On Maximal Families of Binary Polynomials with Pairwise Linear Common Factors

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Abstract

We consider the construction of maximal families of polynomials over the finite field \mathbb{F}_q , all having the same degree n and a nonzero constant term, where the degree of the GCD of any two polynomials is d with $1 \leq d \leq n$. The motivation for this problem lies in a recent construction for subspace codes based on cellular automata. More precisely, the minimum distance of such subspace codes relates to the maximum degree d of the pairwise GCD in this family of polynomials. Hence, characterizing the maximal families of such polynomials is equivalent to determining the maximum cardinality of the corresponding subspace codes for a given minimum distance. We first show a lower bound on the cardinality of such families, and then focus on the specific case where d=1. There, we characterize the maximal families of polynomials over the binary field \mathbb{F}_2 . Our findings prompt several more open questions, which we plan to address in an extended version of this work.

1 Background and Problem Statement

In what follows, we denote by \mathbb{F}_q the finite field of order q, with q being a power of a prime number, while \mathbb{F}_q^n represents the n-dimensional vector space over \mathbb{F}_q . Further, $\mathbb{F}_q[x]$ denotes the the ring of polynomials with coefficients in \mathbb{F}_q in the unknown x.

For all $n \in \mathbb{N}$, we define S_n as follows:

$$S_n := \{ f \in \mathbb{F}_q[x] : \deg(f) = n, f \text{ monic}, f(0) \neq 0 \}$$
.

In other words, S_n is the family of all monic polynomials in $\mathbb{F}_q[x]$ of degree n and with a nonzero constant term. For any k, Let $\mathcal{I}_k \subseteq S_k$ be the set of irreducible polynomials of degree k, and let $I_k := |\mathcal{I}_k|$, which can be computed through Gauss's formula [2].

Furthermore, given $d \in \{0, ..., n\}$, let us define $\mathcal{M}_n^d \subseteq \mathcal{P}(S_n)$ as:

$$\mathcal{M}_n^d := \{ R \subseteq S_n : \forall f \neq g \in R, \deg(\gcd(f,g)) \leq d \}$$
.

Thus, \mathcal{M}_n^d is a family of subsets of polynomials in S_n , such that the degree of the GCD of any two distinct polynomials in it is at most d.

We are interested in the following problem:

Problem 1.1 Let $n \in \mathbb{N}$ and $d \in \{0, \dots, n\}$. What is the size of the largest subset in \mathcal{M}_n^d ?

The motivation for studying Problem 1.1 stems from the construction of subspace codes in the domain of network coding [6]. Subspace codes are essentially families of vector subspaces of \mathbb{F}_q^n . The distance between any two subspaces is defined as the sum of their dimensions, minus twice the dimension of their intersection [3]. A general research question is then to investigate the trade-off between the cardinality of a subspace code \mathcal{C} and its minimum distance d_{min} , i.e. the minimum distance between any two subspaces belonging to \mathcal{C} .

Recently, the second and the third author of this abstract proposed in [5] a new construction of subspace codes based on linear cellular automata (CA), which can be seen as a specific kind

of linear mappings $F: \mathbb{F}_q^n \to \mathbb{F}_q^m$ with *shift-invariant* output coordinates, that can also be defined by polynomials in $\mathbb{F}_q[x]$. An interesting finding of [5] is that the minimum distance of these subspace codes is related to the maximum degree of the GCD of any two polynomials defining the corresponding families of CA; in particular, the higher the degree, the smaller is the minimum distance of the resulting code.

Problem 1.1 has already been solved in [5] for the case d = 0, by leveraging the construction of pairwise coprime polynomials exhibited in [4]. In fact, this construction results in the partial spread code that the authors of [1] used to define a new family of bent functions.

The contributions of this extended abstract are as follows:

- We first show a lower bound on the cardinality of the maximal families in \mathcal{M}_n^d .
- Then, we build a maximal family for the specific case of q=2 and d=1.

Remark that the case q = 2 corresponds to the construction of subspace codes with binary linear CA, which can be seen as a specific kind of (linear) vectorial Boolean functions. This represents also an interesting case from the point of view of the practical applications to network coding.

