]0, x_i]$, (1.1) present a d -dimensional analog of the partial integral process of time series, leading to a limit RF indexed by $\mathbf{x} \in \mathbb{R}_+^d$.

A stationary RF X on \mathbb{R}^d with finite variance with is said *long-range dependent* (LRD) if its covariance function $r_X(\mathbf{t}) := \text{Cov}(X(\mathbf{0}), X(\mathbf{t}))$ is not integrable: $\int_{\mathbb{R}^d} |r_X(\mathbf{t})| d\mathbf{t} = \infty$. It is well-known that LRD RFs can display a variety of Gaussian and non-Gaussian limit behaviors and scaling limits, see [6, 20, 36, 31] and the references therein. The framework in (1.1) can be modified by replacing the isotropic scaling $\mathbf{t} \rightarrow \mathbf{t}/\lambda$ in the test function with *anisotropic scaling* $\mathbf{t} \rightarrow \lambda^{-\Gamma} \mathbf{t}$, where $\Gamma = \text{diag}(\gamma_1, \dots, \gamma_d)$, $(\gamma_1, \dots, \gamma_d) = \boldsymbol{\gamma} \in \mathbb{R}_+^d$. It was observed in [34, 35, 27, 28, 41, 42] that in dimensions $d = 2, 3$ and a rich class of LRD RFs (including Gaussian and linear RFs) the corresponding anisotropic limits in (1.1) exist for any $\boldsymbol{\gamma} \in \mathbb{R}_+^d$, $\phi \in \Phi_{\text{rec},d}$ and depend on $\boldsymbol{\gamma}$. Particularly, when $d = 2$ a *scaling transition* may occur meaning that there exists a critical $\gamma^0 > 0$ depending on internal parameters of RF X such that the limit RFs do not depend on $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$ for $\gamma_2/\gamma_1 > \gamma^0$ and $\gamma_2/\gamma_1 < \gamma^0$ and exhibit a trichotomy of the (anisotropic) scaling behavior [34, 35, 28, 42].

The above scaling procedures can be extended to include aggregation, as follows. Let $X_i, i = 1, 2, \dots, M$ be independent copies of RF X in (1.1). Consider the limit distribution of the ‘aggregate’

$$X_{\lambda,M}(\phi) := \sum_{i=1}^M \int_{\mathbb{R}^d} X_i(\mathbf{t}) \phi(\mathbf{t}/\lambda) d\mathbf{t} \quad (1.2)$$

as $\lambda \rightarrow \infty$ and M increases with λ at a certain rate. In as follows, we refer to the limit distribution of the sums in (1.2) as the *scaling limit with aggregation*. By concretising M as $M = \lfloor \lambda^\gamma y \rfloor$ for some $\gamma > 0, y > 0$ we see that (1.2) may be regarded as a discretized version of the integral in (1.1) for stationary RF X' on $\mathbb{R}^d \times \mathbb{Z}$ given by $X'(\mathbf{t}, s) := X_s(\mathbf{t}), (\mathbf{t}, s) \in \mathbb{R}^d \times \mathbb{Z}$, corresponding to (partly) anisotropic scaling $(\mathbf{t}, s) \rightarrow (\mathbf{t}/\lambda, s/\lambda^\gamma)$ and test function $\phi'(\mathbf{t}, s) := \phi(\mathbf{t})\mathbb{I}(s \in]0, y])$. Scaling limits with aggregation in dimension $d = 1$ were studied in [11, 16, 19, 22, 23, 26, 25, 30] and other works, in connection with applications in communications and econometrics. As noted in [19], the trichotomy of the limit distribution observed in these papers can be interpreted as a scaling transition for RFs on the plane. Iterated limits of (1.2) when $M \rightarrow \infty$ (first) and $\lambda \rightarrow \infty$ (second) and/or vice versa, were discussed in [2, 33, 35] and other above cited references. A particularly simple form of aggregation occurs in models based on Poisson process as in (1.3) where summing over M independent copies reduces to multiplying by M the intensity of the

underlying Poisson process [3, 4, 15, 16, 19, 22].

Unsurprisingly, most existing results on scaling limits of LRD RFs ($d \geq 2$) (with or without aggregation) apply to linear models. A notable exception is Gaussian subordinated RFs (written as a nonlinear function $G(X(\mathbf{t}))$ of a Gaussian LRD RF X), treated via Hermite expansion in the classical work [6]. For more recent developments on chaos expansions and limit theorems under Gaussian subordination, see [24]. This is in contrast to the one-dimensional case $d = 1$, where the martingale approach developed in [14] is applicable to nonlinear functions and statistics of LRD moving averages. See monographs [8, 13]. Scaling limits of polynomial functions G of LRD moving-average RFs X are discussed in [38, 28]. The case of indicator functions $G(x) = \mathbb{I}(x \leq y)$ for the same class of RFs was considered in [7]. See [17] for statistical application.

A class of LRD RFs whose trajectories and distribution is very different from Gaussian and moving-average RFs are *random grain* (RG) models, defined as follows. Let $\{\mathbf{u}_j; j \geq 1\} \subset \mathbb{R}^d$ be a Poisson point process with intensity $d\mathbf{u}$. Let be given a bounded Borel set $\Xi^0 \subset \mathbb{R}^d$ ('generic grain') and an i.i.d. sequence $R, R_j; j \geq 1$ of r.v.s with values in \mathbb{R}_+ , distribution $F(dr)$, finite expectation $ER = \int_{\mathbb{R}_+} rF(dr) < \infty$, and independent of the Poisson process. The RG RF is obtained by counting at each point \mathbf{t} the number of randomly dilated grains $\mathbf{u}_j + R_j^{1/d}\Xi^0$ 'centered' at \mathbf{u}_j which cover it, viz.,

$$X(\mathbf{t}) := \sum_{j=1}^{\infty} \mathbb{I}(\mathbf{t} - \mathbf{u}_j \in R_j^{1/d}\Xi^0), \quad \mathbf{t} \in \mathbb{R}^d. \quad (1.3)$$

For $d = 1$ and $\Xi^0 =]0, 1]$, (1.3) is the number of customers in a stationary M/G/ ∞ queue with service time distribution $F(dr)$. The RF in (1.3) is infinitely divisible and has a Poisson stochastic integral representation

$$X(\mathbf{t}) = \int_{\mathbb{R}^d \times \mathbb{R}_+} \mathbb{I}(\mathbf{t} - \mathbf{u} \in r^{1/d}\Xi^0) \mathcal{N}(d\mathbf{u}, dr), \quad \mathbf{t} \in \mathbb{R}^d, \quad (1.4)$$

where $\mathcal{N}(d\mathbf{u}, dr)$ is a Poisson random measure with mean $E\mathcal{N}(d\mathbf{u}, dr) = d\mathbf{u}F(dr)$. RF in (1.4) is stationary and has marginal Poisson distribution with mean $\mu := EX(\mathbf{t}) = \int_{\mathbb{R}^d} \mathbb{P}(\mathbf{t} - \mathbf{u} \in R^{1/d}\Xi^0) d\mathbf{u} = \text{Leb}_d(\Xi^0)ER$. It is well-known [15] that under mild conditions on Ξ^0 the RG model is LRD if the distribution F varies regularly at infinity with exponent $\alpha \in (1, 2)$. The conditions on F and Ξ^0 in this paper (Assumption LRD in Sec.2) imply that

$$r_X(\mathbf{t}) \sim \|\mathbf{t}\|^{-d(\alpha-1)} \ell\left(\frac{\mathbf{t}}{\|\mathbf{t}\|}\right), \quad \|\mathbf{t}\| \rightarrow \infty, \quad 1 < \alpha < 2$$

where $\ell(\mathbf{z}), \|\mathbf{z}\| = 1$ is a continuous (angular) function on the unit sphere of \mathbb{R}^d given in (2.3) and $\int_{\mathbb{R}^d} |r_X(\mathbf{t})| d\mathbf{t} = \infty$ holds for any $d \geq 1, \alpha \in (1, 2)$.

Scaling limits with aggregation of RG model in (1.4) were discussed in [15] (see also [3, 4]).

Let

$$X_M(\mathbf{t}) := \int_{\mathbb{R}^d \times \mathbb{R}_+} \mathbb{I}(\mathbf{t} - \mathbf{u} \in r^{1/d} \Xi^0) \mathcal{N}_M(d\mathbf{u}, dr), \quad \mathbf{t} \in \mathbb{R}^d \quad (1.5)$$

be random grain model with Poisson intensity $M d\mathbf{u} F(dr)$, a multiple of the intensity in (1.4), where $M > 0$. [15] proved that when M increases with λ at certain rate (e.g., $M = \lambda^\gamma$ for some $\gamma > 0$), scaling limits of (1.5) exhibit a trichotomy depending on γ (described in sec. 2).

The main object of this paper are scaling limits of subordinated RFs

$$Y(\mathbf{t}) = G(X(\mathbf{t})) \quad \text{and} \quad Y_M(\mathbf{t}) := G\left(\frac{X_M(\mathbf{t}) - \mathbb{E}X_M(\mathbf{t})}{M^{1/2}}\right), \quad \mathbf{t} \in \mathbb{R}^d, \quad (1.6)$$

where $X(\mathbf{t}), X_M(\mathbf{t})$ are as in (1.4), (1.5) and $G(x)$ is a nonlinear function with $\mathbb{E}G(X(\mathbf{0}))^2 < \infty$ and $\mathbb{E}Y_M^2(\mathbf{0}) = \mathbb{E}G_M^2(X_M(\mathbf{0})) < \infty, G_M(x) := G((x - M\mu)/M^{1/2})$. In other words, we discuss distributional limits of normalized integrals

$$Y_\lambda(\phi) = \int_{\mathbb{R}^d} Y(\mathbf{t}) \phi(\mathbf{t}/\lambda) d\mathbf{t} \quad \text{and} \quad Y_{\lambda, M}(\phi) = \int_{\mathbb{R}^d} Y_M(\mathbf{t}) \phi(\mathbf{t}/\lambda) d\mathbf{t} \quad (1.7)$$

for a large class of test functions ϕ , as $\lambda \rightarrow \infty$ and $M = \lambda^\gamma$ for some $\gamma > 0$. Our main result - Theorem 2 - says that the scaling limits of (1.6) and (1.5) are essentially the same (including the trichotomy of the limit distribution) *provided the Hermite rank of G is 1*, or

$$h_{G, \mu}(1) := \mathbb{E}[G(Z_\mu)Z_\mu] \neq 0, \quad Z_\mu \sim N(0, \mu), \quad (1.8)$$

in which case the difference between the scaling limits of (1.6) and (1.5) reduces to the multiplicative factor in (1.8). We also prove a similar result for scaling limits of $Y(\mathbf{t}) = G(X(\mathbf{t}))$ with (1.8) replaced by the first coefficient of the expansion of G in Charlier polynomials (Proposition 2).

