ON PICARD'S THEOREM VIA NEVANLINNA THEORY

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Abstract. We study a long-standing Picard's problem for non-compact complete Kähler manifolds with non-negative Ricci curvature. When the manifold carries a positive Green function, a positive answer is given to the Picard's problem, i.e., it is showed that every meromorphic function on such a manifold reduces to a constant if it omits three distinct values.

Contents

2010 *Mathematics Subject Classification.* 32H30; 32H25.
Key words and phrases. Nevanlinna theory; defect relation; Picard type theorem.

1. Introduction

1.1. Motivations and Techniques.

It is well-known that the famous Picard's little theorem asserts that every meromorphic function on the complex Euclidean space reduces to a constant, if it omits three distinct values. All the time, many authors have been trying to generalize this theorem to a more general non-compact complex manifold. Due to S. T. Yau [44], each holomorphic function on a non-compact complete Kähler manifold (Hermitian manifold, more general) with non-negative Ricci curvature has Liouville property. So, a natural problem is that does Picard's little theorem still hold on such a manifold? It is called the Picard's problem. To our knowledge, this is a long-standing problem. In what follows, we shall give an accurate statement.

Let (M, g) be a non-compact complete Kähler manifold with non-negative Ricci curvature, see examples for such a manifold in Sha-Yang [34] and Tian-Yau [39, 40]. The well-known Picard's problem asks that

Picard's Problem. *Is every meromorphic function on M necessarily a constant if it omits 3 distinct values?*

The first result in this direction could be traced back to 1970. S. Kobayashi [20] obtained some Picard-type theorems for holomorphic mappings between complex manifolds from the viewpoint of complex hyperbolicity. As a special consequence, he showed that

Theorem A (Picard's Theorem, Kobayashi). *Every meromorphic function on M on which, a complex Lie group acts transitively, must be a constant if it omits 3 distinct values.*

Kobayashi's result is limited to certain complex manifolds which are acted on transitively by a complex Lie group. In 1975, Goldberg-Ishihara-Petridis [17] presented some Picard-type theorems on harmonic mappings of bounded dilatation on a locally flat Riemannian manifold, refer to N. Petridis [27] also. In particular, they showed that

Theorem B (Picard's Theorem, Goldberg-Ishihara-Petridis). *Assume that M is locally flat. Every holomorphic mapping $f : M \to \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ of bounded dilatation must be a constant mapping.*

It does not seem to be a quite satisfactory answer to the Picard's problem, for some restrictions such as local flatness and bounded dilatation. The first result without any restrictions for M is due to S. T. Yau [44], who considered the Liouville's problem based on a method of gradient estimates at the same time, which is viewed as a weaker version of the Picard's problem. He showed that

Theorem C (Liouville's Theorem, Yau)**.** *Every holomorphic function on M must be a constant if it is bounded.*

By S. T. Yau [44], Liouville's theorem still holds for all harmonic functions on M (see Cheng-Yau [9] also). The gradient estimation method shows great power in many problems from geometric analysis (see, e.g., [35]), but it seems to be unable to give a solution to the Picard's problem. Until 2010, A. Atsuji [2] developed a Nevanlinna-type theory based on heat diffusions, and he gave a positive answer to the Picard's problem for M, as a meromorphic function on M is of a slow growth rate. In the following, we introduce his basic ideas. Let α, Δ be the Kähler form and Laplace-Beltrami operator of M associated to g, respectively. One can endow $\mathbb{P}^1(\mathbb{C})$ with Fubini-Study metric ω_{FS}. Let $f: M \to \mathbb{P}^1(\mathbb{C})$ be a meromorphic function. Using the Kählerness of M, the Hilbert-Schmidt norm of differential df (with respect to metrics α, ω_{FS}) can be written as the form

$$\|df\|^2 = 4m\frac{f^*\omega_{FS}\wedge\alpha^{m-1}}{\alpha^m} = \Delta\log\|f\|^2.$$

He showed that

Theorem D (Picard's Theorem, Atsuji)**.** *Let $f: M \to \mathbb{P}^1(\mathbb{C})$ be a meromorphic function. Assume that f satisfies the growth condition*

$$\int_1^\infty e^{-\epsilon r^2}dr\int_{B(r)}\|df\|^2dv < \infty, \qquad {}^\forall\epsilon > 0,$$

where $B(r)$ is a geodesic ball centered at a fixed point o with radius r in M. Then, f must be a constant if it omits 3 distinct values.

Atsuji's trick [2] (see [11] also) is employing the probabilistic approach (via Brownian motions), who introduced the so-called Nevanlinna-type functions $\tilde{T}_f(t,\omega_{FS}), \tilde{m}_f(t,a)$ and $\tilde{N}_f(t,a)$, in which t is the time of a Brownian motion X_t on M. By using Itô's formula (see, e.g., [18]) and estimates of curvatures, he established a heat diffusion version of Second Main Theorem, which leads to the above Theorem D. It is apparent that M is stochastically complete (or X_t is conservative) since M has non-negative Ricci curvature by Grigor'yan's criterion [15]. However, in order to make $\tilde{T}_f(t,\omega_{FS})$ and $\tilde{N}_f(t,a)$ meaningful in spirit of Nevanlinna's settings, the following conditions are necessary:

- $\tilde{T}_f(t,\omega_{FS}) < \infty$ for $t > 0$;
- $\tilde{T}_f(t,\omega_{FS}) \to \infty$ as $t \to \infty$;
- $\tilde{N}_f(t,a) = 0$ if f omits a.

To do so, f has to satisfy certain growth assumptions. That is why a growth condition is needed in Atsuji's theorem. Unfortunately, the growth condition is so strong that few meromorphic functions can satisfy it.

In the present paper, we wish to settle the Picard's problem by developing Nevanlinna theory (see, e.g., [24, 26, 29]) to complete Kähler manifolds with non-negative Ricci curvature. As we know, Nevanlinna theory studies value distribution of meromorphic mappings between complex spaces, which has been developed for a long time since R. Nevanlinna founded two fundamental theorems for meromorphic functions on the complex plane in 1925. During this period, rich results have been achieved, we may refer the reader to L. V. Ahlfors [4], H. Cartan [6], Carlson-Griffths-King [7, 16], J. Noguchi [23, 24], E. I. Nochka [25], M. Ru [28, 29], B. Shiffman [30], B. Shabat [31], F. Sakai [32, 33], W. Stoll [37, 38], P. Vojta [36], H. Wu [43] and etc., and refer also to [1, 2, 3, 11, 12, 13] and their references therein. To our knowledge, almost all theorems from Nevanlinna theory are based only on Kähler manifolds which admit a complete Kähler metric of non-positive sectional curvature, while we know very little if the domain manifolds are not non-positively curved, since we know little about how to estimate Green functions (satisfying Dirichelet boundary condition) for bounded domains under a Ricci curvature condition.

In 2023, the author [14] obtained the first result for the value distribution of meromorphic mappings on a complete Kähler manifold with non-negative Ricci curvature. As a consequence, the author gave a positive answer to the Picard's problem if M is of maximal volume growth saying that

$$\liminf_{r\to\infty}\frac{V(r)}{r^{2m}}>0,$$

where $V(r)$ denotes the Riemannian volume of geodesic ball $B(r)$ centered at a fixed reference point o with radius r in M.

The original technique in [14] is to construct a family of relatively compact domains $\{\Delta(r)\}_{r>0}$ exhausting M through the global Green functions for M, based on an asymptotic estimate of minimal positive global Green function obtained by Colding-Minicozzi [8]. With an optimal estimate for local Green function for $\Delta(r)$ in terms of integral forms, the author (see [14]) established a Second Main Theorem of meromorphic mappings on M. Let us introduce the main contributions or key techniques in [14]. Let $G(o,x)$ be the minimal positive global Green function of $\Delta/2$ for M. Colding-Minicozzi [8] (see [21] also) obtained the asymptotic behavior of $G(o,x)$: there exists a constant $A=A(m)>0$ such that

$$\lim_{x\to\infty}\frac{(2m-2)G(o,x)}{\rho(x)^{2-2m}}=A,\qquad {}^\forall m\geq 2,$$

where $\rho(x)$ denotes the Riemannian distance function of x from o. By means of the asymptotic estimate of $G(o,x)$, the author defined the domain:

$$\Delta(r)=\left\{x\in M:\ G(o,x)>A\int_r^\infty t^{1-2m}dt\right\},\qquad {}^\forall r>0.$$

Thus, the Green function $g_r(o,x)$ of $\Delta/2$ for $\Delta(r)$ with a pole at o satisfying Dirichlet boundary condition can be written as

$$g_r(o,x) = G(o,x) - A\int_r^\infty t^{1-2m}dt.$$

According to [14], for any $\epsilon>0$, there exists $r_\epsilon>0$ such that

$$(A-\epsilon)\int_{\rho(x)}^r t^{1-2m}dt \leq g_r(o,x) \leq (A+\epsilon)\int_{\rho(x)}^r t^{1-2m}dt$$

holds for all $x\in M$ satisfying $\rho(x)\geq r_\epsilon$. Based on this estimate, the Calculus Lemma can be established. The situation where $m=1$ is trivial for that M is conformally equivalent to $\mathbb{C}$ under two assumptions of non-negative Ricci curvature and maximal volume growth (see [21]). By employing the standard arguments, the author established a Second Main Theorem of meromorphic mappings from M into a complex projective manifold, which gives a Picard's theorem:

Theorem E (Picard's Theorem, Dong)**.** *Assume that M is of maximal volume growth. Every meromorphic function on M must be a constant if it omits* 3 *distinct values.*

However, the technique in [14] depends on Colding-Minicozzi's asymptotic estimate. Unfortunately, there are no asymptotic estimates for $G(o,x)$ if M is not of maximal volume growth.

