

THE DP-COLORING OF THE SQUARE OF SUBCUBIC GRAPHS

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ABSTRACT. The 2-distance coloring of a graph G is equivalent to the proper coloring of its square graph G^2 , it is a special distance labeling problem. DP-coloring (or “Correspondence coloring”) was introduced by Dvořák and Postle in 2018, to answer a conjecture of list coloring proposed by Borodin. In recent years, many researches pay attention to the DP-coloring of planar graphs with some restriction in cycles. We study the DP-coloring of the square of subcubic graphs in terms of maximum average degree $\text{mad}(G)$, and by the discharging method, we showed that: for a subcubic graph G , if $\text{mad}(G) < 9/4$, then G^2 is DP-5-colorable; if $\text{mad}(G) < 12/5$, then G^2 is DP-6-colorable. And the bound in the first result is sharp.

Keywords. DP-coloring, subcubic graphs, maximum average degree, square graphs
Mathematics Subject Classification. 05C15

1. INTRODUCTION

For simple graph $G = (V, E)$, we denote $d_G(v), \Delta(G), \delta(G)$ to be the degree, maximum degree, minimum degree of G , respectively. A cubic graph is a graph with all vertices have degree 3, a subcubic graph is a graph with $\Delta(G) \leq 3$. The square graph G^2 of G is the graph obtained by joining any pair of vertices of distance 2 with a new edge. Let $\text{mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$ be the maximum average degree of G . And the girth g of the G is the length of the shortest cycle in G . A k -vertex is a vertex of degree k , a k -cycle is a cycle of length k , a k -face is a face with a k -cycle be its boundary cycle. We use $E[X, Y]$ for the set consists of the edges of G with one endpoint in $X \subseteq V$ and another in $Y \subseteq V$.

The **proper k -coloring** of G is a mapping $c : V \rightarrow \{1, 2, \dots, k\}$, satisfying $c(v_0) \neq c(v_1)$ for any $v_0v_1 \in E$. The **chromatic number**, denoted by $\chi(G)$, is the smallest k such that G has a proper k -coloring. Moreover, if $c(v_0) \neq c(v_2)$ for any $v_2 \in V$ has a common neighbor with v_0 , we say c is a **2-distance k -coloring** of G . We define $\chi_2(G)$ as the smallest k such that G admits a 2-distance k -coloring. Apparently, the proper coloring of G^2 is equivalent to the 2-distance coloring of G , that is to say: $\chi_2(G) = \chi(G^2)$.

For a list assignment L defined on V , if there exist a proper coloring c , such that $c(v) \in L(v), v \in V$, we call c a L -coloring of G or G is L -colorable. Graph G is **k -choosable** if G is L -colorable for any L with $|L(v)| \geq k, v \in V$. The minimum k such that G is k -choosable is the **list chromatic number**, denoted by $\chi_l(G)$.

As the generalization of proper coloring, $\chi_l(G)$ is no less than $\chi(G)$. In contrast to all the planar graphs are 4-colorable (4-Color Theorem), Thomassen[1] showed that they are 5-choosable. Naturally, there are many articles concerning the 4-choosability and 3-choosability of some planar graphs without certain cycles. In 1996, Borodin[2] proved that: every planar graph without cycles between 4 and 9 is 3-colorable. Then in 2013, Borodin[3] proposed a problem that: prove that every planar graph without cycles of length from 4 to 8 is 3-choosable [Problem 8.1]. Dvořák and Postle[4] gave an affirmative answer by introducing a new concept: **Correspondence Coloring**. It was renamed DP-coloring by Bernshtepn, Kostochka and Pron[5], and they repharsed the definition as following:

Definition 1. For graph $G = (V, E)$ and the list assignment L , the **cover H_L** of G is a graph that satisfy the following conditions:

- (1) $V(H_L) = \{(u, c_i) : u \in V, c_i \in L(u)\}$;

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- (2) The induced subgraph of $\{u\} \times L(u) = \{(u, c_i) : c_i \in L(u)\}$ is a clique;
- (3) $E[\{u\} \times L(u), \{v\} \times L(v)]$, $uv \in E$ form a matching of H_L , denoted by M_{uv} (may be empty);
- (4) $E[\{u\} \times L(u), \{v\} \times L(v)] = \emptyset$, $uv \notin E$.

