Sampling Theorem and interpolation formula for non-vanishing signals

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Abstract

The paper establishes an analog Whittaker-Shannon-Kotelnikov sampling theorem with with fast decreasing coefficient, as well as a new modification of the corresponding interpolation formula applicable for general type non-vanishing bounded continuous signals.

Key words: non-vanishing signals, band-limited signals, Sampling Theorem, interpolation, Whittaker-Shannon-Kotelnikov interpolation formula

1 Introduction

Problems of recovery of signals from incomplete observations were studied intensively in different settings. This includes recovering signals from samples. The most important tools used for signal processing are based on the representation of signal processes in the frequency domain. This includes, in particular, the conditions of data recoverability. In general, possibility or recovery a continuous time signal from a sample is usually associated with restrictions on the class of underlying signals such as restrictions on the spectrum. The classical Nyquist–Shannon sampling theorem establishes that a band-limited signal vanishing on $\pm\infty$ can be recovered without error from a discrete sample taken with a sampling rate that is at least twice the maximum frequency of the signal (the Nyquist critical rate). In particular, a band-limited signal $x(t) \in C(\mathbf{R}) \cap L_2(\mathbf{R})$ with the spectrum contained in the interval $[-\pi, \pi]$ can be recovered from its sample $\{x(k)\}_{k=-\infty}^{\infty}$ as

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{\sin(\pi(k-t))}{\pi(k-t)} x(k). \qquad (1)$$

This is celebrated Whittaker-Shannon-Kotelnikov interpolation formula, also known as Whittaker–Shannon interpolation formula, Shannon's interpolation formula, and Whittaker's interpolation formula. It can be observed that since the coefficients of this interpolation formula are decreasing as $\sim 1/k$, it covers only signals such that $x(t) \to 0$ with a certain rate as $|t| \to +\infty$.

There are many works devoted to generalization of the Sampling Theorem; see e.g. the reviews in [4, 6, 8, 9, 10] and literature therein. However, the extension of this result on non-vanishing signals has not been obtained yet.

The purpose of the present paper is to obtain an interpolation formula similar to (1) but applicable to non-vanishing continuous signals.

It can be observed that a non-vanishing signal from $L_\infty(\mathbf{R})$ can be modified to a signal from $L_1(\mathbf{R})$ without any loss of information, for example, by replacement $x(t)$ by $e^{-|t|}x(t)$. However, at least for the case of signals from $L_1(\mathbf{R})$, these damping transformations represent the convolutions in the frequency domain, with smoothing kernels. Unfortunately, a band-limited signal will be transformed into a non-band-limited one along the way. For the general type two-sided processes from $L_\infty(\mathbf{R})$, one could expect a similar impact of the damping transformations on the spectrum. Therefore, it was essential to develop a special approach for extrapolation of non-vanishing signals from their samples.

The paper present an analog of Sampling Theorem and a modification of Whittaker–Shannon-Kotelnikov interpolation formula for non-vanishing signals (Theorem 2.2 and formula (3) in Section 2). The kth coefficients for this new interpolation formula (3) are decreasing as $\sim 1/k^2$. Some numerical experiments are described in Section 3.

2 The main result: Sampling theorem and interpolation formula

Some notations

Let $\mathbf{R}$ and $\mathbf{C}$, be the set of all real and complex numbers, respectively, and let $\mathbb{Z}$ be the set of all integers.

We denote by $L_\infty(\mathbf{R})$ the standard space of all functions $x : \mathbf{R} \to \mathbf{C}$, considered up to equivalency, such that $\|x\|_{L_\infty(\mathbf{R})} := \operatorname{ess\,sup}_{t\in\mathbf{R}} |x(t)| < +\infty$.

For $r \in [1, \infty)$, we denote by $L_r(\mathbf{R})$ the standard space of all functions $x : \mathbf{R} \to \mathbf{C}$, considered up to equivalency, , such that $\|x\|_{L_r(\mathbf{R})} := \left(\int_{-\infty}^{\infty} |x(t)|^r dt\right)^{1/r} < +\infty$.

We denote by $C(\mathbf{R})$ the standard linear space of continuous functions $f : \mathbf{R} \to \mathbf{C}$ with the uniform norm $\|f\|_C := \sup_t |f(t)|$.

Definition 2.1 *For a Borel measurable set $D \subset \mathbf{R}$ with non-empty interior, let $x \in L_\infty(\mathbf{R})$ be such that $\int_{-\infty}^{\infty} x(t)y(t)dt = 0$ for any $y \in L_1(\mathbf{R})$ such that $Y|_{\mathbf{R}\setminus D} \equiv 0$, where Y is the Fourier* *transform of y. In this case, we say that D is a spectral gap of $x \in L_\infty(\mathbf{R})$.*

For $\Omega \in (0, +\infty)$, we denote by $\mathcal{V}(\Omega)$ the set of all signals $x \in L_\infty(\mathbf{R})$ with the spectral gap $\mathbf{R} \setminus (-\Omega, \Omega)$. We call these signals band-limited.