The proofs of our results are rather simple, relying on expansion of bivariate Poisson distribution in orthogonal Charlier polynomials and a Poissonian analog of Mehler's formula (Lemma 1). We hope that this approach can be useful to other Poisson-based RF models and nonlinear triangular arrays. Some open problems are indicated in Remarks 5, 6, 7, and 8.

In this paper, the limit results are applied to two subordinated RG models. The first one is $G(x) = x \wedge 1$, $x \in \mathbb{N}$, or $\hat{X}(\mathbf{t}) := X(\mathbf{t}) \wedge 1, \mathbf{t} \in \mathbb{R}^d$, referred to below as the *Boolean model*. By taking $\phi(\mathbf{t}) = \mathbb{I}(\mathbf{t} \in A)$, where $A \subset \mathbb{R}^d$ is a bounded Borel set, we see that

$$\int_{\mathbb{R}^d} \mathbb{I}(\mathbf{t}/\lambda \in A) \hat{X}(\mathbf{t}) d\mathbf{t} = \text{Leb}_d(\mathcal{X} \cap \lambda A) =: \hat{X}_\lambda(A) \quad (1.9)$$

is the 'volume' of the intersection of the *Boolean set* $\mathcal{X} := \bigcup_{j=1}^\infty (\mathbf{u}_j + R_j \Xi^0) \subset \mathbb{R}^d$ with large 'inflated' set λA . We remark that the Boolean model is a basic model in stereology and stochastic

geometry [37]. According to Corollary 2, under the above assumptions on (1.3), the limit distribution of (1.9) is asymmetric α -stable for a general set A . The second example is the exponential function $G(x) = e^{ax}, x \in \mathbb{R}$, where $a \neq 0$ is a real parameter. We have that $h_{G,\mu}(1) = ae^{a^2\mu/2}$ satisfies (1.8) and Theorem 2 applies to the above G , see Example 1. The case $G(x) = e^{ax}$ is particular interest to the study of scaling limits of statistical solution of Burgers' equation with random linear data as in (1.5) discussed in the last Section 4.

Notation. In this paper, \xrightarrow{d} (respectively, $\xrightarrow{\text{fdd}}$) denote respectively the weak convergence of distributions (respectively, finite dimensional distributions). C stands for a generic positive constant which may assume different values at various locations and whose precise value has no importance. $\mathbb{R}^d := \{\mathbf{t} = (t_1, \dots, t_d) : t_i \in \mathbb{R}, i = 1, \dots, d\}$, $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^d$, $\mathbb{R}_+^d := (0, \infty)^d$. $\mathbb{I}(A)$ denotes the indicator function of a set A .

2 Scaling limits of RG model

Most results in this section either belong to [15], or are variations of the latter work. For reader's convenience, short complete proofs are included. The following assumption is used throughout this paper without further notice.

Assumption LRD $\Xi^0 \subset \mathbb{R}^d$ is a bounded Borel set whereas $F(dr) = f(r)dr$ has a density function such that

$$f(r) \sim c_f r^{-1-\alpha}, \quad r \rightarrow \infty \quad (\exists c_f > 0, \alpha \in (1, 2)). \quad (2.1)$$

Moreover, the function $(r, \mathbf{z}) \mapsto \text{Leb}_d(\Xi^0 \cap (\Xi^0 - r^{-1/d} \mathbf{z}))$ is continuous in $(r, \mathbf{z}) \in \mathbb{R}_+ \times \{\|\mathbf{z}\| = 1\}$ and nontrivial.

The class of test functions in (1.1) and elsewhere in this paper is

$$\Phi = L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d), \quad (2.2)$$

where $L^1(\mathbb{R}^d)$ ($L^\infty(\mathbb{R}^d)$) stand for the linear space of all Borel functions $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $\int_{\mathbb{R}^d} |\phi(\mathbf{t})| d\mathbf{t} < \infty$ (respectively, such that $\sup_{\mathbf{t} \in \mathbb{R}^d} |\phi(\mathbf{t})| < \infty$). The proofs of Theorems 1 and 2 use the fact that any Borel function is a.e. continuous on \mathbb{R}^d (Lusin's theorem).

Proposition 1 (i) Relation (1.5) holds with bounded, continuous and nonnegative angular function

$$\ell(\mathbf{z}) := c_f \int_{\mathbb{R}_+} \text{Leb}_d(\Xi^0 \cap (\Xi^0 - r^{-1/d} \mathbf{z})) r^{-\alpha} dr. \quad (2.3)$$

(ii) For any $\phi \in \Phi$ as $\lambda \rightarrow \infty$

$$c_\lambda(\phi) := \int_{\mathbb{R}^{2d}} \phi(\mathbf{t}_1/\lambda)\phi(\mathbf{t}_2/\lambda)r_X(\mathbf{t}_1 - \mathbf{t}_2)d\mathbf{t}_1d\mathbf{t}_2 \sim \lambda^{d(3-\alpha)}c(\phi), \quad (2.4)$$

where

$$c(\phi) := \int_{\mathbb{R}^{2d}} \phi(\mathbf{t}_1)\phi(\mathbf{t}_2)\ell\left(\frac{\mathbf{t}_1 - \mathbf{t}_2}{\|\mathbf{t}_1 - \mathbf{t}_2\|}\right)\frac{d\mathbf{t}_1d\mathbf{t}_2}{\|\mathbf{t}_1 - \mathbf{t}_2\|^{d(\alpha-1)}} \quad (2.5)$$

and the integral on the r.h.s. of (2.5) converges. Moreover,

$$\int_{\mathbb{R}^{2d}} \phi(\mathbf{t}_1/\lambda)\phi(\mathbf{t}_2/\lambda)(1 \wedge r_X^2(\mathbf{t}_1 - \mathbf{t}_2))d\mathbf{t}_1d\mathbf{t}_2 = \begin{cases} O(\lambda^d), & \alpha > 3/2, \\ O(\lambda^{2d(2-\alpha)}), & \alpha < 3/2, \\ O(\lambda^d \log \lambda), & \alpha = 3/2. \end{cases} \quad (2.6)$$

Proof. (i) From (1.5) we have that

$$\begin{aligned} \|\mathbf{t}\|^{d(\alpha-1)}r_X(\mathbf{t}) &= \|\mathbf{t}\|^{d(\alpha-1)}\int_0^\infty \text{Leb}_d(r^{1/d}\Xi^0 \cap (r^{1/d}\Xi^0 - \mathbf{t}))f(r)dr \\ &= \int_0^\infty \text{Leb}_d(\Xi^0 \cap (\Xi^0 - r^{-1/d}\frac{\mathbf{t}}{\|\mathbf{t}\|}))\tilde{f}(r; \|\mathbf{t}\|^d)r^{-\alpha}dr \end{aligned} \quad (2.7)$$

where, for any $r > 0$, according to (2.1)

$$\tilde{f}(r; \lambda) := (\lambda r)^{1+\alpha}f(\lambda r) \rightarrow c_f \quad (\lambda \rightarrow \infty). \quad (2.8)$$

By boundedness of Ξ^0 , $\text{Leb}_d(\Xi^0 \cap (\Xi^0 - r^{-1/d}\mathbf{z})) \leq \text{Leb}_d(\Xi^0)$ is bounded and vanishes for $r > 0$ small enough uniformly in $\|\mathbf{z}\| = 1$. Thus, (1.5) follows from (2.7) and (2.8) by the dominated convergence theorem. Properties of $\ell(\mathbf{z})$ follow easily from its definition and Assumption LRD.

(ii) Following (2.7), write

$$\begin{aligned} \frac{c_\lambda(\phi)}{\lambda^{d(3-\alpha)}} &= \int_{\mathbb{R}^{2d}} \phi(\mathbf{t}_1)\phi(\mathbf{t}_2)\|\mathbf{t}_1 - \mathbf{t}_2\|^{d(1-\alpha)}d\mathbf{t}_1d\mathbf{t}_2 \\ &\quad \times \int_0^\infty \text{Leb}_d\left(\Xi^0 \cap \left(\Xi^0 - r^{-1/d}\frac{\mathbf{t}_1 - \mathbf{t}_2}{\|\mathbf{t}_1 - \mathbf{t}_2\|}\right)\right)\tilde{f}(r; \lambda\|\mathbf{t}_1 - \mathbf{t}_2\|)r^{-\alpha}dr \end{aligned}$$

and use (2.8) and the argument as in the proof of (i) to conclude (2.4).

The proof of (2.6) is similar but simpler. Indeed, by boundedness of Ξ^0 and (2.1) we have that $|r_X(\mathbf{t})| \leq C(1 \wedge \|\mathbf{t}\|^{d(1-\alpha)})$, c.f. (2.7), so that the l.h.s. of (2.6) does not exceed $C \int_{\mathbb{R}^{2d}} |\phi(\mathbf{t}_1/\lambda)\phi(\mathbf{t}_2/\lambda)|(1 \wedge \|\mathbf{t}_1 - \mathbf{t}_2\|^{2d(1-\alpha)})d\mathbf{t}_1d\mathbf{t}_2$ whose evaluation by the r.h.s. in (2.6) for $\phi \in \Phi$ in (2.2) is elementary. \square

Introduce a Gaussian RF $B_\alpha(\phi)$ indexed by functions $\phi \in \Phi$ as stochastic integral

$$B_\alpha(\phi) := \int_{\mathbb{R}_+ \times \mathbb{R}^d} W_\alpha(dr, d\mathbf{u}) \int_{\mathbb{R}^d} \phi(\mathbf{t})\mathbb{I}(\mathbf{t} - \mathbf{u} \in r^{1/d}\Xi^0)d\mathbf{t}, \quad \phi \in \Phi \quad (2.9)$$

w.r.t. Gaussian white noise W_α with zero mean and variance $\mathbb{E}W_\alpha(dr, d\mathbf{u})^2 = c_f r^{-1-\alpha} dr d\mathbf{u}$.

