arXiv:2304.03847v2 [hep-ph] 12 Apr 2023

# Light-front puzzles

W. N. Polyzou*
*Department of Physics and Astronomy*
*The University of Iowa*
*Iowa City, IA 52242, USA*
(Dated: April 14, 2023)

Light-front formulations of quantum field theories have many advantages for computing electroweak matrix elements of strongly interacting systems and other quantities that are used to study hadronic structure. The theory can be formulated in Hamiltonian form so non-perturbative calculations of the strongly interacting initial and final states are in principle reduced to linear algebra. These states are needed for calculating parton distribution functions and other types of distribution amplitudes that are used to understand the structure of hadrons. Light-front boosts are kinematic transformations so the strongly interacting states can be computed in any frame. This is useful for computing current matrix elements involving electroweak probes where the initial and final hadronic states are in different frames related by the momentum transferred by the probe. Finally in many calculations the vacuum is trivial so the calculations can be formulated in Fock space.

The advantages of light front-field theory would not be interesting if the light-front formulation was not equivalent to the covariant or canonical formulations of quantum field theory. Many of the distinguishing properties of light-front quantum field theory are difficult to reconcile with canonical or covariant formulations of quantum field theory.

This paper discusses the resolution of some of the apparent inconsistencies in canonical, covariant and light-front formulations of quantum field theory. The puzzles that will be discussed are (1) the problem of inequivalent representations (2) the problem of the trivial vacuum (3) the problem of ill-posed initial value problems (4) the problem of rotational covariance (5) the problem of zero modes and (6) the problem of spontaneously broken symmetries.

## I. INTRODUCTION

Light-front formulations of quantum field theory [1][2][3][4][5][6] have a number of advantages for studying the structure of hadrons. Among the most important advantages are (1) it has a Hamiltonian formulation [7] so hadronic state vectors can in principle be found using methods of linear algebra. The Hamiltonian formulation has the advantage that it is non-perturbative. The resulting hadronic wave-functions can be used to compute form factors, parton distribution functions, generalized parton distribution functions and related quantities that are important for unraveling hadronic structure (2) there is a three-parameter group of interaction-independent boosts that is useful for studying electroweak and gravitational probes of hadronic systems, where the momentum transferred by the probe puts the initial and final hadronic states in different reference frames. Because the light-front boosts form a subgroup the light-front spins do not Wigner rotate under light-front boosts. (3) The algebra of fields restricted to the light front is irreducible and the vacuum is trivial. This implies that calculations can be formulated directly on the free field Fock space of the theory.

These advantages are in contrast to properties of canonical or covariant formulations of quantum field theory. The light-front formulation would not be interesting if it was not equivalent to these conventional formulations of quantum field theory, however some the distinguishing properties of light-front field theory are difficult to reconcile with properties of canonical or covariant formulations of quantum field theory. Some of these puzzles include:

- The problem of inequivalent representations of the canonical commutation relations.
- The problem of the trivial vacuum.
- The problem of the initial value problem.
- The problem of rotational covariance.
- The problem of zero modes.

*Electronic address: polyzou@uiowa.edu; This work supported by the U.S. Department of Energy, Office of Science, Grant #DE-SC0016457

- The problem spontaneously broken symmetries.

A brief description of each of these problems is given below.

1) The problem of inequivalent representations: For systems with a finite number of degrees of freedom the Stone-Von Neumann theorem [8][9][10] implies that canonical transformations can be realized by unitary transformations. Unitary transformations preserve the canonical commutation relations, $[p_i, q_j] = -i\delta_{ij}$ and $[p_i, p_j] = [q_i, q_j] = 0$. For harmonic oscillators with different frequencies the unitary transformation relating the momentum and coordinate of one oscillator to the momentum and coordinate of the other oscillator transforms the creation and annihilation operators for one oscillator to linear combinations of creation and annihilation operators for the other oscillator. It also transforms the vacuum vector, which is annihilated by the transformed annihilation operator. Canonical field theories are canonical systems with an infinite number of degrees of freedom. Canonical coordinates and momenta can be obtained by integrating the fields and generalized momenta at fixed time with an orthonormal set of basis functions. The Stone-Von Neumann theorem does not hold for systems with an infinite number of degrees of freedom. Haag [11] has given a simple example which demonstrates the breakdown of this theorem for free field theories with different masses. What happens is that the generator of the unitary transformation that relates the creation and annihilation operators and ground states for systems of a finite number of degree of freedom maps the ground state of one system to a vector whose norm becomes infinite in the limit of an infinite number of degrees of freedom. On the other hand a simple change of variables transforms the canonical fields to light-front fields. In the light-front case there is a kinematic representation of the irreducible Weyl algebra (exponential form of the canonical commutation relations), and the Weyl algebra and ground states for free field theories with different masses are unitarily equivalent [12]. Thus, while the vacua of two canonical free field theories with different masses are vectors in inequivalent Hilbert space representations of the canonical commutation relations, in the light-front representation the Hilbert space representations are unitarily equivalent. This does not seem to be consistent with the assumption that the canonical and light-front formulations are equivalent.

2) The problem of the triviality of the light-front vacuum: The spectral condition $P^+ \geq 0$ in light-front quantum field theory suggests that interactions cannot change the vacuum. This follows because the interaction must commute with $P^+$ which means that the interaction applied to the vacuum is an eigenstate of $P^+$ with eigenvalue 0. This implies that the interaction cannot change the vacuum. This is consistent with the observation that vacuum diagrams are suppressed by boosting to a frame with large momentum [13][14]. On the other hand it is known that in canonical formulations of quantum field theory for Hamiltonians that are quadratic in the generalized momenta the vacuum determines the Hamiltonian [15][16]. Similarly, in algebraic formulations of quantum field theory, all that is known about the fields is that they are local operator valued distributions that transform covariantly. The vacuum is a linear functional on the algebra of smeared field operators. The dynamics is given by specifying the Wightman distributions of the theory which are vacuum expectation values of products of the fields. Different vacuum functionals acting on the same local field algebra result in different Wightman distributions, which result in unitarily inequivalent theories. One interpretation of these observations is that the vacuum defines the dynamics in conventional formulations of quantum field theory, which is in contrast to the light-front case where all theories have equivalent vacua.

3) The problem of an ill-posed initial value problem: A fixed time hyperplane is a good initial value surface. The fields and their time derivatives (generalized momenta) at fixed time are an irreducible set of operators. They can be used to construct initial data. These operators can be evolved in time using the Heisenberg field equations. A light front is a hyperplane that is tangent to the light cone. It includes points that are causally related by a light signal, so it is not a suitable initial value surface. While massive particles do not move at the speed of light, there is still missing information that must be supplied on the light front in order to get a well-posed initial value problem. On the other hand, because the fields restricted to a light front are irreducible they can also be used to represent any operator, including all of the Poincaré generators. Initial data can also be represented by applying operators in the irreducible light-front field algebra to the light-front vacuum. The evolution of this data normal to the light front is given by a unitary one-parameter subgroup of the Poincaré group, which suggests that the initial value problem is well posed.

4) The problem of rotational covariance. The Poincaré Lie algebra has 10 infinitesimal generators. In light-front representations there are 7 interaction-independent generators (kinematic generators) and three dynamical generators. The dynamical generators can be chosen as the light-front Hamiltonian $P^- := H - \mathbf{P} \cdot \hat{\mathbf{z}}$ and 2 transverse rotation $\hat{\mathbf{z}} \times \mathbf{J}$ generators. The light-front Hamiltonian and the kinematic generators form a closed Lie algebra. This means that the light-front Hamiltonian does not intrinsically have information on how to formulate rotational covariance. In addition the construction of the Poincaré Lie algebra is not unique. Specifically if $J^1$ and $J^2$ are a pair of transverse rotation operators that complete the Poincaré Lie algebra and $W$ is a unitary operator that commutes with $P^-$ and the kinematic generators, then $J^{1\prime} := WJ^1W^\dagger$ and $J^{2\prime} := WJ^2W^\dagger$ are a different set of transverse rotation operators that complete the Poincaré Lie algebra. On the other hand Noether's theorem on the light front results in explicit expressions for the transverse rotation generators. This means that eigenstates of $P^-$ are not intrinsically constrained

by rotational covariance. This results in a problem in assigning spins to states that are degenerate in light-front magnetic quantum numbers.

5) The problem zero modes: In light front-quantum field theory there are the standard divergences associated with large momentum and new divergences that appear as $p^+ \to 0$. These divergences are related by both rotational covariance and space reflection symmetry. This is because the divergences of a quantum field theory associated with $p^3 \to -\infty$ become divergences as $p^+ \to 0$ in light-front quantum field theory. Both the large momentum and small $p^+$ divergences need to be removed. The remaining finite parts are constrained by rotational covariance and space reflection symmetry. This follows because in light-front quantum theories full rotational invariance is equivalent to the requirement that changing the orientation of the light front leaves all physical observables unchanged [17][18] [19][20][21][22] [23][24]. Changing the orientation of the light front implies that the $p^+ \to 0$ divergence associated with one light front is transformed a large momentum divergence in an equivalent theory with a different light front. After renormalization the finite parts must be consistent with rotational covariance.

Similar considerations apply to a reflection about the $x - y$ plane. This is a dynamical transformation that transforms $P^+$ to $P^-$. Space reflection relates the $p^+ \to 0$ divergences to $p^3 \to \infty$ divergences. Thus space reflection symmetry, like rotational symmetry, implies that the renormalization of the $p^+ \to 0$ and $|\mathbf{p}| \to \infty$ divergences cannot be performed independently; they are constrained by rotational covariance and space reflection symmetry.

These considerations imply that separately renormalizing the $p^+ \to 0$ and large-momentum singularities in $P^-$ is not sufficient to restore rotational covariance and space reflection symmetry. It is known that additional zero-mode contributions are required to reconcile some calculations in light-front quantum field theory with calculations based on covariant perturbation theory [25] [26][27] where both rotational and space reflection symmetry are satisfied.

6) The problem of spontaneous symmetry breaking: In canonical field theories spontaneous symmetry breaking is due to a charge operator that couples the vacuum to a massless Goldstone boson, while in the light-front case the spectral condition $P^+ \geq 0$ implies that the light-front charge operator leaves the vacuum invariant.

The advantages of light-front quantum, field theory and its relation to covariant and canonical quantum field theory as well as the analysis of some of the problems listed above have been the subject of many papers in the literature [1] [2] [28] [29] [3] [4] [5] [6] [12] [25] [30] [17] [18] [31] [32] [19] [20] [21] [23] [33] [34] [35] [22] [36] [37] [26] [27] [38] [24] [39] [40] [41] [42], [43] [44] [45] [46] [47] [48] [49] [50] [51] [52] [53] [54] [55] [56] [57] [58] [59] [60] [61]. The purpose of this work is reconcile some of these puzzles; many of which have been previously discussed in theses references.

Free field theories provide examples of solvable quantum field theories where some of these problems can be investigated. Real applications require interacting field theories. These are best understood using perturbation theory. Axiomatic approaches can provide some insight to some of these problems that are not tied to perturbation theory, but as a practical matter there are unresolved questions to how to renormalize light-front theories and restore rotational covariance and reflection symmetry outside of perturbation theory.

A brief summary of the observations and resolutions of some of the puzzles discussed above is given below. They will be discussed in more detail in what follows.

1. In a local quantum field theory a dense set of Hilbert space vectors are constructed by applying polynomials of local field operators smeared with Schwartz test functions [62] in four space-time variables to the vacuum. A pre-Hilbert space inner product of two of these states can be expressed as the vacuum expectation value of another polynomial in the smeared fields. In general this space will have zero-norm vectors. To pass to a Hilbert space representation it is necessary to replace vectors by equivalence classes of vectors that differ by zero norm vectors and add new vectors defined by Cauchy sequences of vectors to make the space complete. There are standard methods to do this [63]. Any transformation that leaves these vacuum expectation values unchanged preserves this Hilbert space inner product and is by definition a unitary transformation. The polynomials in the smeared fields are an algebra of operators; they are closed under addition, multiplication by complex numbers and operator products. This algebra will be referred to as the local Heisenberg algebra. The vacuum can be understood as a positive linear functional on this algebra. Vacuum functionals that do not preserve this inner product lead to inequivalent Hilbert space representations of this algebra of operators. Another algebra is the algebra of fields restricted to the light front, called the light-front Fock algebra. It is defined by replacing fields smeared with 4-dimensional Schwartz test functions by test functions of the form $\delta(x^+)f(x^-, \mathbf{x}_\perp)$ where $f(x^-, \mathbf{x}_\perp)$ is a Schwartz test function in three variables with vanishing $x^-$ derivative [12]. The vacuum functionals for all free field theories are defined on this algebra and give the same vacuum expectation values. For free fields there is a mapping from the local Heisenberg algebra to a *sub-algebra* of the light-front Fock algebra. This maps polynomials of fields smeared with 4-dimensional Schwartz functions to polynomials of fields on the light front by enforcing the "free-field mass shell" condition to eliminate the $x^+$ dependence. This defines a sub-algebra of the light-front Fock algebra. This will be referred to as the light-front mass $m$ sub-algebra. The test functions $f$ are mapped into functions, $\tilde{f}$, of light-front variables; these functions will be referred to as the light-front mass $m$ test functions. If $\tilde{f}_i$ are the light-front mass $m$ test functions associated with the four dimensional Schwartz

functions $f_i$ then

$$ {}_{m_1}\langle 0|\phi_{m_1}(f_1)\cdots\phi_{m_1}(f_n)|0\rangle_{m_1} = {}_{m_1}\langle 0|\phi_{m_1}(\tilde{f}_1)\cdots\phi_{m_1}(\tilde{f}_n)|0\rangle_{m_1} = {}_{m_2}\langle 0|\phi_{m_2}(\tilde{f}_1)\cdots\phi_{m_2}(\tilde{f}_n)|0\rangle_{m_1} \tag{1} $$

where the subscripts $m_1$ and $m_2$ refer to free fields and vacuum functionals associated with masses $m_1$ and $m_2$. This means that free field theories with different masses are unitarily equivalent on the light-front mass $m$ sub-algebra. Equation (1) shows that the Hilbert space inner product of the theory is equal to the vacuum expectation value of elements of light-front mass $m$ sub-algebra using free field theories of *any* mass. In this case all of the dynamical information is contained in the choice of light-front mass $m$ sub-algebra, rather than the vacuum. The mapping from the Heisenberg algebra to the light-front mass $m$ sub-algebra is not invertible since it cannot distinguish four-dimensional Schwartz functions whose Fourier transform agree on the mass shell.

Theories with different masses are associated with different sub-algebras. Light-front vacuum expectation values for *different* sub-algebras do not agree. This resolves the problem of inequivalent representation for free fields.

In the interacting case there is an irreducible algebra of asymptotic fields [64][65][66][67]. These behave like free fields with physical masses [68][69], except the creation and annihilation operators transform $N$-particle scattering states to $N \pm 1$ particle scattering states. Vacuum expectation values of polynomials of smeared asymptotic fields satisfy

$$ \langle 0|\phi_{IN}(f_1)\cdots\phi_{IN}(f_n)|0\rangle = \langle 0|\phi_{OUT}(f_1)\cdots\phi_{OUT}(f_n)|0\rangle = {}_{m_p}\langle 0|\phi_{m_p}(f_1)\cdots\phi_{m_p}(f_n)|0\rangle_{m_P} \tag{2} $$

where the fields in the third factor are free fields with *physical masses* and the corresponding free-field vacuum. The physics is in the relation of the IN fields to the OUT fields or the IN or OUT fields to the Heisenberg fields. It follows by combining (1) and (2) that the vacuum expectation values of products of $IN$ fields are equal to light-front vacuum expectation values of products of free fields smeared with light-front mass $m_p$ test functions:

$$ \langle 0|\phi_{IN}(f_1)\cdots\phi_{IN}(f_n)|0\rangle = {}_{m_p}\langle 0|\phi_{m_p}(\tilde{f}_1)\cdots\phi_{m_p}(\tilde{f}_n)|0\rangle_{m_P}. \tag{3} $$

Since the smeared IN fields are an irreducible set of Hilbert space operators (assuming that the field theory is asymptotically complete), smeared Heisenberg fields can be expanded in normal products of the IN fields [11][70][71] [72]. This means the vacuum expectation values of polynomials of smeared interacting Heisenberg fields can be expressed as limits of vacuum expectation values of polynomials in smeared IN fields which in turn can be expressed as limits of vacuum expectation values of operators in the light-front mass $m_P$ sub algebra.

