# QUANTIZATION OF CANTOR-LIKE SET ON THE REAL PROJECTIVE LINE 

A. HOSSAIN, A. BANERJEE, AND MD. N. AKHTAR


#### Abstract

In this article, an iterated function system (IFS) is considered on the real projective line $\mathbb{R P}^{1}$ so that the attractor is a Cantor-like set. Hausdorff dimension of this attractor is estimated. The existence of a probability measure associated with this IFS on $\mathbb{R P}^{1}$ is also demonstrated. It is shown that the $n$-th quantization error of order $r$ for the push-forward measure is a constant multiple of the $n$-th quantization error of order $r$ of the original measure. Finally, an upper bound for the $n$-th quantization error of order 2 for this measure is provided.


Keywords: Real projective line, Real projective iterated function system, Probability measure, Quantization error.

MSC Classification 37A50, 28A80

## 1. INTRODUCTION

Given a probability measure on a measurable space, the quantization process involves finding a discrete set of points in the space, each point associated with a probability, such that the resulting discrete probability measure is the close approximation of the original one. Quantization error is the difference between the continuous probability measure and its discretized representation. Graf-Luschgy [7], studied the $n$-th quantization error for an invariant probability measure. Also, Roychowdhury [9], estimated the quantization dimension for the self-similar measure using the quantization error of this measure. Most of the authors studied the quantization theory in Euclidean spaces [6, 10]. Barany et al. [2] studied the Furstenberg measure, which plays an important role in information theory. The authors used the Lyapunov exponents to determine the upper bound for the Hausdorff dimension of the Furstenberg measure on the real projective space. Also, Jurga et al. [4], studied the dimension of the attractor of an iterated function system induced by the projective action on the real projective line. In particular, they generalized a recent result of Solomyak and Takahashi [11] by showing that the Hausdorff dimension of the attractor is given by the minimum of 1 and the critical exponent.

The objective of this article is to explore an Iterated Function System (IFS) operating on the real projective line and investigate the quantization theory concerning the probability measure associated with this IFS. To achieve this, we consider a RPIFS on the real projective line $\mathbb{R} \mathbb{P}^{1}$ so that it has an attractor. Then, employing methodologies outlined in prior works such as [4, 11], we estimate the Hausdorff dimension of this attractor. Also, we prove
the existence of a probability measure associated with this RPIFS on $\mathbb{R P}^{1}$. Furthermore, we demonstrate that the quantization error of order $r$ for the push-forward measure is a constant multiple of the quantization error of order $r$ of the original measure. We end the chapter by providing an upper bound of $n$-th quantization error of order 2 for this probability measure.

## 2. Preliminaries

In this section, we give basic definitions of the real projective line, generating cone, and projective metric and introduce some notations related to this article.
2.1. Real projective line. Recall the definition of the real projective line, which is denoted by $\mathbb{R} \mathbb{P}^{1}$, is the quotient of the set $\mathbb{R}^{2} \backslash\{(0,0)\}$ of non-zero vectors by the equivalence relation " $x \sim y$ if and only if $x=c y$ for some $c \in \mathbb{R}^{*}$ (non-zero reals)". The real projective line may be identified with the line $\mathbb{R}$ extended by a point at infinity. More precisely, the line $\mathbb{R}$ may be identified with the subset of $\mathbb{R} \mathbb{P}^{1}$ given by

$$
\mathbb{R P}^{*}=\{[x: 1]: x \in \mathbb{R}\}
$$

This subset covers all points of $\mathbb{R} \mathbb{P}^{1}$ except one, which is the point at infinity, $\infty:=[1: 0]$. Thus

$$
\mathbb{R} \mathbb{P}^{1}=\mathbb{R} \mathbb{P}^{*} \cup\{\infty\}
$$

For $\left[x_{1}: 1\right],\left[x_{2}: 1\right] \in \mathbb{R} \mathbb{P}^{*}$, define

$$
\begin{aligned}
& {\left[x_{1}: 1\right] \oplus\left[x_{2}: 1\right]=\left[x_{1}+x_{2}: 1\right]} \\
& {\left[x_{1}: 1\right] \star\left[x_{2}: 1\right]=\left[x_{1} x_{2}: 1\right]}
\end{aligned}
$$

and the scalar multiplication of an element $[x: 1] \in \mathbb{R} \mathbb{P}^{*}$ with $c \in \mathbb{R}$ is defined by $c \odot[x:$ $1]=[c x: 1]$. The difference between two elements $\left[x_{1}: 1\right],\left[x_{2}: 1\right] \in \mathbb{R} \mathbb{P}^{*}$ is defined by

$$
\left[x_{1}: 1\right] \ominus\left[x_{2}: 1\right]=\left[x_{1}-x_{2}: 1\right] .
$$

Definition 2.1.1 (Projective metric on $\left.\mathbb{R P}^{*}\right)$. For $\left[x_{1}: 1\right],\left[x_{2}: 1\right] \in \mathbb{R} \mathbb{P}^{*}$, define a metric $d_{\mathbb{P}}$ on $\mathbb{R} \mathbb{P}^{*}$ as follows:

$$
\begin{equation*}
d_{\mathbb{P}}\left(\left[x_{1}: 1\right],\left[x_{2}: 1\right]\right):=\left|x_{1}-x_{2}\right| \tag{1}
\end{equation*}
$$

Definition 2.1.2 (Generating cone). Let $[x],[y] \in \mathbb{R P}^{*}$. Then the point $[x],[y]$ generates two line segments in $\mathbb{R P}^{1}$. We consider a segment that does not intersect the point at infinity $[1: 0]$, and we denote it by $C_{\overline{x y}}$, and call it cone generated by $[x]$, $[y]$ (see Figure 11. A cone $C$ is said to be multi-cone if $C$ is the disjoint union of finite numbers of generating cones.

Definition 2.1.3 (Real projective iterated function system (RPIFS) on $\mathbb{R} \mathbb{P}^{1}$ ). Given a finite set $\mathcal{P} \subset G L(2, \mathbb{R})$, the associated RPIFS is denoted by $\mathscr{W}_{\mathcal{P}}=\left\{\mathbb{R}^{1} \mathbb{P}^{1} ; w_{A}: A \in \mathcal{P}\right\}$, where the projective transformations $w_{A}: \mathbb{R}^{P^{1}} \rightarrow \mathbb{R} \mathbb{P}^{1}$ are given by $w_{A}[x]=[A x]$.


Figure 1. Cone generated by the points $[x],[y]$ in $\mathbb{R} \mathbb{P}^{*}$.
Definition 2.1.4 (Oriented RPIFS). The RPIFS $\mathscr{W}_{\mathcal{P}}=\left\{\mathbb{R}^{1} ; w_{A}: A \in \mathcal{P}\right\}$ is said to be orientation preserving if $\mathcal{P} \subset G L^{+}(2, \mathbb{R})=\{A \in G L(2, \mathbb{R}): \operatorname{det}(A)>0\}$ or $\mathcal{P} \subset$ $G L^{-}(2, \mathbb{R})=\{A \in G L(2, \mathbb{R}): \operatorname{det}(A)<0\}$. We denotes the corresponding RPIFS by $\mathscr{W}_{\mathcal{P}}^{+}$ or $\mathscr{W}_{\mathcal{P}}^{-}$if $\mathcal{P} \subset G L^{+}(2, \mathbb{R})$ or $\mathcal{P} \subset G L^{-}(2, \mathbb{R})$ respectively.

