

Chiral perturbative relation for neutrino masses in the type-I seesaw mechanism

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In this letter, we perform a perturbative analysis by the lightest singular value m_{D1} of the Dirac mass matrix m_D in the type-I seesaw mechanism. A mass relation $M_1 = m_{D1}^2/|(m_\nu)_{11}|$ is obtained for the lightest mass M_1 of the right-handed neutrino ν_{R1} and the mass matrix of the left-handed neutrinos m_ν in the diagonal basis of m_D . This relation is rather stable under renormalization because it is gauge-invariant in the SM and associates with the approximate chiral symmetry of ν_{R1} .

If diagonalization of the Yukawa matrices of leptons $Y_{\nu,e}$ has only small mixings, the element $(m_\nu)_{11}$ is close to the effective mass m_{ee} of the neutrinoless double beta decay. By assuming $m_{D1} \simeq m_{u,e} \simeq 0.5$ MeV, the lightest mass is about $M_1 \gtrsim O(100)$ TeV in the normal hierarchy and $M_1 \sim O(10)$ TeV in the inverted hierarchy. Such a ν_{R1} with a tiny Yukawa coupling $y_{\nu 1} \sim O(10^{-5})$ can indirectly influence various observations.

On the other hand, the famous bound of the thermal leptogenesis $M_1 \gtrsim 10^9$ GeV that requires $m_{D1} \gtrsim 30$ MeV seems to be difficult to reconcile with a simple unified theory without a special condition.

I. INTRODUCTION

Chiral symmetries have played an important role in the development of particle physics [1–3]. Even in the flavor physics, chiral symmetries are associated with small fermion masses of the Standard Model (SM). Due to the tiny singular values of the first generation, diagonalized Yukawa matrices $Y_{u,d,e}^{\text{diag}}$ have approximate chiral symmetries $U(1)_{1L} \times U(1)_{1R}$;

$$R(\theta_L) Y_{u,d,e}^{\text{diag}} R(\theta_R) \simeq Y_{u,d,e}^{\text{diag}}, \quad (1)$$

where $R(\theta) \equiv \text{diag}(e^{i\theta}, 1, 1)$ and $\theta_{L,R}$ are arbitrary real parameters.

Breaking of these chiral symmetries is sufficiently small because the ratios of SM fermion masses m_f for a given renormalization scale are about $O(10^{-2})$ [4],

$$\frac{m_u}{m_c} \sim \frac{1}{500}, \quad \frac{m_d}{m_s} \sim \frac{1}{20}, \quad \frac{m_e}{m_\mu} \sim \frac{1}{200}. \quad (2)$$

With some grand unified theories in mind, it is natural to assume that the Dirac mass matrix m_D also has such approximate chiral symmetries. Thus, in this letter, we perform a chiral perturbative analysis [5] by the smallest singular value m_{D1} of m_D in the type-I seesaw mechanism [6–9].

II. CHIRAL PERTURBATIVE ANALYSIS BY m_{D1}

For the singular value decomposition (SVD) $m_D = U_D m_D^{\text{diag}} V_D^T$, it does not lose generality to consider the diagonal basis of m_D by field redefinitions of unitary matrices U_D and V_D [10]. If the mass matrix m_ν of left-handed neutrinos ν_{Li} is regular and invertible, the mass matrix M_R of right-handed neutrinos

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ν_{Ri} in the type-I seesaw mechanism is reconstructed as

$$M_R = m_D^T m_\nu^{-1} m_D = \begin{pmatrix} m_{D1} & 0 & 0 \\ 0 & m_{D2} & 0 \\ 0 & 0 & m_{D3} \end{pmatrix} m_\nu^{-1} \begin{pmatrix} m_{D1} & 0 & 0 \\ 0 & m_{D2} & 0 \\ 0 & 0 & m_{D3} \end{pmatrix}, \quad (3)$$

where m_{Di} are the singular values of m_D .