Let $M_L = \{M_{uv} : uv \in E\}$ be the matching assignment of H_L . For any list assignment L with $|L(u)| \geq k, u \in V$, We say G is **DP- k -colorable** if there exist an independent set I with $|I| = |V|$ in H_L . The **DP-chromatic number** is the minimum k such that G is DP- k -colorable, denoted by $\chi_{DP}(G)$.

For graph G and list assignment L , if G is DP- k -colorable for matching assignment $M_L = \{(u, c_i)(v, c_i) : uv \in E, c_i \in L(u) \cap L(v)\}$, then the independent set $I = \{(u, c(u)) : u \in V, c(u) \in L(u)\}$ is corresponding to a L -coloring c of G . That is to say, DP-coloring is a generalization of list coloring, and $\chi_{DP}(G) \geq \chi_l(G)$. For example, $\chi_{DP}(C_4) = 3, \chi_l(C_4) = 2$.

In this article, we mainly concentrate on the DP-coloring to the square of the subcubic graphs. In terms of proper coloring: Wegner[6] proved that the square of the cubic graph is 8-colorable in 1977; Thomassen[7] showed that the square of a planar cubic graph is 7-colorable. In terms of list coloring, Dvořak: For subcubic graph G , Skrekovski and Tancer[8] showed that: $\chi_l(G^2) \leq 4$ if $\text{mad}(G) < \frac{24}{11}$ and without 5-cycles, $\chi_l(G^2) \leq 5$ if $\text{mad}(G) < \frac{7}{3}$, and $\chi_l(G^2) \leq 6$ if $\text{mad}(G) < \frac{5}{2}$; Havet[10] improved the bound of the third result to $\frac{18}{7}$; Cranston and Kim[9] showed that $\chi_l(G^2) \leq 8$ if G is not Petersen graph (and the bound is sharp). We showed that:

Theorem 1. For subcubic graph $G, \chi_{DP}(G^2) \leq 5$ if $\text{mad}(G) < \frac{9}{4}$.

Theorem 2. For subcubic graph $G, \chi_{DP}(G^2) \leq 6$ if $\text{mad}(G) < \frac{12}{5}$.

Let us now consider the planar subcubic graphs G under girth g constraints. Montassier and Raspaud[11] showed that G^2 is 5-colorable for $g \geq 14$, 6-colorable for $g \geq 10$ and 7-colorable for $g \geq 8$; Dvořak, Skrekovski and Tancer[8] obtained some results about list coloring: G^2 is 4-choosable for $g \geq 24$, 5-choosable for $g \geq 14$, 6-choosable for $g \geq 10$; Cranston and Kim[9] improved the girth bound by showing that: G^2 is 6-choosable for $g \geq 9$ (this result was also obtained by Havet[10]), 7-choosable for $g \geq 7$. And later the girth bound of 5-choosable was improved to 13 by Havet[10], and 12 by Borodin & Invanova[12]. Borodin & Invanova[13] also proved that G^2 is 4-colorable if $g \geq 23$, and they improved this result to 22 in [14], now 21 is the best result which was given by Hoang La and Montassier[15].

By Euler's Formula, there is a folklore result of planar graph G with girth at least g says: $\text{mad}(G) < \frac{2g}{g-2}$. Together with above theorems, the following results can be obtained easily:

Corollary 3. Let G be a planar subcubic graph of girth g , then $\chi_{DP}(G^2) \leq 5$ if $g > 18$.

Corollary 4. Let G be a planar subcubic graph of girth g , then $\chi_{DP}(G^2) \leq 6$ if $g > 12$.

2. TERMINOLOGY

A graph G is **k -minimal** if G^2 is not DP- k -colorable but each of its proper subgraph does. So k -minimal graph is obviously connected. A configuration is **k -reducible** if it can not appear in the k -minimal graph G . A **l -thread** is a path induced by l vertices of degree 2 in G , and a 0-thread is a 3-vertex. A l -thread is called **longest** if its two endpoints are both 3-vertices. Let $Y_{a,b,c}$ be a 3-vertex incident with three threads of length a, b, c . Let $G_2(v)$ be the subgraph induced by a vertex and the longest threads it incident with.

