

MICROPHYSICAL REGULATION OF NON-IDEAL MHD IN WEAKLY-IONIZED SYSTEMS: DOES THE HALL EFFECT MATTER?

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ABSTRACT

The magnetohydrodynamics (MHD) equations plus “non-ideal” (Ohmic, Hall, ambipolar) resistivities are widely used to model weakly-ionized astrophysical systems. We show that if gradients in the magnetic field become too steep, the implied charge drift speeds become much faster than microphysical signal speeds, invalidating the assumptions used to derive both the resistivities and MHD equations themselves. Generically this situation will excite microscale instabilities that suppress the drift and current. We show this could be relevant at low ionization fractions especially if Hall terms appear significant, external forces induce supersonic motions, or dust grains become a dominant charge carrier. Considering well-established treatments of super-thermal drifts in laboratory, terrestrial, and Solar plasmas as well as conduction and viscosity models, we generalize a simple prescription to rectify these issues, where the resistivities are multiplied by a correction factor that depends only on already-known macroscopic quantities. This is generalized for multi-species and weakly-ionized systems, and leaves the equations unchanged in the drift limits for which they are derived, but restores physical behavior (driving the system back towards slow drift by diffusing away small-scale gradients in the magnetic field) if the limits are violated. This has important consequences: restoring intuitive behaviors such as the system becoming hydrodynamic in the limit of zero ionization; suppressing magnetic structure on scales below a critical length which can be comparable to circumstellar disk sizes; limiting the maximum magnetic amplification; and suppressing the effects of the Hall term in particular. This likely implies that the Hall term does not become dynamically important under most conditions of interest in these systems.

Subject headings: magnetohydrodynamics (MHD) – magnetic fields – protoplanetary/circumstellar disks – ISM: clouds – star formation – planet formation – methods: numerical

1. INTRODUCTION

Magnetohydrodynamics (MHD) is fundamental to almost all astrophysical systems. In particular, the last decade has seen a flurry of simulations and detailed models of MHD in weakly-ionized systems such as circumstellar/protoplanetary disks, disk and cool-star winds, or planet formation (Flock et al. 2011; Bai 2011; Zhu & Stone 2014; Keith & Wardle 2014; Tomida et al. 2015; Simon et al. 2015; Wurster et al. 2016; Xu et al. 2019; Wang et al. 2019; Hennebelle et al. 2020; Lee et al. 2021). A large fraction of this literature adopts a common single-fluid “non-ideal” MHD equation for the neutrals plus charge carriers (Cowling 1976; Ichimaru 1978; Nakano & Umebayashi 1986).

These simulations have increasingly pushed to smaller scales and seen a variety of potentially important behaviors in many different regimes, emphasizing the importance of the different Ohmic, Hall, and ambipolar diffusion terms in systems with potentially strong shocks and gravitational and/or radiative forces (for reviews see Teyssier & Commerçon 2019; Wurster 2021; Tsukamoto et al. 2022), as well as the importance of micron-sized dust grains as (sometimes dominant) charge carriers (Mestel & Spitzer 1956; Elmegreen 1979; Zhu et al. 2015; Tsukamoto & Okuzumi 2022; Kawasaki et al. 2022; Kobayashi et al. 2023; Marchand et al. 2023). But this can lead to some non-intuitive behaviors: while some properties become more “hydrodynamic-like” as the degree of ionization decreases ever smaller, others appear to show unique behaviors (especially in the Hall-dominated regime) which are qualitatively unlike hydrodynamics (Krasnopolsky et al. 2011; Braiding & Wardle 2012; Tsukamoto et al. 2015;

Zhao et al. 2020; Wurster et al. 2021). Moreover, as the ionization degree decreases, the implicitly assumed charge-carrier drift velocities can become extremely large, generating situations that are almost certainly highly unstable on unresolved plasma “microscales” or even unphysical (McBride et al. 1972; Mouschovias 1974; Drake et al. 1981; Kulsrud 2005). This can violate the foundational assumptions under which the usual non-ideal MHD expressions (both the equations themselves and the usual expressions for the various coefficients or currents) are derived. Such conditions will trigger micro-scale instabilities which induce strong “anomalous resistivities” far in excess of the “classical” resistivities usually adopted (Buneman 1958; Papadopoulos 1977; Rowland et al. 1982; Galeev & Sagdeev 1984; Norman & Heyvaerts 1985; Wahlund et al. 1992; Zweibel & Yamada 2009), but these have largely been ignored in more recent astrophysical work (though see Krasnopolsky et al. 2010; Che 2017).

In this paper, we follow the usual derivation of the single-fluid non-ideal astrophysical MHD equations in order to derive the corresponding drift velocities, and highlight terms usually neglected that cannot be safely dropped in the limit of super-thermal drift. We show how they can be approximately incorporated into modified non-ideal coefficients, and discuss the consequences of this, most notably in strongly suppressing gradients in the magnetic field below a critical scale ℓ_{crit} . This will limit the effects of the Hall term, likely causing it to lose its dynamical importance in most poorly-ionized plasmas of astrophysical interest.

2. THEORETICAL BACKGROUND

2.1. Derivation

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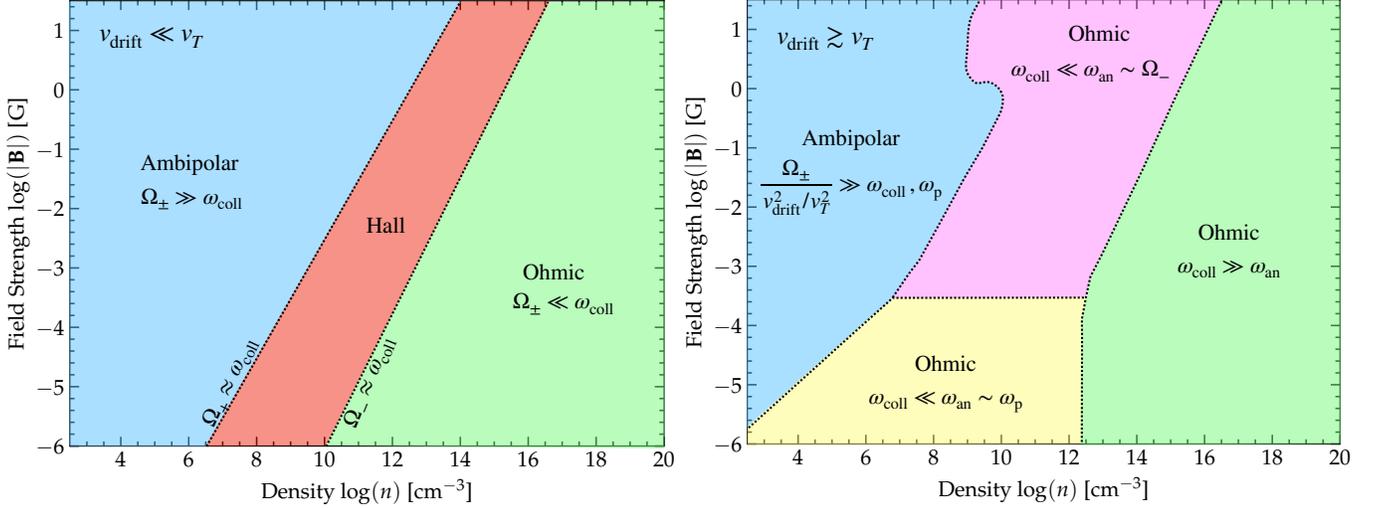


FIG. 1.— Illustration of the regimes in density $\bar{n} = \bar{\rho}/m_p$ and magnetic field strength $\bar{B} \equiv |\bar{\mathbf{B}}|$, in a weakly-ionized gas ($n_{\pm} \ll n$), where different non-ideal MHD terms derived in § 2.1 (Ohmic resistivity $\eta_O \mathbf{J}$, Hall $\eta_H \mathbf{J} \times \mathbf{b}$, and ambipolar drift $\eta_A \mathbf{J} \times \mathbf{b} \times \mathbf{b}$) would each be relatively more important in the induction equation. Quantities like temperature and ionization fractions at each B, n are estimated with a simple 5-species toy model (§ 2.4.3). *Left*: The case with “classical” non-ideal coefficients (§ 2.2, Eq. 8), which are valid only if the drift velocity between charge carriers $v_{\text{drift}} \sim |\mathbf{J}|/n_- |q_-| \sim cB/4\pi n_- |q_-| \ell_B$ ($\ell_B \sim |\mathbf{B}|/|\nabla \mathbf{B}|$) and “slip” velocity between neutrals and carriers (Eq. 9) are much less than the thermal velocities v_T . This in turn requires that the magnetic gradient scale length ℓ_B is much larger than some ℓ_{crit} (Eq. 13). *Right*: The case with corrected “effective” coefficients (§ 2.3; Eqs. 15–16), which account for superthermal drift and its effect both on direct particle collision rates and effective collisions via scattering (anomalous resistivity), assuming drift velocities are superthermal (here $v_{\text{drift}} = 10 v_T$ for illustration), or $\ell_B \lesssim \ell_{\text{crit}}$. When the drift becomes superthermal, the effective Ohmic term is always larger than the Hall term, and increases greatly in relative importance even at low densities. Where it predominates, we label whether the anomalous (ω_{an}) or direct collision (ω_{coll}) term is most important.

We begin with a derivation of the non-ideal MHD equations, following e.g. Cowling (1976); Ichimaru (1978); Nakano & Umebayashi (1986). We follow the usual kinetic approach by computing moments of the distribution function of each species, assuming infinitesimally small averaging volumes (but implicitly larger than the interparticle separation and Debye length), with bulk velocities $\ll c$, quasi-neutrality, and collisional/exchange reactions being mass and momentum conserving (e.g. rest mass loss via photons is negligible). For an arbitrary set of species j (which should formally include all isotopes/masses/ionization states/charges) this gives,

$$\begin{aligned} \frac{\partial(\rho_j \mathbf{u}_j)}{\partial t} + \nabla \cdot (\rho_j \langle \mathbf{v}_j \mathbf{v}_j \rangle) &= \rho_j \mathbf{a}_{\text{ext}, j} \\ + \frac{q_j \rho_j}{m_j} \left[\mathbf{E} + \frac{\mathbf{u}_j}{c} \times \mathbf{B} \right] + \sum_i &[\mathbf{f}_{ji} - \dot{\rho}_{ji} \mathbf{u}_j + \dot{\rho}_{ij} \mathbf{u}_i], \end{aligned} \quad (1)$$

in addition to the associated continuity equations $\partial_t \rho_j + \nabla \cdot (\rho_j \mathbf{u}_j) = \sum_i \dot{\rho}_{ji} - \dot{\rho}_{ji}$. Here $\mathbf{u}_j = \langle \mathbf{v}_j \rangle_{\text{DF}}$ is the local mean velocity of component j averaged over its distribution function (DF); $\rho_j = m_j n_j$ is the mass density of species j in terms of its particle mass m_j and number density n_j ; $\mathbf{a}_{\text{ext}, j}$ represents any external forces from e.g. gravity or radiation; \mathbf{E} and \mathbf{B} are the electric and magnetic fields; and \mathbf{f}_{ji} and $\dot{\rho}_{ji}$ represent collisions and source/sink terms (via e.g. charge exchange, recombination, ionization) for each species. We can define these in terms of some effective rate coefficients: $\mathbf{f}_{ji} \equiv \rho_j \rho_i \gamma_{ji} (\mathbf{u}_i - \mathbf{u}_j)$ where $\rho_i \gamma_{ji} \equiv \rho_i \langle \sigma v \rangle_{\text{DF}}^{ji} / (m_j + m_i) = \omega_{ji}^{\text{coll}}$, and $\dot{\rho}_{ji} \equiv \rho_j \omega_{ji}^{\text{ion/rec}}$. Note that ω^{coll} (or γ) and $\omega^{\text{ion/rec}}$ can in principle be arbitrarily complicated functions of any other variables here including the various \mathbf{u}_j (the exact dependence determined by the physics and particle types involved).

With the definition of total density $\rho \equiv \sum_j \rho_j$ and mean fluid velocity $\mathbf{U} \equiv \rho^{-1} \sum_j \rho_j \mathbf{u}_j$, as well as current $\mathbf{J} \equiv \sum_j q_j n_j \mathbf{u}_j$, we can sum over all species to give the familiar total momen-

tum equation,

$$\frac{\partial(\rho \mathbf{U})}{\partial t} + \nabla \cdot (\rho \mathbf{U} \mathbf{U}) = -\nabla \cdot \mathbf{\Pi} + \rho \mathbf{a}_{\text{ext}} + \frac{\mathbf{J} \times \mathbf{B}}{c}, \quad (2)$$

and continuity equation, $\partial_t \rho + \nabla \cdot (\rho \mathbf{U}) = 0$, where $\rho \mathbf{a}_{\text{ext}} \equiv \sum_j \rho_j \mathbf{a}_{\text{ext}, j}$, and $\mathbf{\Pi} \equiv \sum_a \mathbf{\Pi}_a$ with $\mathbf{\Pi}_a \equiv \rho_a \langle (\mathbf{v}_a - \mathbf{U})(\mathbf{v}_a - \mathbf{U}) \rangle$ is the total stress tensor. Note that the stress tensors are defined in the comoving frame of \mathbf{U} , which means $\mathbf{\Pi}_a = \rho_a \{ \delta \mathbf{u}_a \delta \mathbf{u}_a + \langle \delta \mathbf{v}_a \delta \mathbf{v}_a \rangle \}$ in terms of the drift velocity $\delta \mathbf{u}_a \equiv \mathbf{u}_a - \mathbf{U}$ (aka Reynolds stress or bulk flow pressure) and actual velocity dispersions $\delta \mathbf{v}_a \equiv \mathbf{v}_a - \mathbf{u}_a$.

