

# NEW DEVELOPMENTS TOWARD THE GONEK CONJECTURE ON THE HURWITZ ZETA-FUNCTION

MASAHIRO MINE

ABSTRACT. In this paper, we prove a version of the universality theorem for the Hurwitz zeta-function in the case where the parameter is algebraic and irrational. Then we apply the result to show that many of such Hurwitz zeta-functions have infinitely many zeros in the right half of the critical strip.

## 1. INTRODUCTION AND STATEMENTS OF RESULTS

Let  $s = \sigma + it$  be a complex variable. Denote by  $\zeta(s, \alpha)$  the Hurwitz zeta-function with a parameter  $0 < \alpha \leq 1$ . It is defined by the Dirichlet series

$$(1.1) \quad \zeta(s, \alpha) = \sum_{n=0}^{\infty} (n + \alpha)^{-s}$$

on the half-plane  $\sigma > 1$  and can be continued meromorphically to the whole complex plane  $\mathbb{C}$ . Throughout this paper, let  $D$  and  $\mathcal{A}$  denote

$$D = \{s \in \mathbb{C} \mid 1/2 < \sigma < 1\},$$

$$\mathcal{A} = \{0 < \alpha < 1 \mid \alpha \text{ is algebraic and irrational}\},$$

and  $\text{meas}\{\cdot\}$  stands for the usual Lebesgue measure of a measurable set  $\{\cdot\}$  in  $\mathbb{R}$ . Gonek conjectured in his thesis [9, p. 122] that  $\zeta(s, \alpha)$  has a universality property for any  $\alpha \in \mathcal{A}$  in the following sense.

**Conjecture (Gonek).** *Let  $\alpha \in \mathcal{A}$ . Let  $K$  be a compact subset of the strip  $D$  with connected complement. Let  $f$  be a continuous function on  $K$  which is analytic in the interior of  $K$ . Then, for every  $\epsilon > 0$ , we have*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] \mid \sup_{s \in K} |\zeta(s + i\tau, \alpha) - f(s)| < \epsilon \right\} > 0.$$

The history of universality is briefly summarized in Section 1.1. Here, we recall that the universality theorem of  $\zeta(s, \alpha)$  has already been proved in the case where  $\alpha$  is transcendental or rational. Furthermore, in the case  $\alpha \in \mathcal{A}$ , some progress was made by Sourmelidis and Steuding [26]. They succeeded to show an effective but weak form of universality of  $\zeta(s, \alpha)$  in the same manner as in [8]. See Theorem 1.3 for the details. The main result of this paper also presents a version of universality of  $\zeta(s, \alpha)$  for any  $\alpha \in \mathcal{A}$ , which is quite different from [26]. It provides another piece of evidence for Gonek's conjecture.

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**Theorem 1.** *Let  $\alpha \in \mathcal{A}$ . Let  $K$  be a compact subset of the strip  $D$  with connected complement. Let  $f$  be a continuous function on  $K$  which is analytic in the interior of  $K$ . Then there exists a sequence  $\{\alpha_k\}$  of elements in  $\mathcal{A}$  depending on  $\alpha, f, K$  with the following property: for every  $\epsilon > 0$ , there exists a number  $k_0(\epsilon)$  such that*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] \mid \sup_{s \in K} |\zeta(s + i\tau, \alpha_k) - f(s)| < \epsilon \right\} > 0$$

and  $|\alpha_k - \alpha| < \epsilon$  for all  $k \geq k_0(\epsilon)$ .

Notice that the sequence  $\{\alpha_k\}$  of Theorem 1 converges to  $\alpha$  as  $k \rightarrow \infty$ . If one can take  $\alpha_k = \alpha$  for any  $k$ , then Gonek's conjecture is true.

Theorem 1 yields a new result on the distribution of zeros of  $\zeta(s, \alpha)$ . The study of zeros of the Hurwitz zeta-function has a long history since the classical work of Davenport and Heilbronn [4]. They proved that  $\zeta(s, \alpha)$  has infinitely many zeros in the half-plane  $\sigma > 1$  if the parameter  $\alpha$  is transcendental or rational with  $\alpha \neq 1, 1/2$ . Then, Cassels [3] extended the result to the case  $\alpha \in \mathcal{A}$ . (Actually, his original proof contains an error, and it is corrected in [21].) Studying the zeros in the strip  $D$  is more difficult since (1.1) is not available. To this day, we know that  $\zeta(s, \alpha)$  has infinitely many zeros in  $D$  only if  $\alpha$  is transcendental or rational with  $\alpha \neq 1, 1/2$ . Garunkštis [7] showed that there exist infinitely many  $\alpha \in \mathcal{A}$  such that the Hurwitz zeta-function  $\zeta(s, \alpha)$  has an arbitrary finite number of zeros in  $D$ . In this paper, we refine the result so that  $\zeta(s, \alpha)$  has infinitely many zeros in  $D$ .

**Theorem 2.** *There exist infinitely many  $\alpha \in \mathcal{A}$  such that the Hurwitz zeta-function  $\zeta(s, \alpha)$  has infinitely many zeros in the strip  $D$ .*

In fact, Theorems 1 and 2 are proved in a stronger form. For example, we obtain a lower bound for the number of the zeros of  $\zeta(s, \alpha)$  in the rectangle  $\sigma_1 \leq \sigma \leq \sigma_2$ ,  $0 \leq t \leq T$  for any  $1/2 < \sigma_1 < \sigma_2 < 1$ . We postpone the exact statements, which appear in Section 1.2.

**1.1. Universality theorems.** The notion of universality in analysis first appeared in the classical result of Fekete reported in [23]. He proved that there exists a real formal power series  $\sum_{n=1}^{\infty} a_n x^n$  with the following property: for every continuous function  $f$  on  $[-1, 1]$  satisfying  $f(0) = 0$ , there exists an increasing sequence  $\{m_k\}$  of positive integers such that

$$\sup_{x \in [-1, 1]} \left| \sum_{n=1}^{m_k} a_n x^n - f(x) \right| \rightarrow 0$$

as  $k \rightarrow \infty$ . Some similar approximation results were also obtained in [2, 18]. Then such a kind of property was named universality by Marcinkiewicz [18]. While these results in [2, 18, 23] do not provide explicit examples of an object with universality, Voronin [29] achieved the universality theorem for the Riemann zeta-function  $\zeta(s)$  as follows.

**Theorem 1.1 (Voronin).** *Let  $0 < r < 1/4$ . Let  $f$  be a non-vanishing continuous function on the disc  $|s| \leq r$  which is analytic in the interior. Then, for every  $\epsilon > 0$ , there exists a positive real number  $\tau = \tau(\epsilon)$  such that*

$$\sup_{|s| \leq r} \left| \zeta \left( s + \frac{3}{4} + i\tau \right) - f(s) \right| < \epsilon.$$

There are several methods for the proof of the universality of  $\zeta(s)$ . See Bagchi [1], Gonek [9], and Good [10]. The Linnik–Ibragimov conjecture vaguely asserts that any reasonable Dirichlet series would have a universality property. Then the universality of  $\zeta(s)$  has been extended to many zeta and  $L$ -functions: Dirichlet  $L$ -functions [29]; Dedekind zeta-functions [24];  $L$ -functions attached to certain cusp forms [17], and so on. For more information, see Matsumoto’s survey [19]. The universality of the Hurwitz zeta-function was proved by Bagchi [1] and Gonek [9] independently, where the parameter is transcendental or rational.

**Theorem 1.2** (Bagchi, Gonek). *Let  $0 < \alpha \leq 1$  be a transcendental or rational number with  $\alpha \neq 1, 1/2$ . Let  $K$  be a compact subset of the strip  $D$  with connected complement. Let  $f$  be a continuous function on  $K$  which is analytic in the interior of  $K$ . Then, for every  $\epsilon > 0$ , we have*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] \mid \sup_{s \in K} |\zeta(s + i\tau, \alpha) - f(s)| < \epsilon \right\} > 0.$$

If  $\alpha$  is transcendental, then the real numbers  $\log(n_1 + \alpha), \dots, \log(n_k + \alpha)$  are linearly independent over  $\mathbb{Q}$  for any distinct integers  $n_1, \dots, n_k \geq 0$ . Therefore the Kronecker–Weyl theorem is available to show that  $((n_1 + \alpha)^{i\tau}, \dots, (n_k + \alpha)^{i\tau})$  is uniformly distributed for  $\tau \in \mathbb{R}$ . This is a key step in the proof of Theorem 1.2 in the transcendental case. If  $\alpha$  is rational, then we have the formula

$$(1.2) \quad \zeta(s, \alpha) = \frac{q^s}{\phi(q)} \sum_{\chi \bmod q} \bar{\chi}(a) L(s, \chi)$$

for  $\alpha = a/q$  with  $a, q \in \mathbb{Z}_{>0}$ , where  $\phi(q)$  denotes Euler’s totient function, and  $\chi$  runs through the Dirichlet characters of modulo  $q$ . Then the hybrid joint universality of Dirichlet  $L$ -functions implies Theorem 1.2 in the rational case.

If  $\alpha \in \mathcal{A}$ , then the linear independence of  $\log(n_1 + \alpha), \dots, \log(n_k + \alpha)$  is not necessary valid, and no formula similar to (1.2) is known. These difficulties prevent us from resolving the universality of  $\zeta(s, \alpha)$  in this case. The result of Sourmelidis and Steuding [26] provided the first progress on this problem. Let  $0 < c < 1$  and  $d \geq 1$ . Then we define the set  $\mathcal{A}(c, d)$  as

$$\mathcal{A}(c, d) = \{\alpha \in \mathcal{A} \mid c \leq \alpha \leq 1 \text{ and } \deg(\alpha) \leq d\},$$

where  $\deg(\alpha)$  denotes the degree of  $\alpha$ . Let  $\eta = 4.45$  and  $\theta = 4/(27\eta^2) \approx 0.00748$ . We determine a positive real number  $\iota$  by the equation  $\iota^3 - 2\iota^2 = 2\theta$ . Then we have  $\iota \approx 2.003$ . Furthermore, we put  $\xi = \theta/\iota^2 \approx 0.00186$ .

**Theorem 1.3** (Sourmelidis–Steuding). *Let  $0 < \nu < \xi$ ,  $1 - \xi + \nu \leq \sigma_0 \leq 1$ , and*

$$\mu \leq \left( \frac{\theta}{1 - \sigma_0 + \nu} \right)^{1/2}.$$

*Let  $f$  be a continuous function on  $\mathcal{K} = \{s \in \mathbb{C} \mid |s - s_0| \leq r\}$  which is analytic in the interior of  $\mathcal{K}$ , where  $s_0 = \sigma_0 + it_0$  with  $t_0 \in \mathbb{R}$  and  $r > 0$ . Let  $0 < c < 1$  and  $d = \mu^2 - \theta/\mu^2 + \nu$ . Then, for every  $\epsilon \in (0, |f(s_0)|)$ , there exists a finite subset  $\mathcal{E} \subset \mathcal{A}(c, d)$  with the following property: for any  $\alpha \in \mathcal{A}(c, d) \setminus \mathcal{E}$ , there exist real numbers  $\tau \in [T, 2T]$  and  $\delta = \delta(\epsilon, f, T) > 0$  such that*

$$\max_{|s - s_0| \leq \delta r} |\zeta(s + i\tau, \alpha) - f(s)| < \epsilon,$$

where  $T = T(\epsilon, f, \alpha)$  is a large real number which can be effectively computable. The subset  $\mathcal{E}$  is described by several effective constants depending on  $\epsilon$  and  $f$ . Lastly, the real number  $\delta$  is also effectively computable by choosing it to satisfy

$$\max_{|s-s_0|=r} |\zeta(s+i\tau, \alpha)| \frac{\delta^N}{1-\delta} \leq \frac{1}{3}(2 - e^{\delta r})\epsilon$$

for sufficiently large  $N$ .

In fact, we can drop the dependence of  $\delta$  on  $T$  if  $r$  satisfies  $0 < r < \sigma_0 - 1/2$ . This was suggested by Y. Lee, and a sketch of his idea was described at the end of Section 4 of [26]. It seems impossible to drop the remaining dependence of  $\delta$  by their method.

**1.2. Statements of results.** Let  $0 < c < 1$ . Then we define the set  $\mathcal{A}_\rho(c)$  as

$$\mathcal{A}_\rho(c) = \{\alpha \in \mathcal{A} \mid |\alpha - c| \leq \rho\},$$

where  $\rho$  satisfies  $0 < \rho < \min\{c, 1 - c\}$ . In this paper, we prove the universality of  $\zeta(s, \alpha)$  for all but finitely many  $\alpha \in \mathcal{A}_\rho(c)$ .

**Theorem 1.4.** *Let  $0 < c < 1$ . Let  $K$  be a compact subset of the strip  $D$  with connected complement. Let  $f$  be a continuous function on  $K$  which is analytic in the interior of  $K$ . Then, for every  $\epsilon > 0$ , there exist a positive real number  $\rho$  and a finite subset  $\mathcal{E} \subset \mathcal{A}$  such that*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] \mid \sup_{s \in K} |\zeta(s+i\tau, \alpha) - f(s)| < \epsilon \right\} > 0$$

for any  $\alpha \in \mathcal{A}_\rho(c) \setminus \mathcal{E}$ , where  $\rho$  and  $\mathcal{E}$  depend on  $c, \epsilon, f, K$ .

This is not an effective result such as Theorem 1.3, but we can extend a type of universality of  $\zeta(s, \alpha)$  to functions on any compact subset  $K$  of  $D$  with connected complement. It is worth noting that  $K$  is independent to  $\epsilon, f, T$  in Theorem 1.4, which is a major improvement from the previous work. The proof of Theorem 1.3 in [26] was derived by incorporating the ideas of Good [10] and Voronin [30]. Then it was necessary to show a joint denseness theorem for the values of derivatives of the Hurwitz zeta-function. See [26, Theorem 1]. To prove Theorem 1.4, we do not need such a result. Instead, we deduce from Theorem 1.4 the following result.

**Theorem 1.5.** *Let  $0 < c < 1$ . Let  $1/2 < \sigma_0 < 1$  and  $\underline{z} = (z_0, \dots, z_N) \in \mathbb{C}^{N+1}$ . Then, for every  $\epsilon > 0$ , there exist a positive real number  $\rho$  and a finite subset  $\mathcal{E} \subset \mathcal{A}$  such that*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] \mid \max_{0 \leq n \leq N} |\zeta^{(n)}(\sigma_0 + i\tau, \alpha) - z_n| < \epsilon \right\} > 0$$

for any  $\alpha \in \mathcal{A}_\rho(c) \setminus \mathcal{E}$ , where  $\rho$  and  $\mathcal{E}$  depend on  $c, \epsilon, \sigma_0, \underline{z}$ .

Another consequence of Theorem 1.4 concerns the estimate for the number of zeros of  $\zeta(s, \alpha)$ . Denote by  $N_\alpha(\sigma_1, \sigma_2, T)$  the number of the zeros of  $\zeta(s, \alpha)$  in the rectangle  $\sigma_1 \leq \sigma \leq \sigma_2, 0 \leq t \leq T$  counted with multiplicity. Then an upper bound for  $N_\alpha(\sigma_1, \sigma_2, T)$  is well-known for any  $0 < \alpha \leq 1$ . Indeed, we apply Littlewood's lemma [27, p. 220] to obtain

$$N_\alpha(\sigma_1, \sigma_2, T) \ll T$$

as  $T \rightarrow \infty$  for any  $1/2 < \sigma_1 < \sigma_2 < 1$ . On the other hand, we have a lower bound for  $N_\alpha(\sigma_1, \sigma_2, T)$  of the same magnitude for all but finitely many  $\alpha \in \mathcal{A}_\rho(c)$ .

**Theorem 1.6.** *Let  $0 < c < 1$  and  $1/2 < \sigma_1 < \sigma_2 < 1$ . Then there exist a positive real number  $\rho$  and a finite subset  $\mathcal{E} \subset \mathcal{A}$  such that*

$$N_\alpha(\sigma_1, \sigma_2, T) \gg T$$

as  $T \rightarrow \infty$  for any  $\alpha \in \mathcal{A}_\rho(c) \setminus \mathcal{E}$ , where  $\rho$  and  $\mathcal{E}$  depend on  $c, \sigma_1, \sigma_2$ .

Here, we check that the above results imply Theorems 1 and 2 described before. Let  $\alpha, f, K$  be as in the statement of Theorem 1. Then we apply Theorem 1.4 to see that there exist a positive real number  $\rho$  and a finite subset  $\mathcal{E} \subset \mathcal{A}$  such that

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] \mid \sup_{s \in K} |\zeta(s + i\tau, \alpha') - f(s)| < \frac{1}{k} \right\} > 0$$

for any  $\alpha' \in \mathcal{A}_\rho(\alpha) \setminus \mathcal{E}$ , where  $\rho$  and  $\mathcal{E}$  depend on  $\alpha, f, K$ , and  $k$ . For any  $k \geq 1$ , we can take an element  $\alpha_k$  in  $\mathcal{A}_\rho(\alpha) \setminus \mathcal{E}$  with  $|\alpha - \alpha_k| < 1/k$  since  $\mathcal{E}$  is a finite set. Therefore, Theorem 1 follows if we take  $k_0(\epsilon)$  so that  $k_0(\epsilon) \geq 1/\epsilon$  is satisfied. Let  $1/2 < \sigma_1 < \sigma_2 < 1$ . To deduce Theorem 2, we note that the number of the zeros of  $\zeta(s, \alpha)$  in  $D$  is greater than  $N_\alpha(\sigma_1, \sigma_2, T)$  for any  $T > 0$ . Since  $N_\alpha(\sigma_1, \sigma_2, T) \rightarrow \infty$  as  $T \rightarrow \infty$  for infinitely many  $\alpha \in \mathcal{A}$  by Theorem 1.6, we obtain the conclusion.

**1.3. Key ideas for the proof.** Bagchi's method for the proof of the universality of  $\zeta(s)$  in [1] begins with constructing a certain random element  $\zeta(s, \mathbb{X})$  related to the distribution of  $\zeta(s)$ . Then the probability measure

$$(1.3) \quad P_T(A) = \frac{1}{T} \text{meas} \{ \tau \in [0, T] \mid \zeta(s + i\tau) \in A \}$$

converges weakly as  $T \rightarrow \infty$  to the law of  $\zeta(s, \mathbb{X})$ . Furthermore, by showing that any function  $f$  with nice properties is approximated by  $\zeta(s, \mathbb{X})$  with positive probability, we obtain the universality theorem of  $\zeta(s)$ .

The proof of Theorem 1.4 is derived by a modification of Bagchi's method. First, we construct a random Dirichlet series  $\zeta(s, \mathbb{X}_\alpha)$  related to the distribution of  $\zeta(s, \alpha)$ , according to the method of the previous paper [20]. Then we prove that a probability measure defined analogously to (1.3) for  $\zeta(s, \alpha)$  converges weakly as  $T \rightarrow \infty$  to the law of  $\zeta(s, \mathbb{X}_\alpha)$ . See Theorem 3.1. Then Gonek's conjecture would follow if we could show that  $f$  is approximated by  $\zeta(s, \mathbb{X}_\alpha)$  with positive probability.

Let  $\zeta_N(s, \mathbb{X}_\alpha)$  denote a random Dirichlet polynomial obtained as a partial sum of  $\zeta(s, \mathbb{X}_\alpha)$ . It is quite hard to study  $\zeta(s, \mathbb{X}_\alpha)$  and  $\zeta_N(s, \mathbb{X}_\alpha)$  for  $\alpha \in \mathcal{A}$  due to the lack of the linear independence of  $\log(n_1 + \alpha), \dots, \log(n_k + \alpha)$  as described before. For this reason, we introduce another random Dirichlet polynomial  $\zeta_N(s, \mathbb{Y}_\alpha)$  to show that any function  $f$  with nice properties is approximated by  $\zeta_N(s, \mathbb{Y}_\alpha)$  with positive probability. See Theorem 4.1 and Corollary 4.9. Then, we observe that the error in a replacement of  $\zeta_N(s, \mathbb{X}_\alpha)$  with  $\zeta_N(s, \mathbb{Y}_\alpha)$  can be ignored in some sense except for a finite number of  $\alpha \in \mathcal{A}$ . This is a key idea in this paper for studying  $\zeta(s, \mathbb{X}_\alpha)$  and  $\zeta_N(s, \mathbb{X}_\alpha)$ . The finite subset  $\mathcal{E}$  in the statement of Theorem 1.4 comes from the exceptional  $\alpha \in \mathcal{A}$  in the replacement of  $\zeta_N(s, \mathbb{X}_\alpha)$  with  $\zeta_N(s, \mathbb{Y}_\alpha)$ .

**1.4. Organization of the paper.** The main content of this paper is to establish Theorem 1.4. The proof is divided into the following three parts.

1. *Limit theorem for  $\zeta(s, \alpha)$  in the function space.* (Section 2 & Section 3)

Let  $H(D)$  be the space of analytic functions on  $D$ . The first step to the proof of universality is to study the distribution of  $\zeta(s + i\tau, \alpha)$  for  $\tau \in \mathbb{R}$  in the space  $H(D)$ . For this, we introduce in Section 2 a sequence  $\{\mathbb{X}_\alpha(n)\}$  of random variables valued on the unit circle of  $\mathbb{C}$  such that  $\mathbb{X}_\alpha(n_1), \dots, \mathbb{X}_\alpha(n_k)$  are independent if and only if the real numbers  $\log(n_1 + \alpha), \dots, \log(n_k + \alpha)$  are linearly independent over  $\mathbb{Q}$ . Then we define a random Dirichlet series  $\zeta(s, \mathbb{X}_\alpha)$  valued on  $H(D)$  by using  $\{\mathbb{X}_\alpha(n)\}$ . In Section 3, we prove the limit theorem (Theorem 3.1) which asserts that a probability measure defined analogously to (1.3) for  $\zeta(s, \alpha)$  converges weakly to the law of  $\zeta(s, \mathbb{X}_\alpha)$ .