The complication is that there are asymptotic fields associated with each stable particle of the theory, which includes stable composite systems, so the IN fields with different masses involve mappings to different sub-algebras of the light-front Fock algebra. The basic result is that the vacuum expectation values of polynomials of smeared Heisenberg fields is equal to light-front vacuum expectation values of elements of a sub-algebra of the light-front Fock algebra. While the interacting case is more complicated, the result is the same as in the free-field case. The physics is in the sub-algebra of the light-front Fock algebra. The resulting light-front vacuum expectation value is independent of the vacuum functional used to evaluate the vacuum expectation value.

For case of QCD the Heisenberg fields for quarks and gluons field cannot be expanded in terms of the algebra asymptotic fields. See [73] for a possible way to introduce "asymptotic fields" for confined particles. An expansion in asymptotic fields is not necessary for the vacuum expectation values involving smeared quark and gluon Heisenberg fields to be expressible in terms of a sub-algebra of quark and gluon fields in the light-front Fock algebra. The unsolved problem is to find an appropriate sub algebra.

2. The physical vacuum expectation values of products of free fields can also be computed using products of suitably smeared canonical pairs of fields. Formally there is a mapping from the Heisenberg field algebra to the algebra canonical fields restricted to a fixed time surface. In this case the canonical field sub-algebras are different for different masses and, unlike the light-front case, the vacuum functionals on the sub-algebras are different. They result in inequivalent representations of the canonical commutation relations. In the light-front case the corresponding map to the light-front sub-algebra discussed in 1) carries all of the dynamical information of the theory and the inequivalence is due to the different sub-algebras rather that due to the different light-front vacuum functionals.

3. In the light-front case Noether's theorem on the light front results in expressions for all ten Poincaré generators expressed as elements of the irreducible light-front Fock algebra. The mapping discussed in 1) maps Schwartz test functions in four variables to functions on the light front that vanish faster than any power of $p^+$ as $p^+ \to 0$. If the Schwartz functions are restricted to the dense set of Schwartz functions having Fourier transforms with

compact support, this dense set is mapped to a subspace of vectors on the light-front Fock space, and the power series expansion of $e^{-iP^- x^+/2}$ converges on this subspace of vectors. This means that the light front mass-m sub-algebra enforces boundary conditions that result in a well-defined initial valued problem.

4. Formally the results of any dynamical calculation should be independent of the orientation of the light front. If the scattering operator is independent of the orientation of the light front then it is possible to use the scattering wave functions from both representations to construct a unitary representation of the rotation group. It follows that rotational covariance is equivalent to independence on the orientation of the light front. This is not entirely trivial to achieve because it requires ensuring the rotational covariance of the asymptotic conditions, which is exactly the problem that appears when degenerate bound state solutions have the same magnetic quantum numbers. Replacing the light-front Hamiltonian, $P^-$, by one of the transverse rotation generators has the property that all of the generators can be constructed using the commutation relations with the kinematic generators. While this results in a consistent set of generators, it is still non-trivial because there are linear and non-linear constraints on the transverse rotation generator that come from Poincaré covariance. What makes this problem difficult in Hamiltonian formulations of quantum field theory is that it is related to renormalization. $p^+ \to 0$ divergences with respect to one light front become $p^3 \to \infty$ divergences using a different orientation of the light front. Invariance under change of orientation of the light front implies that the renormalization of the $p^+ \to 0$ and $\mathbf{p} \to \infty$ divergences of the theory cannot be performed independently. Note that restoring rotational covariance may not result in the same representation of the rotation group associated with the Noether charges, but the resulting theory will be unitarily equivalent to the one derived from Noether's theorem. At the perturbative level this can be solved by appealing to the connection with covariant perturbation theory [58], but how to achieve this at the non-perturbative level remains an open problem.

5. The discussion of rotational invariance led to two important observations. The first is that the full rotational covariance is not encoded in the dynamical $P^-$. The second observation is the a proper renormalization of $P^-$ must be consistent with rotational covariance, which implies that the large $p$ and small $p^+$ divergences must be renormalized consistently. Space reflection symmetry also relates these two kinds of divergences. This means that the constraints on the renormalization of the $p^+ \to 0$ and $p \to \infty$ divergences that are needed to define a theory consistent rotational covariance and space reflection symmetry do not come directly from the structure of $P^-$. In perturbation theory the missing information can be determined by appealing to the covariant formulation of the theory. In general the problem is difficult to separate from the problem of how to consistently renormalize all of the dynamical Poincaré generators outside of perturbation theory.

6. The signal for spontaneous symmetry breaking is the presence of a 0 mass Goldstone boson. Coleman [74] gives the following condition for the presence of a 0 mass Goldstone boson

$$\lim_{R\to\infty} \int_{|\mathbf{x}|<R} d\mathbf{x} \langle 0|[j^0(x), \phi(y)]|0\rangle \neq 0 \tag{4}$$

This is non-vanishing if the cutoff charge couples the vacuum to the Goldstone boson. This only assumes that the current is a local operator valued distribution; this does not require the existence a charge operator. Central to the above argument is the locality requirement that $[j^0(x), \phi(y)] = 0$ for space-like separated $x$ and $y$ which cuts off the integral for sufficiently large $R$. This argument fails on the light-front because separated points on the light-front are not all space-like. While this condition can in principle be formulated on the light-front, in a dynamical theory the structure of the light-front sub-algebra associated with an interacting field is more complicated than the mass $m$ light-front Fock algebra. The symmetry breaking must involve the selection of a particular sub-algebra of the light-front Fock algebra.

These results are discussed in more detail in what follows. The next section reviews Poincaré invariance in quantum theories. This interpretation, due to Wigner [75], is relevant to understanding Dirac's forms of dynamics which are relevant for Hamiltonian formulations of quantum field theories and are reviewed in section 3. Inequivalent representations of the canonical commutation relations are discussed in section 4. The irreducibility of covariant, canonical and light-front Fock algebras is discussed in section 5. This section also discusses the relation between the Heisenberg, canonical, and light-front Fock algebras for free fields. The initial value problem is discussed in section 6. The triviality of the vacuum is discussed in section 7. Rotational covariance is discussed in section 8. Comments on interacting theories are given in section 9. Spontaneous symmetry breaking is discussed in section 10. A summary and conclusion is given in section 11. There are two appendices. The first summarizes the conventions used in this paper and the second gives the form of the light-front Poincaré generators as operators in the light-front Fock algebra.

## II. POINCARÉ INVARIANCE AND HAMILTONIAN FORMULATIONS OF RELATIVISTIC QUANTUM THEORIES

The relativistic invariance of a quantum theory requires a unitary (ray) representation, $U(\Lambda, a)$, of the component of the Poincaré group connected to the identity [75]. Here $(\Lambda, a)$ represents the semi-direct product of a Lorentz transformation, $\Lambda$, followed by a spacetime translation by a fixed four vector $a$. This ensures that quantum probabilities, expectation values and ensemble averages are independent of the inertial coordinate system. The Poincaré group is a 10 parameter group. It is generated by four one-parameter subgroups of space-time translations and six one-parameter subgroups of Lorentz transformations. The infinitesimal generators of these unitary one-parameter subgroups are self-adjoint operators [76]. They include the Hamiltonian, $P^0 = H$, the linear momentum, $\mathbf{P}$, the angular momentum, $\mathbf{J}$, and the rotationless boost generators, $\mathbf{K}$. The group representation property

$$U(\Lambda_2, a_2)U(\Lambda_1, a_1) = U(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2) \tag{5}$$

implies that the infinitesimal generators satisfy the commutation relations:

$$[P^\mu, P^\nu] = 0, \qquad [J^i, P^j] = i\epsilon^{ijk}P^k, \qquad [J^i, J^j] = i\epsilon^{ijk}J^k,$$

$$[J^i, K^j] = i\epsilon^{ijk}K^k, \qquad [K^i, K^j] = -i\epsilon^{ijk}J^k$$

$$[K^i, P^i] = i\delta^{ij}H \qquad [K^i, H] = iP^i. \tag{6}$$

Light-front generators are linear combinations of these operators. The relativistic analog of diagonalizing the Hamiltonian is to decompose $U(\Lambda, a)$ into a direct integral of irreducible representations. This is equivalent to simultaneously diagonalizing the mass and spin Casimir operators of the Lie algebra

$$M^2 = (P^0)^2 - \mathbf{P}^2 \qquad \text{and} \qquad \mathbf{S}^2 = W^2/M^2 \tag{7}$$

where $W^\mu$ is the Pauli-Lubanski vector

$$W^\mu = (\mathbf{P}\cdot\mathbf{J}, H\mathbf{J} + \mathbf{P}\times\mathbf{K}). \tag{8}$$

The transformation properties of states in each irreducible subspace is fixed by group theoretical considerations.

## III. DIRAC'S FORMS OF DYNAMICS

This section discusses Dirac's three forms of Hamiltonian dynamics, and in particular the front-form of the dynamics. The starting point in discussing Dirac's forms of dynamics [77] is the assumption that the non-interacting and interacting theories are formulated on the same representation of the Hilbert space as the interacting theory. While this is always true for quantum theories of a finite number of particles, it is not generally true in for theories with an infinite number of degrees of freedom [60]. It is not even true for free field theories. Dirac's forms of dynamics are used in perturbative quantum field theory where the dynamics is introduced as a perturbation of the non-interacting theory. As long as there are cutoffs both theories can be formulated on the same Hilbert space representation. As the cutoffs are removed the interacting theory passes to a different Hilbert space representation. In this work the existence of both an interacting and non-interacting unitary representation of the Poincaré group that act on the same representation of the Hilbert space will be assumed.

Given the assumption that the free and interacting unitary representations of the Poincaré group act on the same representation of the Hilbert space, the non-interacting representation of the Poincaré group also has a set of infinitesimal generators that are self-adjoint operators on the same representation of the Hilbert space. The non-interacting generators are denoted with a "0" subscript:

$$\{H_0, \mathbf{P}_0, \mathbf{J}_0, \mathbf{K}_0\}. \tag{9}$$

For the linear momentum, $\mathbf{P}_0$, angular momentum $\mathbf{J}_0$, and rotationless boost generators, $\mathbf{K}_0$ the spectrum of the operators is fixed by group representation properties and is identical in the non-interacting and interacting theories. These nine operators cannot be related by a single unitary transformation because the commutator

$$[K^i, P^j] = i\delta^{ij}H \tag{10}$$

would then require that $H$ and $H_0$ would be unitarily equivalent. This is certainly not true if $H$ has bound states.

Dirac pointed out that for closed sub-algebras that do not involve the Hamiltonian it is possible to find a representation where all of the operators in the sub-algebra are identical in the free and interacting representations of the Poincaré group. It follows from the commutation relations (6) that two sub-algebras with this property are the three-dimensional Euclidean algebra (instant representation)

$$[P^i, P^j] = 0, \qquad [J^i, P^j] = i\epsilon^{ijk} P^k, \qquad [J^i, J^j] = i\epsilon^{ijk} J^k, \tag{11}$$

and the Lorentz algebra (point representation)

$$[J^i, K^j] = i\epsilon^{ijk} K^k, \qquad [K^i, K^j] = -i\epsilon^{ijk} J^k, \qquad [J^i, J^j] = i\epsilon^{ijk} J^k. \tag{12}$$

Each of these algebras has 6 generators. In the instant representation $H \neq H_0$, and $\mathbf{K} \neq \mathbf{K}_0$. In the point representation $H \neq H_0$, and $\mathbf{P} \neq \mathbf{P}_0$. Dirac identified a third representation where the sub-algebra that generates the seven-parameter subgroup of the Poincaré group that maps the light-front hyperplane, $x^+ = x^0 + x^3 = 0$, to itself is the same in the non-interacting and interacting theory. In this case the non-interacting generators are the following linear combinations of the generators (6):

$$P^1, P^2, P^+ = P^0 + P^3, J^3, K^3, \mathbf{E}_\perp := \mathbf{K}_\perp - \hat{\mathbf{z}} \times \mathbf{J} \tag{13}$$

while the generators

$$P^- = P^0 - P^3 \neq P^-_0; \qquad \text{and} \qquad \mathbf{F}_\perp := \mathbf{K}_\perp + \hat{\mathbf{z}} \times \mathbf{J} \neq \mathbf{F}_{\perp 0} \qquad \text{or} \qquad \mathbf{J}_\perp = \hat{\mathbf{z}} \times \mathbf{J} \neq \hat{\mathbf{z}} \times \mathbf{J}_{0\perp} \tag{14}$$

involve interactions. The dynamical generators can be taken as $P^-, F^1, F^2$ or equivalently $P^-, J^1, J^2$. The operators $K^3$ and $\mathbf{E}_\perp$, which generate light-front preserving boosts, form a closed sub-algebra.

In all three of Dirac's forms of dynamics the dynamical problem is to simultaneously diagonalize the mass $M$ and spin $\mathbf{S}^2$ Casimr operators of the Poincare Lie algebra. Sokolov and Shatnyi [78][79] established the equivalence of these three representations of the Poincaré group when the kinematic and dynamical representations of the Poincaré group act on the same representation of the Hilbert space.

While Dirac's forms of dynamics are normally formulated in terms of initial value surfaces, if the light front hypersurface, $x^+ = x^- + \hat{\mathbf{z}} \cdot \mathbf{x} = 0$, is replaced by $x^+ = x^- + \hat{\mathbf{z}} \cdot \mathbf{x} = c \neq 0$, the subgroup that leaves the $x^+$-evolved surface invariant is only a six parameter group. However generators that are defined on the $x^+ = 0$ are all that is needed. This motivates the preference for discussing forms of dynamics in terms of kinematic subgroups rather than initial value surfaces.

## IV. INEQUIVALENT REPRESENTATIONS OF THE CANONICAL COMMUTATION RELATIONS

For a quantum theory of 1 degree of freedom any operator can be formally expressed in the form

$$O = \int dadb f(a,b) e^{iqa} e^{ipb} \qquad e^{iqa} e^{ipb} = e^{ipb} e^{iqa} e^{-iab} \tag{15}$$

where $p$ and $q$ are canonically conjugate momentum and coordinate operators and $f(a,b)$ is a function of 2 real variables. The operators $e^{iqa}$ and $e^{ipb}$ are an irreducible set of operator on the Hilbert space of square integrable functions in $q$ (or $p$). Any operator that commutes with both of them is a constant multiple of the identity.

A key difference with canonical and light-front quantum theories is that while both theories have irreducible sets of field operators, in canonical field theories with different masses the vacuum functionals generate inequivalent Hilbert space representations of the canonical commutation relations, while in the light-front case there is a common representation of the Weyl algebra (exponential form of the canonical commutation relations (15)) and both vacuum functionals agree on this algebra. General considerations are discussed in this section. The resolution is discussed in the following section.

The Stone Von Neumann's theorem [8][9][10] that demonstrated the equivalence of the Schrödinger and Heisenberg pictures of quantum mechanics breaks down for theories of an infinite number of degree of freedom. Haag [11] provided a simple example that illustrates this breakdown for free fields.