Now consider a three-component system with a dominant set of neutrals, positive and negative charge carriers (labeled $n, +$, and $-$, respectively). We can repeat this derivation for an arbitrary set of species, but all of the salient dimensional scalings and intuition will be identical so the expressions are much more straightforward in this limit. Define an “ambipolar current” $\mathbf{J}_d \equiv -(q_- \rho_c / m_-) (\mathbf{u}_+ - \mathbf{u}_-)$, where $\rho_c \equiv \rho_+ + \rho_-$. We can then express the three velocities $\mathbf{u}_{n,+,-}$ in terms of \mathbf{U}, \mathbf{J} , and \mathbf{J}_d : $\mathbf{u}_- = \mathbf{U} - (m_- / q_- \rho) (\mathbf{J} - \mathbf{J}_d)$, $\mathbf{u}_+ = \mathbf{U} - (m_- / q_- \rho) [\mathbf{J} + (\rho_n / \rho_c) \mathbf{J}_d]$, $\mathbf{u}_n = \mathbf{U} - (m_- / q_- \rho) [-\{(\rho_+ + \rho_n) / \rho_-\} \mathbf{J} + (\rho_n / \rho_c) \mathbf{J}_d]$ (Cowling 1976; Pinto et al. 2008), and rewrite our three Eq. 1 for the evolution of $\mathbf{u}_{n,+,-}$ with an equivalent set for $\mathbf{U}, \mathbf{J}, \mathbf{J}_d$ via substitution. With Eq. 2 for \mathbf{U} , we have:

$$\begin{aligned} D_t \mathbf{J} &= \frac{|q_-|}{m_-} \nabla \cdot (\mathbf{\Pi}_- - \epsilon \mathbf{\Pi}_+) + |q_-| n_- \Omega_1 \left(\frac{c \mathbf{E}}{B} + \mathbf{U} \times \mathbf{b} \right) \\ &- \mathbf{G}_1 - \Omega_2 \mathbf{J} \times \mathbf{b} - \omega_1 \mathbf{J} + \Omega_3 \mathbf{J}_d \times \mathbf{b} + \omega_2 \mathbf{J}_d, \\ \mathbf{G}_1 &\equiv q_+ n_+ (\mathbf{a}_{e,-} - \mathbf{a}_{e,+}), \\ \Omega_1 &\equiv (1 + \epsilon) \Omega_-, \quad \Omega_2 \equiv \Omega_- [1 - (1 + \epsilon) \xi_-], \quad \Omega_3 \equiv \xi_n \Omega_+, \\ \omega_1 &\equiv (1 + \epsilon) \omega_{-+}^{\text{coll}} + \omega_{-n}^{\text{coll}} + \omega_{-n}^{\text{rec}}, \quad \omega_2 \equiv \frac{\epsilon (\omega_{-n}^{\text{coll}} - \omega_{+n}^{\text{coll}})}{1 + \epsilon}, \end{aligned} \quad (3)$$

where $D_t \mathbf{X} \equiv \partial_t \mathbf{X} + \nabla \cdot (\mathbf{U} \mathbf{X} + \mathbf{X} \mathbf{U})$, $\mathbf{b} \equiv \mathbf{B}/|\mathbf{B}|$, $\epsilon \equiv$

$(q_+ m_-)/(|q_-| m_+)$, $\xi_j \equiv \rho_j/\rho$, $\Omega_{\pm} \equiv |q_{\pm}|B/(m_{\pm}c)$ are the gyro frequencies, and

$$D_t [\xi_n(\mathbf{J}_d - \mathbf{J})] = -\mathbf{G}_0 + \Omega_4 \mathbf{J} \times \mathbf{b} + \omega_3 \mathbf{J} - \omega_4 \mathbf{J}_d, \quad (4)$$

$$\mathbf{G}_0 \equiv \frac{|q_-|}{m_-} [\xi_n \nabla \cdot \mathbf{\Pi} - \nabla \cdot \mathbf{\Pi}_n + \xi_n (\mathbf{a}_{e,-\rho_-} + \mathbf{a}_{e,+\rho_+} - \mathbf{a}_{e,n\rho_c})]$$

$$\Omega_4 \equiv \xi_n \Omega_-, \quad \omega_3 \equiv \omega_{-n}^{\text{coll}} + \xi_n [(1 + \epsilon) \omega_{n-}^{\text{ion}} + \omega_{-n}^{\text{rec}}],$$

$$\omega_4 = \frac{\epsilon \omega_{-n}^{\text{coll}} + \omega_{+n}^{\text{coll}}}{1 + \epsilon} + \xi_n \omega_{-n}^{\text{rec}} + (1 - \xi_n) \omega_{n-}^{\text{ion}}.$$

Note these are similar to Eqs. 34 & 35 in Pinto et al. (2008), except we have retained more general ω , charge-exchange, and external acceleration terms.

Eqs. 3-4 can be rearranged to give an expression for \mathbf{E} :

$$\mathbf{E} = -\frac{\mathbf{U}}{c} \times \mathbf{B} + A_0 \left[\omega_O \mathbf{J} + \omega_H \mathbf{J} \times \mathbf{b} - \omega_A \mathbf{J} \times \mathbf{b} \times \mathbf{b} \right. \\ \left. + \mathbf{D}_{t0} + \mathbf{G}_2 - \frac{|q_-|}{m_-} \nabla \cdot (\mathbf{\Pi}_- + \epsilon \mathbf{\Pi}_+) \right], \quad (5)$$

$$\mathbf{D}_{t0} \equiv D_t \mathbf{J} + \frac{\omega_2}{\omega_4} D_t [\xi_n(\mathbf{J}_d - \mathbf{J})] + \frac{\Omega_3}{\omega_4} D_t [\xi_n(\mathbf{J}_d - \mathbf{J})] \times \mathbf{b},$$

$$A_0 \equiv \frac{B}{|q_-| n_- c \Omega_1}, \quad \mathbf{G}_2 \equiv \mathbf{G}_1 + \frac{\omega_2}{\omega_4} \mathbf{G}_0 + \frac{\Omega_3}{\omega_4} \mathbf{G}_0 \times \mathbf{b},$$

$$\omega_O \equiv \omega_1 - \frac{\omega_3 \omega_2}{\omega_4}, \quad \omega_H \equiv \Omega_2 - \frac{\Omega_4 \omega_2 + \omega_3 \Omega_3}{\omega_4}, \quad \omega_A \equiv \frac{\Omega_4 \Omega_3}{\omega_4}.$$

Here \mathbf{D}_{t0} collects the time dependence and advection terms and \mathbf{G}_2 collects the battery and diamagnetic terms. This \mathbf{E} can then be inserted into the induction equation $\partial_t \mathbf{B} = -c \nabla \times \mathbf{E}$. It is convenient for this to define the variables:

$$a_i \equiv A_0 \omega_i = \frac{B}{|q_-| n_- c \Omega_1} \omega_i. \quad (6)$$

Note that ω_O is similar to an effective collision frequency ω^{coll} , ω_H to the gyrofrequency $\sim \Omega_-$, and ω_A scales as $\sim \Omega_- \Omega_+ / \omega^{\text{coll}}$.

Eq. 5 (with $\partial_t \mathbf{B} = -c \nabla \times \mathbf{E}$) formally gives us an expression for the evolution of \mathbf{B} , but this is clearly not solveable exactly without kinetic methods that can resolve the plasma “micro” scales (e.g. gyro radii, collisional mean-free paths, etc.) and separately predict terms like the stress tensors $\mathbf{\Pi}_j$, time derivatives $D_t \mathbf{J}$, $D_t \mathbf{J}_d$, battery terms, etc. Even if one adopted some effective closure relations for these terms in Eqs. 1-5 to attempt to explicitly integrate a “three fluid” method, the equations could, in many cases of interest, develop micro-scale (kinetic) or meso-scale (small-scale fluid) fluctuations in \mathbf{B} or \mathbf{J} . As we are interested in the dynamics of large (“macro”) scales in the system, these could also contribute importantly to the large-scale dynamics. Thus, a crucial set of assumptions made in order to justify dropping certain terms in Eq. 5 and therefore render it integrable on “macro” scales $\ell \sim \Delta x_{\text{macro}}$, $T \sim \Delta t_{\text{macro}}$ is to assume some scale separation of macro/meso/micro scales, and then to assume that fluctuations on the micro and meso-scales are negligible for the macro-scale quantities. Mathematically, one can define the large-scale average of some quantity \mathbf{X} as $\bar{\mathbf{X}} \equiv \langle \mathbf{X} \rangle_{\mathcal{V}} = (\int_{\mathcal{V}} \mathbf{X} d^3 \mathbf{x} dt) / (\int_{\mathcal{V}} d^3 \mathbf{x} dt)$, integrating over some macro-scale hyperspace domain \mathcal{V} which is large compared to micro/meso scales, but still small compared to global length scales of the problem. Then define $\mathbf{X} \equiv \bar{\mathbf{X}} + \delta \mathbf{X}$ where $\delta \mathbf{X}(\mathbf{x}, t)$ represents mesoscale fluctuations (and $\langle \delta \mathbf{X} \rangle_{\mathcal{V}} = 0$). One can then expand the induction

equation:

$$\partial_t \bar{\mathbf{B}} = -c [\nabla \times \langle \mathbf{E} \rangle_{\mathcal{V}} + \langle \nabla \times \delta \mathbf{E} \rangle_{\mathcal{V}}] \\ = \nabla \times [\bar{\mathbf{U}} \times \bar{\mathbf{B}} + \langle \delta \mathbf{U} \times \delta \mathbf{B} \rangle_{\mathcal{V}} + \bar{a}_O \bar{\mathbf{J}} + \langle \delta a_O \delta \mathbf{J} \rangle_{\mathcal{V}} + \dots] \\ + \langle \nabla \times [\delta \mathbf{U} \times \bar{\mathbf{B}} + \bar{\mathbf{U}} \times \delta \mathbf{B} + \dots] \rangle_{\mathcal{V}} \quad (7)$$

where the “...” represents the appropriate (and extensive) expansion of Eq. 5 for all fluctuating terms. Note that, as in large-scale dynamo theory (Rincon 2019), this is not a series expansion, so we cannot prima facie assume the δ terms are small and drop them. Also of importance is that, given some state vector $\Psi \equiv (\mathbf{U}, \mathbf{B}, \mathbf{J}, \mathbf{J}_d, \dots)$, variables like ω_i or a_i will not satisfy $\bar{a}_i = a_i(\bar{\Psi})$, because they are non-linear functions of the state variables. Evaluating terms like $\langle \delta \mathbf{U} \times \delta \mathbf{B} \rangle_{\mathcal{V}}$ requires integrals over the entire spatial and time spectrum of fluctuations knowing the appropriate N -th order correlation functions between the fluctuations, so we stress that Eq. 7 is no more integrable than Eq. 5, unless we impose additional assumptions.

2.2. The “Standard” Non-Ideal MHD Approximations

To arrive at the standard formulation of one/two-fluid non-ideal MHD, we simplify Eq. 7 by making the following assumptions (see Ichimaru 1978; Norman & Heyvaerts 1985; Nakano & Umebayashi 1986; Wardle & Ng 1999; Tassis & Mouschovias 2005, 2007; Kunz & Mouschovias 2009). (1) Assume (given dimensional arguments since velocities are non-relativistic: $|\mathbf{U}|, c_s, v_{\text{drift}}, v_A \ll c$) that displacement currents can be neglected so $\mathbf{J} \rightarrow \mathbf{J}_A \equiv (c/4\pi) \nabla \times \mathbf{B}$. (2) Drop all terms from the second-line of Eq. 5. (a) Assume temporal/advection frequencies D_t are much smaller than the collision+gyro frequencies ω_i ($|D_t| \ll \omega_i$), usually justified by assuming $|D_t| \sim 1/\Delta t_{\text{macro}}$ varies only on “macroscopic” timescales because all currents come into local steady-state ($D_t \bar{\mathbf{X}} \rightarrow 0$) rapidly. (b) Assume that the battery terms are negligible, i.e. external accelerations are negligible compared to the typical non-ideal terms, $|\mathbf{a}_{\text{ext}}| \ll |\omega_i v_{\text{drift}}|$, ionization fractions are small, and the difference between external acceleration/pressure forces on neutrals and charge carriers is small. (c) Assume the stresses are similarly negligible. Noting $\nabla \cdot (\mathbf{\Pi}_- + \epsilon \mathbf{\Pi}_+) \sim \rho_- \delta v^2 / \lambda$ for some fluctuations on scale λ (with velocity dispersion δv including thermal-drift/bulk components), this is equivalent to assuming $|\delta v^2| / \lambda \ll |\omega_i v_{\text{drift}}|$. This is usually justified by assuming the drift is sub-thermal, and the $\rho_j \langle (\mathbf{v}_j - \langle \mathbf{v}_j \rangle) (\mathbf{v}_j - \langle \mathbf{v}_j \rangle) \rangle$ component of the charge carrier pressure tensors come into thermal equilibrium (are Maxwellian and isotropic with a single temperature, $\delta v \sim v_T$) and vary only on macroscopic scales ($\lambda \sim \ell_{\text{macro}}$). (3) In the various ω_i terms, assume collisions with any non-neutrals can be neglected ($\omega_{in} \gg \omega_{ij}$ for $j \neq n$), and ignore inelastic and charge-exchange reactions ($\omega_{\text{rec/ion}} \ll \omega^{\text{coll}}$). (4) Assume the drift ($v_{\text{drift}} \equiv \mathbf{u}_+ - \mathbf{u}_-$) and slip ($v_{\text{slip}}^{\pm} = \mathbf{u}_{\pm} - \mathbf{u}_n$) velocities are vanishingly small compared to the thermal and Alfvén and other wave speeds of all species. This means (a) that the non-isotropic/Reynolds/bulk-flow components of the pressure tensors $\mathbf{\Pi}_j^{\text{aniso}} = \rho_j (\mathbf{u}_j - \mathbf{U}) (\mathbf{u}_j - \mathbf{U})$ can be neglected, (b) that certain instabilities and anomalous resistivities can be neglected (discussed below), and (c) that coefficients like $\langle \sigma v \rangle_{\text{DF}}^{ji}$ (equivalently γ_{ji} or $\omega_{ji}^{\text{coll}}$) can be taken to have their values for zero drift/slip, which (by definition) are independent of the drift speeds and therefore no longer complicated functions of $\mathbf{J}, \mathbf{J}_d, \mathbf{U}$, and their derivatives. (5)

Assume that fluctuations on unresolved (micro/meso) scales are vanishingly small, so all $\delta\mathbf{X}$ terms in Eq. 7 can be neglected and all coefficients like ω_i are functions only of the mean macro-scale average values $\bar{\mathbf{B}}$, $\bar{\mathbf{J}}$, etc. Note that this assumption is often implied to be equivalent to the scale hierarchy assumption $\Delta x_{\text{macro}} \gg \Delta x_{\text{micro}}$, but from the discussion in § 2.1 it is obvious that is not sufficient.

With these assumptions we finally obtain a relatively simple equation for $\bar{\mathbf{B}}$:

$$\partial_t \bar{\mathbf{B}} \approx \nabla \times \left[\bar{\mathbf{U}} \times \bar{\mathbf{B}} - \eta_O^0 \bar{\mathbf{J}}_A - \frac{\eta_H^0}{\bar{B}} \bar{\mathbf{J}}_A \times \bar{\mathbf{B}} + \frac{\eta_A^0}{\bar{B}^2} \bar{\mathbf{J}}_A \times \bar{\mathbf{B}} \times \bar{\mathbf{B}} \right],$$

$$\bar{\mathbf{J}}_A \equiv \nabla \times \bar{\mathbf{B}}, \quad \eta_O^0 = \frac{c^2 a_O^0}{4\pi}, \quad \eta_H^0 = \frac{c^2 a_H^0}{4\pi}, \quad \eta_A^0 = \frac{c^2 a_A^0}{4\pi}. \quad (8)$$

The a_i^0 notation denotes the value of $a_i(\bar{\Psi})$ assuming all state variables have their macroscopic mean values, and dropping terms and simplifying by assuming negligible drift/slip speeds per assumptions (1)-(5).

We can now define the drift speeds implied by this formulation in terms of the variables that are evolved in the fluid model:

$$\mathbf{v}_{\text{drift}} \equiv \bar{\mathbf{u}}_+ - \bar{\mathbf{u}}_- = \frac{\bar{\mathbf{J}}}{n_- |q_-|} = \frac{c}{4\pi \bar{n}_- |q_-|} \nabla \times \bar{\mathbf{B}}, \quad (9)$$

$$\mathbf{v}_{\text{slip}}^+ \equiv \bar{\mathbf{u}}_+ - \bar{\mathbf{u}}_n = \frac{\epsilon}{(1+\epsilon) \bar{n}_+ q_+} \bar{\mathbf{J}}_d = \frac{\epsilon \omega_{-n}^0 \mathbf{v}_{\text{drift}}}{\epsilon \omega_{-n}^0 + \omega_{+n}^0} + \mathbf{v}_{\text{slip},\perp},$$

$$\mathbf{v}_{\text{slip}}^- \equiv \bar{\mathbf{u}}_- - \bar{\mathbf{u}}_n = \mathbf{v}_{\text{slip}}^+ - \mathbf{v}_{\text{drift}} = -\frac{\omega_{+n}^0 \mathbf{v}_{\text{drift}}}{\epsilon \omega_{-n}^0 + \omega_{+n}^0} + \mathbf{v}_{\text{slip},\perp},$$

$$\mathbf{v}_{\text{slip},\perp} \equiv \frac{\bar{\rho}}{\bar{\rho}_n} \frac{\eta_A^0}{\bar{B}^2} (\nabla \times \bar{\mathbf{B}}) \times \bar{\mathbf{B}}.$$

The ω^0 notation follows a^0, η^0 , etc. The ‘‘effective’’ maximum absolute value among the drift/slip speeds is given by:

$$v_{\text{drift}, \text{max}} \approx \sqrt{|\mathbf{v}_{\text{drift}}|^2 + |\mathbf{v}_{\text{slip},\perp}|^2}. \quad (10)$$

For dimensional analysis below, it is convenient to note that we can write the non-ideal terms as:

$$\eta_O^0 \bar{\mathbf{J}}_A = \frac{\omega_O^0}{\Omega_1^0} \mathbf{v}_{\text{drift}} |\bar{\mathbf{B}}| \sim \frac{\omega_{\text{coll}}}{\Omega_-} v_{\text{drift}} B, \quad (11)$$

$$\frac{\eta_H^0}{\bar{B}} \bar{\mathbf{J}}_A \times \bar{\mathbf{B}} = \frac{\omega_H^0}{\Omega_1^0} \mathbf{v}_{\text{drift}} \times \bar{\mathbf{B}} \sim \pm v_{\text{drift}} B,$$

$$\frac{\eta_A^0}{\bar{B}^2} \bar{\mathbf{J}}_A \times \bar{\mathbf{B}} \times \bar{\mathbf{B}} = \frac{\omega_A^0}{\Omega_1^0} (\mathbf{v}_{\text{drift}} \times \mathbf{b}) \times \bar{\mathbf{B}} \sim \frac{\Omega_+}{\omega_{\text{coll}}} v_{\text{drift}} B,$$

where the latter dimensional scalings come from taking the mostly-neutral limit for $\omega_O^0/\Omega_1^0 \approx \omega_{\text{coll}}^{O,0}/\bar{\Omega}_-$ with $\omega_{\text{coll}}^{O,0} = \omega_{+n}^0 \omega_{-n}^0 / \omega_{\text{coll}}^{A,0}$ and $\omega_{\text{coll}}^{A,0} = \omega_{+n}^0 + \epsilon \omega_{-n}^0$; $\omega_H^0/\Omega_1^0 \approx (\omega_{+n}^0 - \epsilon \omega_{-n}^0) / \omega_{\text{coll}}^{A,0} \sim \pm 1$; and $\omega_A^0/\Omega_1^0 \approx \bar{\Omega}_+ / \omega_{\text{coll}}^{A,0}$.