For simplicity, we assume that $\mathcal{P} \subset G L^{+}(2, \mathbb{R})$. Then the action of $G L^{+}(2, \mathbb{R})$ factors through the $S L(2, \mathbb{R})$ action, via $A \rightarrow \frac{A}{\sqrt{\operatorname{det}(A)}}$. Hence it is enough to work on $S L(2, \mathbb{R})$. In this chapter, we assume that $\mathcal{P} \subset S L(2, \mathbb{R})$ and we use the same notation $\mathscr{W}_{\mathcal{P}}^{+}$for this case. We denote $\mathcal{P}^{n}$, as all the products of $n$ matrices from $\mathcal{P}$. Then $\mathcal{P}^{*}:=\bigcup_{n=1}^{\infty} \mathcal{P}^{n}$ form a semi-group generated by $\mathcal{P}$. Given a matrix $A=\left(a_{i j}\right) \in \mathcal{P}$, define

$$
\|A\|=\max _{i j}\left\{\left|a_{i j}\right|\right\}
$$

Definition 2.1.5. Let $\mathcal{P} \subset S L(2, \mathbb{R})$ be finite. Then $\mathcal{P}$ is called semi-discrete if $I d \notin \overline{\mathcal{P}^{*}}$, where the closure is taken over $S L(2, \mathbb{R})$.

Let $\mathcal{P}=\left\{A_{i}: i \in \mathcal{I}\right\} \subset S L(2, \mathbb{R})$, where $\mathcal{I}=\{1,2, \ldots, m\}$. Write $A_{\mathbf{i}}=A_{i_{1}} A_{i_{2}} \cdots A_{i_{n}}$ for $\mathbf{i}=i_{1} i_{2} \cdots i_{n} \in \mathcal{I}^{n}$.

Definition 2.1.6. Let $\mathcal{P}=\left\{A_{i}: i \in \mathcal{I}\right\}$ be a finite collection of matrices in $S L(2, \mathbb{R})$ and $d$ be a left-invariant Riemannian metric on $S L(2, \mathbb{R})$. Then the set $\mathcal{P}$ is said to be Diophantine if there exists $c>0$ such that for all $n \in \mathbb{N}$, we have

$$
\begin{equation*}
\mathbf{i}, \mathbf{j} \in \mathcal{I}^{n}, \quad A_{\mathbf{i}} \neq A_{\mathbf{j}} \Longrightarrow d\left(A_{\mathbf{i}}, A_{\mathbf{j}}\right)>c^{n} . \tag{2}
\end{equation*}
$$

The set $\mathcal{P}$ is said to be strongly Diophantine if there exists $c>0$ such that for all $n \in \mathbb{N}$, we have

$$
\begin{equation*}
\mathbf{i}, \mathbf{j} \in \mathcal{I}^{n}, \quad \mathbf{i} \neq \mathbf{j} \Longrightarrow d\left(A_{\mathbf{i}}, A_{\mathbf{j}}\right)>c^{n} . \tag{3}
\end{equation*}
$$

Definition 2.1.7. A finite set $\mathcal{P} \subset S L(2, \mathbb{R})$ is called uniformly hyperbolic if there exists $\lambda>1$ and a constant $c>0$, such that

$$
\begin{equation*}
\|A\| \geq c \lambda^{n}, \quad \text { for all } A \in \mathcal{P}^{n} \text { and } n \in \mathbb{N} \tag{4}
\end{equation*}
$$

The next theorem can be obtained by the results from [1], [3] and [4].
Theorem 2.1.1. The following statements are equivalent.
(1) The RPIFS $\mathscr{W}_{\mathcal{P}}$ has an attractor $F_{\mathcal{P}}$ that avoids a hyperplane.
(2) $\mathcal{P}$ is uniformly hyperbolic.
(3) There is a non-empty open set $V \subset \mathbb{R}^{1}$ such that $W_{\mathcal{P}}$ is contractive on $\bar{V}$.
(4) There exits a multi-cone $C$ such that $W_{\mathcal{P}}(\overline{\operatorname{int}(C)}) \subsetneq \operatorname{int}(C)$.

Given a finite or a countable set $\mathcal{P}=\left\{A_{i}: i \in \mathcal{I}\right\} \subset S L(2, \mathbb{R})$, define the zeta function $\zeta_{\mathcal{P}}:[0, \infty] \rightarrow \mathbb{R} \cup\{\infty\}$ by

$$
\begin{equation*}
\zeta_{\mathcal{P}}(t):=\sum_{n=1}^{\infty} \sum_{\mathbf{i} \in \mathcal{I}^{n}}\left(\left\|A_{\mathbf{i}}\right\|\right)^{-2 t} \tag{5}
\end{equation*}
$$

and its critical exponent

$$
\begin{equation*}
\xi_{\mathcal{P}}:=\inf \left\{t>0: \zeta_{\mathcal{P}}(t)<\infty\right\} . \tag{6}
\end{equation*}
$$

If $\zeta_{\mathcal{P}}(t)$ is divergent for all $t \geq 0$, then define $\xi_{\mathcal{P}}=\infty$.
Theorem 2.1.2 ( $[4])$. Let $\mathcal{P}=\left\{A_{i}: i \in \mathcal{I}\right\}$ be a finite collection of matrices in $S L(2, \mathbb{R})$ which is Diophantine and semi-discrete. If the attractor $F_{\mathcal{P}}$ of the corresponding RPIFS $\mathscr{W}_{\mathcal{P}}^{+}$is not singleton, then

$$
\operatorname{dim}_{H}\left(F_{\mathcal{P}}\right)=\min \left\{1, \xi_{\mathcal{P}}\right\}
$$

where $\operatorname{dim}_{H}$ denotes the Hausdorff dimension.
Definition 2.1.8. A set $\mathcal{P} \subset S L(2, \mathbb{R})$ is said to be strongly irreducible, if each map in $\mathscr{W}_{\mathcal{P}}^{+}$does not preserve any finite subset of $\mathbb{R P}^{1}$. A set $\mathcal{P} \subset S L(2, \mathbb{R})$ is said to be irreducible if each map in $\mathscr{W}_{\mathcal{P}}^{+}$does not have a common fixed point in $\mathbb{R P}^{1}$.

Given a finite set $\mathcal{P}=\left\{A_{i}: i \in \mathcal{I}\right\} \subset S L(2, \mathbb{R})$ and non-degenerate probability vector $\left(p_{i}\right)_{i \in \mathcal{I}}$, one can consider the probability measure $\mu$ on $S L(2, \mathbb{R})$ whose support is $\mathcal{P}$ as follows:

$$
\begin{equation*}
\mu=\sum_{i \in I} p_{i} \mathcal{X}_{A_{i}}, \tag{7}
\end{equation*}
$$

where $\mathcal{X}_{A}$ denotes the characteristic function on $S L(2, \mathbb{R})$.
Theorem 2.1.3 ( $[5,|8|)$. If $\mathcal{P}=\left\{A_{i}: i \in \mathcal{I}\right\} \subset S L(2, \mathbb{R})$ is strongly irreducible and generates an unbounded semi-group, then there exists a unique probability measure $P$ on $\mathbb{R P}^{1}$ such that