Observe that the variance

$$\begin{aligned} \mathbb{E}B_\alpha(\phi)^2 &= c_f \int_{\mathbb{R}_+ \times \mathbb{R}^d} r^{-1-\alpha} dr d\mathbf{u} \left(\int_{\mathbb{R}^d} \phi(\mathbf{t}) \mathbb{I}(\mathbf{t} - \mathbf{u} \in r^{1/d} \Xi^0) d\mathbf{t} \right)^2 \\ &= c_f \int_{\mathbb{R}^{2d}} \phi(\mathbf{t}) \phi(\mathbf{t}') d\mathbf{t} d\mathbf{t}' \int_{\mathbb{R}_+ \times \mathbb{R}^d} \mathbb{I}(\mathbf{t} - \mathbf{u} \in r^{1/d} \Xi^0, \mathbf{t}' - \mathbf{u} \in r^{1/d} \Xi^0) r^{-1-\alpha} dr d\mathbf{u} \\ &= c(\phi) \end{aligned} \quad (2.10)$$

coincides with (2.5). Let be given a centered Poisson random measure $\tilde{N}_\alpha(dr, d\mathbf{u}) = N_\alpha(dr, d\mathbf{u}) - \mathbb{E}N_\alpha(dr, d\mathbf{u})$ on $\mathbb{R}^d \times \mathbb{R}_+$ with variance $\mathbb{E}\tilde{N}_\alpha(dr, d\mathbf{u})^2 = \mathbb{E}N_\alpha(dr, d\mathbf{u}) = c_f r^{-1-\alpha} dr d\mathbf{u}$ the same as the variance of W_α , and an α -stable random measure $L_\alpha(d\mathbf{u})$ on \mathbb{R}^d with the characteristic function

$$\begin{aligned} \mathbb{E}e^{i\theta L_\alpha(A)} &:= \exp \left\{ \text{Leb}_d(A) c_f \int_0^\infty (e^{i\theta r} - 1 - i\theta r) r^{-1-\alpha} dr \right\} \\ &= \exp \left\{ -\text{Leb}_d(A) \sigma_\alpha |\theta|^\alpha (1 - i \text{sgn}(\theta) \tan(\pi\alpha/2)) \right\}, \quad \theta \in \mathbb{R}, \end{aligned} \quad (2.11)$$

where $A \subset \mathbb{R}^d$ is any Borel set with $\text{Leb}_d(A) < \infty$ and $\sigma_\alpha := \dots$. Introduce RFs L_α and J_α as stochastic integrals

$$\begin{aligned} L_\alpha(\phi) &:= \int_{\mathbb{R}^d} \phi(\mathbf{u}) L_\alpha(d\mathbf{u}), \\ J_\alpha(\phi) &:= \int_{\mathbb{R}_+ \times \mathbb{R}^d} \tilde{N}_\alpha(dr, d\mathbf{u}) \int_{\mathbb{R}^d} \phi(\mathbf{t}) \mathbb{I}(\mathbf{t} - \mathbf{u} \in r^{1/d} \Xi^0) d\mathbf{t} \end{aligned} \quad (2.12)$$

w.r.t. to the above random measures which are well-defined for any $\phi \in \Phi$. Clearly, $\mathbb{E}J_\alpha(\phi)^2 = \mathbb{E}B_\alpha(\phi)^2$. Denote $\Psi(z) := e^{iz} - 1 - iz, z \in \mathbb{R}$.

Theorem 1 *Let $X_{\lambda, M}(\phi) = \int_{\mathbb{R}^d} X_M(\mathbf{t}) \phi(\mathbf{t}/\lambda) d\mathbf{t}$ with X_M as in (1.5), $M = \lambda^\gamma$ ($\gamma > 0$). Then for any $\phi \in \Phi$ as $\lambda \rightarrow \infty$*

$$\lambda^{-H(\gamma)} (X_{\lambda, M}(\phi) - \mathbb{E}X_{\lambda, M}(\phi)) \xrightarrow{d} \begin{cases} B_\alpha(\phi), & \gamma > d(\alpha - 1), \quad H(\gamma) = \frac{\gamma + (3-\alpha)d}{2}, \\ L_\alpha(\phi), & \gamma < d(\alpha - 1), \quad H(\gamma) = \frac{\gamma + d}{\alpha}, \\ J_\alpha(\phi), & \gamma = d(\alpha - 1), \quad H(\gamma) = d. \end{cases} \quad (2.13)$$

Proof. Let $j_\lambda(\theta) := \log \mathbb{E} \exp\{i\theta \lambda^{-H(\gamma)} (X_{\lambda, M}(\phi) - \mathbb{E}X_{\lambda, M}(\phi))\}$, $\theta \in \mathbb{R}$ denote the log-characteristic functional of the l.h.s. in (2.13). We need to show that it converges to the corresponding functional $j(\theta)$ of the r.h.s. as $\lambda \rightarrow \infty$.

Case $\gamma > d(\alpha - 1)$. Set $\tilde{j}_\lambda(\theta) := -(\theta^2/2)\text{Var}(\lambda^{-H(\gamma)}X_{\lambda,M}(\phi))$, $j(\theta) := \log \mathbb{E} \exp\{i\theta B_\alpha(\phi)\} = -(\theta^2/2)\mathbb{E}B_\alpha(\phi)^2$. Then $\tilde{j}_\lambda(\theta) \rightarrow j(\theta)$ ($\lambda \rightarrow \infty$), see (2.4), (2.5), (2.10). Next,

$$j_\lambda(\theta) - \tilde{j}_\lambda(\theta) = \lambda^\gamma \int_{\mathbb{R}^d \times \mathbb{R}_+} d\mathbf{u} f(r) dr \left\{ \Psi\left(\frac{\theta}{\lambda^{H(\gamma)}} \int_{\mathbb{R}^d} \phi(\mathbf{t}/\lambda) \mathbb{I}(\mathbf{t} - \mathbf{u} \in r^{1/d}\Xi^0) d\mathbf{t}\right) + (1/2) \left(\frac{\theta}{\lambda^{H(\gamma)}} \int_{\mathbb{R}^d} \phi(\mathbf{t}/\lambda) \mathbb{I}(\mathbf{t} - \mathbf{u} \in r^{1/d}\Xi^0) d\mathbf{t}\right)^2 \right\}$$

Using $|\Psi(z) + (1/2)z^2| \leq |z|^3$ and (2.4) we get

$$\begin{aligned} |j_\lambda(\theta) - \tilde{j}_\lambda(\theta)| &\leq C\lambda^\gamma \int_{\mathbb{R}^d \times \mathbb{R}_+} \left| \lambda^{-H(\gamma)} \int_{\mathbb{R}^d} \phi(\mathbf{t}/\lambda) \mathbb{I}(\mathbf{t} - \mathbf{u} \in r^{1/d}\Xi^0) d\mathbf{t} \right|^3 d\mathbf{u} f(r) dr \\ &\leq C\lambda^{-(\gamma/2)-(3/2)(3-\alpha)d} \int_{\mathbb{R}^d} |\phi(\mathbf{t}/\lambda)| d\mathbf{t} \text{Var}(X_M(\phi)) \\ &\leq C\lambda^{-(\gamma-d(\alpha-1))/2} = o(1) \end{aligned}$$

since $|\int_{\mathbb{R}^d} \phi(\mathbf{t}/\lambda) \mathbb{I}(\mathbf{t} - \mathbf{u} \in r^{1/d}\Xi^0) d\mathbf{t}| \leq \int_{\mathbb{R}^d} |\phi(\mathbf{t}/\lambda)| d\mathbf{t} < C\lambda^d$ uniformly in $\mathbf{u} \in \mathbb{R}^d$. Thus, $j_\lambda(\theta) \rightarrow j(\theta) \forall \theta \in \mathbb{R}$.

Case $\gamma < d(\alpha - 1)$. We have

$$j_\lambda(\theta) = \int_{\mathbb{R}^d \times \mathbb{R}_+} \Psi\left(\theta \lambda' \int_{\mathbb{R}^d} \phi(\mathbf{t} + \mathbf{u}) \mathbb{I}(\mathbf{t} \in (r/\lambda')^{1/d}\Xi^0) d\mathbf{t}\right) \tilde{f}(r; \lambda^{\frac{\gamma+d}{\alpha}}) \frac{d\mathbf{u} dr}{r^{\alpha+1}}, \quad (2.14)$$

where $\lambda' := \lambda^{d-\frac{\gamma+d}{\alpha}} \rightarrow \infty$ and \tilde{f} is as in (2.8). Since ϕ is a.e. continuous and $h_\lambda(\mathbf{u}, r) := \lambda' \int_{\mathbb{R}^d} \phi(\mathbf{t} + \mathbf{u}) \mathbb{I}(\mathbf{t} \in (\frac{r}{\lambda'})^{1/d}\Xi^0) d\mathbf{t} \rightarrow r\phi(\mathbf{u}) \text{Leb}_d(\Xi^0) =: h(\mathbf{u}, r)$ at each continuity point \mathbf{u} of ϕ , we get that $h_\lambda(\mathbf{u}, r) \rightarrow h(\mathbf{u}, r)$ for a.e. $\mathbf{u} \in \mathbb{R}^d$ and each $r > 0$. This fact together with (2.14) and (2.8) suggest that

$$\begin{aligned} j_\lambda(\theta) \rightarrow j(\theta) &= c_f \int_{\mathbb{R}^d \times \mathbb{R}_+} (e^{i\theta r \phi(\mathbf{u}) \text{Leb}_d(\Xi^0)} - 1 - i\theta r \phi(\mathbf{u}) \text{Leb}_d(\Xi^0)) \frac{d\mathbf{u} dr}{r^{\alpha+1}} \\ &= \log \mathbb{E} \exp\{i\theta L_\alpha(\phi)\}, \end{aligned} \quad (2.15)$$

see (2.11), (2.12). The convergence in (2.15) can be justified using Pratt's lemma [32], as follows.

It suffices to show that

$$\int_{\mathbb{R}^d} \Psi(h_\lambda(\mathbf{u}, r)) d\mathbf{u} \rightarrow \int_{\mathbb{R}^d} \Psi(h(\mathbf{u}, r)) d\mathbf{u}, \quad \forall r > 0, \quad (2.16)$$

$$\int_{\mathbb{R}^d} |\Psi(h_\lambda(\mathbf{u}, r))| d\mathbf{u} \leq C(r \wedge r^2), \quad (2.17)$$

$$\int_0^\infty (r \wedge r^2) \tilde{f}(r; \lambda^{\frac{\gamma+d}{\alpha}}) \frac{dr}{r^{\alpha+1}} \rightarrow c_f \int_0^\infty (r \wedge r^2) \frac{dr}{r^{\alpha+1}}. \quad (2.18)$$

Relations (2.16) and (2.18) are rather easy. To show (2.17), use $|\Psi(z)| \leq 2(|z| \wedge |z|^2)$ and boundedness of Ξ^0 . Thus, the l.h.s. of (2.17) does not exceed $C \int_{\mathbb{R}^d} \min\left(\lambda' \int_{\mathbb{R}^d} |\phi(\mathbf{t} + \mathbf{u})| \mathbb{I}(\|\mathbf{t}\| \leq$

$C(\frac{r}{\lambda^\gamma})^{\frac{1}{d}}dt, (\lambda' \int_{\mathbb{R}^d} |\phi(\mathbf{t} + \mathbf{u})| \mathbb{I}(\|\mathbf{t}\| \leq C(\frac{r}{\lambda^\gamma})^{\frac{1}{d}}) dt)^2 d\mathbf{u} \leq C \min \left(\lambda' \int_{\mathbb{R}^d} (\int_{\mathbb{R}^d} |\phi(\mathbf{t} + \mathbf{u})| d\mathbf{u}) \mathbb{I}(\|\mathbf{t}\| \leq C(\frac{r}{\lambda^\gamma})^{\frac{1}{d}}) dt, (\lambda')^2 \int_{\mathbb{R}^{2d}} (\int_{\mathbb{R}^d} |\phi(\mathbf{t}_1 + \mathbf{u}) \phi(\mathbf{t}_2 + \mathbf{u})| d\mathbf{u}) \mathbb{I}(\|\mathbf{t}_1\| \leq C(\frac{r}{\lambda^\gamma})^{\frac{1}{d}}, \|\mathbf{t}_2\| \leq C(\frac{r}{\lambda^\gamma})^{\frac{1}{d}}) dt_1 dt_2 \right) \leq C \|\phi\|_{L^1} (\lambda' \int_{\mathbb{R}^d} \mathbb{I}(\|\mathbf{t}\| \leq C(\frac{r}{\lambda^\gamma})^{\frac{1}{d}}) dt) \wedge (\lambda' \int_{\mathbb{R}^d} \mathbb{I}(\|\mathbf{t}\| \leq C(\frac{r}{\lambda^\gamma})^{\frac{1}{d}}) dt)^2 \leq C(r \wedge r^2)$. This proves (2.17) and (2.15).