In this paper, we will give a positive answer under a much weaker volume growth condition:

$$\int_1^\infty \frac{t}{V(t)}dt<\infty,$$

which is equivalent to that M carries a positive global Green function. Refer to N. Varopoulos [41, 42] (or see Li-Tam-Wang [19, 21]), there exists uniquely a minimal positive global Green function of $\Delta/2$ for M, denoted by $G(o,x)$. Our original method is the construction of domain $\Delta(r)$ and the application of Li-Yau's estimates on heat kernels and Green functions.

According to the estimate for $G(o,x)$ obtained by Li-Yau [22], there exist constants $A,B>0$ such that

$$A\int_{\rho(x)}^\infty \frac{t}{V(t)}dt \leq G(o,x) \leq B\int_{\rho(x)}^\infty \frac{t}{V(t)}dt, \quad {}^\forall x\in M.$$

Re-define $\Delta(r)$ by

$$\Delta(r) = \left\{x\in M:\ G(o,x) > A\int_r^\infty \frac{t}{V(t)}dt\right\}, \quad {}^\forall r>0.$$

In further, we have

$$g_r(o,x)=G(o,x)-A\int_r^\infty\frac{t}{V(t)}dt,$$

which defines the Green function of $\Delta/2$ for $\Delta(r)$ with a pole at o satisfying Dirichlet boundary condition. There is a natural relationship between $\Delta(r)$ and $g_r(o,x)$ (see Lemma 4.1): for $0<t\leq r$, we have

$$g_r(o,x)=A\int_t^r\frac{s}{V(s)}ds, \qquad {}^\forall x\in\partial\Delta(t),$$

which plays an important role in the establishment of Nevanlinna theory in this paper.

In the study of Nevanlinna theory, Negative Curvature Method and Logarithmic Derivative Lemma Method are considered as two standard research approaches. In our investigations, Logarithmic Derivative Lemma Method is used. In doing so, it is necessary to establish the so-called Calculus Lemma, which turns out to be a workable road to the Logarithmic Derivative Lemma.

Let us introduce the main techniques: Gradient Estimation (see Theorem 4.2) and Calculus Lemma (see Theorem 5.2) as follows.

Gradient Estimation. *For any $\epsilon>0$, there exists $r_\epsilon>0$ such that*

$$\|\nabla g_r(o,x)\|\leq(B+\epsilon)\frac{\rho(x)}{V(\rho(x))}, \qquad {}^\forall x\in\partial\Delta(r)$$

holds for all $r\geq r_\epsilon$.

The above gradient estimate of $g_r(o,x)$ gives an estimate of the harmonic measure π_r on $\partial\Delta(r)$ with respect to o: for any $\epsilon>0$, there exists $r_\epsilon>0$ such that (see Corollary 4.3)

$$d\pi_r(x)\leq\frac{B+\epsilon}{2}\frac{r}{V(r)}d\sigma_r(x), \qquad {}^\forall x\in\partial\Delta(r)$$

holds for all $r\geq r_\epsilon$.

Calculus Lemma. *Let $k\geq 0$ be a locally integrable function on M. Assume that k is locally bounded at o. Then for any $\delta>0$, there exist a constant $C>0$ and a subset $E_\delta\subseteq(0,\infty)$ of finite Lebesgue measure such that*

$$\int_{\partial\Delta(r)}kd\pi_r\leq Cr^{(2m-1)\delta}\left(\int_{\Delta(r)}g_r(o,x)kdv\right)^{(1+\delta)^2}$$

holds for all $r>0$ outside E_δ.

Let ψ be a nonconstant meromorphic function on M. The Calculus Lemma can lead to Logarithmic Derivative Lemma (see Theorem 5.6): for any $\delta>0$, there exists a subset $E_\delta \subseteq (0,\infty)$ of finite Lebesgue measure such that

$$m\Big(r, \frac{\|\nabla\psi\|}{|\psi|}\Big) \leq \frac{2+(1+\delta)^2}{2}\log^+ T(r,\psi) + \frac{(2m-1)\delta}{2}\log r$$

holds for all $r>0$ outside E_δ.

1.2. **Main results.**

Let X be a complex projective manifold with $\dim_{\mathbb{C}} X \leq m$, over which we can put a Hermitian positive line bundle (L,h) with Chern form $c_1(L,h)>0$. Fix a reduced divisor $D\in|L|$, where $|L|$ denotes the complete linear system of L. Given a meromorphic mapping $f: M\to X$, we have the Nevanlinna's functions $T_f(r,L), m_f(r,D), N_f(r,D)$ and $\overline{N}_f(r,D)$, see definition in Section 3.2 of this paper. Assume that M carries a positive global Green function, i.e., it satisfies the volume growth condition:

$$\int_1^\infty \frac{t}{V(t)}dt<\infty.$$

Let $\mathscr{R} := -dd^c\log\det(g_{i\bar{j}})$ be the Ricci form of M associated with the metric g, and K_X be the canonical line bundle over X. The characteristic function of $\mathscr{R}$ is defined as

$$T(r,\mathscr{R}) = \frac{\pi^m}{(m-1)!}\int_{\Delta(r)} g_r(o,x)\mathscr{R}\wedge\alpha^{m-1}.$$

The first main result is the following Second Main Theorem.

Theorem 1.1 (=Theorem 6.1)**.** *Let $f: M\to X$ be a differentiably non-degenerate meromorphic mapping. Let $D\in|L|$ be a reduced divisor of simple normal crossing type. Then for any $\delta>0$, there exists a subset $E_\delta\subseteq(0,\infty)$ of finite Lebesgue measure such that*

$$T_f(r,L)+T_f(r,K_X)+T(r,\mathscr{R}) \leq \overline{N}_f(r,D) + O\left(\log^+ T_f(r,L)+\delta\log r\right)$$

holds for all $r>0$ outside E_δ.

For a divisor $D\in|L|$, the *simple defect* of f with respect to D is defined by

$$\bar{\delta}_f(D) = 1-\limsup_{r\to\infty}\frac{\overline{N}_f(r,D)}{T_f(r,L)}.$$

Again, put

$$\left[\frac{c_1(K_X^*)}{c_1(L)}\right] = \inf\left\{s\in\mathbb{R}:\ \omega_2\leq s\omega_1;\ \ ^\exists\omega_1\in c_1(L),\ ^\exists\omega_2\in c_1(K_X^*)\right\}.$$

Theorem 1.1 gives a defect relation:

Corollary 1.2 (=Corollary 6.4)**.** *Assume the same conditions as in Theorem* 6.1. *Then*

$$\bar{\delta}_f(D) \leq \left[\frac{c_1(K_X^*)}{c_1(L)}\right] - \liminf_{r\to\infty} \frac{T(r,\mathscr{R})}{T_f(r,L)}.$$

Let $R_{\rm sc}$ be the scalar curvature of M. Set

$$\nu(f) = \liminf_{r\to\infty} \frac{\int_{\Delta(r)} g_r(o,x) R_{\rm sc} dv}{\int_{\Delta(r)} g_r(o,x)\Delta \log(1+|f|^2) dv}.$$

We obtain a Picard's theorem:

Corollary 1.3 (=Corollary 6.11)**.** *Every meromorphic function f on M must be a constant if it omits* $\max\{[3-2\nu(f)],0\}$ *distinct values, where* $[3-2\nu(f)]$ *denotes the maximal integer not greater than* $3-2\nu(f)$.

Corollary 1.3 shows that each meromorphic function f satisfying $\nu(f)>0$ on M must be a constant, if it omits 2 distinct values. Actually, this theorem gives a quantitative solution to the Picard's problem.

2. Some Facts from Differential Geometry and Geometric Analysis

2.1. **Volume Comparison Theorem.**

A *space form* is defined as a complete Riemannian manifold with constant sectional curvature. Given a simply-connected space form M^K with constant sectional curvature K, of dimension n. Let $V(K,r)$ denote the Riemannian volume of a geodesic ball with radius r in M^K. Besides, let M be a complete Riemannian manifold with Ricci curvature ${\rm Ric}_M$, of dimension n. Fix a point $o\in M$. Denote by $V(r)$ the Riemannian volume of a geodesic ball centered at o with radius r.