2. *Covering theorem for the Bergman space.* (Section 4)

Let  $A^2(U)$  be the Bergman space for a bounded domain  $U$  with  $\overline{U} \subset D$ . We introduce another sequence  $\{\mathbb{Y}_\alpha(n)\}$  of random variables valued on the unit circle of  $\mathbb{C}$ , which satisfies that  $\mathbb{Y}_\alpha(n_1), \dots, \mathbb{Y}_\alpha(n_k)$  are independent for any distinct integers  $n_1, \dots, n_k \geq 0$ . Then we define a random Dirichlet polynomial  $\zeta_N(s, \mathbb{Y}_\alpha)$  valued on  $A^2(U)$  by using  $\{\mathbb{Y}_\alpha(n)\}$ . In Section 4, we prove a covering theorem (Theorem 4.1) which asserts that  $A^2(U)$  is covered by neighborhoods of certain sets of Dirichlet polynomials related to  $\zeta_N(s, \mathbb{Y}_\alpha)$ . This theorem yields that any function  $f \in A^2(U)$  can be approximated by  $\zeta_N(s, \mathbb{Y}_\alpha)$  with positive probability (Corollary 4.9), if  $\rho$  is sufficiently small and  $N$  is sufficiently large.

3. *Estimate of a conditional square mean value.* (Section 5)

Corollary 4.9 implies that any  $f \in A^2(U)$  is approximated by the Dirichlet polynomial  $\zeta_N(s, \mathbb{Y}_\alpha)(\omega_0)$  with some sample  $\omega_0$ . Then we define an event  $\Omega_0$  so that each of  $\mathbb{X}_\alpha(n)$  is close to  $\mathbb{Y}_\alpha(n)(\omega_0)$  for  $0 \leq n \leq N$ . By definition,  $f$  can be approximated by  $\zeta_N(s, \mathbb{X}_\alpha)(\omega)$  if  $\omega \in \Omega_0$ . In Section 5, we prove an upper bound for the expected value of  $\|\zeta(s, \mathbb{X}_\alpha) - \zeta_N(s, \mathbb{X}_\alpha)\|^2$  under the condition  $\omega \in \Omega_0$  for  $\alpha \in \mathcal{A} \setminus \mathcal{E}$ , where  $\mathcal{E}$  is a finite subset. The key idea described in Section 1.3 is closely related to this result. See Proposition 5.2 and Remark 5.3 for the details. Then we deduce that any function  $f \in H(D)$  is approximated by  $\zeta(s, \mathbb{X}_\alpha)$  with positive probability. Finally, we obtain the desired approximation of  $f$  as in the statement of Theorem 1.4 by applying Theorem 3.1 and the Mergelyan approximation theorem.

Theorems 1.5 and 1.6 are deduced from Theorem 1.4 by standard methods in the theory of universality. The proofs are completed in Section 6.

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## 2. RANDOM DIRICHLET SERIES RELATED TO $\zeta(s, \alpha)$

Let  $H(D)$  denote the space of analytic functions on  $D$  equipped with the topology of uniform convergence on compact subsets. It is known that  $H(D)$  is a complete

metric space by the distance  $d$  defined as follows. Let

$$(2.1) \quad K_\nu = \left\{ s \in \mathbb{C} \mid \frac{1}{2} + \frac{1}{5\nu} \leq \sigma \leq 1 - \frac{1}{5\nu}, \quad -\nu \leq t \leq \nu \right\}$$

for  $\nu \geq 1$ . Then  $\{K_\nu\}$  is a sequence of compact subsets of  $D$  such that  $K_\nu \subset K_{\nu+1}$  and  $D = \bigcup_{\nu=1}^{\infty} K_\nu$ . Furthermore, if  $K$  is a compact subset of  $D$ , then  $K \subset K_\nu$  holds for some  $\nu \geq 1$ . For  $f, g \in H(D)$ , we define

$$(2.2) \quad d(f, g) = \sum_{\nu=1}^{\infty} 2^{-\nu} \frac{d_\nu(f, g)}{1 + d_\nu(f, g)}, \quad d_\nu(f, g) = \sup_{s \in K_\nu} |f(s) - g(s)|.$$

Denote by  $\mathcal{B}(H(D))$  the Borel algebra of  $H(D)$  with the topology described above. The purpose of this section is to introduce two random elements

$$(2.3) \quad \zeta(s, \mathbb{X}_\alpha) = \sum_{n=0}^{\infty} \frac{\mathbb{X}_\alpha(n)}{(n + \alpha)^s} \quad \text{and} \quad \zeta(s, \mathbb{Y}_\alpha) = \sum_{n=0}^{\infty} \frac{\mathbb{Y}_\alpha(n)}{(n + \alpha)^s}$$

valued on the space  $H(D)$ . Here,  $\{\mathbb{X}_\alpha(n)\}$  and  $\{\mathbb{Y}_\alpha(n)\}$  are sequences of certain random variables valued on the unit circle  $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ .

**2.1. Random variables  $\mathbb{X}_\alpha(n)$  and  $\mathbb{Y}_\alpha(n)$ .** Let  $\alpha$  be an algebraic number with  $0 < \alpha \leq 1$ . The construction of the random variable  $\mathbb{X}_\alpha(n)$  is essentially the same as in the previous paper [20]. Let  $\mathcal{K} = \mathbb{Q}(\alpha)$  be the algebraic field generated by  $\alpha$ , and denote by  $\mathcal{O}_\mathcal{K}$  the ring of integers of  $\mathcal{K}$ . Let  $h$  be the class number of  $\mathcal{K}$ . Recall that  $\mathfrak{a}^h$  is a principal ideal of  $\mathcal{K}$  for any ideal  $\mathfrak{a}$ . Then, for any prime ideal  $\mathfrak{p}$ , we fix an element  $\varpi_\mathfrak{p} \in \mathcal{O}_\mathcal{K}$  such that  $\mathfrak{p}^h = (\varpi_\mathfrak{p})$  and  $\varpi_\mathfrak{p} > 0$ . Let  $y \in \mathcal{K}$  with  $y > 0$ . The fractional principal ideal  $(y)$  has the decomposition

$$(y) = \mathfrak{p}_1^{a_1} \cdots \mathfrak{p}_k^{a_k},$$

where  $\mathfrak{p}_j$  are prime ideals, and  $a_j \in \mathbb{Z}$  are uniquely determined by  $y$ . It yields

$$(y)^h = \mathfrak{p}_1^{ha_1} \cdots \mathfrak{p}_k^{ha_k} = (\varpi_{\mathfrak{p}_1}^{a_1} \cdots \varpi_{\mathfrak{p}_k}^{a_k}).$$

Next, we fix a fundamental system  $(u_1, \dots, u_d)$  of units of  $\mathcal{O}_\mathcal{K}$  such that  $u_j > 0$  for every  $j$ . Then  $y^h$  has the decomposition

$$(2.4) \quad y^h = \varpi_{\mathfrak{p}_1}^{a_1} \cdots \varpi_{\mathfrak{p}_k}^{a_k} u_1^{b_1} \cdots u_d^{b_d},$$

where  $b_j \in \mathbb{Z}$  are uniquely determined by  $y$ . Let

$$\Lambda_1 = \{\varpi_\mathfrak{p} \mid \mathfrak{p} \text{ is a prime ideal of } \mathcal{K}\} \cup \{u_1, u_2, \dots, u_d\}.$$

For  $\lambda \in \Lambda_1$ , we define  $\text{ord}(y, \lambda)$  as

$$(2.5) \quad \text{ord}(y, \lambda) = \begin{cases} a_j & \text{if } \lambda = \varpi_{\mathfrak{p}_j} \text{ for some } \varpi_{\mathfrak{p}_j} \text{ in (2.4),} \\ b_j & \text{if } \lambda = u_j \text{ for some } u_j \text{ in (2.4),} \\ 0 & \text{otherwise.} \end{cases}$$

By the uniqueness of  $a_j$  and  $b_j$ , we have the identity

$$(2.6) \quad \text{ord}(y_1 y_2, \lambda) = \text{ord}(y_1, \lambda) + \text{ord}(y_2, \lambda)$$

for any  $\lambda \in \Lambda_1$  and any  $y_1, y_2 \in \mathcal{K}$  with  $y_1, y_2 > 0$ . We are now ready to define the random variable  $\mathbb{X}_\alpha(n)$ . For any  $s, t \in \mathbb{R}$  with  $0 < t - s \leq 2\pi$ , we define

$$\mathbf{m}(A(s, t)) = \frac{t - s}{2\pi},$$

where  $A(s, t)$  is the arc of the unit circle  $S^1$  defined as

$$(2.7) \quad A(s, t) = \{e^{i\theta} \mid s < \theta < t\}.$$

Then  $\mathbf{m}$  is extended to a probability measure on  $(S^1, \mathcal{B}(S^1))$ , where  $\mathcal{B}(S^1)$  denotes the Borel algebra of  $S^1$  with the usual topology. Let  $\Omega_1 = \prod_{\lambda \in \Lambda_1} S^1$ , and denote by  $\mathcal{F}_1$  the product  $\sigma$ -algebra  $\bigotimes_{\lambda \in \Lambda_1} \mathcal{B}(S^1)$ . By the Kolmogorov extension theorem, there exists a probability measure  $\mathbf{P}_1$  on  $(\Omega_1, \mathcal{F}_1)$  such that

$$(2.8) \quad \mathbf{P}_1(\{\omega = \{\omega_\lambda\} \in \Omega_1 \mid \omega_{\lambda_j} \in A(s_j, t_j) \text{ for } j = 1, \dots, k\}) = \prod_{j=1}^k \mathbf{m}(A(s_j, t_j))$$

for any finite subset  $\{\lambda_1, \dots, \lambda_k\} \subset \Lambda_1$  and any  $s_j, t_j \in \mathbb{R}$  with  $0 < t_j - s_j \leq 2\pi$ . Denote by  $\mathcal{X}(\lambda) : \Omega_1 \rightarrow S^1$  the  $\lambda$ -th projection  $\omega \mapsto \omega_\lambda$ . Then we define

$$\mathbb{X}_\alpha(n) = \prod_{\lambda \in \Lambda_1} \mathcal{X}(\lambda)^{\text{ord}(n+\alpha, \lambda)}$$

for  $n \geq 0$  by using  $\text{ord}(y, \lambda)$  defined as (2.5). By definition, it is an  $S^1$ -valued random variable for any  $n \geq 0$ .

On the other hand, the random variable  $\mathbb{Y}_\alpha(n)$  is defined as follows. Let

$$\Lambda_2 = \{n \in \mathbb{Z} \mid n \geq 0\}.$$

Let  $\Omega_2 = \prod_{n \in \Lambda_2} S^1$ , and denote by  $\mathcal{F}_2$  the product  $\sigma$ -algebra  $\bigotimes_{n \in \Lambda_2} \mathcal{B}(S^1)$ . By the Kolmogorov extension theorem, there exists a probability measure  $\mathbf{P}_2$  on  $(\Omega_2, \mathcal{F}_2)$  such that

$$\mathbf{P}_2(\{\omega = \{\omega_n\} \in \Omega_2 \mid \omega_{n_j} \in A(s_j, t_j) \text{ for } j = 1, \dots, k\}) = \prod_{j=1}^k \mathbf{m}(A(s_j, t_j))$$

for any finite subset  $\{n_1, \dots, n_k\} \subset \Lambda_2$  and any  $s_j, t_j \in \mathbb{R}$  with  $0 < t_j - s_j \leq 2\pi$ . Then we define the  $S^1$ -valued random variable  $\mathbb{Y}_\alpha(n)$  as the  $n$ -th projection  $\omega \mapsto \omega_n$  for  $n \geq 0$ .

For convenience, we regard  $\mathbb{X}_\alpha(n)$  and  $\mathbb{Y}_\alpha(n)$  as random variables defined on the same probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  as follows. Let  $\Omega = \Omega_1 \times \Omega_2$ ,  $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$ , and  $\mathbf{P} = \mathbf{P}_1 \otimes \mathbf{P}_2$ . Then we put

$$\mathbb{X}_\alpha(n)(\omega^1, \omega^2) = \mathbb{X}_\alpha(n)(\omega^1) \quad \text{and} \quad \mathbb{Y}_\alpha(n)(\omega^1, \omega^2) = \mathbb{Y}_\alpha(n)(\omega^2)$$

for  $(\omega^1, \omega^2) \in \Omega$  with  $\omega^1 \in \Omega_1$  and  $\omega^2 \in \Omega_2$ . In general, the expected value of a  $\mathbb{C}$ -valued random variable  $\mathcal{X}$  defined on  $(\Omega, \mathcal{F}, \mathbf{P})$  is denoted by

$$\mathbf{E}[\mathcal{X}] = \int_{\Omega} \mathcal{X} d\mathbf{P}$$

as usual. We abbreviate  $\mathbf{P}(\{\omega \in \Omega \mid \mathcal{X}(\omega) \in A\})$  as  $\mathbf{P}(\mathcal{X} \in A)$  for simplicity, which is called the law of a random element  $\mathcal{X}$ .

**Lemma 2.1.** *Let  $\alpha$  be an algebraic number with  $0 < \alpha \leq 1$ . Then we have*

$$\mathbf{E}[\mathbb{X}_\alpha(n_1)^{m_1} \cdots \mathbb{X}_\alpha(n_k)^{m_k}] = \begin{cases} 1 & \text{if } (n_1 + \alpha)^{m_1} \cdots (n_k + \alpha)^{m_k} = 1, \\ 0 & \text{otherwise} \end{cases}$$



for any integers  $n_1, \dots, n_k \geq 0$  and  $m_1, \dots, m_k \in \mathbb{Z}$ . Furthermore,

$$\mathbf{E} [\mathbb{Y}_\alpha(n_1)^{m_1} \cdots \mathbb{Y}_\alpha(n_k)^{m_k}] = \begin{cases} 1 & \text{if } m_1 = \cdots = m_k = 0, \\ 0 & \text{otherwise} \end{cases}$$

for any distinct integers  $n_1, \dots, n_k \geq 0$  and  $m_1, \dots, m_k \in \mathbb{Z}$ .

*Proof.* The first result has already been proved in [20], but we present the full proof here because of its importance. Put

$$y = (n_1 + \alpha)^{m_1} \cdots (n_k + \alpha)^{m_k}.$$

Applying (2.6), we obtain

$$\text{ord}(y, \lambda) = m_1 \text{ord}(n_1 + \alpha, \lambda) + \cdots + m_k \text{ord}(n_k + \alpha, \lambda)$$

for any  $\lambda \in \Lambda_1$ . Therefore the formula

$$\begin{aligned} \mathbb{X}_\alpha(n_1)^{m_1} \cdots \mathbb{X}_\alpha(n_k)^{m_k} &= \prod_{\lambda \in \Lambda_1} \mathcal{X}(\lambda)^{m_1 \text{ord}(n_1 + \alpha, \lambda) + \cdots + m_k \text{ord}(n_k + \alpha, \lambda)} \\ &= \prod_{\lambda \in \Lambda_1} \mathcal{X}(\lambda)^{\text{ord}(y, \lambda)} \end{aligned}$$

holds by the definition of  $\mathbb{X}_\alpha(n_j)$ . Then the expected value is calculated as

$$\begin{aligned} (2.9) \quad \mathbf{E} [\mathbb{X}_\alpha(n_1)^{m_1} \cdots \mathbb{X}_\alpha(n_k)^{m_k}] &= \int_{\Omega_1} \prod_{\lambda \in \Lambda_1} \mathcal{X}(\lambda)^{\text{ord}(y, \lambda)} d\mathbf{P}_1 \\ &= \prod_{\lambda \in \Lambda_1} \int_{S_1} \omega_\lambda^{\text{ord}(y, \lambda)} d\mathbf{m}(\omega_\lambda) \end{aligned}$$

by (2.8) and Fubini's theorem. We see that

$$(2.10) \quad \int_{S_1} \omega_\lambda^q d\mathbf{m}(\omega_\lambda) = \frac{1}{2\pi} \int_0^{2\pi} e^{iq\theta} d\theta = \begin{cases} 1 & \text{if } q = 0, \\ 0 & \text{otherwise} \end{cases}$$

for any  $q \in \mathbb{Z}$ . Hence, the right-hand side of (2.9) is

$$\prod_{\lambda \in \Lambda_1} \int_{S_1} \omega_\lambda^{\text{ord}(y, \lambda)} d\mathbf{m}(\omega_\lambda) = \begin{cases} 1 & \text{if } \text{ord}(y, \lambda) = 0 \text{ for any } \lambda \in \Lambda_1, \\ 0 & \text{otherwise.} \end{cases}$$

The condition that  $\text{ord}(y, \lambda) = 0$  for any  $\lambda \in \Lambda_1$  is equivalent to  $y^h = 1$  by (2.4). Note that  $y^h = 1$  if and only if  $y = 1$  since  $y$  is a positive real number. Therefore, the first result follows. The second result is proved more easily. We have

$$\begin{aligned} \mathbf{E} [\mathbb{Y}_\alpha(n_1)^{m_1} \cdots \mathbb{Y}_\alpha(n_k)^{m_k}] &= \int_{\Omega_2} \prod_{j=1}^k \mathbb{Y}_\alpha(n_j)^{m_j} d\mathbf{P}_2 \\ &= \prod_{j=1}^k \int_{S_1} \omega_{n_j}^{m_j} d\mathbf{m}(\omega_{n_j}) \end{aligned}$$

by the definition of  $\mathbb{Y}_\alpha(n_j)$ . Using (2.10), we obtain the conclusion.  $\square$

Moreover, it was proved in [20] that  $\mathbb{X}_\alpha(n)$  is uniformly distributed on  $S^1$  if  $0 < \alpha < 1$ , and that  $\mathbb{X}_\alpha(n_1), \dots, \mathbb{X}_\alpha(n_k)$  are independent if and only if the real numbers  $\log(n_1 + \alpha), \dots, \log(n_k + \alpha)$  are linearly independent over  $\mathbb{Q}$ . It is rather clear that  $\mathbb{Y}_\alpha(n)$  is uniformly distributed on  $S^1$ , and that  $\mathbb{Y}_\alpha(n_1), \dots, \mathbb{Y}_\alpha(n_k)$  are independent for any distinct integers  $n_1, \dots, n_k \geq 0$ .

**2.2. Random Dirichlet series  $\zeta(s, \mathbb{X}_\alpha)$  and  $\zeta(s, \mathbb{Y}_\alpha)$ .** Using the random variables  $\mathbb{X}_\alpha(n)$  and  $\mathbb{Y}_\alpha(n)$  constructed in Section 2.1, we define the Dirichlet polynomials

$$\zeta_N(s, \mathbb{X}_\alpha) = \sum_{n=0}^N \frac{\mathbb{X}_\alpha(n)}{(n + \alpha)^s} \quad \text{and} \quad \zeta_N(s, \mathbb{Y}_\alpha) = \sum_{n=0}^N \frac{\mathbb{Y}_\alpha(n)}{(n + \alpha)^s}$$

for  $N \geq 0$ . To begin with, we check that they are random elements valued on the space  $H(D)$ .

**Lemma 2.2.** *Let  $\alpha$  be an algebraic number with  $0 < \alpha \leq 1$ . Then  $\zeta_N(s, \mathbb{X}_\alpha)$  and  $\zeta_N(s, \mathbb{Y}_\alpha)$  are random elements valued on  $H(D)$  for any  $N \geq 0$ .*

*Proof.* Note that the map  $\psi : \prod_{n=0}^N S^1 \rightarrow H(D)$  defined as

$$(\gamma_0, \dots, \gamma_N) \mapsto \sum_{n=0}^N \frac{\gamma_n}{(n + \alpha)^s}$$

is continuous. Since  $\zeta_N(s, \mathbb{X}_\alpha)$  and  $\zeta_N(s, \mathbb{Y}_\alpha)$  are obtained as compositions of  $\psi$  and the  $S^1$ -valued random variables  $\mathbb{X}_\alpha(n)$  and  $\mathbb{Y}_\alpha(n)$ , they are random elements valued on  $H(D)$ .  $\square$

To consider the convergences of the random Dirichlet series  $\zeta(s, \mathbb{X}_\alpha)$  and  $\zeta(s, \mathbb{Y}_\alpha)$  as in (2.3), we apply the following version of the Menshov–Rademacher theorem.

