## A NOTE ON CONTINUITY AND CONSISTENCY OF MEASURES OF RISK AND VARIABILITY

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ABSTRACT. In this short note, we show that every convex, order bounded above functional on a Banach lattice is automatically norm continuous. This improves a result in [21] and applies to many deviation and variability measures. We also show that an order-continuous, law-invariant functional on an Orlicz space is strongly consistent everywhere, extending a result in [18].

### 1. Automatic Continuity

Since its introduction in the landmark paper Artzner et al [3], the axiomatic theory of risk measures has been a fruitful area of research. Among many topics, one particular direction is to investigate automatic continuity of risk measures. In general, automatic continuity has long been an interesting research topic in mathematics and probably originates from the fact that a real-valued convex function on an open interval is continuous. This well-known fact was later extended to the following theorem for real-valued convex functionals on general Banach lattices.

**Theorem** (Ruszczyński and Shapiro [21]). A real-valued, convex, monotone functional on a Banach lattice is norm continuous.

Recall that a functional  $\rho$  on a vector space  $\mathcal{X}$  is said to be *convex* if  $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$  for any  $X, Y \in \mathcal{X}$  and any  $\lambda \in [0, 1]$ . Recall also that a *Banach lattice*  $\mathcal{X}$  is a real Banach space with a linear order that is compatible with norm, i.e.,  $|X| \leq |Y|$  in  $\mathcal{X}$  implies  $||X|| \leq ||Y||$  (see [2] for standard terminology and facts regarding Banach lattices). A functional  $\rho$  on a Banach lattice  $\mathcal{X}$  is said to be *increasing* if  $\rho(X) \leq \rho(Y)$  whenever  $X \leq Y$  in  $\mathcal{X}$ .  $\rho$  is said to be *decreasing* if  $-\rho$  is increasing.  $\rho$  is said to be *monotone* if it is either increasing or decreasing.

The above celebrated result of Ruszczyński and Shapiro has drawn extensive attention in optimization, operations research and risk management. We refer the reader to

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Biagini and Frittelli [7] for a version on Frechet lattices and Farkas, Koch-Medina and Munari [12] for further literature on automatic norm continuity properties.

With law invariance, other types of continuity properties beyond norm continuity can be established. The theorem below is striking. Recall first that a functional  $\rho$  is said to be *law invariant* if  $\rho(X) = \rho(Y)$  whenever X and Y have the same distribution. Recall also that a functional  $\rho$  on a set  $\mathcal{X}$  of random variables is said to have the *Fatou* property if  $\rho(X) \leq \liminf_n \rho(X_n)$  whenever  $X_n \xrightarrow{o} X$  in  $\mathcal{X}$ . Here  $X_n \xrightarrow{o} X$  in  $\mathcal{X}$ , termed as order convergence in  $\mathcal{X}$ , is used in the literature to denote dominated a.s. convergence in  $\mathcal{X}$ , i.e.,  $X_n \xrightarrow{a.s.} X$  and there exists  $Y \in \mathcal{X}$  such that  $|X_n| \leq Y$  a.s. for any  $n \in \mathbb{N}$ . The Fatou property is therefore just order lower semicontinuity.

**Theorem** (Jouini et al [17]). A real-valued, convex, monotone, law-invariant functional on  $L^{\infty}$  over a non-atomic probability space has the Fatou property. Consequently, it is  $\sigma(L^{\infty}, L^1)$  lower semicontinuous and admits a dual representation via  $L^1$ .

This theorem was recently extended by Chen et al [10] to general rearrangementinvariant spaces. See [10, Theorem 2.2., Theorem 4.3, Theorem 4.7] for details.

In this section, we aim at extending the above theorem of Ruszczyński and Shapiro on norm continuity of convex functionals. Specifically, we show that the monotonicity assumption can be significantly relaxed to the following notion on order boundedness.