In the limit of $m_{D1} \rightarrow 0$, m_D and M_R have $U(1)_{1R}$ chiral symmetry associated with the lepton number of ν_{R1} [11, 12];

$$m_D R(\theta_R) = m_D, \quad M_R R(\theta_R) = M_R. \quad (4)$$

These conditions are equivalent to m_D and M_R having no eigenvector components in the massless modes [13];

$$m_D \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = M_R \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (5)$$

Conversely, if M_R has a certain chiral symmetry, m_D has the same symmetry [13]. The proof is as follows. SVDs of two matrices (with rank two) $M_R = V M_R^{\text{diag}} V^T$ and $m_D = U_D m_D^{\text{diag}} V_D^T$ lead to

$$V \begin{pmatrix} 0 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix} V^T = V_D \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_{D2} & 0 \\ 0 & 0 & m_{D3} \end{pmatrix} U_D^T m_\nu^{-1} U_D \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_{D2} & 0 \\ 0 & 0 & m_{D3} \end{pmatrix} V_D^T, \quad (6)$$

where M_i are singular values of M_R . By performing production of matrices between two m_D^{diag} ,

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix} = V^\dagger V_D \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} V_D^T V^*, \quad (7)$$

where * denotes any matrix element. Since this is also a SVD of M_R , $V^\dagger V_D$ must be a unitary matrix in the 2-3 subspace;

$$V^\dagger V_D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}. \quad (8)$$

Therefore, the first eigenvector of V and V_D coincide in this limit, and the two mass matrices share the same chiral symmetry. Of course this $U(1)_{1R}$ symmetry must be broken because the massless ν_{R1} contradicts observations.

On the other hand, it appears unreasonable that the kernels of m_D and M_R , which can be given arbitrarily in a model, must coincide. This point can rather be considered as a constraint on the seesaw mechanism. In the basis where M_R is diagonal, a parameterization of m_D

$$m_D = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix} \equiv (\mathbf{A}, \mathbf{B}, \mathbf{C}), \quad (9)$$

yields the natural representation [14] of m_ν given by

$$m_\nu = m_D M_R^{-1} m_D^T = \frac{1}{M_1} \mathbf{A} \otimes \mathbf{A}^T + \frac{1}{M_2} \mathbf{B} \otimes \mathbf{B}^T + \frac{1}{M_3} \mathbf{C} \otimes \mathbf{C}^T. \quad (10)$$

If the kernels of m_D and M_R do not coincide, such as limits $A_i \rightarrow 0$ and $M_2 \rightarrow 0$, the matrix m_ν have two extremely different mass scales. If we expect m_ν to be non-hierarchical, magnitudes of $A_i A_j / M_1$

and $B_i B_j / M_2$ must be comparable. This means that the chiral symmetries associated with the first generation must coincide, and moreover, its breaking parameter ϵ_R is common to some extent;

$$\frac{A_i^2}{B_i^2} \sim \frac{M_1}{M_2} \sim \epsilon_R^2. \quad (11)$$

Next we analyze the mass matrix M_R (3) treating m_{D1} as a perturbation. For $(M_R)_{ij} = m_{Di}(m_\nu^{-1})_{ij}m_{Dj}$, let M_{R0} be the unperturbed mass matrix with $m_{D1} = 0$. Clearly matrix elements of M_{R0} are limited to the 2-3 submatrix, and a unitary matrix V_0 that diagonalizes M_{R0} rotates the subspace.

The full matrix M_R in the diagonalized basis of M_{R0} is

$$V_0^\dagger M_R V_0^* = \begin{pmatrix} (\delta^2 M_R)_{11} & (\delta M_R)_{12} & (\delta M_R)_{13} \\ (\delta M_R)_{12} & M_2^{(0)} & 0 \\ (\delta M_R)_{13} & 0 & M_3^{(0)} \end{pmatrix} \equiv M', \quad (12)$$

where $M_i^{(0)}$ are the singular values of M_{R0} without perturbation and

$$(\delta^2 M_R)_{11} = m_{D1}^2 (m_\nu^{-1})_{11}, \quad (13)$$

$$(\delta M_R)_{1(2,3)} = m_{D1} (m_\nu^{-1})_{12} m_{D2} (V_0)_{2(2,3)}^* + m_{D1} (m_\nu^{-1})_{13} m_{D3} (V_0)_{3(2,3)}^*. \quad (14)$$

From these expressions, corrections of the diagonalization occur in the first order of m_{D1} , and the lightest mass M_1 does in the second order. As long as the smallness of perturbation m_{D1} allows the approximation $M_{2,3} \simeq M_{2,3}^{(0)}$, the diagonalization of Eq. (12) is evaluated in a similar way to the seesaw mechanism,

$$M_1 \simeq \left| (\delta^2 M_R)_{11} - \frac{(\delta M_R)_{12}^2}{M_2^{(0)}} - \frac{(\delta M_R)_{13}^2}{M_3^{(0)}} \right|. \quad (15)$$

This is consistent with a formal solution of perturbed SVD to the second order [13]. Although this expression diverges in the limit of $M_{2,3}^{(0)} \rightarrow 0$, the approximation $M_{2,3}^{(0)} \simeq M_{2,3}$ does not hold in this situation, and the diagonalization must be done correctly. The applicability of this perturbation theory will be discussed later.