3. PROOFS OF RESULTS

We proof the main results by contradiction. If Theorem 1 is not true, there must exist some subcubic graphs with maximum average degree less than $\frac{9}{4}$ while their square graphs are not DP-5-colorable. Let G be the counterexample with the fewest vertices, named 5-minimal graph. Then the square of any proper subcubic graph of G is DP-5-colorable. The same goes for Theorem 2. We will show that the k -minmimal graph for $k = 5, 6$ with the assumption about $\text{mad}(G)$ is actually not exist.

3.1. Reducible Configurations. To show that configuration R is k -reducible in k -minimal graph G , it suffices to show any DP- k -coloring of the square of $G - R$ can be extended to G^2 . By definition, which means there always exist an independent set I with $|I| = |V|$, for each list assignment L with $|L(u)| = k, u \in V$ and matching assignment M_L . That is contradict the minimality of G . Now we present some k -reducible configurations for $k = 5, 6$.

Lemma 5. $d_G(v) \geq 2, v \in V$.

Proof. For k -minimal graph G , its square graph G^2 is not DP- k -colorable. So there is no independent set I with $|I| = |V|$ under a matching assignment M_L . Assume that v is a 1-vertex in G , then $(G - v)^2 = G^2 - v$ is DP- k -colorable by minimality of G . Let I' be the independent set with $|I'| = |V| - 1$. The degree of v is at most 3 in G^2 , so there are at least two colors $c_1, c_2 \in L(v)$, satisfying $(v, c_i)(u, c) \notin E(M_{uv}), (u, c) \in I'$, in which u is any neighbor of v in G^2 . Hence, $I' \cup \{(v, c_i)\}$ is an independent set of G^2 with cardinality $|V|$. Which means G^2 is DP- k -colorable, that is a contradiction. \square

Stated differently, there are only 2-vertices and 3-vertices in G .

Lemma 6. *If F is a m -face $v_1 v_2 \cdots v_m v_1 (m \geq 3)$, and only v_1 is a 3-vertex, then F is k -reducible.*

Proof. For k -minimal graph G , let M_L be a matching assignment such that the cover of G^2 does not contain the independent set of cardinality $|V|$. If configuration F is contained in G , then $(G - F)^2$ is DP- k -colorable, so there must exist an independent set I' of cardinality $|V| - m$. Assume that 3-vertex v_1 is adjacent to u_1 in $G - F$, and the neighbors of u_1 are v_1, u_2, u_3 for $d_G(v) = 3$ (or v_1, u_2 for $d_G(u_1) = 2$). Denote:

$$L^*(v_i) = L(v_i) \setminus \bigcup_{u_j v_i \in E[R^2, G^2 - R^2]} \{c' \in L(v_i) : (u_j, c)(v_i, c') \in M_{u_j, v_i}, (u_j, c) \in I'\}$$

v_1 has at most 3 neighbors in $G^2 - R^2$, and $|L(v_1)| \geq 5$, so $|L^*(v_1)| \geq 2$. Similarly, $|L^*(v_2)| \geq 4, |L^*(v_m)| \geq 4, |L^*(v_i)| \geq 4, i = 3, 4, \dots, m - 1$. We can obtain an independent set $I^* = \{(v_i, c_i), i = 1, 2, \dots, m\}$ by coloring $v_1, v_2, v_m, v_3, \dots, v_{m-1}$ in order. Now $I' \cup I^*$ is an independent set of cardinality $|V|$ in the cover of G^2 . Therefore G^2 is DP- k -colorable, a contradiction. \square

The next lemmata are concerning specific 5-reducible and 6-reducible configurations of G . The method of prove configuration R is k -reducible is similar to the proof of Lemma 6: Let $V(R) = \{v_i : i = 1, 2, \dots, n_1\}, V(G - R) = \{u_j : j = 1, 2, \dots, n_2\}$, in which $n_1 + n_2 = |V|$. If H_L is the cover of G^2 with list assignment L and matching assignment M_L . By the minimality, $G^2 - R^2$ is DP- k -colorable, namely there exist an independent set I' of cardinality n_2 in the cover of $G^2 - R^2$. For any $v_i \in V(R)$, denote:

$$L^*(v_i) = L(v_i) \setminus \bigcup_{u_j v_i \in E[R^2, G^2 - R^2]} \{c' \in L(v_i) : (u_j, c)(v_i, c') \in M_{u_j, v_i}, (u_j, c) \in I'\}$$

Finally, we show the existence of independent set I^* of cardinality n_1 in $H_L[V(R)]$, by giving an order of color $V(R)$. Now $I' \cup I^*$ is an independent set of cardinality $|V|$ in the cover of G^2 . Therefore G^2 is DP- k -colorable.