**Lemma 2.3** (Menshov–Rademacher theorem). *Let  $\{\mathcal{X}_n\}$  be a sequence of  $\mathbb{C}$ -valued random variables defined on the probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ . Then the random series*

$$\sum_{n=1}^{\infty} a_n \mathcal{X}_n$$

*converges almost surely if the following conditions are satisfied.*

- (i)  $\mathbf{E}[\mathcal{X}_n] = 0$  for any  $n \geq 1$ .
- (ii) The sequence  $\{\mathcal{X}_n\}$  is orthonormal in the sense that

$$\mathbf{E}[\mathcal{X}_m \overline{\mathcal{X}_n}] = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

- (iii) We have

$$\sum_{n=1}^{\infty} |a_n|^2 (\log n)^2 < \infty.$$

*Proof.* See [15, Theorem B.10.5] for a proof.  $\square$

**Proposition 2.4.** *Let  $\alpha$  be an algebraic number with  $0 < \alpha \leq 1$ . Then  $\zeta(s, \mathbb{X}_\alpha)$  and  $\zeta(s, \mathbb{Y}_\alpha)$  are random elements valued on  $H(D)$ .*

*Proof.* By Lemma 2.1, we can check that the sequence  $\{\mathbb{X}_\alpha(n)\}$  satisfies conditions (i) and (ii) of Lemma 2.3. Furthermore, we have

$$\sum_{n=0}^{\infty} \left| \frac{1}{(n+\alpha)^s} \right|^2 (\log n)^2 = \sum_{n=0}^{\infty} \frac{(\log n)^2}{(n+\alpha)^{2\sigma}} < \infty$$

on the half-plane  $\sigma > 1/2$ . Hence Lemma 2.3 yields that the random Dirichlet series  $\zeta(s, \mathbb{X}_\alpha)$  converges for  $\sigma > 1/2$  almost surely. In other words, if we define

$$\Omega_c = \{\omega \in \Omega \mid \zeta_N(s, \mathbb{X}_\alpha)(\omega) \text{ converges as } N \rightarrow \infty \text{ for } \sigma > 1/2\},$$

then  $\mathbf{P}(\Omega_c) = 1$ . For  $\delta > 0$ , we also define

$$\Omega_{u,\delta} = \{\omega \in \Omega \mid \zeta_N(s, \mathbb{X}_\alpha)(\omega) \text{ converges uniformly as } N \rightarrow \infty \text{ for } \sigma \geq 1/2 + \delta\}.$$

Then  $\Omega_c \subset \Omega_{u,\delta}$  is valid for any  $\delta > 0$ . Indeed, if a Dirichlet series converges for  $\sigma > 1/2$ , then it converges uniformly for  $\sigma \geq 1/2 + \delta$ . See [16, Corollary 2.1.3] for a proof. Let  $\Omega_u = \bigcap_{\delta > 0} \Omega_{u,\delta}$ . Then

$$\zeta(s, \mathbb{X}_\alpha)(\omega) = \lim_{N \rightarrow \infty} \zeta_N(s, \mathbb{X}_\alpha)(\omega)$$

belongs to the space  $H(D)$  for any  $\omega \in \Omega_u$ . Furthermore, we obtain that  $\mathbf{P}(\Omega_u) = 1$  since  $\mathbf{P}(\Omega_c) = 1$  and  $\Omega_c \subset \Omega_{u,\delta}$  for any  $\delta > 0$ . Thus  $\zeta(s, \mathbb{X}_\alpha)$  defines a map from  $\Omega$  to  $H(D)$ , which is a random element since  $\zeta_N(s, \mathbb{X}_\alpha)$  is a random element for any  $N \geq 0$ . The proof for  $\zeta(s, \mathbb{Y}_\alpha)$  is completely the same as that for  $\zeta(s, \mathbb{X}_\alpha)$ .  $\square$

### 3. LIMIT THEOREM FOR $\zeta(s, \alpha)$ IN THE SPACE $H(D)$

Let  $\alpha$  be an algebraic number with  $0 < \alpha \leq 1$ . We define

$$P_{\alpha,T}(A) = \frac{1}{T} \text{meas} \{\tau \in [0, T] \mid \zeta(s + i\tau, \alpha) \in A\},$$

$$Q_\alpha(A) = \mathbf{P}(\zeta(s, \mathbb{X}_\alpha) \in A)$$

for  $A \in \mathcal{B}(H(D))$ , where  $\zeta(s, \mathbb{X}_\alpha)$  is the random Dirichlet series defined in Section 2. Then  $P_{\alpha,T}$  and  $Q_\alpha$  are probability measures on  $(H(D), \mathcal{B}(H(D)))$ . In this section, we prove the following limit theorem.

**Theorem 3.1.** *Let  $\alpha$  be an algebraic number with  $0 < \alpha \leq 1$ . Then the probability measure  $P_{\alpha,T}$  converges weakly to  $Q_\alpha$  as  $T \rightarrow \infty$ .*

This is a generalization of Bagchi's limit theorem of [1], which was proved for the Riemann zeta-function  $\zeta(s)$ . The proof of Theorem 3.1 is largely based on Bagchi's method, while we adopt new ideas from Kowalski [14, 15] in part.

**3.1. Fourier analysis on a torus.** Let  $\mathcal{T}^k$  be the  $k$ -dimensional torus given by

$$\mathcal{T}^k = \prod_{j=1}^k S^1.$$

Denote by  $\mathcal{B}(\mathcal{T}^k)$  the Borel algebra of  $\mathcal{T}^k$  with the product topology. Let  $\mu$  be any probability measure on  $(\mathcal{T}^k, \mathcal{B}(\mathcal{T}^k))$ , and define its Fourier transform as

$$g(\underline{m}) = \int_{\mathcal{T}^k} \prod_{j=1}^k \gamma_j^{m_j} d\mu(\underline{\gamma})$$

for  $\underline{m} = (m_1, \dots, m_k) \in \mathbb{Z}^k$ . Then the following result holds for the torus  $\mathcal{T}^k$ .

**Lemma 3.2.** *Let  $\{\mu_n\}$  be a sequence of probability measures on  $(\mathcal{T}^k, \mathcal{B}(\mathcal{T}^k))$  whose Fourier transforms are denoted by  $g_n(\underline{m})$ . If the limit*

$$g(\underline{m}) = \lim_{n \rightarrow \infty} g_n(\underline{m})$$

*exists for any  $\underline{m} \in \mathbb{Z}^k$ , then there exists a probability measure  $\nu$  on  $(\mathcal{T}^k, \mathcal{B}(\mathcal{T}^k))$  such that the following properties are satisfied.*

- (i) *The probability measure  $\mu_n$  converges weakly to  $\nu$  as  $n \rightarrow \infty$ .*
- (ii) *The Fourier transform of  $\nu$  is equal to  $g(\underline{m})$ .*

*Proof.* This is a special case of a more general result [11, Theorem 1.4.2] proved for every locally compact Abelian group. See also [16, Theorem 1.3.19].  $\square$

Let  $\alpha$  be an algebraic number with  $0 < \alpha \leq 1$ . Define the probability measures  $\mu_{\alpha, T}$  and  $\nu_\alpha$  on  $(\mathcal{T}^k, \mathcal{B}(\mathcal{T}^k))$  by letting

$$\begin{aligned} \mu_{\alpha, T}(A) &= \frac{1}{T} \text{meas} \{ \tau \in [0, T] \mid ((n_1 + \alpha)^{-i\tau}, \dots, (n_k + \alpha)^{-i\tau}) \in A \}, \\ \nu_\alpha(A) &= \mathbf{P}((\mathbb{X}_\alpha(n_1), \dots, \mathbb{X}_\alpha(n_k)) \in A) \end{aligned}$$

for  $A \in \mathcal{B}(\mathcal{T}^k)$ , where  $n_1, \dots, n_k$  are non-negative integers. We begin by the proof of the following auxiliary result.

**Proposition 3.3.** *Let  $\alpha$  be an algebraic number with  $0 < \alpha \leq 1$ . Then the probability measure  $\mu_{\alpha, T}$  converges weakly to  $\nu_\alpha$  as  $T \rightarrow \infty$ .*

*Proof.* Denote by  $g_{\alpha, T}(\underline{m})$  the Fourier transform of  $\mu_{\alpha, T}$ . Then it is calculated as

$$\begin{aligned} g_{\alpha, T}(\underline{m}) &= \int_{\mathcal{T}^k} \prod_{j=1}^k \gamma_j^{m_j} d\mu_{\alpha, T}(\underline{\gamma}) \\ &= \frac{1}{T} \int_0^T \{(n_1 + \alpha)^{m_1} \cdots (n_k + \alpha)^{m_k}\}^{-i\tau} d\tau \end{aligned}$$

for any  $\underline{m} = (m_1, \dots, m_k) \in \mathbb{Z}^k$ . If  $(n_1 + \alpha)^{m_1} \cdots (n_k + \alpha)^{m_k} = 1$ , then we have

$$g_{\alpha, T}(\underline{m}) = \frac{1}{T} \int_0^T d\tau = 1$$

for any  $T > 0$ . Otherwise, we obtain that

$$g_{\alpha, T}(\underline{m}) = \frac{1}{-i\Delta} \frac{\{(n_1 + \alpha)^{m_1} \cdots (n_k + \alpha)^{m_k}\}^{-iT} - 1}{T},$$

where  $\Delta = m_1 \log(n_1 + \alpha) + \cdots + m_k \log(n_k + \alpha) \neq 0$ . As a result, we derive the limit formula

$$\lim_{T \rightarrow \infty} g_{\alpha, T}(\underline{m}) = g_\alpha(\underline{m}) := \begin{cases} 1 & \text{if } (n_1 + \alpha)^{m_1} \cdots (n_k + \alpha)^{m_k} = 1, \\ 0 & \text{otherwise} \end{cases}$$

for any  $\underline{m} = (m_1, \dots, m_k) \in \mathbb{Z}^k$ . Hence, by Lemma 3.2, the probability measure  $\mu_{\alpha, T}$  converges weakly to some probability measure on  $(\mathcal{T}^k, \mathcal{B}(\mathcal{T}^k))$  whose Fourier transform is equal to  $g_\alpha(\underline{m})$ . Furthermore, the Fourier transform of  $\nu_\alpha$  is

$$\int_{\mathcal{T}^k} \prod_{j=1}^k \gamma_j^{m_j} d\nu_\alpha(\underline{\gamma}) = \mathbf{E}[\mathbb{X}_\alpha(n_1)^{m_1} \cdots \mathbb{X}_\alpha(n_k)^{m_k}] = g_\alpha(\underline{m})$$

for any  $\underline{m} = (m_1, \dots, m_k) \in \mathbb{Z}^k$  by Lemma 2.1. Therefore, the limit measure of  $\mu_{\alpha, T}$  is equal to  $\nu_\alpha$ , and the proof is completed.  $\square$

**3.2. Approximation by smoothed partial sums.** From now, we fix an infinitely differentiable function  $\phi : [0, \infty) \rightarrow [0, 1]$  with compact support such that  $\phi(0) = 1$ . The Mellin transform of  $\phi$  is defined by

$$\widehat{\phi}(w) = \int_0^\infty \phi(x) x^w \frac{dx}{x}$$

for  $\operatorname{Re}(w) > 0$ . Then we collect standard properties of  $\widehat{\phi}(w)$  as follows. For proofs, see [15, Proposition A.3.1]. Firstly,  $\widehat{\phi}(w)$  extends to a meromorphic function on the half-plane  $\operatorname{Re}(w) > -1$  only with a simple pole at  $w = 0$  with residue 1. Secondly,  $\widehat{\phi}(w)$  has rapid decay on the strip  $a \leq \operatorname{Re}(w) \leq b$  with  $a > -1$  in the following sense. Let  $k \geq 1$  and  $b > a > -1$ . Then we have

$$(3.1) \quad |\widehat{\phi}(w)| \ll (|v| + 1)^{-k}$$

for any  $w = u + iv \in \mathbb{C}$  with  $a \leq u \leq b$  and  $|v| \geq 1$ , where the implied constant depends only on  $a, b, k$ . Lastly, the inversion formula

$$(3.2) \quad \phi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \widehat{\phi}(w) x^{-w} dw$$

holds for any  $c > 0$ . Let  $\alpha$  be an algebraic number with  $0 < \alpha \leq 1$ . Then we define

$$Z_N(s, \alpha) = \sum_{n=0}^{\infty} (n + \alpha)^{-s} \phi\left(\frac{n + \alpha}{N}\right),$$

$$Z_N(s, \mathbb{X}_\alpha) = \sum_{n=0}^{\infty} \frac{\mathbb{X}_\alpha(n)}{(n + \alpha)^s} \phi\left(\frac{n + \alpha}{N}\right)$$

for  $N \geq 1$ . Since  $\phi((n + \alpha)/N) = 0$  for sufficiently large  $n$ , we see that  $Z_N(s, \alpha)$  is a function belonging to the space  $H(D)$ , and that  $Z_N(s, \mathbb{X}_\alpha)$  is a random element valued on  $H(D)$ . Then we prove two preliminary lemmas.

**Lemma 3.4.** *Let  $\alpha$  be an algebraic number with  $0 < \alpha \leq 1$ . Let  $s = \sigma + it$  be a complex number with  $1/2 < \sigma \leq 1$ , and suppose  $0 < \delta < \sigma - 1/2$ . Then we have*

$$\zeta(s, \alpha) = Z_N(s, \alpha) - \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta+i\infty} \zeta(s+w, \alpha) \widehat{\phi}(w) N^w dw - \widehat{\phi}(1-s) N^{1-s}$$

for any  $N \geq 1$ . Furthermore, we have

$$\zeta(s, \mathbb{X}_\alpha) = Z_N(s, \mathbb{X}_\alpha) - \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta+i\infty} \zeta(s+w, \mathbb{X}_\alpha) \widehat{\phi}(w) N^w dw$$

almost surely for any  $N \geq 1$ .

*Proof.* Let  $c$  be a positive real number with  $\sigma + c > 1$ . By inversion formula (3.2), the function  $Z_N(s, \alpha)$  is represented as

$$Z_N(s, \alpha) = \sum_{n=0}^{\infty} (n + \alpha)^{-s} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \widehat{\phi}(w) \left(\frac{n + \alpha}{N}\right)^{-w} dw$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{c-i\infty}^{c+i\infty} (n + \alpha)^{-(s+w)} \widehat{\phi}(w) N^w dw.$$

By Fubini's theorem, we obtain

$$Z_N(s, \alpha) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta(s+w, \alpha) \widehat{\phi}(w) N^w dw$$

since  $\operatorname{Re}(s+w) > 1$  for  $\operatorname{Re}(w) = c$ . Then we shift the contour to  $\operatorname{Re}(w) = -\delta$  with  $0 < \delta < \sigma - 1/2$ . Note that we come across poles of  $\zeta(s+w, \alpha) \widehat{\phi}(w) N^w$  only at  $w = 0, 1-s$ , which are simple. Calculating the residues, we derive

$$Z_N(s, \alpha) = \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta+i\infty} \zeta(s+w, \alpha) \widehat{\phi}(w) N^w dw + \zeta(s, \alpha) + \widehat{\phi}(1-s) N^{1-s}.$$

This yields the desired formula of  $\zeta(s, \alpha)$ . As for  $Z_N(s, \mathbb{X}_\alpha)$ , we have

$$Z_N(s, \mathbb{X}_\alpha)(\omega) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta(s+w, \mathbb{X}_\alpha)(\omega) \widehat{\phi}(w) N^w dw$$

for  $\omega \in \Omega_u$  in a similar way, where  $\Omega_u$  is the same as in the proof of Proposition 2.4. Note that  $\zeta(s+w, \mathbb{X}_\alpha)(\omega)$  remains holomorphic for any  $\omega \in \Omega_u$ , while shifting the contour to  $\operatorname{Re}(w) = -\delta$  with  $0 < \delta < \sigma - 1/2$ . Therefore, the formula

$$Z_N(s, \mathbb{X}_\alpha)(\omega) = \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta+i\infty} \zeta(s+w, \mathbb{X}_\alpha)(\omega) \widehat{\phi}(w) N^w dw + \zeta(s, \mathbb{X}_\alpha)(\omega)$$

holds for  $\omega \in \Omega_u$ . Since  $\mathbf{P}(\Omega_u) = 1$ , we obtain the conclusion.  $\square$

**Lemma 3.5.** *Let  $\alpha$  be an algebraic number with  $0 < \alpha \leq 1$ . Let  $s = \sigma + it$  be a complex number with  $1/2 < \sigma < 1$ . Then we have*

$$\frac{1}{T} \int_0^T |\zeta(s + i\tau, \alpha)| d\tau \ll \frac{1}{\sqrt{2\sigma-1}} \left(1 + \frac{|t|}{T}\right) + \frac{1}{1-\sigma} \left(1 + \frac{|t|}{T}\right)$$

for any  $T \geq 3$ . Furthermore, we have

$$\mathbf{E} [|\zeta(s, \mathbb{X}_\alpha)|] \ll \frac{1}{\sqrt{2\sigma-1}}.$$

Here, the implied constants depend only on  $\alpha$ .

*Proof.* Let  $1/2 < \sigma < 1$  be a real number. To begin with, we show that

$$(3.3) \quad \frac{1}{V} \int_{-V}^V |\zeta(\sigma + iv, \alpha)| dv \ll \frac{1}{\sqrt{2\sigma-1}} + \frac{1}{1-\sigma}$$

for any  $V \geq 3$ , where the implied constant depends only on  $\alpha$ . We use the following approximate formula of  $\zeta(\sigma + iv, \alpha)$ . For any  $2\pi \leq |v| \leq \pi V$ , we have

$$\zeta(\sigma + iv, \alpha) = \sum_{0 \leq n \leq V} \frac{1}{(n+\alpha)^{\sigma+iv}} + \frac{V^{1-(\sigma+iv)}}{\sigma+iv-1} + O(V^{-\sigma})$$

with an absolute implied constant. See [12, Theorem III.2.1] for a proof. Then we obtain the estimate

$$\begin{aligned}
(3.4) \quad & \frac{1}{V} \int_{2\pi}^V |\zeta(\sigma + iv, \alpha)| dv \\
& \leq \frac{1}{V} \int_{2\pi}^V \left| \sum_{0 \leq n \leq V} \frac{1}{(n + \alpha)^{\sigma + iv}} \right| dv + \frac{1}{V} \int_{2\pi}^V \left| \frac{V^{1 - (\sigma + iv)}}{\sigma + iv - 1} \right| dv + CV^{-\sigma} \\
& \leq \left\{ \frac{1}{V} \int_0^V \left| \sum_{0 \leq n \leq V} \frac{1}{(n + \alpha)^{\sigma + iv}} \right|^2 dv \right\}^{1/2} + (\log V)V^{-\sigma} + CV^{-\sigma}
\end{aligned}$$

by using the Cauchy–Schwarz inequality, where  $C$  is a positive absolute constant. Furthermore, we apply Hilbert’s inequality [22, Corollary 2] to see that

$$\int_0^V \left| \sum_{0 \leq n \leq V} \frac{1}{(n + \alpha)^{\sigma + iv}} \right|^2 dv \leq \sum_{0 \leq n \leq V} \frac{1}{(n + \alpha)^{2\sigma}} (V + 3\pi\delta_n^{-1}),$$

where  $\delta_n = \min_{m \neq n} |\log(n + \alpha) - \log(m + \alpha)|$ . We have

$$\delta_n \geq \log \left( \frac{n + 1 + \alpha}{n + \alpha} \right) \geq \frac{1}{n + 2} \gg \frac{1}{V}$$

for any  $0 \leq n \leq V$ . Hence we derive

$$\frac{1}{V} \int_0^V \left| \sum_{0 \leq n \leq V} \frac{1}{(n + \alpha)^{\sigma + iv}} \right|^2 dv \ll \sum_{0 \leq n \leq V} \frac{1}{(n + \alpha)^{2\sigma}}.$$

Furthermore, the last sum is estimated as

$$\sum_{0 \leq n \leq V} \frac{1}{(n + \alpha)^{2\sigma} } \leq \alpha^{-1} + \sum_{n=1}^{\infty} n^{-2\sigma} \ll \frac{1}{2\sigma - 1},$$

where the implied constant depends only on  $\alpha$ . By (3.4), we obtain

$$(3.5) \quad \frac{1}{V} \int_{2\pi}^V |\zeta(\sigma + iv, \alpha)| dv \ll \frac{1}{\sqrt{2\sigma - 1}}$$

since  $(\log V)V^{-\sigma} + V^{-\sigma} \ll 1$  and  $1/\sqrt{2\sigma - 1} \gg 1$ . Furthermore, (3.5) implies

$$(3.6) \quad \frac{1}{V} \int_{-V}^{-2\pi} |\zeta(\sigma + iv, \alpha)| dv \ll \frac{1}{\sqrt{2\sigma - 1}}$$

by the identity  $\zeta(\sigma - iv, \alpha) = \overline{\zeta(\sigma + iv, \alpha)}$ . Then, we recall that  $\zeta(s, \alpha)$  has a simple pole at  $s = 1$ . Thus we deduce

$$(3.7) \quad \frac{1}{V} \int_{-2\pi}^{2\pi} |\zeta(\sigma + iv, \alpha)| dv \ll \sup_{|v| \leq 2\pi} |\zeta(\sigma + iv, \alpha)| \ll \frac{1}{1 - \sigma},$$

where the implied constant depends only on  $\alpha$ . Combining (3.5), (3.6), and (3.7), we obtain (3.3). Then, we prove the first part of the lemma. Let  $s = \sigma + it$  be a

complex number with  $1/2 < \sigma < 1$ . We have

$$\frac{1}{T} \int_0^T |\zeta(s + i\tau, \alpha)| d\tau = \frac{1}{T} \int_t^{T+t} |\zeta(\sigma + i\tau, \alpha)| d\tau \leq \frac{1}{T} \int_{-(T+|t|)}^{T+|t|} |\zeta(\sigma + i\tau, \alpha)| d\tau$$

since  $T + t \leq T + |t|$  and  $t \geq -(T + |t|)$ . Applying (3.3) with  $V = T + |t|$ , we obtain

$$\begin{aligned} \frac{1}{T} \int_0^T |\zeta(s + i\tau, \alpha)| d\tau &\leq \left(1 + \frac{|t|}{T}\right) \frac{1}{T + |t|} \int_{-(T+|t|)}^{T+|t|} |\zeta(\sigma + i\tau, \alpha)| d\tau \\ &\ll \frac{1}{\sqrt{2\sigma - 1}} \left(1 + \frac{|t|}{T}\right) + \frac{1}{1 - \sigma} \left(1 + \frac{|t|}{T}\right) \end{aligned}$$

as desired. The second part is proved as follows. By the Cauchy–Schwarz inequality, we have

$$\mathbf{E} [|\zeta(s, \mathbb{X}_\alpha)|]^2 \leq \mathbf{E} [|\zeta(s, \mathbb{X}_\alpha)|^2] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\mathbf{E}[\mathbb{X}_\alpha(m) \overline{\mathbb{X}_\alpha(n)}]}{(m + \alpha)^s (n + \alpha)^{\bar{s}}}.$$

Lemma 2.1 yields that  $\mathbf{E}[\mathbb{X}_\alpha(m) \overline{\mathbb{X}_\alpha(n)}] = 0$  for  $m \neq n$ . As a result, we deduce

$$\mathbf{E} [|\zeta(s, \mathbb{X}_\alpha)|] \leq \left\{ \sum_{n=0}^{\infty} \frac{1}{(n + \alpha)^{2\sigma}} \right\}^{1/2} \ll \frac{1}{\sqrt{2\sigma - 1}},$$

where the implied constant depends only on  $\alpha$ . Hence the proof is completed.  $\square$

Let  $d$  be the distance of  $H(D)$  as in (2.2). Since  $x/(1+x) \leq \min\{x, 1\}$  for  $x > 0$ , we obtain the inequality

$$(3.8) \quad d(f, g) \leq \sum_{\nu=1}^M 2^{-\nu} d_\nu(f, g) + \sum_{\nu=M+1}^{\infty} 2^{-\nu} \leq d_M(f, g) + 2^{-M}$$

for any  $f, g \in H(D)$  and any  $M \geq 1$ . Then, applying Lemmas 3.4 and 3.5, we prove the following result.