**Definition 1.1.** Let  $\mathcal{X}$  be a Banach lattice. For  $U, V \in \mathcal{X}$  with  $U \leq V$ , the order interval [U, V] is defined by

$$[U,V] = \{ X \in \mathcal{X} : U \le X \le V \}.$$

A functional  $\rho : \mathcal{X} \to \mathbb{R}$  is said to be *order bounded above* if it is bounded above on all order intervals.

While risk measures are usually assumed to be monotone, many important functionals used in finance, insurance and other disciplines are not necessarily monotone.

**Example 1.2.** General deviation measures were introduced in Rockafellar et al [20] as convex functionals satisfying certain conditions. They are usually not monotone, but may be order bounded above. A specific example is standard deviation and semideviations. Recall that for a random variable  $X \in L^2$ , its standard deviation, upper and lower semideviations are given by

$$\sigma(X) = \|X - \mathbb{E}[X]\|_{L^2}, \ \sigma_+(X) = \|(X - \mathbb{E}[X])^+\|_{L^2}, \ \sigma_-(X) = \|(X - \mathbb{E}[X])^-\|_{L^2},$$

respectively. They are well known to be convex. It is also easy to see that they are not increasing or decreasing. We show that they are all order bounded above on  $L^2$ .

Indeed, take any order interval  $[U, V] \subset L^2$  and any  $X \in [U, V]$ . The desired order boundedness property is immediate by the following inequalities.

$$U - \mathbb{E}[V] \leq X - \mathbb{E}[X] \leq V - \mathbb{E}[U]$$
$$0 \leq (X - \mathbb{E}[X])^+ \leq (V - \mathbb{E}[U])^+$$
$$0 \leq (X - \mathbb{E}[X])^- \leq (U - \mathbb{E}[V])^+$$

**Example 1.3.** General variability measures were introduced in Bellini et al [4]. Many of them are also order bounded above, although usually not monotone. In fact, all the three one-parameter families of variability measures in [4, Section 2.3] are easily seen to be order bounded above but not monotone.

Our main result in this section is as follows.

**Theorem 1.4.** Let  $\mathcal{X}$  be a Banach lattice. Let  $\rho : \mathcal{X} \to \mathbb{R}$  be a convex, order bounded above functional. Then  $\rho$  is norm continuous.

It is obvious that a monotone functional is order bounded above. Hence, Theorem 1.4 includes the preceding theorem of Ruszczyński and Shapiro as a special case.

Proof of Theorem 1.4. Let  $(X_n)$  and X be such that  $||X_n - X|| \to 0$  in  $\mathcal{X}$ . We want to show that  $\rho(X_n) \to \rho(X)$ . Suppose otherwise that  $\rho(X_n) \not\to \rho(X)$ . By passing to a subsequence, we may assume that

(1.1) 
$$|\rho(X_n) - \rho(X)| > \varepsilon_0 \text{ for some } \varepsilon_0 > 0 \text{ and any } n \in \mathbb{N}.$$

Since  $||X_n - X|| \to 0$ , by passing to a further subsequence, we may assume that

$$||X_n - X|| \le \frac{1}{n2^n}$$
 for any  $n \in \mathbb{N}$ .

Then  $\sum_{n=1}^{\infty} ||n|X_n - X||| < \infty$  so that  $Y := \sum_{n=1}^{\infty} n|X_n - X|$  exists in  $\mathcal{X}$ . Note that (1.2)  $|X_n - X| \le \frac{1}{n}Y$  for any  $n \in \mathbb{N}$ .

Moreover, since  $\rho$  is order bounded above on [X - Y, X + Y] = X + [-Y, Y], there exists a real number M > 0 such that

(1.3) 
$$\rho(X+Z) \le M$$
 for any  $Z \in [-Y,Y]$ , i.e., whenever  $|Z| \le Y$ .