Case $\gamma = d(\alpha - 1)$. The expression in (2.14) with $\lambda' = 1$ for $j_\lambda(\theta)$ is valid. The proof of (2.13) in this case is similar to that when $\gamma < d(\alpha - 1)$ and we omit the details. \square

Remark 1 Theorem 1 applies also to $\gamma = 0$ or integrals in (1.1) of the RG model in (1.4) with Poisson intensity $d\mathbf{u}F(dr)$. Indeed, the argument in the case $\gamma < d(\alpha - 1)$ applies without change when $\gamma = 0$, yielding the stable limit

$$\lambda^{-d/\alpha}(X_\lambda(\phi) - \mathbb{E}X_\lambda(\phi)) \xrightarrow{d} L_\alpha(\phi), \quad \forall \phi \in \Phi. \quad (2.19)$$

Remark 2 Condition $M = \lambda^\gamma$ in Theorem 1 can be weakened [15]. Particularly, it can be replaced by $M = y\lambda^\gamma$ for some $y > 0$, in which case the control measure of the limit RFs in (2.13) contain the extra multiplicative factor y . For $d = 1, \phi_s(t) := \mathbb{I}(t \in]0, s])$, $t \in \mathbb{R}, (s, y) \in \mathbb{R}_+^2$ we have that $X_{\lambda, M}(\phi_s) = \int_0^{\lambda s} X_M(t) dt$, where $X_M(t) = \sum_{j \in \mathbb{Z}} \mathbb{I}(u_j(M) < t \leq u_j(M) + R_j)$ is the number of customers at time t in the M/G/ ∞ queue with service time distribution R and a Poisson arrival process $\{u_j(M); j \in \mathbb{Z}\}$ with intensity $M du = y\lambda^\gamma du$. Then

$$\lambda^{-H(\gamma)} \int_0^{\lambda s} (X_M(t) - \mathbb{E}X_M(t)) dt \xrightarrow{\text{fdd}} \begin{cases} B_\alpha(s, y), & \gamma > \alpha - 1, H(\gamma) = \frac{\gamma + 3 - \alpha}{2}, \\ L_\alpha(s, y), & \gamma < \alpha - 1, H(\gamma) = \frac{\gamma + 1}{\alpha}, \\ J_\alpha(s, y), & \gamma = \alpha - 1, H(\gamma) = 1, \end{cases} \quad (2.20)$$

where the limits are RFs indexed by $(s, y) \in \mathbb{R}_+^2$ written as stochastic integrals w.r.t. Gaussian, α -stable and Poisson random measures on $\mathbb{R} \times \mathbb{R}_+^2$ analogously to (2.9), (2.12). See [19] for details. Particularly, $\{B_\alpha(s, v); (s, v) \in \mathbb{R}_+^2\}$ is a fractional Brownian sheet with Hurst parameters $(H_1, H_2) = (\frac{3-\alpha}{2}, \frac{1}{2})$, $\{L_\alpha(s, v); (s, v) \in \mathbb{R}_+^2\}$ is an α -stable Lévy sheet, and $\{J_\alpha(s, v); (s, v) \in \mathbb{R}_+^2\}$ is the Telecom RF defined in [19] as a RF extension of the corresponding Telecom process in [16]. As noted above, Theorem 1 is essentially due to [15] while (2.20) is a version of the results in [22, 16] and other previous work.

3 Charlier polynomials and Mehler's formula

The derivations in this sec. can be compared to the discussion of Hermite polynomials and expansions in the case of Gaussian distribution [13, pp.22-26]. See [9] for classical Mehler's formula for Hermite polynomials.

Recall from [39, 24] the definition of Charlier polynomials $P_k(x; \mu)$ of discrete variable $x \in \mathbb{N}$ through the generating function:

$$\mathcal{P}(u; x, \mu) := \sum_{k=0}^{\infty} \frac{u^k}{k!} P_k(x; \mu) = (1+u)^x e^{-u\mu}, \quad (3.1)$$

where the series is convergent for any $x \in \mathbb{N}, \mu > 0$ and any (complex) $u \in \mathbb{C}$. We have $P_0(x; \mu) = 1, P_1(x; \mu) = x - \mu, P_2(x; \mu) = x^2 - (2\mu + 1)x + \mu^2$ and

$$P_k(x; \mu) = (-1)^k \mu^k p(x; \mu)^{-1} D_-^k p(x; \mu), \quad k \in \mathbb{N} \quad (3.2)$$

where $D_-^k := D_- D_-^{k-1}$ is the backward difference operator, $D_- G(x) := G(x) - G(x-1)\mathbb{I}(x \geq 1), D_-^0 G(x) = G(x)$ and

$$p(x; \mu) = e^{-\mu \frac{\mu^x}{x!}}, \quad x \in \mathbb{N} \quad (3.3)$$

is the distribution of Poisson r.v. N with mean μ . Relation (3.2) follows from (3.1) using the identity $(1+u)\partial\mathcal{P}(u; x, \mu)/\partial u = (x - (1+u)\mu)\mathcal{P}(u; x, \mu)$ [39]. We have

$$\mathbb{E}P_k(N; \mu) = 0, \quad \mathbb{E}P_k(N; \mu)^2 = k! \mu^k, \quad k = 1, 2, \dots, \quad (3.4)$$

$$\mathbb{E}P_k(N; \mu)P_\ell(N; \mu) = 0, \quad k \neq \ell = 0, 1, \dots.$$

Facts (3.4) follow from multiplying the series in (3.1) at the points u and v and taking the expectation of the product:

$$\begin{aligned} \sum_{k, \ell=0}^{\infty} \frac{u^k v^\ell}{k! \ell!} \mathbb{E}P_k(N; \mu)P_\ell(N; \mu) &= e^{-(u+v)\mu} \mathbb{E}[(1+u)(1+v)]^N \\ &= e^{\mu uv} = \sum_{k=0}^{\infty} \frac{(\mu uv)^k}{k!} \end{aligned}$$

and equating the coefficients of $u^k v^\ell, k, \ell \in \mathbb{N}$ of the power series on both sides.

Any $G = G(x), x \in \mathbb{N}$ with $\mathbb{E}G^2(N) < \infty$ can be uniquely expanded in Charlier polynomials

$$G(x) = \sum_{k=0}^{\infty} \frac{c_G(k; \mu)}{k!} P_k(x; \mu), \quad x \in \mathbb{N} \quad (3.5)$$

where

$$c_G(k; \mu) := \mu^{-k} \mathbb{E}G(N)P_k(N; \mu), \quad k \in \mathbb{N} \quad (3.6)$$

are *Charlier coefficients* of G in (4.12). (3.2) and summation by parts yields another expression for these coefficients

$$c_G(k; \mu) = \mathbb{E}D_+^k G(N), \quad k \in \mathbb{N}, \quad (3.7)$$

where $D_+^k := D_+ D_+^{k-1}$ is the forward difference operator, $D_+ G(x) := G(x+1) - G(x), D_+^2 G(x) = D_+ G(x+1) - D_+ G(x) = G(x+2) - 2G(x+1) + G(x)$ etc. (3.6) and (3.4) yield the bound

$$|c_G(k; \mu)| \leq \mu^{-k} \sqrt{\mathbb{E}[G^2(N)] \mathbb{E}[P_k^2(N; \mu)]} = C(k!/\mu^k)^{1/2}, \quad C = \sqrt{\mathbb{E}[G(N)^2]}. \quad (3.8)$$

Lemma 1 Let $M_i, i = 1, 2, 3$, be independent Poisson distributed r.v.s with respective means $EM_1 = \mu_1 - \mu_3, EM_2 = \mu_2 - \mu_3, EM_3 = \mu_3, 0 \leq \mu_3 < \mu_1 \wedge \mu_2$, and

$$N_i := M_i + M_3, \quad i = 1, 2.$$

Let

$$p(x, y; \mu_1, \mu_2, \mu_3) := P(N_1 = x, N_2 = y), \quad (x, y) \in \mathbb{N}^2 \quad (3.9)$$

denote the joint distribution of (N_1, N_2) .

(i) (Orthogonality property): For any $k, \ell \in \mathbb{N}$

$$EP_k(N_1; \mu_1)P_\ell(N_2; \mu_2) = \begin{cases} 0, & k \neq \ell, \\ \mu_3^k k!, & k = \ell, \end{cases} \quad (3.10)$$

with the convention $0^0 := 1$.

(ii) Let $G_i = G_i(x), x \in \mathbb{N}, i = 1, 2$ be given functions such that

$$EG_i^2(N_i) < \infty, \quad i = 1, 2. \quad (3.11)$$

Then

$$EG_1(N_1)G_2(N_2) = \sum_{k=0}^{\infty} \frac{c_{G_1}(k; \mu_1)c_{G_2}(k; \mu_2)}{k!} \mu_3^k. \quad (3.12)$$

(iii) (Mehler's formula):

$$\begin{aligned} p(x, y; \mu_1, \mu_2, \mu_3) &= \sum_{k=0}^{\infty} \frac{\mu_3^k}{k!} D_-^k p(x; \mu_1) D_-^k p(y; \mu_2) \\ &= p(x; \mu_1)p(y; \mu_2) \sum_{k=0}^{\infty} \frac{\rho_{12}^k}{k!} P_k(x; \mu_1)P_k(y; \mu_2), \end{aligned} \quad (3.13)$$

where $\rho_{12} := \mu_3/\sqrt{\mu_1\mu_2} = \text{Corr}(N_1, N_2)$ is the correlation coefficient.