$$
P=\sum_{i \in \mathcal{I}} p_{i} P \circ A_{i}^{-1} .
$$

Definition 2.1.9 (Bernoulli measure). Let $\mathcal{I}^{*}=\{1,2, \ldots, m\}^{\mathbb{N}}$ be the sequence space and for $\mathbf{i}=\left(i_{1}, i_{2}, \ldots\right) \in \mathcal{I}^{*}$, define $\left.\mathbf{i}\right|_{n}=\left(i_{1}, i_{2}, \ldots, i_{n}\right)$. Then the cylinder set on $\mathcal{I}^{*}$, is defined by $\left[i_{1}, i_{2}, \ldots, i_{n}\right]=\left\{\mathbf{j} \in \mathcal{I}^{*}:\left.\mathbf{j}\right|_{n}=\left(i_{1}, i_{2}, \ldots, i_{n}\right)\right\}$. Let $\left(p_{1}, p_{2}, \ldots, p_{m}\right)$ be the
probability vector and $\mathcal{B}\left(\mathcal{I}^{*}\right)$ be the $\sigma$-algebra on $\mathcal{I}^{*}$ generated by the cylinder sets and given a cylinder set $\left[i_{1}, i_{2}, \ldots, i_{n}\right]$, the measure $\mu$ on $\mathcal{B}\left(\mathcal{I}^{*}\right)$ is defined by

$$
\begin{equation*}
\mu\left(\left[i_{1}, i_{2}, \ldots, i_{n}\right]\right)=\prod_{k=1}^{n} p_{i_{k}} \tag{8}
\end{equation*}
$$

This measure $\mu$ is known as Bernoulli measure.
Theorem 2.1.4 ( [2, 8] ). If $P$ be the probability measure associated with the RPIFS $\mathscr{W}_{\mathcal{P}}=\left\{\mathbb{R P}^{1} ; w_{A}: A \in \mathcal{P}\right\}$ and $\mu$ be the Bernoulli measure on $\mathcal{I}^{*}=\{1,2, \ldots, m\}^{\mathbb{N}}$, then

$$
\begin{equation*}
P=\mu \circ \Pi^{-1} \tag{9}
\end{equation*}
$$

where $\Pi: \mathcal{I}^{*} \rightarrow F_{\mathcal{P}}$ be the coordinate map given by $\Pi(\mathbf{i})=\lim _{n \rightarrow \infty} w_{A_{i_{1}}} w_{A_{i_{2}}} \cdots w_{A_{i_{n}}}([x])$, for arbitrary $[x] \in \mathbb{R P}^{1}$.

## 3. Cantor-like set on the real projective line

In this section, we consider the set $\mathcal{P}=\left\{A_{1}=\left(\begin{array}{cc}\frac{1}{3} & -\frac{2}{3} \\ 0 & 1\end{array}\right), A_{2}=\left(\begin{array}{cc}\frac{1}{3} & \frac{2}{3} \\ 0 & 1\end{array}\right)\right\} \subset G L(2, \mathbb{R})$ and the corresponding RPIFS on $\mathbb{R P}^{1}$. We estimate the Hausdorff dimension of the attractor of it and we see the existence of an invariant probability measure whose support is the attractor of this RPIFS. Here, we get the following results.

Theorem 3.0.1. The RPIFS $\mathscr{W}_{\mathcal{P}}=\left\{\mathbb{R P}^{1} ; w_{A_{1}}, w_{A_{2}}\right\}$ associated with the set $\mathcal{P}$ has an attractor.

Proof. Since $\operatorname{det}\left(A_{1}\right)=\operatorname{det}\left(A_{2}\right)=\frac{1}{3}>0$, so, we consider the corresponding RPIFS $\mathscr{W}_{\mathcal{P}}^{+}$, where $\mathcal{P}=\left\{A_{1}=\left(\begin{array}{cc}\frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{3}} \\ 0 & \sqrt{3}\end{array}\right), A_{2}=\left(\begin{array}{cc}\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ 0 & \sqrt{3}\end{array}\right)\right\} \subset S L(2, \mathbb{R})$. It can be seen that for all $A \in \mathcal{P}^{n}$ and $n \in \mathbb{N}$,

$$
A=\left(\begin{array}{cc}
\left(\frac{1}{\sqrt{3}}\right)^{n} & *  \tag{10}\\
0 & (\sqrt{3})^{n}
\end{array}\right) .
$$

So, $\|A\| \geq(\sqrt{3})^{n}$. Therefore, $\mathcal{P}$ is uniformly hyperbolic. Therefore, by Theorem 2.1.1 the RPIFS $\mathscr{W}_{\mathcal{P}}^{+}=\left\{\mathbb{R P}^{1} ; w_{A_{1}}, w_{A_{2}}\right\}$ has an attractor. Since for $i=1,2, w_{A_{i}}$ 's are the projective transformations. So, $\mathscr{W}_{\mathcal{P}}^{+}$and $\mathscr{W}_{\mathcal{P}}$ have the same attractor. Therefore, $\mathscr{W}_{\mathcal{P}}$ has an attractor in $\mathbb{R} \mathbb{P}^{1}$.

Here, we see the step-by-step construction of the projective cantor set:
Let us consider the cone $C_{\overline{a b}}$, generated by the points $[a]=[-1: 1]$ and $[b]=[1: 1]$ (see Fig 2). Now

$$
\begin{array}{ll}
w_{A_{1}}[a]=\left[A_{1} a\right]=[-1: 1] ; \quad w_{A_{1}}[b]=\left[A_{1} b\right]=\left[-\frac{1}{3}: 1\right] \\
w_{A_{2}}[a]=\left[A_{2} a\right]=\left[\frac{1}{3}: 1\right] ; \quad w_{A_{2}}[b]=\left[A_{2} b\right]=[1: 1] .
\end{array}
$$



Figure 2. Cone generated by $[a]$ and $[b]$.
Since projective transformation preserves the collinearity. So, $w_{A_{1}}\left(C_{\overline{a b}}\right)=C_{\overline{a_{1} b_{1}}}$, is the cone generated by the points $\left[a_{1}\right]=[a]=[-1: 1]$ and $\left[b_{1}\right]=\left[-\frac{1}{3}: 1\right]$. Similarly, $w_{A_{2}}\left(C_{\overline{a b}}\right)=$ $C_{\overline{a_{2} b_{2}}}$, is the cone generated by the points $\left[a_{2}\right]=\left[\frac{1}{3}: 1\right]$ and $\left[b_{2}\right]=[b]=[1: 1]$. Therefore, $W_{\mathcal{P}}\left(C_{\overline{a b}}\right)=C_{\overline{a_{1} b}} \cup C_{\overline{a_{2} b_{2}}}$ (see Fig 35$)$. To continue this process, we get the attractor of this RPIFS (see Fig 4). We are calling it projective Cantor set.