**Proposition 3.6.** *Let  $\alpha$  be an algebraic number with  $0 < \alpha \leq 1$ . Then there exists an absolute constant  $c > 0$  such that*

$$\frac{1}{T} \int_0^T d(\zeta(s + i\tau, \alpha), Z_N(s + i\tau, \alpha)) d\tau \ll \exp(-c\sqrt{\log N}) + \frac{N(\log N)^3}{T}$$

for any  $T \geq 3$  and sufficiently large  $N$ . Furthermore, we have

$$\mathbf{E} [d(\zeta(s, \mathbb{X}_\alpha), Z_N(s, \mathbb{X}_\alpha))] \ll \exp(-c\sqrt{\log N})$$

for sufficiently large  $N$ . Here, the implied constants depend only on  $\alpha$ .

*Proof.* Let  $M$  be a positive integer chosen later. By (3.8), we have

$$(3.9) \quad \begin{aligned} &\frac{1}{T} \int_0^T d(\zeta(s + i\tau, \alpha), Z_N(s + i\tau, \alpha)) d\tau \\ &\leq \frac{1}{T} \int_0^T \sup_{s \in K_M} |\zeta(s + i\tau, \alpha) - Z_N(s + i\tau, \alpha)| d\tau + 2^{-M}. \end{aligned}$$

Let  $s \in K_M$  and  $\tau \in \mathbb{R}$ . Then Cauchy's integral formula is available to obtain

$$\zeta(s + i\tau, \alpha) - Z_N(s + i\tau, \alpha) = \frac{1}{2\pi i} \oint_{\partial K_{M+1}} \frac{\zeta(z + i\tau, \alpha) - Z_N(z + i\tau, \alpha)}{z - s} dz.$$



By the definition of  $\{K_\nu\}$ , we see that  $|z - s| \geq (5M(M + 1))^{-1}$  for any  $s \in K_M$  and any  $z \in \partial K_{M+1}$ . Therefore we obtain

$$(3.10) \quad \begin{aligned} & \sup_{s \in K_M} |\zeta(s + i\tau, \alpha) - Z_N(s + i\tau, \alpha)| \\ & \leq \frac{5}{2\pi} M(M + 1) \oint_{\partial K_{M+1}} |\zeta(z + i\tau, \alpha) - Z_N(z + i\tau, \alpha)| |dz|. \end{aligned}$$

For any  $z \in \partial K_{M+1}$ , we have  $1/2 < \operatorname{Re}(z + i\tau) < 1$ . Then, we apply Lemma 3.4 to derive the estimate

$$\begin{aligned} & |\zeta(z + i\tau, \alpha) - Z_N(z + i\tau, \alpha)| \\ & \leq \frac{1}{2\pi} \int_{-\delta - i\infty}^{-\delta + i\infty} |\zeta(z + w + i\tau, \alpha)| |\widehat{\phi}(w)| N^{\operatorname{Re}(w)} |dw| + |\widehat{\phi}(1 - (z + i\tau))| N^{1 - \operatorname{Re}(z)} \\ & \ll N^{-\delta} \int_{-\delta - i\infty}^{-\delta + i\infty} |\zeta(z + w + i\tau, \alpha)| |\widehat{\phi}(w)| |dw| + N^{1/2} |\widehat{\phi}(1 - z - i\tau)| \end{aligned}$$

for any  $z \in \partial K_{M+1}$ , where  $\delta$  satisfies  $0 < \delta < \operatorname{Re}(z) - 1/2$ . Inserting this estimate to (3.10), we obtain

$$\begin{aligned} & \sup_{s \in K_M} |\zeta(s + i\tau, \alpha) - Z_N(s + i\tau, \alpha)| \\ & \ll M^2 N^{-\delta} \oint_{\partial K_{M+1}} \int_{-\delta - i\infty}^{-\delta + i\infty} |\zeta(z + w + i\tau, \alpha)| |\widehat{\phi}(w)| |dw| |dz| \\ & \quad + M^2 N \oint_{\partial K_{M+1}} |\widehat{\phi}(1 - z - i\tau)| |dz|. \end{aligned}$$

Using (3.9) and changing the orders of integrals, we arrive at

$$(3.11) \quad \begin{aligned} & \frac{1}{T} \int_0^T d(\zeta(s + i\tau, \alpha), Z_N(s + i\tau, \alpha)) d\tau \\ & \ll M^2 N^{-\delta} \oint_{\partial K_{M+1}} \int_{-\delta - i\infty}^{-\delta + i\infty} \left( \frac{1}{T} \int_0^T |\zeta(z + w + i\tau, \alpha)| d\tau \right) |\widehat{\phi}(w)| |dw| |dz| \\ & \quad + M^2 N \oint_{\partial K_{M+1}} \left( \frac{1}{T} \int_0^T |\widehat{\phi}(1 - z - i\tau)| d\tau \right) |dz| + 2^{-M}. \end{aligned}$$

Let  $z \in \partial K_{M+1}$ , and put  $\delta = (10(M + 1))^{-1}$  so that  $0 < \delta < \operatorname{Re}(z) - 1/2$  is satisfied. Furthermore, we have  $1/2 + (10(M + 1))^{-1} \leq \operatorname{Re}(z + w) \leq 1 - (5(M + 1))^{-1}$  on the line  $\operatorname{Re}(w) = -\delta$ . By Lemma 3.5, we obtain

$$(3.12) \quad \begin{aligned} & \frac{1}{T} \int_0^T |\zeta(z + w + i\tau, \alpha)| d\tau \ll M \left( 1 + \frac{|\operatorname{Im}(z + w)|}{T} \right) \\ & \ll M^2 (|\operatorname{Im}(w)| + 1) \end{aligned}$$

for any  $T \geq 3$ , where the implied constant depends only on  $\alpha$ . On the other hand, we have  $0 \leq \operatorname{Re}(1 - z - i\tau) \leq 1/2$ . Furthermore,  $|\operatorname{Im}(1 - z - i\tau)| \geq \tau/2 \geq 1$  is satisfied if  $\tau \in [2(M + 1), T]$ . Hence we deduce from (3.1) that  $|\widehat{\phi}(1 - z - i\tau)| \ll \tau^{-2}$  for any  $\tau \in [2(M + 1), T]$ , where the implied constant is absolute. Hence the integral

of  $|\widehat{\phi}(1 - z - i\tau)|$  is evaluated as

$$\begin{aligned} \frac{1}{T} \int_0^T |\widehat{\phi}(1 - z - i\tau)| d\tau &\ll \frac{1}{T} \int_{2(M+1)}^T \tau^{-2} d\tau + \frac{1}{T} \int_0^{2(M+1)} |\widehat{\phi}(1 - z - i\tau)| d\tau \\ &\leq \frac{1}{T} + \frac{2(M+1)}{T} \sup_{0 \leq \tau \leq 2(M+1)} |\widehat{\phi}(1 - z - i\tau)| \end{aligned}$$

for any  $T \geq 2(M+1)$ . Note that the same estimate is valid even if  $T < 2(M+1)$ , since we have

$$\frac{1}{T} \int_0^T |\widehat{\phi}(1 - z - i\tau)| d\tau \leq \frac{2(M+1)}{T} \sup_{0 \leq \tau \leq 2(M+1)} |\widehat{\phi}(1 - z - i\tau)|$$

for any  $T < 2(M+1)$ . By the definition of the Mellin transform, we have

$$\begin{aligned} |\widehat{\phi}(1 - z - i\tau)| &\leq \int_0^\infty \phi(x) x^{1 - \operatorname{Re}(z)} \frac{dx}{x} \\ &\leq \int_1^\infty \phi(x) x^{1/2} \frac{dx}{x} + \int_0^1 \phi(x) x^{1/(5(M+1))} \frac{dx}{x} \\ &\leq \widehat{\phi}\left(\frac{1}{2}\right) + \widehat{\phi}\left(\frac{1}{5(M+1)}\right) \end{aligned}$$

for any  $z \in \partial K_{M+1}$  and  $\tau \in \mathbb{R}$ . Since  $\widehat{\phi}(w)$  has a simple pole at  $w = 0$ , we obtain that  $\widehat{\phi}(1/(5(M+1))) \ll M$ . Hence  $|\widehat{\phi}(1 - z - i\tau)| \ll M$  follows. From the above, we arrive at

$$(3.13) \quad \frac{1}{T} \int_0^T |\widehat{\phi}(1 - z - i\tau)| d\tau \ll \frac{M^2}{T}$$

for any  $T \geq 3$ , where the implied constant is absolute. Inserting (3.12) and (3.13) to (3.11), we deduce that

$$\begin{aligned} &\frac{1}{T} \int_0^T d(\zeta(s + i\tau, \alpha), Z_N(s + i\tau, \alpha)) d\tau \\ &\ll M^4 N^{-\delta} \oint_{\partial K_{M+1}} |dz| \int_{-\delta - i\infty}^{-\delta + i\infty} (|\operatorname{Im}(w)| + 1) |\widehat{\phi}(w)| |dw| \\ &\quad + \frac{M^4 N}{T} \oint_{\partial K_{M+1}} |dz| + 2^{-M} \\ &\ll M^5 N^{-\delta} \int_{-\delta - i\infty}^{-\delta + i\infty} (|\operatorname{Im}(w)| + 1) |\widehat{\phi}(w)| |dw| + \frac{M^5 N}{T} + 2^{-M}, \end{aligned}$$

where the last line is derived from  $\oint_{\partial K_{M+1}} |dz| \ll M$ . Recall that  $\delta = (10(M+1))^{-1}$ . Hence we have  $-1/2 \leq -\delta \leq 0$ . Applying (3.1) with  $k = 3$ , we obtain

$$\begin{aligned} \int_{-\delta - i\infty}^{-\delta + i\infty} (|\operatorname{Im}(w)| + 1) |\widehat{\phi}(w)| |dw| &\ll \int_{|v| \geq 1} |v|^{-2} dv + \int_{|v| \leq 1} |\widehat{\phi}(-\delta + iv)| dv \\ &\ll M \end{aligned}$$

since  $\widehat{\phi}(-\delta + iv) \ll M$  for any  $|v| \leq 1$ , which follows from the fact that  $\widehat{\phi}(w)$  has a simple pole at  $w = 0$ . Here, we choose the positive integer  $M$  as  $M = \lfloor \sqrt{\log N} \rfloor$  for

$N \geq 3$ . We finally obtain

$$\begin{aligned} \frac{1}{T} \int_0^T d(\zeta(s + i\tau, \alpha), Z_N(s + i\tau, \alpha)) d\tau &\ll M^6 N^{-\delta} + \frac{M^5 N}{T} + 2^{-M} \\ &\ll \exp\left(-c\sqrt{\log N}\right) + \frac{N(\log N)^3}{T} \end{aligned}$$

for sufficiently large  $N$ , where  $c$  is an absolute constant. Then, we prove the second part of the proposition. In a similar way that we derive (3.11), we also obtain

$$\begin{aligned} \mathbf{E} [d(\zeta(s, \mathbb{X}_\alpha), Z_N(s, \mathbb{X}_\alpha))] \\ \ll M^2 N^{-\delta} \oint_{\partial K_{M+1}} \int_{-\delta-i\infty}^{-\delta+i\infty} \mathbf{E} [|\zeta(s, \mathbb{X}_\alpha)|] |\widehat{\phi}(w)| |dw| |dz| + 2^{-M} \end{aligned}$$

by noting the disappearance of the term  $\widehat{\phi}(1-s)N^{1-s}$  in the formula for  $\zeta(s, \mathbb{X}_\alpha)$  of Lemma 3.4. Applying Lemma 3.5, we have

$$\begin{aligned} \mathbf{E} [d(\zeta(s, \mathbb{X}_\alpha), Z_N(s, \mathbb{X}_\alpha))] &\ll M^3 N^{-\delta} \oint_{\partial K_{M+1}} |dz| \int_{-\delta-i\infty}^{-\delta+i\infty} |\widehat{\phi}(w)| |dw| + 2^{-M} \\ &\ll M^5 N^{-\delta} + 2^{-M} \\ &\ll \exp\left(-c\sqrt{\log N}\right) \end{aligned}$$

for sufficiently large  $N$ . From the above, we obtain the conclusion.  $\square$

**3.3. Proof of Theorem 3.1.** Let  $S$  be a metric space with distance  $d$ . We say that  $F : S \rightarrow \mathbb{C}$  is a Lipschitz function if there exists a constant  $\mathcal{L} > 0$  such that

$$|F(x) - F(y)| \leq \mathcal{L}d(x, y)$$

for any  $x, y \in S$ . The constant  $\mathcal{L}$  is called a Lipschitz constant for  $F$ . By definition, we know that any Lipschitz function is continuous. Denote by  $\mathcal{B}(S)$  the Borel algebra of  $S$  with the topology induced from  $d$ .

**Lemma 3.7** (Portmanteau theorem). *Let  $\{P_n\}$  be a sequence of probability measures on  $(S, \mathcal{B}(S))$ . Let  $Q$  be a probability measure on  $(S, \mathcal{B}(S))$ . Then the followings are equivalent.*

- (i)  $P_n$  converges weakly to  $Q$  as  $n \rightarrow \infty$ .
- (ii) For any bounded Lipschitz function  $F : S \rightarrow \mathbb{C}$ , we have

$$\lim_{n \rightarrow \infty} \int_S F dP_n = \int_S F dQ.$$

- (iii) For any open set  $A$  of  $S$ , we have

$$\liminf_{n \rightarrow \infty} P_n(A) \geq Q(A).$$

*Proof.* See [13, Theorem 13.16] for a proof.  $\square$

From the above preparations, we finally prove Theorem 3.1. By the definitions of  $P_{\alpha,T}$  and  $Q_\alpha$ , the integrals with respect to these probability measures are

$$\int_{H(D)} F dP_{\alpha,T} = \frac{1}{T} \int_0^T F(\zeta(s+i\tau, \alpha)) d\tau,$$

$$\int_{H(D)} F dQ_\alpha = \mathbf{E}[F(\zeta(s, \mathbb{X}_\alpha))]$$

for any measurable function  $F : H(D) \rightarrow \mathbb{C}$ .

*Proof of Theorem 3.1.* Let  $F : H(D) \rightarrow \mathbb{C}$  be a bounded Lipschitz function with a Lipschitz constant  $\mathcal{L}$ . Then we derive by Proposition 3.6 that

$$(3.14) \quad \left| \frac{1}{T} \int_0^T F(\zeta(s+i\tau, \alpha)) d\tau - \frac{1}{T} \int_0^T F(Z_N(s+i\tau, \alpha)) d\tau \right|$$

$$\leq \frac{\mathcal{L}}{T} \int_0^T d(\zeta(s+i\tau, \alpha), Z_N(s+i\tau, \alpha)) d\tau$$

$$\leq \mathcal{L}_\alpha \left\{ \exp(-c\sqrt{\log N}) + \frac{N(\log N)^3}{T} \right\}$$

for any  $T \geq 3$  and sufficiently large  $N$ , where  $\mathcal{L}_\alpha > 0$  is a constant depending only on  $\alpha$ . Analogously, we obtain

$$(3.15) \quad \left| \mathbf{E}[F(\zeta(s, \mathbb{X}_\alpha))] - \mathbf{E}[F(Z_N(s, \mathbb{X}_\alpha))] \right| \leq \mathcal{L}_\alpha \exp(-c\sqrt{\log N})$$

for sufficiently large  $N$ . Since the function  $\phi$  is compactly supported, there exists an integer  $k = k(\alpha, N) \geq 1$  such that  $\phi((n+\alpha)/N) = 0$  for any integer  $n \geq k$ . Let  $\psi_N : \mathcal{T}^k \rightarrow H(D)$  be a continuous map defined as

$$\psi_N(\underline{\gamma}) = \sum_{n=0}^{k-1} \frac{\gamma_n}{(n+\alpha)^s} \phi\left(\frac{n+\alpha}{N}\right)$$

for  $\underline{\gamma} = (\gamma_0, \dots, \gamma_{k-1}) \in \mathcal{T}^k$ . Let  $\mu_{\alpha,T}$  and  $\nu_\alpha$  denote the probability measures on  $(\mathcal{T}^k, \mathcal{B}(\mathcal{T}^k))$  as in Proposition 3.3, where we put  $n_j = j-1$  for  $j = 1, \dots, k$ . Then we obtain

$$\frac{1}{T} \int_0^T F(Z_N(s+i\tau, \alpha)) d\tau = \int_{\mathcal{T}^k} (F \circ \psi_N) d\mu_{\alpha,T},$$

$$\mathbf{E}[F(Z_N(s, \mathbb{X}_\alpha))] = \int_{\mathcal{T}^k} (F \circ \psi_N) d\nu_\alpha.$$

Note that  $F \circ \psi_N$  is a bounded continuous function on  $\mathcal{T}^k$  for any  $N \geq 1$ . Hence, we derive the limit formula

$$(3.16) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T F(Z_N(s+i\tau, \alpha)) d\tau = \mathbf{E}[F(Z_N(s, \mathbb{X}_\alpha))]$$

for any  $N \geq 1$  by Proposition 3.3. Let  $\epsilon$  be any positive real number. By (3.14) and (3.15), there exists an integer  $N_0 = N_0(\epsilon, \alpha) \geq 3$  such that

$$\left| \frac{1}{T} \int_0^T F(\zeta(s+i\tau, \alpha)) d\tau - \frac{1}{T} \int_0^T F(Z_{N_0}(s+i\tau, \alpha)) d\tau \right| < \frac{\epsilon}{3},$$

$$\left| \mathbf{E}[F(\zeta(s, \mathbb{X}_\alpha))] - \mathbf{E}[F(Z_{N_0}(s, \mathbb{X}_\alpha))] \right| < \frac{\epsilon}{3}$$

for any  $T \geq T_1 := (6/\epsilon)\mathcal{L}_\alpha N_0(\log N_0)^3$ . Furthermore, we derive by (3.16) that there exists a real number  $T_2 = T_2(\epsilon, \alpha) \geq 3$  such that

$$\left| \frac{1}{T} \int_0^T F(Z_{N_0}(s + i\tau, \alpha)) d\tau - \mathbf{E}[F(Z_{N_0}(s, \mathbb{X}_\alpha))] \right| < \frac{\epsilon}{3}$$

for any  $T \geq T_2$ . As a result, we have

$$\left| \frac{1}{T} \int_0^T F(\zeta(s + i\tau, \alpha)) d\tau - \mathbf{E}[F(\zeta(s, \mathbb{X}_\alpha))] \right| < \epsilon$$

for any  $T \geq \max\{T_1, T_2\}$ . By Lemma 3.7, the proof is completed.  $\square$

#### 4. COVERING THEOREM FOR THE SPACE $A^2(U)$

Let  $U$  be a bounded domain with  $\bar{U} \subset D$ . Define the inner product and norm as

$$\langle f, g \rangle = \iint_U f(s)\overline{g(s)} d\sigma dt \quad \text{and} \quad \|f\| = \sqrt{\langle f, f \rangle}$$

for measurable functions  $f, g : U \rightarrow \mathbb{C}$ . The Bergman space  $A^2(U)$  is defined as the space of all analytic functions  $f : U \rightarrow \mathbb{C}$  such that  $\|f\| < \infty$ . Then it is a complex Hilbert space with the inner product described above. Throughout this section, we suppose that the boundary  $\partial U$  is a Jordan curve. Then the subspace

$$(4.1) \quad \mathcal{P} = \{a_0 + a_1s + \cdots + a_Ns^N \mid a_k \in \mathbb{C} \text{ for } 0 \leq k \leq N \text{ with } N \geq 0\}$$

is dense in  $A^2(U)$ . See [6] for a proof. Let  $X$  be any subset of  $A^2(U)$ . For every  $\epsilon > 0$ , we denote by  $X^{(\epsilon)}$  the  $\epsilon$ -neighborhood of  $X$  defined as

$$X^{(\epsilon)} = \{f \in A^2(U) \mid \exists g \in X \text{ such that } \|f - g\| < \epsilon\}.$$

Furthermore, we define

$$\Gamma(\alpha, N) = \left\{ \sum_{n=0}^N \frac{\gamma_n}{(n + \alpha)^s} \mid |\gamma_n| = 1 \text{ for } 0 \leq n \leq N \right\}$$

for  $0 < \alpha \leq 1$  and  $N \geq 0$ . In this section, we prove the following covering theorem for the space  $A^2(U)$ .