For two quantum mechanical harmonic oscillators with different frequencies, $\omega_i$, the creation and annihilation operators are related to the canonical coordinates and momenta by

$$q = \frac{1}{\sqrt{2\omega_i}} \left( a_i + a_i^\dagger \right) \qquad p = -i\sqrt{\frac{\omega_i}{2}} \left( a_i - a_i^\dagger \right) \tag{16}$$

where $\omega_i$ is the angular frequency of the $i^{th}$ oscillator. Solving for one set of creation and annihilation operators in terms of the other gives the canonical transformation

$$a_2 = \cosh(\eta)a_1 + \sinh(\eta)a_1^\dagger \tag{17}$$

where $\eta$ is defined by

$$\cosh(\eta) := \frac{1}{2}\left(\sqrt{\frac{\omega_2}{\omega_1}} + \sqrt{\frac{\omega_1}{\omega_2}}\right) \qquad \sinh(\eta) := \frac{1}{2}\left(\sqrt{\frac{\omega_2}{\omega_1}} - \sqrt{\frac{\omega_1}{\omega_2}}\right). \tag{18}$$

This canonical transformation can be realized by a unitary transformation $e^{iG}$ with infinitesimal generator

$$G = -\frac{i}{2}\eta(a_1 a_1 - a_1^\dagger a_1^\dagger). \tag{19}$$

The ground state vectors of the two oscillators are related by this unitary transformation

$$|0\rangle_2 = e^{iG}|0\rangle_1. \tag{20}$$

For free scalar fields with different masses, the canonical coordinates and momenta are replaced by the fields $\phi(0,\mathbf{x})$ and $\pi(0,\mathbf{x})$. They can be expressed in terms of creation and annihilation operators by

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2\omega_{m_i}(\mathbf{p})}} \left(e^{ip\cdot x}a_i(\mathbf{p}) + e^{-ip\cdot x}a_i^\dagger(\mathbf{p})\right) \tag{21}$$

$$\pi(x) = -\frac{i}{(2\pi)^{3/2}} \int d\mathbf{p}\sqrt{\frac{\omega_{m_i}(\mathbf{p})}{2}} \left(e^{ip\cdot x}a_i(\mathbf{p}) - e^{-ip\cdot x}a_i^\dagger(\mathbf{p})\right) \tag{22}$$

where $\omega_m(\mathbf{p}) := \sqrt{m^2 + \mathbf{p}^2}$ is the energy of a particle of mass $m$.

Following the construction in the one degree of freedom case, the creation and annihilation operators in fields $\phi_m(0,\mathbf{x})$ with different masses are related by a canonical transformation of the form

$$a_2(\mathbf{p}) = \cosh(\eta(\mathbf{p}))a_1(\mathbf{p}) + \sinh(\eta(\mathbf{p}))a_1^\dagger(\mathbf{p}) \tag{23}$$

where

$$\cosh(\eta(\mathbf{p})) := \frac{1}{2}\left(\sqrt{\frac{\omega_{m_2}(\mathbf{p})}{\omega_{m_1}(\mathbf{p})}} + \sqrt{\frac{\omega_{m_1}(\mathbf{p})}{\omega_{m_2}(\mathbf{p}}}\right) \tag{24}$$

$$\sinh(\eta(\mathbf{p})) := \frac{1}{2}\left(\sqrt{\frac{\omega_{m_2}(\mathbf{p})}{\omega_{m_1}(\mathbf{p})}} - \sqrt{\frac{\omega_{m_1}(\mathbf{p})}{\omega_{m_2}(\mathbf{p}}}\right). \tag{25}$$

If this canonical transformation was implemented by a unitary transformation the generator, $G$, of the unitary operator, $U = e^{iG}$, would be

$$G = -\frac{i}{2}\int \eta(\mathbf{p})(a_1(\mathbf{p})a_1(\mathbf{p}) - a_1^\dagger(\mathbf{p})a_1^\dagger(\mathbf{p}))d\mathbf{p} \tag{26}$$

and the vacuum vectors in the two theories would be related by

$$|0\rangle_2 = e^{iG}|0\rangle_1, \tag{27}$$

however a straightforward calculation gives

$$\|G|0\rangle_1\|^2 = \frac{1}{4}\int \eta(\mathbf{p})^2 d\mathbf{p}\delta(0) = \infty. \tag{28}$$

Since the norm of any vector is a linear combination of products of contractions multiplied by ${}_2\langle 0|0\rangle_2$ the generator $G$ has an empty domain on the Hilbert space generated by the vacuum $a_1(\mathbf{p})|0\rangle_1 = 0$ of $\phi_1(,\mathbf{x}), \pi_1(0,\mathbf{x})$.

The conclusion is that even for free canonical field theories the vacuum vectors for theories with different masses live in inequivalent representations of the Hilbert space.

In the canonical case, assuming that the Hamiltonian is quadratic in the generalized momentum, the vacuum actually determines the Hamiltonian [15][16]. This also follows because while the fields for any mass satisfy the same canonical commutation relations, the choice $a_i(\mathbf{p})|0\rangle_i = 0$ of vacuum determines the mass. This is in contrast to the light-front case where vacuum expectation values of fields on the light front have no dynamical information.

Another way to understand this inequivalence is to recall that in quantum field theory the vacuum is a linear functional on an algebra of field operators; where this algebra is generated by fields smeared with Schwartz [62] test functions in all four spacetime variables. For free fields the inner product of two one-particle states has the form

$$\langle f|g\rangle = \sum_{ij} \int f_i^*(x)\langle 0|\phi_i^\dagger(x)\phi_j(y)|0\rangle g_j(y) d^4x d^4y. \tag{29}$$

The Fourier transform of $\langle 0|\phi_i^\dagger(x)\phi_j(y)|0\rangle$ is the product of a spinor matrix [63], a mass-shell delta function and a Heaviside function that selects the positive-energy branch of the mass shell. This means that two test functions with Fourier transforms that agree on the mass shell represent the same vector, so Hilbert space vectors are represented by equivalence classes of test functions. If the mass is changed, two functions in the same equivalence class with respect to the first mass do not necessarily have Fourier transforms in the same equivalence class with respect to the second mass. This means that there is no correspondence of equivalence classes associated with one mass with equivalence classes associated with a second mass. For free fields all Wightman functions are products of two point functions. Given the same field algebra, different two point functions are determined by the vacuum functional that gives the Wightman functions.

The resolution of the vacuum problem is alluded to in [12]. In general a given theory has one vacuum, which is a linear functional on an algebra of local Heisenberg fields. The algebra can be generated polynomials of fields smeared with Schwartz functions in 3+1 variables or linear combinations of bounded functions of fields smeared with Schwartz functions in 3+1 variables:

$$e^{i\phi(f_i)}.$$

It is also possible that a given vacuum functional may still be defined by taking limits that put the support of the test functions on different hypersurfaces of Minkowski space. What is pointed out in [12] is that when the algebra of free field theories with different masses is restricted to fields smeared with test functions that give the same value to two point Wightman functions, then the theories defined on this sub-algebra become unitarily equivalent. In this case the different vacuum functionals agree on this sub-algebra, but they do not agree when the fields are smeared with arbitrary Schwartz functions. This limited algebra does not have enough operators to distinguish the two theories.

This suggests that while different field theories have different vacuum functionals, the vacuum functionals may agree when the space of test functions is limited to functions supported on the light front, although in the light-front case the different vacuum functionals are only defined for fields smeared with test functions supported on the light front with Fourier transforms that vanish at $p^+ = 0$, which corresponds to infinite momentum [12]. These will be referred to as Schlieder-Seiler test functions.

In the light-front case, for free fields, the condition $\int a(\tilde{\mathbf{p}})f(\tilde{\mathbf{p}})d\tilde{\mathbf{p}}|0\rangle_{LF} = 0$, where $\tilde{\mathbf{p}} := (p^+, p^1, p^2)$ and $f(\tilde{\mathbf{p}})$ is a Schlieder-Seiler test function, does not have enough test functions to distinguish vacuum functionals for different free field theories. The spaces generated by applying elements of this algebra to different vacua are unitarily equivalent [12].

The conclusion is that while each field theory will have a different vacuum vector, defined as a linear functional on the Heisenberg field algebra, different vacuum functionals can agree on a sub-algebra. For the case of free fields the test functions for the light front and canonical field algebras are not sub-algebras of the Heisenberg algebra because the test functions are not four dimensional Schwartz functions, but the vacuum functionals are defined on three-dimensional hypersurfaces because the vacuum functionals are only sensitive to the values of the Fourier transforms of the test functions on the mass shells, so the different vacuum functionals can still make sense on these algebras. This will be shown explicitly in the next section.

## V. IRREDUCIBILITY

In this section it is shown that the Hilbert space generated by the vacuum and the local Heisenberg algebra of free field operators of a given mass can alternatively represented by the vacuum and an algebra of canonical fields at fixed time or the vacuum and an algebra of fields on the light front. It follows that vacuum expectation values of elements of a sub-algebra of fields on the light front can be used to construct the Wightman functions of the field theory. These Wightman functions can also be constructed using the algebra of canonical fields restricted to a fixed time surface. For free fields this demonstrates the equivalence of all three representations.

To show this the irreducibility of the canonical or light-front fields is used to express creation and annihilation operators in terms of fields on the light front or fixed time surface. In this way the Heisenberg field can be expressed in terms of fields restricted to the light front or a fixed time surface. With this connection the covariant Poincaré generators get mapped to light-front generators that act on a Hilbert space generated by applying a (sub)algebra of fields restricted to light-front hyperplanes to the vacuum.

The canonical algebra is generated by polynomials in

$$\phi(f) = \int d\mathbf{x}\phi(t=0,\mathbf{x})f(\mathbf{x}) \qquad \pi(f) = \int d\mathbf{x}\pi(t=0,\mathbf{x})f(\mathbf{x}) \qquad f(\mathbf{x}) \in S(\mathbb{R}^3) \tag{30}$$

where the $f(\mathbf{x})$'s are Schwartz functions.

The light-front Fock algebra is generated by polynomials in

$$\phi(f) = \int \frac{dx^- d^2\mathbf{x}_\perp}{2}\phi(x^+=0,x^-,\mathbf{x}_\perp)f(x^-,\mathbf{x}_\perp) \qquad f((x^-,\mathbf{x}_\perp) \in S(\mathbb{R}^3) \qquad \frac{\partial f}{\partial x^-} = 0. \tag{31}$$

The test functions in (31) are Schlieder-Seiler functions.

Normally irreducibility is discussed in terms of the exponential form (15) of the canonical commutation relations which avoids the use of unbounded operators, but for practical purposes what is needed is to be able to separately extract the creation and annihilation operators from the field operators, since any operator can be expressed in terms of creation and annihilation operators.

The creation and annihilation operators for free fields restricted to a fixed-time surface can be extracted from the fields (21) and the generalized momenta (22) by taking Fourier transforms:

$$a(\mathbf{p}) = \frac{1}{\sqrt{2\omega_{m_i}(\mathbf{p})}}(\omega_{m_i}(\mathbf{p})\hat{\phi}(\mathbf{p})_{x^0=0} + i\hat{\pi}(-\mathbf{p})_{x^0=0}), \tag{32}$$

$$a^\dagger(\mathbf{p}) = \frac{1}{\sqrt{2\omega_{m_i}(\mathbf{p})}}(\omega_{m_i}(\mathbf{p})\hat{\phi}(\mathbf{p})_{x^0=0} - i\hat{\pi}(-\mathbf{p})_{x^0=0}), \tag{33}$$

where

$$\hat{\phi}(\mathbf{p})_{x^0=0} := \int \frac{d\mathbf{x}}{(2\pi)^{3/2}}e^{-i\mathbf{p}\cdot\mathbf{x}}\phi(x^0=0,\mathbf{x}) \qquad \hat{\pi}(\mathbf{p})_{x^0=0} := \int \frac{d\mathbf{x}}{(2\pi)^{3/2}}e^{-i\mathbf{p}\cdot\mathbf{x}}\pi(x^0=0,\mathbf{x}) \tag{34}$$

are Fourier transforms of the fields at time $t=0$ and $\omega_m(\mathbf{p}) = \sqrt{m^2+\mathbf{p}^2}$ is the energy. The operators $a(\mathbf{p})$ and $a^\dagger(\mathbf{p})$ are operator valued distributions. In this case the linear combination of the field operators that give the creation and annihilation operators depend on the mass.

The free Heisenberg field (21) can also be expressed in light-front variables by replacing integrals over the light front components of the momentum, $\mathbf{p}$ in (21) by integrals over $\tilde{\mathbf{p}} := (p^+, \mathbf{p}_\perp)$. Using

$$|\frac{\partial(\tilde{\mathbf{p}})}{\partial(\mathbf{p})}| = \frac{p^+}{\omega_m(\mathbf{p})} \qquad a(\tilde{\mathbf{p}}) := a(\mathbf{p})\sqrt{\frac{\omega_m(\mathbf{p})}{p^+}} \tag{35}$$

and

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}'] = \delta(\mathbf{p}-\mathbf{p}') \tag{36}$$

gives

$$[a(\tilde{\mathbf{p}}), a^\dagger(\tilde{\mathbf{p}}'] = \delta(\tilde{\mathbf{p}}-\tilde{\mathbf{p}}') = \delta(\mathbf{p}_\perp - \mathbf{p}'_\perp)\delta(p^+ - p^{+\prime}) \tag{37}$$

and the following expression for the field in terms of light-front variables

$$\phi(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d\mathbf{p}_\perp dp^+\theta(p^+)}{\sqrt{2p^+}}\left(e^{ip\cdot x}a(\tilde{\mathbf{p}}) + e^{-ip\cdot x}a^\dagger(\tilde{\mathbf{p}})\right) \tag{38}$$

where

$$p^- = \frac{m^2 + \mathbf{p}_\perp^2}{p^+} \qquad p \cdot x = -\frac{1}{2}(p^+ x^- + p^- x^+) + \mathbf{p}_\perp \cdot \mathbf{x}_\perp . \tag{39}$$

Both (38) and (21) are different ways of writing the *same* field operator. The Fourier transform of the field (38) restricted to the light front is

$$\hat{\phi}(\tilde{\mathbf{p}})_{x^+=0} := \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{x}_\perp dx^-}{2} \phi(\mathbf{x}_\perp, x^-, x^+ = 0) e^{i(\frac{p^+ x^-}{2} - \mathbf{p}_\perp \cdot \mathbf{x}_\perp)} = \tag{40}$$

$$\frac{1}{\sqrt{2p^+}} \theta(p^+) a(\tilde{\mathbf{p}}) + \frac{1}{\sqrt{-2p^+}} \theta(-p^+) a^\dagger(-\tilde{\mathbf{p}}) \tag{41}$$

which express both the creation and annihilation operators

$$a(\tilde{\mathbf{p}}) = \sqrt{2p^+}\theta(p^+)\hat{\phi}(\tilde{\mathbf{p}})_{x^+=0} \qquad a^\dagger(\tilde{\mathbf{p}}) = \sqrt{2p^+}\theta(p^+)\hat{\phi}(-\tilde{\mathbf{p}})_{x^+=0} \tag{42}$$

in terms of the field restricted to the light front, without using any information normal to the light front. This is in contrast to the canonical case where $\pi(x) = \dot{\phi}(x)$ involves a derivative normal to the $t = constant$ surface.

Since the free local Heisenberg field can be expressed in terms of creation annihilation operators and the creation and annihilation operators can be expressed in terms of either $\phi(t = 0, \mathbf{x})$ and $\pi(t = 0, \mathbf{x})$ or $\phi(x^+ = 0, x^-, \mathbf{x}_\perp)$ it follows that the Heisenberg field $\phi(x)$ can be expressed in terms of $\phi(t = 0, \mathbf{x})$ and $\pi(t = 0, \mathbf{x})$ or $\phi(x^+ = 0, x^-, \mathbf{x}_\perp)$.

A dense set of vectors in the physical Hilbert space of a quantum field theory is constructed by applying polynomials, $A, B, \cdots$ of smeared Heisenberg field operators of the form

$$\phi(f) := \int d^4x \sum_i \phi_i(x) f_i(x), \tag{43}$$

to the vacuum of the theory, where $f_i(x)$ are 4-variable Schwartz functions. The physical vacuum is a positive linear functional $L$ on this algebra of polynomials. Vectors are be represented by elements $A, B$ of this algebra with inner product

$$\langle A|B\rangle = L(A^\dagger B) = \langle 0|A^\dagger B|0\rangle \tag{44}$$

(after eliminating zero norm vectors and including Cauchy sequences).

In both the canonical and light-front cases there are also algebras generated by operators of the form

$$\phi_C(f) := \int d^3x \sum_i \phi_i(t = 0, \mathbf{x}) f_i(\mathbf{x}) \qquad \pi_C(f) := \int d^3x \sum_i \pi_i(t = 0, \mathbf{x}) f_i(\mathbf{x}) \tag{45}$$

and

$$\phi_{LF}(f) := \int \frac{d^2\mathbf{x}_\perp dx^-}{2} \sum_i \phi_i(x^+ = 0, x^-, \mathbf{x}_\perp) f_i(x^-, \mathbf{x}_\perp). \tag{46}$$

where in the canonical case $f_i$ are 3-variable Schwartz test functions and in the light-front case the test functions are 3-variable Schwartz functions with vanishing $x^-$ derivative [12] .