Bishop-Gromov (see, e.g., [5]) gave an upper bound of $V(r)$:

Theorem 2.1 (Volume Comparison Theorem)**.** *If* ${\rm Ric}_M \geq (n-1)K$ *for a constant K, then the volume ratio $V(r)/V(K,r)$ is non-increasing in $r>0$, and it tends to* 1 *as $r\to\infty$. Hence, we have*

$$V(r)\leq V(K,r)$$

holds for all $r\geq 0$.

In particular, if $M^K=\mathbb{R}^n$, then we obtain:

Corollary 2.2. *Assume that M is non-compact. If* ${\rm Ric}_M\geq 0$, *then we have*

$$V(r)\leq \omega_n r^n$$

holds for all $r\geq 0$, where ω_n is the volume of a unit ball in $\mathbb{R}^n$.

Calabi-Yau (see, e.g., [35]) gave a lower bound of $V(r)$:

Theorem 2.3. *Assume that M is non-compact. If $\mathrm{Ric}_M \geq 0$, then M has an infinite volume. More precisely, there exists a constant $C = C(n, V(1)) > 0$ such that*

$$V(r) \geq Cr$$

holds for all $r > 2$.

2.2. **Estimates of Heat Kernels.**

Let M be a complete Riemannian manifold of dimension n, with Laplace-Beltrami operator Δ. The heat kernel $p(t, x, y)$ of $\Delta/2$ for M is the minimal positive fundamental solution of the following heat equation

$$\Big(\frac{\partial}{\partial t} - \frac{1}{2}\Delta\Big)u(t, x) = 0.$$

Fix any reference point $o \in M$. Let $\rho(x)$ be the Riemannian distance function of x from o.

Li-Yau [22] gave an estimate for $p(t, o, x)$:

Theorem 2.4. *Assume that M has non-negative Ricci curvature. Then for any $0 < \epsilon < 1$, there exist constants $C_1 = C_1(\epsilon, n) > 0$ and $C_2 = C_2(\epsilon, n) > 0$ such that*

$$C_1 V(t)^{-1} e^{-\frac{\rho(x)^2}{(4-\epsilon)t}} \leq p(t, o, x) \leq C_2 V(t)^{-1} e^{-\frac{\rho(x)^2}{(4+\epsilon)t}}, \qquad ^{\forall} t > 0$$

holds for all $x \in M$.

Set

$$G(o, x) = \int_0^\infty p(t, o, x)dt.$$

If the right-hand integral converges , then $G(o, x)$ defines a minimal positive global Green function of $\Delta/2$ for M with a pole at o, i.e.,

$$-\frac{1}{2}\Delta G(o, x) = \delta(x); \qquad G(o, x) > 0; \qquad \lim_{x \to \infty} G(o, x) = 0, \tag{1}$$

where δ is the Dirac's delta function with a pole at o. It is evident that such a Green function is also unique if (1) is satisfied.

In further, Li-Yau [22] obtained an estimate for $G(o, x)$:

Theorem 2.5. *Assume that M has non-negative Ricci curvature. If $G(o, x)$ exists, then there exist constants $C_1, C_2 > 0$ depending only on n such that*

$$C_1 \int_{\rho(x)}^\infty \frac{t}{V(t)}dt \leq G(o, x) \leq C_2 \int_{\rho(x)}^\infty \frac{t}{V(t)}dt$$

holds for all $x \in M$.

3. Nevanlinna's Functions and First Main Theorem

Let (M, g) be a non-compact complete Kähler manifold with non-negative Ricci curvature of complex dimension m, whose Kähler form is defined by

$$\alpha = \frac{\sqrt{-1}}{\pi} \sum_{i,j=1}^{m} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$$

in a local holomorphic coordinate $(z_1, \cdots, z_m)$. Fix a reference point $o \in M$. Let $B(r)$ stand for the geodesic ball centered at o with radius r in M. Denote by $V(r)$ the Riemannian volume of $B(r)$.

3.1. **Construction of $\Delta(r)$.**

Assume that M is satisfied the condition of volume growth:

$$\int_1^{\infty} \frac{t}{V(t)} dt < \infty.$$

It implies that M is non-parabolic and there exists a unique minimal positive global Green function $G(o, x)$ for M satisfying

$$-\frac{1}{2} \Delta G(o, x) = \delta_o(x),$$

where Δ denotes the Laplace-Beltrami operator and δ_o is the Dirac function with a pole at o. Let $\rho(x)$ be the Riemannian distance function of x from o. According to Li-Yau [22], there exist constants $A, B > 0$ such that

$$A \int_{\rho(x)}^{\infty} \frac{t}{V(t)} dt \leq G(o, x) \leq B \int_{\rho(x)}^{\infty} \frac{t}{V(t)} dt, \quad {}^{\forall} x \in M. \tag{2}$$

Define

$$\Delta(r) = \left\{ x \in M : \ G(o, x) > A \int_r^{\infty} \frac{t}{V(t)} dt \right\}, \quad {}^{\forall} r > 0.$$

It is evident that $\Delta(r)$ is relatively compact for any $r > 0$, and the sequence $\{\Delta(r_n)\}_{n=1}^{\infty}$ exhausts M if $0 < r_1 < r_2 < \cdots < r_n < \cdots \to \infty$. According to Sard's theorem, the boundary $\partial \Delta(r)$ of $\Delta(r)$ is a submanifold of M for almost every $r > 0$. Set

$$g_r(o, x) = G(o, x) - A \int_r^{\infty} \frac{t}{V(t)} dt,$$

which is the positive Green function of $\Delta/2$ for $\Delta(r)$ with a pole at o satisfying Dirichelet boundary condition, i.e.,

$$-\frac{1}{2} \Delta g_r(o, x) = \delta_o(x), \quad {}^{\forall} x \in \Delta(r); \qquad g_r(o, x) = 0, \quad {}^{\forall} x \in \partial \Delta(r).$$

Moreover, we denote by π_r the harmonic measure on $\partial\Delta(r)$ with respect to o, which is defined by

$$d\pi_r(x)=\frac{1}{2}\frac{\partial g_r(o,x)}{\partial\vec{\nu}}d\sigma_r(x), \qquad {}^\forall x\in\partial\Delta(r),$$

where $\partial/\partial\vec{\nu}$ is the inward normal derivative on $\partial\Delta(r)$, $d\sigma_r$ is the Riemannian area element of $\partial\Delta(r)$.

3.2. **Nevanlinna's Functions.**

In what follows, we will introduce Nevanlinna's functions. Let $f:M\to X$ be a meromorphic mapping, where X is a complex projective manifold. Let (L,h) be a Hermitian holomorphic line bundle over X, with the Chern form $c_1(L,h)=-dd^c\log h$, where

$$d=\partial+\overline{\partial}, \qquad d^c=\frac{\sqrt{-1}}{4\pi}(\overline{\partial}-\partial).$$

Fix a divisor $D\in|L|$, where $|L|$ is the complete linear system of L. Let s_D be the canonical section associated to D, i.e., s_D is a holomorphic section of L over X with zero divisor D. The characteristic function, proximity function, counting function and simple counting function of f are respectively defined by

$$\begin{aligned}
T_f(r,L) &= -\frac{1}{4}\int_{\Delta(r)}g_r(o,x)\Delta\log(h\circ f)dv, \\
m_f(r,D) &= \int_{\partial\Delta(r)}\log\frac{1}{\|s_D\circ f\|}d\pi_r, \\
N_f(r,D) &= \frac{\pi^m}{(m-1)!}\int_{f^*D\cap\Delta(r)}g_r(o,x)\alpha^{m-1}, \\
\overline{N}_f(r,D) &= \frac{\pi^m}{(m-1)!}\int_{\mathrm{Supp}(f^*D)\cap\Delta(r)}g_r(o,x)\alpha^{m-1},
\end{aligned}$$

where dv is the volume element of M.