Lemma 7. *The following configurations are 5-reducible:*

- (1) 3-thread;
- (2) 3-face with only two 3-vertices, and each 3-vertex is incident to a 1-thread;
- (3) 4-face with only two adjacent 3-vertices, and each 3-vertex is incident to a 1-thread;
- (4) two 3-faces share one common edge, and only the endpoints of common edge are 3-vertices;
- (5) two 4-faces share one common edge, and only the endpoints of common edge are 3-vertices;
- (6) 3-face shares one common edge with 4-face, and only the endpoints of common edge are 3-vertices.

- Proof.* (1) For 3-thread $v_1v_2v_3$. If v_1 and v_3 have a common neighbor, then $|L^*(v_1)| \geq 3, |L^*(v_2)| \geq 4, |L^*(v_3)| \geq 3$; otherwise $|L^*(v_1)| \geq 2, |L^*(v_2)| \geq 3, |L^*(v_3)| \geq 2$. In both cases, we can extend the DP-5-coloring of $G^2 - R^2$ to G^2 by greedily color v_1, v_3, v_2 in order.
- (2) For 3-face $v_1v_2v_3v_1$, in which 3-vertex v_i is adjacent to a 2-vertex v'_i for $i = 1, 2$. Then $|L^*(v_3)| = 5, |L^*(v_1)| \geq 4, |L^*(v_2)| \geq 4, |L^*(v'_1)| \geq 2, |L^*(v'_2)| \geq 2$. We can extend the DP-5-coloring of $G^2 - R^2$ to G^2 by greedily color $v'_1, v'_2, v_1, v_2, v_3$ in order.
- (3) For 4-face $v_1v_2v_3v_4v_1$, in which 3-vertex v_i is adjacent to a 2-vertex v'_i for $i = 1, 2$. Then $|L^*(v_3)| = |L^*(v_4)| = 5, |L^*(v_1)| \geq 4, |L^*(v_2)| \geq 4, |L^*(v'_1)| \geq 2, |L^*(v'_2)| \geq 2$. We can extend the DP-5-coloring of $G^2 - R^2$ to G^2 by greedily color $v'_2, v'_1, v_1, v_2, v_3, v_4$ in order.
- (4) For configuration contains two adjacent 3-faces uvw_1u and uvw_2u . Its square graph is isomorphism to complete graph K_4 , which is obviously DP-5-colorable.
- (5) For configuration consist of two adjacent 4-faces $v_1v_2v_3v_4v_1$ and $v_1v_2v_5v_6v_1$, in which only v_1, v_2 are 3-vertices. Then $|L^*(v_i)| = 5, i = 1, 2, \dots, 6$. We can extend the DP-5-coloring of $G^2 - R^2$ to G^2 by greedily color v_1, v_2, \dots, v_6 in order.
- (6) Let R be the configuration with one 3-face $v_1v_2v_3v_1$ adjacent to a 4-face $v_1v_2v_4v_5v_1$, in which only v_1, v_2 are 3-vertices. Then its square graph is isomorphic to complete graph K_5 , which is obviously DP-5-colorable. □

Lemma 8. *The following configurations are 6-reducible:*

- (1) 2-thread;
- (2) 4-face with only two nonadjacent 3-vertices;
- (3) $F_{2,3}$ depicted in Figure 1.

- Proof.* (1) For 2-thread v_1v_2 , if v_1, v_2 have a common neighbor, then $|L^*(v_1)| \geq 4, |L^*(v_2)| \geq 4$; otherwise, $|L^*(v_1)| \geq 2, |L^*(v_2)| \geq 2$. We can extend the DP-6-coloring of $G^2 - R^2$ to G^2 by greedily color v_1, v_2 in order.
- (2) For 4-face $v_1v_2v_3v_4v_1$ with only two 3-vertices v_1, v_3 . Then in $(G - \{v_1, v_2, v_3, v_4\})^2$, $|L^*(v_2)| \geq 4, |L^*(v_4)| \geq 4, |L^*(v_1)| \geq 3, |L^*(v_3)| \geq 3$. We can greedily color v_1, v_2 in order to obtain a DP-6-coloring.
- (3) Let $F_{2,3}$ depicted in Figure 1 be the configuration consists of three 2-vertices u_1, u_2, u_3 and their two common neighbors v_1, v_2 with $d_G(v_1) = d_G(v_2) = 3$. Then its square graph is isomorphic to complete graph K_5 , which is obviously DP-6-colorable. Thus $F_{2,3}$ is a 6-reducible configuration. □