**Theorem 4.1.** *Let  $0 < c < 1$ . Then, for every  $\epsilon > 0$ , there exists a positive real number  $\rho$  such that*

$$A^2(U) = \bigcup_{N_0 \geq 0} \bigcap_{N > N_0} \bigcap_{\alpha \in \mathcal{A}_\rho(c)} \Gamma(\alpha, N)^{(\epsilon)},$$

where  $\rho$  depends only on  $c, \epsilon, U$ .

**4.1. Preliminary lemmas.** Let  $H$  be a complex Hilbert space. Recall that any continuous linear functional  $f : H \rightarrow \mathbb{C}$  is represented as  $f(x) = \langle x, y \rangle$  with some  $y \in H$  by the Riesz representation theorem. We say that a subset  $K \subset H$  is convex if  $tx + (1 - t)y \in K$  for any  $x, y \in K$  and any  $0 \leq t \leq 1$ .

**Lemma 4.2.** *Let  $H$  be a complex Hilbert space. Let  $K$  be any closed convex subset of  $H$ , and suppose  $x \in H \setminus K$ . Then there exist an element  $y \in H$  and a constant  $\eta \in \mathbb{R}$  such that*

$$\operatorname{Re} \langle z, y \rangle \leq \eta < \operatorname{Re} \langle x, y \rangle$$

for all  $z \in K$ .

*Proof.* This is a special case of the Hahn–Banach separation theorem, which holds for every locally convex vector space. See [5, Theorem 8.73] for a proof.  $\square$

For a subspace  $L$  of  $H$ , we denote by  $L^\perp$  the orthogonal complement, that is,

$$L^\perp = \{x \in H \mid \forall y \in L, \langle x, y \rangle = 0\}.$$

If  $L$  is a closed subspace, then every element  $x \in H$  has a unique representation  $x = y + z$  such that  $y \in L$  and  $z \in L^\perp$ . Thus we obtain  $H = L \oplus L^\perp$ .

**Lemma 4.3.** *Let  $H$  be a complex Hilbert space. Then a subspace  $L$  is dense in  $H$  if and only if  $L^\perp = \{0\}$ .*

*Proof.* Note that  $M = \overline{L}$  is a closed subspace of  $H$ . Then the result follows from the decomposition  $H = M \oplus M^\perp$ .  $\square$

**Lemma 4.4.** *Let  $H$  be a complex Hilbert space, and take  $x_1, \dots, x_n \in H$  arbitrarily. Let  $\beta_1, \dots, \beta_n$  be complex numbers with  $|\beta_j| \leq 1$  for  $1 \leq j \leq n$ . Then we have*

$$\left\| \sum_{j=1}^n \beta_j x_j - \sum_{j=1}^n \gamma_j x_j \right\|^2 \leq 4 \sum_{j=1}^n \|x_j\|^2$$

with some complex numbers  $\gamma_1, \dots, \gamma_n$  with  $|\gamma_j| = 1$  for  $1 \leq j \leq n$ .

*Proof.* See [16, Lemma 6.1.15] for a proof.  $\square$

Let  $F(z)$  be an entire function. We say that  $F(z)$  is of exponential type if

$$\limsup_{r \rightarrow \infty} \frac{\log |F(re^{i\theta})|}{r} < \infty$$

uniformly in  $\theta \in \mathbb{R}$ . The following lemmas are also used to prove Theorem 4.1.

**Lemma 4.5.** *Let  $F(z)$  be an entire function of exponential type. Let  $\{\lambda_m\}$  be a sequence of complex numbers. If there exist positive real numbers  $\alpha, \beta, \delta$  such that*

- (a)  $\limsup_{y \rightarrow \infty} \frac{\log |F(\pm iy)|}{y} \leq \alpha$ ,
- (b)  $|\lambda_m - \lambda_n| \geq \delta |m - n|$ ,
- (c)  $\lim_{m \rightarrow \infty} \frac{\lambda_m}{m} = \beta$ ,
- (d)  $\alpha\beta < \pi$ ,

then we have

$$\limsup_{m \rightarrow \infty} \frac{\log |F(\lambda_m)|}{|\lambda_m|} = \limsup_{r \rightarrow \infty} \frac{\log |F(r)|}{r}.$$

*Proof.* See [16, Theorem 6.4.12] for a proof.  $\square$

**Lemma 4.6.** *Let  $\mu$  be a complex measure on  $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$  whose support is compact and contained in the half-plane  $\sigma > \sigma_0$ . If the function  $F(z)$  defined by*

$$F(z) = \int_{\mathbb{C}} e^{zs} d\mu(s)$$

for  $z \in \mathbb{C}$  does not vanish everywhere, then we have

$$\limsup_{r \rightarrow \infty} \frac{\log |F(r)|}{r} > \sigma_0.$$

*Proof.* See [16, Lemma 6.4.10] for a proof.  $\square$

**4.2. Proof of Theorem 4.1.** Before proving Theorem 4.1, we show two auxiliary results by applying the lemmas in Section 4.1. The first result is a denseness result for the space  $A^2(U)$ . Note that a similar result is also used in Bagchi's method for the proof of universality in [1].

**Proposition 4.7.** *Let  $0 < c < 1$ . Then the set*

$$B(c, M) = \left\{ \sum_{M < n \leq N} \frac{\beta_n}{(n+c)^s} \mid |\beta_n| \leq 1 \text{ for } M < n \leq N \text{ with } N \geq M+1 \right\}$$

is dense in  $A^2(U)$  for any  $M \geq 0$ .

*Proof.* Suppose  $\overline{B(c, M)} \neq A^2(U)$  for some  $M \geq 0$ . Then we can take a function  $f$  in  $A^2(U)$  such that  $f \notin \overline{B(c, M)}$ . By definition, the set  $B(c, M)$  is convex. Thus the closure  $\overline{B(c, M)}$  is also a convex subset, which is closed in  $A^2(U)$ . By Lemma 4.2, there exist a function  $g \in A^2(U)$  and a constant  $\eta \in \mathbb{R}$  such that

$$(4.2) \quad \operatorname{Re} \langle h, g \rangle \leq \eta < \operatorname{Re} \langle f, g \rangle$$

for all  $h \in \overline{B(c, M)}$ . Put

$$h_N(s) = \sum_{M < n \leq N} \frac{|\langle (n+c)^{-s}, g(s) \rangle|}{\langle (n+c)^{-s}, g(s) \rangle} \frac{1}{(n+c)^s}$$

for  $N \geq M+1$ . It is obviously an element of the set  $\overline{B(c, M)}$ . By (4.2), we have

$$\operatorname{Re} \langle h_N, g \rangle = \sum_{M < n \leq N} |\langle (n+c)^{-s}, g(s) \rangle| < \operatorname{Re} \langle f, g \rangle$$

for any  $N \geq M+1$ . Therefore, we find that

$$(4.3) \quad \sum_{n=0}^{\infty} |\langle (n+c)^{-s}, g(s) \rangle| < \infty$$

must be satisfied. Using the function  $g$ , we define  $F_g(z) = \langle e^{-zs}, g(s) \rangle$  for  $z \in \mathbb{C}$ . Put  $\alpha = \max\{|s| \mid s \in \overline{U}\}$ . Then the Cauchy-Schwarz inequality yields that

$$(4.4) \quad |F_g(re^{i\theta})| \leq \left\{ \int_U |\exp(-re^{i\theta}s)|^2 d\sigma dt \right\}^{1/2} \left\{ \int_U |g(s)|^2 d\sigma dt \right\}^{1/2} \\ \ll \exp(\alpha r)$$

uniformly in  $\theta \in \mathbb{R}$ . Hence  $F_g(z)$  is an entire function of exponential type. Furthermore, it does not vanish everywhere. Indeed, if  $F_g(z) = 0$  for any  $z \in \mathbb{C}$ , then

$$F_g(0) = \langle 1, g(s) \rangle = 0 \quad \text{and} \quad \left. \frac{d^k}{dz^k} F_g(z) \right|_{z=0} = (-1)^k \langle s^k, g(s) \rangle = 0$$

would hold for all  $k \geq 1$ . These imply that  $g \in \mathcal{P}^\perp$ , where  $\mathcal{P}$  is the subspace of  $A^2(U)$  as in (4.1). However, since  $\mathcal{P}$  is dense in  $A^2(U)$ , we get  $g = 0$  by Lemma 4.3. This contradicts inequality (4.2). Then, we prove

$$(4.5) \quad \limsup_{r \rightarrow \infty} \frac{\log |F_g(r)|}{r} \leq -1$$

by applying Lemma 4.5. We put  $\beta = \pi/(2\alpha)$  so that condition (d) of Lemma 4.5 is satisfied. By (4.4), we see that condition (a) is also satisfied. Define  $A_\beta$  as the

set of all integers  $m \geq 1$  such that  $|F_g(r)| < e^{-r}$  holds with some  $r \in \mathbb{R}$  satisfying  $m\beta < r < (m + 1/2)\beta$ . Let

$$C_\beta(m) = \left\{ n \in \mathbb{Z}_{\geq 0} \mid m\beta < \log(n + c) < \left(m + \frac{1}{2}\right)\beta \right\}$$

for  $m \geq 1$ . By the definitions of  $A_\beta$  and  $C_\beta(m)$ , we have

$$\sum_{n=0}^{\infty} |F_g(\log(n + c))| \geq \sum_{m \notin A_\beta} \sum_{n \in C_\beta(m)} |F_g(\log(n + c))| \geq \sum_{m \notin A_\beta} \sum_{n \in C_\beta(m)} \frac{1}{n + c}.$$

Furthermore, the series of the left-hand side is finite by (4.3), and therefore,

$$(4.6) \quad \sum_{m \notin A_\beta} \sum_{n \in C_\beta(m)} \frac{1}{n + c} < \infty.$$

If  $m$  is sufficiently large, then the inner sum is evaluated as

$$(4.7) \quad \sum_{n \in C_\beta(m)} \frac{1}{n + c} \geq \sum_{e^{m\beta} < n < e^{(m+1/2)\beta} - 1} \frac{1}{n + 1} = \frac{\beta}{2} + o(1)$$

by the well-known formula  $\sum_{n \leq x} 1/n = \log x + \gamma + o(1)$  as  $x \rightarrow \infty$ . Here,  $\gamma$  is the Euler constant. By (4.6) and (4.7), we find that the complement of  $A_\beta$  must be a finite set. Denote by  $a_m$  the  $m$ -th integer in the set  $A_\beta$ . Then we have  $a_m/m \rightarrow 1$  as  $m \rightarrow \infty$ . By the definition of  $A_\beta$ , there exists  $\lambda_m \in \mathbb{R}$  for any  $m \geq 1$  such that

$$|F_g(\lambda_m)| < e^{-\lambda_m} \quad \text{and} \quad a_m\beta < \lambda_m < \left(a_m + \frac{1}{2}\right)\beta.$$

Hence  $\lambda_m$  satisfies condition (c) of Lemma 4.5. Furthermore, condition (b) holds with  $\delta = \beta/2$  since

$$|\lambda_m - \lambda_n| > (a_m - a_n)\beta - \frac{\beta}{2} \geq (m - n)\beta - \frac{\beta}{2} \geq \frac{\beta}{2}(m - n)$$

for any  $m > n$ . As a result, we deduce from Lemma 4.5 that

$$\limsup_{r \rightarrow \infty} \frac{\log |F_g(r)|}{r} = \limsup_{m \rightarrow \infty} \frac{\log |F_g(\lambda_m)|}{|\lambda_m|}.$$

Then we finally obtain (4.5), since  $|F_g(\lambda_m)| < e^{-\lambda_m}$  holds for any  $m \geq 1$ . On the other hand, the function  $F_g(z)$  is represented as

$$F_g(z) = \int_{\mathbb{C}} e^{zs} d\mu_g(s),$$

where  $d\mu_g(s) = 1_U(-s)\overline{g(-s)}d\sigma dt$ . Note that  $\mu_g$  is a complex measure on  $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$  which is supported on  $-\overline{U} = \{-s \mid s \in \overline{U}\}$ . The assumption  $\overline{U} \subset D$  implies that  $-\overline{U}$  is contained in the half-plane  $\sigma > -1$ . Hence Lemma 4.6 yields

$$(4.8) \quad \limsup_{r \rightarrow \infty} \frac{\log |F_g(r)|}{r} > -1,$$

and we obtain contradiction by (4.5) and (4.8). As a result,  $\overline{B(c, M)} = A^2(U)$  holds for any  $M \geq 0$ , that is, the set  $B(c, M)$  is dense in  $A^2(U)$  for any  $M \geq 0$ .  $\square$



The second result ensures that we can approximate an element in  $\Gamma(\alpha, N)$  by an element in  $\Gamma(c, N)$  uniformly for  $\alpha \in \mathcal{A}_\rho(c)$  if  $\rho$  is sufficiently small. The condition  $|\alpha - c| \leq \rho$  in the definition of  $\mathcal{A}_\rho(c)$  is required here.

**Proposition 4.8.** *Let  $0 < c < 1$ . Then, for every  $\epsilon > 0$ , there exists a positive real number  $\rho$  such that*

$$\sup_{\alpha \in \mathcal{A}_\rho(c)} \left\| \sum_{n=0}^N \frac{\gamma_n}{(n+\alpha)^s} - \sum_{n=0}^N \frac{\gamma_n}{(n+c)^s} \right\| < \epsilon$$

for any  $N \geq 0$  if  $|\gamma_n| = 1$  for  $0 \leq n \leq N$ , where  $\rho$  depends only on  $c, \epsilon, U$ .

*Proof.* First, we have

$$(4.9) \quad \left\| \sum_{n=0}^N \frac{\gamma_n}{(n+\alpha)^s} - \sum_{n=0}^N \frac{\gamma_n}{(n+c)^s} \right\| \leq \sum_{n=0}^N \left\| (n+\alpha)^{-s} - (n+c)^{-s} \right\| \\ = \sum_{n=0}^N \left\| \int_c^\alpha (-s)(n+u)^{-s-1} du \right\|$$

since  $|\gamma_n| = 1$  for  $0 \leq n \leq N$ . Recall that  $|\alpha - c| \leq \rho$  is valid for any  $\alpha \in \mathcal{A}_\rho(c)$ . Hence the inequality

$$\left| \int_c^\alpha (-s)(n+u)^{-s-1} du \right| \leq \rho |s| \sup_{u \in (c-\rho, c+\rho)} |(n+u)^{-s-1}|$$

holds for any  $\alpha \in \mathcal{A}_\rho(c)$ . Assume that  $\rho$  satisfies  $0 < \rho \leq \min\{c, 1-c\}/2$ . Then we obtain

$$\left| \int_c^\alpha (-s)(n+u)^{-s-1} du \right| \leq \rho |s| (n+c/2)^{-3/2}$$

for any  $\alpha \in \mathcal{A}_\rho(c)$  and any  $s \in U$ . Inserting this inequality to (4.9), we deduce

$$\left\| \sum_{n=0}^N \frac{\gamma_n}{(n+\alpha)^s} - \sum_{n=0}^N \frac{\gamma_n}{(n+c)^s} \right\| \leq \rho \sqrt{K} \sum_{n=0}^N (n+c/2)^{-3/2} \leq \rho \sqrt{K} \zeta(3/2, c/2),$$

where  $K = \iint_U |s|^2 d\sigma dt$  is a positive constant determined only by  $U$ . Hence, if we assume further that  $\rho$  satisfies  $0 < \rho < \epsilon \{\sqrt{K} \zeta(3/2, c/2)\}^{-1}$ , then we have

$$\left\| \sum_{n=0}^N \frac{\gamma_n}{(n+\alpha)^s} - \sum_{n=0}^N \frac{\gamma_n}{(n+c)^s} \right\| < \epsilon$$

for any  $\alpha \in \mathcal{A}_\rho(c)$  and any  $N \geq 0$ . This is the desired result.  $\square$

*Proof of Theorem 4.1.* Let  $0 < c < 1$  and  $f \in A^2(U)$ . Let  $M$  be a positive integer chosen later. Then the function

$$g(s) = f(s) - \sum_{n=0}^M \frac{1}{(n+c)^s}$$

belongs to the space  $A^2(U)$ . Thus Proposition 4.7 yields  $g \in \overline{B(c, M)}$ , that is, there exists an element  $h \in B(c, M)$  with  $\|g - h\| < \epsilon/3$  for every  $\epsilon > 0$ . Hence, by the

definition of  $B(c, M)$ , there exists an integer  $N_0 \geq M + 1$  such that

$$\left\| \left( f(s) - \sum_{n=0}^M \frac{1}{(n+c)^s} \right) - \sum_{M < n \leq N_0} \frac{\beta_n}{(n+c)^s} \right\| < \frac{\epsilon}{3}$$

with some  $|\beta_n| \leq 1$  for  $M < n \leq N_0$ . Note that the integer  $N_0$  depends only on  $c, \epsilon, f, U$ , and  $M$ . Put  $\beta_n = 0$  for  $N_0 < n \leq N$  with any integer  $N > N_0$ . Then we obtain the inequality

$$(4.10) \quad \left\| f(s) - \sum_{n=0}^M \frac{1}{(n+c)^s} - \sum_{M < n \leq N} \frac{\beta_n}{(n+c)^s} \right\| < \frac{\epsilon}{3}.$$

We also deduce from Lemma 4.4 that

$$\left\| \sum_{M < n \leq N} \frac{\beta_n}{(n+c)^s} - \sum_{M < n \leq N} \frac{\gamma_n}{(n+c)^s} \right\|^2 \leq 4 \sum_{M < n \leq N} \|(n+c)^{-s}\|^2$$

with some  $|\gamma_n| = 1$  for  $M < n \leq N$ . The sum of the right-hand side is estimated as

$$\sum_{M < n \leq N} \|(n+c)^{-s}\|^2 \leq \sum_{M < n \leq N} L(n+c)^{-2\sigma_0} \leq \frac{L}{2\sigma_0 - 1} M^{1-2\sigma_0},$$

where  $\sigma_0 = \min\{\operatorname{Re}(s) \mid s \in \bar{U}\}$  and  $L = \iint_U d\sigma dt$  are positive constants determined only by  $U$ . Here, we choose the integer  $M = M(\epsilon, U)$  so that

$$\frac{L}{2\sigma_0 - 1} M^{1-2\sigma_0} < \frac{\epsilon^2}{36}$$

is satisfied. Then we derive

$$(4.11) \quad \left\| \sum_{M < n \leq N} \frac{\beta_n}{(n+c)^s} - \sum_{M < n \leq N} \frac{\gamma_n}{(n+c)^s} \right\| \leq 2 \left( \frac{L}{2\sigma_0 - 1} M^{1-2\sigma_0} \right)^{1/2} < \frac{\epsilon}{3}.$$

Lastly, by Proposition 4.8, there exists a positive real number  $\rho$  depending only on  $c, \epsilon, U$  such that

$$(4.12) \quad \sup_{\alpha \in \mathcal{A}_\rho(c)} \left\| \sum_{n=0}^N \frac{\gamma_n}{(n+\alpha)^s} - \sum_{n=0}^N \frac{\gamma_n}{(n+c)^s} \right\| < \frac{\epsilon}{3},$$

where we put  $\gamma_n = 1$  for  $0 \leq n \leq M$ . Combining (4.10), (4.11), and (4.12), we arrive at the inequality

$$\left\| f(s) - \sum_{n=0}^N \frac{\gamma_n}{(n+\alpha)^s} \right\| < \epsilon$$

for any  $N > N_0$  and any  $\alpha \in \mathcal{A}_\rho(c)$ . In other words, we obtain that

$$f \in \bigcup_{N_0 \geq 0} \bigcap_{N > N_0} \bigcap_{\alpha \in \mathcal{A}_\rho(c)} \Gamma(\alpha, N)^{(\epsilon)}.$$

Therefore the desired result follows.  $\square$

**4.3. Support of the random Dirichlet polynomial  $\zeta_N(s, \mathbb{Y}_\alpha)$ .** For any function  $f \in H(D)$ , the restriction of  $f$  to  $U$  is an element of  $A^2(U)$ . Then we regard the random Dirichlet polynomial  $\zeta_N(s, \mathbb{Y}_\alpha)$  defined in Section 2 as a random element valued on  $A^2(U)$ . The following result is a simple consequence of Theorem 4.1.