Proof. (i) The proof of (3.10) using the generating function in (3.2) is similar as in the univariate case of (3.4). Consider the expectation

$$\begin{aligned} EP(u; N_1, \mu_1)\mathcal{P}(v; N_2, \mu_2) &= e^{-u\mu_1 - v\mu_2} \mathbf{E}[(1+u)^{N_1}(1+v)^{N_2}] \\ &= e^{-u\mu_1 - v\mu_2} \mathbf{E}[(1+u)^{M_1}] \mathbf{E}[(1+v)^{M_2}] \mathbf{E}[(1+u)(1+v)]^{M_3} \\ &= e^{-u\mu_1 - v\mu_2} e^{(\mu_1 - \mu_3)u} e^{(\mu_2 - \mu_3)v} e^{((1+u)(1+v) - 1)\mu_3} \\ &= e^{uv\mu_3} = \sum_{k=0}^{\infty} \frac{(uv\mu_3)^k}{k!}. \end{aligned} \quad (3.14)$$

On the other hand,

$$E\mathcal{P}(u; N_1, \mu_1)\mathcal{P}(v; N_2, \mu_2) = \sum_{k, \ell=0}^{\infty} \frac{u^k v^\ell}{k! \ell!} E[P_k(N_1; \mu_1)P_\ell(N_2; \mu_2)]$$

and (3.10) follows by equating the coefficients of $u^k v^\ell, k, \ell \in \mathbb{N}$ of the power series on both sides.

(ii) Immediate from (4.12) and (3.10).

(iii) Apply (3.12) to $G_1(x) := \mathbb{I}(x = n), G_2(x) := \mathbb{I}(x = m)$, for given $n, m \in \mathbb{N}$. By (3.7), (3.2), $c_{G_1}(k; \mu_1) = E[D_+^k \mathbb{I}(N_1 = n)] = D_-^k p(n; \mu_1) = (-1)^k \mu_1^{-k} P_k(n; \mu_1) p(n; \mu_1)$, $c_{G_2}(k; \mu_2) = E[D_+^k \mathbb{I}(N_2 = m)] = D_-^k p(m; \mu_2) = (-1)^k \mu_2^{-k} P_k(m; \mu_2) p(m; \mu_2)$. This proves (3.13) and the lemma, too. \square

Remark 3 Let $N = \{N_t; t = 0, 1, \dots\}$ be a stationary Markov process on \mathbb{N} with marginal Poisson distribution $P(N_t = x) = p(x; \mu)$ and transition probabilities

$$p(y|x; \mu) := \frac{p(x, y; \mu)}{p(x; \mu)}, \quad x, y \in \mathbb{N}, \quad (3.15)$$

where $p(x, y; \mu) := p(x, y; \mu, \mu, \mu_3)$ is the joint distribution in (3.9) with $\mu_1 = \mu_2 =: \mu > \mu_3$. This process is well-known in the literature as the *Poisson AR(1)* or *INAR(1)* and is related to M/M/ ∞ queueing system, see e.g. [21]. Substitution of (3.13) into (3.15) yields an expansion of (3.15) into a series of Charlier polynomials. Since the transition probability of the Poisson INAR(1) process is usually written via a different expansion, the coincidence with (3.15) most easily can be verified through the bivariate generating function of (N_1, N_2) in (3.14), see [21, (9)]. We remark that the Poisson INAR(1) process is a particular case of stationary Markov evolutions of non-interacting particle systems with Poisson marginal distribution discussed in [39], closely related to chaos expansions in multiple Poisson stochastic integrals.

Given a $G(x), x \in \mathbb{N}, EG(N)^2 < \infty$ with Charlier expansion in (4.12) we define the *Charlier rank* $k^*(G; \mu)$ of G as the minimal $k \geq 1$ such that $c_G(k; \mu) \neq 0$, viz.,

$$k^*(G; \mu) := \min\{k \geq 1 : c_G(k; \mu) \neq 0\}.$$

Lemma 1 (3.12) and the bound in (3.8) imply the following

Corollary 1 *Let $G_i, N_i, i = 1, 2$, be as in Lemma 1, $\rho_{12} = \text{Corr}(N_1, N_2)$, $k^* := k_C(G_1; \mu_1) \vee k_C(G_2; \mu_2)$. Then*

$$\begin{aligned} \text{Cov}(G_1(N_1), G_2(N_2)) &= \sum_{k=k^*}^{\infty} \frac{c_{G_1}(k; \mu_1)c_{G_2}(k; \mu_2)}{k!} \mu_3^k \\ &= \frac{c_{G_1}(k^*; \mu_1)c_{G_2}(k^*; \mu_2)}{k^*!} \mu_3^{k^*} + R(k^*), \end{aligned}$$

where

$$|R(k^*)| \leq \frac{(\mu_3/\sqrt{\mu_1\mu_2})^{k^*+1}}{1 - (\mu_3/\sqrt{\mu_1\mu_2})} \prod_{i=1}^2 \mathbb{E}^{1/2}[G(N_i)^2].$$

Moreover,

$$\begin{aligned} |\text{Cov}(G_1(N_1), G_2(N_2))| &\leq \sum_{k=k^*}^{\infty} |\rho_{12}|^k \frac{|c_{G_1}(k; \mu_1)c_{G_2}(k; \mu_2)|(\mu_1\mu_2)^{k/2}}{k!} \\ &\leq |\rho_{12}|^{k^*} \prod_{i=1}^2 \left(\sum_{k=k^*}^{\infty} \frac{c_{G_i}^2(k; \mu_i)\mu_i^k}{k!} \right)^{1/2} \\ &\leq |\rho_{12}|^{k^*} \sqrt{\text{Var}(G_1(N_1))\text{Var}(G_2(N_2))}. \end{aligned}$$

Particularly, $\sup\{|\text{Cov}(G_1(N_1), G_2(N_2))| : \text{Var}(G_i(N_i)) = 1, i = 1, 2\} = |\rho_{12}|$ and the last supremum is attained by linear functions $G_i(x) = x/\sqrt{\mu_i}, i = 1, 2$.

4 Scaling limits of nonlinear functions of RG model

We study the limit distribution of $Y_M(\phi), Y_{\lambda, M}(\phi)$ defined in (1.6)-(1.7) as $\lambda \rightarrow \infty$ and $M = \lambda^\gamma \rightarrow \infty$, for some $\gamma > 0$, where $G = G(x), x \in \mathbb{R}$ is a general function satisfying some conditions. We will show that the limit of $Y_{\lambda, M}(\phi)$ is the same as that of the linear integral $X_{\lambda, M}(\phi)$ in Theorem 1 up to the multiplicative constant equal to the first coefficient in the Hermite expansion of G . Since for each $\mathbf{t} \in \mathbb{R}^d$, the quantity inside G in $Y_M(\mathbf{t})$ of (1.6), viz., $(X_M(\mathbf{t}) - \mathbb{E}X_M(\mathbf{t}))/M^{1/2} \xrightarrow{d} Z_\mu \sim N(0, \mu)$ when $M \rightarrow \infty$, we consider the corresponding Hermite expansion

$$G(x) = \sum_{k=0}^{\infty} \frac{h_{G, \mu}(k)}{k!} H_k(x; \mu) \quad (4.1)$$

in Hermite polynomials $H_k(x; \mu), k \in \mathbb{N}, x \in \mathbb{R}$ with generating function $\sum_{k=0}^{\infty} (u^k/k!) H_k(x; \mu) = e^{ux - \mu u^2/2}$ and Hermite coefficients

$$h_{G, \mu}(k) := \mu^{-k} \mathbb{E}[G(Z_\mu) H_k(Z_\mu; \mu)], \quad k \in \mathbb{N}. \quad (4.2)$$

Theorem 2 *Let $Y_{\lambda, M}(\phi)$ be as in (1.6), where $X_M(\mathbf{t})$ satisfies the conditions of Theorem 1 and $G = G(x), x \in \mathbb{R}$ is a Borel function such that*

$$\lim_{M \rightarrow \infty} \mathbb{E}G\left(\frac{X_M(\mathbf{0}) - \mathbb{E}X_M(\mathbf{0})}{M^{1/2}}\right)^2 = \mathbb{E}G(Z_\mu)^2 < \infty. \quad (4.3)$$

Let $M = \lambda^\gamma$ for some $\gamma > 0$. Then for any $\phi \in \Phi$ as $\lambda \rightarrow \infty$

$$\lambda^{(\gamma/2) - H(\gamma)} (Y_{\lambda, M}(\phi) - \mathbb{E}Y_{\lambda, M}(\phi)) \xrightarrow{d} h_{G, \mu}(1) \begin{cases} B_\alpha(\phi), & \gamma > d(\alpha - 1), \\ L_\alpha(\phi), & \gamma < d(\alpha - 1), \\ J_\alpha(\phi), & \gamma = d(\alpha - 1), \end{cases} \quad (4.4)$$

where $H(\gamma), B_\alpha(\phi), L_\alpha(\phi), J_\alpha(\phi)$ are the same as in (2.13), and

$$h_{G,\mu}(1) = \mu^{-1} \mathbb{E}G(Z_\mu)Z_\mu \quad (4.5)$$

is the first coefficient in the Hermite expansion (4.1) of G .

Proof. We have $Y_M(\mathbf{t}) = G_M(X_M(\mathbf{t}))$ where $G_M(x) := G((x - \mu M)/M^{1/2}), x \in \mathbb{N}$ and $X_M(\mathbf{t}) =: N_M$ has a Poisson distribution with mean μM . Following (4.12) consider the expansion

$$G_M(x) = \sum_{k=0}^{\infty} \frac{c_{G,M}(k)}{k!} P_k(x; \mu M), \quad x \in \mathbb{N} \quad (4.6)$$

where

$$c_{G,M}(k) := (\mu M)^{-k} \mathbb{E}[G_M(N_M)P_k(N_M; \mu M)], \quad k \in \mathbb{N} \quad (4.7)$$

Particularly,

$$M^{1/2}c_{G,M}(1) = \frac{M^{1/2}}{\mu M} \mathbb{E}\left[G\left(\frac{N_M - M\mu}{M^{1/2}}\right)(N_M - M\mu)\right] \rightarrow h_{G,\mu}(1) \quad (4.8)$$

as $M \rightarrow \infty$, the limit as in (4.5). The convergence in (4.8) under condition (4.3) can be verified with the help of Pratt's lemma [32], as follows. Let $g(x) := G(x)x, x \in \mathbb{R}, \xi_M := \frac{N_M - M\mu}{M^{1/2}}$, then $M^{1/2}c_{G,M}(1) = \mu^{-1} \mathbb{E}g(\xi_M)\xi_M$ and $\xi_M \xrightarrow{d} Z_\mu (M \rightarrow \infty)$. Since G is a.e. continuous and Z_μ has a continuous distribution, this implies $G(\xi_M) \xrightarrow{d} G(Z_\mu), g(\xi_M) \xrightarrow{d} g(Z_\mu)$. By the Skorohod representation theorem, $\xi_M \rightarrow_p Z_\mu, G(\xi_M) \rightarrow_p G(Z_\mu), g(\xi_M) \rightarrow_p g(Z_\mu)$ in probability. Moreover, $|g(\xi_M)| \leq (1/2)(G(\xi_M)^2 + \xi_M^2) =: \bar{g}_M$ where $\bar{g}_M \rightarrow_p (1/2)(G(Z_\mu)^2 + Z_\mu^2) =: \bar{g}$ and $\mathbb{E}\bar{g}_M \rightarrow \mathbb{E}\bar{g}$ according to (4.3). This and [32] prove (4.8).