While the algebras generated by polynomials in the smeared fields in (45) and (46) are distinct and different from the local Heisenberg field algebra generated by the smeared fields in (43), for free fields any $\phi(f)$ in (43) can be expressed as a linear combination of the fields $\phi_C(g)$ and $\pi_C(g)$ or $\phi_{LF}(g)$ for suitable test functions $g$ related to $f$. In addition the vacuum expectation values (44) of elements in the Heisenberg field algebra can also be expressed as vacuum expectation values of operators in the canonical or light-front field algebras.

While (for the case of free fields) vacuum functionals for different masses are different, they become unitarily equivalent [12] when they are applied to elements of the light-front Fock algebra. The equivalence follows from

$${}_1\langle 0|\phi_1(f_1, x^+ = 0) \cdots \phi_1(f_n, x^+ = 0)|0\rangle_1 =_2 \langle 0|\phi_2(f_1, x^+ = 0) \cdots \phi_2(f_n, x^+ = 0)|0\rangle_2$$

for all $f_n$ test functions on the light front with vanishing $x^-$ derivative.

Next it is shown that the algebra of free Heisenberg fields can be mapped to the canonical or light front field algebras. In this way vacuum expectation values of operators in the Heisenberg field algebra can be expressed as vacuum expectation values of operators in either the canonical or light front field algebras.

To show this use (32) and (33) in (21):

$$\phi(f) = \int f(x)\phi(x)d^4x =$$

$$\int d^4x f(x) \int d\mathbf{y}d\mathbf{p}\frac{1}{(2\pi)^3}\left(\cos(\omega_m(\mathbf{p})x^0)e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\phi(y^0=0,\mathbf{y}) + \frac{\sin(\omega_m(\mathbf{p})x^0)}{\omega_m(\mathbf{p})}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\pi(y^0=0,\mathbf{y})\right) =$$

$$\int d^4x f(x)\int d\mathbf{y}[\mathcal{K}_{\phi m}(x,\mathbf{y})\phi(y^0=0,\mathbf{y}) + \mathcal{K}_{\pi m}(x,\mathbf{y})\pi(y^0=0,\mathbf{y})] \tag{47}$$

which defines the kernels $\mathcal{K}_{\phi m}$ and $\mathcal{K}_{\pi m}$. This can be expressed as

$$\phi(f) = \int d\mathbf{y}(f_\phi(\mathbf{y})\phi_C(\mathbf{y}) + f_\pi(\mathbf{y})\pi_C(\mathbf{y})) \tag{48}$$

where

$$f_\phi(\mathbf{y}) = \int d^4x f(x)\mathcal{K}_{\phi m}(x,\mathbf{y}) \qquad f_\pi(\mathbf{y}) = \int d^4x f(x)\mathcal{K}_{\pi m}(x,\mathbf{y}) \tag{49}$$

with similar relations for $\pi(f)$. This means that the algebra of operators generated by smearing the free Heisenberg field with Schwartz functions in four spacetime variables can be expressed in terms of fixed time canonical pairs of fields smeared with Schwartz functions in three variables. This is because the mass-shell condition eliminates one of the variables.

This exercise can be repeated in the light-front case. In the light-front case the vacuum is not sensitive to the dynamics when applied to operators in the light-front sub-algebra.

The key observation in the light-front case is that the Heisenberg algebra is mapped to a *proper sub-algebra* of the light-front Fock algebra and the dynamical information is contained in the sub-algebra. A smeared Heisenberg field has the form

$$\phi(f) = \int f(x)\phi(x)d^4x =$$

$$\int d^4x f(x)\int \frac{d\tilde{\mathbf{p}}\theta(p^+)}{(2\pi)^{3/2}\sqrt{2p^+}}\left[e^{ip\cdot x}a(\tilde{\mathbf{p}}) + e^{-ip\cdot x}a^\dagger(\tilde{\mathbf{p}})\right].$$

Inserting the expressions (42) for the light-front creation and annihilation operator in terms of fields restricted to the light front gives

$$= \int d^4x f(x)\frac{d\tilde{\mathbf{p}}\theta(p^+)}{(2\pi)^{3/2}\sqrt{2p^+}}\left[e^{ip\cdot x}\sqrt{2p^+}\theta(p^+)\hat{\phi}(\tilde{\mathbf{p}})_{|x^+=0} + e^{-ip\cdot x}\sqrt{2p^+}\theta(p^+)\hat{\phi}(-\tilde{\mathbf{p}})_{|x^+=0}\right] =$$

$$\int d^4x \frac{f(x)d\tilde{\mathbf{p}}\theta(p^+)}{(2\pi)^3}\left[e^{ip\cdot x}e^{i\frac{p^+y^-}{2}-i\mathbf{p}_\perp\cdot\mathbf{y}_\perp} + e^{-ip\cdot x}e^{-i\frac{p^+y^-}{2}+i\mathbf{p}_\perp\cdot\mathbf{y}_\perp}\right]\frac{dy^+d\mathbf{y}_\perp}{2}\phi(\tilde{\mathbf{y}})_{|x^+=0} =$$

$$\int d^4x f(x)\frac{d\tilde{\mathbf{p}}\theta(p^+)}{(2\pi)^3}\left[e^{-ix^+\frac{\mathbf{p}_\perp^2+m^2}{2p^+}-i\frac{p^+\cdot(x^--y^-)}{2}+i\mathbf{p}_\perp\cdot(\mathbf{x}_\perp-\mathbf{y}_\perp)} + e^{ix^+\frac{\mathbf{p}_\perp^2+m^2}{2p^+}+i\frac{p^+\cdot(x^--y^-)}{2}-i\mathbf{p}_\perp\cdot(\mathbf{x}_\perp-\mathbf{y}_\perp)}\right]\frac{dy^+d\mathbf{y}_\perp}{2}\phi(\tilde{\mathbf{y}})_{|x^+=0}. \tag{50}$$

This expression defines a kernel:

$$\mathcal{K}_m(x,\tilde{\mathbf{y}}) := \int \frac{d\tilde{\mathbf{p}}\theta(p^+)}{(2\pi)^3}\left[e^{-ix^+\frac{\mathbf{p}_\perp^2+m^2}{2p^+}-i\frac{p^+\cdot(x^--y^-)}{2}+i\mathbf{p}_\perp\cdot(\mathbf{x}_\perp-\mathbf{y}_\perp)} + e^{ix^+\frac{\mathbf{p}_\perp^2+m^2}{2p^+}+i\frac{p^+\cdot(x^--y^-)}{2}-i\mathbf{p}_\perp\cdot(\mathbf{x}_\perp-\mathbf{y}_\perp)}\right] \tag{51}$$

that is used to define the mass $m$ light-front test function $f_m(\tilde{\mathbf{y}})$ associated with a four-variable Schwartz function $f(x)$:

$$f_m(\tilde{\mathbf{y}}) = \int d^4x f(x)\mathcal{K}_m(x,\tilde{\mathbf{y}}) = \mathcal{K}_m(f,\tilde{\mathbf{y}}). \tag{52}$$

Changing the sign of $\tilde{\mathbf{p}}$ in the second term in (51) gives the following expression for the kernel

$$\mathcal{K}_m(x,\tilde{\mathbf{y}}) := \int \frac{d\tilde{\mathbf{p}}}{(2\pi)^3} e^{-ix^+\frac{\mathbf{p}_\perp^2+m^2}{2p^+}} e^{-i\frac{p^+\cdot(x^- - y^-)}{2}+i\mathbf{p}_\perp\cdot(\mathbf{x}_\perp-\mathbf{y}_\perp)}. \tag{53}$$

In both (47) and (53) if the Klein Gordon operator, $(\Box^2 - m^2)$, is applied to a test function in $S(\mathbb{R}^4)$, integrating by parts and applying the Klein Gordon operator to the kernels in (47) and (53) gives 0. In these expressions the dynamics enters through these kernels, which satisfy the field equations.

These kernels pick out the part of $f(x)$ on the mass shell. These mappings are not invertible since there are many test functions $f(x) \in S(\mathbb{R}^4)$ whose Fourier transforms agree on the mass shell.

The expression for the smeared Heisenberg field becomes

$$\phi(f) = \int \frac{dy^+ d\mathbf{y}_\perp}{2} f_m(\tilde{\mathbf{y}})\phi(\tilde{\mathbf{y}})_{|y^+=0} := \phi_{LF}(f_m) \tag{54}$$

which demonstrates that operators in the Heisenberg field algebra can be expressed as operators in the light-front Fock algebra.

Equation (52-53) can be used to find the Fourier transform $\hat{f}_m(\tilde{\mathbf{p}})$ of $f_m(\tilde{\mathbf{y}})$ assuming that $f(x)$ in the Heisenberg field is a real Schwartz function and $\hat{f}(p) = \hat{f}(p^-, p^+, \mathbf{p}_\perp)$ is its Fourier transform:

$$\hat{f}_m(\tilde{\mathbf{p}}) = \sqrt{2\pi}\hat{f}(\frac{\mathbf{p}_\perp^2+m^2}{p^+}, \tilde{\mathbf{p}}). \tag{55}$$

It follows from (55) for $m \neq 0$ that $\hat{f}_m(\tilde{\mathbf{p}})$ vanishes faster than any power of $p^+$ as $p^+ \to 0$. This is because $\hat{f}(p)$ is also a Schwartz function which vanishes faster that any polynomial in $1/p^-$ for large $p^-$.

This means that the Fourier transforms of the light-front mass $m$ test functions, (52), vanish for light-like light-front momenta. It also ensures that fields on the light front smeared with the $f_m(\tilde{\mathbf{y}})$ generate a proper sub-algebra of the light-front field algebra.

Another observation is that the kernel $\mathcal{K}_m$ in the mixed coordinate-momentum representation

$$\mathcal{K}_m(x,\tilde{\mathbf{p}}) := \int \mathcal{K}_m(x,\tilde{\mathbf{y}}) \frac{dy^+ d\mathbf{y}_\perp}{2} e^{i\tilde{\mathbf{y}}\cdot\tilde{\mathbf{p}}} \tag{56}$$

maps the covariant $x^+$ translation generator

$$2i\frac{\partial}{\partial x^+}\mathcal{K}_m(x,\tilde{\mathbf{p}}) = \mathcal{K}_m(x,\tilde{\mathbf{p}})\frac{\mathbf{p}_\perp^2+m^2}{p^+} \tag{57}$$

to the light-front Hamiltonian in the light-front Fock algebra. More generally the kernel $\mathcal{K}_m$ maps the covariant (differential) form of the Poincaré Lie algebra to the light-front representation of the Poincaré Lie algebra.

In the light-front case the smeared Heisenberg fields (43) can be expressed in terms of fields on the light front smeared with light-front mass $m$ test functions. Polynomials in the smeared Heisenberg fields (43) applied to the vacuum can be expressed as polynomials in the corresponding light-front fields smeared with light-front mass $m$ test functions applied to vacuum. Finally vacuum expectation values of products of smeared Heisenberg fields can be expressed as vacuum expectation values of operators in a sub-algebra of the light-front field algebra:

$$\langle 0|\phi(f)\phi(g)|0\rangle = \langle 0|\phi_{LF}(f_m)\phi_{LF}(g_m)|0\rangle \tag{58}$$

For free fields the Wightman functions are products of two-point functions, so this immediately generalizes to all Wightman functions.

The image of the Heisenberg field algebra under this mapping will be called the mass $m$ light-front Fock algebra. It is a sub-algebra of the light-front Fock algebra.

Note that while the different free-field vacuum functionals are unitarily equivalent when restricted to the light-front Fock algebra, because the mappings from the Heisenberg algebras to the light front sub-algebras are not invertible, this equivalence does not extend to the canonical or Heisenberg algebras. It does however indicate that any one of the unitarily equivalent vacua on the light front can be used with the mass m light front field algebra to construct the free-field Wightman functions. This identification is in the following sense:

$$_1\langle 0|\phi_{1LF}(f_{1m})\cdots\phi_{1LF}(f_{nm})|0\rangle_1 =_2 \langle 0|\phi_{2LF}(f_{1m})\cdots\phi_{2LF}(f_{nm})|0\rangle_2$$

The conclusion is that for free fields the Heisenberg field algebra can be mapped to a sub-algebra of the canonical or light-front Fock algebra. While free field theories are related by unitary transformations when the algebra is restricted to the light-front Fock algebra, the sub-algebras associated with different theories are different and the mappings $\mathcal{K}_m$ are not invertible, so it is not possible to use this unitary equivalence to find unitary transformations that relate free field theories with different masses.

It is useful to summarize these observations:

1. For free fields the vacuum functional of each theory is defined on the light-front Fock algebra. The different vacuum functions are unitarily equivalent when restricted to that algebra.

2. The light-front Fock space is equivalent to a generic Hilbert space of square integrable functions of light-front variables that vanish at $p^+ = 0$. Creation and annihilation operators can be defined that create and annihilate "partons" that are not associated with any mass. The identification with particles comes from restricting the light-front Fock space to a subspace generated by a sub-algebra of the light-front Fock algebra applied to a generic light-front vacuum.

3. The dynamics enters through a mapping from the Heisenberg field algebra to a sub-algebra of the light-front Fock algebra. The Wightman functions of the field theory can be constructed evaluating products of the light-front fields smeared with light-front mass $m$ test functions using any free-field vacuum functional. The kernel $\mathcal{K}_m$, which has the dynamical content of the theory, is a solution the Klein -Gordon equation with mass $m$.

4. Because the mappings $\mathcal{K}_m$ are not invertible, this equivalence does not extend to the Heisenberg algebra. Specifically if A denotes the Heisenberg algebra, $K_m A$ denotes the mass $m$ light-front algebra and $L_m$ denotes the mass $m$ vacuum functional then

$$L_{m_1}[A] = L_{m_1}[K_{m_1}A] = L_{m_2}[K_{m_1}A]$$

$$L_{m_1}[A] \neq L_{m_2}[A] \qquad L_{m_1}[K_{m_1}A] \neq L_{m_1}[K_{m_2}A]$$

$$L_{m_2}[A] = L_{m_2}[K_{m_2}A] = L_{m_1}[K_{m_2}A]$$

   This means that the vacuum functionals can be interchanged *after* the mapping but not before.

5. The kernels $\mathcal{K}_m$ determine the representation of the Poincaré group in the light-front mass m sub-algebra. Specifically the infinitesimal versions of the covariant Poincaré transformations, $x^\mu \to x'^\mu \to \Lambda^\mu{}_\nu + a^\mu$, get mapped into the light-front generators defined as operators on the light-front Fock algebra. The kernels also satisfy the field equations.

6. The kernels, $\mathcal{K}_m$, for different masses are associated with inequivalent representations of the canonical commutation relations. This follows because vacuum expectation values of polynomials in the Heisenberg fields can be expressed in terms of vacuum expectation values of both polynomials in the canonical algebra or vacuum expectation values of polynomials in the light-front mass $m$ sub-algebra.

7. The kernels $\mathcal{K}_m$ determine local nature of the Heisenberg fields. This follow directly from the operator relation (54). Specifically is $f$ and $g$ have space-like separated supports then

$$0 = [\phi(f), \phi(g)]_\pm = [\phi_{LF}(f_m), \phi_{LF}(g_m)]_\pm$$

   where the $+$ is for half integer spin free fields and the $-$ is for integer spin free fields.

8. The kernels map Schwartz functions to functions of light-front variables that vanish faster than any polynomial in $p^+$ near 0. It will be shown that this suppression of light-like momenta in the light-front mass $m$ test functions removes non-causal momenta from the light-front mass $m$ algebra resulting in a well-defined initial value problem.