2.3. The Superthermal Drift Limit

Per § 2.2, the assumptions used in deriving the standard non-ideal formulation explicitly require

$$\text{MAX}(v_{\text{drift}}, v_{\text{slip}}^\pm) \approx v_{\text{drift}, \text{max}} \ll v_T, \quad (12)$$

where v_T is some characteristic slow wavespeed of the different components. Eq. 12 is equivalent to

$$\ell_B \equiv \frac{\bar{B}}{|\nabla \times \bar{\mathbf{B}}|} \gg \ell_{\text{crit}} \equiv \text{MAX} \left[\ell_{\text{crit}}^{\text{drift}}, \ell_{\text{crit}}^{\text{slip}} \right], \quad (13)$$

$$\ell_{\text{crit}}^{\text{drift}} \equiv \frac{\bar{B} c}{4\pi \bar{n}_- |q_-| v_T}, \quad \ell_{\text{crit}}^{\text{slip}} \equiv \frac{\eta_A^0}{\bar{\xi}_n v_T} = \frac{\bar{B}^2 \bar{\xi}_n}{4\pi \alpha_n^0 v_T},$$

where $\alpha_n^0 \equiv \bar{\rho}_c \omega_A^0 = \rho_- \omega_{-n}^0 + \rho_+ \omega_{+n}^0$. We will show below that nothing in the standard formulation of Eq. 8 prevents this condition from being violated. So what will happen if superthermal drifts arise ($\ell_B \lesssim \ell_{\text{crit}}$)?

2.3.1. Modified Collision Rates

Superthermal drift/slip speeds self-evidently violate assumption (4) in § 2.2. This means that the collision rates usually adopted, which assume negligible bulk relative drift/slip velocities of different species (so they depend on thermal velocities but not the various u_{ij}): $\omega_{ij}^0 \equiv \omega_{ij}(u_{ij} \equiv |u_i - u_j| \rightarrow 0)$, are simply incorrect, even if all the other assumptions in § 2.2 held perfectly. In detail, the full collisional and inelastic couplings ω_{ij} are complicated non-linear (even non-monotonic) functions of the different relative velocities u_{ij} , which means they become implicit functions of $\mathbf{U}, \mathbf{J}, \mathbf{J}_d$, etc (Elmegreen 1979; Draine & Salpeter 1979; Pandey & Wardle 2006; Pinto & Galli 2008; Öberg et al. 2023). However, the leading-order behavior in the superthermal limit is straightforward. In the limit $u_{ij} \gg v_{T,i}, v_{T,j}$, then $\omega_{ij} \propto \langle \sigma v_{ij} \rangle \propto |u_{ij}|$ increasingly moves towards the Epstein limit in which the drag force scales with the interparticle velocity. Thus, the collision rates scale with the drift/slip speed $\omega_{ij} \propto n_j \sigma_{ij} \Delta v_{ij}$ where $\Delta v_{ij} \propto u_{ij} = v_{\text{drift/slip}, ij}$. For the Epstein limit specifically as applicable to neutral

hard-sphere collisions, $\omega \propto v_{T, ij} \sqrt{1 + (v_{\text{drift}, ij}/v_{T, \text{eff}})^2}$ with $v_{T, \text{eff}} = (8/3\pi^{1/2}) v_{T, ij}$. But this will directly modify the coefficients η , increasing η_O and decreasing η_A (see also Hillier 2024) proportional to the change in ω .

2.3.2. Anomalous Resistivity: Dropped Terms and Instabilities

Superthermal drifts also imply that many of the dropped terms associated with assumptions (2), (4) & (5) can no longer be dropped. It has been well-known for decades that superthermal drifts are generically unstable on micro and mesoscales (see Kulsrud 2005, and references therein). For systems with non-trivial chemistry (many charged species), there are a potentially infinite number of such instabilities, some relatively well-studied such as the dozen distinct current-driven instabilities (e.g. two-stream, ion-acoustic, ion-cyclotron, whistler, Buneman) reviewed in Kindel & Kennel (1971); Krall & Liewer (1971); Norman & Smith (1978); Norman & Heyvaerts (1985) or tearing (Mestel 1968) or hybrid-mode (Davidson & Gladd 1975) instabilities; others that have only barely been explored, such as the various novel instabilities of mixed neutral-charged systems identified in Huba (1991); Kamaya & Nishi (2000); Mamun & Shukla (2001); Nekrasov (2008b,a), or the families of intrinsically two-fluid (neutrals plus ions+electrons) drift instabilities in Tytarenko et al. (2002); Nekrasov (2009) or the charged dust+gas ‘‘resonant drag instabilities’’ in Squire & Hopkins (2018b); Hopkins & Squire (2018a); Seligman et al. (2019); Hopkins et al. (2020b). These instabilities grow rapidly on micro/meso-scales, pro-

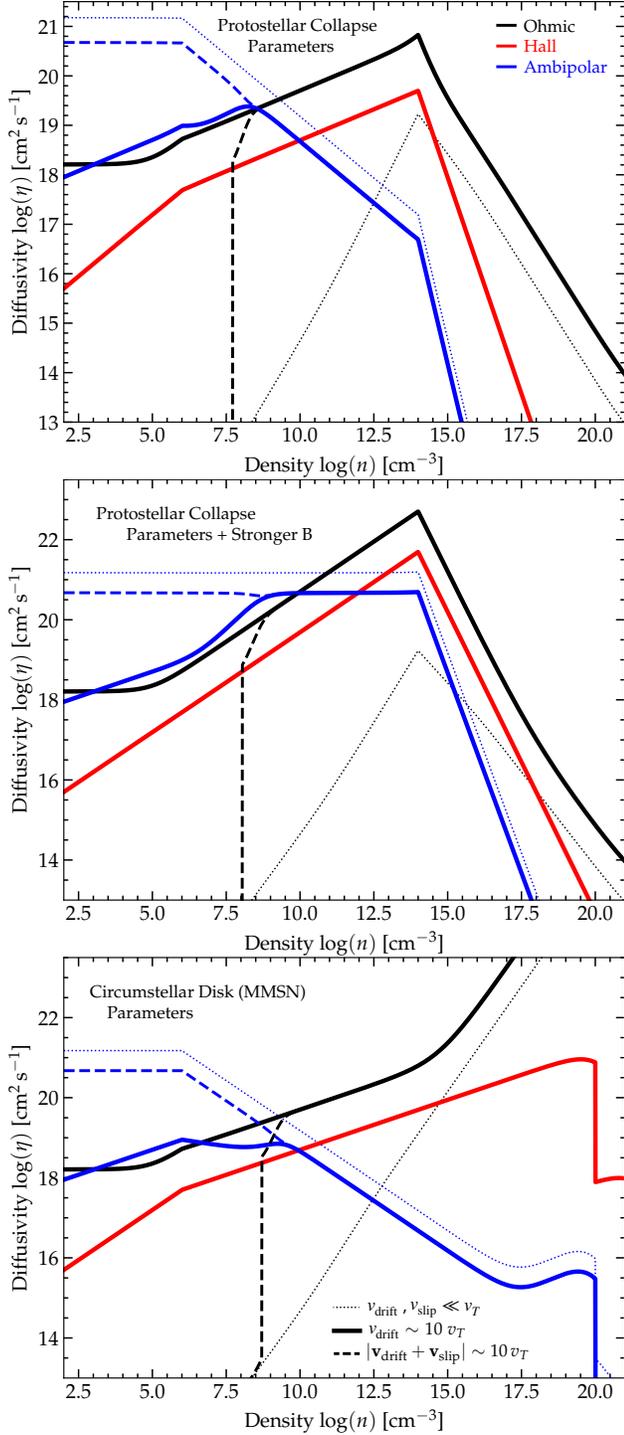


FIG. 2.— Ohmic, Hall, and ambipolar diffusivities $\eta_{O,H,A}$ computed for a multispecies, weakly-ionized gas with total (mostly neutral) density n , assuming simple fitting functions for B , T , ionization, etc. (§ 2.4.3). We compare values for slow drift+slip $v_{\text{drift}}, v_{\text{slip}} \ll v_T$ ($\ell_B \gg \ell_{\text{crit}}$; dotted) vs superthermal drift ($v_{\text{drift}} \sim 10 v_T$, $\ell_B \sim 0.1 \ell_{\text{crit}}^{\text{drift}}$, solid) including the proposed corrections in Eq. 16 to account for enhanced scattering. We also compare $\ell_B \sim 0.1 \ell_{\text{crit}}$ ($v_{\text{drift, max}} = |v_{\text{drift}} + v_{\text{slip}}| \sim 10 v_T$; dashed), which has superthermal slip but not drift at low- n . For Hall, the lines overlap. We consider three toy models: one appropriate for protostellar core-collapse with weak B -fields (top); one with stronger B ($B \propto n^{1/2}$ at all n ; middle); and one with cooler temperatures and weaker ionization better matched to a circumstellar disk (bottom). This has a large effect on the absolute value of $\eta_{O,H,A}$ (“protostellar” models become hot with $T \geq 1000$ K, at $n \geq 10^{14} \text{ cm}^{-3}$, causing rapid ionization and decreasing η), but does not change the behaviors of interest here. With $\ell_B \gg \ell_{\text{crit}}$ (sub-thermal drift), the system transitions from ambipolar to Hall to Ohmic with increasing n . Superthermal drift/slip greatly enhances η_O and eliminates the Hall regime.

ducing nonlinear fluctuations that violate assumptions (2), (4) & (5).

Consider a simple example. An explicit assumption in deriving the standard MHD equations is that the pressure tensors $\mathbf{\Pi}_j = \rho_a(\mathbf{u}_a - \mathbf{U})(\mathbf{u}_a - \mathbf{U}) + \rho_a \langle \delta \mathbf{v}_a \delta \mathbf{v}_a \rangle$ are dominated by an isotropic component ($\langle \delta \mathbf{v}_a \delta \mathbf{v}_a \rangle \approx \mathbb{I} v_{T,a}^2$), i.e. that the Reynolds/bulk component $(\mathbf{u}_a - \mathbf{U})(\mathbf{u}_a - \mathbf{U}) \sim \mathcal{O}(v_{\text{drift}}^2)$ of the tensor can be neglected. If $v_{\text{drift},a} \gtrsim v_{T,a}$ this is self-evidently invalid. Moreover if the instabilities can source non-linear fluctuations in $\mathbf{v}_{\text{drift}}$ on some scale $\ell \ll \ell_B$, then the anisotropic pressure term in Eq. 5, $\langle \nabla \cdot \mathbf{\Pi}_- \rangle_{\mathcal{V}} \sim \langle \rho_- \delta \mathbf{v}_{\text{drift}} \cdot \nabla \delta \mathbf{v}_{\text{drift}} \rangle_{\mathcal{V}}$, will generically give rise to a non-vanishing term of order $|q_-| n_- |v_{\text{drift}}|^2 / \ell$ along the local direction of $\hat{\mathbf{v}}_{\text{drift}}$ and therefore $\hat{\mathbf{J}}$. This will act like an “effective” resistivity. Comparison of the magnitude of this term to the “classical” resistivity term $\omega_O \mathbf{J} \sim \omega_{\text{coll}} |q_-| n_- v_{\text{drift}}$ in Eq. 5 shows their ratio is $\sim v_{\text{drift}} / (\ell \omega_{\text{coll}})$. Thus if we associate a characteristic frequency $\omega_\delta \sim v_T / \ell$ to the fluctuations, we see that this specific term will contribute a resistivity $\sim \omega_\delta (v_{\text{drift}} / v_T)$ which is larger than the classical resistivity if this is larger than the collision frequency. We can straightforwardly obtain similar results for terms like $\langle \delta \mathbf{u}_- \times \delta \mathbf{B} \rangle$, which for non-linear fluctuations could give rise to resistivities as large as the gyro or plasma frequencies (Sagdeev 1966; Frieman & Chen 1982; Yoon & Lui 2006; Graham et al. 2022).

These effects are “anomalous resistivity,” which is well-studied in laboratory fusion plasmas, ionospheric dynamics, and Solar physics (Buneman 1958; Papadopoulos 1977; Rowland et al. 1982; Galeev & Sagdeev 1984; Wahlund et al. 1992). In these applications, it has been shown that when v_{drift} exceeds speeds like the ion thermal speed and/or Alfvén speed, the measured resistivity rises rapidly (often by orders of magnitude as the drift speed increases by a small amount, from e.g. just below the thermal speed to ~ 1.5 times larger than the thermal speed; see Davidson & Gladd 1975; Gentle et al. 1978; Petkaki et al. 2003; Graham et al. 2022). Considerable theoretical and experimental effort has gone into characterizing and modeling it in these fields (see references above and Dobrowolny & Santini 1974; Ugai 1984; Bychenkov et al. 1988; Uzdensky 2003; Yoon & Lui 2006; Lee et al. 2007; Roytershteyn et al. 2012; Beving et al. 2023), but it has largely been neglected in the applications in § 1 (with some exceptions noted there). Detailed predictions in the superthermal regime clearly require explicit kinetic treatments (e.g. PIC simulations), and many of the non-linear outcomes remain poorly understood (particularly so for e.g. circumstellar disks as opposed to fusion plasmas). However, a generic result of these calculations and experiments is that the leading-order terms that arise act to oppose the drift (i.e. they act like resistivity), as the energy dissipated in fluctuations must ultimately come from the driving force (the drift current itself). Further, their “effective” resistivity scales with $\sim \omega_\delta \sim \omega_{\text{max}} W$, where ω_{max} is some fastest rate on which the modes can act (usually the gyro or plasma frequency, or some combination of the two; Davidson & Gladd 1975; Nayar & Revathy 1978; Norman & Heyvaerts 1985; Treumann 2001; Graham et al. 2022)¹ and $W \sim \langle \delta v^2 \rangle / v_0^2$ is defined from some “characteristic” velocity v_0 above which the modes can grow (e.g. the

¹ Note that much of the historical study of anomalous resistivity, as a result of its intended applications in laboratory and Solar plasmas, has focused on the high-density regime where $\omega_{pe} \gg \Omega_e$. Thus it is common to see $\omega_{\text{max}} \sim \omega_{pe} (\delta v / v_T)^2$ as a characteristic estimate. For systems with $\omega_{pj} \ll \Omega_j$ where collective effects are weak, the situation is more analogous to pitch-

thermal-ion or ideal-Alfvén speed, depending on the mode). Generically, such instabilities saturate with $\omega_\delta \sim \omega_{\max}$ (see references above).²

Briefly, we note that while analogous instabilities exist when the neutral-charge carrier “slip” velocity (as opposed to the current drift velocity) becomes superthermal (with v_{drift} still subthermal; see [Tytarenko et al. 2002](#); [Squire & Hopkins 2018b](#)), it is less clear what their non-linear outcomes should be. This is partially for lack of study, but also because they can be stabilized on microscales by pressure effects in the neutral-ionized fluids, and some particle-based simulations of the RDI have argued that saturation can involve grains “running away” (i.e. dust-gas separation rather than stronger coupling; [Moseley et al. 2019](#)).