In view of (4.6), we have the representation

$$\begin{aligned} Y_{\lambda,M}(\phi) - \mathbb{E}Y_{\lambda,M}(\phi) &= c_{G,M}(1)(X_{\lambda,M}(\phi) - \mathbb{E}X_{\lambda,M}(\phi)) + Y_{\lambda,M}^*(\phi), \quad \text{where} \quad (4.9) \\ Y_{\lambda,M}^*(\phi) &:= \int_{\mathbb{R}^d} Y_M^*(\mathbf{t})\phi(\mathbf{t}/\lambda)d\mathbf{t}, \quad Y_M^*(\mathbf{t}) := \sum_{k=2}^{\infty} \frac{c_{G,M}(k)}{k!} P_k(X_M(\mathbf{t}); \mu M). \end{aligned}$$

The convergence in (2.13) follows from (4.4) and (4.8), once we show that $Y_{\lambda,M}^*(\phi)$ in (4.9) is negligible, or

$$\mathbb{E}Y_{\lambda,M}^*(\phi)^2 = o(\lambda^{2H(\gamma)-\gamma}), \quad \lambda \rightarrow \infty \quad (4.10)$$

for $M, H(\gamma)$ as in Theorem 2 and any fixed $\gamma > 0$. Applying Corollary 1 with $k^* = 2, \mu_3 = \text{Cov}(X_M(\mathbf{t}), X_M(\mathbf{0})) = Mr_X(\mathbf{t})$ and the bound $c_{G,M}^2(k) \leq (\mu M)^{-k} k! \mathbb{E}G_M(X_M(\mathbf{0}))^2$, see (3.8),

we get

$$\begin{aligned}
|\mathbb{E}Y_M^*(\mathbf{t})Y_M^*(\mathbf{0})| &\leq \sum_{k=2}^{\infty} \frac{c_{G,M}^2}{(k!)^2} |\mathbb{E}[P_k(X_M(\mathbf{t}); \mu M)P_k(X_M(\mathbf{0}); \mu M)]| \\
&\leq \mathbb{E}G_M(X_M(\mathbf{0}))^2 \sum_{k=2}^{\infty} \frac{(r_X(\mathbf{t})M)^k}{(\mu M)^k} \\
&= \mathbb{E}G_M(X_M(\mathbf{0}))^2 \sum_{k=2}^{\infty} \left(\frac{r_X(\mathbf{t})}{r_X(\mathbf{0})}\right)^k \leq C(1 \wedge r_X^2(\mathbf{t})). \tag{4.11}
\end{aligned}$$

Applying Proposition 1 (2.6), relation (4.10) follows since $\max\{d, 2d(2 - \alpha)\} < \min\{(3 - \alpha)d, \frac{2(\gamma+d)}{\alpha} - \gamma\}$ holds for any $\gamma > 0, \alpha \in (1, 2)$. \square

Remark 4 Condition (4.3) on G involving convergence of the second moments only is rather weak. Using the notation in (1.6), it writes as $\lim_{M \rightarrow \infty} \mathbb{E}Y_M(0)^2 = \mathbb{E}G(Z_\mu)^2 < \infty$. (4.3) can be replaced by a boundedness condition:

$$|G(x)| \leq C_1 e^{C_2|x|}, \quad x \in \mathbb{R} \quad (\exists C_1, C_2 > 0). \tag{4.12}$$

Indeed, verification of (4.3) for $G(x) = C_1 e^{C_2|x|}$ is easy, implying (4.3) for G in (4.12) by Pratt's lemma.

Obviously, Theorem 2 does not hold when $\gamma = 0$ or $M = 1$ is fixed. This case is treated in the following proposition.

Proposition 2 *Let $Y(\mathbf{t}) := G(X(\mathbf{t}))$, $\mathbf{t} \in \mathbb{R}^d$, where $X(\mathbf{t})$ is as in (1.4) and $G(x), x \in \mathbb{N}$ satisfies $\mathbb{E}G(X(\mathbf{0}))^2 < \infty$, $Y_\lambda(\phi) := \int_{\mathbb{R}^d} Y(\mathbf{t})\phi(\mathbf{t}/\lambda)d\mathbf{t}$, $\phi \in \Phi$. Then*

$$\lambda^{-d/\alpha}(Y_\lambda(\phi) - \mathbb{E}Y_\lambda(\phi)) \xrightarrow{d} c_G(1; \mu)L_\alpha(\phi), \tag{4.13}$$

where $c_G(1; \mu) = \mathbb{E}G(X(\mathbf{0}))(X(\mathbf{0}) - \mathbb{E}X(\mathbf{0}))$ and $L_\alpha(\phi)$ is the same α -stable RF as in (2.19).

Proof. Similarly as in the proof of Theorem 2, write $Y_\lambda(\phi) - \mathbb{E}[Y_\lambda(\phi)] = c_G(1; \mu)(X_\lambda(\phi) - \mathbb{E}[X_\lambda(\phi)] + Y_\lambda^*(\phi))$, where $Y_\lambda^*(\phi) := \int_{\mathbb{R}^d} Y^*(\mathbf{t})\phi(\mathbf{t}/\lambda)d\mathbf{t}$ and $Y^*(\mathbf{t}) := \sum_{k=2}^{\infty} \frac{c_G(k; \mu)}{k!} P_k(X(\mathbf{t}); \mu)$, $\mu = \mathbb{E}[X(\mathbf{0})]$ satisfies $|\mathbb{E}Y^*(\mathbf{t})Y^*(\mathbf{0})| \leq C(1 \wedge r_X^2(\mathbf{t}))$ as in (4.11). Then, (4.13) follows in view of (2.19) and (2.6). \square

Remark 5 Theorem 2 and Proposition 2 yield trivial limits if the respective coefficients $h_{G,\mu}(1)$, $c_G(1)$ vanish. The question of the limit distribution of $Y_{\lambda,M}(\phi), Y_\lambda(\phi)$ in such case is open.

EXAMPLE 1 (Scaling limit of the Boolean model.) The Boolean model $\hat{X}(\mathbf{t}) = X(\mathbf{t}) \wedge 1$ corresponds to $Y(\mathbf{t}) = G(X(\mathbf{t}))$ with $G(x) = x \wedge 1, x \in \mathbb{N}$. We have $c_G(0; \mu) = 1 - e^{-\mu}, c_G(k; \mu) = (-1)^{k+1}e^{-\mu} (k \geq 1)$ and the convergence in (4.13) holds with $c_G(1; \mu) = e^{-\mu}$. Let $\phi(\mathbf{x}) = \mathbb{I}(x \in A)$, where $A \subset \mathbb{R}^d$ is a Borel set and $\hat{X}_\lambda(A)$ be as in (1.9). From Proposition 2 and (2.19) it follows

Corollary 2 *Let $A \subset \mathbb{R}^d$ be a bounded Borel set and $\hat{X}_\lambda(A)$ as in (1.9). Then*

$$\lambda^{-d/\alpha}(\hat{X}_\lambda(A) - \mathbb{E}\hat{X}_\lambda(A)) \xrightarrow{d} e^{-\mu}L_\alpha(A), \quad \lambda \rightarrow \infty \quad (4.14)$$

where $L_\alpha(A)$ is α -stable r.v. with characteristic function $\mathbb{E}e^{i\theta L_\alpha(A)} = \exp\{-\sigma_\alpha|\theta|^\alpha \text{Leb}_d(A)(1 - i \text{sgn}(\theta) \tan(\pi\alpha/2))\}, \theta \in \mathbb{R}$.

EXAMPLE 2 (Scaling limits of the Exponential RG model.) We define the *Exponential RG model* as $\mathcal{E}(\mathbf{t}) := e^{aX(\mathbf{t})}, \mathbf{t} \in \mathbb{R}^d$ where $X(\mathbf{t})$ is the RG model in (1.4) and $a \in \mathbb{R}$ a real parameter. We also consider the exponential function of the aggregated RG model

$$\mathcal{E}_M(\mathbf{t}) := e^{a(X_M(\mathbf{t}) - \mathbb{E}X_M(\mathbf{t}))/M^{1/2}}, \quad \mathcal{E}_{\lambda, M}(\phi) := \int_{\mathbb{R}^d} \phi(\mathbf{t}/\lambda) \mathcal{E}_M(\mathbf{t}) d\mathbf{t}, \quad (4.15)$$

where X_M, ϕ are as in (1.5). As noted in the Introduction, the interest in the scaling limits of (4.15) is motivated by application to large-time asymptotics of statistical solution of Burgers' equation discussed in the last section. Obviously, (4.15) is a particular case of (1.6) corresponding to $G(x) = e^{ax}$. Note $D_+^k G(x) = (e^a - 1)^k e^{ax}$ and $c_G(k) = (e^a - 1)^k e^{(e^a - 1)\mu}, k \in \mathbb{N}$. We also find that

$$\begin{aligned} M^{1/2}c_{G, M}(1) &= \exp\{(e^{a/M^{1/2}} - 1 - (a/M^{1/2}))\mu M\} M^{1/2}(e^{a/M^{1/2}} - 1) \\ &\rightarrow ae^{a^2\mu/2} = \mu^{-1}\mathbb{E}[e^{aZ_\mu} Z_\mu] = h_{G, \mu}(1) \end{aligned} \quad (4.16)$$

as $M \rightarrow \infty$, see (4.5). It is easy to see $G(x) = e^{ax}$ that satisfies the conditions of Theorem (2) and the convergences in (4.4) hold for (4.15) with $h_{G, \mu}(1)$ in (4.16). The relevant bound in (4.11) for the above $G(x)$ can be directly obtained from the equality

$$\text{Cov}(\mathcal{E}_M^*(\mathbf{0}), \mathcal{E}_M^*(\mathbf{t})) = (\mathbb{E}\mathcal{E}_M(\mathbf{0}))^2 \{e^{(e^{a/M^{1/2}} - 1)^2 M r_X(\mathbf{t})} - 1 - (e^{a/M^{1/2}} - 1)^2 M r_X(\mathbf{t})\},$$

where $\mathcal{E}_M^*(\mathbf{t}) = (\mathcal{E}_M(\mathbf{t}) - \mathbb{E}\mathcal{E}_M(\mathbf{t})) - c_{G, M}(1)(X_M(\mathbf{t}) - \mathbb{E}X_M(\mathbf{t}))$ as in (4.11).

Remark 6 [3] discusses *small-scale* scaling limits of RG model in (1.5) as $\lambda \rightarrow 0$ and $M \rightarrow 0$ together with λ , under a similar condition (c.f. (2.1)) on the behavior of $f(r)$ as $r \rightarrow 0$ with $\alpha \in (0, 1)$. Extending these results to nonlinear functions in (1.6) is open. Anisotropic small-scale limits (without aggregation) for Lévy driven RFs on \mathbb{R}^2 were studied in [29].