This result can also be considered from another viewpoint. Multiplying Eq. (15) by $M_2^{(0)} M_3^{(0)}$ leads to

$$M_1 M_2^{(0)} M_3^{(0)} \simeq |M'_{11} M'_{22} M'_{33} - M'_{12} M'_{21} M'_{33} - M'_{13} M'_{31} M'_{22}| = |\text{Det } M'| = |\text{Det } M_R|. \quad (16)$$

The determinant and the minor determinant $\det M_{R0}$ which is restricted to the heavier two generations are,

$$|\text{Det } M_R| = M_1 M_2 M_3 = \prod_{i=1}^3 m_{Di}^2 |\text{Det } m_\nu^{-1}|, \quad (17)$$

$$|\det M_{R0}| = M_2^{(0)} M_3^{(0)} = \prod_{i=1}^2 m_{Di}^2 |\det m_\nu^{-1}|. \quad (18)$$

The cofactor of the inverse matrix $\det m_\nu^{-1} = (m_\nu)_{11} / \text{Det } m_\nu$ yields

$$M_1 \simeq \left| \frac{\text{Det } M_R}{\det M_{R0}} \right| = m_{D1}^2 \left| \frac{\text{Det } m_\nu^{-1}}{\det m_\nu^{-1}} \right| = \frac{m_{D1}^2}{|(m_\nu)_{11}|}. \quad (19)$$

Therefore, the lightest mass M_1 is expressed by the matrix element of m_ν in the basis where m_D is diagonal. The right-hand side contains a basis-dependent quantity, because the minor determinant is evaluated in a particular basis.

This relation is equivalent to the result of integrating out heavy neutrinos by $\partial \mathcal{L} / \partial \nu_{R2,3} = 0$. Removing the heavy generations like the seesaw mechanism, we obtain

$$M_1 = \left| (M_R)_{11} - \sum_{\alpha, \beta=2}^3 (M_R)_{1\alpha} (M_{R0}^{-1})_{\alpha\beta} (M_R)_{\beta 1} \right| \quad (20)$$

$$= m_{D1}^2 \left| (m_\nu^{-1})_{11} - \sum_{\alpha, \beta=2}^3 (m_\nu^{-1})_{1\alpha} (m_\nu^{-1})_{\alpha\beta}^{-1} (m_\nu^{-1})_{\beta 1} \right| = \frac{m_{D1}^2}{|(m_\nu)_{11}|}, \quad (21)$$

where $(m_\nu^{-1})_{\alpha\beta}^{-1}$ denotes an inverse matrix of (m_ν^{-1}) when it is restricted to the 2-3 submatrix. By the simple normalization $m_{D1} \sim m_{u,d,e} \sim 1 \text{ MeV}$ and $|(m_\nu)_{11}| \sim 1 \text{ meV}$,

$$M_1 \simeq \left(\frac{m_{D1}}{1 \text{ MeV}} \right)^2 \left(\frac{1 \text{ meV}}{|(m_\nu)_{11}|} \right) 10^6 \text{ GeV}. \quad (22)$$