Equip X with the Kähler metric $c_1(L,h)$. Let $\|df\|$ be the Hilbert-Schmidt norm of differential df. By the Kählerity of M, we have

$$\|df\|^2=4m\frac{f^*c_1(L,h)\wedge\alpha^{m-1}}{\alpha^m}=-\Delta\log(h\circ f).$$

Locally, write $s_D=\tilde{s}_De$, where e is a local holomorphic frame of L and $\tilde{s}_D$ is a holomorphic function. By Poincaré-Lelong formula (see, e.g., [7]), it leads to

$$[D]=dd^c\big[\log|\tilde{s}_D|^2\big]$$

in the sense of currents. So, we obtain the alternative expressions of $T_f(r,L)$ and $N_f(r,D)$ as follows

$$\begin{aligned}
T_f(r,L) &= \frac{\pi^m}{(m-1)!}\int_{\Delta(r)} g_r(o,x)f^*c_1(L,h)\wedge\alpha^{m-1} \\
&= \frac{1}{4}\int_{\Delta(r)} g_r(o,x)\|df\|^2 dv
\end{aligned}$$

and

$$\begin{aligned}
N_f(r,D) &= \frac{\pi^m}{(m-1)!}\int_{\Delta(r)} g_r(o,x)dd^c\big[\log|\tilde{s}_D\circ f|^2\big]\wedge\alpha^{m-1} \\
&= \frac{1}{4}\int_{\Delta(r)} g_r(o,x)\Delta\log|\tilde{s}_D\circ f|^2 dv.
\end{aligned}$$

3.3. **First Main Theorem.**

To establish the First Main Theorem of f, we need Jensen-Dynkin formula (see, e.g., [12, 13]) which states that

Lemma 3.1 (Jensen-Dynkin formula)**.** *Let ϕ be a $\mathscr{C}^2$-class function on M outside a polar set of singularities at most. Assume that $\phi(o)\neq\infty$. Then*

$$\int_{\partial\Delta(r)}\phi(x)d\pi_r(x)-\phi(o)=\frac{1}{2}\int_{\Delta(r)} g_r(o,x)\Delta\phi(x)dv(x).$$

Assume that $f(o)\not\in \mathrm{Supp}D$. Apply Jensen-Dynkin formula to $\log\|s_D\circ f\|$, then we are led to

$$\begin{aligned}
& m_f(r,D)-\log\frac{1}{\|s_D\circ f(o)\|} \\
=\ & \frac{1}{2}\int_{\Delta(r)} g_r(o,x)\Delta\log\frac{1}{\|s_D\circ f\|}dv \\
=\ & -\frac{1}{4}\int_{\Delta(r)} g_r(o,x)\Delta\log h\circ f dv-\frac{1}{4}\int_{\Delta(r)} g_r(o,x)\Delta\log|\tilde{s}_D\circ f|^2 dv \\
=\ & T_f(r,L)-N_f(r,D).
\end{aligned}$$

Therefore, we are led to that

Theorem 3.2 (First Main Theorem)**.** *Assume that $f(o)\not\in \mathrm{Supp}D$. Then*

$$T_f(r,L)+\log\frac{1}{\|s_D\circ f(o)\|}=m_f(r,D)+N_f(r,D).$$

4. Gradient Estimates of Green Functions

Let M be a non-compact complete Kähler manifold of complex dimension m satisfying

$$\int_1^\infty \frac{t}{V(t)}dt < \infty.$$

Theorem 4.1. *We have*

$$g_r(o,x) = A\int_t^r \frac{s}{V(s)}ds, \qquad {}^\forall x \in \partial\Delta(t)$$

holds for all $0 < t \leq r$, *where* A *is given by* (2).

Proof. According to the definition of Green function for $\Delta(r)$, it is immediate that for $0 < t \leq r$

$$\begin{aligned} g_r(o,x) &= G(o,x) - A\int_r^\infty \frac{t}{V(t)}dt \\ &= G(o,x) - A\int_t^\infty \frac{s}{V(s)}ds + A\int_t^r \frac{s}{V(s)}ds \\ &= g_t(o,x) + A\int_t^r \frac{s}{V(s)}ds. \end{aligned}$$

Since

$$g_t(o,x) = 0, \qquad {}^\forall x \in \partial\Delta(t),$$

then we obtain

$$g_r(o,x) = A\int_t^r \frac{s}{V(s)}ds, \qquad {}^\forall x \in \partial\Delta(t).$$

□

Let ∇ denote the gradient operator on M. We obtain an estimate of upper bounds of $\|\nabla g_r(o,x)\|$ as follows:

Theorem 4.2. *For any* $\epsilon > 0$, *there exists* $r_\epsilon > 0$ *such that*

$$\|\nabla g_r(o,x)\| \leq (B+\epsilon)\frac{\rho(x)}{V(\rho(x))}, \qquad {}^\forall x \in \partial\Delta(r)$$

holds for all $r \geq r_\epsilon$, *where* B *is given by* (2).

Proof. It yields from (2) that

$$\limsup_{x\to\infty} \frac{G(o,x)}{\int_{\rho(x)}^\infty tV(t)^{-1}dt} \leq B.$$

On the other hand, we have

$$\lim_{x\to\infty} G(o,x) = 0, \qquad \lim_{x\to\infty}\int_{\rho(x)}^\infty \frac{t}{V(t)}dt = 0.$$

Whence, it concludes that

$$\begin{aligned}
&\limsup_{r\to\infty}\frac{\frac{\partial G(o,x)}{\partial\vec{\nu}}}{\frac{\partial\int_{\rho(x)}^{\infty}tV(t)^{-1}dt}{\partial\vec{\nu}}}\\
&=\limsup_{r\to\infty}\frac{\nabla\|G(o,x)\|}{-\frac{\rho(x)}{V(\rho(x))}\frac{\partial\rho(x)}{\partial\vec{\nu}}}\leq B,\quad {}^{\forall}x\in\partial\Delta(r)\setminus Cut(o),
\end{aligned}$$

where $Cut(o)$ is the cut locus of o and $\partial/\partial\vec{\nu}$ is the inward normal derivative on $\partial\Delta(r)$. By

$$0<-\frac{\partial\rho(x)}{\partial\vec{\nu}}\leq 1,\quad {}^{\forall}x\in\partial\Delta(r)\setminus Cut(o),$$

we see that for any $\epsilon>0$, there exists $r_\epsilon>0$ such that

$$\|\nabla G(o,x)\|\leq(B+\epsilon)\frac{\rho(x)}{V(\rho(x))},\quad {}^{\forall}x\in\partial\Delta(r)\setminus Cut(o)$$

holds for all $r\geq r_\epsilon$. Since $Cut(o)$ has measure 0, with the aid of the continuity of $\|\nabla G(o,x)\|$ on $M\setminus\{o\}$ and $\|\nabla g_r(o,x)\|=\|\nabla G(o,x)\|$ on $\partial\Delta(r)$, one then has the theorem proved. □

As a result, we obtain an estimate for upper bounds of harmonic measure π_r on $\partial\Delta(r)$ with respect to o as follows:

Corollary 4.3. *For any $\epsilon>0$, there exists $r_\epsilon>0$ such that*

$$d\pi_r(x)\leq\frac{B+\epsilon}{2}\frac{r}{V(r)}d\sigma_r(x),\quad {}^{\forall}x\in\partial\Delta(r)$$

holds for all $r\geq r_\epsilon$, where B is given by (2).

Proof. By

$$\begin{aligned}
d\pi_r(x)&=\frac{1}{2}\frac{\partial g_r(o,x)}{\partial\vec{\nu}}d\sigma_r(x)\\
&=\frac{1}{2}\|\nabla g_r(o,x)\|d\sigma_r(x),\quad {}^{\forall}x\in\partial\Delta(r),
\end{aligned}$$

we see from Theorem 4.2 that for any $\epsilon>0$, there exists $r_\epsilon>0$ such that

$$d\pi_r(x)\leq\frac{B+\epsilon}{2}\frac{\rho(x)}{V(\rho(x))}d\sigma_r(x),\quad {}^{\forall}x\in\partial\Delta(r)\tag{3}$$

holds for all $r\geq r_\epsilon$. Again, by (2)

$$\int_{\rho(x)}^{\infty}\frac{t}{V(t)}dt\leq\int_r^{\infty}\frac{t}{V(t)}dt,\quad {}^{\forall}x\in\partial\Delta(r).$$

It implies that

$$\rho(x)\geq r,\quad {}^{\forall}x\in\partial\Delta(r).$$

On the other hand, Corollary 2.2 and Theorem 2.3 conclude that $r/V(r)$ is non-increasing in $r>0$. Hence, we have

$$\frac{\rho(x)}{V(\rho(x))}\leq\frac{r}{V(r)},\qquad {}^{\forall}x\in\partial\Delta(r).$$

That is to say, (3) leads to that

$$d\pi_r(x)\leq\frac{B+\epsilon}{2}\frac{r}{V(r)}d\sigma_r(x),\qquad {}^{\forall}x\in\partial\Delta(r)$$

holds for all $r\geq r_\epsilon$. □

5. Calculus Lemma and Logarithmic Derivative Lemma

Let M be a non-compact complete Kähler manifold of complex dimension m satisfying

$$\int_1^\infty\frac{t}{V(t)}dt<\infty.$$

5.1. Calculus Lemma.

We need the following Borel's lemma (see, e.g., [24]):

Lemma 5.1 (Borel Lemma). *Let $u\geq 0$ be a non-decreasing function on (r_0,∞) with $r_0\geq 0$. Then for any $\delta>0$, there exists a subset $E_\delta\subseteq(r_0,\infty)$ of finite Lebesgue measure such that such that*

$$u'(r)\leq u(r)^{1+\delta}$$

holds for all $r>r_0$ outside E_δ.