Remark 7 A class of RF which lie between Gaussian and RG RFs and are quite popular in applied sciences are *shot-noise* RFs having a representation

$$X(\mathbf{t}) := \sum_{j=1}^{\infty} W_j(\mathbf{t} - \mathbf{u}_j), \quad \mathbf{t} \in \mathbb{R}^d \quad (4.17)$$

w.r.t. the same Poisson point process $\{\mathbf{u}_j\}$ as in (1.3), where $W_j = \{W_j(\mathbf{t}); \mathbf{t} \in \mathbb{R}^d\}$ are i.i.d. copies of (generic) *pulse RF* $W = \{W(\mathbf{t}); \mathbf{t} \in \mathbb{R}^d\}$, all independent of $\{\mathbf{u}_j\}$. We see that (4.17) encompasses (1.3) which correspond to $W(\mathbf{t}) = \mathbb{I}(\mathbf{t} \in R^{1/d}\Xi^0)$. Assuming that the trajectories of W belong to $L^1(\mathbb{R}^d)$ a.s., (4.17) can be written as the Poisson stochastic integral

$$X(\mathbf{t}) = \int_{\mathbb{R}^d \times L^1(\mathbb{R}^d)} w(\mathbf{t} - \mathbf{u}) \mathcal{N}(d\mathbf{u}, dw), \quad \mathbf{t} \in \mathbb{R}^d \quad (4.18)$$

w.r.t. to Poisson random measure $\mathcal{N}(d\mathbf{u}, dw)$ with mean $E\mathcal{N}(d\mathbf{u}, dw) = d\mathbf{u}P(W(\cdot) \in dw)$, $(\mathbf{u}, w) \in \mathbb{R}^d \times L^1(\mathbb{R}^d)$. The integral in (4.18) is a well-defined and stationary RF with finite variance provided $\int_{\mathbb{R}^d} (E|W(\mathbf{t})| + E|W(\mathbf{t})|^2) d\mathbf{t} < \infty$ holds, in which case $EX(\mathbf{t}) = \int_{\mathbb{R}^d} EW(\mathbf{t}) d\mathbf{t}$ and the covariance $\text{Cov}(X(\mathbf{0}), X(\mathbf{t})) = \int_{\mathbb{R}^d} EW(\mathbf{u})W(\mathbf{t} + \mathbf{u}) d\mathbf{u}$, $\mathbf{t} \in \mathbb{R}^d$ may exhibit LRD property under suitable assumption on the pulse RF W . A rather general form of pulse allowing for LRD (see [1, 12, 43]) is given by

$$W(\mathbf{t}) = \eta a(R^{-1/d}\mathbf{t}), \quad \mathbf{t} \in \mathbb{R}^d, \quad (4.19)$$

where $a(\cdot) \in L^2(\mathbb{R}^d)$ is a deterministic function whereas $\eta \in \mathbb{R}, R > 0$ are r.v.s satisfying certain moment conditions. The corresponding covariance function writes as

$$\text{Cov}(X(\mathbf{0}), X(\mathbf{t})) = E[\eta^2 R (a \star a)(R^{-1/d}\mathbf{t})], \quad \mathbf{t} \in \mathbb{R}^d, \quad (4.20)$$

where \star denotes convolution. (4.19) may be interpreted as a ‘typical’ ‘ a -pulse’ with random ‘amplitude’ η and random ‘frequency’ $R^{-1/d}$. [12, Lemma 2] provides general conditions on $\eta, R, a \star a$ implying covariance LRD and regular decay of (4.20) as $\|\mathbf{t}\| \rightarrow \infty$ for $d = 1$ which can be generalized to any $d \geq 1$. Gaussian scaling limits of shot-noise RFs in (4.17), (4.19) were studied in several papers (see, e.g., [12] and the references therein). The recent work [19] discussed scaling limits and a trichotomy similar to (2.20) for one-dimensional ($d = 1$) shot-noise in (4.17) with rescaled intensity $M d\mathbf{u}P(W(\cdot) \in dw)$, $M = \lambda^\gamma$ of the Poisson random measure, for a general class of pulse W . An interesting open problem is to extend the results on nonlinear functionals in Theorem 2 to shot-noise RF in (4.18)-(4.19).

Remark 8 A *Cox* (or doubly stochastic Poisson) point process $U = \{\mathbf{u}_j\}$ is a Poisson point process on \mathbb{R}^d with *random* intensity $\zeta(\mathbf{u})d\mathbf{u}$, where $\{\zeta(\mathbf{u}); \mathbf{u} \in \mathbb{R}^d\} =: \zeta$ is a nonnegative RF,

meaning that the conditional distribution of U given ζ is Poisson with intensity $\zeta(\mathbf{u})d\mathbf{u}$ [37]. Scaling limits of shot-noise RFs driven by Cox process were studied in [10] and other works. LRD property in such RFs may be due to random intensity ζ . Extending some results of this sec. to RG models driven by Cox process seems feasible.

5 Application: scaling limits of solutions of Burgers' equation with initial RG data

Burgers' equation with (random) potential initial data is written as

$$\begin{aligned}\partial\vec{v}(t, \mathbf{x})/\partial t + (\vec{v}(t, \mathbf{x}), \nabla)\vec{v}(t, \mathbf{x}) &= \frac{1}{2}\kappa\Delta\vec{v}(t, \mathbf{x}), \\ \vec{v}(0, \mathbf{x}) &= -\nabla\xi(\mathbf{x}),\end{aligned}\tag{5.1}$$

where $\vec{v}(t, \mathbf{x}) = (v_1(t, \mathbf{x}), \dots, v_d(t, \mathbf{x}))$, $(t, \mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^d$ is a \mathbb{R}^d -valued function (velocity field) and $\xi = \{\xi(\mathbf{x}); \mathbf{x} \in \mathbb{R}^d\}$ is a scalar (potential) RF; $(\vec{v}(t, \mathbf{x}), \nabla) := \sum_{i=1}^d v_i(t, \mathbf{x})\partial/\partial x_i$. The parameter $\kappa > 0$ is usually called the viscosity parameter. Burgers' equation is one of the important equations of mathematical physics. The solution $\vec{v}(t, \mathbf{x})$ with random initial data is a (vector-valued) RF whose behavior as $t \rightarrow \infty$ and/or $\kappa \rightarrow 0$ presents considerable physical and mathematical interest and has been extensively studied in the literature. For a probabilistic approach, we refer to [1] and the review paper [44]. When $\kappa > 0$ is fixed, the natural parabolic scaling leads to the RF $\vec{v}_\lambda(t, \mathbf{x}) := \vec{v}(\lambda^2 t, \lambda\mathbf{x})$ and the problem concerns the limit distribution of RF $\vec{v}_\lambda(t, \mathbf{x}); (t, \mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^d$ as $\lambda \rightarrow \infty$.

The study of Burgers' equation is facilitated by the Hopf-Cole substitution

$$\vec{v}(t, \mathbf{x}) = -\kappa\nabla\log u(t, \mathbf{x}) = -\frac{\kappa\nabla u(t, \mathbf{x})}{u(t, \mathbf{x})}\tag{5.2}$$

with a scalar-valued $u(t, \mathbf{x})$ satisfying the heat equation $\partial u(t, \mathbf{x})/\partial t = \frac{1}{2}\kappa\Delta u(t, \mathbf{x})$ with the (exponential) initial condition $u(0+, \mathbf{x}) = e^{\xi(\mathbf{x})/\kappa}$, $\mathbf{x} \in \mathbb{R}^d$. Thus, (5.2) has an explicit representation through the heat kernel $g(t, \mathbf{x}, \mathbf{y}) := (2\pi\kappa t)^{-d/2} \exp\{-\|\mathbf{x} - \mathbf{y}\|^2/2\kappa t\}$ as the ratio

$$\vec{v}(t, \mathbf{x}) = -\frac{\kappa \int_{\mathbb{R}^d} \nabla g(t, \mathbf{x}, \mathbf{y}) e^{\xi(\mathbf{y})/\kappa} d\mathbf{y}}{\int_{\mathbb{R}^d} g(t, \mathbf{x}, \mathbf{y}) e^{\xi(\mathbf{y})/\kappa} d\mathbf{y}}.\tag{5.3}$$

Using the fact that $\int_{\mathbb{R}^d} \nabla g(t, \mathbf{x}, \mathbf{y}) d\mathbf{y} = 0$, one can replace $e^{\xi(\mathbf{y})/\kappa}$ in the numerator of (5.3) by $e^{\xi(\mathbf{y})/\kappa} - \mathbb{E}e^{\xi(\mathbf{y})/\kappa}$, provided the last expectation is finite and does not depend on \mathbf{y} . As a consequence, the rescaled velocity RF writes as

$$\vec{v}_\lambda(t, \mathbf{x}) = -\frac{\kappa \int_{\mathbb{R}^d} \phi_{t, \mathbf{x}}(\mathbf{y}/\lambda) (G(\xi(\mathbf{y})) - \mathbb{E}G(\xi(\mathbf{y}))) d\mathbf{y}}{\lambda \int_{\mathbb{R}^d} \psi_{t, \mathbf{x}}(\mathbf{y}/\lambda) G(\xi(\mathbf{y})) d\mathbf{y}},\tag{5.4}$$

where $G(x) = e^{x/\kappa}$ and the integrals in the numerator and denominator resemble (1.1) with $\phi(\mathbf{y}) = \phi_{t,\mathbf{x}}(\mathbf{y}) := \nabla g(t, \mathbf{x}, \mathbf{y})$ and $\phi(\mathbf{y}) = \psi_{t,\mathbf{x}}(\mathbf{y}) := g(t, \mathbf{x}, \mathbf{y})$, respectively. Clearly, for any fixed $(t, \mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^d$ the above ϕ 's belong to Φ in (2.2).

As mentioned in the Introduction, our aim is the limit distribution of (5.4) for the initial potential RF

$$\xi(\mathbf{y}) := M^{-1/2}(X_M(\mathbf{y}) - \mathbb{E}X_M(\mathbf{y})), \quad \mathbf{y} \in \mathbb{R}^d, \quad (5.5)$$

where X_M is the (aggregated) RG model as in Theorem 2 and elsewhere in this paper, with intensity $M = \lambda^\gamma$ increasing with λ when $\gamma > 0$. Similarly as in [1] and some other works, we ignore the meaning of the initial condition in (5.1) for ξ in (5.5) and consider scaling properties of the Hopf-Cole solution in (5.3) alone. Note $\mathbb{E}e^{\xi(\mathbf{y})/\kappa} \rightarrow \mathbb{E}e^{Z_\mu/\kappa} = e^{\mu/2\kappa^2}$ ($M \rightarrow \infty$), $\mathbb{E}[e^{Z_\mu/\kappa} Z_\mu]/\mathbb{E}e^{Z_\mu/\kappa} = 1/\kappa$ and $\lambda^{-d} \int_{\mathbb{R}^d} g(t, \mathbf{x}, \mathbf{y}/\lambda) e^{\xi(\mathbf{y})/\kappa} d\mathbf{y} \rightarrow \mathbb{E}e^{Z_\mu/\kappa}$ in probability by the law of large numbers. The application of Theorem 2 with $G(x) = e^{x/\kappa}$, $x \in \mathbb{R}$ and (4.16) (Example 2) yields the following result.