Since the light front fields are still operator valued distributions the fields on the light front must first be smeared with test functions before taking vacuum expectation values. In order to calculate the restriction of the Heisenberg field to the light front it is necessary to take the limit using the kernel $\mathcal{K}_m$. For free fields the result will be [80]

$$\langle 0|\phi(x)\phi(0)|0\rangle \to -i\frac{\epsilon(x^-)\delta(\mathbf{x}_\perp^2)}{4\pi} - \frac{mK_1(m\sqrt{\mathbf{x}_\perp^2})}{4\pi^2\sqrt{\mathbf{x}_\perp^2}}. \tag{59}$$

On the other hand if this is computed directly on the light front without using one of the kernels $\mathcal{K}_m$, then the result will be

$$\langle 0|\phi(x)\phi(0)|0\rangle = \frac{1}{2(2\pi)^3}\int \frac{\theta(q^+)dq^+d\mathbf{q}_\perp}{q^+}e^{-i\frac{q^+}{2}x^- + i\mathbf{q}_\perp\cdot\mathbf{x}_\perp} = \frac{\delta(\mathbf{x}_\perp^2)}{4\pi}\int_0^\infty \frac{dq^+}{q^+}e^{-i\frac{q^+}{2}x^-} \tag{60}$$

again independent of the choice of vacuum.

## VI. INITIAL VALUE PROBLEM

A dense set of Schwartz functions are Fourier transforms of Schwartz functions with compact support. These functions satisfy

$$\{f(x)|\hat{f}(p) := \int \frac{d^4x}{(2\pi)^2}e^{-p\cdot x}f(x), \qquad \hat{f}(p) = 0 \qquad \text{for} \qquad (p^0)^2 + \mathbf{p}^2 < R^2 < \infty\} \tag{61}$$

where $R$ is any finite positive constant. The Heisenberg algebra of fields smeared with these test functions applied to the vacuum generates a dense subspace of the Hilbert space.

If $\hat{f}(p)$ is expressed in terms of light front variables $\hat{f}(p) = \hat{f}(\frac{p^+ + p^-}{2}, \mathbf{p}, \frac{p^+ - p^-}{2}) =: g(p^-, p^+, \mathbf{p}_\perp)$, for $g(p)$ to be non-vanishing requires

$$|p^+|, |p^-| < 2R. \tag{62}$$

The Fourier transform of the light front mass $m$ test function

$$g_m(\tilde{\mathbf{y}}) = \int \mathcal{K}_m(x, \tilde{\mathbf{y}})f(x)d^4x \tag{63}$$

is (55)

$$\hat{g}_m(p^+, \mathbf{p}_\perp) = \sqrt{2\pi}\hat{g}(\frac{\mathbf{p}^2 + m^2}{p^+}, p^+, \mathbf{p}_\perp). \tag{64}$$

The support condition implies

$$\hat{g}(p) = 0 \qquad \text{unless} \qquad (p^0)^2 + \mathbf{p}^2 < R^2 < \infty \tag{65}$$

for some finite $R$. The support condition (62) implies that

$$|p^+| < 2R \qquad \text{and} \qquad |p^-| = \frac{\mathbf{p}^2 + m^2}{p^+} < 2R. \tag{66}$$

These inequalities can be combined to show that the support of $\hat{g}_m(p^+, \mathbf{p}_\perp)$ is for

$$\frac{m^2}{2R} < p^+ < 2R. \tag{67}$$

This condition eliminates light-like momenta with $p^+ = 0$ provided $m > 0$.

The $x^+$ evolution in the covariant representation is a translation of the argument of the test function. The intertwining relations (57) give

$$\int \phi(x^+, \tilde{\mathbf{x}})f(x - a^+)d^4x =$$

$$\int \phi(y^+ + a^+, \tilde{\mathbf{y}}) f(y^+, \tilde{\mathbf{y}}) d^4y =$$

$$\int d^4y d\tilde{\mathbf{p}} f(y) \mathcal{K}_m(y^+ + a^+, \tilde{\mathbf{y}}, \tilde{\mathbf{p}}) \phi_{LF}(-\tilde{\mathbf{p}}) =$$

$$\int d^4y d\tilde{\mathbf{p}} f(y) \sum_{n=0}^{\infty} \frac{(ia^+)^n}{n!} \frac{\partial^n}{\partial(iy^+)^n} \mathcal{K}_m(y^+, \tilde{\mathbf{y}}, \tilde{\mathbf{p}}) \phi_{LF}(-\tilde{\mathbf{p}}). \tag{68}$$

Using (57) gives

$$= \int d^4y d\tilde{\mathbf{p}} f(y) \sum_{n=0}^{\infty} \frac{(-ia^+)^n}{n!} (\frac{\mathbf{p}^2 + m^s}{2p^+})^n \mathcal{K}_m(y^+, \tilde{\mathbf{y}}, \tilde{\mathbf{p}}) \phi_{LF}(-\tilde{\mathbf{p}}) =$$

$$\int d\tilde{\mathbf{p}} e^{-ia^+(\frac{\mathbf{p}^2+m^s}{2p^+})} f_m(\tilde{\mathbf{p}}) \phi_{LF}(-\tilde{\mathbf{p}}) =$$

$$\sqrt{2\pi} \int d\tilde{\mathbf{p}} \sum_{n=0}^{\infty} \frac{(-ia^+)^n}{n!} (\frac{\mathbf{p}^2 + m^s}{2p^+})^n \hat{f}(\frac{\mathbf{p}_\perp^2 + m^2}{p^+}, \tilde{\mathbf{p}}) \phi_{LF}(-\tilde{\mathbf{p}}). \tag{69}$$

The series

$$\sum_{n=0}^{\infty} \frac{(-ia^+)^n}{n!} (\frac{\mathbf{p}_\perp^2 + m^2}{2p^+})^n \hat{f}(\frac{\mathbf{p}_\perp^2 + m^2}{p^+}, \tilde{\mathbf{p}}) \tag{70}$$

converges provided for $m > 0$ since

$$|\sum_{n=0}^{\infty} \frac{(-ia^+)^n}{n!} (\frac{\mathbf{p}_\perp^2 + m^2}{2p^+})^n| \leq \sum_{n=0}^{\infty} \frac{(2Ra^+)^n}{n!} = e^{2Ra^+} < \infty. \tag{71}$$

This shows that the initial value problem for the field is well defined on the light front provided the mass m test functions are derived from this dense set of Schwartz functions.

The relevant observations are:

1 The mapping from the Heisenberg field algebra to the light-front mass $m$ field algebra maps the covariant representation of the Poincaré group to a light-front representation (57).

2. This mapping maps a dense set of vectors in the covariant representation of the Hilbert space to a dense set of vectors in the space generated by the light-front mass m sub-algebra.

3. $x^+$ evolution, defined by the exponential series, converges on this dense set of vectors.

4. The dense set in the Hilbert space can be generated by applying elements of this sub-algebra of the light-front Fock algebra to any free-field vacuum.

## VII. TRIVIALITY OF THE VACUUM

Since the vacuum is invariant with respect to translations that leave the light-front invariant, it follows that

$$P_0^+|0\rangle = 0. \tag{72}$$

Since $P_0^+$ commutes with both $M$ and $M_0$ it commutes with the interaction $V := M - M_0$. It follows that

$$P_0^+ V|0\rangle = V P_0^+|0\rangle = 0. \tag{73}$$

This means that $|0\rangle$ and $V|0\rangle$ are both eigenstates of $P_0^+$ with eigenvalue 0. It follows that

$$\langle 0|V^\dagger V|0\rangle = \int |\langle p_0^+, d|V|0\rangle|^2 d\mu(p_0^+) dd \tag{74}$$

where $d$ represents the remaining degenerate quantum numbers and $d\mu(p_0^+)$ is the spectral measure of $P_0^+$. The spectrum of $P_0^+$, except for the vacuum, is absolutely continuous and non-negative. The only contribution to this integral over intermediate states comes from states that are discrete eigenstates of $p_0^+$ with eigenvalue 0. It follows that

$$V|0\rangle = |0\rangle\langle 0|V|0\rangle \tag{75}$$

which means that $V$ cannot change the vacuum.

If the vacuum is the only discrete normalizable state of the theory that is invariant under translations on the light front then

$$0 = (P^- P^+ - \mathbf{P}_\perp^2)|0\rangle = M^2|0\rangle = (M_0^2 + VM_0 + M_0 V + V^2)|0\rangle =$$

$$V^2|0\rangle = |0\rangle\langle 0|V|0\rangle^2 \tag{76}$$

which means that the constant must vanish.

The problem with this naive argument is that all of the translation generators are unbounded operators. $P^+ = 0$ corresponds to infinite momentum while $P^-$ diverges as $p^+ \to 0$, so $P^+|0\rangle = 0$ does not imply $P^- P^+|0\rangle = 0$. This is clear since the light-front dispersion relation has the form $P^- = (M^2 + \mathbf{P}_\perp^2)/P^+$, which has a $P^+$ in the denominator.

In a dynamical theory the pure creation operator part of the interaction is the part of the interaction that is responsible for the difference between the Heisenberg and Fock vacuums in canonical formulations of quantum field theory. In a $\phi^4(x)$ theory the pure creation part of the interaction term in the light-front Hamiltonian has the form:

$$\int \frac{\theta(p^+)\delta(p^+)dp^+}{(p^+)^2 \prod \xi_i} \prod d\mathbf{p}_{i\perp} d\xi_i \delta(\sum \mathbf{p}_{i\perp})\delta(\sum \xi_i - 1)\times$$

$$a^\dagger(\xi_1 p^+, \mathbf{p}_{\perp 1}) a^\dagger(\xi_2 p^+, \mathbf{p}_{\perp 2}) a^\dagger(\xi_3 p^+, \mathbf{p}_{\perp 3}) a^\dagger(\xi_4 p^+, \mathbf{p}_{\perp 4}).$$

This expression is divergent as $p^+ \to 0$. Multiplication by $p^+$ is not sufficient to cancel the $p^+ = 0$ singularities in the denominator.

It is clear some kind renormalization is necessary to remove this singular behavior. In this next section is will be argued that how this must be done is constrained by rotational covariance. The interacting case will be discussed in the following section.

## VIII. ROTATIONAL COVARIANCE

The light front Hamiltonian $P^-$ and the kinematic generators of the Poincaré group form a closed sub-algebra. This sub-algebra contains no information about transverse rotations. Given this sub-algebra, if $J^1$ and $J^2$ are the transverse rotation generators that complete the Poincaré Lie algebra and $W$ is any unitary operator that commutes with $P^-$ and the kinematic generators then $J'^1 = WJ^1W^\dagger$ and $J'^2 = WJ^2W^\dagger$ also complete the Poincaré Lie algebra with the same $P^-$ and kinematic generators. It follows that $P^-$ and the kinematic generators do not uniquely determine the transverse rotation operators.

Noether's theorem, which assumes the equations of motion, provides conserved currents for all ten independent Poincaré transformations (see Appendix II). Noether charges can be constructed by integrating the current over a light front. Even though the currents are integrated over a light front rather than a fixed time surface, the resulting Noether charges for free fields satisfy the Poincaré commutations relations. In the light-front case both the kinematic and dynamical generators are expressed in terms of the irreducible algebra of fields restricted to the light front. There are explicit expressions for both $P^-$ and the transverse rotation generators, $\mathbf{J}_\perp$. So while $P^-$ and the kinematic generators are not sufficient to complete the Poincaré Lie algebra, Noether's theorem gives explicit expressions for the transverse rotation generators in terms of fields on the light front that correspond to a particular choice of $W$.

The transverse rotation generators do not have this ambiguity. The Poincaré commutation relations imply

$$P^- := P^+ - 2[J^2,[J^2,P^+]] \qquad \text{and} \qquad J^1 := -i[J^2,J^3] \tag{77}$$

which means that given $J^1$ or $J^2$ and the kinematic generators it is possible to use the commutation relations to construct the other two dynamical generators. Since these relations are a consequence of the Poincaré commutation relations they also hold in dynamical theories.

The rotational properties of fields can be solved using the commutation relations. Formally rotations about the $y$ axis can be expressed as an infinite series in the irreducible light front algebra

$$U_y(\theta)\phi(\tilde{\mathbf{x}},0)U_y^\dagger(\theta) = \phi(\frac{1+\cos(\theta)}{2}x^-, -\sin(\theta)x_1, x_2, \frac{1-\cos(\theta)}{2}x^-) =$$

$$\sum_{n=0}^\infty \frac{(i\theta)^n}{n!} \underbrace{[J_2,[J_2,\cdots,[J_2,\phi(\tilde{\mathbf{x}},0)]]]}_{\text{N times}}. \tag{78}$$

where $J^2$, $\phi(\tilde{\mathbf{x}},0)$ and all commutators are all restricted to the light front. The arguments of the coordinates $x^-$ and $\mathbf{x}_\perp$ on the light front can be changed to any values using kinematic translations. Equation (78) uses the fields on the light front and the rotation generators, constructed from fields on the light front, to formally construct the local Heisenberg field algebra.

The explicit structure for $J^2$ (for scalar fields) in terms of field operators restricted to the light front that follows from Noether's theorem is:

$$J^2 = -\int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{4}\left(x^- T^{+1}(x) + x^1(T^{++} - T^{+-}))\right) =$$

$$\int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{4}\left(-x^- 2 : \partial_-\phi(x)\partial_1\phi(x) : -x^1(4 : \partial_-\phi(x)\partial_-\phi(x) : -(: \boldsymbol{\nabla}_\perp\phi(x)\cdot\boldsymbol{\nabla}_\perp\phi(x) : + m^2 : \phi(x)^2 : + 2 : V(\phi(x)) :)\right) \tag{79}$$

where $T^{\mu\nu}$ is the energy momentum tensor density. The relevant observation is that this expression only involves fields on the light front and derivatives of fields that are tangent to the light front.

Equation (78) is formally true for both free and interacting theories, but in the interacting case the operator products in $: V(\phi(x)) :$ have divergences as $p^+ \to 0$ and $p \to \infty$ that require renormalization in order to define $J^2$.

While it is formally sufficient to construct the dynamics using $J^2$ and kinematic transformations following the construction in the free field case, the structure of the dynamical part of $J^2$ is constrained by the Poincaré commutation relations. While Noether's theorem gives the formal expression (79) for $J^2$, the complete definition requires renormalization of the operator products which must be done in a manner that preserves these constraints.

To understand the origin of these constraints it is useful to express the transverse rotation generator, $J^2$, as the sum of a non-interacting generator and an interaction, $J^2 = J_0^2 + J_I^2$, where $J_0^2$ is the kinematic generator of rotations about the 2 axis. The commutation relations imply that the interaction, $J_I^2$, must commute with all of the kinematic generators and in addition it must satisfy the following linear and non-linear dynamical constraints:

$$[J_I^2,[J_0^2,P^1]] = 0 \tag{80}$$

and

$$[J_I^2,[J_I^2,J^3]] + [J_0^2,[J_I^2,J^3]] + i[J_I^2,J_0^1] = 0. \tag{81}$$

These are non-trivial constraints on the allowed interactions in $J^2$.

In the absence of the other dynamical generators the renormalization of $P^-$ is not constrained by rotational covariance. This is the situation that appears in many applications of light-front field theory. The rotational covariance is not encoded in $P^-$ so it must be imposed by hand. For free fields the angular momentum is already encoded in the spinors for the field restricted to the light front. In an interacting theory there can be degenerate composite eigenstates of mass and $\mathbf{S}\cdot\hat{\mathbf{z}}$ where $P^-$ provides no information on how to assign the total spin to these states. The transverse angular momentum operator is needed to separate these states according to their spin. Knowing only $P^-$ is not sufficient.