**Corollary 4.9.** *Let  $0 < c < 1$  and  $f \in A^2(U)$ . Then, for every  $\epsilon > 0$ , there exist a positive real number  $\rho$  and an integer  $N_0 \geq 0$  such that*

$$\mathbf{P}(\|\zeta_N(s, \mathbb{Y}_\alpha) - f(s)\| < \epsilon) > 0$$

for any  $N > N_0$  and any  $\alpha \in \mathcal{A}_\rho(c)$ , where  $\rho$  and  $N_0$  depend only on  $c, \epsilon, f, U$ .

*Proof.* Recall that the random variables  $\mathbb{Y}_\alpha(n_1), \dots, \mathbb{Y}_\alpha(n_k)$  are independent for any distinct integers  $n_1, \dots, n_k \geq 0$ . Indeed, we derive by Fubini's theorem that

$$\begin{aligned} & \mathbf{E}[\phi_1(\mathbb{Y}_\alpha(n_1)) \cdots \phi_k(\mathbb{Y}_\alpha(n_k))] \\ &= \int_{\Omega_2} \prod_{j=1}^k \phi_j(\mathbb{Y}_\alpha(n_j)) d\mathbf{P}_2 = \prod_{j=1}^k \int_{S^1} \phi_j(\omega_{n_j}) d\mathbf{m}(\omega_{n_j}) = \prod_{j=1}^k \mathbf{E}[\phi_j(\mathbb{Y}_\alpha(n_j))] \end{aligned}$$

for any measurable functions  $\phi_1, \dots, \phi_k$  on  $S^1$ . Hence, the support of  $\zeta_N(s, \mathbb{Y}_\alpha)$  is calculated as

$$\text{supp } \zeta_N(s, \mathbb{Y}_\alpha) = \overline{\sum_{n=0}^N \text{supp} \left( \frac{\mathbb{Y}_\alpha(n)}{(n+\alpha)^s} \right)}$$

by [15, Proposition B.10.8], where  $\sum_{n=0}^N S_n$  denotes the set of all points  $x_0 + \cdots + x_N$  with  $x_n \in S_n$  for  $0 \leq n \leq N$ . We have

$$\text{supp} \left( \frac{\mathbb{Y}_\alpha(n)}{(n+\alpha)^s} \right) = \left\{ \frac{\gamma_n}{(n+\alpha)^s} \mid |\gamma_n| = 1 \right\}$$

for any  $0 \leq n \leq N$ . Therefore, by the definition of the set  $\Gamma(\alpha, N)$ , we derive that  $\text{supp } \zeta_N(s, \mathbb{Y}_\alpha) = \overline{\Gamma(\alpha, N)}$ . We see that  $\Gamma(\alpha, N)$  is closed in the space  $A^2(U)$ . Indeed, the continuous map  $\psi : \prod_{n=0}^N S^1 \rightarrow A^2(U)$  defined as

$$(\gamma_0, \dots, \gamma_N) \mapsto \sum_{n=0}^N \frac{\gamma_n}{(n+\alpha)^s}$$

is a closed map since  $\prod_{n=0}^N S^1$  is compact and  $A^2(U)$  is Hausdorff. Hence

$$(4.13) \quad \text{supp } \zeta_N(s, \mathbb{Y}_\alpha) = \overline{\Gamma(\alpha, N)}$$

follows. Let  $0 < c < 1$  and  $f \in A^2(U)$ . By Theorem 4.1, there exist a positive real number  $\rho$  and an integer  $N_0 \geq 0$  such that  $f \in \Gamma(\alpha, N)^{(\epsilon/2)}$  for any  $N > N_0$  and any  $\alpha \in \mathcal{A}_\rho(c)$ . Here,  $\rho$  and  $N_0$  depend at most on  $c, \epsilon, f, U$ . Then, by (4.13), there exists an element  $g \in \text{supp } \zeta_N(s, \mathbb{Y}_\alpha)$  such that  $\|f - g\| < \epsilon/2$ . Note that the condition  $g \in \text{supp } \zeta_N(s, \mathbb{Y}_\alpha)$  implies

$$\mathbf{P} \left( \|\zeta_N(s, \mathbb{Y}_\alpha) - g(s)\| < \frac{\epsilon}{2} \right) > 0.$$

Therefore, we obtain

$$\mathbf{P}(\|\zeta_N(s, \mathbb{Y}_\alpha) - f(s)\| < \epsilon) \geq \mathbf{P} \left( \|\zeta_N(s, \mathbb{Y}_\alpha) - g(s)\| < \frac{\epsilon}{2} \right) > 0,$$

and the proof is completed.  $\square$

## 5. PROOF OF THE MAIN RESULT

Let  $K$  be a compact subset of the strip  $D$  with connected complement. Then there exists a bounded domain  $U$  such that  $K \subset U$  and  $\overline{U} \subset D$ , whose boundary  $\partial U$  is a Jordan curve. In the following, we fix such a domain  $U$ , and put

$$\sigma_0 = \min\{\operatorname{Re}(s) \mid s \in \overline{U}\}.$$

Then we have  $1/2 < \sigma_0 < 1$ . Let  $\omega_0 \in \Omega$  be any sample. Denote by  $\theta_n \in [0, 2\pi)$  the argument of the value  $\mathbb{Y}_\alpha(n)(\omega_0) \in S^1$  for  $0 \leq n \leq N$ . Define the event  $\Omega_0$  as

$$(5.1) \quad \begin{aligned} \Omega_0 &= \Omega_0(\delta; N, \alpha, \omega_0) \\ &= \{\omega \in \Omega \mid \mathbb{X}_\alpha(n)(\omega) \in A(\theta_n - \pi\delta, \theta_n + \pi\delta) \text{ for } 0 \leq n \leq N\} \end{aligned}$$

for  $0 < \delta < 1$ , where  $A(s, t)$  is the arc of  $S^1$  as in (2.7). For any  $\mathbb{C}$ -valued random variable  $\mathcal{X}$  defined on the probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ , we define

$$\mathbf{E}_{\Omega_0}[\mathcal{X}] = \int_{\Omega_0} \mathcal{X} d\mathbf{P}.$$

Before proceeding to the proof of Theorem 1.4, we study the conditional square mean value  $\mathbf{E}_{\Omega_0}[\|\zeta(s, \mathbb{X}_\alpha) - \zeta_N(s, \mathbb{X}_\alpha)\|^2]$  for  $\alpha \in \mathcal{A} \setminus \mathcal{E}$ , where  $\mathcal{E}$  is a certain finite subset chosen suitably.

**5.1. Results on the Beurling–Selberg functions.** Let  $\mathbf{1}_{(s,t)}$  denote the indicator function of an open interval  $(s, t) \subset \mathbb{R}$ . The Beurling–Selberg functions present a nice approximation of  $\mathbf{1}_{(s,t)}$ . Here, we collect several results used later. See [28] for the details of proofs. First, we define the functions  $H(z)$  and  $K(z)$  as

$$H(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{m=-\infty}^{\infty} \frac{\operatorname{sgn}(m)}{(z-m)^2} + \frac{2}{z} \right\} \quad \text{and} \quad K(z) = \left(\frac{\sin \pi z}{\pi z}\right)^2$$

for  $z \in \mathbb{C}$ . They are entire functions of exponential type such that

$$\limsup_{r \rightarrow \infty} \frac{\log |H(re^{i\theta})|}{r} \leq 2\pi, \quad \limsup_{r \rightarrow \infty} \frac{\log |K(re^{i\theta})|}{r} \leq 2\pi$$

uniformly in  $\theta \in \mathbb{R}$ . It is known that the inequalities

$$(5.2) \quad |\operatorname{sgn}(x) - H(x)| \leq K(x) \quad \text{and} \quad |H(x)| \leq 1$$

hold for any  $x \in \mathbb{R}$ . Let  $s, t, \Delta \in \mathbb{R}$  with  $s < t$  and  $\Delta > 1$ . Then we define

$$\begin{aligned} U_{s,t}(z, \Delta) &= \frac{1}{2} \left\{ H\left(\frac{\Delta}{2\pi}(z-s)\right) + H\left(\frac{\Delta}{2\pi}(t-z)\right) \right\}, \\ K_{s,t}(z, \Delta) &= \frac{1}{2} \left\{ K\left(\frac{\Delta}{2\pi}(z-s)\right) + K\left(\frac{\Delta}{2\pi}(t-z)\right) \right\} \end{aligned}$$

for  $z \in \mathbb{C}$ . We deduce from (5.2) the inequality

$$(5.3) \quad |\mathbf{1}_{(s,t)}(x) - U_{s,t}(x, \Delta)| \leq K_{s,t}(x, \Delta)$$

for any  $x \in \mathbb{R}$ . Define the Fourier transforms  $\tilde{U}_{s,t}(\xi, \Delta)$  and  $\tilde{K}_{s,t}(\xi, \Delta)$  as

$$\tilde{U}_{s,t}(\xi, \Delta) = \int_{\mathbb{R}} U_{s,t}(x, \Delta) e^{-ix\xi} dx, \quad \tilde{K}_{s,t}(\xi, \Delta) = \int_{\mathbb{R}} K_{s,t}(x, \Delta) e^{-ix\xi} dx$$

for  $\xi \in \mathbb{R}$ . The Paley–Wiener theorem [25, Theorem 19.3] yields that  $\tilde{U}_{s,t}(\xi, \Delta) = \tilde{K}_{s,t}(\xi, \Delta) = 0$  for any  $|\xi| > \Delta$  since  $U_{s,t}(z, \Delta)$  and  $K_{s,t}(z, \Delta)$  are entire functions of exponential type such that

$$\limsup_{r \rightarrow \infty} \frac{\log |U_{s,t}(re^{i\theta}, \Delta)|}{r} \leq \Delta, \quad \limsup_{r \rightarrow \infty} \frac{\log |K_{s,t}(re^{i\theta}, \Delta)|}{r} \leq \Delta$$

uniformly in  $\theta \in \mathbb{R}$ . Then, we define the functions  $\mathcal{U}_{s,t}(z, \Delta)$  and  $\mathcal{K}_{s,t}(z, \Delta)$  as

$$\mathcal{U}_{s,t}(z, \Delta) = \frac{1}{2\pi} \sum_{|m| \leq \Delta} \tilde{U}_{s,t}(m, \Delta) z^m, \quad \mathcal{K}_{s,t}(z, \Delta) = \frac{1}{2\pi} \sum_{|m| \leq \Delta} \tilde{K}_{s,t}(m, \Delta) z^m$$

for  $z \in S^1$ . We have the Fourier series representation

$$(5.4) \quad \sum_{k \in \mathbb{Z}} U_{s,t}(\theta + 2k\pi, \Delta) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \tilde{U}_{s,t}(m, \Delta) e^{im\theta} = \mathcal{U}_{s,t}(e^{i\theta}, \Delta),$$

$$(5.5) \quad \sum_{k \in \mathbb{Z}} K_{s,t}(\theta + 2k\pi, \Delta) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \tilde{K}_{s,t}(m, \Delta) e^{im\theta} = \mathcal{K}_{s,t}(e^{i\theta}, \Delta)$$

for any  $\theta \in \mathbb{R}$ . Thus  $\mathcal{U}_{s,t}(z, \Delta)$  and  $\mathcal{K}_{s,t}(z, \Delta)$  are real valued functions.

**Lemma 5.1.** *Let  $N \geq 0$  be any integer. Let  $s_n, t_n \in \mathbb{R}$  for  $0 \leq n \leq N$  satisfying  $0 < t_n - s_n \leq 2\pi$ . Let  $\Delta > 3$ . Then we have*

$$\left| \prod_{n=0}^N \mathbf{1}_{A(s_n, t_n)}(z_n) - \prod_{n=0}^N \mathcal{U}_{s_n, t_n}(z_n, \Delta) \right| \ll (\log \Delta)^{N+1} \sum_{n=0}^N \mathcal{K}_{s_n, t_n}(z_n, \Delta)$$

for any  $(z_0, \dots, z_N) \in \prod_{n=0}^N S^1$ , where  $\mathbf{1}_{A(s,t)}$  denotes the indicator function of the arc  $A(s, t)$  as in (2.7). Here, the implied constant is absolute.

*Proof.* Let  $0 \leq n \leq N$ . Note that the formula

$$\mathbf{1}_{A(s_n, t_n)}(e^{i\theta}) = \sum_{k \in \mathbb{Z}} \mathbf{1}_{(s_n, t_n)}(\theta + 2k\pi)$$

holds for any  $\theta \in \mathbb{R}$ . Hence we have

$$(5.6) \quad \left| \mathbf{1}_{A(s_n, t_n)}(e^{i\theta}) - \mathcal{U}_{s_n, t_n}(e^{i\theta}, \Delta) \right| \leq \sum_{k \in \mathbb{Z}} \left| \mathbf{1}_{A(s_n, t_n)}(\theta + 2k\pi) - U_{s_n, t_n}(\theta + 2k\pi, \Delta) \right| \leq \mathcal{K}_{s_n, t_n}(e^{i\theta}, \Delta)$$

by (5.3), (5.4), and (5.5). Then, we prove the desired estimate by induction on  $N$ . Notice that (5.6) derives the result for  $N = 0$ . Furthermore, we obtain

$$\begin{aligned} & \left| \prod_{n=0}^{N+1} \mathbf{1}_{A(s_n, t_n)}(z_n) - \prod_{n=0}^{N+1} \mathcal{U}_{s_n, t_n}(z_n, \Delta) \right| \\ & \leq \prod_{n=0}^N \mathbf{1}_{A(s_n, t_n)}(z_n) \cdot \left| \mathbf{1}_{A(s_{N+1}, t_{N+1})}(z_{N+1}) - \mathcal{U}_{s_{N+1}, t_{N+1}}(z_{N+1}, \Delta) \right| \\ & \quad + \left| \prod_{n=0}^N \mathbf{1}_{A(s_n, t_n)}(z_n) - \prod_{n=0}^N \mathcal{U}_{s_n, t_n}(z_n, \Delta) \right| \cdot \left| \mathcal{U}_{s_{N+1}, t_{N+1}}(z_{N+1}, \Delta) \right| \end{aligned}$$

for any  $(z_0, \dots, z_{N+1}) \in \prod_{n=0}^{N+1} S^1$ . By the inductive assumption, it yields

$$(5.7) \quad \left| \prod_{n=0}^{N+1} \mathbf{1}_{A(s_n, t_n)}(z_n) - \prod_{n=0}^{N+1} \mathcal{U}_{s_n, t_n}(z_n, \Delta) \right| \\ \ll \mathcal{H}_{s_{N+1}, t_{N+1}}(z_{N+1}, \Delta) \\ + (\log \Delta)^{N+1} \sum_{n=0}^N \mathcal{H}_{s_n, t_n}(z_n, \Delta) \cdot \left| \mathcal{U}_{s_{N+1}, t_{N+1}}(z_{N+1}, \Delta) \right|.$$

By the definition of the function  $\mathcal{U}_{s,t}(z, \Delta)$ , we see that

$$(5.8) \quad \left| \mathcal{U}_{s,t}(z, \Delta) \right| \leq \frac{1}{2\pi} \sum_{|m| \leq \Delta} \left| \tilde{U}_{s,t}(m, \Delta) \right|.$$

for any  $s, t \in \mathbb{R}$  with  $0 < t - s \leq 2\pi$  and any  $z \in S^1$ . For  $m = 0$ , we have

$$\tilde{U}_{s,t}(0, \Delta) = \int_{\mathbb{R}} U_{s,t}(x, \Delta) dx \leq \int_{\mathbb{R}} \mathbf{1}_{(s,t)}(x) dx + \int_{\mathbb{R}} K_{s,t}(x, \Delta) dx$$

by (5.3). Here, the first integral of the right-hand side is bounded by  $2\pi$ , and the second integral is

$$(5.9) \quad \int_{\mathbb{R}} K_{s,t}(x, \Delta) dx = \int_{\mathbb{R}} K\left(\frac{\Delta x}{2\pi}\right) dx = \frac{2\pi}{\Delta} \int_{\mathbb{R}} \left(\frac{\sin \pi x}{\pi x}\right)^2 dx \ll \frac{1}{\Delta}$$

by the definition of  $K_{s,t}(x, \Delta)$ . Hence we have  $\tilde{U}_{s,t}(0, \Delta) \ll 1$  with an absolute implied constant. For  $m \neq 0$ , we obtain

$$\tilde{U}_{s,t}(m, \Delta) = -\frac{1}{im} \int_{\mathbb{R}} \left( \frac{\partial}{\partial x} U_{s,t}(x, \Delta) \right) e^{-imx} dx$$

by integrating by parts. Here, we apply the estimate  $H'(x) \ll (1 + |x|)^{-3}$  proved in [28, Theorem 6]. By the definition of  $U_{s,t}(x, \Delta)$ , it yields

$$\frac{\partial}{\partial x} U_{s,t}(x, \Delta) \ll \frac{\Delta}{(1 + \Delta|x - s|)^3} + \frac{\Delta}{(1 + \Delta|x - t|)^3}$$

uniformly for  $x \in \mathbb{R}$ . Hence we have  $\tilde{U}_{s,t}(m, \Delta) \ll 1/|m|$  with an absolute implied constant. By (5.8), we obtain

$$\left| \mathcal{U}_{s,t}(z, \Delta) \right| \ll 1 + \sum_{0 < m \leq \Delta} \frac{1}{m} \ll \log \Delta$$

for any  $s, t \in \mathbb{R}$  with  $0 < t - s \leq 2\pi$  and any  $z \in S^1$ . Inserting this to (5.7), we derive

$$\left| \prod_{n=0}^{N+1} \mathbf{1}_{A(s_n, t_n)}(z_n) - \prod_{n=0}^{N+1} \mathcal{U}_{s_n, t_n}(z_n, \Delta) \right| \\ \ll \mathcal{H}_{s_{N+1}, t_{N+1}}(z_{N+1}, \Delta) + (\log \Delta)^{N+2} \sum_{n=0}^N \mathcal{H}_{s_n, t_n}(z_n, \Delta) \\ \ll (\log \Delta)^{N+2} \sum_{n=0}^{N+1} \mathcal{H}_{s_n, t_n}(z_n, \Delta).$$

This is the result for  $N + 1$ , and hence the proof is completed.  $\square$

**5.2. Estimate of a conditional square mean value.** By the definition of the event  $\Omega_0$  as in (5.1), the indicator function  $\mathbf{1}_{\Omega_0}$  is represented as

$$\mathbf{1}_{\Omega_0} = \prod_{n=0}^N \mathbf{1}_{A(s_n, t_n)}(\mathbb{X}_\alpha(n)),$$

where we put  $s_n = \theta_n - \pi\delta$  and  $t_n = \theta_n + \pi\delta$ . Hence Lemma 5.1 yields

$$(5.10) \quad \mathbf{1}_{\Omega_0} = \prod_{n=0}^N \mathcal{U}_{s_n, t_n}(\mathbb{X}_\alpha(n), \Delta) + O\left((\log \Delta)^{N+1} \sum_{n=0}^N \mathcal{K}_{s_n, t_n}(\mathbb{X}_\alpha(n), \Delta)\right),$$

where the implied constant is absolute. Using this asymptotic formula, we prove the following result which plays an important role in the remaining part of the proof of Theorem 1.4.

**Proposition 5.2.** *Let  $0 < c < 1$  and  $0 < \delta < 1/2$ . Let  $\omega_0 \in \Omega$  be any sample. For any  $L > N \geq 1$  and any  $\Delta > \Delta_0$  with some absolute constant  $\Delta_0$ , there exists a finite subset  $\mathcal{E} \subset \mathcal{A}$  such that*

$$(5.11) \quad \mathbf{P}(\Omega_0) = \delta^{N+1} + O\left(\frac{N(\log \Delta)^{N+1}}{\Delta}\right),$$

$$(5.12) \quad \mathbf{E}_{\Omega_0} [\|\zeta(s, \mathbb{X}_\alpha) - \zeta_N(s, \mathbb{X}_\alpha)\|^2] \ll \mathbf{P}(\Omega_0) N^{1-2\sigma_0} + L^{1-2\sigma_0} + \frac{NL(\log \Delta)^{N+1}}{\Delta}$$

for any  $\alpha \in \mathcal{A} \setminus \mathcal{E}$ , where  $\mathcal{E}$  depends only on  $N$ ,  $L$ , and  $\Delta$ . The implied constants depend only on the compact subset  $K$ .