Proof. The conclusion is true clearly for $u\equiv 0$. Next, we assume that $u\not\equiv 0$. Since $u\geq 0$ is a non-decreasing function, then there exists a number $r_1>r_0$ such that $u(r_1)>0$. The non-decreasing property of u implies that the limit

$$A:=\lim_{r\to\infty}u(r)$$

exists, here $A=\infty$ is allowed. If $A=\infty$, then $A^{-1}=0$. Set

$$E_\delta=\left\{r\in(r_0,\infty):\ u'(r)>u(r)^{1+\delta}\right\}.$$

Note that $u'(r)$ exists for almost every $r\in(r_0,\infty)$. Then, we have

$$E_\delta=\int_{E_\delta}dr\leq\int_{r_0}^{r_1}dr+\int_{r_1}^{\infty}\frac{u'(r)}{u(r)^{1+\delta}}dr=\frac{1}{\delta u(r_1)^\delta}-\frac{1}{\delta A^\delta}+r_1-r_0<\infty.$$

This completes the proof. □

Now, we give the Calculus Lemma:

Theorem 5.2 (Calculus Lemma). *Let $k \geq 0$ be a locally integrable function on M. Assume that k is locally bounded at o. Then for any $\delta > 0$, there exist a constant $C > 0$ and a subset $E_\delta \subseteq (0, \infty)$ of finite Lebesgue measure such that*

$$\int_{\partial\Delta(r)} k d\pi_r \leq C \left(\frac{V(r)}{r}\right)^\delta \left(\int_{\Delta(r)} g_r(o,x) k dv\right)^{(1+\delta)^2}$$

holds for all $r > 0$ outside E_δ.

Proof. Invoking Theorem 4.1, we obtain

$$\begin{aligned}\int_{\Delta(r)} g_r(o,x) k dv &= \int_0^r dt \int_{\partial\Delta(t)} g_r(o,x) k d\sigma_t \\ &= A \int_0^r \left(\int_t^r \frac{s}{V(s)} ds\right) dt \int_{\partial\Delta(t)} k d\sigma_t.\end{aligned}$$

Set

$$\Lambda(r) = \int_0^r \left(\int_t^r \frac{s}{V(s)} ds\right) dt \int_{\partial\Delta(t)} k d\sigma_t.$$

A simple computation leads to

$$\Lambda'(r) = \frac{d\Lambda(r)}{dr} = \frac{r}{V(r)} \int_0^r dt \int_{\partial\Delta(t)} k d\sigma_t.$$

In further, we have

$$\frac{d}{dr}\left(\frac{V(r)\Lambda'(r)}{r}\right) = \int_{\partial\Delta(r)} k d\sigma_r.$$

Now, we apply Borel's lemma to the left hand side of the above equality for twice: one is to $V(r)\Lambda'(r)/r$ and the other is to $\Lambda'(r)$, we conclude that for any $\delta > 0$, there exists a subset $F_\delta \subseteq (0, \infty)$ of finite Lebesgue measure such that

$$\int_{\partial\Delta(r)} k d\sigma_r \leq \left(\frac{V(r)}{r}\right)^{1+\delta} \Lambda(r)^{(1+\delta)^2}$$

holds for all $r > 0$ outside F_δ. On the other hand, Corollary 4.3 implies that there exists $r_0 > 0$ such that

$$d\pi_r(x) \leq \frac{Br}{V(r)} d\sigma_r(x), \quad {}^\forall x \in \partial\Delta(r)$$

holds for all $r \geq r_0$, where B is given by (2). Combining the above, we have

$$\int_{\partial\Delta(r)} k d\pi_r \leq B \left(\frac{V(r)}{r}\right)^\delta \Lambda(r)^{(1+\delta)^2}$$

holds for all $r > 0$ outside $E_\delta := F_\delta \cup (0, r_0)$. Therefore, we have the theorem proved by setting

$$C = BA^{-(1+\delta)^2} > 0.$$

□

As a result, we conclude that

Corollary 5.3. *Let $k \geq 0$ be a locally integrable function on M. Assume that k is locally bounded at o. Assume that M is of maximal volume growth. Then for any $\delta > 0$, there exists a subset $E_\delta \subseteq (0, \infty)$ of finite Lebesgue measure such that*

$$\log^+ \int_{\partial\Delta(r)} k d\pi_r \leq (1+\delta)^2 \log^+ \int_{\Delta(r)} g_r(o,x) k dv + (2m-1)\delta \log r$$

holds for all $r > 0$ outside E_δ.

Proof. Using Corollary 2.2 and Theorem 2.3, there exist constants $C_1, C_2 > 0$ such that

$$C_1 r \leq V(r) \leq C_2 r^{2m}, \quad {}^\forall r \geq 0.$$

It yields that

$$\delta \log C_1 \leq \log^+ \left(\frac{V(r)}{r}\right)^\delta \leq (2m-1)\delta \log r + \delta \log^+ C_2, \quad {}^\forall r \geq 1.$$

By Theorem 5.2 again, then we have the corollary proved if disturbing $\delta > 0$ a little and replacing E_δ by $E_\delta \cup (0,1)$ so that $\log^+ C_2$ is absorbed. □

5.2. **Logarithmic Derivative Lemma.**

Let ψ be a meromorphic function on M. The norm of the gradient $\nabla\psi$ is defined by

$$\|\nabla\psi\|^2 = 2\sum_{i,j=1}^m g^{i\bar{j}} \frac{\partial\psi}{\partial z_i} \overline{\frac{\partial\psi}{\partial z_j}}$$

in a local holomorphic coordinate $(z_1, \cdots, z_m)$, where $(g^{i\bar{j}})$ is the inverse of $(g_{i\bar{j}})$. Again, define the Nevanlinna's characteristic function of ψ by

$$T(r,\psi) = m(r,\psi) + N(r,\psi),$$

where

$$\begin{aligned} m(r,\psi) &= \int_{\partial\Delta(r)} \log^+ |\psi| d\pi_r, \\ N(r,\psi) &= \frac{\pi^m}{(m-1)!} \int_{\psi^*\infty \cap \Delta(r)} g_r(o,x)\alpha^{m-1}. \end{aligned}$$

It is not difficult to show that

$$T\Big(r, \frac{1}{\psi - \zeta}\Big) = T(r,\psi) + O(1).$$

On $\mathbb{P}^1(\mathbb{C})$, one puts a singular metric

$$\Psi = \frac{1}{|\zeta|^2(1+\log^2|\zeta|)}\frac{\sqrt{-1}}{4\pi^2}d\zeta\wedge d\bar{\zeta}.$$

Then, it leads to

$$\int_{\mathbb{P}^1(\mathbb{C})}\Psi = 1.$$

Lemma 5.4. *We have*

$$\frac{1}{4\pi}\int_{\Delta(r)}g_r(o,x)\frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}dv \le T(r,\psi)+O(1).$$

Proof. Note that

$$\frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)} = 4m\pi\frac{\psi^*\Psi\wedge\alpha^{m-1}}{\alpha^m}.$$

Whence, it concludes from Fubini's theorem that

$$\begin{aligned}
&\frac{1}{4\pi}\int_{\Delta(r)}g_r(o,x)\frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}dv \\
&= m\int_{\Delta(r)}g_r(o,x)\frac{\psi^*\Psi\wedge\alpha^{m-1}}{\alpha^m}dv \\
&= \frac{\pi^m}{(m-1)!}\int_{\mathbb{P}^1(\mathbb{C})}\Psi(\zeta)\int_{\psi^*\zeta\cap\Delta(r)}g_r(o,x)\alpha^{m-1} \\
&= \int_{\mathbb{P}^1(\mathbb{C})}N\Big(r,\frac{1}{\psi-\zeta}\Big)\Psi(\zeta) \\
&\le \int_{\mathbb{P}^1(\mathbb{C})}\big(T(r,\psi)+O(1)\big)\Psi \\
&= T(r,\psi)+O(1).
\end{aligned}$$

□

Lemma 5.5. *Assume that $\psi\not\equiv 0$. Then for any $\delta>0$, there exists a subset $E_\delta\subseteq(0,\infty)$ of finite Lebesgue measure such that*

$$\int_{\partial\Delta(r)}\log^+\frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}d\pi_r \le (1+\delta)^2\log^+T(r,\psi)+(2m-1)\delta\log r$$

holds for all $r>0$ outside E_δ.

Proof. The concavity of log implies that

$$\begin{aligned}
&\int_{\partial\Delta(r)}\log^+\frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}d\pi_r \\
\leq\ &\log\int_{\partial\Delta(r)}\left(1+\frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}\right)d\pi_r \\
\leq\ &\log^+\int_{\partial\Delta(r)}\frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}d\pi_r+O(1).
\end{aligned}$$

By this with Corollary 5.3, we show that for any $\delta>0$, there exists a subset $E_\delta\subseteq(0,\infty)$ of finite Lebesgue measure such that

$$\begin{aligned}
&\log^+\int_{\partial\Delta(r)}\frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}d\pi_r \\
\leq\ &(1+\delta)^2\log^+\int_{\Delta(r)}g_r(o,x)\frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}dv+(2m-1)\delta\log r \\
\leq\ &(1+\delta)^2\log^+T(r,\psi)+(2m-1)\delta\log r
\end{aligned}$$

holds for all $r>0$ outside E_δ. Disturbing $\delta>0$ such that $O(1)$ is absorbed. The proof is completed. □

Define

$$m\left(r,\frac{\|\nabla\psi\|}{|\psi|}\right)=\int_{\partial\Delta(r)}\log^+\frac{\|\nabla\psi\|}{|\psi|}d\pi_r.$$

Next, we establish the following Logarithmic Derivative Lemma:

Theorem 5.6 (Logarithmic Derivative Lemma)**.** *Let ψ be a nonconstant meromorphic function on M. Then for any $\delta>0$, there exists a subset $E_\delta\subseteq(0,\infty)$ of finite Lebesgue measure such that*

$$m\Big(r,\frac{\|\nabla\psi\|}{|\psi|}\Big)\leq\frac{2+(1+\delta)^2}{2}\log^+T(r,\psi)+\frac{(2m-1)\delta}{2}\log r$$

holds for all $r>0$ outside E_δ.