2.3.3. Other Assumptions

Note that even in the superthermal limit, assumption (1) from § 2.2, that displacement currents can be safely neglected, is still usually valid (producing a large displacement current would require much larger drift speeds, which we will show below is prevented by the effects discussed in § 2.3.1-2.3.2 in the regimes of interest). We briefly note that the effect of violating (1) would be to decouple \mathbf{B} so it detaches and propagates at c , akin to an even much larger anomalous resistivity. In (2), the neglect of the battery terms, which requires $a_{\text{ext}} \ll \omega_i v_{\text{drift}}$, is not invalidated by superthermal drift (so long as it was already valid with sub-thermal drift, its validity should only be strengthened by superthermal drift). Assumption (3), while common, is completely unnecessary for Eq. 8 – one can simply include the appropriate coefficients $\omega^{\text{ion/rec},0}$ and $\omega_{-+}^{\text{coll}}$ in ω_i^0 .

2.4. An Approximate Treatment in the Fast Drift Limit

2.4.1. Corrected Collision Rates

In the spirit of remaining as close as possible to the traditional non-ideal MHD formulation of § 2.2, note that (excepting trivial cases) $\tilde{\mathbf{J}}, \tilde{\mathbf{J}} \times \tilde{\mathbf{B}}, \tilde{\mathbf{J}} \times \tilde{\mathbf{B}} \times \tilde{\mathbf{B}}$ form a complete basis so it is technically possible to express the information in Eq. 7 via a set of “effective” coefficients:

$$\partial_t \tilde{\mathbf{B}} \approx \nabla \times \left[\tilde{\mathbf{U}} \times \tilde{\mathbf{B}} - \eta_O^{\text{eff}} \tilde{\mathbf{J}}_A - \frac{\eta_H^{\text{eff}}}{\tilde{\mathbf{B}}} \tilde{\mathbf{J}}_A \times \tilde{\mathbf{B}} + \frac{\eta_A^{\text{eff}}}{\tilde{\mathbf{B}}^2} \tilde{\mathbf{J}}_A \times \tilde{\mathbf{B}} \times \tilde{\mathbf{B}} \right], \quad (14)$$

angle scattering where one generically obtains $\omega_\delta \sim \Omega_j (\delta v/v_{\text{eff}})^2$ (with excitation above some “effective” thermal or Alfvén speed if the heavier carriers and neutrals are well-coupled; [Jokipii 1966](#); [Kulsrud & Pearce 1969](#); [Coppi & Mazzucato 1971](#); [Schlickeiser 1989](#)).

² Consider the following heuristic “derivation.” Many of the salient instabilities have dispersion relations for which the growth rate can be (dimensionally) expressed as $\Gamma \sim \omega_{\max} (v_{\text{drift}}/v_{\text{crit}} - 1)$ above some critical drift velocity $v_{\text{crit}} \sim v_T$. In the weakly-ionized case of interest, the dominant damping mechanism is collisions, with frequency ω_{coll} . If $\omega_{\text{coll}} \gg \omega_{\max}$, the instabilities will be suppressed or damped but in this case, one would (by definition) already be well into the “classical” Ohmic limit, and the maximum anomalous resistivity correction ($\propto (1 + \omega_{\max}/\omega_{\text{coll}})$) would be small, so this has no effect on our results. But if $\omega_{\text{coll}} \ll \omega_{\max}$, damping cannot compete with growth. Since the frequency ω_{\max} is much larger than macroscopic frequencies $\sim |\mathbf{u}|/\ell_B$, so long as Ampere’s law holds v_{drift} cannot rapidly self-adjust, so the instabilities can only saturate non-linearly with $W \sim 1$, i.e. by inducing an effective damping (hence collision or scattering rate) of order the growth rate: $\omega_\delta \sim \omega_{\max} v_{\text{drift}}/v_{\text{crit}}$. This therefore must enhance the Ohmic term, which allows \mathbf{B} to diffuse, to reduce the current and v_{drift} until the growth can be shut down by achieving $v_{\text{drift}} \lesssim v_{\text{crit}}$.

where the coefficients η can be dimensionally expressed as:

$$\begin{aligned} \eta_O^{\text{eff}} &\equiv \frac{c^2 a_O^{\text{eff}}}{4\pi} = \frac{c \tilde{\mathbf{B}}}{4\pi |q_-| \tilde{n}_-} \frac{\omega_{\text{coll}}^{\text{eff}, O}}{\tilde{\Omega}_-}, \\ \eta_H^{\text{eff}} &\equiv \frac{c^2 a_H^{\text{eff}}}{4\pi} = \frac{c \tilde{\mathbf{B}}}{4\pi |q_-| \tilde{n}_-} \mathcal{F}_H^{\text{eff}}, \\ \eta_A^{\text{eff}} &\equiv \frac{c^2 a_A^{\text{eff}}}{4\pi} = \frac{c \tilde{\mathbf{B}}}{4\pi |q_-| \tilde{n}_-} \frac{\tilde{\Omega}_+}{\omega_{\text{coll}}^{\text{eff}, A}}. \end{aligned} \quad (15)$$

Here $\mathcal{F}_H^{\text{eff}} \equiv c a_H^{\text{eff}} |q_-| \tilde{n}_-$ is a dimensionless $O(1)$ function and $\omega_{\text{coll}}^{\text{eff}, j}$ are effective collision frequencies. In this form, we have simply re-parameterized our ignorance of the microphysics into the effective collision rates.

Nonetheless, the discussion above suggests that we can capture the leading-order effects from § 2.3 by (1) multiplying all collision rates by $\sim [1 + (v_{\text{drift, slip}}/v_T)^2]^{1/2}$, so they correctly capture the leading-order superthermal corrections to the drag law, and (2) adding an approximate anomalous resistivity as an additional effective collision rate, which “turns on” at $v_{\text{drift}} > v_T$ and therefore self-regulates the system to $v_{\text{drift}} \lesssim v_T$. Noting the dependence of the various η on the collision frequencies in § 2.1, this becomes:

$$\begin{aligned} \eta_O^{\text{eff}} &\rightarrow \left(\eta_O^0 + \eta_O^{\text{an}} \Theta \left[\frac{v_{\text{drift}}}{v_T} \right] \right) \left[1 + \left(\frac{v_{\text{drift}}}{v_T} \right)^2 \right]^{1/2}, \\ \eta_O^{\text{an}} &\equiv \frac{c^2 m_-}{4\pi |q_-|^2 \tilde{n}_-} \omega^{\text{an}}, \\ \eta_H^{\text{eff}} &\rightarrow \eta_H^0, \\ \eta_A^{\text{eff}} &\rightarrow \eta_A^0 \left[1 + \left(\frac{v_{\text{drift, max}}}{v_T} \right)^2 \right]^{-1/4}, \end{aligned} \quad (16)$$

where $\Theta(x)$ can be the Heaviside step function or any similar function (e.g. $e^{-1/x}$), and $v_{\text{drift, max}}$ is defined from Eq. 9 using η_A^0 (i.e. from the usual coefficient estimated without including velocity-dependence). Note (as justified more quantitatively in § A), the dependence of η_O on v_{drift} (as the relevant terms in the collision rates depend more strongly on v_{drift} in the limit where this is large), and the 1/4 slope in η_A^{eff} arises because of the implicit non-linear dependence of v_{slip} on η_A (hence the collision rate $\omega_{\text{coll}}^{\text{eff}, A}$ in Eq. 9; see § A and [Hillier 2024](#)). It is not necessary to drop inelastic reactions in the η^0 terms if they are important. Motivated by the extensive studies discussed § 2.3, we take ω^{an} to be the faster of either the gyro or plasma frequencies, e.g. $\omega^{\text{an}} \sim (\tilde{\Omega}_-^2 + \omega_p^2)^{1/2}$, but we will show that the exact value does not matter so long as it reflects a sufficiently fast frequency of the system.

Prescriptions similar to this are already widely used in laboratory, Solar, and ionospheric plasma physics applications (see references in § 2.3.2 and [Somov & Oreshina 2000](#); [Rousev et al. 2002](#); [Ni et al. 2007](#); [Færder et al. 2023](#), for examples). Likewise, it is directly analogous to widely adopted treatments of similar problems in astrophysical applications of thermal conduction and viscosity, where naively applying the “classical” conduction/viscosity equations ([Spitzer & Härm 1953](#); [Braginskii 1965](#)) can produce sharp gradients that imply unphysically superthermal drift velocities (of e.g. thermal electrons or ions in the “saturated” limit; e.g. [Spitzer 1962](#);

Chapman & Cowling 1970; Cowie & McKee 1977). It has also been widely recognized that for high- β^{plasma} plasmas, the maximum anisotropy of the velocity distribution function (which can be considered similarly to a maximum drift speed) is limited by microscopic instabilities like the whistler, mirror, and firehose, which are activated above certain thresholds (Komarov et al. 2014, 2016, 2018; Kunz et al. 2014; Squire et al. 2016, 2017a,b). These are similarly approximated on macroscopic scales by multiplying the “classical” coefficients by corrections designed to represent some anomalous scattering that limits the velocity anisotropy (see e.g. Riquelme et al. 2016; Roberg-Clark et al. 2016, 2018; Su et al. 2017; Squire et al. 2017c; Hopkins et al. 2020a). We are simply pointing out that similar terms should apply in the other astrophysical cases of interest here (as was also argued by authors like Norman & Heyvaerts 1985), generalizing them to cases with e.g. appreciable neutrals and non-electron-ion plasmas, while also accounting for similar the relevant terms in the ambipolar drift.

2.4.2. Generalization To Arbitrary Numbers of Species

Given the full set of restricted assumptions in § 2.2, one can derive coefficients $\eta_{O,H,A}^0$ for an arbitrary number of species. In principle, as outlined in more detail in Appendix A, one could solve an entire chemical network for species abundances and simultaneously obtain the effective coefficients $\eta_{O,H,A}$ for an arbitrary set of velocity-dependent collision rates and anomalous resistivities ω_{ij}^{an} , based on the relative velocities between all pairs of species. But this would involve fundamental changes to the sorts of chemical networks and solvers usually employed, as well as introducing a number of ambiguities. Instead, much of the attention in modeling weakly ionized systems has focused on using chemical networks to calculate different charged-particle abundances n_i , which are then used in a fully operator-split manner to calculate $\eta_{O,H,A}^0$ from simplified expressions that depend only on n_i (and background properties like T or B , but not drift speeds). So it is important to consider how to generalize our approach to include many species. In this section, we develop a prescription that captures anomalous resistivity effects, but is sufficiently simple to be straightforwardly implemented into standard chemical-network methods.

Per Appendix A, for the applications of interest here (weakly ionized systems), the “standard” coefficients are derived in the multi-species limit by taking Eq. 1, neglecting all terms except Lorentz and neutral collisions, and assuming these are in equilibrium, to solve for $\delta\mathbf{u}_j \equiv \mathbf{u}_j - \mathbf{u}_n$ as a function of $\mathbf{E}' \equiv \mathbf{E} - \mathbf{u}_n \times \mathbf{B}/c$ with $q_j(\mathbf{E}' + \delta\mathbf{u}_j \times \mathbf{B}/c) = m_j\omega_{jn}^{\text{coll}}\delta\mathbf{u}_j$ (this is the local “terminal velocity approximation”). This leads to the well-known result that the relative speeds of different particles scale with their Hall parameter $\propto q_j/m_j\omega_j^{\text{coll}}$, often parameterized as:

$$\tilde{\beta}_j \equiv \frac{q_j B}{m_j c \gamma_j \rho} \quad (17)$$

(Cowling 1976; Nakano & Umebayashi 1986; Wardle & Ng 1999). The contribution of each to the current is then $|\delta\mathbf{J}| \sim n_j q_j \tilde{\beta}_j$.³ Consider e.g. the simple case where there are two negative charge carriers – one “light” and relatively

³ Note since this only appears as a weight function, the choice to multiply by $eB/c\rho$ is arbitrary. One can generalize Eq. 17 further for the case with

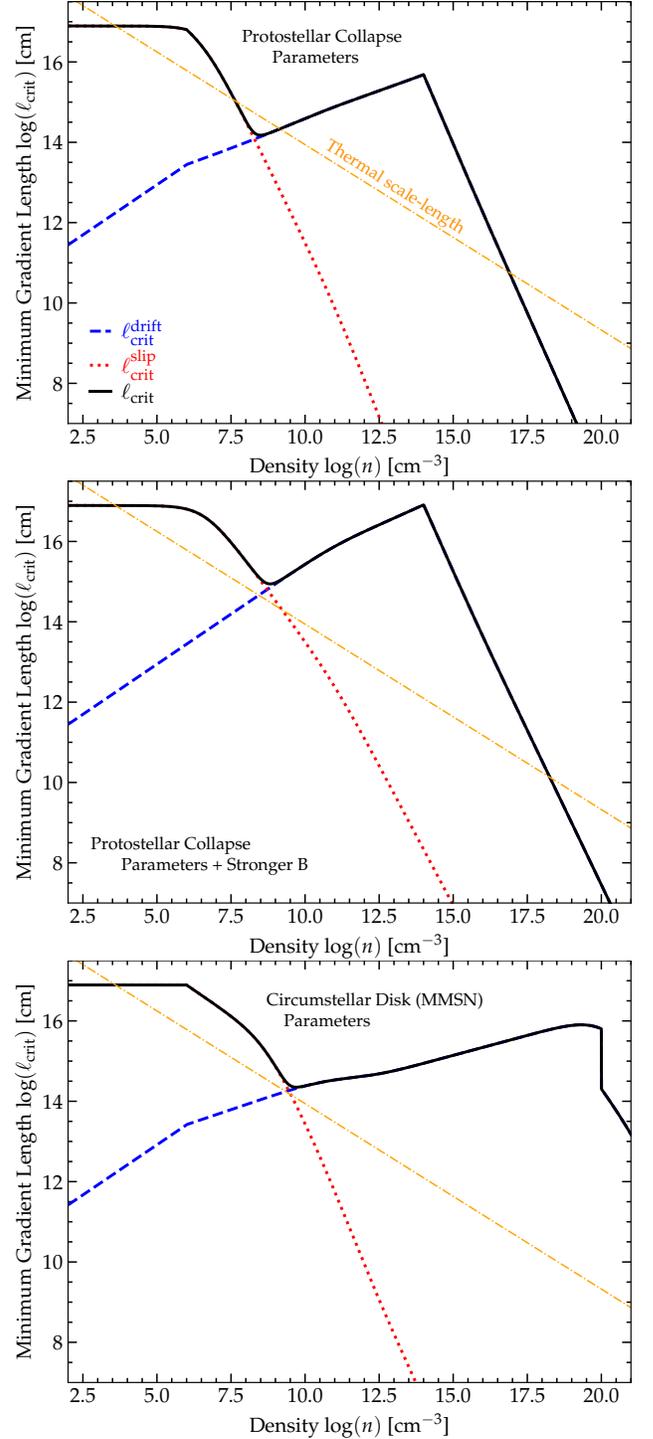


FIG. 3.— Critical magnetic gradient scale-length ℓ_{crit} , below which the drift ($\ell_{\text{crit}}^{\text{drift}}$) or slip ($\ell_{\text{crit}}^{\text{slip}}$) speeds become superthermal, versus density n for the same three model variants as Fig. 2. Strong gradients below this scale will be rapidly erased by enhanced resistivity (Fig. 2; § 4). For comparison we plot the disk scale-height or thermal-pressure scale-length h_T predicted for a MMSN at the radii corresponding to each n . At circumstellar disk radii from $\sim 0.1 - 100$ au (densities $n \sim 10^{10} - 10^{18} \text{cm}^{-3}$), and at all densities where the Hall diffusion would naively dominate absent anomalous correction terms (for $v_{\text{drift}} \ll v_T$, we have $\ell_{\text{crit}} \gg h_T$, so magnetic structures and Hall effects on disk scales will be strongly suppressed).