A necessary condition for light-front formulations of quantum field theories to be equivalent to the covariant formulation is that quantum observables do not depend on the choice of orientation of the light front [17] [18] [19] [20]

[21] [22][23] [24]. Let $U_{\hat{\mathbf{n}}}(g)$, with $g=(\Lambda,a)$ be the unitary representation of the Poincaré group associated with light front $x^0+\hat{\mathbf{n}}\cdot\mathbf{x}=0$ and let $K_{\hat{\mathbf{n}}}$ denote the corresponding kinematic subgroup. The operator

$$Y_{\hat{\mathbf{n}}}(g) := U_0(g)U^\dagger_{\hat{\mathbf{n}}}(g) \tag{82}$$

is unitary and is the identity whenever $g\in K_{\hat{\mathbf{n}}}$. This means that $Y_{\hat{\mathbf{n}}}(g)$ only depends on left cosets of $K_{\hat{\mathbf{n}}}$ in the Poincare group:

$$g' \in [g]_{\hat{\mathbf{n}}} \iff g'=gk \qquad \text{for some} \qquad k\in K_{\hat{\mathbf{n}}}. \tag{83}$$

It follows that (82) can be expressed as $Y_{\hat{\mathbf{n}}}([g]_{\hat{\mathbf{n}}})$. Applying this operator to $U_{\hat{\mathbf{n}}}(g)$ gives a new unitary representation of the Poincaré group with a different kinematic subgroup

$$Y_{\hat{\mathbf{n}}}([g]_{\hat{\mathbf{n}}})U_{\hat{\mathbf{n}}}(g)Y^\dagger_{\hat{\mathbf{n}}}([g]_{\hat{\mathbf{n}}}) = U_{\hat{\mathbf{n}}'}(g) \tag{84}$$

where the new kinematic subgroup is

$$K_{\hat{\mathbf{n}}'} = [g]K_{\hat{\mathbf{n}}}[g^{-1}]. \tag{85}$$

If in addition this transformation leaves the scattering operator unchanged it can be expressed as in terms of wave operators

$$Y^\dagger_{\hat{\mathbf{n}}}([g]_{\hat{\mathbf{n}}}) = \Omega_{+K}\Omega^\dagger_{+[g]K[g^{-1}]} = \Omega_{-K}\Omega^\dagger_{-[g]K[g^{-1}]} \tag{86}$$

which means that the operator can be expressed in terms of complete sets of scattering solutions with different light fronts. Using these relations with (82) for $g=(R,0)$, a pure rotation, gives an explicit expression of the rotation operator with kinematic subgroup $K_{\hat{\mathbf{z}}}$:

$$U_{\hat{\mathbf{z}}}(R,0) = Y^\dagger_{\hat{\mathbf{z}}}([R]_{\hat{\mathbf{z}}})U_0(R) = \Omega_{\pm K_{\hat{\mathbf{z}}}}\Omega^\dagger_{\pm[R]K_{\hat{\mathbf{z}}}[R^{-1}]}U_0(R). \tag{87}$$

This provides an expression for the rotation operator in terms of a complete sets of wave functions for the same state in different S-matrix equivalent representations of the dynamics. These formal relations hide the actual difficulty. In order to calculate the scattering states it is important to know the rotational properties of the asymptotic states. If there are 2 composite one-body solutions with the same mass that are degenerate in the magnetic quantum numbers, it is necessary to know how to consistently assign total spins the these states in the asymptotic region order to assign them to the appropriate scattering channel. Equation (87) assumes that problem has been solved, but it is equivalent to the problem of assigning spins to degenerate bound states in light-front dynamics.

If this is done successfully it gives a dynamical representation of the rotation group. While it is not necessarily that same as the one that comes from the Noether charges, it will be unitarily equivalent.

The important observation about the relation between rotational covariance and invariance with respect to changing the orientation of the light front is that if theories with different light front orientations are scattering equivalent, the $p^+\to 0$ divergences associated with one light front become $p\to\infty$ divergences when associated with a different light front. This means that the $p^+\to 0$ and $p\to\infty$ renormalizations are constrained by rotational covariance. At the perturbative level the connection can be resolved by appealing to the connection to covariant perturbation theory [58], but one of the compelling reasons for using the Hamiltonian formulation of light-front quantum field theory is that it is in principle a non-perturbative formulation of quantum field theory. This means the problem of rotational covariance cannot be separated from the problem of how to perform rotationally invariant renormalization beyond perturbation theory.

## IX. INTERACTING FIELDS

For case of interacting fields, in the absence of non-trivial dynamical solutions, light-front field theories can be studied based on general properties of quantum field theory. This assumes that the problem of how to properly renormalize the theory outside of perturbation theory has been solved.

The first assumption is that the theory is asymptotically complete. This means that it has complete sets of scattering states satisfying IN and OUT asymptotic conditions and these states (along with the vacuum and one-particle states) are a basis for the Hilbert space of the theory. The convergence of the limits that define these scattering states is a consequence of the axioms of quantum field theory provided the theory has one-particle states (point spectrum eigenstates) and a mass gap. The scattering states are expressed as strong limits in the Haag-Ruelle formulation of scattering theory [67][69][81][68].

The Haag-Ruelle formulation of scattering is a field theoretic version of the non-relativistic formulation of time-dependent scattering where the scattering states are defined by the strong limits

$$|f_1 \cdots f_n\rangle_{OUT/IN} = \lim_{t\to\pm\infty} \int \phi_h(\mathbf{x}_1, t)\cdots\phi_n(\mathbf{x}_N, t)|0\rangle \overleftrightarrow{\partial}_t \prod_n f_n(\mathbf{x}_n, t)d\mathbf{x}_n, \tag{88}$$

and the functions,

$$f_n(\mathbf{x}_n, t) = \int \frac{e^{-i\omega_n(\mathbf{p})t+i\mathbf{p}\cdot\mathbf{x}}}{(2\pi)^{3/2}(2\sqrt{\omega_m(\mathbf{p})})} g(\mathbf{p})d\mathbf{p} \tag{89}$$

are positive energy solutions of the Klein Gordon equation. $\phi_h(x)$ is a quasilocal field which is the Fourier transform of

$$\hat{\phi}_h(p) = \hat{\phi}(p)h(p) \tag{90}$$

where $\hat{\phi}(p)$ is the Fourier transform of the field and $h$ is a smooth function that is 1 when $p^2 = -m^2$ and $p^0 > 0$ and vanishes on the rest of the mass spectrum of the theory and for $p^0 < 0$. Here the mass is the physical mass of the one-particle state. If the theory has one-particle states that are not associated with the elementary fields $\hat{\phi}(p)$ can be replaced by the Fourier transform of a local interpolating field [82] that has non-zero matrix elements between the vacuum and one-particle state .

Ruelle showed that the strong limit of (88) exists when $g(\mathbf{p})$ is a smooth function of the 3-momentum with compact support. The integral of the norm of the difference of two sides of (88) has the same kind of $t^{-3/2}$ fall off that occurs in the Cook condition in quantum mechanical scattering, except in this case the fall off is due to cluster properties and the assumed mass gap rather than the range of the potential.

The purpose of providing this detail is the observation that the scattering states can be expressed as an operator applied to the vacuum that is multi-linear in the wave packets $g_i(\mathbf{p})$ which correspond to the momentum distributions of the asymptotic clusters in the scattering reaction. It is also multi-linear in the positive energy solutions $f(\mathbf{x}, t)$ of the Klein Gordon equation. The time derivative, $\overleftrightarrow{\partial}_t$, selects the part of the field that asymptotically looks like a creation operator.

The set of $IN$ or $OUT$ scattering states form a complete orthonormal set of states in the theory. These strong limits can be used to define asymptotic field operators that transform N-particle scattering states to $N \pm 1$ particle scattering states.

$$\Phi_{IN}(f)|f_1 \cdots f_n\rangle_{IN} = |f, f_1 \cdots f_n\rangle_{IN} \tag{91}$$

where

$$\Phi_{IN}(f) = \lim_{t\to-\infty} \int \Phi_{IN}(\mathbf{x}, t) \overleftrightarrow{\partial} f(\mathbf{x}, t)d\mathbf{x}. \tag{92}$$

There are IN (resp. OUT) fields for each one-particle state of the field theory. Mathematically $\Phi_{IN}$ has the same Fourier representation as free fields with physical masses. The difference is that the creation operators create $n + 1$ particle scattering states out of $n$ particle scattering states. The algebra of IN (resp OUT) fields applied to the vacuum generates the Hilbert space of the field theory. Mathematically it is equivalent to a free field Fock space (of a direct sum of free field Fock spaces). Just like ordinary free fields the Fourier transforms of the asymptotic fields can be expressed in terms of light-front variables, and as in the case of free fields the creation and annihilation operators in the asymptotic fields can be expressed in terms of the algebra of asymptotic fields restricted to the light front. This follows from the construction in section V:

$$\int \Phi_{IN}(x)f^*(x)d^4x = \int \mathcal{K}_m(x, \tilde{\mathbf{y}})f^*(x)d^4x = \int \Phi_{IN}(y^+ = 0, \tilde{\mathbf{y}})f_m(\tilde{\mathbf{y}})\frac{dy^- d\mathbf{y}_\perp}{2}. \tag{93}$$

This generates a map from the algebra of asymptotic fields to a sub-algebra of the light front field algebra as in the case of free fields.

The assumed irreducibility of the asymptotic fields means that the Heisenberg fields can be expanded in normal products of the asymptotic fields: [64][70][71][83]:

$$\Phi(f) = \sum \int R_n(x; x_1 \cdots x_n)f(x)d^4x : \Phi_{IN}(x_1)\cdots\Phi_{IN}(x_n) : \prod_k d^4x_k. \tag{94}$$

Using (93) this expansion can be expressed in terms of the algebra of fields on the light front

$$\Phi(f) = \sum \int R_n(x; x_1, \cdots, x_n) f(x) d^4x \prod_k d^4 x_k K_{m_k}(x_k, \tilde{\mathbf{y}}_k) : \Phi_{IN}(y_1^+ = 0, \tilde{\mathbf{y}}_1) \cdots \Phi_{IN}(y_n^+ = 0, \tilde{\mathbf{y}}_n) : \prod_l d\tilde{\mathbf{y}}_l. \quad (95)$$

This expansion defines the analog of the mapping (53) from the Heisenberg field algebra to a sub-algebra of the light-front Fock algebra. Some versions of perturbation theory start with expansions Heisenberg fields in terms of asymptotic fields rather than free fields [84].

Vacuum expectation values of products of the Heisenberg fields can be expressed as light-front expectation values of functions of the asymptotic fields restricted to the light front using the expansions (95). As in the case of free fields, any vacuum restricted to the light front can be used in this computations. The mapping defines the values of the vacuum functional off of the light-front.

If there are asymptotic states with composite particles there will be tensor products of Fock spaces of asymptotic states restricted to the light-front. Since the Fock space are countable they are unitarily equivalent to tensor products of Fock spaces, so vacuum expectation values of products of smeared asymptotic field can still be expressed in terms of vacuum expectation values of free fields on the light-front.

One of the primary purposes of using the Hamiltonian formulation of light-front quantum field theory is applications to QCD. One of the new problems is that the asymptotic fields for QCD are all composite fields. Individual Heisenberg fields for quarks and gluons cannot be represented as expansions in the asymptotic fields of the theory, which are all colorless. Greenberg [73] showed that in a simple model the introduction of "confined asymptotic fields" satisfying certain properties allowed for the expansion of color carrying field in terms of normal products of asymptotic fields. Whether this is applicable to QCD remains an open question. This does not imply that the irreducible algebra of fields on the light-front cannot be used to represent smeared Heisenberg quark fields. The light front is particularly interesting because the dependence on the quark masses disappears for fields restricted to the light front. This consistent with the partonic description of quarks, which have electric charge but no asymptotic mass.

The main observation is that even in an interacting theory, if the theory is properly defined, the Wightman functions of the theory can be expressed in terms of a sub-algebra of fields on the light front applied to any vacuum restricted to the light front. In this case the sub-algebra distinguishes different theories.

This assumes that the full mass spectrum has been determined. In order to use this it is necessary to determine the point spectrum of the theory and express the Heisenberg fields in terms of normal products of the IN or OUT fields.

Just like in the free field case, once the smeared Heisenberg fields are expressed in as elements of a sub-algebra of the light front algebra, vacuum expectation values of products of fields can be evaluated using any vacuum. In this case the vacuum looks trivial because the true vacuum is equivalent to a free field vacuum on the light front sub-algebra associated with the interacting theory.

## X. SPONTANEOUS SYMMETRY BREAKING

One of the puzzling aspects of light-front quantum field theory is how spontaneous symmetry breaking is realized. The signal for spontaneous symmetry breaking is a conserved local current, $j^\mu(x)$, that arises from the symmetry and the presence of a 0 mass Goldstone boson in the spectrum. The presence of a Goldstone boson is confirmed when charge operator has non-zero matrix elements between the vacuum and Goldstone boson. In general the charge operator does not have to make sense, since it is an operator valued distribution evaluated at point $\mathbf{p} = 0, t = 0$.

For a local scalar field theory the following condition [74]

$$\lim_{R \to \infty} \langle 0 | [Q_R, \phi(y)] | 0 \rangle \neq 0, \quad (96)$$

where

$$\langle 0 | [Q_R, \phi(y)] | 0 \rangle := \langle 0 | [\int d\mathbf{x} \chi_R(|\mathbf{x}|) j^0(\mathbf{x}, t), \phi(y)] | 0 \rangle \quad (97)$$

where $\chi_R(|\mathbf{x}|)$ is a smooth function that that is 1 for $|\mathbf{x}| < R$ and 0 for $|\mathbf{x}| > R + \epsilon$ for some finite positive $\epsilon$, which does not require the existence of a charge operator, implies the existence of a 0 mass particle.

The cutoff function $\chi_R$ ensures that the integral converges for large $|\mathbf{x}|$. The commutator vanishes for $x - y$ space-like. For fixed $y$ and $t$ and sufficiently large $R$, if $|\mathbf{x}| > R$ then $x - y$ is space-like and the commutator vanishes. In this case

$$\langle 0 | [Q_R, \phi(y)] | 0 \rangle := \langle 0 | [\int d\mathbf{x} \chi_R(|\mathbf{x}|) j^0(\mathbf{x}, t), \phi(y)] | 0 \rangle = \int d\mathbf{x} \langle 0 | [j^0(\mathbf{x}, t), \phi(y)] | 0 \rangle. \quad (98)$$

Current conservation implies

$$\langle 0|[\int d\mathbf{x}\partial_\mu j^\mu(\mathbf{x},t),\phi(y)]|0\rangle = 0. \tag{99}$$

Inserting a complete set of intermediate states gives

$$0 = \sum \int d\mathbf{x} \left( \langle 0|\partial_\mu j^\mu(\mathbf{x},t)|p,n\rangle \frac{d\mathbf{p}}{2p_n^0} \langle p,n|\phi(y)|0\rangle - \langle 0|\phi(y)|p,n\rangle \frac{d\mathbf{p}}{2p_n^0} \langle p,n|\partial_\mu j^\mu(\mathbf{x},t)|0\rangle \right) =$$

$$\sum \int d\mathbf{x} \left( \langle 0|\partial_\mu j^\mu(\mathbf{0},0)|p_r,n\rangle \frac{d\mathbf{p}}{2p_n^0} \langle p_r,n|\phi(0)|0\rangle e^{ip\cdot(x-y)} - \langle 0|\phi(0)|p_r,n\rangle \frac{d\mathbf{p}}{2p_n^0} \langle p_r,n|\partial_\mu j^\mu(\mathbf{0},0)|0\rangle e^{ip\cdot(y-x)} \right) \tag{100}$$

where Poincaré covariance has been used to remove the non-trivial $x, y$ and $p$ dependence from the matrix elements. $p_r$ is the constant rest four momentum for massive states and a constant light-like vector for massless states. For a scalar field theory the vacuum expectation value of the current vanishes so the vacuum does not appear as an intermediate state. The Lehmann weights that appear in this matrix element

$$\sigma(m_n)m_n^2 = \langle 0|\partial_\mu j^\mu(\mathbf{0},0)|p_{nr}\rangle\langle p_{nr}|\phi(0)|0\rangle \tag{101}$$

and

$$\sigma^*(m_n)m_n^2 = \langle 0|\phi(0)|p_{nr},n\rangle\langle p_{nr},n|\partial_\mu j^\mu(\mathbf{0},0)|0\rangle \tag{102}$$

are functions of the invariant mass of the intermediate states. The factor $m_n^2 = -p_n^2$ is due to the $\partial_\mu j^\mu(\mathbf{0},0)$ assuming that the current is a local function of the scalar field. Current conservation requires that this quantity vanishes which follows if either $\sigma(m_n) = 0$ or $m_n^2 = 0$. Inserting intermediate states in (97) gives a similar result with $\sigma(m_n)m_n^2$ replaced by $p_{nr}^0\sigma(m_n)$ where $p_{nr}^0$ is a non-zero constant.