Proof. Note that

$$\begin{aligned}
m\left(r,\frac{\|\nabla\psi\|}{|\psi|}\right)\leq\ &\frac{1}{2}\int_{\partial\Delta(r)}\log^+\frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}d\pi_r \\
&+\frac{1}{2}\int_{\partial\Delta(r)}\log\left(1+\log^2|\psi|\right)d\pi_r
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\partial\Delta(r)} \log^+ \frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}d\pi_r \\
&\quad + \frac{1}{2}\int_{\partial\Delta(r)} \log\left(1+\Big(\log^+|\psi|+\log^+\frac{1}{|\psi|}\Big)^2\right)d\pi_r \\
&\leq \frac{1}{2}\int_{\partial\Delta(r)} \log^+ \frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}d\pi_r \\
&\quad + \log\int_{\partial\Delta(r)} \Big(\log^+|\psi|+\log^+\frac{1}{|\psi|}\Big)d\pi_r + O(1) \\
&\leq \frac{1}{2}\int_{\partial\Delta(r)} \log^+ \frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}d\pi_r + \log^+ T(r,\psi) + O(1).
\end{aligned}
$$

Using Lemma 5.5 and disturbing $\delta > 0$, we have for any $\delta > 0$, there exists a subset $E_\delta \subseteq (0,\infty)$ of finite Lebesgue measure such that

$$
m\left(r, \frac{\|\nabla\psi\|}{|\psi|}\right) \leq \frac{2+(1+\delta)^2}{2}\log^+ T(r,\psi) + \frac{(2m-1)\delta}{2}\log r
$$

holds for all $r > 0$ outside E_δ. □

6. Second Main Theorem and Defect Relation

Let M be a non-compact complete Kähler manifold of complex dimension m, with non-negative Ricci curvature satisfying

$$
\int_1^\infty \frac{t}{V(t)}dt < \infty.
$$

Set

$$
T(r,\mathscr{R}) = \frac{\pi^m}{(m-1)!}\int_{\Delta(r)} g_r(o,x)\mathscr{R}\wedge\alpha^{m-1}.
$$

6.1. **Second Main Theorem.**

Let $D \in |L|$ be a reduced divisor of simple normal crossing type. We write $D = D_1 + \cdots + D_q$, which is an irreducible decomposition of D. Equip every holomorphic line bundle $\mathscr{O}(D_j)$ with a Hermitian metric h_j, which induces a Hermitian metric $h = h_1 \otimes \cdots \otimes h_q$ on L. Since $L > 0$, then we can assume that $c_1(L,h) > 0$. As a result, $\Omega = c_1^n(L,h)$ gives a volume form on X. Pick $s_j \in H^0(X, \mathscr{O}(D_j)$ such that $(s_j) = D_j$ and $\|s_j\| < 1$. Moreover, we define a singular volume form

$$
\Phi = \frac{\Omega}{\prod_{j=1}^q \|s_j\|^2}
$$

on X. Set

$$
f^*\Phi \wedge \alpha^{m-n} = \xi\alpha^m.
$$

It is clear that

$$\alpha^m = m!\det(g_{i\bar{j}})\bigwedge_{j=1}^m \frac{\sqrt{-1}}{\pi}dz_j\wedge d\bar{z}_j.$$

In further, we have

$$dd^c[\log\xi] \geq f^*c_1(L,h_L) - f^*\mathrm{Ric}(\Omega) + \mathscr{R} - [\mathrm{Red}(f^*D)]$$

in the sense of currents, where $\mathrm{Red}(f^*D)$ is the reduced divisor of f^*D, and $\mathscr{R} = -dd^c\log\det(g_{i\bar{j}})$ is the Ricci form of M. Therefore, it yields that

$$\begin{aligned} &\frac{1}{4}\int_{\Delta(r)} g_r(o,x)\Delta\log\xi dv \\ &\geq T_f(r,L) + T_f(r,K_X) + T(r,\mathscr{R}) - \overline{N}_f(r,D). \end{aligned} \tag{4}$$

Let us establish a Second Main Theorem as follows:

Theorem 6.1 (Second Main Theorem)**.** *Let X be a complex projective manifold of complex dimension not greater than that of M. Let $D\in|L|$ be a reduced divisor of simple normal crossing type, where L is a positive line bundle over X. Let $f:M\to X$ be a differentiably non-degenerate meromorphic mapping. Then for any $\delta>0$, there exists a subset $E_\delta\subseteq(0,\infty)$ of finite Lebesgue measure such that*

$$T_f(r,L) + T_f(r,K_X) + T(r,\mathscr{R}) \leq \overline{N}_f(r,D) + O\left(\log^+ T_f(r,L) + \delta\log r\right)$$

holds for all $r>0$ outside E_δ.

Proof. The argument of proof is standard (see [14]), but for the completeness of this paper, we still give a proof. Since D has only simple normal crossings, then there exist a finite open covering $\{U_\lambda\}$ of X and finitely many rational functions $w_{\lambda1},\cdots,w_{\lambda n}$ on X such that $w_{\lambda1},\cdots,w_{\lambda n}$ are holomorphic on U_λ satisfying that

$$dw_{\lambda1}\wedge\cdots\wedge dw_{\lambda n}(x)\neq 0, \quad {}^\forall x\in U_\lambda;$$
$$D\cap U_\lambda = \left\{w_{\lambda1}\cdots w_{\lambda h_\lambda}=0\right\}, \quad {}^\exists h_\lambda\leq n.$$

In addition, we can require that $\mathscr{O}(D_j)|_{U_\lambda}\cong U_\lambda\times\mathbb{C}$ for λ, j. On U_λ, write

$$\Phi = \frac{e_\lambda}{|w_{\lambda1}|^2\cdots|w_{\lambda h_\lambda}|^2}\bigwedge_{k=1}^n\frac{\sqrt{-1}}{\pi}dw_{\lambda k}\wedge d\bar{w}_{\lambda k},$$

where e_λ is a positive smooth function on U_λ. Let $\{\phi_\lambda\}$ be a partition of the unity subordinate to $\{U_\lambda\}$. Set

$$\Phi_\lambda = \frac{\phi_\lambda e_\lambda}{|w_{\lambda1}|^2\cdots|w_{\lambda h_\lambda}|^2}\bigwedge_{k=1}^n\frac{\sqrt{-1}}{\pi}dw_{\lambda k}\wedge d\bar{w}_{\lambda k}.$$

Again, put $f_{\lambda k}=w_{\lambda k}\circ f$. On $f^{-1}(U_\lambda)$, we have

$$
\begin{aligned}
f^*\Phi_\lambda &= \frac{\phi_\lambda\circ f\cdot e_\lambda\circ f}{|f_{\lambda 1}|^2\cdots|f_{\lambda h_\lambda}|^2}\bigwedge_{k=1}^{n}\frac{\sqrt{-1}}{\pi}df_{\lambda k}\wedge d\bar f_{\lambda k}\\
&= \phi_\lambda\circ f\cdot e_\lambda\circ f\sum_{1\le i_1\neq\cdots\neq i_n\le m}\frac{\left|\frac{\partial f_{\lambda 1}}{\partial z_{i_1}}\right|^2}{|f_{\lambda 1}|^2}\cdots\frac{\left|\frac{\partial f_{\lambda h_\lambda}}{\partial z_{i_{h_\lambda}}}\right|^2}{|f_{\lambda h_\lambda}|^2}\left|\frac{\partial f_{\lambda(h_\lambda+1)}}{\partial z_{i_{h_\lambda+1}}}\right|^2\\
&\quad\cdots\left|\frac{\partial f_{\lambda n}}{\partial z_{i_n}}\right|^2\Big(\frac{\sqrt{-1}}{\pi}\Big)^n dz_{i_1}\wedge d\bar z_{i_1}\wedge\cdots\wedge dz_{i_n}\wedge d\bar z_{i_n}.
\end{aligned}
$$

Fix a $x_0\in M$, one can take a local holomorphic coordinate $(z_1,\cdots,z_m)$ near x_0 and a local holomorphic coordinate $(\zeta_1,\cdots,\zeta_n)$ near $f(x_0)$ such that

$$
\alpha|_{x_0}=\frac{\sqrt{-1}}{\pi}\sum_{j=1}^m dz_j\wedge d\bar z_j,\quad c_1(L,h)|_{f(x_0)}=\frac{\sqrt{-1}}{\pi}\sum_{j=1}^n d\zeta_j\wedge d\bar\zeta_j.
$$