arbitrary neutral density, where for the fully-ionized case the weights $\tilde{\beta}_j \propto q_j/m_j$, but in this case the various classical η terms are generally negligible compared to the ideal MHD terms.

weakly collisionally coupled, and one “heavy” and much more strongly collisionally coupled (both with $q \sim -e$). This is often the case with electrons and charged grains, where the assumption of such a mass hierarchy is well justified. This gives the intuitive result that even if $n_{\text{grains}} > n_e$, the current is still primarily carried by the electrons, unless $n_{\text{grains}}/n_e \gg (m_{\text{grain}}\gamma_{\text{grain}})/(m_e\gamma_{en}) \sim 10^{9.3}(R_{\text{grain}}/\mu\text{m})^2$. For the systems of interest (e.g. protostellar cores, circumstellar/protostellar disks, planetary atmospheres), the latter condition is rarely met, so even though grains can dominate the total charge, they do not actually carry most of the current. This means the current is being carried by relatively fewer free electrons, implying even higher drift speeds.⁴

To calculate v_{drift} , we therefore take Eq. 9 ($v_{\text{drift}} = (c\nabla \times \bar{\mathbf{B}})/(4\pi\bar{n}_+q_+) = (c\nabla \times \bar{\mathbf{B}})/(4\pi\bar{n}_-|q_-|)$) and replace it with the weighted version:

$$v_{\text{drift}} \rightarrow \frac{c\nabla \times \bar{\mathbf{B}}}{4\pi\langle\bar{n}_-|q_-|\rangle}, \quad (18)$$

$$\langle\bar{n}_-|q_-|\rangle = \langle\bar{n}_+q_+\rangle \rightarrow \frac{\sum_j n_j |q_j| |\tilde{\beta}_j|}{\sum_j |\tilde{\beta}_j|}.$$

while $v_{\text{slip}, \perp}$ (and hence $v_{\text{drift}, \text{max}}$) can be calculated directly from Eq. 9 as $v_{\text{slip}, \perp} = (\bar{\rho}/\bar{\rho}_n)(\eta_A^0/\bar{B}^2)\nabla \times \bar{\mathbf{B}} \times \bar{\mathbf{B}}$ using the total density and total density of (all) neutrals. Note that Eq. 18 immediately reduces to our expression from § 2.2 ($\langle\bar{n}_-|q_-|\rangle = n_-|q_-| = n_+q_+$ when we just have two dominant charge carriers (one positive one negative), as it should. We similarly replace $n_-|q_-|$ or $n_i q_i$ by $\langle\bar{n}_-|q_-|\rangle$ (Eq. 18) in definitions like Eq. 13 for ℓ_{crit} .

For η_O^{an} , note that we can now write:

$$\eta_O^{\text{an}} \sim \frac{c\bar{B}}{4\pi\langle\bar{n}_-|q_-|\rangle} \left[1 + \frac{4\pi\bar{\rho}_-c^2}{\bar{B}^2} \right]^{1/2}, \quad (19)$$

so simply make the same replacement for $\bar{n}_-q_- \rightarrow \langle\bar{n}_-|q_-|\rangle \sim en_e$ in the prefactor, and note $\bar{\rho}_- \equiv m_- \langle n_- \rangle \sim m_e n_e$ should reflect the same dominant current-carrying species as $\langle\bar{n}_-|q_-|\rangle$. Also note that in the multi-species treatment, the pre-factors $\propto (n_-|q_-|)^{-1}$ in Eq. 15 are replaced by a similar weighted factor, so the two are consistent.

The generalization of v_T is less obvious. Of course per § 2.3 the exact correct value here depends on the exact parameters and microphysics by way of exactly which instabilities and collision and interaction terms dominate on which scales. We can define an effective isothermal soundspeed for a multi-component plasma as:

$$v_T^2 \equiv \frac{P_{\text{eff}}}{\rho_{\text{eff}}} \equiv \frac{\sum_j \psi_j n_j k_B T_j}{\sum_j \psi_j m_j n_j}, \quad (20)$$

where ψ_j is some weight. In most applications of interest here, since it is already assumed in the standard formulation, we can take $T_j = T$ (a single temperature system).

Naively, a sensible weight might be $\psi_j \sim \text{constant}$ for charged species and $= 0$ for neutrals, or $\psi_j = |q_j|$. For an ion-

⁴ Consider, for example, the limit of extremely large grains or “rocks.” Even if these somehow accumulated free electrons, clearly their electromagnetic forces are negligible and they would remain anchored to their positions, with Coulomb forces accumulating an opposite charge “locked” to their location. Thus even if they do not recombine, it is more appropriate to consider these effectively part of the neutral background, while the current and MHD dynamics of the fluid on scales large compared to the inter-grain separation are carried entirely by the remaining free electrons.

electron plasma, this would give the expected answer of the thermal-ion speed (up to an $O(1)$ prefactor, which is degenerate with our uncertainties in the “most salient” v_T in any case). But this choice of ψ would not correctly capture the case where one or both carriers are dust grains: with that definition $v_T \sim (k_B T/m_{\text{grain}})^{1/2} \sim 0.01 \text{ cm s}^{-1} (T/10 \text{ K})^{1/2} (a_{\text{grain}}/\mu\text{m})^{-3/2}$ would become vanishingly small. This produces a number of results that might technically be valid if the *only* species were dust grains (i.e. no neutrals or ions or electrons present – an extreme “dusty plasma” case) but do not make sense in the context of interest here (where the dust thermal speed is extremely slow compared to that of other important species in the plasma). We can formulate a ψ to capture realistic dusty plasmas by noting that in the limits of interest where dust is a dominant charge carrier, we are always in the weakly-ionized limit (large neutral density) with the dust strongly coupled to said neutrals ($\omega_{\text{dust}, n}^{\text{coll}} \gg \Omega_{\text{dust}}$). This means the neutrals should be included in Eq. 20, so $v_T \sim v_{T, n}$, and instabilities are excited when v_{drift} exceeds the neutral thermal speed. This also accords with the analysis of dust streaming instabilities/RDIs in Squire & Hopkins (2018b,a); Hopkins & Squire (2018b), where $v_{T, n}$ is the speed grains must be moving to excite strong instabilities on small scales. We can capture these cases appropriately by taking:

$$\psi_j \equiv \begin{cases} |q_j| & (q_j \neq 0) \\ \frac{1}{\sum_i |q_i| n_i} \sum_i \frac{\omega_{ij}^{\text{coll}} |q_i| n_i}{\omega_{ij}^{\text{coll}} + \Omega_i} & (q_j = 0). \end{cases} \quad (21)$$

Note the latter sum is nearly identical to sums that already need to be computed for $\eta_{O, H, A}^0$ in the standard multi-species formalism (Wardle & Ng 1999) so entails no additional computational expense. In the weakly-ionized limit with any modest neutral coupling, this will tend to $v_T \rightarrow v_{T, n}$, while in the strongly ionized limit, this will tend to the thermal ion speed, as desired.

As noted above, another critical wavespeed of interest is the effective Alfvén speed v_A^{eff} (with various instabilities excited for $v_{\text{drift}} > v_A^{\text{eff}}$; § 2.3.2), so we should more generally take

$$v_T \rightarrow \text{MIN} [v_T^{\text{therm}}, v_A^{\text{eff}}]$$

$$v_A^{\text{eff}} \equiv \frac{B}{\sqrt{4\pi \sum_j \psi_j m_j n_j}} \quad (22)$$

where v_T^{therm} is given by Eq. 20 and the weights in v_A^{eff} follow from the same argument as above for v_T^{therm} . Qualitatively, most super-Alfvénic instabilities are similar to non-relativistic CR streaming instabilities (Wentzel 1968) with growth rates $\sim \Omega_e (v_{\text{drift}}/v_A - 1)$. If all species have $\omega^{\text{coll}} \gg \Omega$, these instabilities and the excited Alfvén waves will be strongly-damped, but this case is well into the classical Ohmic regime where the anomalous resistivity would be relatively small anyway. If all species have $\omega^{\text{coll}} \ll \Omega$, i.e. the classical ambipolar regime, then $v_A \rightarrow B/\sqrt{4\pi\rho_i}$ becomes the “ion Alfvén speed,” but this is $\gg v_T$ in weakly-ionized systems. But in the classical Hall regime ($\Omega_+ \ll \omega \ll \Omega_-$), this gives interesting behavior: the effective Alfvén speed is the ideal $v_A^{\text{eff}} \approx B/\sqrt{4\pi\rho}$ because the wavelengths of interest are sufficiently large that the Alfvén wave frequency is much smaller than ion-neutral collisions, but the growth rate for the fast carriers (e.g. electrons) is much larger than the damping/collision rates. In this limit, $v_A^{\text{eff}} < v_T^{\text{therm}}$ is equivalent to $\beta^{\text{plasma}} > 1$. So for systems

in the classical Hall regime, with $\beta^{\text{plasma}} > 1$, the inclusion of v_A^{eff} in v_T can be a non-negligible correction. While, for the examples we plot below this makes almost no difference to the results, the most interesting consequence in this limit is that $\ell_{\text{crit}}^{\text{drift}} \rightarrow 73 \text{au} (n/10^{15} \text{cm}^{-3})^{1/2} (n_e/\text{cm}^{-3})^{-1}$, independent of B or T and dependent only on the total density and free electron density of the system.

2.4.3. Specific Examples for Calculation

The relative values of the standard resistivity (η) coefficients are shown in Figs. 1, 2, & 3, for relatively simple assumptions of a multi-species system. In order to calculate actual values of the coefficients, of course, we need to make a number of specific assumptions about the scalings of $B = |\mathbf{B}|$, $n \equiv \rho/m_p$, ionization fractions, species abundances, temperatures, etc. These figures are intended for illustrative purposes only, so we adopt an extremely simple toy model for the quantities needed to compute the values shown. Specifically, we take (with all units here in CGS) similar parameters to those usually assumed in models of initial “warm” protostellar core collapse with temperature $T = 10\sqrt{1 + (n/10^{n_{T,1}})^{0.8} (1 + n/10^{n_{T,2}})^{-0.3} (1 + n/10^{n_{T,3}})^{0.57}}$ for $(n_{T,1}, n_{T,2}, n_{T,3}) = (11, 16, 21)$ from Machida et al. (2006); and $B = 10^{-6} n^{1/2}$ for $n < n_{0,B}$, $B = 10^{-6} (n n_{0,B})^{1/4}$ for $n \geq n_{0,B}$ with $n_{0,B} = 10^6$ from Wardle & Ng (1999). We adopt a simple five-species model of: molecular neutrals with Solar abundances ($\xi_n \approx 1$; $m_n = 2.3m_p$); electrons ($n_e \sim 0.01$ for $n < 10^{n_{e,1}}$ and $n_e \sim 0.01(n/10^{n_{e,1}})^2$ for $n \geq 10^{n_{e,1}}$, with $n_{e,1} = 14$); atomic+molecular ions ($n_i = n_e$, $q_i = e$, $\langle m_i \rangle = 29m_p$); negative and positive small dust grains ($n_{g+} = n_{g-} \sim 10^{-13}n$, $q_{g+} = -q_{g-} = e$, $R_{\text{grain}} = 3 \times 10^{-6}$). These fitting functions give a rough approximation to the results from the chemical model using similar temperature and B assumptions from NICIL (Wurster et al. 2021). And we take the zero-drift cross sections to have values $\langle \sigma v \rangle_{in} = 1.5 \times 10^{-9}$, $\langle \sigma v \rangle_{en} = 4.7 \times 10^{-10} T^{1/2}$, $\langle \sigma v \rangle_{gn} = 5.1 \times 10^{-6} T^{1/2}$, $\langle \sigma v \rangle_{ei} = 51 T^{-3/2}$ (Spitzer & Härm 1953) and neglect any other reactions.

In addition to the above, we compare with two alternative models. The first is identical except we assume B continues to rise more rapidly with n (setting $n_{0,B} = 10^{22}$). For the second we adopt temperatures and ionization fractions more appropriate for a circumstellar disk, specifically adjusting the model parameters above to give similar results to the minimum mass Solar nebula (MMSN) model from Chiang & Youdin (2010), taking $(n_{T,1}, n_{T,2}, n_{T,3}) \rightarrow (10.5, 12.5, 19.5)$, $n_{e,1} \rightarrow 20$, and $n_{g\pm} \rightarrow 10^{-15}n$ at $n < 10^{n_{e,1}}$.

In Fig. 1, we estimate which of the Ohmic/Hall/ambipolar coefficients is largest in magnitude as a function of B and n (taking just the default model for abundances, T , etc., above, but allowing B to freely vary at fixed values of all other properties at each n). In Fig. 2 and Fig. 3, we show the values of $\eta_{O,H,A}$ and the critical gradient length scales $\ell_{\text{crit}}^{\text{drift}}$ and $\ell_{\text{crit}}^{\text{slip}}$ (Eq. 13) for each of the three models above. In Figs. 1 & 2, we compare the standard non-ideal formulation for $\eta_{O,H,A} = \eta_{O,H,A}^0$ (Eq. 8), which ignores anomalous corrections and is therefore only valid when $\ell_B \gg \ell_{\text{crit}}$ ($v_{\text{drift}} \ll v_T$), to the corrected coefficients for the case of an ℓ_B which would otherwise give $v_{\text{drift}} \sim 10 v_T$ (using the implied v_{drift} from the standard formulation, Eq. 9), i.e. $\ell_B \sim 0.1 \ell_{\text{crit}}^{\text{drift}}$. So Fig. 3 plots the dividing line between these cases, in terms of ℓ_B .

3. WHERE MIGHT THE “STANDARD” FORMULATION BE PROBLEMATIC?

Where might the correction terms in § 2.4 be important? Consider the case with $v_{\text{drift}} > v_{\text{slip}}$, and note we can write $\ell_{\text{crit}}^{\text{drift}} \approx (v_{A,i}/v_T) d_i$, where $v_{A,i} \equiv B/\sqrt{4\pi\rho_i}$ and $d_i \equiv c/\omega_{pi} = v_{A,i}/\Omega_{c,i} = r_{g,i}/\sqrt{\beta_i^{\text{plasma}}}$ are the ion Alfvén speed and inertial length (and $\beta_i^{\text{plasma}} \equiv (c_s/v_{A,i})^2 \equiv P_{\text{therm},i}/2P_B \equiv 4\pi n_i k_B T_i/B^2 \equiv f_i \beta^{\text{plasma}}$ is the ion plasma β_i^{plasma} , with $r_{g,i}$ the ion gyro radius). This means that for well-ionized plasmas, so long as $\ell_B \gg d_i$ (for $v_T \sim v_A$) or $\gg d_i/\beta_i^{\text{plasma}^{1/2}} = r_g/\beta_{\text{plasma}}$ (for $v_T \sim v_{\text{thermal}}$), the system will be “safely” in the $v_{\text{drift}} \ll v_{T,i}$ regime.⁵

However, in weakly-ionized plasmas such as dense molecular cloud cores, circum-stellar and circum-planetary disks (including protostellar and protoplanetary disks), planetary atmospheres, and some cool-star outflows, f_i can become extremely small (e.g. $\sim 10^{-15}$; Galli & Shu 1993; Basu & Mouschovias 1994; Kunz & Mouschovias 2010; Dapp & Basu 2010). For example, scaling to parameters expected at distances of order ~ 1 au in a circumstellar disk, we have $\ell_{\text{crit}} \sim 400 \text{ au} (B/\text{G}) (0.1 \text{ km s}^{-1}/v_{T,i}) (0.1 \text{ cm}^{-3}/n_i)$ (or even larger, in the specific regime noted in § 2.4.2 where $v_A < v_T^{\text{therm}}$), much larger than disk scale-heights ~ 0.05 au. The values of ℓ_{crit} one might expect at different densities are illustrated more directly in Fig. 3.