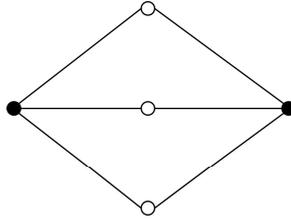


FIGURE 1. The graph $F_{2,3}$

Furtherly, we descript the k -minimal graph for $k = 5, 6$.

Lemma 9. *For 5-minimal graph G and 3-vertex $v \in V$, we have:*

- (1) if v is not adjacent to any 3-vertex, then $G_2(v)$ is isomorphic to the subgraph of $Y_{2,2,2}$;
- (2) if v is adjacent to a 3-vertex u , then $G_2(v), G_2(u)$ are both isomorphic to the subgraph of $Y_{0,2,2}$.

Proof. (1) If 3-vertex $v \in V$ only have neighbors of degree 2, and 3-vertices $v_i(i = 1, 2, 3)$ are connected to v by l_i -thread($i = 1, 2, 3$) respectively. By Lemma 6, the m -face with only one 3-vertex is reducible, so v_1, v_2, v_3 can not coincide with v . Without loss of generality, we may assume that $l_1 \leq l_2 \leq l_3$. Then $l_3 \leq 2$, by Lemma 7(1). Thus, $G_2(v)$ is isomorphic to the subgraph of $Y_{2,2,2}$.

(2) If 3-vertex $v \in V$ only have one neighbor of degree 3, say u . And assume 3-vertices $v_i(i = 1, 2)$ are connected to v by l_i -thread($i = 1, 2$); 3-vertices $u_i(i = 1, 2)$ are connected to u by l'_i -thread($i = 1, 2$). Without loss of generality, we may assume $l_1 \leq l_2 \leq l'_2, l'_1 \leq l'_2$.

Case 1: none of u_1, u_2 coincide with v , and none of v_1, v_2 coincide with u . By Lemma 7(1), $l'_2 \leq 2$. Thus $G_2(v), G_2(u)$ are both isomorphic to the subgraph of $Y_{0,2,2}$;

Case 2: vertex u_1 is coincide with v , while u_2 is not coincide with v . Then u, v (or u_1), u_2, v_1, v_2 and l_i -thread, l'_i -thread($i = 1, 2$) may form the induced subgraph isomorphic to the configurations in Figure 2(a). Which are both 5-reducible by Lemma 7(2)(3).

Case 3: both u_1 and u_2 are coincide with v . Then u (or v_1, v_2), v (or u_1, u_2) and l_i -thread, l'_i -thread($i = 1, 2$) may form the induced subgraph isomorphic to the configurations in Figure 2(b). Which are both 5-reducible by Lemma 7(4)(5)(6).

In conclusion, $G_2(v), G_2(u)$ are both isomorphic to the subgraph of $Y_{0,2,2}$. □