Put

$$
f^*\Phi_\lambda\wedge\alpha^{m-n}=\xi_\lambda\alpha^m.
$$

Then, we have $\xi=\sum_\lambda\xi_\lambda$ and at x_0:

$$
\begin{aligned}
\xi_\lambda &= \phi_\lambda\circ f\cdot e_\lambda\circ f\sum_{1\le i_1\neq\cdots\neq i_n\le m}\frac{\left|\frac{\partial f_{\lambda 1}}{\partial z^{i_1}}\right|^2}{|f_{\lambda 1}|^2}\cdots\frac{\left|\frac{\partial f_{\lambda h_\lambda}}{\partial z^{i_{h_\lambda}}}\right|^2}{|f_{\lambda h_\lambda}|^2}\left|\frac{\partial f_{\lambda(h_\lambda+1)}}{\partial z^{i_{h_\lambda+1}}}\right|^2\cdots\left|\frac{\partial f_{\lambda n}}{\partial z^{i_n}}\right|^2\\
&\le \phi_\lambda\circ f\cdot e_\lambda\circ f\sum_{1\le i_1\neq\cdots\neq i_n\le m}\frac{\|\nabla f_{\lambda 1}\|^2}{|f_{\lambda 1}|^2}\cdots\frac{\|\nabla f_{\lambda h_\lambda}\|^2}{|f_{\lambda h_\lambda}|^2}\\
&\quad\cdot\|\nabla f_{\lambda(h_\lambda+1)}\|^2\cdots\|\nabla f_{\lambda n}\|^2.
\end{aligned}
$$

Define a non-negative function ϱ on M by

$$
f^*c_1(L,h)\wedge\alpha^{m-1}=\varrho\alpha^m. \tag{5}
$$

Again, set $f_j=\zeta_j\circ f$ for $1\le j\le n$. Then, at x_0:

$$
f^*c_1(L,h)\wedge\alpha^{m-1}=\frac{(m-1)!}{2}\sum_{j=1}^m\|\nabla f_j\|^2\alpha^m.
$$

That is, at x_0:

$$
\varrho=(m-1)!\sum_{i=1}^n\sum_{j=1}^m\left|\frac{\partial f_i}{\partial z_j}\right|^2=\frac{(m-1)!}{2}\sum_{j=1}^n\|\nabla f_j\|^2
$$

Combining the above, we are led to

$$
\xi_\lambda\le\frac{\phi_\lambda\circ f\cdot e_\lambda\circ f\cdot(2\varrho)^{n-h_\lambda}}{(m-1)!^{n-h_\lambda}}\sum_{1\le i_1\neq\cdots\neq i_n\le m}\frac{\|\nabla f_{\lambda 1}\|^2}{|f_{\lambda 1}|^2}\cdots\frac{\|\nabla f_{\lambda h_\lambda}\|^2}{|f_{\lambda h_\lambda}|^2}
$$

on $f^{-1}(U_\lambda)$. Note that $\phi_\lambda\circ f\cdot e_\lambda\circ f$ is bounded on M, whence it yields from $\log^+\xi\leq\sum_\lambda\log^+\xi_\lambda+O(1)$ that

$$\log^+\xi\leq O\Big(\log^+\varrho+\sum_{k,\lambda}\log^+\frac{\|\nabla f_{\lambda k}\|}{|f_{\lambda k}|}\Big)+O(1). \tag{6}$$

By Jensen-Dynkin formula

$$\frac{1}{4}\int_{\Delta(r)}g_r(o,x)\Delta\log\xi dv=\frac{1}{2}\int_{\partial\Delta(r)}\log\xi d\pi_r+O(1). \tag{7}$$

Combining (6) with (7) and using Theorem 5.6, we obtain

$$\begin{aligned}&\frac{1}{4}\int_{\Delta(r)}g_r(o,x)\Delta\log\xi dv\\ &\leq O\bigg(\sum_{k,\lambda}m\Big(r,\frac{\|\nabla f_{\lambda k}\|}{|f_{\lambda k}|}\Big)+\log^+\int_{\partial\Delta(r)}\varrho d\pi_r\bigg)+O(1)\\ &\leq O\bigg(\sum_{k,\lambda}\log^+T(r,f_{\lambda k})+\log^+\int_{\partial\Delta(r)}\varrho d\pi_r\bigg)+O(1)\\ &\leq O\bigg(\log^+T_f(r,L)+\log^+\int_{\partial\Delta(r)}\varrho d\pi_r\bigg)+O(1).\end{aligned}$$

Using Theorem 5.2 and (5), for any $\delta>0$, there exists a subset $E_\delta\subseteq(0,\infty)$ of finite Lebesgue measure such that

$$\log^+\int_{\partial\Delta(r)}\varrho d\pi_r\leq O\big(\log^+T_f(r,L)+\delta\log r\big)$$

holds for all $r>0$ outside E_δ. Thus, we conclude that

$$\frac{1}{4}\int_{\Delta(r)}g_r(o,x)\Delta\log\xi dv\leq O\big(\log^+T_f(r,L)+\delta\log r\big)$$

for all $r>0$ outside E_δ. Combining this with (4), we prove the theorem. □

Let S be a compact Riemann surface of genus g. Let L_0 be the holomorphic line bundle over S defined by a point $a\in S$. Note that L_0 is independent of the choice of a. It yields from $c_1(K_S)=-(2+2g)c_1(L_0)$ that

Corollary 6.2. *Let $a_1,\cdots,a_q$ be distinct points in S Let $f:M\to S$ be a nonconstant meromorphic mapping. Then for any $\delta>0$, there exists a subset $E_\delta\subseteq(0,\infty)$ of finite Lebesgue measure such that*

$$(q-2-2g)T_f(r,L_0)+T(r,\mathscr{R})\leq\sum_{j=1}^q\overline{N}_f(r,a_j)+O\left(\log^+T_f(r,L_0)+\delta\log r\right)$$

holds for all $r>0$ outside E_δ.

Let $\mathscr{O}(1)$ be the hyperplane line bundle over $\mathbb{P}^n(\mathbb{C})$ and whose Chern form is given by ω_{FS}. It yields from $c_1(K_{\mathbb{P}^n(\mathbb{C})}) = -(n+1)c_1(\mathscr{O}(1))$ that

Corollary 6.3. *Let $H_1, \cdots, H_q$ be hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. Let $f : M \to \mathbb{P}^n(\mathbb{C})$ be a differentiably non-degenerate meromorphic mapping. Then for any $\delta > 0$, there exists a subset $E_\delta \subseteq (0, \infty)$ of finite Lebesgue measure such that*

$$(q-n-1)T_f(r, \mathscr{O}(1)) + T(r, \mathscr{R}) \leq \sum_{j=1}^{q} \overline{N}_f(r, H_j) + O\left(\log^+ T_f(r, \mathscr{O}(1)) + \delta \log r\right)$$

holds for all $r > 0$ outside E_δ.

6.2. **Defect Relation.**

For any divisor $D \in |L|$, the *defect* and *simple defect* of f with respect to D are defined respectively by

$$\delta_f(D) = 1 - \limsup_{r\to\infty} \frac{N_f(r, D)}{T_f(r, L)},$$
$$\bar{\delta}_f(D) = 1 - \limsup_{r\to\infty} \frac{\overline{N}_f(r, D)}{T_f(r, L)}.$$

Using the First Main Theorem, we see that

$$0 \leq \delta_f(D) \leq \bar{\delta}_f(D) \leq 1.$$

For two holomorphic line bundles L_1, L_2 over X, we define (see [7])

$$\left[\frac{c_1(L_2)}{c_1(L_1)}\right] = \inf\left\{s \in \mathbb{R} :\ \omega_2 \leq s\omega_1;\ \ ^\exists\omega_1 \in c_1(L_1),\ ^\exists\omega_2 \in c_1(L_2)\right\}.$$

By (2), Corollary 2.2 and Theorem 2.3, we obtain $T_f(r, L) \geq O(\log r)$ for a nonconstant meromorphic mapping f.