III. APPLICABLE LIMIT OF PERTURBATIVITY

This mass relation diverges in the limit of $(m_\nu)_{11} \rightarrow 0$, indicating that the perturbation theory is not valid for too small $(m_\nu)_{11}$. Since this limit corresponds to $\det m_\nu^{-1} = 0$ and $M_2^{(0)}$ (or $M_3^{(0)}$) = 0, the second (or third) term in Eq. (15) causes the divergence. First, let us consider the applicability of the chiral perturbation theory from general observation. By using dimensionless parameters $\epsilon_{L,R}$ representing the chiral symmetry breaking of m_ν and M_R , Eq. (3) is denoted as

$$M_R = \begin{pmatrix} \epsilon_R^2 & \epsilon_R & \epsilon_R \\ \epsilon_R & * & * \\ \epsilon_R & * & * \end{pmatrix} = \begin{pmatrix} m_{D1} & 0 & 0 \\ 0 & m_{D2} & 0 \\ 0 & 0 & m_{D3} \end{pmatrix} \begin{pmatrix} \epsilon_L^2 & \epsilon_L & \epsilon_L \\ \epsilon_L & * & * \\ \epsilon_L & * & * \end{pmatrix}^{-1} \begin{pmatrix} m_{D1} & 0 & 0 \\ 0 & m_{D2} & 0 \\ 0 & 0 & m_{D3} \end{pmatrix}, \quad (23)$$

where $O(1)$ coefficients are omitted. Because of the constraint $\epsilon_R \epsilon_L \propto m_{D1}$, the smaller breaking parameter ϵ_L yields the larger ϵ_R . Therefore, for the approximate chiral symmetry of M_R to be a good description, the smallness of m_{D1} must not be cancelled by m_ν^{-1} . This property is due to the commutativity of the left and right chiral transformations in the diagonalized basis of m_D .

More specifically, this validity of the perturbation theory can be shown as conditions on the matrix elements. To this end, we examine another mass relation of M_2 by a similar perturbative analysis for $m_{D2} \ll m_{D3}$;

$$M_2 \simeq \frac{|\det m_D|^2 |\det m_\nu^{-1}|}{m_{D3}^2 |(m_\nu^{-1})_{33}|} = m_{D2}^2 \left| \frac{\det m_\nu^{-1}}{(m_\nu^{-1})_{33}} \right| = m_{D2}^2 \left| \frac{(m_\nu)_{11}}{\det' m_\nu} \right|, \quad (24)$$

where \det' denotes a minor determinant restricted to 1-2 submatrix. A normalization for $m_{D2} \sim 100 \text{ MeV}$ leads to

$$M_2 \sim \left(\frac{m_{D2}}{100 \text{ MeV}} \right)^2 \left| \frac{(m_\nu)_{11} 10 \text{ meV}}{\det' m_\nu} \right| 10^9 \text{ GeV}. \quad (25)$$

Indeed, the limit of $m_{11} \rightarrow 0$ yields $M_2 \rightarrow 0$ and breaks the perturbation theory (of the first generation). Conditions to prevent such a breakdown are

$$\frac{M_1}{M_2} \simeq \frac{m_{D1}^2}{m_{D2}^2} \left| \frac{\det' m_\nu}{(m_\nu)_{11}^2} \right| \lesssim 0.1, \quad \frac{m_{D1}}{m_{D2}} \lesssim 0.3 \left| \frac{(m_\nu)_{11}}{\sqrt{\det' m_\nu}} \right|. \quad (26)$$

In order to make M_1 larger, $(m_\nu)_{11}$ and $\det' m_\nu$ in Eq. (26) must be small simultaneously. In this case, the 1-2 submatrix of m_ν approaches singular;

$$m_\nu \sim (m_\nu)_{22} \begin{pmatrix} \epsilon^2 & \epsilon & * \\ \epsilon & 1 & * \\ * & * & * \end{pmatrix}, \quad (27)$$

where $\epsilon \sim |(m_\nu)_{11}|/\sqrt{\det' m_\nu}$ is a small dimensionless parameter and $O(1)$ coefficients are ignored. Since the perturbability for M_1/M_3 provides a similar restriction for the 1-3 element, a heavy M_1 requires that m_ν^{-1} compensates for the smallness of m_{D1} as

$$m_\nu^{-1} \sim \begin{pmatrix} \epsilon^{-2} & \epsilon^{-1} & \delta^{-1} \\ \epsilon^{-1} & * & * \\ \delta^{-1} & * & * \end{pmatrix}, \quad \epsilon \gtrsim \frac{m_{D1}}{m_{D2}}, \quad \delta \gtrsim \frac{m_{D1}}{m_{D3}}, \quad (28)$$

where δ is another small parameter. Eventually, M_1 will only be heavy when approximate chiral symmetries of m_D and m_ν are almost identical;

$$R(\theta_L) m_\nu \simeq m_\nu, \quad R(\theta_L) m_D \simeq m_D. \quad (29)$$

If the breaking parameter $\epsilon(\delta)$ is smaller than $m_{D1}/m_{D2(3)}$, the perturbation theory for $M_1/M_{2(3)}$ is no longer valid. This feature is similar to the discussion around Eq. (11), and it indicates that M_R and/or m_ν share the chiral symmetry associated with m_{D1} .