*Proof.* First, we evaluate the probability  $\mathbf{P}(\Omega_0)$ . Using (5.10), we obtain the formula

$$(5.13) \quad \mathbf{P}(\Omega_0) = \mathbf{E} \left[ \prod_{n=0}^N \mathcal{U}_{s_n, t_n}(\mathbb{X}_\alpha(n), \Delta) \right] + O\left((\log \Delta)^{N+1} \sum_{n=0}^N \mathbf{E}[\mathcal{K}_{s_n, t_n}(\mathbb{X}_\alpha(n), \Delta)]\right).$$

By the definition of the function  $\mathcal{U}_{s, t}(z, \Delta)$ , the first term of the right-hand side is calculated as

$$(5.14) \quad \mathbf{E} \left[ \prod_{n=0}^N \mathcal{U}_{s_n, t_n}(\mathbb{X}_\alpha(n), \Delta) \right] = \left(\frac{1}{2\pi}\right)^{N+1} \sum_{|m_0| \leq \Delta} \cdots \sum_{|m_N| \leq \Delta} \prod_{n=0}^N \tilde{U}_{s_n, t_n}(m_n, \Delta) \mathbf{E} \left[ \prod_{n=0}^N \mathbb{X}_\alpha(n)^{m_n} \right].$$

Define the set  $\mathcal{E}_1$  as

$$\mathcal{E}_1 = \bigcup_{\substack{(m_0, \dots, m_N) \in \mathbb{Z}^{N+1} \setminus \{0\} \\ \forall n, |m_n| \leq \Delta}} \left\{ \alpha \in \mathcal{A} \mid \prod_{n=0}^N (n + \alpha)^{m_n} = 1 \right\}.$$

Then  $\mathcal{E}_1$  is a finite subset of  $\mathcal{A}$  which is determined only by  $N$  and  $\Delta$ . Suppose  $\alpha \in \mathcal{A} \setminus \mathcal{E}_1$ . By the definition of  $\mathcal{E}_1$ , we see that  $\prod_{n=0}^N (n + \alpha)^{m_n} \neq 1$  is satisfied

unless  $m_0 = \dots = m_N = 0$  for any  $|m_n| \leq \Delta$ . Hence Lemma 2.1 yields

$$\mathbf{E} \left[ \prod_{n=0}^N \mathbb{X}_\alpha(n)^{m_n} \right] = \begin{cases} 1 & \text{if } m_0 = \dots = m_N = 0 \\ 0 & \text{otherwise} \end{cases}$$

for any  $|m_n| \leq \Delta$ . Inserting this to (5.14), we derive

$$\mathbf{E} \left[ \prod_{n=0}^N \mathcal{U}_{s_n, t_n}(\mathbb{X}_\alpha(n), \Delta) \right] = \left( \frac{1}{2\pi} \right)^{N+1} \prod_{n=0}^N \tilde{U}_{s_n, t_n}(0, \Delta).$$

Then we evaluate  $\tilde{U}_{s_n, t_n}(0, \Delta)$  for  $0 \leq n \leq N$ . Note that  $t_n - s_n = 2\pi\delta$  is satisfied by the setting of  $s_n$  and  $t_n$ . Applying (5.3), we have

$$\begin{aligned} \tilde{U}_{s_n, t_n}(0, \Delta) &= \int_{\mathbb{R}} U_{s_n, t_n}(x, \Delta) dx \\ &= \int_{\mathbb{R}} \mathbf{1}_{(s_n, t_n)}(x) dx + O \left( \int_{\mathbb{R}} K_{s_n, t_n}(x, \Delta) dx \right) \\ &= 2\pi\delta + O \left( \frac{1}{\Delta} \right), \end{aligned}$$

where the last line follows from (5.9). As a result, we obtain

$$(5.15) \quad \begin{aligned} \mathbf{E} \left[ \prod_{n=0}^N \mathcal{U}_{s_n, t_n}(\mathbb{X}_\alpha(n), \Delta) \right] &= \left( \frac{1}{2\pi} \right)^{N+1} \prod_{n=0}^N \left( 2\pi\delta + O \left( \frac{1}{\Delta} \right) \right) \\ &= \delta^{N+1} + O \left( \frac{N}{\Delta} \right) \end{aligned}$$

for any  $\Delta > \Delta_0$  with a large absolute constant  $\Delta_0$ . On the other hand, we have

$$\mathbf{E}[\mathcal{K}_{s_n, t_n}(\mathbb{X}_\alpha(n), \Delta)] = \frac{1}{2\pi} \sum_{|m| \leq \Delta} \tilde{K}_{s_n, t_n}(m, \Delta) \mathbf{E}[\mathbb{X}_\alpha(n)^m] = \frac{1}{2\pi} \tilde{K}_{s_n, t_n}(0, \Delta)$$

since Lemma 2.1 yields  $\mathbf{E}[\mathbb{X}_\alpha(n)^m] = 0$  unless  $m = 0$ . Furthermore,

$$\tilde{K}_{s_n, t_n}(0, \Delta) = \int_{\mathbb{R}} K_{s_n, t_n}(x, \Delta) dx \ll \frac{1}{\Delta}$$

by (5.9). Thus  $\mathbf{E}[\mathcal{K}_{s_n, t_n}(\mathbb{X}_\alpha(n), \Delta)] \ll 1/\Delta$  follows. Then we arrive at

$$(5.16) \quad (\log \Delta)^{N+1} \sum_{n=0}^N \mathbf{E}[\mathcal{K}_{s_n, t_n}(\mathbb{X}_\alpha(n), \Delta)] \ll \frac{N(\log \Delta)^{N+1}}{\Delta}.$$

Combining (5.13), (5.15), and (5.16), we see that (5.11) holds for any  $\alpha \in \mathcal{A} \setminus \mathcal{E}_1$ .

Next, we consider the conditional square mean value of (5.12). We divide it into the following two expected values:

$$(5.17) \quad \begin{aligned} &\mathbf{E}_{\Omega_0} [\|\zeta(s, \mathbb{X}_\alpha) - \zeta_N(s, \mathbb{X}_\alpha)\|^2] \\ &\leq \mathbf{E}_{\Omega_0} \left[ \left( \|\zeta(s, \mathbb{X}_\alpha) - \zeta_L(s, \mathbb{X}_\alpha)\| + \|\zeta_L(s, \mathbb{X}_\alpha) - \zeta_N(s, \mathbb{X}_\alpha)\| \right)^2 \right] \\ &\leq 2\mathbf{E}_{\Omega_0} [\|\zeta(s, \mathbb{X}_\alpha) - \zeta_L(s, \mathbb{X}_\alpha)\|^2] + 2\mathbf{E}_{\Omega_0} [\|\zeta_L(s, \mathbb{X}_\alpha) - \zeta_N(s, \mathbb{X}_\alpha)\|^2]. \end{aligned}$$



The first expected value is evaluated as

$$\begin{aligned} \mathbf{E}_{\Omega_0} [\|\zeta(s, \mathbb{X}_\alpha) - \zeta_L(s, \mathbb{X}_\alpha)\|^2] &= \mathbf{E} [\mathbf{1}_{\Omega_0} \cdot \|\zeta(s, \mathbb{X}_\alpha) - \zeta_L(s, \mathbb{X}_\alpha)\|^2] \\ &\leq \mathbf{E} [\|\zeta(s, \mathbb{X}_\alpha) - \zeta_L(s, \mathbb{X}_\alpha)\|^2]. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \mathbf{E} [\|\zeta(s, \mathbb{X}_\alpha) - \zeta_L(s, \mathbb{X}_\alpha)\|^2] &= \sum_{m, n > L} \mathbf{E}[\mathbb{X}_\alpha(m) \overline{\mathbb{X}_\alpha(n)}] \langle (m + \alpha)^{-s}, (n + \alpha)^{-s} \rangle \\ &= \sum_{n > L} \|(n + \alpha)^{-s}\|^2 \end{aligned}$$

since  $\mathbf{E}[\mathbb{X}_\alpha(m) \overline{\mathbb{X}_\alpha(n)}] = 0$  for  $m \neq n$  by Lemma 2.1. Then we see that

$$(5.18) \quad \sum_{n > L} \|(n + \alpha)^{-s}\|^2 \leq \sum_{n > L} M(n + \alpha)^{-2\sigma_0} \leq \frac{M}{2\sigma_0 - 1} L^{1-2\sigma_0},$$

where  $M = \iint_U d\sigma dt$  is a positive constant determined only by  $K$ . From the above, we deduce

$$(5.19) \quad \mathbf{E}_{\Omega_0} [\|\zeta(s, \mathbb{X}_\alpha) - \zeta_L(s, \mathbb{X}_\alpha)\|^2] \ll L^{1-2\sigma_0}$$

with an implied constant depending only on  $K$ . On the other hand, we obtain

$$\begin{aligned} (5.20) \quad \mathbf{E}_{\Omega_0} [\|\zeta_L(s, \mathbb{X}_\alpha) - \zeta_N(s, \mathbb{X}_\alpha)\|^2] &= \sum_{N < n_1, n_2 \leq L} \mathbf{E}_{\Omega_0}[\mathbb{X}_\alpha(n_1) \overline{\mathbb{X}_\alpha(n_2)}] \langle (n_1 + \alpha)^{-s}, (n_2 + \alpha)^{-s} \rangle \\ &= \mathbf{P}(\Omega_0) \sum_{N < n \leq L} \|(n + \alpha)^{-s}\|^2 \\ &\quad + \sum_{\substack{N < n_1, n_2 \leq L \\ n_1 \neq n_2}} \mathbf{E}_{\Omega_0}[\mathbb{X}_\alpha(n_1) \overline{\mathbb{X}_\alpha(n_2)}] \langle (n_1 + \alpha)^{-s}, (n_2 + \alpha)^{-s} \rangle. \end{aligned}$$

In a similar way that we obtain (5.18), the first term is estimated as

$$(5.21) \quad \mathbf{P}(\Omega_0) \sum_{N < n \leq L} \|(n + \alpha)^{-s}\|^2 \ll \mathbf{P}(\Omega_0) N^{1-2\sigma_0},$$

where the implied constant depends only on  $K$ . Let  $N < n_1, n_2 \leq L$  with  $n_1 \neq n_2$ . By (5.10), we obtain the formula

$$\begin{aligned} (5.22) \quad \mathbf{E}_{\Omega_0}[\mathbb{X}_\alpha(n_1) \overline{\mathbb{X}_\alpha(n_2)}] &= \mathbf{E} \left[ \mathbb{X}_\alpha(n_1) \overline{\mathbb{X}_\alpha(n_2)} \prod_{n=0}^N \mathcal{U}_{s_n, t_n}(\mathbb{X}_\alpha(n), \Delta) \right] \\ &\quad + O \left( (\log \Delta)^{N+1} \sum_{n=0}^N \mathbf{E}[\mathcal{U}_{s_n, t_n}(\mathbb{X}_\alpha(n), \Delta)] \right). \end{aligned}$$

Here, the first term on the right-hand side is calculated as

$$\begin{aligned}
(5.23) \quad & \mathbf{E} \left[ \mathbb{X}_\alpha(n_1) \overline{\mathbb{X}_\alpha(n_2)} \prod_{n=0}^N \mathcal{U}_{s_n, t_n}(\mathbb{X}_\alpha(n), \Delta) \right] \\
&= \left( \frac{1}{2\pi} \right)^{N+1} \sum_{|m_0| \leq \Delta} \cdots \sum_{|m_N| \leq \Delta} \\
&\quad \times \prod_{n=0}^N \tilde{\mathcal{U}}_{s_n, t_n}(m_n, \Delta) \mathbf{E} \left[ \mathbb{X}_\alpha(n_1) \overline{\mathbb{X}_\alpha(n_2)} \prod_{n=0}^N \mathbb{X}_\alpha(n)^{m_n} \right].
\end{aligned}$$

Define the set  $\mathcal{E}_2$  as

$$\mathcal{E}_2 = \bigcup_{\substack{(m_0, \dots, m_N) \in \mathbb{Z}^{N+1} \\ \forall n, |m_n| \leq \Delta}} \bigcup_{\substack{N < n_1, n_2 < L \\ n_1 \neq n_2}} \left\{ \alpha \in \mathcal{A} \mid \prod_{n=0}^N (n + \alpha)^{m_n} = \frac{n_2 + \alpha}{n_1 + \alpha} \right\}.$$

Then  $\mathcal{E}_2$  is a finite subset of  $\mathcal{A}$  which is determined only by  $N$ ,  $L$ , and  $\Delta$ . Suppose  $\alpha \in \mathcal{A} \setminus \mathcal{E}_2$ . By the definition of  $\mathcal{E}_2$ , we see that

$$(n_1 + \alpha)(n_2 + \alpha)^{-1} \prod_{n=0}^N (n + \alpha)^{m_n} = 1$$

never holds for any  $|m_n| \leq \Delta$ . Therefore, Lemma 2.1 yields that all expected values appearing in (5.23) are equal to 0. Thus we find that

$$(5.24) \quad \mathbf{E} \left[ \mathbb{X}_\alpha(n_1) \overline{\mathbb{X}_\alpha(n_2)} \prod_{n=0}^N \mathcal{U}_{s_n, t_n}(\mathbb{X}_\alpha(n), \Delta) \right] = 0.$$

By (5.24) and (5.16), it is deduced from (5.22) that

$$\mathbf{E}_{\Omega_0}[\mathbb{X}_\alpha(n_1) \overline{\mathbb{X}_\alpha(n_2)}] \ll \frac{N(\log \Delta)^{N+1}}{\Delta}$$

for any  $N < n_1, n_2 \leq L$  with  $n_1 \neq n_2$ , where the implied constant is absolute. Hence we obtain

$$\begin{aligned}
& \sum_{\substack{N < n_1, n_2 \leq L \\ n_1 \neq n_2}} \mathbf{E}_{\Omega_0}[\mathbb{X}_\alpha(n_1) \overline{\mathbb{X}_\alpha(n_2)}] \langle (n_1 + \alpha)^{-s}, (n_2 + \alpha)^{-s} \rangle \\
& \ll \frac{N(\log \Delta)^{N+1}}{\Delta} \sum_{\substack{N < n_1, n_2 \leq L \\ n_1 \neq n_2}} |\langle (n_1 + \alpha)^{-s}, (n_2 + \alpha)^{-s} \rangle|.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\sum_{\substack{N < n_1, n_2 \leq L \\ n_1 \neq n_2}} |\langle (n_1 + \alpha)^{-s}, (n_2 + \alpha)^{-s} \rangle| &\leq \sum_{N < n_1, n_2 \leq L} \|(n_1 + \alpha)^{-s}\| \|(n_2 + \alpha)^{-s}\| \\
&\leq \left( \sum_{N < n \leq L} \|(n + \alpha)^{-s}\| \right)^2
\end{aligned}$$

by the Cauchy–Schwarz inequality. Here, the last sum is evaluated as

$$\sum_{N < n \leq L} \|(n + \alpha)^{-s}\| \leq \sum_{N < n \leq L} \sqrt{M}(n + \alpha)^{-1/2} \leq 2\sqrt{M}\sqrt{L},$$

where we put  $M = \iint_U d\sigma dt$  as before. Therefore we obtain

$$(5.25) \quad \sum_{\substack{N < n_1, n_2 \leq L \\ n_1 \neq n_2}} \mathbf{E}_{\Omega_0}[\mathbb{X}_\alpha(n_1)\overline{\mathbb{X}_\alpha(n_2)}] \langle (n_1 + \alpha)^{-s}, (n_2 + \alpha)^{-s} \rangle \ll \frac{NL(\log \Delta)^{N+1}}{\Delta},$$

where the implied constant depends only on  $K$ . By (5.21) and (5.25), it is deduced from (5.20) that

$$(5.26) \quad \mathbf{E}_{\Omega_0} [\|\zeta_L(s, \mathbb{X}_\alpha) - \zeta_N(s, \mathbb{X}_\alpha)\|^2] \ll \mathbf{P}(\Omega_0)N^{1-2\sigma_0} + \frac{NL(\log \Delta)^{N+1}}{\Delta}.$$

Combining (5.17), (5.19), and (5.26), we derive (5.12) for any  $\alpha \in \mathcal{A} \setminus \mathcal{E}_2$ . Then we obtain the desired result by letting  $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ .  $\square$

**Remark 5.3.** Let  $\delta, N, \alpha, \omega_0$  be in Proposition 5.2. Define the event  $\Omega'_0 \subset \Omega$  as

$$\begin{aligned} \Omega'_0 &= \Omega'_0(\delta; N, \alpha, \omega_0) \\ &= \{\omega \in \Omega \mid \mathbb{Y}_\alpha(n)(\omega) \in A(\theta_n - \pi\delta, \theta_n + \pi\delta) \text{ for any } 0 \leq n \leq N\} \end{aligned}$$

as an analogue of (5.1). The probability  $\mathbf{P}(\Omega'_0)$  is calculated as

$$\mathbf{P}(\Omega'_0) = \prod_{n=0}^N \mathbf{P}(\mathbb{Y}_\alpha(n) \in A(\theta_n - \pi\delta, \theta_n + \pi\delta)) = \delta^{N+1}$$

for any  $\alpha \in \mathcal{A}$  since  $\mathbb{Y}_\alpha(0), \dots, \mathbb{Y}_\alpha(N)$  are independent. Furthermore, we have

$$\begin{aligned} \mathbf{E}_{\Omega'_0} [\|\zeta(s, \mathbb{Y}_\alpha) - \zeta_N(s, \mathbb{Y}_\alpha)\|^2] &= \mathbf{E}[\mathbf{1}_{\Omega'_0}] \cdot \mathbf{E} [\|\zeta(s, \mathbb{Y}_\alpha) - \zeta_N(s, \mathbb{Y}_\alpha)\|^2] \\ &\ll \mathbf{P}(\Omega'_0)N^{1-2\sigma_0} \end{aligned}$$

for any  $\alpha \in \mathcal{A}$ . Proposition 5.2 means that similar results are valid for all but finitely many  $\alpha \in \mathcal{A}$  if we replace the above  $\mathbb{Y}_\alpha(n)$  with  $\mathbb{X}_\alpha(n)$ .

**5.3. Proof of Theorem 1.4.** In this section, we finally complete the proof of the main result. The last lemma required for the proof is the Mergelyan theorem, which is a complex analogue of the Weierstrass approximation theorem.

**Lemma 5.4** (Mergelyan theorem). *Let  $K$  be a compact subset of  $\mathbb{C}$  with connected complement. Let  $f$  be a continuous function on  $K$  which is analytic in the interior of  $K$ . Then, for every  $\epsilon > 0$ , there exists a polynomial  $p(s)$  such that*

$$\sup_{s \in K} |f(s) - p(s)| < \epsilon.$$

*Proof.* See [25, Theorem 20.5] for a proof.  $\square$

*Proof of Theorem 1.4.* Let  $K, f, \epsilon$  be as in the statement of Theorem 1.4. Then we deduce from Lemma 5.4 that

$$\sup_{s \in K} |f(s) - p(s)| < \frac{\epsilon}{2}$$

with some polynomial  $p(s)$ . Remark that  $p(s)$  is obviously an element of  $H(D)$ . For any  $\alpha \in \mathcal{A}$ , we obtain

$$(5.27) \quad \begin{aligned} & \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] \mid \sup_{s \in K} |\zeta(s + i\tau, \alpha) - f(s)| < \epsilon \right\} \\ & \geq \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] \mid \sup_{s \in K} |\zeta(s + i\tau, \alpha) - p(s)| < \frac{\epsilon}{2} \right\} \\ & = P_{\alpha, T}(A), \end{aligned}$$

where the subset  $A \subset H(D)$  is defined as

$$A = \left\{ g \in H(D) \mid \sup_{s \in K} |g(s) - p(s)| < \frac{\epsilon}{2} \right\}.$$

Here, we check that it is an open set of  $H(D)$ . Let  $\{g_n\}$  be any sequence of functions in  $H(D) \setminus A$  such that  $g_n$  converges to  $g \in H(D)$  as  $n \rightarrow \infty$ . Then  $g_n(s)$  converges uniformly on  $K$  by the definition of the topology of  $H(D)$ . Thus we obtain

$$\sup_{s \in K} |g(s) - p(s)| = \lim_{n \rightarrow \infty} \sup_{s \in K} |g_n(s) - p(s)| \geq \frac{\epsilon}{2},$$

which implies  $g \notin A$ . Hence  $H(D) \setminus A$  is closed, and equivalently,  $A$  is open. Then it is deduced from Theorem 3.1 and Lemma 3.7 that

$$(5.28) \quad \liminf_{T \rightarrow \infty} P_{\alpha, T}(A) \geq Q_{\alpha}(A) = \mathbf{P} \left( \sup_{s \in K} |\zeta(s, \mathbb{X}_{\alpha}) - p(s)| < \frac{\epsilon}{2} \right).$$

Let  $s_0 \in K$ , and put  $R = \min\{|z - s| \mid z \in \partial U, s \in K\}$ . Here,  $U$  is a fixed domain such that  $K \subset U$  and  $\bar{U} \subset D$ , whose boundary  $\partial U$  is a Jordan curve. Then Cauchy's integral formula yields that

$$\begin{aligned} |\zeta(s_0, \mathbb{X}_{\alpha})(\omega) - p(s_0)| &= \left| \frac{1}{2\pi i} \oint_{|z-s_0|=r} \frac{\zeta(z, \mathbb{X}_{\alpha})(\omega) - p(z)}{z - s_0} dz \right| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \left| \zeta(re^{i\theta}, \mathbb{X}_{\alpha})(\omega) - p(re^{i\theta}) \right| d\theta \end{aligned}$$

for  $\omega \in \Omega_u$  and  $0 < r < R$ , where  $\Omega_u$  is the same as in the proof of Proposition 2.4. Therefore, we obtain

$$\begin{aligned} |\zeta(s_0, \mathbb{X}_{\alpha})(\omega) - p(s_0)| &\leq \frac{1}{\pi R^2} \int_0^R \left( \int_0^{2\pi} \left| \zeta(re^{i\theta}, \mathbb{X}_{\alpha})(\omega) - p(re^{i\theta}) \right| d\theta \right) r dr \\ &\leq \frac{1}{\pi R^2} \iint_U |\zeta(s, \mathbb{X}_{\alpha})(\omega) - p(s)| d\sigma dt \\ &\leq \frac{\sqrt{M}}{\pi R^2} \|\zeta(s, \mathbb{X}_{\alpha})(\omega) - p(s)\| \end{aligned}$$

by the Cauchy–Schwarz inequality, where  $M = \iint_U d\sigma dt$ . Hence we have

$$\sup_{s \in K} |\zeta(s, \mathbb{X}_{\alpha})(\omega) - p(s)| \leq C_K \|\zeta(s, \mathbb{X}_{\alpha})(\omega) - p(s)\|$$

with a positive constant  $C_K$  depending only on  $K$ . It yields the inequality

$$(5.29) \quad \mathbf{P} \left( \sup_{s \in K} |\zeta(s, \mathbb{X}_{\alpha}) - p(s)| < \frac{\epsilon}{2} \right) \geq \mathbf{P} \left( \|\zeta(s, \mathbb{X}_{\alpha}) - p(s)\| < \frac{\epsilon}{2C_K} \right).$$