If one only models “ideal” or Braginskii/Spitzer MHD, there is nothing to prevent such strong currents from forming. However in weakly-ionized environments, the “classical” non-ideal terms in Eq. 8 (Ohmic $\eta_{O,0}$, Hall $\eta_{H,0}^0$, and ambipolar $\eta_{A,0}^0$) need to be accounted for. Dimensionally, the coefficients η scale as $\mathcal{O}(\eta^0) \sim B c F(\Psi)/4\pi e n_i$ (where $F(\Psi) \gtrsim 1$ is a dimensionless function of the different frequencies, see above), so the ratio of the non-ideal to ideal ($\mathbf{U} \times \mathbf{B}$) term is $\sim F(\Psi) \ell_{\text{crit}} v_{T,i} / \ell_B |\mathbf{U}|$. Since the Ohmic and ambipolar terms are fundamentally diffusive, this means they will generally tend to mitigate against super-thermal drifts: so long as the bulk motions of the fluid are subsonic ($|\mathbf{U}| \ll v_{T,i}$), then if the drift becomes large ($\ell_B \lesssim \ell_{\text{crit}}$), the diffusive term will become large in the induction equation, diffusing the gradient away and restoring subsonic drift speeds. However, even in the Ohmic/ambipolar regime, superthermal drifts can still arise in principle if we only include the “classical” terms η^0 and external forces (e.g. gravity or radiation) drive superthermal compressions/outflows/shocks.

The problem in the classical regime is much more severe if the Hall term dominates, because its effect is fundamentally *not* diffusive. Indeed, simulations of Eq. 8 in weakly-ionized systems show that the Hall term drives the formation of thin current sheets (thinner than thermal scale lengths) causing ℓ_B to shrink to very small values at these interfaces (Bejarano et al. 2011; Kunz & Lesur 2013; Lesur et al. 2014; Bai & Stone 2017). This would thus *increase* v_{drift} . And Zhao et al. (2018, 2021) explicitly show in dense core collapse and circumstellar

⁵ For example, in the diffuse interstellar, circumgalactic, and intergalactic medium, stellar photospheres, HII regions, and OB winds, the ratio of typical ℓ_B for order-unity changes in B (i.e. the Alfvén scale of turbulence) to ℓ_{crit} is $\sim 10^{10} - 10^{15}$, and it reaches $\gg 10^{20}$ in stellar interiors. Even in extremely low- β^{plasma} environments (as low as $< 10^{-6}$), such as those proposed for quasar accretion disks in Hopkins et al. (2023a,c,b) this ratio is still $\gg 10^9$.

disk simulations (using standard non-ideal MHD formulation and coefficients $\eta_{O,H,A}^0$) that the implied drift velocities reach several km s^{-1} at temperatures $T \sim 10\text{--}100\text{ K}$ (thermal speeds $\lesssim 0.1\text{ km s}^{-1}$). So superthermal drift is clearly occurring in at least some calculations. This is illustrated in Figs. 1-3, and discussed in § 4.5 below, where we find no regime in which Hall can dominate on global scales.

A third potentially problematic regime is when dust grains become one of the primary charge carriers by absorbing free electrons (and/or ions). This can technically make the system a “dusty plasma” (as distinct from the more typical “dust-laden” plasma in most astrophysical environments; see Melzer et al. 2021a,b; Beckers et al. 2023). It is not always obvious that the other assumptions used in § 2 to derive Eq. 8 hold in this regime, since grains can develop internal currents and fields (being complicated electrostatic and dielectric media themselves) and can undergo near continuous charge exchange reactions (so we might need to include terms like $\partial_t q_{\pm}$ which are ignored in MHD derivations), but such questions are beyond the scope of our study. Even if we simply treat grains as “heavy” ions (the usual approximation), their masses are enormous ($\gtrsim 10^{13} m_p$), and the “light” carriers are either heavy molecules or other grains. This means the effective thermal “ion” speeds can become extremely small and (correspondingly) they can develop extreme anisotropy in their distribution functions which usually enhances the instabilities reviewed above, allowing them to persist down to extremely small scales (Squire & Hopkins 2018b). Moreover, since grains are so heavy, an even greater current-carrying requirement (and therefore even higher drift speeds) is implicitly placed on the (fewer) remaining free electrons.

4. SOME CONSEQUENCES

Now consider some of the important consequences of Eq. 14, compared to the “standard” zero-drift/slip coefficients $\eta_{O,H,A}^0$ (Eq. 8). We illustrate these in Figs. 1, 2, & 3.

4.1. Ordering and Scaling of Terms

From Eq. 15 the relative importance of the non-ideal η terms are given by ratios $\Omega_c/\omega_{\text{coll}}$ of gyro to collision frequencies, so it is useful to consider three limits in turn, illustrated in Fig. 1.

The regime where ambipolar diffusion would dominate, in the “standard” or zero-drift-assumption limit ($\Omega_{c,\pm} \gg \omega_{\text{coll}}^0$), generically has $v_{\text{drift}} \ll v_{\text{slip}}$, meaning current-driven instabilities will not be important. While there can be pure slip-driven instabilities, their effect is less clear (see § 2.3.2). So when the slip becomes superthermal, we just have the Epstein-type correction for $\langle \sigma v \rangle$ accounting for the superthermal relative velocity of charge carriers versus neutrals in the collision rates, which decreases η_A and increases η_O , but only by a linear factor of $v_{\text{drift/slip}}/v_T$. Thus it will not usually change the relative ordering of the non-ideal terms.

In the regime where Ohmic resistivity would strongly dominate with zero drift ($\Omega_{c,\pm} \ll \omega_{\text{coll}}$), $v_{\text{drift}} \gg v_{\text{slip}}$. If the drift becomes superthermal, current-driven instabilities could appear but in this limit, the anomalous resistivity $\eta_O^{\text{an}} \sim \eta_O^0 (\Omega_{c-,p-}/\omega_{\text{coll}}^0) \ll \eta_O^0$ would be relatively small because the collision frequency driving the resistivity is already the fastest frequency of interest (so even if collisions strongly damp these instabilities in this limit, it makes no practical difference). The dominant correction will again simply be

the proper accounting for superthermal drift in the Epstein correction to the collision rates, boosting η_O by a factor of $\sim v_{\text{drift}}/v_T$.

The regime where the Hall term would dominate with zero drift ($\Omega_{c,+} \ll \omega_{\text{coll}} \ll \Omega_{c,-}$, $v_{\text{drift}} \sim v_{\text{slip,-}} \gg v_{\text{slip,+}}$) is perhaps most interesting, despite the fact that the Hall term *itself* is independent of any of these correction factors (being independent of the effective collision frequency except for an $O(1)$ prefactor that depends only on the dimensionless ratios of different collision rates). In this regime, by definition, the zero-drift Ohmic term is relatively small $\eta_O^0 \sim |\eta_H^0| \omega_{\text{coll}}/\Omega_{c-} \ll |\eta_H^0|$. But if the drift becomes superthermal, current-driven instabilities will appear and be weakly or negligibly damped. Thus $\eta_O \rightarrow (\eta_O^0 + \eta_O^{\text{an}}) (v_{\text{drift}}/v_T)$, where $\eta_O^{\text{an}} \sim |\eta_H|$ and $v_{\text{drift}}/v_T \gtrsim 1$. So $\eta_O \sim |\eta_H| (v_{\text{drift}}/v_T) \gtrsim |\eta_H|$ and the excited Ohmic term always becomes larger than the Hall term. The outcome is that one can never have a dominant Hall term *and* superthermal drift speeds (see also Choueiri 1999; Shay et al. 1999).

4.2. Driving the System Towards Sub-Thermal Drift Speeds and Smooth Magnetic Gradients

Consider what happens if superthermal drift arises, i.e. something induces gradients with $\ell_B \ll \ell_{\text{crit}}$. As discussed in § 3, if we only adopt the zero-drift coefficients $\eta_{O,H,A}^0$, then especially if the Hall term dominates, there is no guarantee that the non-ideal terms will actually reduce the drift speed (or increase ℓ_B) because it is non-diffusive. But per § 4.1, we see that with the corrected coefficients, η_H can never dominate in this limit, so we are ensured that the nonideal terms behave diffusively (see Fig. 2). This means that they will always act to smooth out the gradients ℓ_B until subthermal drift is restored. The timescale for this to occur will be $\sim \ell_B^2/\text{MAX}[\eta_O, \eta_A] \sim (\ell_B/v_T) (v_{\text{drift}}/v_T)^{-2} \mathcal{F}_{\text{max}}^{-1}$ where $v_{\text{drift}}/v_T \gtrsim 1$ and $\mathcal{F}_{\text{max}} \equiv \text{MAX}[1, \Omega_+/\omega_{\text{coll}}, \omega_{p-}/\Omega_-, \omega_{\text{coll}}/\Omega_-] \gtrsim 1$. So generically this will occur on a timescale much shorter than the sound crossing time.

So unless there are strongly supersonic extrinsic motions driving the “ideal” term $\mathbf{U} \times \mathbf{B}$ to compress ℓ_B – essentially, strong (supersonic and super-Alfvénic) transverse shocks with a large energy source “pumping” the currents – the corrected coefficients will generically ensure ℓ_B rapidly expands to restore subthermal drift ($\ell_B \gtrsim \ell_{\text{crit}}$). Equivalently, strong magnetic field gradients are diffused away on scales $\lesssim \ell_{\text{crit}}$, shown in Fig. 3.

Note here that $\ell_B \equiv |\bar{\mathbf{B}}|/|\nabla \times \bar{\mathbf{B}}|$ is the global gradient length scale; so *weak* magnetic field gradients on scales $\lambda \ll \ell_{\text{crit}}$ are still allowed. If we consider some small-amplitude structure with $|\nabla \times \bar{\mathbf{B}}| \sim \delta B/\lambda$ and $\delta B \ll B$; then $\ell_B \sim \lambda |B|/|\delta B| \gtrsim \ell_{\text{crit}}$. In this limit we can thus think of the physics above as setting an upper limit to the strength of field perturbations on scale λ , i.e. $|\delta B|/|B| \lesssim \lambda/\ell_{\text{crit}}$. This implies that the modification to MHD waves from the Hall effect will remain, but that these waves will experience non-linear damping above a certain (wavelength-dependent) amplitude owing to induced super-thermal drift enhancing the Ohmic resistivity. Recalling that $\ell_{\text{crit}}^{\text{drift}} \sim r_g/(f_{\text{ion}} \beta^{\text{plasma}})$, this becomes $|\delta B|/|B| \lesssim (\lambda/r_g) f_{\text{ion}} \beta^{\text{plasma}}$, we see that this places a serious constraint in very low f_{ion} plasmas for short-wavelength MHD waves and other perturbations. This is similar to effects known in high- β^{plasma} collisionless plasmas due to analogous instability-mediated “anomalous viscosities”: Alfvén waves

cannot propagate (Squire et al. 2017c) and sound waves are modified (Kunz et al. 2014) above a β^{plasma} -dependent threshold.

4.3. Behavior in the Zero-Charge Limit

A peculiar feature of the “standard” formulation of non-ideal MHD (Eq. 8) is that it is *not* guaranteed to reduce to hydrodynamics as charge vanishes ($q_i n_i \rightarrow 0$). The total momentum equation (Eq. 2) depends only on $\mathbf{J} \times \mathbf{B} \propto \nabla \times \mathbf{B} \times \mathbf{B}$ and is independent of $q_i n_i$, so the system can only reduce to hydrodynamics if these magnetic gradient terms vanish. If the Ohmic or ambipolar terms dominate over the Hall terms in the non-ideal diffusivities, they are diffusive and scale $\propto 1/q_+ n_+$, so the diffusivity would become infinite and ℓ_B would increase, ensuring the fields rapidly escape and cannot couple strongly in the momentum equation, as desired. But recall the Hall term is non-diffusive and strong field gradients (i.e. strong forces on the neutrals) can persist indefinitely or even become sharper. And in the standard formulation, *whether* the Hall term dominates depends only on the ratio of the gyro to collision frequencies, independent of $q_i n_i$. So one can have arbitrarily strong MHD forces on neutrals sourced by a vanishingly small number of charged particles, which are implicitly assumed to be moving at infinitely fast speeds to carry the current. This is obviously unphysical.

Our proposed correction terms in Eq. 15 restore physical behavior by ensuring that as $q_i n_i \rightarrow 0$ (which causes $v_{\text{drift}} \rightarrow \infty$), the Ohmic term always dominates, causing the field to decay resistively. Thus, the magnetic diffusivity $\propto v_{\text{drift}}/q_i n_i \propto 1/(q_i n_i)^2$ becomes infinitely large more rapidly than any other term in the MHD equations.

4.4. Maximum Magnetic Field Amplification

Imagine some collapsing gas wherein \mathbf{B} is amplified via flux-freezing and/or some local dynamo, with global length scale $\sim R$. To maintain flux-freezing or have a dynamo on scales $\ll R$ obviously requires that the medium not be highly resistive, which requires (at a minimum) that the drifts remain subthermal, i.e. ℓ_{crit} cannot exceed R . This sets an effective maximum \bar{B} , from Eq. 9:

$$\bar{B} \lesssim B_{\text{max}} \sim \frac{4\pi e v_T}{c} n_e R \sim 0.2 \text{G} \left(\frac{T}{100\text{K}} \right)^{1/2} \left(\frac{n_e R}{\text{cm}^{-3} \text{au}} \right) \quad (23)$$

(where we have used $\bar{n}_- |q_-| \sim e n_e$ for most regimes of interest; § 2.4.2). This has the interesting observationally testable consequence that $B_{\text{max}} \propto v_T n_e R$, i.e. the maximum field is proportional to the free electron column/dispersion measure. For well-ionized systems ($n_e \sim n$) this is generally uninteresting as B_{max} is enormous (equivalently ℓ_{crit} is tiny; § 3), and if we assume spherical isothermal collapse ($n \propto R^{-3}$) $B_{\text{max}} \propto n^{2/3}$ scales as steeply as isotropic spherical flux-freezing and/or supersonic dynamo amplification (Su et al. 2017, 2018). But for weakly-ionized systems like circumstellar disks, Eq. 23 is quite constraining. Moreover if we assume weakly-ionized ($n_e \ll n$) isothermal spherical collapse, with a constant ionization rate per neutral ζ and simple ion-electron recombination ($\dot{n}_e \sim n_n \zeta - \alpha_{\text{rec}} n_e n_i$) so $n_e \propto n^{1/2}$, then we obtain $B_{\text{max}} \propto n^{1/6}$. Note that for systems with a strong mean field, we might expect cylindrical collapse along an axis R perpendicular to \mathbf{B} , so $n \propto R^{-2}$, which for the same ionization assumption gives $B_{\text{max}} \propto n_e R \propto n^{1/2} R \propto R^0 \propto n^0$ – i.e. no

amplification at all. In either case, B_{max} increases very weakly (far weaker than flux-freezing) with density.

4.5. Is Non-Ideal Hall MHD Ever Important?

From the above (§ 4.1 & Figs. 1-2), the non-ideal Hall term can only dominate in the induction equation when $\Omega_+ \ll \omega_{\text{coll}} \ll \Omega_-$ and $|\mathbf{U}| \ll v_{\text{drift}} \ll v_T$. This is not theoretically impossible; it is however challenging in practice to imagine in equilibrium on large scales. For example, it is only possible if $|\mathbf{U}| \ll v_T$, i.e. if we have some flow with negligible gas (subthermal) gas motions on some relevant global scale ℓ_B , within some modest window of density and field strength (Fig. 1), with $\ell_B \gg \ell_{\text{crit}}$.