Moreover, Eq. (29) requires the zero eigenvector \mathbf{v}_0 of m_D is close to that of m_ν . In the basis where m_ν is diagonalized by the MNS matrix, \mathbf{v}_0 is close to $(0, 1, -1)$ or $(-2, 1, 1)$ for the normal hierarchy (NH) or the inverted hierarchy (IH). Since this is equivalent to m_D having no \mathbf{v}_0 component, the form of m_D in the case of NH is

$$m_D \simeq \begin{pmatrix} A_2 & B_2 & C_2 \\ A_2 & B_2 & C_2 \\ A_2 & B_2 & C_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ A_3 & B_3 & C_3 \\ -A_3 & -B_3 & -C_3 \end{pmatrix}. \quad (30)$$

Thus, it is difficult to impose a strong hierarchy of $|(m_D)_{33}| \gg |(m_D)_{ij}|$.

The mass relation (22) leads to significant phenomenological consequences.

1. For the thermal leptogenesis [15] by ν_{R1} , the famous lower limit of the mass $M_1 \gtrsim 10^9$ GeV [16, 17] implies that

$$m_{D1} \gtrsim 30 \text{ MeV}. \quad (31)$$

Therefore, except in a special condition that amplifies the mass M_1 , a simple unified theory with $m_{D1} \sim 1$ MeV and the type-I seesaw mechanism seems to be difficult to reconcile with the thermal leptogenesis by ν_{R1} . This feature is expected in wide parameter regions of many models¹.

2. If diagonalization of the Yukawa matrices of leptons $Y_{\nu,e}$ has only small mixings, the value $(m_\nu)_{11}$ is close to the effective mass m_{ee} of the neutrinoless double beta decay [19]. Although NH has a canceling region $m_{ee} \simeq 0$, there is no chiral symmetry because $m_1 \sim 3$ meV, and the chiral perturbation theory simply breaks down. Since the lepton mass is not susceptible to renormalization, m_{D1} is expected to be about $m_{D1} \simeq 0.5$ MeV from singular values at the GUT scale $m_u \simeq m_e \simeq 0.5$ MeV. Therefore, the lightest mass is about $M_1 \gtrsim O(100)$ TeV in NH and $M_1 \sim O(10)$ TeV in IH.
3. Although the lightest TeV-scale right-handed neutrino ν_{R1} only has a tiny Yukawa coupling $y_{\nu 1} \sim O(10^{-5})$ and a very weak Higgs interaction, such a ν_{R1} may be indirectly involved in the anomaly called IceCube gap [20].

If m_ν has a good chiral symmetry, this relation can be applied for m_ν . A similar discussion for the seesaw formula $m_\nu = m_D^{\text{diag}} M_R^{-1} m_D^{\text{diag}}$ and perturbatively small m_{D1} leads to ,

$$m_1 = \left| \frac{\text{Det } m_\nu}{\det m_{\nu 0}} \right| = m_{D1}^2 \left| \frac{\text{Det } M_R^{-1}}{\det M_R^{-1}} \right| = \frac{m_{D1}^2}{|(M_R)_{11}|}, \quad (32)$$

and $|(M_R)_{11}| \sim 1$ PeV holds for $m_1 \sim 1$ meV. However, we need to be careful about this argument. For a strongly hierarchical M_R with $M_3 \gg M_{1,2}$, the lightest mass m_1 comes from $1/M_3$ by the sequential dominance [21] and Eq. (32) is not the correct relationship. Since M_R seems to be much closer to a singular matrix than m_ν , the mass relation appears safer to consider only for m_ν^{-1} .

In both cases of NH and IH, the eigenvectors associated with the lightest mass $m_{1 \text{ or } 3}$ are not in the direction $(1, 0, 0)$. Since chiral symmetries of left-handed fields seem not to be shared between m_D and m_ν , it is natural to think that the smallness of m_{D1} rather ensures the hierarchy of m_2/m_3 in NH.