Note 3.0.1. Observe that the attractor $F_{\mathcal{P}}$ (say), of the RPIFS $\mathscr{W}_{\mathcal{P}}$ is a Cantor-like set on $\mathbb{R} \mathbb{P}^{1}$.
Theorem 3.0.2. If $F_{\mathcal{P}}$ is the attractor of the RPIFS $\mathscr{W}_{\mathcal{P}}^{+}$, then $\operatorname{dim}_{H}\left(F_{\mathcal{P}}\right)=\frac{\log 2}{\log 3}$.
Proof. Since

$$
\begin{equation*}
\mathbf{i}, \mathbf{j} \in \mathcal{I}^{n}, \quad \mathbf{i} \neq \mathbf{j} \Longrightarrow d\left(A_{\mathbf{i}}, A_{\mathbf{j}}\right) \geq\left(\frac{2}{\sqrt{3}}\right)^{n} \tag{11}
\end{equation*}
$$

So the set $\mathcal{P}$ is Diophantine. Also $I d \notin \overline{A^{*}}$. Therefore, the set $\mathcal{P}$ is non-discrete. Hence by Theorem 2.1.2

$$
\begin{equation*}
\operatorname{dim}_{H}\left(F_{\mathcal{P}}\right)=\min \left\{1, \xi_{\mathcal{P}}\right\} \tag{12}
\end{equation*}
$$

Now, all the matrices in $\mathcal{P}$ have at least one non-zero eigenvalue, so, here we consider the spectral norm. That is for $A \in S L(2, \mathbb{R}),\|A\|=|\lambda|$, where $|\lambda|$ is the spectral radius of $A$. Since for any $\mathbf{i} \in \mathcal{I}^{\mathbf{n}}$, we have

$$
A_{\mathbf{i}}=\left(\begin{array}{cc}
\left(\frac{1}{\sqrt{3}}\right)^{n} & *  \tag{13}\\
0 & (\sqrt{3})^{n}
\end{array}\right)
$$



Figure 3. Multi cone generated by the points $\left[a_{1}\right],\left[b_{1}\right]$ and $\left[a_{2}\right],\left[b_{2}\right]$.


Figure 4. Projective Cantor set on $\mathbb{R P}^{1}$.
So, $\left\|A_{\mathbf{i}}\right\|=(\sqrt{3})^{n}$. Therefore,

$$
\begin{aligned}
\zeta_{\mathcal{P}}(t):=\sum_{n=1}^{\infty} \sum_{\mathbf{i} \in \mathcal{I}^{n}}\left(\left\|A_{\mathbf{i}}\right\|\right)^{-2 t} & =\sum_{n=1}^{\infty} 2^{n}\left((\sqrt{3})^{n}\right)^{-2 t} \\
7 & =\sum_{n=1}^{\infty}\left(\frac{2}{3^{t}}\right)^{n}
\end{aligned}
$$

This series is convergent if $\frac{2}{3^{t}}<1$. That is $\frac{\log 2}{\log 3}<t$. And it is divergent if $\frac{\log 2}{\log 3} \geq t$. Therefore, $\xi_{\mathcal{P}}=\frac{\log 2}{\log 3}$. Hence from (12), we have $\operatorname{dim}_{H}\left(F_{\mathcal{P}}\right)=\frac{\log 2}{\log 3}$.

The following theorem proves the existence of an invariant probability measure associated with $\mathscr{W}_{\mathcal{P}}^{+}$.

Theorem 3.0.3. Let $p=\left(\frac{1}{2}, \frac{1}{2}\right)$ be the probability vector and
$\mathcal{P}=\left\{A_{1}=\left(\begin{array}{cc}\frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{3}} \\ 0 & \sqrt{3}\end{array}\right), A_{2}=\left(\begin{array}{cc}\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ 0 & \sqrt{3}\end{array}\right)\right\}$. Let $\mathscr{W}_{\mathcal{P}}^{+}=\left\{\mathbb{R P}^{1} ; w_{A_{1}}, w_{A_{2}}\right\}$ be the associated RPIFS. Then there exists unique probability measure $P$ on $\mathbb{R P}^{1}$ such that

$$
\begin{equation*}
P=\frac{1}{2} P \circ w_{A_{1}}^{-1}+\frac{1}{2} P \circ w_{A_{2}}^{-1} . \tag{14}
\end{equation*}
$$

In particular, if $\mu$ is the Bernoulli measure on the sequence $\{1,2\}^{\mathbb{N}}$ with associated probability vector $p=\left(\frac{1}{2}, \frac{1}{2}\right)$, then

$$
\begin{equation*}
P(E)=\mu \circ \Pi^{-1}(E) \quad \text { for } E \subset \mathbb{R} \mathbb{P}^{*}, \tag{15}
\end{equation*}
$$

where $\Pi:\{1,2\}^{\mathbb{N}} \rightarrow F_{\mathcal{P}}$ is given by $\Pi(\mathbf{i})=\lim _{n \rightarrow \infty} A_{i_{1}} A_{i_{2}} \cdots A_{i_{n}}([x])$, for arbitrary $[x] \in \mathbb{R P}^{1}$.
Proof. Since $A_{1}$ fixes the unique point $[a]=[-1: 1]$ and $A_{2}$ fixes the unique point $[b]=$ [1:1]. So the matrices $A_{1}$ and $A_{2}$ do not preserve any finite subset of $\mathbb{R P}^{1}$ simultaneously. Hence the set $\mathcal{P}$ is strongly irreducible. Also, for all $A \in \mathcal{P}^{n}$,

$$
\begin{equation*}
\|A\| \geq(\sqrt{3})^{n} \tag{16}
\end{equation*}
$$

So, $\mathcal{P}$ generates the unbounded semi-group. Therefore, by Theorem 2.1.3, there exists a unique probability measure $P$ which satisfies the equation

$$
P=\frac{1}{2} P \circ w_{A_{1}}^{-1}+\frac{1}{2} P \circ w_{A_{2}}^{-1} .
$$

The last part of the proof follows from Theorem 2.1.4.

## 4. Voronoi partition and Quantization error

In this section, we define the Voronoi region on the real projective line $\mathbb{R P}^{1}$ and the $n$-th quantization error for a probability distribution. We see the action of a projective transformation on the Voronoi region in $\mathbb{R P}^{1}$. Also, we prove results related to $n$-th quantization error of order $r$ of an invariant probability measure.
Definition 4.0.1. A set $A \subset \mathbb{R}^{1}$ is said to be a locally finite set in the sense that for any bounded set $B \subset \mathbb{R}^{1}$, the number of elements in $A \cap B$ is finite and a collection $\mathscr{A}$ of subsets of $\mathbb{R}^{1}$ is called locally finite if the number of elements in $\mathscr{A}$ intersecting any bounded subset of $\mathbb{R}^{1}$ is finite.

Definition 4.0.2. Let $\Delta$ be a locally finite subset of $\mathbb{R} \mathbb{P}^{*}$ and let the set

$$
\mathbb{W}([a] \mid \Delta)=\left\{[x] \in \mathbb{R}^{*}: d_{\mathbb{P}}([x],[a])=\min _{[b] \in \Delta} d_{\mathbb{P}}([x],[b])\right\}
$$

is the Voronoi region generated by $[a]$.
Remark 4.0.1. The Voronoi diagram $\{\mathbb{W}([a] \mid \Delta):[a] \in \Delta\}$ is locally finite covering of $\mathbb{R} \mathbb{P}^{*}$.

Here, we provide some results concerning the general projective transformation on the real projective line.
Theorem 4.0.1. Let $A=\left(\begin{array}{cc}a_{11} & a_{12} \\ v & 1\end{array}\right) \in G L(2, \mathbb{R})$ and $T_{A}: \mathbb{R P}^{1} \rightarrow \mathbb{R} \mathbb{P}^{1}$ be the corresponding projective transformation. Then the following holds:
(1) If $v=0$, then $\mathbb{W}\left(T_{A}[a] \mid T_{A}(\Delta)\right)=T_{A}(\mathbb{W}([a] \mid \Delta))$ for all $[a] \in \Delta$.
(2) If $T_{A}(\Delta) \subset \mathbb{R P}^{*}$ and $\mathbb{W}\left(T_{A}[a] \mid T_{A}(\Delta)\right)=T_{A}(\mathbb{W}([a] \mid \Delta))$ for all $[a] \in \Delta$, then $v=0$.