Now fix any  $\varepsilon > 0$ . Put  $N = \lfloor \frac{1}{\varepsilon} \rfloor + 1$ . By (1.2),

(1.4) 
$$\frac{1}{\varepsilon}|X_n - X| \le Y \quad \text{for any } n \ge N.$$

On one hand, by the convexity of  $\rho$  and the following identity

$$X_n = (1 - \varepsilon)X + \varepsilon \left(X + \frac{1}{\varepsilon}(X_n - X)\right),$$

we have

$$\rho(X_n) \leq (1-\varepsilon)\rho(X) + \varepsilon\rho\left(X + \frac{1}{\varepsilon}(X_n - X)\right),$$

implying that

$$\rho(X_n) - \rho(X) \le \varepsilon \Big( \rho \Big( X + \frac{1}{\varepsilon} (X_n - X) \Big) - \rho(X) \Big).$$

This together with (1.4) and (1.3) implies that

(1.5) 
$$\rho(X_n) - \rho(X) \le \varepsilon \left( M - \rho(X) \right) \quad \text{for any } n \ge N.$$

On the other hand, by the convexity of  $\rho$  and

$$2X - X_n = (1 - \varepsilon)X + \varepsilon \left(X + \frac{1}{\varepsilon}(X - X_n)\right),$$

we have as before that

$$\rho(2X - X_n) \leq (1 - \varepsilon)\rho(X) + \varepsilon\rho\left(X + \frac{1}{\varepsilon}(X - X_n)\right)$$
$$\leq (1 - \varepsilon)\rho(X) + \varepsilon M,$$

for any  $n \geq N$ . In particular,

$$\rho(2X - X_n) - \rho(X) \le \varepsilon(M - \rho(X)) \quad \text{for any } n \ge N.$$

By  $X = \frac{1}{2}X_n + \frac{1}{2}(2X - X_n)$  and the convexity of  $\rho$ , we also get

$$\rho(X) \le \frac{1}{2}\rho(X_n) + \frac{1}{2}\rho(2X - X_n)$$

so that

$$\rho(X) - \rho(X_n) \le \rho(2X - X_n) - \rho(X).$$

It follows that

(1.6) 
$$\rho(X) - \rho(X_n) \le \varepsilon(M - \rho(X)) \quad \text{for any } n \ge N.$$

Combining (1.5) and (1.6), we have

$$|\rho(X) - \rho(X_n)| \le \varepsilon (M - \rho(X))$$
 for any  $n \ge N$ .

Hence,  $\rho(X_n) \to \rho(X)$ . This contradicts (1.1) and completes the proof.

**Remark 1.5.** Theorem 1.4 is also valid on Frechet lattices. The major technical difference lies in constructing Y satisfying (1.2). This can be achieved using the techniques in the proof of Biagini and Frittelli [7, Theorem 1].

#### 2. Strong Consistency

In this section, we discuss the strong consistency of estimating the risk  $\rho(X)$  using historical data or Monte Carlo simulations.

Throughout this section, fix a nonatomic probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Let  $L^0$  be the space of all random variables on  $\Omega$ , with a.s. equal random variables identified as the same. Let  $\mathcal{X}$  be a subset of  $L^0$ . Denote the set of distributions of all random variables in  $\mathcal{X}$  by

$$\mathcal{M}(\mathcal{X}) = \{ \mathbb{P} \circ X^{-1} : X \in \mathcal{X} \}.$$

Recall that a law-invariant functional  $\rho$  on  $\mathcal{X}$  induces a natural mapping  $\mathcal{R}_{\rho}$  on  $\mathcal{M}(\mathcal{X})$  by

$$\mathcal{R}_{\rho}(\mathbb{P} \circ X^{-1}) = \rho(X), \quad \text{for any } X \in \mathcal{X}$$