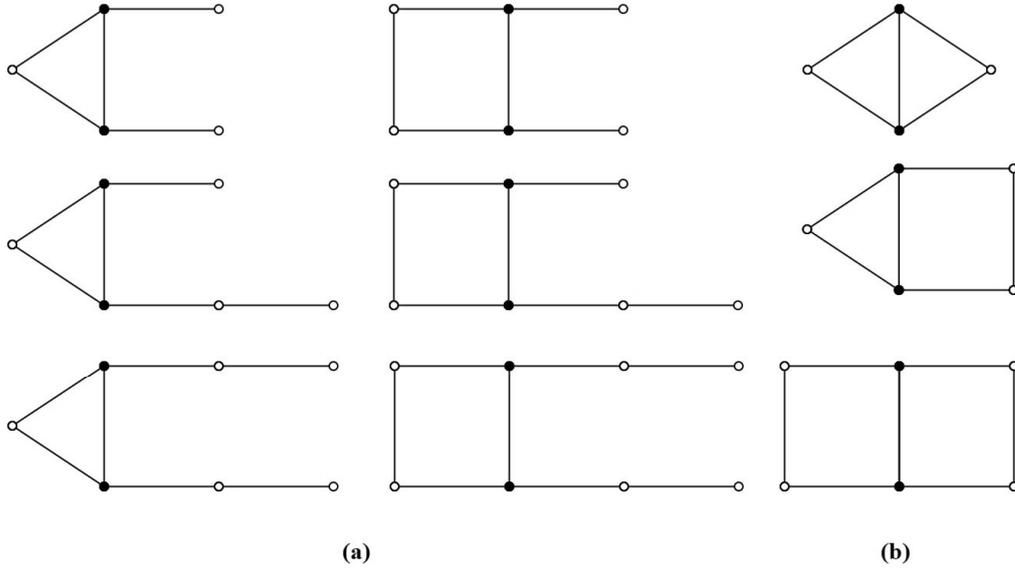


FIGURE 2. The 5-reducible configurations

Lemma 10. For 6-minimal graph G and 3-vertex v , $G_2(v)$ is isomorphic to $Y_{1,1,1}$ or subgraph of $Y_{0,1,1}$.

Proof. Let $v_i(i = 1, 2, 3)$ be the 3-vertices connected to v by l_i -thread($i = 1, 2, 3$) respectively, and assume $l_1 \leq l_2 \leq l_3$. By Lemma 8(1), $l_3 \leq 1$. For $l_1 = 0$, if v_2, v_3 are coincide, then $G_2(v)$ will form a 4-face with two nonadjacent 3-vertices. Which is 6-reducible by Lemma 8(2), so $G_2(v)$ is isomorphic to subgraph of $Y_{0,1,1}$. If $l_1 = 1$, v_1, v_2, v_3 can not coincide by Lemma 6 and Lemma 8(2)(3), thus $G_2(v)$ is isomorphic to $Y_{1,1,1}$. □

3.2. Main Results. Now we complete the proof by discharging method. Let the initial charge of k -vertex be its degree k , and let $ch^*(v)$ be the final charge of v .

Proof of Theorem 1. For 5-minimal graph G , we only need one discharging rule:

R1: Each 3-vertex gives $\frac{1}{4}$ to each adjacent 2-vertex.

We show that after the discharging procedure, the final charge of any vertex is at least $\frac{9}{4}$, which is in contradiction to the assumption of the maximum average degree of 5-minimal graph G .

For any 3-vertex v , it has at most three neighbors of degree 2 by Lemma 9. Thus, $ch^*(v) \geq 3 - 3 \cdot \frac{1}{4} = \frac{9}{4}$. For any 2-vertex v , it must be on a 1-thread or a 2-thread. In the former case, two neighbors of v are both 2-vertices, then $ch^*(v) = 2 + 2 \cdot \frac{1}{4} = \frac{10}{4} = \frac{5}{2}$; in the latter case, v is only adjacent to one 3-vertex, thus $ch^*(v) = 2 + \frac{1}{4} = \frac{9}{4}$. \square

In fact, the bound we give above is tight. Consider the graph $F_{2,6}$ depicted in Figure 3, which has average degree $\frac{9}{4}$. But the square of the subgraph induced by $\{u_1, u_2, v_1, v_2, w_1, w_2\}$ in $(F_{2,6})^2$ is isomorphic to complete graph K_6 , which is not DP-5-colorable.

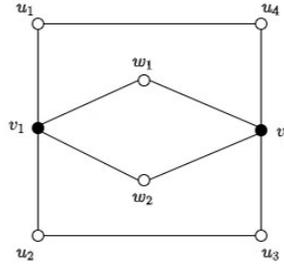


FIGURE 3. The graph $F_{2,6}$

Proof of Theorem 2. For 6-minimal graph G , the only discharging rule is as follows:

R2: Each 3-vertex gives $\frac{1}{5}$ to each adjacent 2-vertex.

We show that if we redistribute charges by the rule above, the final charge of any vertex is at least $\frac{12}{5}$, which also leads to a contradiction.

For any 3-vertex v , it has at most three neighbors of degree 2 by Lemma 10. Then we obtain $ch^*(v) \geq 3 - 3 \cdot \frac{1}{5} = \frac{12}{5}$. For any 2-vertex v , it is only adjacent to 3-vertices by Lemma 8(1). Thus $ch^*(v) = 2 + 2 \cdot \frac{1}{5} = \frac{12}{5}$. \square

Unfortunately, we don't know whether this bound is tight or not.

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