Proof. (1) If $v=0$. Since $\operatorname{det}(A) \neq 0$, so $a_{11} \neq 0$ and for all $[x: 1] \in \mathbb{R P}^{*}, T_{A}[x: 1]=$ $[A(x, 1)]=\left[a_{11} x+a_{12}: 1\right] \in \mathbb{R} \mathbb{P}^{*}$. Therefore, $T_{A}$ restricted on $\mathbb{R P}^{*}$ is invertible. Then

$$
\begin{aligned}
\mathbb{W}\left(T_{A}[a] \mid T_{A}(\Delta)\right) & =\left\{[x] \in \mathbb{R} \mathbb{P}^{*}: d_{\mathbb{P}}\left([x], T_{A}[a]\right)=\min _{[b] \in T_{A}(\Delta)} d_{\mathbb{P}}([x],[b])\right\} \\
& =\left\{T_{A}[x] \in \mathbb{R P}^{*}: d_{\mathbb{P}}\left(T_{A}[x], T_{A}[a]\right)=\min _{[d] \in \Delta} d_{\mathbb{P}}\left(T_{A}[x], T_{A}[d]\right)\right\} .
\end{aligned}
$$

Now, if $[c]=[c: 1] \in \mathbb{R P}^{*}$, then

$$
\begin{aligned}
d_{\mathbb{P}}\left(T_{A}[x], T_{A}[c]\right) & =d_{\mathbb{P}}\left(\left[a_{11} x+a_{12}: 1\right],\left[a_{11} c+a_{12}: 1\right]\right) \\
& =\left|a_{11}\right||x-c| \\
& =\left|a_{11}\right| d_{\mathbb{P}}([x],[c]) .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbb{W}\left(T_{A}[a] \mid T_{A}(\Delta)\right) & =\left\{T_{A}[x] \in \mathbb{R P}^{*}:\left|a_{11}\right| d_{\mathbb{P}}([x],[a])=\min _{[d] \in \Delta}\left|a_{11}\right| d_{\mathbb{P}}([x],[d])\right\} \\
& =\left\{T_{A}[x] \in \mathbb{R} \mathbb{P}^{*}: d_{\mathbb{P}}([x],[a])=\min _{[d] \in \Delta} d_{\mathbb{P}}([x],[d])\right\} \\
& =T_{A}(\mathbb{W}([a] \mid \Delta)) .
\end{aligned}
$$

(2) If possible let $v \neq 0$. Then $\left[-\frac{1}{v}: 1\right] \in \mathbb{R}^{*}$. Since $\mathbb{R P}^{*}=\bigcup_{[a] \in \Delta} \mathbb{W}([a] \mid \Delta)$, so, $\left[-\frac{1}{v}: 1\right] \in \mathbb{W}([c] \mid \Delta)$ for some $[c] \in \Delta$. Therefore, $T_{A}\left[-\frac{1}{v}: 1\right]=\left[-\frac{1}{v} a_{11}+a_{12}:\right.$ $0] \in T_{A}(\mathbb{W}([c] \mid \Delta))$. Given that $T_{A}(\Delta) \subset \mathbb{R P}^{*}$, so $[a] \neq\left[-\frac{1}{v}: 1\right]$ for all $[a] \in \Delta$. Therefore, $\mathbb{R P}^{*}=\bigcup_{[a] \in \Delta} \mathbb{W}\left(T_{A}[a] \mid T_{A}(\Delta)\right)$. Hence $\left[-\frac{1}{v}: 1\right] \notin \mathbb{W}\left(T_{A}[c] \mid T_{A}(\Delta)\right)$. This gives a contradiction to $\mathbb{W}\left(T_{A}[c] \mid T_{A}(\Delta)\right)=T_{A}(\mathbb{W}([c] \mid \Delta))$. Hence the proof.

Definition 4.0.3. Let $\mu$ be the probability distribution on $\mathbb{R} \mathbb{P}^{*}$ and $[x: 1]$ be the random variable on $\mathbb{R} \mathbb{P}^{*}$ with distribution $\mu$. Let
$\mathscr{F}_{n}=\left\{f: \mathbb{R} \mathbb{P}^{*} \rightarrow \mathbb{R} \mathbb{P}^{*}\right.$ such that $f$ is measurable map with $\left.\left|f\left(\mathbb{R} \mathbb{P}^{*}\right)\right| \leq n\right\}$.

The elements of $\mathscr{F}_{n}$ are called $n$-quantizers. Let $r \geq 1$ and assume that

$$
\int_{\mathbb{R} \mathbb{P}^{*}} d_{\mathbb{P}}([x: 1],[0: 1])^{r} d \mu[x]<\infty .
$$

The $n$-th quantization error for $\mu$ of order $r$ is defined by

$$
\mathbb{V}_{n, r}(\mu)=\inf _{f \in \mathscr{F}_{n}} \int_{\mathbb{R}^{*}} d_{\mathbb{P}}([x], f[x])^{r} d \mu[x] .
$$

A quantizer $f \in \mathscr{F}_{n}$ is said to be $n$-optimal for $\mu$ of order $r$ if

$$
\mathbb{V}_{n, r}(\mu)=\int_{\mathbb{R}^{*}} d_{\mathbb{P}}([x], f[x])^{r} d \mu[x]
$$

## Lemma 4.0.1.

$$
\begin{aligned}
\mathbb{V}_{n, r}(\mu)= & \inf _{\substack{\Delta \subset \mathbb{R} \mathbb{P}^{*}}} \int_{\mathbb{R P}^{*}[a] \in \Delta} \min ^{|\Delta|<n} \text { 刑 }([x],[a])^{r} d \mu[x] .
\end{aligned}
$$

Proof. Let $\Delta=f\left(\mathbb{R P}^{*}\right)$ for a fixed $f \in \mathscr{F}_{n}$. Then $|\Delta| \leq n$. For $[a] \in \Delta$, let

$$
\mathbb{A}_{[a]}=\left\{[x] \in \mathbb{R P}^{*}: f[x]=[a]\right\}
$$

Then $\mathbb{A}_{[a]}$ is non-empty and $\bigcup_{[a] \in \Delta} \mathbb{A}_{[a]}=\mathbb{R} \mathbb{P}^{*}$. Now,

$$
\begin{aligned}
\int_{\mathbb{R P}^{*}} d_{\mathbb{P}}([x], f[x])^{r} d \mu[x] & =\sum_{[a] \in \Delta} \int_{\mathbb{A}_{[a]}} d_{\mathbb{P}}([x],[a])^{r} d \mu[x] \\
& \geq \sum_{[a] \in \Delta} \int_{\mathbb{A}_{[a]}} \min _{[b] \in \Delta} d_{\mathbb{P}}([x],[b])^{r} d \mu[x] \\
& =\int_{\mathbb{R} \mathbb{P}^{*}} \min _{[b] \in \Delta} d_{\mathbb{P}}([x],[b])^{r} d \mu[x] .
\end{aligned}
$$