Note that the Hall factor $(\nabla \times \mathbf{B}) \times \mathbf{B}$ is the same term that appears in the total momentum equation: so for the Hall term to be important in induction on large scales (e.g. to drive formation of current sheets or magnetic switches, or “torques up” the disk, as observed in the Hall-dominated regime using the zero-drift coefficients), it must necessarily appear in the momentum equation as a magnetic pressure or tension force. Further it is easy to verify that the interesting Hall terms generically correspond to compressible fluctuations ($\nabla \cdot [(\nabla \times \mathbf{B}) \times \mathbf{B}] \neq 0$). So we should compare both restoring magnetic-tension forces (with characteristic timescales $\sim \ell_B/v_A$) and pressure forces (with timescales $\sim \ell_B/v_T$), to the characteristic Hall induction timescale $\sim \ell_B^2/|\eta_H|$. Noting the scaling of $|\eta_H| \sim v_{\text{drift}} \ell_B$, the Hall timescale is $\sim \ell_B/v_{\text{drift}}$, so can only be shorter than thermal timescales if $v_{\text{drift}} \gg v_T$, but in that superthermal limit the Ohmic term always becomes larger than Hall. So only rather contrived Hall-driven effects would be expected to persist.

Another way of saying this is to note $|\eta_H| \sim v_{\text{drift}} \ell_B \sim v_T \ell_{\text{crit}}$, so the effective propagation speed of Hall effects is $v_H \sim v_T \ell_{\text{crit}}/\ell_B$ and is self-limiting to $v_H \lesssim v_T$. So there will always be faster speeds from e.g. restoring pressure forces, if those are involved, let alone if any other super-sonic speeds appear in the problem. In contrast, the diffusive Ohmic and ambipolar speeds (v_O and v_A) are v_H modified by additional factors of $\omega^{\text{coll}}/\Omega$ (plus anomalous terms) or $\Omega/\omega^{\text{coll}}$ respectively, so can be much larger than thermal speeds in principle.

For weak perturbations, as noted above (§ 4.2), the Hall term could still be important. Consider an *incompressible* magnetic fluctuation which is also weak (amplitude $\delta B \ll (\lambda/\ell_{\text{crit}}) |B|$) and on sufficiently short wavelengths $\lambda \ll \ell_{\text{crit}} \ll \ell_B$ such that we can boost to a frame moving with the mean \mathbf{U} and have only small fluctuations $\delta \mathbf{U}(\lambda)$ on scale λ . If we are in the limit where the “classical” Hall term is larger than the Ohmic and ambipolar terms, then in this limit we will just recover the usual linearized Hall-MHD equations: i.e. that the Hall term modifies short-wavelength Alfvén waves to ion-cyclotron and whistler waves. So our intuition for these is unchanged, except that when we consider finite-amplitude waves, their amplitude must remain very small with $|\delta B| \ll (\lambda/\ell_{\text{crit}}) |B|$ to avoid exciting current-driven instabilities.

So in short, while it is not strictly impossible to imagine a parameter space where the non-ideal, weakly-ionized Hall MHD term could be important for order-unity magnetic field changes/fluctuations, it is highly restricted. It seems likely in particular that many of the conclusions regarding e.g. thin current sheets in circumstellar disks (with short-axis width less than thermal pressure scale-lengths), or global “disk torquing” by the Hall effect, would need to be revised (Fig. 3).

4.6. Behavior in the Well-Ionized Limit

Briefly, consider the fully-ionized limit in some more detail. Returning to § 2.1, if we have a fully-ionized electron plasma we can quickly see $a_O \rightarrow \omega_{-+}^{\text{coll}}/\Omega_-$, $a_H \rightarrow (1 - \epsilon)/(1 + \epsilon) \approx 1$, $a_A \rightarrow 0$. Thus the ambipolar term vanishes, and the Hall term is the largest non-ideal term in the “classical” ($v_{\text{drift}} \ll v_T$) limit when $\Omega_- \gg \omega_{-+}^{\text{coll}}$ (where $\omega_{-+}^{\text{coll}}$ reflects e.g. the Spitzer/collisionless Coulomb interactions), which is easily satisfied in most low-density well-ionized astrophysical systems (except at extremely high densities and low temperatures). In this limit the term inside the induction equation becomes $\approx (\bar{\mathbf{U}} - \mathbf{v}_{\text{drift}}) \times \bar{\mathbf{B}} \approx \bar{\mathbf{u}}_e \times \bar{\mathbf{B}}$, as expected.

In this limit, then, the Hall term can become important on some (small) scale λ so long as $|\delta\mathbf{U}(\lambda)| \lesssim v_{\text{drift}} \lesssim v_T$. In the fully ionized limit the drift velocities are small, $v_{\text{drift}} \sim v_T \ell_{\text{crit}}/\ell_B \sim (v_T/\beta^{\text{plasma}})(r_g/\ell_B) \sim (v_T/\beta^{\text{plasma}})(r_g/\lambda)(|\delta B|/|B|)$. Thus the Hall term being important requires $|\delta\mathbf{U}(\lambda)|/v_T \ll (r_g/\beta^{\text{plasma}}\lambda)(|\delta B|/|B|) \ll 1$. Since MHD is only valid on scales $\lambda \gg r_g$, the latter inequality ($v_{\text{drift}} \ll v_T$ being sub-thermal) is essentially always satisfied, while the former inequality ($|\delta\mathbf{U}(\lambda)| \ll v_{\text{drift}}$) is valid on sufficiently small scales λ . Note that on large scales in ISM and other applications, $|\delta\mathbf{U}(\lambda)|/v_T \gtrsim 1$, i.e. bulk motions are trans or super-sonic, and $\lambda \gg r_g$ by a huge factor while $|\delta B| \lesssim |B|$, which means that the Hall term is completely negligible compared to the ideal MHD term $\mathbf{U} \times \mathbf{B}$ – this is just a restatement of the fact that ideal MHD is an excellent approximation on large scales in well-ionized astrophysical plasmas. But this again leads to the usual expected small-scale behaviors that Hall terms will modify short-wavelength Alfvén waves as expected in a well-ionized plasma, though other effects (e.g. electron inertia) can also become important on small scales (Schekochihin et al. 2009).

5. CONCLUSIONS

The commonly-adopted astrophysical MHD equations are no longer internally self-consistent if gradients in \mathbf{B} become too steep (gradient scale-length ℓ_B smaller than some critical ℓ_{crit}), because the implied drift speeds (v_{drift}) become much faster than thermal speeds (v_T). This can invalidate foundational assumptions of the equations, rendering the expressions generally adopted for certain terms (like the non-ideal Ohmic/Hall/ambipolar coefficients) incorrect. Most importantly, it will excite microscale instabilities that suppress the drift, potentially modifying the effective collision frequencies by orders of magnitude. We show that this could be relevant in weakly ionized systems like cold cores, circumstellar/planetary disks, or planetary atmospheres, especially if (1) the Hall effect is significant, (2) external forces (like gravity) induce highly supersonic motions, or (3) dust grains become an important charge carrier. The “classical” treatments could then predict unphysical behaviors. We derive a simple pre-

scription to remedy this, extending treatments well-established in the fusion, Solar, and ionospheric plasma literature, and directly analogous to well-established treatments for similar problems in thermal conduction and viscosity. The treatment amounts to simply multiplying the Ohmic and ambipolar coefficients by factors involving only macroscopic (already-calculated) quantities. Implementing these correction terms in simulations therefore entails negligible cost or complexity. They also leave the behavior unchanged in the limit of slow drift for which the classical expressions are derived, but will restore the physical behavior if this limit is violated, driving the system back towards slow drift.

We show that this leads to several important behaviors in weakly-ionized systems. First, the Hall term can never dominate among non-ideal (weakly-ionized) terms if the drifts are superthermal, which ensures the behavior of these terms is diffusive. Second, this rapidly drives the system back to sub-thermal drifts by diffusing away any strong gradients in the magnetic field on length scales smaller than ℓ_{crit} , which can be quite large (larger than protostellar disk sizes). Third, this ensures intuitive behaviors such as the system becoming hydrodynamic if the ionization fraction vanishes, which is not strictly ensured by the classical equations. Fourth, this can strongly restrict the maximum magnetic-field amplification during collapse such that it scales well below the flux-freezing estimate. And fifth, we show that under these restrictions, it may be difficult (if not impossible) for weakly-ionized systems to ever achieve quasi-steady-state conditions where the Hall term could be dynamically important on large scales, as we show these generically require superthermal drift speeds.

In future work, it will be important to study the practical effects of Eq. 14 in numerical simulations of weakly-ionized media. But of course this will be problem-specific and the effects could, by construction, vary from negligible to complete magnetic decoupling. Our hope here is that the simple prescription provided can inform future simulations of weakly ionized systems, being used as a rough approximation to maintain consistency and probe both where the microphysics should be important and how it should limit macroscopic consequences. Of course, it will also be valuable to explore PIC or MHD-PIC type simulations of these behaviors, especially in the particularly ambiguous dust-charge-dominated limits (see Ji et al. 2022; Ji & Hopkins 2022). Such studies are needed better inform the precise scalings of η_{an} especially in more extreme regimes.

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APPENDIX

A. ADDITIONAL MULTI-SPECIES DETAILS

In the main text, for the arbitrary multi-species case (§ 2.4.2), we propose a simple weighting of the effective

Hall/Ohmic/ambipolar coefficients to capture the leading-order anomalous resistivity effects. Here we discuss the standard multi-species derivation of the non-ideal terms in more detail, to show that our proposed corrects capture the relevant behavior in the limits of interest.

A.1. Overview and Classical Derivation

Consider the case with an arbitrary number N of charged species. In order to obtain any closed set of equations on macro scales, impose from § 2.2 assumptions (1) (Ampere’s law), (2) (drop D_t , stress, and battery terms), and (5) (neglect micro/meso fluctuations), and assume one dominant neutral species (for multiple neutrals some other equations must be introduced to close their relative slip velocities). The momentum equation for each charged species j then reduces to $\mathbf{E} \approx (B/c)\mathbf{b} \times \mathbf{u}_j + (1/n_j q_j) \sum_i [\rho_j \omega_{ji}(\mathbf{u}_i - \mathbf{u}_j) - \dot{\rho}_{ji} \mathbf{u}_j + \dot{\rho}_{ij} \mathbf{u}_i]$. Define the $3N$ -element vectors $\mathbf{V} \equiv \{u_0^x, u_0^y, u_0^z, u_1^x, u_1^y, u_1^z, u_2^x, \dots\}$ ($V_n = u_i^\alpha$ with $i = \lfloor n/3 \rfloor$ the species and $\alpha = n - \lfloor n/3 \rfloor$ the directional component) and $\tilde{\mathbf{E}} \equiv \{E^x, E^y, E^z, E^x, \dots\}$ ($\tilde{E}_n = E_\alpha$). Then we can write the momentum equation $\tilde{\mathbf{E}} = \mathbf{S}\mathbf{V}$ where \mathbf{S} is $3N \times 3N$ and invertible, so $\mathbf{V} = \mathbf{S}^{-1}\tilde{\mathbf{E}}$. Because of the repetitive nature of $\tilde{\mathbf{E}}$ we can decompose $\mathbf{S}^{-1} = \mathbf{S}'_L \mathbf{S}'_R$ into the $3N \times 3$ and $3 \times 3N$ left/right blocks \mathbf{S}'_L and \mathbf{S}'_R , respectively, with $\mathbf{S}'_R \cdot \tilde{\mathbf{E}} = \mathbf{E}$, so $\mathbf{V} = \mathbf{S}'_L \cdot \mathbf{E}$. Then we impose $\sum_j n_j q_j \mathbf{u}_j = \mathbf{J} = \mathbf{J}_A$, which we can write $\mathbf{A}_R \mathbf{V} = \mathbf{J}_A$ where \mathbf{A}_R is $3 \times 3N$, giving $(\mathbf{A}_R \mathbf{S}'_L) \mathbf{E} = \mathbf{J}_A$, where $(\mathbf{A}_R \mathbf{S}'_L)$ is a 3×3 invertible matrix (even though \mathbf{A}_R and \mathbf{S}'_L are not invertible). So we obtain the solutions for both \mathbf{E} and the various \mathbf{u}_j as $\mathbf{E} = (\mathbf{A}_R \mathbf{S}'_L)^{-1} \cdot \mathbf{J}_A$ and $\mathbf{V} = \mathbf{S}'_L (\mathbf{A}_R \mathbf{S}'_L)^{-1} \cdot \mathbf{J}_A$. Finally noting that we can write the basis vector $\hat{\mathbf{J}}' \equiv \{\hat{\mathbf{J}}, \hat{\mathbf{J}} \times \mathbf{b}, \hat{\mathbf{J}} \times \mathbf{b} \times \mathbf{b}\} = \mathbf{R}\hat{\mathbf{x}}$, we can express \mathbf{E} in terms of the components $\hat{\mathbf{J}}'$ as $\mathbf{E}' \rightarrow (\mathbf{R}^{-1})^T \mathbf{E}$ to write the MHD induction equation in terms of some effective Ohmic, Hall, and ambipolar coefficients⁶ $\eta_{O,H,A}$.

In principle, one could take this approach and simply use the drift/slip-velocity-dependent form of all the collision rates $\omega_{ij} \rightarrow \omega_{ij}(\mathbf{V}, \dots) + \omega_{ij}^{\text{an}}(\mathbf{V}, \dots)$ (and similarly modify the inelastic and charge exchange rates $\dot{\rho}$ to be appropriate functions of \mathbf{V} as well), i.e. account for both the classical dependence of the direct collision rates $\omega_{ij}(\mathbf{V}, \dots)$ on all the relative velocities \mathbf{u}_j (in \mathbf{V} ; e.g. Pinto & Galli 2008) as well as the anomalous resistivity/collision rate of each species with respect to others (itself also a function of \mathbf{V} through e.g. functions like $\Theta(\mathbf{u}_j - \mathbf{u}_i)$ in Eq. 16). But then the terms appearing in \mathbf{S} are highly non-linear functions of the different components of \mathbf{V} and (implicitly) \mathbf{E} and \mathbf{J} , and multi-dimensional iterative root-finding methods are needed. And even if we took the “zero drift/slip” coefficients ($\omega_{ij}(\mathbf{V}) \rightarrow \omega_{ij}(\mathbf{V} = \mathbf{0})$) so \mathbf{S} were independent of \mathbf{V} , if we retain general cross-interaction terms ω_{ij} and charge-exchange/inelastic terms then the matrix solve remains quite complicated. Still, for methods that actually numerically solve a chemical network on-the-fly for the abundances of N directly, this is not necessarily more expensive than the network solve itself, so may be preferred. However, most implementations either pre-compute chemical tables or otherwise simplify to only use some end product of this series of inversions. Moreover, even if we followed the non-linear procedure above, for general $\omega_{ij}(\mathbf{V})$ and N , there is no guarantee of a *unique* solution \mathbf{V} for given \mathbf{J} (there can be multiple viable sets of \mathbf{u}_j , which would require evolving $D_t \mathbf{u}_j$ to distinguish).

We can now see the primary motivation for the common literature assumptions (3) (assume dominant neutrals with $\mathbf{U} \approx \mathbf{u}_n$, and neglect all $\dot{\rho}_{ij}$ and non-neutral collision ω_{ij} terms) and

⁶ If we locally define coordinate axes such that $\mathbf{b} = \{0, 0, 1\}$ and $\mathbf{J} = |\mathbf{J}| \{\sin \theta, 0, \cos \theta\}$, then $\eta_O = (c^2/4\pi|\mathbf{J}|) E_z / \cos \theta$, $\eta_H = -(c^2/4\pi|\mathbf{J}|) E_y / \sin \theta$, $\eta_A = (c^2/4\pi|\mathbf{J}|) E_x / \sin \theta - \eta_O$.