Let  $\mathcal{X}$  be a subset of  $L^0$  containing  $L^{\infty}$ . Take any  $X \in \mathcal{X}$ . Let  $(X_n)$  be a stationary and ergodic sequence of random variables with the same distribution as X (see [9, Section 6.7] for the definition of a stationary and ergodic process). We denote the *empirical distribution* of X arising from  $X_1, \ldots, X_n$  by

$$\widehat{m}_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i};$$

here  $\delta_x$  is the Dirac measure on  $\mathbb{R}$  at x. Since  $L^{\infty} \subset \mathcal{X}$ ,  $\widehat{m}_n \in \mathcal{M}(L^{\infty}) \subset \mathcal{M}(\mathcal{X})$ . This allows us to consider the corresponding *empirical estimate* for  $\rho(X)$ :

$$\widehat{\rho}_n := \mathcal{R}_{\rho}(\widehat{m}_n);$$

We say that  $\rho$  is *strongly consistent* at X if for any stationary and ergodic sequence of random variables with the same distribution as X,

$$\widehat{\rho}_n = \mathcal{R}_{\rho}(\widehat{m}_n) \xrightarrow{a.s.} \rho(X)$$

We refer to Krätschmer, Schied and Zähle [18, 19] and the references therein for literature on strong (and weak) consistency of risk measures. In particular, the proof of [18, Theorem 2.6] gives the following result.

**Theorem** (Krätschmer et al [18]). A norm-continuous, law-invariant functional on an Orlicz heart is strongly consistent everywhere.

Let's first recall the definitions of Orlicz spaces and hearts. A function  $\Phi : [0, \infty) \rightarrow [0, \infty)$  is called an *Orlicz function* if it is non-constant, convex, increasing, and  $\Phi(0) = 0$ . The *Orlicz space*  $L^{\Phi}$  is the space of all  $X \in L^0$  such that the *Luxemburg norm* is finite:

$$|X||_{\Phi} := \inf\left\{\frac{1}{\lambda} > 0 : \mathbb{E}\left[\Phi(\lambda|X|)\right] \le 1\right\} < \infty.$$

The Orlicz heart  $H^{\Phi}$  is a subspace of  $L^{\Phi}$  defined by

$$H^{\Phi} := \left\{ X \in L^0 : \mathbb{E} \left[ \Phi(\lambda |X|) \right] < \infty \text{ for any } \lambda > 0 \right\}.$$

We refer to [11] for standard terminology and facts on Orlicz spaces. Risk measures on Orlicz spaces have been studied extensively; see, e.g., [5, 6, 8, 13, 14, 15, 16] and the references therein.

The above theorem in conjunction with Theorem 1.4 immediately yields the following result, which improves [18, Theorem 2.6].

**Corollary 2.1.** A convex, law-invariant, order bounded above functional on an Orlicz heart is strongly consistent everywhere.

The above theorem of Krätschmer et al is essentially due to the fact that for any  $X \in H^{\Phi}$ , and for a.e.  $\omega \in \Omega$ , there exist a random variable  $X^{\omega}$  on  $\Omega$  with same distribution as X and a sequence of random variables  $(X_n^{\omega})$  on  $\Omega$  with distributions  $\widehat{m}_n(\omega)$ 's such that

(2.1) 
$$||X_n^{\omega} - X^{\omega}||_{\Phi} \to 0.$$

This, however, does not hold for arbitrary random variables in a general Orlicz space  $L^{\Phi}$ . Specifically, when  $\Phi$  fails the  $\Delta_2$ -condition, there exists  $X \in L^{\Phi} \setminus H^{\Phi}$ . For this X, (2.1) must fail:  $X_n^{\omega}$  takes only at most n values and thus is a simple random variable lying in  $H^{\Phi}$ ; therefore, (2.1) would imply  $X \in H^{\Phi}$  as well.

We extend the theorem of Krätschmer et al as follows. Recall first that on a set  $\mathcal{X} \subset L^0$ , a functional  $\rho : \mathcal{X} \to \mathbb{R}$  is said to be *order continuous* at  $X \in \mathcal{X}$  if  $\rho(X_n) \to \rho(X)$  whenever  $X_n \xrightarrow{o} X$  in  $\mathcal{X}$ . In the literature, order continuity is also termed as the *Lebesgue property*.