Therefore,

$$
\begin{gather*}
\mathbb{V}_{n, r}(\mu)=\inf _{f \in \mathscr{F}_{n}} \int_{\mathbb{R}^{*}} d_{\mathbb{P}}([x], f[x])^{r} d \mu[x] \geq \inf _{\substack{\Delta \subset \mathbb{R} \mathbb{P}^{*}}} \int_{\mathbb{R} \mathbb{P}^{*}} \min _{[a] \in \Delta} d_{\mathbb{P}}([x],[a])^{r} d \mu[x] .<n \tag{17}
\end{gather*}
$$

Conversely, let $\Delta \subset \mathbb{R P}^{*}$ with $|\Delta| \leq n$ and $\mathbb{A}_{[a]}=\mathbb{W}([a] \mid \Delta)$. For $[a] \in \Delta$, define $f: \mathbb{R} \mathbb{P}^{*} \rightarrow$ $\mathbb{R P}^{*}$ such that $f=\sum_{[a] \in \Delta}[a] \chi_{\mathbb{A}_{[a]}}$, where the summation is over $\oplus$ and $\chi_{\mathbb{A}_{[a]}}: \mathbb{R} \mathbb{P}^{*} \rightarrow \mathbb{R} \mathbb{P}^{*}$ is the characteristic function on $\mathbb{A}_{[a]}$. Then $f \in \mathscr{F}_{n}$ and

$$
\int_{\mathbb{R}^{*}} \min _{[a] \in \Delta} d_{\mathbb{P}}([x],[a])^{r} d \mu[x]=\sum_{[a] \in \Delta} \int_{\mathbb{A}_{[a]}} \min _{[a] \in \Delta} d_{\mathbb{P}}([x],[a])^{r} d \mu[x]
$$

Sincc $[x] \in \mathbb{A}_{[a]}=\mathbb{W}([a] \mid \Delta)$. Therefore,

$$
\begin{aligned}
\int_{\mathbb{R}^{*}{ }^{*}[a] \in \Delta} \min _{\mathbb{P}}([x],[a])^{r} d \mu[x] & =\sum_{[a] \in \Delta} \int_{\mathbb{A}_{[a]}} d_{\mathbb{P}}([x],[a])^{r} d \mu[x] \\
& =\int_{\mathbb{R} \mathbb{P}^{*}} d_{\mathbb{P}}([x], f[x])^{r} d \mu[x] \\
& \geq \inf _{f \in \mathscr{F}_{n}} \int_{\mathbb{R}^{*}} d_{\mathbb{P}}([x], f[x])^{r} d \mu[x]=\mathbb{V}_{n, r}(\mu) .
\end{aligned}
$$

So,

$$
\begin{equation*}
\inf _{\substack{\Delta \subset \mathbb{R} \mathbb{P}^{*}}} \int_{\mathbb{R P}^{*}} \min _{|\Delta| \in \Delta} d_{\mathbb{P}}([x],[a])^{r} d \mu[x] \geq \mathbb{V}_{n, r}(\mu) \tag{18}
\end{equation*}
$$

Hence from (17) and (18), we get the desired result.
Definition 4.0.4. A set $\Delta \subset \mathbb{R} \mathbb{P}^{*}$ with $|\Delta|<n$ is said to be $n$-optimal set of centers for the probability distribution $\mu$ of order $r$ if

$$
\mathbb{V}_{n, r}(\mu)=\int_{\mathbb{R}^{*}} \min _{[a] \in \Delta} d_{\mathbb{P}}([x],[a])^{r} d \mu[x]
$$

Let $\mathscr{C}_{n, r}(\mu)$ be the collection of all $n$-optimal set of centers for $\mu$ of order $r$. Then we get the following.
Theorem 4.0.2. Let $A=\left(\begin{array}{cc}a_{11} & a_{12} \\ v & 1\end{array}\right) \in G L(2, \mathbb{R})$ and $T_{A}: \mathbb{R} \mathbb{P}^{1} \rightarrow \mathbb{R P}^{1}$ be the corresponding projective transformation. Then the following holds:
(1) If $v=0$, then $\mathbb{V}_{n, r}\left(T_{A} \mu\right)=\left|a_{11}\right|^{r} \mathbb{V}_{n, r}(\mu)$.
(2) If $v=0$, then $\mathscr{C}_{n, r}\left(T_{A} \mu\right)=T_{A}\left(\mathscr{C}_{n, r}(\mu)\right)$,
where $T_{A} \mu$ is the push-forward measure of $\mu$.
Proof. (1) If $v=0$. Let $\Omega \subset T_{A}\left(\mathbb{R P}^{*}\right)$ such that $|\Omega| \leq n$. Since $v=0$, so, $\left.T_{A}\right|_{\mathbb{R} \mathbb{P}^{*}}$ is a non-singular map on $\mathbb{R} \mathbb{P}^{*}$. Therefore, there is $\Delta$ in $\mathbb{R P}^{*}$ such that $T_{A}(\Delta)=\Omega$ and $|\Delta|=|\Omega| \leq n$. Then

$$
\begin{aligned}
\int_{T_{A}\left(\mathbb{R P}^{*}\right)} \min _{[a] \in \Omega} d_{\mathbb{P}}([y],[a])^{r} d\left(\mu \circ T_{A}^{-1}\right)[y] & =\int_{\mathbb{R P}^{*}} \min _{[b] \in \Delta} d_{\mathbb{P}}\left(T_{A}[x], T_{A}[b]\right)^{r} d \mu[x] \\
& =\left|a_{11}\right|^{r} \int_{\mathbb{R} \mathbb{R}^{*}} \min _{[b] \in \Delta} d_{\mathbb{P}}([x],[b])^{r} d \mu[x] \\
& \geq\left|a_{11}\right|^{r} \mathbb{V}_{n, r}(\mu) .
\end{aligned}
$$

Therefore, taking infimum over $\Omega$ with $|\Omega| \leq n$, we get

$$
\mathbb{V}_{n, r}\left(T_{A} \mu\right) \geq\left|a_{11}\right|^{r} \mathbb{V}_{n, r}(\mu)
$$

Now, it is an easy exercise to see that $T_{A}^{-1}=T_{A^{-1}}$, where $A^{-1}=\left(\begin{array}{cc}\frac{1}{a_{11}} & -\frac{a_{12}}{a_{11}} \\ 0 & 1\end{array}\right)$. So, if we replace $T_{A}$ by $T_{A}^{-1}$ in the beginning and proceed as above then we get

$$
\mathbb{V}_{n, r}\left(T_{A}^{-1} \mu\right) \geq \frac{1}{\left|a_{11}\right|^{r}} \mathbb{V}_{n, r}(\mu)
$$

That is

$$
\begin{equation*}
\left|a_{11}\right|^{r} \mathbb{V}_{n, r}(\mu) \geq \mathbb{V}_{n, r}\left(T_{A} \mu\right) \tag{20}
\end{equation*}
$$

Combining (19) and (20), result follows.
(2) Let $\Omega \subset T_{A}\left(\mathbb{R} \mathbb{P}^{*}\right)$ such that $\Omega \in \mathscr{C}_{n, r}\left(T_{A} \mu\right)$. Since $T_{A}$ is invertible and $v=0$, so, there is $\Delta$ in $\mathbb{R P}^{*}$ such that $\Omega=T_{A}(\Delta)$ and $|\Delta|=|\Omega| \leq n$. We claim that $\Delta$ is also an $n$-optimal set of the centers for $\mu$. Since $\Omega \in \mathscr{C}_{n, r}\left(T_{A} \mu\right)$, therefore,

$$
\begin{aligned}
\mathbb{V}_{n, r}\left(T_{A} \mu\right) & =\int_{T_{A}\left(\mathbb{R P}^{*}\right)} \min _{[a] \in \Omega} d_{\mathbb{P}}([y],[a])^{r} d\left(\mu \circ T_{A}^{-1}\right)[y] \\
& =\int_{\mathbb{R P}^{*}} \min _{[b] \in \Delta} d_{\mathbb{P}}\left(T_{A}[x], T_{A}[b]\right)^{r} d \mu[x] \\
& =\left|a_{11}\right|^{r} \int_{\mathbb{R} \mathbb{P}^{*}} \min _{[b] \in \Delta} d_{\mathbb{P}}([x],[b])^{r} d \mu[x]
\end{aligned}
$$