Since this does not include the $m^2$ factor, it vanishes by current conservation unless $\sigma(m_n)$ contains a $\delta(m_n)$. Thus for (96) to be non-vanishing the sum over intermediate states must have a contribution from a 0 mass particle. This is the non-perturbative form of Goldstone's theorem.

In [74] Coleman avoids directly computing the charge operator. The current is an operator valued distribution, so there is no reason to expect that integrating against the constant 1 will result in a well defined operator.

While the non-vanishing of (96) is a sign of spontaneous symmetry breaking, the argument used above does not work when it is applied to the light-front charge. The problem is that there is no compact region on the light front where outside of that region $(x-y)$ is always space-like for fixed $y$ and $x^+ = 0$. In the light-front case, if the charge exists, $Q$ is invariant with respect to translations on the light front, which means that it commutes with $P^+$. It follows that $P^+Q|0\rangle = QP^+|0\rangle = 0$. Because of the spectral condition all of the individual $p_i^+$, $p_i^+Q|0\rangle = 0$. Multiplying $Q|0\rangle$ by the completeness relation implies that the only non-zero term in the sum is the vacuum

$$Q|0\rangle = |0\rangle\langle 0|Q|0\rangle \tag{103}$$

which means that it cannot couple the vacuum to a another state.

In the light-front representation it is still possible to define $Q_R$ on a fixed-time surface and the analysis above still applies. To do this the field and current in the matrix element (96)(97) must be mapped to the sub-algebra of the light-front Fock algebra. Since spontaneous symmetry breaking is a dynamical problem, both the current and field would have to be expanded in asymptotic fields as in section IX.

## XI. CONCLUSION

The purpose of this work is to understand the relation between light-front and more conventional formulations of quantum field theory. Most of the conclusions are based on known properties of free fields and properties of axiomatic formulations of field theories.

The first question is whether the vacuum of light front quantum field theories is trivial.

> Each theory has one vacuum. It is a linear functional on the Heisenberg field algebra generated by covariant local fields smeared with Schwartz test functions in four variables. The Wightman functions, which define the dynamics, are constructed by applying the vacuum functional to products of fields in the Heisenberg field

algebra. In quantum field theory the Wightman functions are the kernel of the Hilbert space inner product. The covariance of the fields and invariance of the vacuum functional define a unitary representation of the Poincaré group on this representation of the Hilbert space. Theories that have the same Wightman functions preserve the Hilbert space inner product and are unitarily equivalent. Vacuum functionals that result in different Wightman functions do not preserve the Hilbert space inner product and define unitarily inequivalent theories.

Different vacuum functionals can agree on a sub-algebra of the Heisenberg algebra without agreeing on the whole algebra. In section V and IX it was shown that vacuum functionals for free and asymptotic fields theories are defined and agree on the light-front Fock algebra. In (54) and (93) it was shown that Wightman functions for the free and asymptotic Heisenberg field algebras could be expressed as vacuum expectation values of elements of sub-algebras of the light-front Fock algebra (The test functions in the light-front Fock algebra are restricted to Schwartz functions in the light-front coordinates that have vanishing $x^-$ derivative). All of the vacuum functionals agree on this sub-algebra since they agree on the full light front Fock algebra.

The assumed irreducibility of the asymptotic fields means that the Heisenberg fields can be expanded in a series of normal products of asymptotic fields; the vacuum does not change. It then follows that the Wightman functions of the field theory can be formally expressed in terms of vacuum expectation values of the asymptotic fields which can in turn be expressed in terms vacuum expectation values of elements of a sub-algebra of the light-front Fock algebra. Any of the vacuum functionals have the same value when applied to elements of this sub-algebra and can be used to evaluate the Wightman functions of the theory.

The asymptotic fields generate a complete set of one-body and many-body scattering states that diagonailze the field theoretic Hamiltonian. The masses of these fields encode the mass and spin spectrum of the field theoretic Poincaré generators. Haag-Ruelle scattering theory defines the asymptotic fields in terms of Heisenberg fields that satisfy the Wightman axioms. The dynamical difficulties appear in the inverse problem of how to define the Heisenberg algebra satisfying these axioms in terms of the algebra of asymptotic fields. This is where all of the problems with renormalization appear. The relevant expansion is given in [70] where the coefficients of the expansion are only sensitive to the asymptotic fields on their mass shell. The input to the expansion is nested commutators of the Heisenberg fields with asymptotic fields.

This suggest that the free field Fock vacuum can be used provided the dynamical generators can be expressed as well-defined operators in the light-front Fock algebra. In this case the light-front Hamiltonian determines the dynamical sub-algebra of the light-front Fock algebra.

The next question is the problem of inequivalent representations of the canonical commutation relations.

As discussed above the Wightman functions for different theories can be expressed as vacuum expectation values of elements of a sub-algebra of the light-front Fock algebra. While vacuum expectation values of elements of the sub-algebra are independent of the choice of vacuum functional, functionals applied to different sub-algebras are associcted with Wightman functions for different theories. The sub-algebras define the dynamics; they contain all of the information about the mass and spin spectrum of the interacting theory. Inequivalent theories are associated with different sub-algebras of the light-front Fock algebra. There is no relation between the vacuum expectation values of different sub-algebras of the light-front Fock algebra. This is independent of the vacuum functional used to compute them.

The restriction of the fields to the light front has no dynamical information. In addition fields on the light front are still operator valued distributions so they must first be smeared with suitable test functions. Dynamical information enters when the test functions are restricted to the test functions constructed by mapping the four-dimensional test functions to the light front. This is demonstrated in (59) and (60) relating the two-point functions to two point functions on the light front. The mapping encodes the correct approach to the light front in different theories.

The initial value problem

The mapping from the local Heisenberg field algebra to the sub-algebra of the light-front Fock algebra maps covariant Poincaré generators to light-front Poincaré generators (see (57)). For theories with a mass gap there is a dense set of vectors in the Hilbert space generated by the Heisenberg field algebra that gets mapped into a set of vectors generated by a sub-algebra of the light-front Fock algebra where the power series for the $x^+$ evolution operator converges (71). This is because the mapping to the sub-algebra enforces boundary conditions that eliminate light-like momenta.

The problem of rotational covariance

The Hamiltonian formulation of light-front quantum field theory defines infinitesimal generators of the Poincaré group by integrating the energy momentum and angular momentum tensors over the light front. This results in formal expressions for the Poincaré generators expressed in term of fields on the light front and derivatives of fields tangent to the light front. These formal expression are classical. When the fields are replaced by operator valued distributions the expression for the dynamical generators become ill-defined. These have to be renormalized in a manner that preserves the commutation relations and agrees with the representation obtained by mapping the covariant representations of the generators to the light-front Fock algebra (see (57)).

The light-front Hamiltonian, $P^-$ and the generators of the kinematic subgroup form a closed Lie algebra which does not uniquely determine the transverse rotation properties of the theory. The light-front Hamiltonian is only defined after the formal expression for the light-front Hamiltonian is renormalized.

The transverse rotational properties can be fixed by either replacing the light-front Hamiltonian by one transverse rotation generator or by demonstrating the invariance of the predictions of the theory by changing the orientation of the light front (see (87)). In the first case there are non-linear constraints on acceptable transverse rotation generators while in the second case the equivalence requires matching correct scattering asymptotic conditions in theories with different light fronts. This reduces to the problem of assigning the total spin to degenerate states with the same magnetic quantum numbers.

In light-front Hamiltonian forms of quantum field theories the problem of rotational invariance is connected with the problem of how to consistently renormalize both the $p \to \infty$ and $p^+ \to 0$ divergences outside of perturbation theory.

In perturbation theory the rotational properties of the light-front formulation follow from the mapping from the Heisenberg algebra to the sub-algebra of the light-front Fock algebra. In perturbation theory there are no composite states; the spins are fixed by the spinor representations of the fields. The rotational properties can be determined by comparing light-front perturbation theory to covariant perturbation theory. The problem with rotational covariance is related to the problem of assigning spins for degenerate composite states. This is relevant outside of perturbation theory.

Rotational covariance remains a problem for non-perturbative applications.

The problem of zero modes

The light-front Hamiltonian and the kinematic generators do not fix the transverse rotational properties of the theory. This means that in order to get a light-front dynamics that is equivalent to the covariant form of the theory, additional information that is not contained in the light-front Hamiltonian is needed.

Rotational invariance is equivalent to invariance with respect to changing the orientation of the light front. The formal expressions for the Poincaré generators constructed using Noether's theorem are generally ill-defined. They have both $p \to \infty$ and $p^+ \to 0$ divergences. These divergence mix under changing orientation of the light front. This means that the in order ensure rotational covariance the $p \to \infty$ and $p^+ \to 0$ renormalizations cannot be done independently. This means that some of the information missing from the light-front Hamiltonian is how to consistently treat the $p \to \infty$ and $p^+ \to 0$ renormalization. Similar problems occur with the space reflection operator, which already appears in $1+1$ dimensional applications.

This indicates that the renormalization of the $p^+ \to 0$ divergences and $p \to \infty$ divergences in the light-front Hamiltonian are constrained by both rotational covariance and space reflection invariance. This requires information that is not contained in the formal expressions for the light-front Hamiltonian. A complete treatment requires a non-perturbative renormalization.

In the perturbative case the connection with covariant formulations can be used to determine the needed $p^+ \to 0$ corrections to light-front perturbation theory [58].

The problem of spontaneous symmetry breaking

The signal for spontaneous symmetry breaking is the presence of a massless Goldstone boson in the mass spectrum. While it is usually expressed in terms of the charge operator, the charge operator is the current density evaluated at a point $t = 0$, $\mathbf{p} = 0$ in a mixed time momentum representation. It is an operator valued distribution and there is no reason to expect that it exists. There is an unambiguous non-perturbative condition for the presence of a Goldstone boson that involves that vacuum expectation value of the commutator of a field and a current density integrated over a ball of finite volume in the limit that the volume $\to \infty$. This works because locality eventually cuts off the integral when the ball gets sufficiently large, so there is never an integral over infinite volume. This cannot be done with the corresponding light-front charge operator because locality is not available to cutoff the integral.

Vacuum expectation values of the commutator of a cutoff current at fixed time with a field smeared over a compact region can be mapped to the light-front sub-algebra. The non-vanishing of the light-front vacuum expectation values of operator in the limit of sufficiently large cutoff will be evidence of a Goldstone boson. While this can in principle be formulated in the light front sub-algebra, it is not the same as using the light-front charge operator.

Because spontaneous symmetry breaking is a non-pertrubative dynamical phenomena, it is not surprising that it does not manifest itself on the light-front Fock algebra. This consistent with treatments that formulate spontaneous symmetry breaking in terms of light-front limits of Heisenberg fields [39] or in term of commutators with dynamical generators [50].

The covariant and light-front formulations are two equivalent representations of quantum field theory. The vacuum in both theories is the same and non-trivail, however when the fields are properly mapped to the light front, the vacuum is unitarily equivalent to the vacuum any free field vacuum. This means that the vacuum can be treated as a trivial vacuum for calculations on the light-front. Noether's theorem on the light front (see Appendix 2) gives formal expressions for the Poincaré generators as operators in the light-front Fock algebra. The formal expression for these operators are ill-defined and require a consistent renormalization. These operators generate automorphisms in the light-front Fock algebra that can be used to build the $x^+$ dependence of the fields.

An important advantage of the light-front representation over the covariant representation is that it has a natural Hamiltonian formulation, which implies that many non-perturbative problems can be formally reduced to linear algebra. What is missing in the light-front case is a non-perturbative way to renormalize the theory in a manner that is consistent with rotational covariance and space reflection symmetry. Assuming that this program can be performed, this work suggests that the vacuum can still be treated as trivial. Issues with the dynamics and 0 modes appear in the process of removing the divergences and renormalizing the theory consistent with rotational covariance and space reflection invariance.

This research was supported by the US Department of Energy, Office of Science, grant number DE-SC0016457. The author would like to thank John Collins, Stan Brodsky, , Chueng Ji, and Peter Lowdon for useful comments on this work.

## XII. APPENDIX 1

Light front conventions. The light-front components of the spacetime coordinate $x$ are

$$x^{\pm} = x^0 \pm x^3 = -(x_0 \mp x_3) \qquad \mathbf{x}_\perp = (x_1, x_2)$$

$$x^0 = \frac{1}{2}(x^+ + x^-) \qquad x^3 = \frac{1}{2}(x^+ - x^-).$$

With these conventions the Lorentz invariant scalar product of two four vectors is

$$x \cdot p = \mathbf{x}_\perp \cdot \mathbf{p}_\perp - \frac{1}{2}(x^- p^+ + x^+ p^-).$$

The covariant components of $x^\mu$ are

$$x_+ = -\frac{1}{2}x^- \qquad x_- = -\frac{1}{2}x^+$$

so

$$x_+ x^+ + x_- x^- + x_1 x^1 + x_2 x^2 = -\frac{1}{2}x^+ x^- - \frac{1}{2}x^- x^+ + \mathbf{x}_\perp \cdot \mathbf{x}_\perp = -x^{02} + \mathbf{x} \cdot \mathbf{x}.$$

The 4-volume element and partial derivatives are

$$d^4x = \frac{1}{2}dx^+ dx^- d^2\mathbf{x}_\perp$$

$$\frac{\partial}{\partial x^0} = \frac{\partial x^+}{\partial x^0}\frac{\partial}{\partial x^+} + \frac{\partial x^-}{\partial x^0}\frac{\partial}{\partial x^-} = \frac{\partial}{\partial x^+} + \frac{\partial}{\partial x^-} \qquad \frac{\partial}{\partial x^3} = \frac{\partial x^+}{\partial x^3}\frac{\partial}{\partial x^+} + \frac{\partial x^-}{\partial x^3}\frac{\partial}{\partial x^-}\frac{\partial}{\partial x^+} - \frac{\partial}{\partial x^-}.$$

With these conventions the Lagrangian density for a scalar field theory has the form

$$\mathcal{L} = 2\partial_+(x)\phi\partial_-\phi(x) - \frac{1}{2}\partial_i(x)\phi\partial_i\phi(x) - \frac{m^2}{2}\phi^2(x) - V(\phi(x)).$$

## XIII. APPENDIX 2

Poincaré invariance of the action leads to formal expressions for the infinitesimal generators of the Poincaré group using Noether's theorem. The conserved currents are the energy momentum and angular momentum tensors

$$T^{\mu\nu}(x) = \eta^{\mu\nu}\mathcal{L} - (\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))})\partial_\alpha\phi(x)\eta^{\alpha\nu}) \tag{104}$$

$$M^{\mu\alpha\beta} = T^{\mu\beta}x^\alpha - T^{\mu\alpha}x^\beta. \tag{105}$$

Poincaré invariance of the action and Lagrange's equation implies that these currents are conserved:

$$\partial_\mu T^{\mu\nu}(x) = 0 \qquad \text{and} \qquad \partial_\mu M^{\mu\alpha\beta}(x) = 0. \tag{106}$$

For a scalar field with Lagrangian density

$$\mathcal{L}(x) = -\frac{1}{2}\partial^\mu\phi(x)\partial_\mu\phi(x) - \frac{m^2}{2}\phi(x)^2 - V(\phi(x)). \tag{107}$$

The field equations can be derived from the principle of stationary action

$$-\partial^\mu\partial_\mu\phi(x) + m^2\phi(x) + \frac{\partial V(\phi(x))}{\partial\phi(x)} = 0. \tag{108}$$

The Lagrangian density and field equations can be expressed in terms of light front variables as

$$\mathcal{L}(x) = 2\partial_-\phi(x)\partial_+\phi(x) - \frac{1}{2}\partial^i\phi(x)\partial_i\phi(x) - \frac{m^2}{2}\phi(x)^2 - V(\phi(x)). \tag{109}$$

and

$$4\partial_-\partial_+\phi(x) - \partial^i\partial_i\phi(x) + m^2\phi(x) + \frac{\partial V(\phi(x))}{\partial\phi(x)} = 0 \tag{110}$$