Corollary 3 *Let $\vec{v}_\lambda(t, \mathbf{x})$ be as in (5.4), (5.5), with X_M, M satisfying the conditions of Theorem 2. Then, as $\lambda \rightarrow \infty$, for any $\gamma > 0$*

$$\lambda^{1+d+\frac{\gamma}{2}-H(\gamma)} \vec{v}_\lambda(t, \mathbf{x}) \xrightarrow{\text{fdd}} \begin{cases} B_\alpha(\nabla g(t, \mathbf{x}, \cdot)), & \gamma > d(\alpha - 1), \\ L_\alpha(\nabla g(t, \mathbf{x}, \cdot)), & \gamma < d(\alpha - 1), \\ J_\alpha(\nabla g(t, \mathbf{x}, \cdot)), & \gamma = d(\alpha - 1), \end{cases} \quad (5.6)$$

where $H(\gamma)$ and the limit RFs are the same as in (2.13)

Let us remark that convergence to α -stable limit in (5.6) holds also for $\gamma = 0$ or $\xi(\mathbf{y})$ in (5.5) replaced by $\xi(\mathbf{y}) = X(\mathbf{y})$ in (1.4). From Proposition 2 we conclude the following result.

Corollary 4 *Let $\vec{v}_\lambda(t, \mathbf{x})$ be as in (5.4) with $\xi(\mathbf{y}) = X(\mathbf{y})$ given in (1.4). Then, as $\lambda \rightarrow \infty$*

$$\lambda^{1+d-\frac{d}{\alpha}} \vec{v}_\lambda(t, \mathbf{x}) \xrightarrow{\text{fdd}} \kappa(e^{1/\kappa} - 1) L_\alpha(\nabla g(t, \mathbf{x}, \cdot)), \quad (5.7)$$

where L_α is defined in (2.11)-(2.12).

We remark that in dimension $d = 1$, a similar result to (5.7) was proved in [40, Thm.1.1(iii)] for initial (potential) data $\xi(y), y \in \mathbb{R}$ given by a piecewise-constant renewal-reward process with renewal distribution having α -tail with $\alpha \in (1, 2)$.

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References

- [1] Albeverio, S., Molchanov, S.A. and Surgailis, D. (1994) Stratified structure of the Universe and Burgers' equation - a probabilistic approach. *Probab. Th. Rel. Fields* 100, 457–484.
- [2] Barczy, M., Nedényi, F. and Pap, G. (2017) Iterated scaling limits for aggregation of randomized INAR(1) processes with idiosyncratic Poisson innovations. *J. Math. Anal. Appl.* 451, 524–543.
- [3] Biermé, H., Estrade, A. and Kaj, I. (2010) Self-similar random fields and rescaled random balls models. *J. Theoret. Probab.* 23, 1110–1141.
- [4] Biermé, H., Durieu, O. and Wang, Y. (2018) Generalized operator-scaling random ball model. *ALEA, Lat. Am. J. Probab. Math. Stat.* 15, 1401–1429.
- [5] Dobrushin, R.L. (1980) Automodel generalized random fields and their renormgroup. In: R.L. Dobrushin and Ya.G. Sinai (Eds.), *Multicomponent Random Systems*, pp. 153–198. Dekker, New York.
- [6] Dobrushin, R.L. and Major, P. (1979) Non-central limit theorems for non-linear functionals of Gaussian fields. *Z. Wahrsch. verw. Geb.* 50, 27–52.
- [7] Doukhan, P., Lang, G. and Surgailis, D. (2002) Asymptotics of weighted empirical processes of linear random fields with long range dependence. *Annales d'Institute de H. Poincaré* 38, 879–896.
- [8] Doukhan, P., Oppenheim, G. and Taqqu, M.S. (Eds.) (2003) *Theory and Applications of Long-Range Dependence*. Birkhäuser, Boston.
- [9] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G. (1953) *Higher Transcendental Functions*, vol.2. McGraw-Hill, New York.
- [10] Funaki, T., Surgailis, D. and Woyczynski, W.A. (1995) Gibbs-Cox random fields and Burgers' turbulence. *Ann. Appl. Probab.* 5, 701–735.
- [11] Gaigalas, R. and Kaj, I. (2003) Convergence of scaled renewal processes and a packet arrival model. *Bernoulli* 9, 671–703.
- [12] Giraitis, L., Molchanov, S.A. and Surgailis, D. (1992) Long memory shot noises and limit thoerems with application to Burgers' equation. In: D. Brillinger et al. (Eds.) *New Directions in Time Series Analysis, Part II*. IMA Volumes in Mathematics and its Applications, vol. 46, pp. 153–176. Springer, New York.
- [13] Giraitis, L., Koul, H.L. and Surgailis, D. (2012) *Large Sample Inference for Long Memory Processes*. Imperial College Press, London.
- [14] Ho, H.-C. and Hsing, T. (1997) Limit theorems for functionals of moving averages. *Ann. Probab.* 25, 1636–1669.
- [15] Kaj, I., Leskelä, L., Norros, I. and Schmidt, V. (2007) Scaling limits for random fields with long-range dependence. *Ann. Probab.* 35, 528–550.
- [16] Kaj, I. and Taqqu, M.S. (2008) Convergence to fractional Brownian motion and to the Telecom process: the integral representation approach. In: M.E. Vares and V. Sidoravicius (Eds.) *In and Out of Equilibrium 2*. Progress in Probability, vol. 60, pp. 383–427. Birkhäuser, Basel.
- [17] Koul, H.L., Mimoto, N. and Surgailis, D. (2016) Goodness-of-fit tests for marginal distribution of linear random fields with long memory. *Metrika* 79, 165–193.

- [18] Lahiri, S.N. and Robinson, P.M. (2016) Central limit theorems for long range dependent spatial linear processes. *Bernoulli* 22, 345–375.
- [19] Leipus, R., Pilipauskaitė, V. and Surgailis, D. (2023) Aggregation of network traffic and anisotropic scaling of random fields. *Theor. Probab. Math. Statist.* 108, 77–126.
- [20] Leonenko, N.N. (1999) *Random Fields with Singular Spectrum*. Kluwer, Dordrecht.
- [21] McKenzie, E. (2003) Discrete variate time series. In: D.N. Shanbhag and C.R. Rao (Eds.) *Handbook of Statistics*, vol. 21, pp. 573–606. Elsevier, Amsterdam.
- [22] Mikosch, T., Resnick, S., Rootzén, H. and Stegeman, A. (2002) Is network traffic approximated by stable Lévy motion or fractional Brownian motion? *Ann. Appl. Probab.* 12, 23–68.
- [23] Mikosch, T. and Samorodnitsky, G. (2007) Scaling limits for cumulative input processes. *Math. Oper. Res.* 32, 890–918.
- [24] Peccati, G. and Taqqu, M.S. (2011) *Wiener Chaos: Moments, Cumulants and Diagrams*. Springer, New York.
- [25] Pilipauskaitė, V., Skorniakov, V. and Surgailis, D. (2020) Joint temporal and contemporaneous aggregation of random-coefficient AR(1) processes with infinite variance. *Adv. Appl. Probab.* 52, 237–265.
- [26] Pilipauskaitė, V. and Surgailis, D. (2014) Joint temporal and contemporaneous aggregation of random-coefficient AR(1) processes. *Stochastic Process. Appl.* 124, 1011–1035.
- [27] Pilipauskaitė, V. and Surgailis, D. (2016) Anisotropic scaling of random grain model with application to network traffic. *J. Appl. Probab.* 53, 857–879.
- [28] Pilipauskaitė, V. and Surgailis, D. (2017) Scaling transition for nonlinear random fields with long-range dependence. *Stochastic Process. Appl.* 127, 2751–2779.
- [29] Pilipauskaitė, V. and Surgailis, D. (2022) Local scaling limits of Lévy driven fractional random fields. *Bernoulli* 28, 2833–2861.
- [30] Pipiras, V., Taqqu, M.S. and Levy, L.B. (2004) Slow, fast, and arbitrary growth conditions for renewal reward processes when the renewals and the rewards are heavy-tailed. *Bernoulli* 10, 121–163.
- [31] Pipiras, V. and Taqqu, M.S. (2017) *Long-Range Dependence and Self-Similarity*. Cambridge Univ. Press, Cambridge.
- [32] Pratt, J.W. (1960) On interchanging limits and integrals. *Ann. Math. Statist.* 31, 74–77.
- [33] Puplinskaitė, D. and Surgailis, D. (2010) Aggregation of random coefficient AR(1) process with infinite variance and idiosyncratic innovations. *Adv. Appl. Probab.* 42, 509–527.
- [34] Puplinskaitė, D. and Surgailis, D. (2015) Scaling transition for long-range dependent Gaussian random fields. *Stoch. Process. Appl.* 125, 2256–2271.
- [35] Puplinskaitė, D. and Surgailis, D. (2016) Aggregation of autoregressive random fields and anisotropic long-range dependence. *Bernoulli* 22, 2401–2441.
- [36] Samorodnitsky, G. (2016) *Stochastic Processes and Long Range Dependence*. Springer, New York.
- [37] Stoyan, D., Kendall, W.S. and Mecke, J. (1989) *Stochastic Geometry and Its Applications*. Akademie-Verlag, Berlin.

- [38] Surgailis, D. (1982) Zones of attraction of self-similar multiple integrals. *Lithuanian Math. J.* 22, 185–201.
- [39] Surgailis, D. (1984) On multiple Poisson stochastic integrals and associated Markov semigroups. *Probab. Math. Statist.* 3, 217–239.
- [40] Surgailis, D. (1996) Asymptotic of solutions of Burgers’ equation with random piecewise constant data, In: S.A. Molchanov and W.A. Woyczynski (Eds.) *Stochastic Models in Geosystems*, IMA Volumes in Mathematics and Its Applications vol. 85, pp. 427–442, Springer, New York.
- [41] Surgailis, D. (2019) Anisotropic scaling limits of long-range dependent linear random fields on \mathbb{Z}^3 . *J. Math. Anal. Appl.* 472, 328–351.
- [42] Surgailis, D. (2020) Scaling transition and edge effects for negatively dependent linear random fields on \mathbb{Z}^2 . *Stochastic Process. Appl.* 130, 7518–7546.
- [43] Surgailis, D. and Woyczynski, W.A. (1994) Burgers’ equation with non-local shot noise data, *J. Appl. Probab.* 31A, 351–362.
- [44] Surgailis, D. and Woyczynski, W.A. (2003) Limit theorems for the Burgers equation initialized by data with long-range dependence. In: P. Doukhan, G. Oppenheim and M.S. Taqqu (Eds.) *Long Range Dependence: Theory and Applications*, pp. 507–523. Birkhäuser, Boston.