**Theorem 2.2.** An order-continuous, law-invariant functional on an Orlicz space is strongly consistent everywhere.

For the proof of Theorem 2.2, we need to establish a few technical lemmas, which along the way also reveal why order continuity is the most natural condition for general Orlicz spaces. For an Orlicz function  $\Phi$ , the Young class is defined by

$$Y^{\Phi} := \left\{ X \in L^{0} : \mathbb{E}\left[\Phi\left(|X|\right)\right] < \infty \right\}.$$

It is easy to see that  $H^{\Phi} \subset Y^{\Phi} \subset L^{\Phi}$ . As in [18], we use the term  $\Phi$ -weak topology in place of the  $\Phi(|\cdot|)$ -weak topology on  $\mathcal{M}(Y^{\Phi})$  for brevity. This topology is metrizable. Moreover, for a sequence  $(\mu_n) \subset \mathcal{M}(Y^{\Phi})$  and  $\mu_0 \in \mathcal{M}(Y^{\Phi})$ ,  $(\mu_n)$  converges  $\Phi$ -weakly to  $\mu_0$ , written as  $\mu_n \xrightarrow{\Phi$ -weakly}{} \mu\_0, iff

$$\mu_n \xrightarrow{\text{weakly}} \mu \text{ and } \int \Phi(|x|)\mu_n(dx) \to \int \Phi(|x|)\mu_0(dx).$$

The following Skorohod representation for  $\Phi$ -weak convergence is a general order version of [18, Theorem 3.5] and [19, Theorem 6.1] beyond the Orlicz heart and without any restrictions on  $\Phi$ .

- **Lemma 2.3.** (i) Let  $(\mu_n)$  be a sequence in  $\mathcal{M}(Y^{\Phi})$  that converges  $\Phi$ -weakly to some  $\mu_0 \in \mathcal{M}(Y^{\Phi})$ . Then there exist a subsequence  $(\mu_{n_k})$  of  $(\mu_n)$ , a sequence  $(X_k)$  in  $Y^{\Phi}$  and  $X \in Y^{\Phi}$  such that  $X_k$  has distribution  $\mu_{n_k}$  for each  $k \in \mathbb{N}$ , Xhas distribution  $\mu_0$ , and  $X_k \xrightarrow{o} X$  in  $Y^{\Phi}$ .
  - (ii) Let  $(X_n)$  in  $Y^{\Phi}$  and  $X \in Y^{\Phi}$  be such that  $X_n \xrightarrow{o} X$  in  $Y^{\Phi}$ . Then  $\mu_n \xrightarrow{\Phi-weakly} \mu_0$ , where  $\mu_n$ 's are the distributions of  $X_n$ 's and  $\mu_0$  is the distribution of X, respectively.

*Proof.* We start with the following observation. Since  $\Phi$  is continuous and increasing, for any sequence  $(X_n)$  in  $Y^{\Phi}$  we have

(2.2) 
$$\mathbb{E}\left[\Phi\left(\sup_{n\in\mathbb{N}}|X_n|\right)\right] = \mathbb{E}\left[\sup_{n\in\mathbb{N}}\Phi(|X_n|)\right]$$

(i). Take  $(\mu_n)$  in  $\mathcal{M}(Y^{\Phi})$  that converges  $\Phi$ -weakly to  $\mu_0 \in \mathcal{M}(Y^{\Phi})$ . Since the probability space is nonatomic, the classical Skorohod representation yields  $(X_n) \subset Y^{\Phi}$  and  $X \in Y^{\Phi}$  such that  $X_n \sim \mu_n$  for every  $n \in \mathbb{N}, X \sim \mu_0$ , and  $X_n \xrightarrow{a.s.} X$ . Clearly,

(2.3) 
$$\mathbb{E}[\Phi(|X|)] = \int \Phi(|x|)\mu_0(dx) = \lim_n \int \Phi(|x|)\mu_n(dx) = \lim_n \mathbb{E}[\Phi(|X_n|)] < \infty.$$