2 Lower Bound in the General Case

We first start by proving a lower bound on the cardinality of the maximal families in \mathcal{M}_n^d . Consider the family of polynomials in S_n built as follows. Note that we work under the implicit assumption that d < n/2. The following construction can be easily adapted if this assumption does not hold.

Construction-Lower-Bound(n, d)

- 1. Take all irreducible polynomials of degree n, namely all the elements in \mathcal{I}_n .
- 2. For all $i \in \{1, ..., d\}$, for all $h \in \mathcal{I}_{n-i}$, pick a $g \in \mathcal{I}_i$ and take gh.
- 3. For all $i \in \{d+1, \ldots, \lfloor (n-1)/2 \rfloor\}$, for all $g \in \mathcal{I}_i$, pick a $h \in \mathcal{I}_{n-i}$ not previously used and take gh.
- 4. If n is even, for all $g \in \mathcal{I}_{n/2}$, take g^2 .
- 5. For all $i \in \{1, \ldots, d\}$, for all $g \in \mathcal{I}_i$, pick a $h \in \mathcal{I}_{n-\lfloor n/i \rfloor i}$ and take $g^{\lfloor n/i \rfloor} h$.

In steps 1, 3, and 4, we proceed as in the coprime case studied in [4]. In particular, we take all irreducible polynomials as they always fit the condition to be in a family of \mathcal{M}_n^d . Then, we combine each irreducible of degree d+1 with a distinct irreducible of degree n-d-1, and so on also for degrees $d+2,\ldots,\lfloor (n-1)/2\rfloor$. It is always possible to find these distinct polynomials as I_i is monotonically non-decreasing in i (see Lemma 3 of [4]), and thus $I_{n-i} \geq I_i$ for $i \leq n/2$. However, in this case we are allowed to have a common factor of degree at most d. Hence, when considering combinations of the form gh with $g \in \mathcal{I}_i$ and $h \in \mathcal{I}_{n-i}$ for $i \leq d$ (in step 2), we can pick the elements in \mathcal{I}_i multiple times, as their degree is at most d. This allows us to index these combinations over the irreducibles in \mathcal{I}_{n-i} , which are more in number than the ones in \mathcal{I}_i . Moreover, we can get all the smooth combinations of small factors of degrees $\leq d$. To make sure to avoid big common factors, in step 5 we use powers of individual irreducibles of degree $\leq d$, combined with a suitable h of degree < d.

Hence, we have shown that the set generated by the Construction-Lower-Bound (n, d) is indeed a member of the family \mathcal{M}_n^d . The cardinality of such set is given by

$$\sum_{i=1}^{\lfloor n/2 \rfloor} I_i + \sum_{i=n-d}^{n-1} I_i + I_n .$$

The first sum is due to steps 3, 4, and 5, the second sum is due to step 2, while I_n is due to step 1. We have thus obtained a lower bound for the cardinality of the maximal family in \mathcal{M}_n^d . In the next section, we show that this family is maximal at least in a specific case.

3 Maximal Family for Linear Common Factor

We now show that the construction proposed in the previous section is maximal when considering it on the binary field \mathbb{F}_2 , for the specific case of d=1. This means allowing for a common factor of degree at most 1, namely 1 or x + 1. Let us first adapt the construction for this case.

Construction-Maximal(n)

- 1. Take all irreducible polynomials of degree n, namely all the elements in \mathcal{I}_n .
- 2. For all $g \in \mathcal{I}_{n-1}$, take (x+1)g.
- 3. For all $i \in \{2, \ldots, \lfloor (n-1)/2 \rfloor\}$, for all $g \in \mathcal{I}_i$, pick a $h \in \mathcal{I}_{n-i}$ not previously used and take qh.
- 4. If n is even, for all $g \in \mathcal{I}_{n/2}$, take g^2 .
- 5. Take $(x+1)^n$.