Corollary 6.4 (Defect Relation)**.** *Assume the same conditions as in Theorem* 6.1. *Then*

$$\bar{\delta}_f(D) \leq \left[\frac{c_1(K_X^*)}{c_1(L)}\right] - \liminf_{r\to\infty} \frac{T(r, \mathscr{R})}{T_f(r, L)}.$$

Endowing X with Kähler metric $c_1(L, h)$. Let $\|df\|$ be the Hilbert-Schmidt norm of differential df with respect to metrics $\alpha, c_1(L, h)$. Denote by $R_{\rm sc}$ the scalar curvature of M. Take a local holomorphic coordinate $(z_1, \ldots, z_m)$, the Ricci curvature tensor of M can be written as the form

$$\mathrm{Ric} = \sum_{i,j=1}^{m} R_{i\bar{j}} dz_i \otimes d\bar{z}_j,$$

where

$$R_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log\det(g_{i\bar{j}}).$$

Using the definition of scalar curvature, we have

$$R_{\rm sc} = \sum_{i,j=1}^{m} g^{i\bar{j}} R_{i\bar{j}} = -\frac{1}{2}\Delta \log\det(g_{i\bar{j}}).$$

In further, it yields that

$$\begin{aligned} T(r,\mathscr{R}) &= \frac{\pi^m}{(m-1)!}\int_{\Delta(r)} g_r(o,x)\mathscr{R}\wedge\alpha^{m-1} \\ &= -\frac{1}{4}\int_{\Delta(r)} g_r(o,x)\Delta\log\det(g_{i\bar{j}})dv \\ &= \frac{1}{2}\int_{\Delta(r)} g_r(o,x)R_{\rm sc}dv. \end{aligned}$$

Corollary 6.5. *Let $f: M \to X$ be a meromorphic mapping. If there exists a small number $\epsilon_0 > 0$ such that f satisfies the growth-curvature condition*

$$\frac{R_{\rm sc}(x)}{\|df(x)\|^2} \geq \frac{1}{2}\left[\frac{c_1(K_X^*)}{c_1(L)}\right] + \epsilon_0, \qquad {}^{\forall}x \in M \setminus \Delta(r_0)$$

for sufficiently large $r_0 > 0$, then f must be differentiably degenerate.

Proof. Assume on the contrary that f is differentiably non-degenerate. Then, we have $T_f(r,L) \to \infty$ as $r \to \infty$. The growth-curvature condition leads to

$$\begin{aligned} & \liminf_{r\to\infty} \frac{T(r,\mathscr{R})}{T_f(r,L)} \\ = {} & 2\liminf_{r\to\infty} \frac{\int_{\Delta(r)} g_r(o,x)R_{\rm sc}dv}{\int_{\Delta(r)} g_r(o,x)\|df\|^2dv} \\ = {} & 2\liminf_{r\to\infty} \frac{\int_{\Delta(r)\setminus\Delta(r_0)} g_r(o,x)R_{\rm sc}dv + \int_{\Delta(r_0)} g_r(o,x)R_{\rm sc}dv}{\int_{\Delta(r)\setminus\Delta(r_0)} g_r(o,x)\|df\|^2dv + \int_{\Delta(r_0)} g_r(o,x)\|df\|^2dv} \\ = {} & 2\liminf_{r\to\infty} \frac{\int_{\Delta(r)\setminus\Delta(r_0)} g_r(o,x)R_{\rm sc}dv}{\int_{\Delta(r)\setminus\Delta(r_0)} g_r(o,x)\|df\|^2dv} \\ \geq {} & \left[\frac{c_1(K_X^*)}{c_1(L)}\right] + 2\epsilon_0 \\ > {} & \left[\frac{c_1(K_X^*)}{c_1(L)}\right]. \end{aligned}$$

Thus, we obtain $\bar{\delta}_f(D) < 0$ due to Corollary 6.4, but which contradicts with $\bar{\delta}_f(D) \geq 0$. □

Treat a compact Riemann surface S of genus g, we have:

Corollary 6.6. *Let $f : M \to S$ be a nonconstant meromorphic mapping. Let $a_1, \cdots, a_q$ be distinct points in S. Then*

$$\sum_{j=1}^{q} \delta_f(a_j) \leq 2 - 2g - 2\liminf_{r\to\infty} \frac{\int_{\Delta(r)} g_r(o,x) R_{\mathrm{sc}} dv}{\int_{\Delta(r)} g_r(o,x)\|df\|^2 dv}.$$

For a meromorphic mapping

$$f = [f_0 : f_1 : \cdots : f_n] : \ M \to \mathbb{P}^n(\mathbb{C}),$$

we have

$$\|df\|^2 = \Delta \log \|f\|^2 := \Delta \log(|f_0|^2 + |f_1|^2 + \cdots + |f_n|^2)$$

with respect to metrics α, ω_{FS}.

Corollary 6.7. *Let $f : M \to \mathbb{P}^n(\mathbb{C})$ be a differentiably non-degenerate meromorphic mapping. Let $H_1, \cdots, H_q$ be hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. Then*

$$\sum_{j=1}^{q} \delta_f(H_j) \leq n + 1 - 2\liminf_{r\to\infty} \frac{\int_{\Delta(r)} g_r(o,x) R_{\mathrm{sc}} dv}{\int_{\Delta(r)} g_r(o,x)\Delta \log \|f\|^2 dv}.$$

A meromorphic function f on M can be seen as a meromorphic mapping $f = f_1/f_0 = [f_0 : f_1] : M \to \mathbb{P}^1(\mathbb{C})$. Set

$$\mu(f) = 2\liminf_{r\to\infty} \frac{\int_{\Delta(r)} g_r(o,x) R_{\mathrm{sc}} dv}{\int_{\Delta(r)} g_r(o,x)\Delta \log(|f_0|^2 + |f_1|^2) dv}.$$

We obtain a Picard's theorem:

Corollary 6.8 (Picard Theorem)**.** *Every meromorphic function on M must be a constant if it omits* $\max\{[3-\mu(f)], 0\}$ *distinct values, where* $[3-\mu(f)]$ *denotes the maximal integer not greater than* $3-\mu(f)$.

6.3. **Picard's Theorem.**

In the following, we investigate a more exact form of the Picard's theorem given in Corollary 6.8. On $\mathbb{P}^1(\mathbb{C})$, we introduce the spherical distance

$$\|a, b\| = \begin{cases} \frac{|a-b|}{\sqrt{1+|a|^2}\sqrt{1+|b|^2}}, & a \neq \infty, \ b \neq \infty; \\ \frac{1}{\sqrt{1+|a|^2}}, & a \neq \infty, \ b = \infty; \\ 0, & a = \infty, \ b = \infty. \end{cases}$$

Let f be a meromorphic function on M, which is regarded as a meromorphic mapping

$$f = \frac{f_1}{f_0} = [f_0 : f_1] : \ M \to \mathbb{P}^1(\mathbb{C}).$$

In terms of the spherical distance, the proximity function of f with respect to $a \in \mathbb{P}^1(\mathbb{C}) = \overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ can be defined as

$$m_{f,a}(r) = \int_{\partial\Delta(r)} \log \frac{1}{\|f,a\|} d\pi_r.$$

We use the Ahlfors-Shimizu's characteristic function of f:

$$T_f(r) = \frac{1}{4} \int_{\Delta(r)} g_r(o,x) \Delta \log(1+|f|^2) dv.$$

Moreover, we define

$$N_{f,a}(r) = N_f(r,a), \quad \overline{N}_{f,a}(r) = \overline{N}_f(r,a).$$

It is not hard to show the First Main Theorem:

$$T_f(r) + O(1) = m_{f,a}(r) + N_{f,a}(r).$$

Theorem 6.9 (Second Main Theorem)**.** *Let $a_1, \cdots, a_q$ be distinct values in $\overline{\mathbb{C}}$. Let f be a nonconstant meromorphic function on M. Then for any $\delta > 0$, there exists a subset $E_\delta \subseteq (0, \infty)$ of finite Lebesgue measure such that*

$$(q-2)T_f(r) + T(r, \mathscr{R}) \leq \sum_{j=1}^{q} \overline{N}_{f,a_j}(r) + O\left(\log^+ T_f(r) + \delta \log r\right)$$

holds for all $r > 0$ outside E_δ.

Proof. By

$$\Delta \log \|f\|^2 = \Delta \log(1+|f|^2) + \Delta \log |f_0|^2$$

in the sense of distributions, we deduce that

$$T_f(r) \leq T_f(r, \mathscr{O}(1)) = T_f(r) + N_{f,\infty}(r) \leq 2T_f(r) + O(1).$$

Combining this with Corollary 6.3, it gives immediately

$$(q-2)T_f(r) + T(r, \mathscr{R}) \leq \sum_{j=1}^{q} \overline{N}_{f,a_j}(r) + O\left(\log^+ T_f(r) + \delta \log r\right) \ \|_{E_\delta}.$$

□

We re-define the simple defect of f with respect to a by

$$\bar{\delta}_{f,a} = 1 - \liminf_{r \to \infty} \frac{\overline{N}_{f,a}(r)}{T_f(r)}.$$

Set

$$\nu(f) = \liminf_{r \to \infty} \frac{\int_{\Delta(r)} g_r(o,x) R_{\mathrm{sc}} dv}{\int_{\Delta(r)} g_r(o,x) \Delta \log(1+|f|^2) dv}.$$

Corollary 6.10 (Defect Relation). *Assume the same conditions as in Theorem* 6.9. *Then*

$$\sum_{j=1}^{q}\bar{\delta}_{f,a_j}\leq 2-2\nu(f).$$

Consequently, we derive a more exact Picard's theorem:

Corollary 6.11 (Picard's Theorem). *Every meromorphic function on M must be a constant if it omits* $\max\{[3-2\nu(f)],0\}$ *distinct values, where* $[3-2\nu(f)]$ *denotes the maximal integer not greater than* $3-2\nu(f)$.

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