Finally, this mass relation must be almost stable against quantum corrections because it is associated with the approximate chiral symmetry [3] of the right-handed neutrino ν_{R1} and the gauge charges of SM cancel out between m_ν and m_D . Thus, this is considered a general constraint on the type-I seesaw mechanism.

¹ Note that such a bound is inconsistent with the existence of long-lived particles [18], and it does not immediately rule out the possibility of leptogenesis.

IV. SUMMARY

In this letter, we perform a perturbative analysis by the lightest singular value m_{D1} of the Dirac mass matrix m_D in the type-I seesaw mechanism. The lightest mass M_1 of the right-handed neutrino ν_{R1} is expressed as $M_1 = m_{D1}^2/|(m_\nu)_{11}|$ by the mass matrix of the left-handed neutrinos m_ν in the diagonal basis of m_D . A similar relationship $M_2 \propto m_{D2}$ is also obtained for the second generation.

This chiral perturbation theory breaks down when m_ν^{-1} cancels the hierarchy of m_D and it corresponds to a situation where an approximate chiral symmetry m_ν and m_D for the left-hand field ν_{L1} is almost identical.

Since $m_{D1} \sim 0.5$ MeV leads to $M_1 \gtrsim O(100)$ TeV for NH and $M_1 \sim O(10)$ TeV for IH, such a light TeV-scale right-handed neutrino with a tiny Yukawa coupling of $y_{\nu 1} \sim O(10^{-5})$ can indirectly influence various observations. On the other hand, the famous bound of the thermal leptogenesis $M_1 \gtrsim 10^9$ GeV that requires $m_{D1} \gtrsim 30$ MeV seems to be difficult to reconcile with a simple unified theory without a special condition.

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- [1] S. Weinberg, Phys. Rev. **166**, 1568 (1968).
 - [2] E. Witten, Nucl. Phys. B **145**, 110 (1978).
 - [3] G. 't Hooft, NATO Adv.Study Inst.Ser.B Phys. **59**, 135 (1980).
 - [4] Z.-z. Xing, H. Zhang, and S. Zhou, Phys. Rev. D **86**, 013013 (2012), arXiv:1112.3112.
 - [5] J. Gasser and H. Leutwyler, Nucl. Phys. B **250**, 465 (1985).
 - [6] P. Minkowski, Phys. Lett. **67B**, 421 (1977).
 - [7] M. Gell-Mann, P. Ramond, and R. Slansky, Conf. Proc. **C790927**, 315 (1979).
 - [8] T. Yanagida, Conf. Proc. **C7902131**, 95 (1979).
 - [9] R. N. Mohapatra and G. Senjanovic, Phys. Rev. Lett. **44**, 912 (1980).
 - [10] T. Endoh, S. Kaneko, S. K. Kang, T. Morozumi, and M. Tanimoto, Phys. Rev. Lett. **89**, 231601 (2002), arXiv:hep-ph/0209020.
 - [11] G. C. Branco, W. Grimus, and L. Lavoura, Nucl. Phys. B **312**, 492 (1989).
 - [12] R. Adhikari and A. Raychaudhuri, Phys. Rev. D **84**, 033002 (2011), arXiv:1004.5111.
 - [13] M. J. S. Yang, (2022), arXiv:2211.15101.
 - [14] V. Barger, D. A. Dicus, H.-J. He, and T.-j. Li, Phys. Lett. B **583**, 173 (2004), arXiv:hep-ph/0310278.
 - [15] M. Fukugita and T. Yanagida, Phys. Lett. **B174**, 45 (1986).
 - [16] S. Davidson and A. Ibarra, Phys. Lett. **B535**, 25 (2002), arXiv:hep-ph/0202239.
 - [17] K. Hamaguchi, H. Murayama, and T. Yanagida, Phys. Rev. **D65**, 043512 (2002), arXiv:hep-ph/0109030.
 - [18] M. Y. Khlopov and A. D. Linde, Phys. Lett. B **138**, 265 (1984).
 - [19] J. D. Vergados, Phys. Rept. **133**, 1 (1986).
 - [20] IceCube, M. G. Aartsen *et al.*, Phys. Rev. Lett. **113**, 101101 (2014), arXiv:1405.5303.
 - [21] S. F. King, Nucl. Phys. B **562**, 57 (1999), arXiv:hep-ph/9904210.