Since $a_{11} \neq 0$, using item (1), it follows that

$$
\mathbb{V}_{n, r}(\mu)=\int_{\mathbb{R}^{*}} \min _{[b] \in \Delta} d_{\mathbb{P}}([x],[b])^{r} d \mu[x]
$$

This proves our claim. Hence $\Delta \in \mathscr{C}_{n, r}(\mu)$. Therefore, $\Omega=T_{A}(\Delta) \in T_{A}\left(\mathscr{C}_{n, r}(\mu)\right)$. This implies that $\mathscr{C}_{n, r}\left(T_{A} \mu\right) \subseteq T_{A}\left(\mathscr{C}_{n, r}(\mu)\right)$. To prove the converse part, we replace $T_{A}$ by $T_{A}^{-1}$ in the beginning and proceed as above. This completes the proof.

To estimate the upper bound of the $n$-th quantization error of a probability measure, we recall the RPIFS $\mathscr{W}_{\mathcal{P}}=\left\{\mathbb{R P}^{1} ; w_{A_{1}}, w_{A_{2}}\right\}$ and the associated invariant probability measure $P$ defined as above. Let $C_{\overline{a b}}$, be the cone generated by the points $[a]=[-1: 1]$ and $[b]=[1: 1]$. For $\omega=\left(\omega_{1}, \omega_{2}, \ldots, \omega_{l}\right) \in\{1,2\}^{l}$, define $w_{A_{\omega}}=w_{A_{\omega_{1}}} \circ w_{A_{\omega_{2}}} \circ \cdots \circ w_{A_{\omega_{l}}}$. Let $C_{\omega}=w_{A_{\omega}}\left(C_{\overline{a b}}\right)$ and $[c],[d]$ be the boundary points of the cone $C_{\omega}$. Let $m_{\omega} \in C_{\omega}$ be the point which lies in the equal distance from the points $[c]$ and $[d]$ concerning the metric $d_{\mathbb{P}}$. That is the mid-point of the cone $C_{\omega}$. The collection $\left(C_{\omega}\right)_{\omega \in\{1,2\}^{l}}$ consists $2^{l}$ number of cones in the $l$-th level of the construction of Cantor-like set on $\mathbb{R} \mathbb{P}^{1}$. The cones $C_{\omega 1^{*}}$, $C_{\omega 2^{*}}$ are said to be children of $C_{\omega}$ into which $C_{\omega}$ is split up at $(l+1)$-th level. Then the attractor $F_{\mathbb{P}}=\bigcap_{l \in \mathbb{N}} \bigcup_{\omega \in\{1,2\}^{l}} C_{\omega}$ is the Cantor-like set on $\mathbb{R} \mathbb{P}^{1}$.

Let $n \in \mathbb{N}$ with $n \geq 1$, and $k(n) \in \mathbb{N}$ be the unique number such that $2^{k(n)} \leq n<2^{k(n)+1}$. Let $\mathbb{I} \subset\{1,2\}^{k(n)}$ with $\operatorname{card}(\mathbb{I})=n-2^{k(n)}$, and let $\Delta_{n}$ be the set consisting all points $m_{\omega}$
of the cone $C_{\omega}$ with $\omega \in\{1,2\}^{k(n)} \backslash \mathbb{I}$ and all mid-points $m_{\omega 1^{*}}, m_{\omega 2^{*}}$ of the children of $C_{\omega}$ with $\omega \in \mathbb{I}$. That is

$$
\Delta_{n}=\left\{m_{\omega}: \omega \in\{1,2\}^{k(n)} \backslash \mathbb{I}\right\} \cup\left\{m_{\omega 1^{*}}: \omega \in \mathbb{I}\right\} \cup\left\{m_{\omega 2^{*}}: \omega \in \mathbb{I}\right\}
$$

Let $D_{n}=\frac{1}{2} \frac{1}{18^{k(n)}}\left[2^{k(n)+1}-n+\frac{1}{9}\left(n-2^{k(n)}\right)\right]$. Then we get the following:
Theorem 4.0.3. The upper bound of the $n$-th quantization error for $P$ of order 2 is $D_{n}$.
To prove the above theorem, first, we prove the following lemmas.
Lemma 4.0.2. If $S: \mathbb{R}^{*} \mathbb{P}^{*} \rightarrow \mathbb{R}_{+}$is Borel measurable and $k \in \mathbb{N}$, then

$$
\int S[x] d P[x]=\frac{1}{2^{l}} \sum_{\mathbf{i} \in\{1,2\}^{l}} \int S \circ w_{A_{\mathbf{i}}}[x] d P[x]
$$

Proof. From (14),

$$
\begin{align*}
\int S[x] d P[x] & =\int S[x] d\left(\frac{1}{2} P \circ w_{A_{1}}^{-1}+\frac{1}{2} P \circ w_{A_{2}}^{-1}\right)[x] \\
& =\frac{1}{2} \sum_{\mathbf{i} \in\{1,2\}} \int S[x] d P \circ w_{A_{\mathbf{i}}}^{-1}[x] \\
& =\frac{1}{2} \sum_{\mathbf{i} \in\{1,2\}} \int S \circ w_{A_{\mathbf{i}}}[x] d P[x] . \tag{22}
\end{align*}
$$

Now, by repeated applications of the equation (14) in (22), we get the required result.
Lemma 4.0.3. For $[0: 1] \in \mathbb{R} \mathbb{P}^{*}$,

$$
\int d_{\mathbb{P}}([x: 1],[0: 1])^{2} d P[x]=\frac{1}{2}
$$

Proof. For $[0: 1] \in \mathbb{R P}^{*}$, using lemma 4.0.2

$$
\begin{aligned}
\int d_{\mathbb{P}}([x: 1],[0: 1])^{2} d P[x] & =\frac{1}{2} \int d_{\mathbb{P}}\left(w_{A_{1}}[x: 1],[0: 1]\right)^{2} d P[x]+\frac{1}{2} \int d_{\mathbb{P}}\left(w_{A_{2}}[x: 1],[0: 1]\right)^{2} d P[x] \\
& =\frac{1}{2} \int d_{\mathbb{P}}\left(\left[\frac{x}{3}-\frac{2}{3}: 1\right],[0: 1]\right)^{2} d P[x]+\frac{1}{2} \int d_{\mathbb{P}}\left(\left[\frac{x}{3}+\frac{2}{3}: 1\right],[0: 1]\right)^{2} d P[x] \\
& =\frac{1}{2}\left[\int\left(\frac{x}{3}-\frac{2}{3}\right)^{2} d P[x]+\int\left(\frac{x}{3}+\frac{2}{3}\right)^{2} d P[x]\right] \\
& =\frac{1}{2}\left[2 \int \frac{x^{2}}{9} d P[x]+2 \int \frac{4}{9} d P[x]\right] \\
& =\frac{1}{9} \int x^{2} d P[x]+\frac{4}{9} \\
& =\frac{1}{9} \int d_{\mathbb{P}}([x: 1],[0: 1])^{2} d P[x]+\frac{4}{9} .
\end{aligned}
$$