Since  $\Phi$  is continuous, we also have that  $\Phi(|X_n|) \xrightarrow{a.s.} \Phi(|X|)$ . This combined with (2.3) yields (see [1, Theorem 31.7]) that

$$\left\| \Phi(|X_n|) - \Phi(|X|) \right\|_{L^1} \to 0.$$

Passing to a subsequence we may assume that

$$\sum_{n=1}^{\infty} \left\| \Phi(|X_n|) - \Phi(|X|) \right\|_{L^1} < \infty$$

so that

$$\sum_{n=1}^{\infty} \left| \Phi(|X_n|) - \Phi(|X|) \right| \in L^1.$$

In particular,

$$\sup_{n \in \mathbb{N}} \left| \Phi(|X_n|) - \Phi(|X|) \right| \in L^1.$$

It follows from  $\Phi(|X_n|) \leq |\Phi(|X_n|) - \Phi(|X|)| + \Phi(|X|)$  that  $\sup_{n \in \mathbb{N}} \Phi(|X_n|) \in L^1$ . Hence, by (2.2),  $\mathbb{E}[\Phi(\sup_{n \in \mathbb{N}} |X_n|)] = \mathbb{E}[\sup_{n \in \mathbb{N}} \Phi(|X_n|)] < \infty$ . That is,  $\sup_{n \in \mathbb{N}} |X_n| \in Y^{\Phi}$ ; equivalently,  $(X_n)$  is dominated in  $Y^{\Phi}$ . In particular, we have  $X_n \xrightarrow{o} X$  in  $Y^{\Phi}$  (ii). Let  $(X_n)$  be such that  $X_n \xrightarrow{o} X$  in  $Y^{\Phi}$  and  $\mu_n$  be the distribution of  $X_n$  for each  $n, \mu_0$  be the distribution of X. We clearly have  $\mu_n \xrightarrow{\text{weakly}} \mu$  and by the continuity of  $\Phi$ , we get  $\Phi(|X_n|) \xrightarrow{a.s.} \Phi(|X|)$ . Since  $(X_n)$  is dominated in  $Y^{\Phi}$ ,  $\sup_{n \in \mathbb{N}} |X_n| \in Y^{\Phi}$ . Thus in view of (2.2), we get

$$\mathbb{E}\left[\sup_{n\in\mathbb{N}}\Phi(|X_n|)\right] = \mathbb{E}\left[\Phi\left(\sup_{n\in\mathbb{N}}|X_n|\right)\right] < \infty,$$

i.e.,  $\sup_{n\in\mathbb{N}} \Phi(|X_n|) \in L^1$ . By the dominated convergence theorem, we get

$$\int \Phi(|x|)\mu_0(dx) = \mathbb{E}[\Phi(|X|)] = \lim_n \mathbb{E}[\Phi(|X_n|)] = \lim_n \int \Phi(|x|)\mu_n(dx).$$

This proves  $\mu_n \xrightarrow{\Phi\text{-weakly}} \mu_0$ .

The lemma below reveals the essential and natural role of order continuity.

# **Lemma 2.4.** Let $\rho: Y^{\Phi} \to \mathbb{R}$ be law invariant. The following are equivalent.

- (i)  $\mathcal{R}_{\rho}$  is continuous on  $\mathcal{M}(Y^{\Phi})$  with the  $\Phi$ -weak topology.
- (ii)  $\rho$  is order continuous on  $Y^{\Phi}$ .