(4) (assume vanishing drift/slip so $\omega_{ij} \rightarrow \omega_{ij}(\mathbf{V} = \mathbf{0})$). With these simplifications, \mathbf{S} has a trivial block form and the equation for each species is separable: $\mathbf{W} \equiv (c/B)(\mathbf{E} + \mathbf{U} \times \mathbf{B}/c) \approx \mathbf{b} \times \delta \mathbf{u}_{jn} + \tilde{\beta}_j^{-1} \delta \mathbf{u}_{jn}$ (where $\delta \mathbf{u}_{jn} \equiv \mathbf{u}_j - \mathbf{u}_n$ are the slip velocities and $\tilde{\beta}_j \equiv q_j B / m_j c \omega_{jn}$). This gives the familiar result that the relative contributions to the current for each species depend only on $\tilde{\beta}_j$, and coefficients $\eta_O = (cB/4\pi)\tilde{\sigma}_O^{-1}$, $\eta_H = (cB/4\pi)\tilde{\sigma}_H/\tilde{\sigma}_\perp^2$, $\eta_A = (cB/4\pi)(\tilde{\sigma}_P/\tilde{\sigma}_\perp^2 - 1/\tilde{\sigma}_O)$ where $\tilde{\sigma}_O \equiv \sum n_j q_j \tilde{\beta}_j$, $\tilde{\sigma}_H \equiv n_j q_j / (1 + \tilde{\beta}_j^2)$, $\tilde{\sigma}_P \equiv n_j q_j \tilde{\beta}_j / (1 + \tilde{\beta}_j^2)$, $\tilde{\sigma}_\perp^2 \equiv \tilde{\sigma}_H^2 + \tilde{\sigma}_P^2$. With the slip velocities⁷ $\delta \mathbf{u}_j$, we can calculate each pairwise drift ($\delta \mathbf{u}_j - \delta \mathbf{u}_i$), and again in principle modify ω_{ij} to account for each drift+slip speed. But this again introduces the difficulties above: we then modify $\tilde{\beta}_j$ so must iteratively re-calculate the relative drift/slip speeds, are not ensured a unique solution, and (conceptually) must evaluate “which drifts matter.”⁸

A.2. Application to Scalings In the Text

Here we comment on how well the approximate multi-species scalings in the text § 2.4.2 capture the previous discussion. Despite the challenges of solving the full system in generality with superthermal drifts and anomalous terms, it is nonetheless possible to make some robust statements.

First, the leading-order contribution to the charged drift speeds $|\mathbf{u}_i - \mathbf{u}_j|$ always scales as $\propto \tilde{\beta}_i - \tilde{\beta}_j$ (whether the various $\tilde{\beta}_{i,j}$ are $\ll 1$ or $\gg 1$), and therefore contributions to the current scale $\propto \tilde{\beta}_j$ as we noted in § 2.4.2. More quantitatively, one can show that our approximate expression for the “effective” drift v_{drift} (Eq. 18) and slip (Eq. 9) $v_{\text{slip, max}}$ velocities in the text is exact (for the dominant current-carrying species) in the limits of either (a) two dominant species containing most charge and current (one positive one negative); (b) one dominant species in the current (one species with $|\tilde{\beta}_j|$ or $|n_j q_j \mathbf{u}_j|$ much larger than any other of the salient charge-carriers); or (c) all salient $|\tilde{\beta}_j|$ small. And even in the worst-case scenario (all $|\tilde{\beta}_j|$ large with several species carrying comparable current) these expressions still approximate the mean drift/slip velocity to within a factor of ~ 2 . Given the ambiguity in these cases and our approach of even defining “which” v_{drift} matters and the order-unity ambiguity in defining the threshold drift for anomalous resistivity (depending on precisely which instabilities are active and how), this uncertainty is more than adequate for our purposes.

We can also see that if we multiply all collision rates by some factor ψ , then $\eta_O \rightarrow \eta_O \psi$, while for all salient limits (when the various dominant charge-carrier $|\tilde{\beta}_j|$ are $\ll 1$ [Ohmic], or $\gg 1$ [ambipolar], or some are $\ll 1$ and some $\gg 1$ [Hall], or when there is a single dominant carrier), $\eta_A \rightarrow \eta_A/\psi$, while η_H

⁷ The slip velocities are given in a frame where $\mathbf{b} = \{0, 0, 1\} = \hat{\mathbf{z}}$ by:

$$\begin{aligned} \delta \mathbf{u}_{jn,x} &= \frac{\tilde{\beta}_j (W_x + \tilde{\beta}_j W_y)}{(1 + \tilde{\beta}_j^2)} = \frac{\tilde{\beta}_j [J_x (\tilde{\sigma}_P - \tilde{\beta}_j \tilde{\sigma}_H) + J_y (\tilde{\sigma}_H + \tilde{\beta}_j \tilde{\sigma}_P)]}{(1 + \tilde{\beta}_j^2) \sigma_\perp^2}, \\ \delta \mathbf{u}_{jn,y} &= \frac{\tilde{\beta}_j (W_y - \tilde{\beta}_j W_x)}{(1 + \tilde{\beta}_j^2)} = \frac{\tilde{\beta}_j [J_y (\tilde{\sigma}_P - \tilde{\beta}_j \tilde{\sigma}_H) - J_x (\tilde{\sigma}_H + \tilde{\beta}_j \tilde{\sigma}_P)]}{(1 + \tilde{\beta}_j^2) \sigma_\perp^2}, \\ \delta \mathbf{u}_{jn,z} &= \tilde{\beta}_j W_z = \frac{\tilde{\beta}_j J_z}{\tilde{\sigma}_O}. \end{aligned} \quad (\text{A1})$$

⁸ For example, even if electrons carry most of the current, since we only retain their effective collision rate with neutrals here, do we include anomalous resistivity if their drift is subthermal relative to a dominant ion in the current but superthermal relative to a more trace ion, like charged grains?

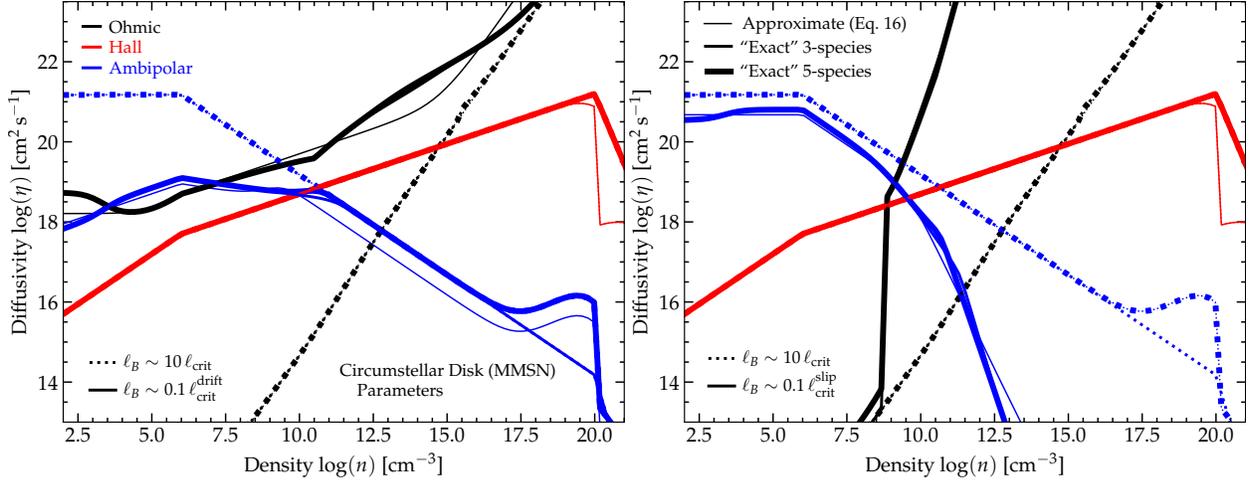


FIG. 4.— Ohmic, Hall, and ambipolar diffusivities $\eta_{O,H,A}$ as a function of density as in Fig. 2 (for the “circumstellar disk/MMSN” parameters therein), but for the specific numerical cases in § A.3. *Left*: Comparison of a generally-subthermal case ($\ell_B = 10 \ell_{\text{crit}}$; *dotted*) to a superthermal drift case ($\ell_B = 0.1 \ell_{\text{crit}}^{\text{drift}}$; *solid*). *Right*: Same but for the superthermal slip case $\ell_B = 0.1 \ell_{\text{crit}}^{\text{slip}}$. For each, we compare (1) the prediction from the simple parameterized rescaling we propose in Eq. 16 (for the multi-species scalings in § 2.4.2; “Approximate” in *thin* lines); to (2) the case where ignore charged dust grains as a current carrier (treat them as part of the neutral inertia taking $\beta_{\text{grain}} \rightarrow 0$) and solve the full non-linear three-species problem with the exact coefficients (§ 2.1; “Exact” 3-species in *intermediate* lines); and (2) the results of exactly solving the non-linear multi-species equations as described in § A.1 with all possible pairs of species featuring simple Epstein-type scalings and anomalous terms (§ A.3; “Exact” 5-species in *thick* lines). The 3 and 5-species methods give excellent agreement, indicating that grain current and differences between exact velocity-dependent scalings in the superthermal regime make little difference. The closed-form analytic approximate scaling from Eq. 16 reproduces these reasonably well in all regimes of interest. Key qualitative results (e.g. vanishing of the Hall-dominated regime) are robust.

is unmodified. Again even in the worst-case scenario (many nearly-equal β_i), this at worst equates to an $O(1)$ uniform difference in the pre-factor of η_H . This justifies the correction in the text for the Epstein drag limit (see also Hillier 2024, who consider Epstein drag corrections with a more restricted set of assumptions for just ambipolar drag in the two-fluid limit). Finally, if we return to the original equation and consider adding a constant ω^{an} which represents a collision frequency between *charge carriers* (either between them and each-other, or some comoving Alfvén wavepacket, or similar concept), we can see that in the limits above or any limit where the Ohmic term would dominate ($|\tilde{\beta}| \ll 1$), in σ_O we should simply take $\tilde{\beta}_j \rightarrow q_j B / m_j c \omega_j^{\text{tot}}$ where $\omega_j^{\text{tot}} \equiv \sum_i \omega_{ji}$ includes *all* collision terms influencing j . This is explicit in our three-species derivation in the text, where ω^{tot} includes terms like ω_{\pm} between charge-carriers. This means $\eta_O \propto \omega^{\text{tot}} \sim \omega^{\text{classical}} + \omega^{\text{an}}$ for whatever dominant species j contributes the most in σ_O , i.e. that we can justify the addition of ω^{an} as in Eq. 15 in the text.

In contrast, we find for any limit where the ambipolar term is dominant in the classical limit ($\tilde{\beta} \gg 1$ ignoring ω^{an}), that *only* collisions with neutrals contribute to leading order in η_A . Again, this is explicit in our three-species derivation (only $\omega_{\pm n}$ terms appear), and obvious in the sense that the ambipolar limit physically represents neutral-charge carrier slip independent of the charge-carrier drift velocities amongst themselves. As noted in the text, there certainly are instabilities that grow when the slip becomes superthermal, but their non-linear outcomes are less clear. Nonetheless one could add them if desired to the equivalent β , if desired. For the reasons discussed above, the Hall term is again unmodified in either limit by these additions, and even in the worst-case scenario is only modified by an $O(1)$ prefactor, as in our three-species derivation.

A.3. A Specific Example

To numerically validate the above and our approach, we revisit our simple example in the text of Fig. 2, where we

calculate the $\eta_{O,H,A}$ coefficients according to the simple five-species model from § 2.4.3, but solving the generalized non-linear multidimensional problem exactly as described above (§ A.1; making assumptions (1), (2), (5) to close the system, but no others). We now have interactions between electrons, heavy ions, neutrals, positively and negatively charged grains. Per § A.1, to even write down the equations for \mathbf{E} in this limit we must define the velocity-dependent interaction terms between all pairs of particles, each of which can have different instability thresholds and qualitative behaviors of their anomalous terms (many of which remain poorly understood in detail). So we necessarily adopt a simple toy model, with the purpose not of providing any sort of “first principles” calculation, but simply to validate the simple approximate scalings discussed above. We neglect charge-exchange reactions, but consider all effective collisional cross terms. This includes both a “classical” (but correctly velocity-dependent) rate which we approximate with a simple hard-sphere-scattering type expression $\omega_{ji}^{\text{coll}} = \omega_{ji}^{\text{coll}}(\delta \mathbf{u}_{ji} \rightarrow 0) \times \left[1 + |\delta \mathbf{u}_{ji}|^2 / v_{T,ij}^2\right]^{1/2}$ (for which we use the values of $\langle \sigma v(\delta \mathbf{u}_{ji} \rightarrow 0) \rangle$ in text, supplemented with the zero-drift ion-grain and electron-grain cross sections from Draine & Sutin 1987), plus a pairwise anomalous term to capture different instabilities that can occur between each pair of interacting charged species, $\omega_{ji}^{\text{an}} = \text{MAX}[\Omega_j, \Omega_i, \omega_{pj}, \omega_{pi}] \Theta(|\delta \mathbf{u}_{ji}| / v_{T,ij})$ for $j, i \neq n$, with $v_{T,ij} = \text{MIN}[v_{T,i}, v_{T,j}]$ (per Eqs. 20-22, with $\psi_j = 0$ for $j \neq i, n$, so just the effective speeds of the individual carriers plus potentially-entrained neutrals for each). Like in Fig. 2 we consider two limits, which we define by the magnetic-field gradient $\ell_B = 10 \ell_{\text{crit}}$ and $\ell_B = 0.1 \ell_{\text{crit}}^{\text{drift}}$ or $\ell_B = 0.1 \ell_{\text{crit}}^{\text{slip}}$ (with $v_T = \text{MIN}[v_{T,n}, v_{A,\text{ideal}}]$ in ℓ_{crit}). In the model in the text, these correspond one-to-one to the subthermal ($v_{\text{drift}} \sim 0.1 v_T$) and super-thermal ($v_{\text{drift}} \sim 10 v_T$) limits, but since there is no single drift velocity to speak of in the more general case, they must be defined in terms of

ℓ_B (which is independent of the collision rates and species involved). Because of the non-linear dependencies involved, we must also specify the geometry, namely $\cos \theta \equiv \hat{\mathbf{J}} \cdot \mathbf{b}$, for which we take a representative $\theta = 30^\circ$, but note the results are very weakly dependent on θ , except if very close to parallel ($|\cos \theta| \approx 1$) where the ambipolar and Hall terms trivially vanish. Again, we emphasize the various coefficients and rates above are not meant to be exact, but simply to provide some calculable illustration.

We further consider one more comparison. If we treat the dust grains as simply a “charge sink” and assume their mass to be anchored to the neutrals (adding to neutral inertia to leading order), then our system reduces to a three-component (neutrals, ions, electrons) system. We take this, plus the full velocity-dependent scattering rates from [Pinto & Galli \(2008\)](#) for each component, with a single ion-electron anomalous resistivity added to the ion-electron collision rate (where present), and otherwise adopt the scalings above. Then we can apply our solution from the main text (§ 2.1) and exactly solve the non-linear problem with all terms from § 2.1 included.

In [Fig. 4](#) we see that our extremely simplified proposed rescaling of the zero-drift coefficients ([Eq. 16](#)) provides a reasonable approximation to either of these more complicated calculations (both of which agree very well with each other), for both large and small $\ell_B/\ell_{\text{crit}}$, at all densities plotted. The most important reason for this is that for all the regimes of interest here, even when grains contain most of the charge, the current is always primarily being carried by the free elec-

trons. In the limit of a single dominant current carrier, all that matters to leading order is the effective collision rate for that species, whose drift velocity must be set by ℓ_B (nearly independently of the collision rates) if Ampere’s law is to hold. The similarity between the two more complicated numerical cases (the five- and three-species models) further emphasizes that the dust current is not important.

The residual deviations between the simple analytic approximation and the more complicated calculations at superthermal drift speeds primarily owe to subtleties of where the various drift and slip terms appear. For example, at high- n , the drift and slip corrections to η_A cannot be trivially combined into a single effective coefficient as we do in [Eq. 16](#), and this leads to the systematic offset between [Eq. 16](#) and the more general solutions. Similarly, dust effects play some role in η_A at the highest densities, and η_H has offsetting terms in the velocity-dependent case that make its “jump” at $n \gtrsim 10^{20} \text{ cm}^{-3}$ more smooth. But all of these differences are in the regimes where those terms are much smaller than the Ohmic term, so do not dynamically matter. For the Ohmic term the deviations at high densities owe to a mix of the above effects, as well as the fact that when the anomalous and classical resistivities are comparable, they add nonlinearly in the more exact formalisms (like that in § 2.1 and as implemented in the 3 or 5-species models).

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