Therefore,

$$
\int d_{\mathbb{P}}([x: 1],[0: 1])^{2} d P[x]=\frac{1}{2}
$$

Proof of Theorem 4.0.3.

$$
\begin{aligned}
\int_{\mathbb{R P}^{*}} \min _{[a] \in \Delta_{n}} d_{\mathbb{P}}([x],[a])^{2} d P[x]= & \sum_{\omega \in\{1,2\}^{k(n)} \backslash \mathbb{I}} \int_{C_{\omega}} d_{\mathbb{P}}([x],[a])^{2} d P[x]+\sum_{\omega \in \mathbb{I}} \int_{C_{\omega 1^{*}}} d_{\mathbb{P}}([x],[a])^{2} d P[x] \\
& +\sum_{\omega \in \mathbb{I}} \int_{C_{\omega 2^{*}}} d_{\mathbb{P}}([x],[a])^{2} d P[x] .
\end{aligned}
$$

For every $l \in \mathbb{N}$ and for every $\sigma \in\{1,2\}^{l}$, from Lemma 4.0.2 and Lemma 4.0.3, we get

$$
\begin{aligned}
\int_{C_{\sigma}} d_{\mathbb{P}}\left([x],\left[m_{\sigma}\right]\right)^{2} d P[x] & =\frac{1}{2^{l}} \int d_{\mathbb{P}}\left(w_{A_{\sigma}}[x],\left[m_{\sigma}\right]\right)^{2} d P[x] \\
& =\frac{1}{2^{l}} \int d_{\mathbb{P}}\left(w_{A_{\sigma}}[x: 1], w_{A_{\sigma}}[0: 1]\right)^{2} d P[x]
\end{aligned}
$$

Now, $w_{A_{\sigma}}=\left(\begin{array}{cc}\frac{1}{3^{l}} & * \\ 0 & 1\end{array}\right)$. Therefore, $d_{\mathbb{P}}\left(w_{A_{\sigma}}[x: 1], w_{A_{\sigma}}[0: 1]\right)^{2}=\left(\frac{1}{3^{l}}\right)^{2} d_{\mathbb{P}}([x: 1],[0: 1])^{2}$. So,

$$
\begin{aligned}
\int_{C_{\sigma}} d_{\mathbb{P}}\left([x],\left[m_{\sigma}\right]\right)^{2} d P[x] & =\frac{1}{18^{l}} \int d_{\mathbb{P}}([x: 1],[0: 1])^{2} d P[x] \\
& =\frac{1}{2} \frac{1}{18^{l}} .
\end{aligned}
$$

Since $\omega 1^{*}$ and $\omega 2^{*}$ are the elements of $\{1,2\}^{k(n)+1}$, therefore,

$$
\begin{aligned}
\int_{\mathbb{R P}^{*}[a] \in \Delta_{n}} d_{\mathbb{P}}([x],[a])^{2} d P[x]= & \frac{1}{2} \frac{1}{18^{k(n)}} \sum_{\omega \in\{1,2\}^{k(n)} \backslash \mathbb{I}} \int_{C_{\omega}} d P[x]+\frac{1}{2} \frac{1}{18^{k(n)+1}} \sum_{\omega \in \mathbb{I}} \int_{C_{\omega 1^{*}}} d P[x] \\
& +\frac{1}{2} \frac{1}{18^{k(n)+1}} \sum_{\omega \in \mathbb{I}} \int_{C_{\omega 2^{*}}} d P[x] \\
= & \frac{1}{2} \frac{1}{18^{k(n)}}\left[\operatorname{card}\left(\{1,2\}^{k(n)} \backslash \mathbb{I}\right)+\frac{1}{9} \operatorname{card}(\mathbb{I})\right] \\
= & \frac{1}{2} \frac{1}{18^{k(n)}}\left[2^{k(n)}-\left(n-2^{k(n)}\right)+\frac{1}{9}\left(n-2^{k(n)}\right)\right] \\
= & \frac{1}{2} \frac{1}{18^{k(n)}}\left[2^{k(n)+1}-n+\frac{1}{9}\left(n-2^{k(n)}\right)\right]=D_{n} .
\end{aligned}
$$

Hence

$$
\mathbb{V}_{n, r}(P) \leq D_{n}
$$

This completes the proof.

## Conclusion

In this article, an IFS is considered on the real projective line $\mathbb{R} \mathbb{P}^{1}$ so that the attractor is a Cantor-like set on it and estimated the Hausdorff dimension of such an attractor. Then, we have shown the existence of a probability measure on $\mathbb{R P}^{1}$ associated with this IFS and estimated the upper bound of the $n$-th quantization error for this measure. In the future, it may be possible to discover the optimal set and calculate the quantization dimension of the Cantor-like set in $\mathbb{R} \mathbb{P}^{1}$.

## References

[1] A. Avila, J. Bochi, and J.-C. Yoccoz, Uniformly hyperbolic finite-valued sl (2, (r))-cocycles, Commentarii Mathematici Helvetici, 85 (2010), pp. 813-884.
[2] B. BÁrány, M. Pollicott, and K. Simon, Stationary measures for projective transformations: the blackwell and furstenberg measures, J. Stat. Phys., 148 (2012), pp. 393-421.
[3] M. F. Barnsley and A. Vince, Real projective iterated function systems, J. Geom. Anal., 22 (2012), pp. 1137-1172.
[4] A. Christodoulou and N. Jurga, The hausdorff dimension of self-projective sets, ArXiv. /abs/2007.064301, 0 (2020), pp. 1-29.
[5] H. Furstenberg, Noncommuting random products, Trans. Amer. Math. Soc., 108 (1963), pp. 377428.
[6] S. Graf and H. Luschgy, The quantization of the cantor distribution, Mathematische Nachrichten, 183 (1997), pp. 113-133.
[7] S. Graf and H. Luschgy, Foundations of quantization for probability distributions, Springer, 2007.
[8] J. E. Hutchinson, Fractals and self similarity, Indiana University Mathematics Journal, 30 (1981), pp. 713-747.
[9] M. K. Roychowdhury, Quantization dimension and temperature function for recurrent self-similar measures, Chaos, Solitons \& Fractals, 44 (2011), pp. 947-953.
[10] M. K. Roychowdhury, Quantization dimension and temperature function for bi-lipschitz mappings, Israel Journal of Mathematics, 192 (2012), pp. 473-488.
[11] B. Solomyak and Y. Takahashi, Diophantine property of matrices and attractors of projective iterated function systems in $\mathbb{R}^{1} \mathbb{P}^{1}$, Int. Math. Res. Not., 2021 (2021), pp. 12639-12669.

Department of Mathematics, Presidency University, 86/1, College Street, Kolkata, 700 073, West Bengal, India

Email address: hossain4791@gmail.com
Department of Mathematics, Presidency University, 86/1, College Street, Kolkata, 700 073, West Bengal, India

Email address: akash.mapping@gmail.com
Department of Mathematics, Presidency University, 86/1, College Street, Kolkata, 700 073, West Bengal, India

Email address: nasim.iitm@gmail.com