Proof. (ii)  $\implies$  (i). Suppose that (ii) holds but (i) fails. Recall that the  $\Phi$ -weak topology is metrizable. Thus we can find a sequence  $(\mu_n)$  and  $\mu_0$  in  $\mathcal{M}(Y^{\Phi})$  such that  $\mu_n \xrightarrow{\Phi\text{-weakly}} \mu_0$  but  $\mathcal{R}_{\rho}(\mu_n) \not\rightarrow \mathcal{R}_{\rho}(\mu_0)$ . Passing to a subsequence, we may assume that

(2.4) 
$$|\mathcal{R}_{\rho}(\mu_n) - \mathcal{R}_{\rho}(\mu_0)| \ge \varepsilon_0,$$

for some  $\varepsilon_0 > 0$  and all  $n \in \mathbb{N}$ . By Lemma 2.3(i), there exist a subsequence  $(\mu_{n_k})$  of  $(\mu_n)$ , a sequence  $(X_k)$  in  $Y^{\Phi}$  and  $X \in Y^{\Phi}$  such that  $X_k$  has distribution  $\mu_{n_k}$  for each  $k \in \mathbb{N}$ , X has distribution  $\mu_0$ , and  $X_k \xrightarrow{o} X$  in  $Y^{\Phi}$ . (ii) implies that

$$\mathcal{R}_{\rho}(\mu_{n_k}) = \rho(X_k) \to \rho(X) = \mathcal{R}_{\rho}(\mu_0)$$

This contradicts (2.4) and proves (ii)  $\implies$  (i). The reverse implication (i)  $\implies$  (ii) is immediate by Lemma 2.3(ii).

We now present the proof of Theorem 2.2.

Proof of Theorem 2.2. Suppose that  $\rho: L^{\Phi} \to \mathbb{R}$  is law invariant and order continuous. Take any  $X \in L^{\Phi}$  and any stationary and ergodic sequence of random variables with the same distribution as X. Denote by  $\mu_0$  their distribution. Let  $\lambda > 0$  be such that  $\mathbb{E}[\Phi(\lambda|X|)] < \infty$ . Put  $\Phi_{\lambda}(\cdot) := \Phi(\lambda \cdot)$ . Arguing similarly as in the proof of [18, Theorem 2.6], by applying Birkhoff's ergodic theorem, one obtains a measurable subset  $\Omega_0$  of  $\Omega$  such that  $\mathbb{P}(\Omega_0) = 1$  and for every  $\omega \in \Omega_0$ ,

$$\widehat{m}_n(\omega) \xrightarrow{\Phi_\lambda \text{-weakly}} \mu_0$$

Since  $\rho$  is order continuous on  $L^{\Phi}$  and  $Y^{\Phi_{\lambda}} \subset L^{\Phi}$ ,  $\rho$  is also order continuous on  $Y^{\Phi_{\lambda}}$ . By Lemma 2.4,  $\mathcal{R}_{\rho}$  is continuous on  $\mathcal{M}(Y^{\Phi_{\lambda}})$  with the  $\Phi_{\lambda}$ -weak topology. Thus  $\hat{\rho}_{n}(\omega) = \mathcal{R}_{\rho}(\hat{m}_{n}(\omega)) \to \mathcal{R}_{\rho}(\mu_{0}) = \rho(X)$  for every  $\omega \in \Omega_{0}$ . This proves that  $\rho$  is strongly consistent at X.

We end this note with the following remark that improves the implication (b)  $\implies$  (a) in [18, Theorem 2.8] due to our Theorem 1.4.

**Corollary 2.5.** Suppose that  $\Phi$  satisfies the  $\Delta_2$ -condition. Let  $\rho$  be any convex, lawinvariant, order bounded above functional on  $L^{\Phi}$ . Then  $\mathcal{R}_{\rho}$  is continuous on  $\mathcal{M}(L^{\Phi})$ for the  $\Phi$ -weak topology.

Proof. By Theorem 1.4,  $\rho$  is norm continuous. When  $\Phi$  satisfies the  $\Delta_2$ -condition, order convergence in  $L^{\Phi}$  implies norm convergence. Thus  $\rho$  is also order continuous. Under the  $\Delta_2$ -condition, we also have  $H^{\Phi} = Y^{\Phi} = L^{\Phi}$ . Now apply Lemma 2.4.

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