Combining (5.27), (5.28), and (5.29), we derive

$$(5.30) \quad \liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] \mid \sup_{s \in K} |\zeta(s + i\tau, \alpha) - f(s)| < \epsilon \right\} \\ \geq \mathbf{P} \left( \|\zeta(s, \mathbb{X}_\alpha) - p(s)\| < \frac{\epsilon}{2C_K} \right)$$

for any  $\alpha \in \mathcal{A}$ . Then, we prove that the last probability is positive if  $\alpha \in \mathcal{A}_\rho(c) \setminus \mathcal{E}$ , where  $\rho$  is as in Corollary 4.9, and  $\mathcal{E}$  is as in Proposition 5.2. As a consequence of Corollary 4.9, there exists a sample  $\omega_0 \in \Omega$  such that

$$\|\zeta_N(s, \mathbb{Y}_\alpha)(\omega_0) - p(s)\| < \frac{\epsilon}{8C_K}$$

for any  $\alpha \in \mathcal{A}_\rho(c)$ , where  $N > N_0$  is an integer chosen later. Here, we can assume that  $\rho$  satisfies  $0 < \rho \leq \min\{c, 1 - c\}/2$ . For  $\alpha \in \mathcal{A}_\rho(c)$ , we denote by  $\Omega_0$  the event defined as (5.1). If  $\omega \in \Omega_0$ , then we have  $|\mathbb{X}_\alpha(n)(\omega) - \mathbb{Y}_\alpha(n)(\omega_0)| \leq 2\pi\delta$  for any  $0 \leq n \leq N$ . It derives

$$\|\zeta_N(s, \mathbb{X}_\alpha)(\omega) - \zeta_N(s, \mathbb{Y}_\alpha)(\omega_0)\| \leq \sum_{n=0}^N |\mathbb{X}_\alpha(n)(\omega) - \mathbb{Y}_\alpha(n)(\omega_0)| \|(n + \alpha)^{-s}\| \\ \leq \sum_{n=0}^N 2\pi\delta \sqrt{M} (n + c/2)^{-1/2} \\ \leq \delta B_1 \sqrt{N}$$

for  $\omega \in \Omega_0$ , where  $B_1 = B_1(c, K)$  is a positive constant depending only on  $c$  and  $K$ . Assume that  $\delta = \delta(N; \epsilon, K, B_1)$  satisfies

$$0 < \delta < \min \left\{ \frac{1}{B_1 \sqrt{N}} \frac{\epsilon}{8C_K}, \frac{1}{2} \right\}.$$

We can choose such a  $\delta$  depending only on  $c, \epsilon, K$ , and  $N$ . Then, we see that the norm  $\|\zeta_N(s, \mathbb{X}_\alpha)(\omega) - p(s)\|$  is evaluated as

$$\|\zeta_N(s, \mathbb{X}_\alpha)(\omega) - p(s)\| \\ \leq \|\zeta_N(s, \mathbb{X}_\alpha)(\omega) - \zeta_N(s, \mathbb{Y}_\alpha)(\omega_0)\| + \|\zeta_N(s, \mathbb{Y}_\alpha)(\omega_0) - p(s)\| \\ < \frac{\epsilon}{4C_K}$$

for any  $\omega \in \Omega_0$ . Hence we derive the inequality

$$(5.31) \quad \mathbf{P} \left( \|\zeta(s, \mathbb{X}_\alpha) - p(s)\| < \frac{\epsilon}{2C_K} \right) \\ \geq \mathbf{P} \left( \Omega_0 \cap \left\{ \|\zeta(s, \mathbb{X}_\alpha) - \zeta_N(s, \mathbb{X}_\alpha)\| < \frac{\epsilon}{4C_K} \right\} \right) \\ = \mathbf{P}(\Omega_0) - \mathbf{P} \left( \Omega_0 \cap \left\{ \|\zeta(s, \mathbb{X}_\alpha) - \zeta_N(s, \mathbb{X}_\alpha)\| \geq \frac{\epsilon}{4C_K} \right\} \right).$$

By Chebyshev's inequality and Proposition 5.2, we have

$$\begin{aligned}
(5.32) \quad & \mathbf{P} \left( \Omega_0 \cap \left\{ \|\zeta(s, \mathbb{X}_\alpha) - \zeta_N(s, \mathbb{X}_\alpha)\| \geq \frac{\epsilon}{4C_K} \right\} \right) \\
& \leq \left( \frac{4C_K}{\epsilon} \right)^2 \mathbf{E}_{\Omega_0} [\|\zeta(s, \mathbb{X}_\alpha) - \zeta_N(s, \mathbb{X}_\alpha)\|^2] \\
& \leq B_2 \left\{ \mathbf{P}(\Omega_0)N^{1-2\sigma_0} + L^{1-2\sigma} + \frac{NL(\log \Delta)^{N+1}}{\Delta} \right\}
\end{aligned}$$

for any  $\alpha \in \mathcal{A}_\rho(c) \setminus \mathcal{E}$ , where  $B_2 = B_2(\epsilon, K)$  is a positive constant depending only on  $\epsilon$  and  $K$ . Then it is deduced from (5.31) and (5.32) that

$$\begin{aligned}
(5.33) \quad & \mathbf{P} \left( \|\zeta(s, \mathbb{X}_\alpha) - p(s)\| < \frac{\epsilon}{2C_K} \right) \\
& \geq (1 - B_2N^{1-2\sigma_0})\mathbf{P}(\Omega_0) - B_2L^{1-2\sigma_0} - B_2\frac{NL(\log \Delta)^{N+1}}{\Delta}.
\end{aligned}$$

We choose the integer  $N = N(\sigma_0, N_0, B_2) > N_0$  so that  $B_2N^{1-2\sigma_0} \leq 1/2$  is satisfied. Note that  $\sigma_0$  depends only on  $K$ , and that  $N_0$  of Corollary 4.9 depends only on  $c, \epsilon, f, K$ . Thus the integer  $N$  depends only on  $c, \epsilon, f, K$ . Then we obtain

$$(1 - B_2N^{1-2\sigma_0})\mathbf{P}(\Omega_0) \geq \frac{1}{2}\mathbf{P}(\Omega_0) \geq \frac{1}{2}\delta^{N+1} - B_3\frac{N(\log \Delta)^{N+1}}{\Delta}$$

for any  $\alpha \in \mathcal{A}_\rho(c) \setminus \mathcal{E}$  by Proposition 5.2, where  $B_3 = B_3(K)$  is a positive constant depending only on  $K$ . Furthermore, we choose  $L = L(\delta, \sigma_0, N, B_2) > N$  so that

$$B_2L^{1-2\sigma_0} < \frac{1}{4}\delta^{N+1}$$

is satisfied. Lastly, we choose a real number  $\Delta = \Delta(\delta, N, L, B_2, B_3) > \Delta_0$  so that

$$B_2\frac{NL(\log \Delta)^{N+1}}{\Delta} + B_3\frac{N(\log \Delta)^{N+1}}{\Delta} < \frac{1}{4}\delta^{N+1}$$

is satisfied. As a result, we derive by (5.33) that

$$(5.34) \quad \mathbf{P} \left( \|\zeta(s, \mathbb{X}_\alpha) - p(s)\| < \frac{\epsilon}{2C_K} \right) > 0$$

for any  $\alpha \in \mathcal{A}_\rho(c) \setminus \mathcal{E}$ . From the above setting,  $N, L$ , and  $\Delta$  depend only on  $c, \epsilon, f, K$ , and therefore, the finite set  $\mathcal{E}$  is determined only by  $c, \epsilon, f, K$ . By (5.30) and (5.34), we obtain the desired result.  $\square$

## 6. CONSEQUENCES OF THE MAIN RESULT

Let  $f \in H(D)$  and  $1/2 < \sigma_0 < 1$ . By Cauchy's integral formula, we have

$$(6.1) \quad f^{(n)}(\sigma_0) = \frac{n!}{2\pi i} \oint_{|s-\sigma_0|=r} \frac{f(s)}{(s-\sigma_0)^{n+1}} ds$$

for  $n \geq 0$ , where  $r$  satisfies  $0 < r < \min\{\sigma_0 - 1/2, 1 - \sigma_0\}$ . Applying this formula, we prove that Theorem 1.4 implies Theorem 1.5.

*Proof of Theorem 1.5.* Denote by  $K$  the disc  $K = \{s \in \mathbb{C} \mid |s - \sigma_0| \leq r\}$ , where  $r$  is taken as  $r = \min\{\sigma_0 - 1/2, 1 - \sigma_0\}/2$ . Then  $K$  is a compact subset of the strip  $D$  with connected complement. Define the function  $f$  as

$$f(s) = \sum_{n=0}^N \frac{z_n}{n!} (s - \sigma_0)^n$$

by using  $\underline{z} = (z_0, \dots, z_N) \in \mathbb{C}^{N+1}$ . Note that  $f^{(n)}(\sigma_0) = z_n$  by definition. Therefore, we deduce from (6.1) that

$$\begin{aligned} |\zeta^{(n)}(\sigma_0 + i\tau, \alpha) - z_n| &= \left| \frac{n!}{2\pi i} \oint_{|s-\sigma_0|=r} \frac{\zeta(s + i\tau, \alpha) - f(s)}{(s - \sigma_0)^{n+1}} ds \right| \\ &\leq \frac{n!}{r^n} \sup_{s \in K} |\zeta(s + i\tau, \alpha) - f(s)|. \end{aligned}$$

For every  $\epsilon > 0$ , we put

$$\epsilon' = \epsilon \cdot \left( 1 + \frac{1}{r} + \dots + \frac{N!}{r^N} \right)^{-1}.$$

Then we see that  $|\zeta^{(n)}(\sigma_0 + i\tau, \alpha) - z_n| < \epsilon$  is satisfied for any  $0 \leq n \leq N$  if we suppose  $\sup_{s \in K} |\zeta(s + i\tau, \alpha) - f(s)| < \epsilon'$ . Since the function  $f$  is continuous on  $K$  and analytic in the interior of  $K$ , we can apply Theorem 1.4. Hence, there exist a positive real number  $\rho$  and a finite subset  $\mathcal{E} \subset \mathcal{A}_\rho(c)$  depending on  $c, \epsilon', f, K$  such that

$$\begin{aligned} &\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] \mid \max_{0 \leq n \leq N} |\zeta^{(n)}(\sigma_0 + i\tau, \alpha) - z_n| < \epsilon \right\} \\ &\geq \liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] \mid \sup_{s \in K} |\zeta(s + i\tau, \alpha) - f(s)| < \epsilon' \right\} > 0 \end{aligned}$$

for any  $\alpha \in \mathcal{A}_\rho(c) \setminus \mathcal{E}$ . Recall that  $\epsilon', f, K$  are determined only by  $\epsilon, \sigma_0, \underline{z}$ . Thus  $\rho$  and  $\mathcal{E}$  depend on  $c, \epsilon, \sigma_0, \underline{z}$ , and we obtain the conclusion.  $\square$

For the proof of Theorem 1.6, we apply Rouché's theorem [25, Theorem 10.43]. Let  $f, g \in H(D)$  and  $1/2 < \sigma_0 < 1$ . Suppose that the inequality

$$(6.2) \quad \sup_{|s-\sigma_0|=r} |g(s) - f(s)| < \inf_{|s-\sigma_0|=r} |f(s)|$$

holds, where  $r$  satisfies  $0 < r < \min\{\sigma_0 - 1/2, 1 - \sigma_0\}$ . Then  $g$  has the same number of zeros as that of  $f$  in the region  $|s - \sigma_0| < r$ .

*Proof of Theorem 1.6.* For  $1/2 < \sigma_1 < \sigma_2 < 1$ , we take the real numbers  $\sigma_0$  and  $r$  as  $\sigma_0 = (\sigma_1 + \sigma_2)/2$  and  $r = (\sigma_2 - \sigma_1)/4$ . Then  $K = \{s \in \mathbb{C} \mid |s - \sigma_0| \leq r\}$  is a compact subset of the strip  $D$  with connected complement. Furthermore, we define the function  $f$  as

$$f(s) = s - \sigma_0.$$

Obviously, it is continuous on  $K$  and analytic in the interior of  $K$ . Therefore we can apply Theorem 1.4. Then there exist a positive real number  $\rho$  and a finite subset

$\mathcal{E} \subset \mathcal{A}_\rho(c)$  such that

$$(6.3) \quad \liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] \mid \sup_{s \in K} |\zeta(s + i\tau, \alpha) - f(s)| < r \right\} > 0$$

for any  $\alpha \in \mathcal{A}_\rho(c) \setminus \mathcal{E}$ . Define  $\mathfrak{S}(\alpha)$  as the set of all  $\tau \in \mathbb{R}_{\geq 0}$  such that the inequality  $\sup_{s \in K} |\zeta(s + i\tau, \alpha) - f(s)| < r$  is satisfied. Then, for any  $\alpha \in \mathcal{A}_\rho(c) \setminus \mathcal{E}$ , there exists a sequence  $\{\tau_n\}$  of elements in  $\mathfrak{S}(\alpha)$  such that

$$\tau_{n+1} \geq \tau_n + 2r \quad \text{and} \quad \mathfrak{S}(\alpha) \subset \bigcup_{n=1}^{\infty} [\tau_n - r, \tau_n + r]$$

by (6.3). Furthermore, we see that

$$\sup_{|s - \sigma_0| = r} |\zeta(s + i\tau_n, \alpha) - f(s)| < r = \inf_{|s - \sigma_0| = r} |f(s)|$$

for such  $\tau_n$ . Hence (6.2) holds with  $g(s) = \zeta(s + i\tau_n, \alpha)$ . By Rouché's theorem, the function  $\zeta(s + i\tau_n, \alpha)$  has exactly one zero in  $|s - \sigma_0| < r$ , that is,  $\zeta(s, \alpha)$  has exactly one zero in the region

$$U_n = \{s \in \mathbb{C} \mid |s - (\sigma_0 + i\tau_n)| < r\}$$

for any  $n \geq 1$ . Note that the regions  $U_n$  are distinct by  $\tau_{n+1} \geq \tau_n + 2r$ . As a result, we obtain

$$(6.4) \quad N_\alpha(\sigma_1, \sigma_2, T) \geq n(T) := \max \{n \mid \tau_n + r \leq T\}$$

since  $\bigcup_{n=1}^{n(T)} U_n$  is included in the rectangle  $\sigma_1 \leq \sigma \leq \sigma_2$ ,  $0 \leq t \leq T$ . On the other hand, the inequality

$$(6.5) \quad \text{meas}(\mathfrak{S}(\alpha) \cap [0, T]) \leq \text{meas} \left( \bigcup_{n=1}^{n(T)+1} [\tau_n - r, \tau_n + r] \right) = 2r(n(T) + 1)$$

holds since  $\mathfrak{S}(\alpha) \cap [0, T]$  is covered by  $\bigcup_{n=1}^{n(T)+1} [\tau_n - r, \tau_n + r]$ . Combining (6.3), (6.4), and (6.5), we have

$$N_\alpha(\sigma_1, \sigma_2, T) \gg \text{meas}(\mathfrak{S}(\alpha) \cap [0, T]) \gg T$$

as  $T \rightarrow \infty$  for any  $\alpha \in \mathcal{A}_\rho(c) \setminus \mathcal{E}$ . Here, we recall that  $\rho$  and  $\mathcal{E}$  depend on  $c, r, f, K$ . Since  $r, f, K$  are determined only by  $\sigma_1$  and  $\sigma_2$ , they depend on  $c, \sigma_1, \sigma_2$ . Therefore the proof is completed.  $\square$

## REFERENCES

- [1] B. Bagchi, *The statistical behaviour and universality properties of the Riemann zeta function and other allied Dirichlet series*, Indian Statistical Institute, Calcutta, 1981, Thesis (Ph.D.).
- [2] G. D. Birkhoff, *Démonstration d'un théorème élémentaire sur les fonctions entières*, C. R. Acad. Sci., Paris **189** (1929), 473–475.
- [3] J. W. S. Cassels, *Footnote to a note of Davenport and Heilbronn*, J. London Math. Soc. **36** (1961), 177–184. MR 146359
- [4] H. Davenport and H. Heilbronn, *On the Zeros of Certain Dirichlet Series*, J. London Math. Soc. **11** (1936), no. 3, 181–185. MR 1574345
- [5] M. Einsiedler and T. Ward, *Functional analysis, spectral theory, and applications*, Graduate Texts in Mathematics, vol. 276, Springer, Cham, 2017. MR 3729416
- [6] O. J. Farrell, *On approximation to an analytic function by polynomials*, Bull. Amer. Math. Soc. **40** (1934), no. 12, 908–914. MR 1562998



- [7] R. Garunkštis, *Note on the zeros of the Hurwitz zeta-function*, Voronoi's impact on modern science. Book III. Proceedings of the 3rd Voronoi conference on analytic number theory and spatial tessellations., Adv. Stud. Pure Math., vol. 49, Kyiv: Institute of Mathematics, 2005, pp. 10–12.
- [8] R. Garunkštis, A. Laurinčikas, K. Matsumoto, J. Steuding, and R. Steuding, *Effective uniform approximation by the Riemann zeta-function*, Publ. Mat. **54** (2010), no. 1, 209–219. MR 2603597
- [9] S. M. Gonek, *Analytic properties of zeta and L-functions*, ProQuest LLC, Ann Arbor, MI, 1979, Thesis (Ph.D.)—University of Michigan. MR 2628587
- [10] A. Good, *On the distribution of the values of Riemann's zeta function*, Acta Arith. **38** (1980/81), no. 4, 347–388. MR 621007
- [11] H. Heyer, *Probability measures on locally compact groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 94, Springer-Verlag, Berlin-New York, 1977. MR 0501241
- [12] A. A. Karatsuba and S. M. Voronin, *The Riemann zeta-function*, De Gruyter Expositions in Mathematics, vol. 5, Walter de Gruyter & Co., Berlin, 1992, Translated from the Russian by Neal Koblitz. MR 1183467
- [13] A. Klenke, *Probability theory—a comprehensive course*, Universitext, Springer, Cham, 2020, Third edition. MR 4201399
- [14] E. Kowalski, *Bagchi's theorem for families of automorphic forms*, Exploring the Riemann zeta function, Springer, Cham, 2017, pp. 181–199. MR 3700042
- [15] ———, *An introduction to probabilistic number theory*, Cambridge Studies in Advanced Mathematics, vol. 192, Cambridge University Press, Cambridge, 2021. MR 4274079
- [16] A. Laurinčikas, *Limit theorems for the Riemann zeta-function*, Mathematics and its Applications, vol. 352, Kluwer Academic Publishers Group, Dordrecht, 1996. MR 1376140
- [17] A. Laurinčikas and K. Matsumoto, *The universality of zeta-functions attached to certain cusp forms*, Acta Arith. **98** (2001), no. 4, 345–359. MR 1829777
- [18] J. Marcinkiewicz, *Sur les nombres dérivés*, Fundam. Math. **24** (1935), 305–308.
- [19] K. Matsumoto, *A survey on the theory of universality for zeta and L-functions*, Number theory, Ser. Number Theory Appl., vol. 11, World Sci. Publ., Hackensack, NJ, 2015, pp. 95–144. MR 3382056
- [20] M. Mine, *A random variable related to the Hurwitz zeta-function with algebraic parameter*, 2022, preprint, <https://arxiv.org/abs/2210.15252>.
- [21] ———, *An upper bound for the number of smooth values of a polynomial and its applications*, 2024, preprint, <https://arxiv.org/abs/2410.09558>.
- [22] H. L. Montgomery and R. C. Vaughan, *Hilbert's inequality*, J. London Math. Soc. (2) **8** (1974), 73–82. MR 337775
- [23] J. Pál, *Zwei kleine Bemerkungen*, Tôhoku Math. J. **6** (1915), 42–43.
- [24] A. Reich, *Werteverteilung von Zetafunktionen*, Arch. Math. (Basel) **34** (1980), no. 5, 440–451. MR 593771
- [25] W. Rudin, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987. MR 924157
- [26] A. Sourmelidis and J. Steuding, *On the value-distribution of Hurwitz zeta-functions with algebraic parameter*, Constr. Approx. **55** (2022), no. 3, 829–860. MR 4434025
- [27] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, second ed., The Clarendon Press, Oxford University Press, New York, 1986, Edited and with a preface by D. R. Heath-Brown. MR 882550
- [28] J. D. Vaaler, *Some extremal functions in Fourier analysis*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), no. 2, 183–216. MR 776471
- [29] S. M. Voronin, *Theorem on the “universality” of the Riemann zeta-function*, Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), no. 3, 475–486. MR 0472727
- [30] ———,  *$\Omega$ -theorems of the theory of the Riemann zeta-function*, Izv. Akad. Nauk SSSR Ser. Mat. **52** (1988), no. 2, 424–436, 448. MR 941684

GLOBAL EDUCATION CENTER, WASEDA UNIVERSITY, 1-6-1 NISHIWASEDA, SHINJUKU-KU, TOKYO 169-8050, JAPAN

Email address: m-mine@aoni.waseda.jp