As shown in Section 2, the set produced by Construction-Maximal (n) belongs to the family \mathcal{M}_n^1 . Moreover, it has cardinality

$$\sum_{i=1}^{\lfloor n/2 \rfloor} I_i + I_{n-1} + I_n .$$

We show that this set is also maximal as follows.

Theorem 3.1 In \mathbb{F}_2 , a maximal element of \mathcal{M}_n^1 has cardinality

$$N_n := \sum_{i=1}^{\lfloor n/2 \rfloor} I_i + I_{n-1} + I_n .$$

Proof: Let $A \in \mathcal{M}_n^1$ be a maximal element. We know by construction that there exists an element in \mathcal{M}_n^1 with cardinality N_n , meaning that $|A| \geq N_n$. Clearly, A must contain all irreducible polynomials of degree n, \mathcal{I}_n . Let $B := A \setminus \mathcal{I}_n$. Given a polynomial $f \in \mathbb{F}_q[x]$, let l(f) be its irreducible factor of lowest degree (if more than one are present, pick the first one in lexicographical order). For any $f \in B$, we define the function

$$L(f) := \begin{cases} l(f) & \text{if } x + 1 \nmid f \\ l(f/(x+1)) & \text{if } x + 1 \mid f \end{cases}$$

First, we prove that the image of L is a subset of $\bigcup_{i=1}^{\lfloor n/2 \rfloor} \mathcal{I}_i \cup \mathcal{I}_{n-1}$. In the first case of the definition of L, since f is reducible we have that $deg(l(f)) \leq |n/2|$ (also $deg(l(f)) \geq 2$). In the second case, let g such that f = (x+1)g, then we have the following two cases:

- if $x + 1 \mid g$, then l(f/(x + 1)) = l(g) = x + 1;
- if $x+1 \nmid g$, then we consider the following cases:
 - if g is irreducible, then l(g) = g, thus deg(l(g)) = n 1;
 - if g is reducible, then $\deg(l(g)) \leq \lfloor (n-1)/2 \rfloor$ (also $\deg(l(g)) \geq 2$).

Thus the image of L is a subset of $\bigcup_{i=1}^{\lfloor n/2 \rfloor} \mathcal{I}_i \cup \mathcal{I}_{n-1}$. Now, we prove that $L: B \to \bigcup_{i=1}^{\lfloor n/2 \rfloor} \mathcal{I}_i \cup \mathcal{I}_{n-1}$ is injective. Let $f_1 \neq f_2 \in B$ such that $L(f_1) = L(f_2).$

- If $deg(L(f_1)) > 1$, then f_1 and f_2 have a common factor of degree higher than 1, leading to contradiction.
- If $\deg(L(f_1)) = 1$, then $L(f_1) = x + 1$, which can only be possible if we are in the second case of the definition of L, namely $x + 1 \mid f_1$. But then $f_1 = (x + 1)^2 g_1$ for some g_1 and the same would hold for f_2 . Thus, f_1 and f_2 would share at least a quadratic factor $(x + 1)^2$, leading to contradiction.

Since L is injective, we have that $|B| \leq \sum_{i=1}^{\lfloor n/2 \rfloor} I_i + I_{n-1}$ and $|A| \leq \sum_{i=1}^{\lfloor n/2 \rfloor} I_i + I_{n-1} + I_n = N_n$, hence $|A| = N_n$.

Characterization of maximal families Note that in the proof above we also got that $L: B \to \bigcup_{i=1}^{\lfloor n/2 \rfloor} \mathcal{I}_i \cup \mathcal{I}_{n-1}$ is a bijection. We can use this function to characterize all maximal families in \mathcal{M}_n^1 as follows:

- Obviously \mathcal{I}_n is part of any maximal family.
- Assuming n > 2. If n is even and $g \in \mathcal{I}_{n/2}$, then $L^{-1}(g) = gh$ for some $h \in \mathcal{I}_{n/2}$. If $h \neq g$ then $L^{-1}(h) \neq gh = L^{-1}(g)$, and $gcd(L^{-1}(h), L^{-1}(g)) = h$ and its degree is greater than 1, leading to contradiction. Thus g = h.
- For $i \in \{1, \ldots, \lfloor (n-1)/2 \rfloor\}$, if $g \in \mathcal{I}_i$, then we have four cases:
 - (only applicable if $i \mid n$) $L^{-1}(g) = g^a$ with a = n/i
 - (only applicable if $i \mid n-1$) $L^{-1}(g) = (x+1)g^a$ with a = (n-1)/i
 - $-L^{-1}(g) = g^a h$ for some $a \ge 1$, ai < n, $g \nmid h$, $\deg(h) > 1$, and $x + 1 \nmid h$. Note that h must be irreducible and ai < n/2, in particular $h \in \mathcal{I}_{n-ai}$.

Proof: If h is reducible or if ai < n/2, then $\deg(l(h)) \le \lfloor n/2 \rfloor$, thus we can apply L^{-1} to l(h). Then l(h) would be a common divisor of $L^{-1}(g)$ and $L^{-1}(l(h))$. Note that $l(h) \ne g$ since $g \nmid h$ and thus $\gcd(g,h) = 1$, hence $L^{-1}(g) \ne L^{-1}(l(h))$. Note that $\deg(l(h)) > 1$ since $x + 1 \nmid h$. Thus we have a contradiction.

Also note that h does not divide any other $f' \in A$. Otherwise we would have a common factor of degree strictly greater than 1.

- (with $\deg(g) = i \geq 2$) $L^{-1}(g) = (x+1)g^a h$ for some $a \geq 1$, ai < n, $x+1 \nmid g$, $g \nmid h$, $\deg(h) > 1$, and $x+1 \nmid h$. Note that h must be irreducible and ai-1 < n/2, in particular $h \in \mathcal{I}_{n-ai-1}$.

Proof: Similar to the previous one.

Hence, we have proven that a set $A \in \mathcal{M}_n^1$ is maximal if and only if:

- A contains \mathcal{I}_n ;
- if n is even then A contains $\{g^2 : g \in \mathcal{I}_{n/2}\};$
- for every $g \in \mathcal{I}_i$ with 1 < i < n/2 there exists a unique $f \in A$ such that $g \mid f$, and this f is either of the form
 - $-f = q^a$ with a = n/i, or
 - $f = (x+1)q^a$ with a = (n-1)/i, or
 - $f = q^a h$ with ai < n/2 and $h \in \mathcal{I}_{n-ai}$, or
 - $f = (x+1)g^a h$ with ai < n/2 + 1 and $h \in \mathcal{I}_{n-ai-1}$;

and in the last two cases h does not divide any other $f' \in A$;

• either $(x+1)^n \in A$ or $(x+1)^a h \in A$ for 1 < a < n/2 and some $h \in \mathcal{I}_{n-a}$.

4 Conclusions and Future Works

In this abstract, we considered the problem of characterizing the maximal families of polynomials of degree n and nonzero constant term over \mathbb{F}_q where the degree of the GCD of any two polynomials is at most d. The motivation for studying this problem originates from a recent construction of subspace codes based on linear cellular automata. After providing a general lower bound, we focused our attention on the specific case where q=2 and d=1, i.e. each pair of polynomials in the family has a linear common factor. We proved the formula for the cardinality of a maximal element of \mathcal{M}_n^1 , and then gave a characterization of the corresponding maximal families.

Clearly, Problem 1.1 is still far from being completely solved, and there are several avenues for future research on this subject. We plan to address the following questions in an extended version of this abstract:

- The most natural direction to explore is to generalize our counting results to a larger degree d of the pairwise GCD and to larger finite fields. Clearly, since the higher is d the lower becomes the minimum distance of the resulting subspace code, it makes sense to consider only small values of d for practical applications.
- It would be interesting to investigate more closely the trade-off between the cardinality of the maximal families and the degree of the GCD. This will give useful information for the design of subspace codes based on linear CA depending on the application's requirements.
- Finally, after determining their size, an interesting research question is to count the *number* of maximal families, by varying the degree of the GCD and the order of the ground field.

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