The Noether charges are constructed by integrating the + component of Noether currents over the $x^+ = 0$ light front where the + components of the currents are

$$T^{++} = 4 : \partial_-\phi(x)\partial_-\phi(x) : \tag{111}$$

$$T^{+-} = -2\mathcal{L} + 4\partial_-\phi(x))\partial_+\phi(x) =: \boldsymbol{\nabla}_\perp\phi(x)\cdot\boldsymbol{\nabla}_\perp\phi(x) : +m^2 : \phi(x)^2 : +2 : V(\phi(x)) : \tag{112}$$

$$T^{+i} = -2 : \partial_-\phi(x)\partial_i\phi(x) : \tag{113}$$

and

$$M^{+\alpha\beta}(x) = x^\alpha T^{+\beta}(x) - x^\beta T^{+\alpha}(x) = x^\alpha T^{+\beta}(x) - x^\beta T^{+\alpha}(x) \tag{114}$$

Using these expressions the light front Poincaré generators can be expressed in terms of the algebra of field operators restricted to the light front. Note that all derivatives in these expression are normal to the light front:

$$P^+ = \int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{2} T^{++}(x) = \int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{2} 4 : \partial_-\phi(x)\partial_-\phi(x) : \tag{115}$$

$$P^i = \int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{2} T^{+i} = -\int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{2} 2 : \partial_- \phi(x) \partial_i \phi(x) : \qquad i \in \{1, 2\} \tag{116}$$

$$E^i = \int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{2} T^{++} x^i = \int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{2} 4 : \partial_- \phi(x) \partial_- \phi(x) : x^i \tag{117}$$

$$J^3 = \int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{2} \left( x^1 T^{+2}(x) - x^2 T^{+1} \right) =$$

$$\int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{2} \left( -2x^1 : \partial_- \phi(x) \partial_2 \phi(x) : +2x^2 : \partial_- \phi(x) \partial_1 \phi(x) : \right) \tag{118}$$

$$K^3 = \int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{2} T^{++}(x) x^- = \int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{2} 4 : \partial_- \phi(x) \partial_- \phi(x) : x^- \tag{119}$$

The dynamical generators are:

$$P^- = \int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{2} T^{+-}(x) = \tag{120}$$

$$\int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{2} \left( : \boldsymbol{\nabla}_\perp \phi(x) \cdot \boldsymbol{\nabla}_\perp \phi(x) : +m^2 : \phi(x)^2 : +2 : V(\phi(x)) : \right) \tag{121}$$

$$J^1 = \int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{4} \left( x^2 (T^{++}(x) - T^{+-}) + x^- T^{+2} \right) =$$

$$\int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{4} \left( x^2 (4 : \partial_- \phi(x) \partial_- \phi(x) : - : \boldsymbol{\nabla}_\perp \phi(x) \cdot \boldsymbol{\nabla}_\perp \phi(x) : -m^2 : \phi(x)^2 : -2 : V(\phi(x)) :) - x^- 2 : \partial_- \phi(x) \partial_2 \phi(x) : \right) \tag{122}$$

$$J^2 = -\int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{4} \left( x^- T^{+1}(x) + x^1 (T^{++} - T^{+-})) \right) =$$

$$\int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{4} \left( -x^- 2 : \partial_- \phi(x) \partial_1 \phi(x) : \; -x^1 (4 : \partial_- \phi(x) \partial_- \phi(x) : -(: \boldsymbol{\nabla}_\perp \phi(x) \cdot \boldsymbol{\nabla}_\perp \phi(x) : +m^2 : \phi(x)^2 : +2 : V(\phi(x)) :) \right) \tag{123}$$

For the case of free fields these operators can be expressed in terms of light-front creation and annihilation operators

$$P^+ = \int d\tilde{\mathbf{p}} \theta(p^+) a^\dagger(\tilde{\mathbf{p}}) p^+ a(\tilde{\mathbf{p}}) \tag{124}$$

$$P^- = \int d\tilde{\mathbf{p}} \theta(p^+) a^\dagger(\tilde{\mathbf{p}}) \frac{\mathbf{p}_\perp + m^2}{p^+} a(\tilde{\mathbf{p}}) \tag{125}$$

$$P^i = \int d\tilde{\mathbf{p}} \theta(p^+) a^\dagger(\tilde{\mathbf{p}}) p^i a(\tilde{\mathbf{p}}) \tag{126}$$

$$E^i = \int d\tilde{\mathbf{p}} \theta(p^+) a^\dagger(\tilde{\mathbf{p}}) (-ip^+ \frac{\partial}{\partial p^i}) a(\tilde{\mathbf{p}}) \tag{127}$$

$$K^3 \int d\tilde{\mathbf{p}}\theta(p^+)a^\dagger(\tilde{\mathbf{p}})(i\{p^+, \frac{\partial}{\partial p^+}\})a(\tilde{\mathbf{p}}) \tag{128}$$

$$J^3 = \int d\tilde{\mathbf{p}}\theta(p^+)a^\dagger(\tilde{\mathbf{p}})(x^2(i\partial_{p_1}) - x^1(i\partial_{p_2}))a(\tilde{\mathbf{p}}) \tag{129}$$

$$\frac{1}{2}\int d\tilde{\mathbf{p}}\theta(p^+)a^\dagger(\tilde{\mathbf{p}}) = (\{\frac{\mathbf{p}_\perp^2 + m^2}{p^+}, (i\partial_{pi})\})a(\tilde{\mathbf{p}})$$

$$J^1 = \int d\tilde{\mathbf{p}}\theta(p^+)a^\dagger(\tilde{\mathbf{p}})\left(\frac{1}{2}p^+(i\partial_{p_2}) - \frac{1}{4}\{\frac{\mathbf{p}_\perp^2 + m^2}{p^+}, i\partial_{p_2}\} - \frac{1}{2}(2ip^2\partial_{p+})\right)a(\tilde{\mathbf{p}}) \tag{130}$$

$$J^2 = \int d\tilde{\mathbf{p}}\theta(p^+)a^\dagger(\tilde{\mathbf{p}})\left(-\frac{1}{2}p^+(i\partial_{p_1}) + \frac{1}{4}\{\frac{\mathbf{p}_\perp^2 + m^2}{p^+}, i\partial_{p_1}\} + \frac{1}{2}(2ip^1\partial_{p+})\right)a(\tilde{\mathbf{p}}). \tag{131}$$

The three dynamical generators have a dependence on the mass $m$ and only involve derivatives tangent to the light front.


[1] S.-J. Chang and S.-K. Ma, Phys. Rev. **180**, 1506 (1969), URL https://link.aps.org/doi/10.1103/PhysRev.180.1506.
[2] J. B. Kogut and D. E. Soper, Phys. Rev. D **1**, 2901 (1970), URL https://link.aps.org/doi/10.1103/PhysRevD.1.2901.
[3] S.-J. Chang, R. G. Root, and T.-M. Yan, Phys.Rev. **D7**, 1133 (1973).
[4] S.-J. Chang and T.-M. Yan, Phys. Rev. D **7**, 1147 (1973), URL https://link.aps.org/doi/10.1103/PhysRevD.7.1147.
[5] T.-M. Yan, Phys. Rev. D **7**, 1760 (1973), URL https://link.aps.org/doi/10.1103/PhysRevD.7.1760.
[6] T.-M. Yan, Phys. Rev. D **7**, 1780 (1973), URL https://link.aps.org/doi/10.1103/PhysRevD.7.1780.
[7] J. Vary, H. Honkanen, J. Li, P. Maris, S. Brodsky, A. Harindranath, G. de Teramond, P. Sternberg, E. Ng, and C. Yang, Nuclear Physics B - Proceedings Supplements **199**, 64 (2010), ISSN 0920-5632, proceedings of the International Workshop Light Cone 2009 (LC2009): Relativistic Hadronic and Particle Physics, URL https://www.sciencedirect.com/science/article/pii/S0920563210000289.
[8] M. H. Stone, Proceedings of the National Academy of Sciences of the United States of America **vol. 16**, 172–175 (1930).
[9] J. von Neumann, Mathematische Annalen **vol. 104**, 570–578 (1931).
[10] J. von Neumann, Annals of Mathematics **33**, 567–573 (1932).
[11] R. Haag, Mat-Fys. Medd. K. Danske Vidensk. Selsk. **29**, 1 (1955).
[12] S. Schlieder and E. Seiler, Commun. Math. Phys. **25**, 62 (1972).
[13] S. Weinberg (2009), 0903.0568.
[14] S. Weinberg, Phys. Rev. **158**, 1638 (1967), URL https://link.aps.org/doi/10.1103/PhysRev.158.1638.
[15] H. Araki, J. Math. Phys. **1**, 492 (1964).
[16] F. Coester and R. Haag, Phys. Rev. **117**, 1137 (1960).
[17] V. Karmanov, Sov. Phys. JETP **44**, 210 (1976).
[18] V. Karmanov, Sov. Phys. JETP **48**, 598 (1976).
[19] V. Karmanov, Sov. Phys. JETP **76**, 1884 (1979).
[20] V. Karmanov, Sov. Phys. JETP **49**, 954 (1979).
[21] V. Karmanov, Nucl. Phys. B **166**, 378 (1980).
[22] M. Fuda, Annals Phys. **231**, 1 (1994).
[23] M. Fuda, Annals Phys. **197**, 265 (1990).
[24] W. N. Polyzou, Few Body Syst. **27**, 57 (1999).
[25] T. Maskawa and K. Yamawaki, Prog. Theor. Phys. **56**, 270 (1976).
[26] K. Yamawaki (1998), hep-th/9802037.
[27] H.-M. Choi and C.-R. Ji, Phys. Rev. D **58**, 071901(R) (1998), URL https://link.aps.org/doi/10.1103/PhysRevD.58.071901.
[28] H. Leutwyler, J. R. Klauder, and L. Streit, Nuovo Cim. **A66**, 536 (1970).
[29] F. Rohrlich, Acta Phys.Austriaca Suppl. **8**, 277 (1971).
[30] N. Nakanishi and H. Yabuki, Lett. Math. Phys. **1**, 371 (1977).
[31] N. Nakanishi and K. Yamawaki, Nucl. Phys. **B122**, 15 (1977).
[32] H. Leutwyler and J. Stern, Ann. Phys. **112**, 94 (1978).
[33] F. Coester, Progress in Particle and Nuclear Physics **29**, 1 (1992).
[34] F. Coester and W. Polyzou, Found. Phys. **24**, 387 (1994), nucl-th/9305005.
[35] K. G. Wilson, T. S. Walhout, A. Harindrarath, W.-M. Zhang, R. J. Perry, and S. D. Glazek, Physical Review D **49**, 6720 (1994).

[36] A. Bylev, J. Phys. G **22**, 1553 (1996).
[37] S. Brodsky, H.-C. Pauli, and S. Pinsky, Physics Reports **301**, 299 (1998).
[38] S. Tsujimaru and K. Yamawaki, Phys.Rev. **D57**, 4942 (1998), hep-th/9704171.
[39] F. Lenz, Nucl. Phys. (Proc. Suppl.) **B90**, 46 (2000).
[40] T. Heinzl, Lec. Notes Ohys. **572**, 55 (2001), hep-th/0008096.
[41] M. Burkardt, F. Lenz, and M. Thies, Phys. Rev. D **65**, 125002 (2002), URL https://link.aps.org/doi/10.1103/PhysRevD.65.125002.
[42] P. P. Srivastava and S. J. Brodsky, Phys. Rev. D **64**, 045006 (2001), URL https://link.aps.org/doi/10.1103/PhysRevD.64.045006.
[43] P. P. Srivastava and S. J. Brodsky, Phys. Rev. D **66**, 045019 (2002), URL https://link.aps.org/doi/10.1103/PhysRevD.66.045019.
[44] T. Heinzl (2003), hep-th/0310165.
[45] P. Ullrich and E. Werner, J. Phys. A **39**, 6057 (2006).
[46] L. Martinovič and P. Grange, Modern Physics Letters A **23**, 417 (2008).
[47] B. L. Bakker, Few Body Syst. **49**, 177 (2011).
[48] H.-M. Choi and C.-R. Ji, Phys. Lett. B **696**, 518 (2011), 1101.3035.
[49] H.-M. Choi and C.-R. Ji, Few Body Syst. **55**, 435 (2014), 1311.0552.
[50] S. R. Beane, Annals Phys. **337**, 111 (2013), 1302.1600.
[51] S. S. Chabysheva and J. R. Hiller, Annals of Physics **340**, 188 (2014), ISSN 0003-4916.
[52] M. Herrmann and W. N. Polyzou, Phys. Rev. D **91**, 085043 (2015), 1502.01230.
[53] S. J. Brodsky, G. F. de Téramond, H. G. Dosch, and J. Erlich, Physics Reports **584**, 1 (2015), ISSN 0370-1573, light-front holographic QCD and emerging confinement, URL https://www.sciencedirect.com/science/article/pii/S0370157315002306.
[54] J. Hiller, Prog. Part. Nucl. Phys. **90**, 75 (2016), hep-th/1606.08348.
[55] C.-R. Ji, Few Body Syst. **58**, 42 (2017).
[56] C.-R. Ji, Z. Li, B. Ma, and A. T. Suzuki, Phys. Rev. D **98**, 036017 (2018), URL https://link.aps.org/doi/10.1103/PhysRevD.98.036017.
[57] J. Collins (2018), 1801.03960.
[58] P. D. Mannheim, P. Lowdon, and S. J. Brodsky, Phys. Rept. **891**, 1 (2021), 2005.00109.
[59] B. Ma and C.-R. Ji, Phys. Rev. D **104**, 036004 (2021), URL https://link.aps.org/doi/10.1103/PhysRevD.104.036004.
[60] W. N. Polyzou, Phys. Rev. **D103**, 105017 (2021).
[61] S. J. Brodsky, A. Deur, and C. Roberts, Nature Rev. Phys. **4**, 489 (2022).
[62] I. M. Gelfand, G. E. Shilov, M. I. Graev, N. Y. Vilenkin, and I. I. Pyatetskii-Shapiro, *Generalized functions*, AMS Chelsea Publishing (Academic Press, New York, NY, 1964), trans. from the Russian, Moscow, 1958, URL https://cds.cern.ch/record/105396.
[63] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That* (Princeton Landmarks in Physics, 1980).
[64] R. Haag, Phys. Rev. **112**, 669 (1958).
[65] W. Brenig and R. Haag, Fort. der Physik **7**, 183 (1959).
[66] D. Ruelle, Helv. Phys. Acta. **35**, 147 (1962).
[67] R. Jost, *The General Theory of Quantized Fields (Lectures in Applied Mathematics, Volume IV* (American Mathematical Society, Providence, Rhode Island, 1965., 1965).
[68] F. Strocchi, *An Introduction to Non-Perturbative Foundations of Quantum Field Theory* (Oxford University Press, 2013).
[69] M. Reed and B. Simon, *Methods of Modern mathematical Physics*, vol. III Scattering Theory (Academic Press, 1979).
[70] V. Glaser, H. Lehmann, and W. Zimmermann, Il Nuovo Cimento **6**, 1122 (1957).
[71] W. O. Greenberg, Phys. Rev. **139**, B1038 (1965).
[72] K. Nishijima, Phys. Rev. **111**, 995 (1958).
[73] O. W. Greenberg, Phys. Rev. D **17**, 2576 (1978).
[74] S. Coleman, *Quantum Field Theory, Lectures of Sidney Coleman* (World Scientific Pub., Hackensack, New Jersey, 2019).
[75] E. P. Wigner, Annals Math. **40**, 149 (1939).
[76] M. Reed and B. Simon, *Methods of Modern mathematical Physics*, vol. I Functional Analysis (Academic Press, 1972).
[77] P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).
[78] S. N. Sokolov and A. N. Shatnyi, Theor. Math. Phys. **37**, 1029 (1979).
[79] B. D. Keister and W. N. Polyzou, J. Comput. Phys. **134**, 231 (1997), nucl-th/9611049.
[80] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the theory of quntized fields* (Wiley, 1959).
[81] H. Araki, *Mathematical theory of quantum fields* (Oxford University Press, 1999).
[82] W. Zimmermann, Il Nuove Cimento **X**, 597 (1958).
[83] C. Yang and D. Feldman, Phys. Rev. **79**, 972 (1950).
[84] M. Veltman, *Diagrammatica* (Cambridge University Press, Melbourne, Australia, 1994).