We use the terms "spectral gap" and "band-limited" because, for a signal $x \in L_2(\mathbf{R})$, Definition 2.1 means that $X(\omega) = 0$ for $\omega \in D$, where X the Fourier transform of x. The standard Fourier transform is not applicable for general type non-vanishing signals from $L_\infty(\mathbf{R}$, however, we will use the terms "spectral gap" and "band-limited" for them as well. It is shown in Section 4.3 below that this is still justified with respect to the spectral properties of these signal.

Let $\Omega \in (\Omega, \pi)$ be given. Let some even integer number N be selected such that

$$N > \frac{\Omega}{\pi - \Omega}.$$

Further, let some $\Omega_1 \in (\Omega, \pi)$ be selected such that

$$N \geq \frac{\Omega_1}{\pi - \Omega_1}.$$

Clearly, such Ω_1 exists.

For $t \in [N, N+1)$ and $\tau = t - N \in [0, 1)$, let us select

$$g(t) = \frac{\pi \lfloor t \rfloor}{t} = \frac{\pi N}{N + \tau}.$$

It is easy to see that, for any $t \in [N, N+1)$, we have $g(t) \geq \pi N (N+1)^{-1} \geq \Omega_1$ and $g(t) t = \pi N$.

Assume that the function $g(t)$ is extended periodically from $[N, N+1)$ to $g : \mathbf{R} \to [\Omega_1, \pi]$. This function is right-continuous. In addition, $g(m) = \pi$ and $(t-m) g(t) = \pi N$ for any integer m and any $t \in [N+m, N+m+1)$.

Theorem 2.2 *For any continuous bounded band-limited signal $x \in C(\mathbf{R}) \cap \mathcal{V}(\Omega)$, for any integer m and any $t \in [N+m, N+m+1)$, we have that*

$$x(t) = \sum_{k \in \mathbb{Z}} a_k(t) x(k), \tag{2}$$

where $a_m(t) = 1 - \frac{g(t)}{\pi}$, and

$$a_k(t) = \frac{(t-m) \sin[g(t)(k-m)]}{\pi (k-m)(k-t)}, \qquad k \neq m. \tag{3}$$

The corresponding series is absolutely convergent.

In particular, formula (3) implies that $a_k(k) = 1$, and that $a_k(l) = 0$ for any integers k and l such that $k \neq l$.

It can be noted that, under the assumptions of Theorem 2.2, we have that

$$a_k(t) = \frac{(t-m)g(t)\mathrm{sinc}[g(t)(k-m)]}{\pi(k-t)} = \frac{N\mathrm{sinc}[g(t)(k-m)]}{k-t}, \quad k \neq m.$$

We used here that $(t-m)g(t) = \pi N$.

Corollary 2.3 *Theorem 2.2 implies that a non-vanishing process from $x \in \mathcal{V}(\mathbf{R}) \cap C(\mathbf{R})$, i.e., a band-limited process with the spectral gap $\mathbf{R} \setminus (-\Omega, \Omega)$, is uniquely defined for any $\theta \in \mathbf{R}$ by its one-sided sample $\{x(k)\}_{k \in \mathbb{Z}, k \leq \theta}$.*

3 Some numerical experiments

In some straightforward numerical experiments, we applied truncated interpolation classical interpolation formula (1) and our formula (3) for simulated band limited signals, with summation over $\{k \in \mathbb{Z} : \ |k| \leq L\}$, for some large enough L. For these experiments, we used $\Omega = \frac{5}{12}\pi$ and preselected $\Omega_1 = (\Omega + \pi)/2$. The number N was selected as the smallest even number such that $N > \Omega_1/(\pi - \Omega_1)$. It can be noted that these choices define the simulated signal uniquely. We experimented with band-limited vanishing signals from $L_2(\mathbf{R})$ as well as with non-vanishing signals from $L_\infty(\mathbf{R})$.

For band-limited vanishing signals from $L_2(\mathbf{R})$, we found that the results were indistinguishable for both formulae (1) and (3). In particular, we considered a band-limited signal $x(t) = A[\mathrm{sinc}(M\pi t) + \mathrm{sinc}(M\pi(t-1)/2]$, with $M = 256$ and $A = \sqrt{M4/5}$. This signal has been used for numerical examples in [6], p.30.

We estimated $x(t)$ at several arbitrarily selected single points. For example, for $t = 47830.4$, we found the following.

- The error for both interpolation for $L = 10^3$ was about 10^{-5}.
- The error for both interpolation for $L = 10^4$ was about 10^{-6}.
- The error for both interpolation for $L = 10^5$ was about 10^{-7}. More precisely, the error for interpolation (1) was $1.3503206299415515963 \cdot 10^{-7}$, and the error for interpolation (3) was $7.6562947048617628244 \cdot 10^{-7}$.

In addition, we tested these interpolation formulae for some non-vanishing processes. In particular, we considered signal $x(t) = \cos(\Omega t - L/2))$. Again, we estimated $x(t)$ at several arbitrarily selected single points. For $t = 47830.4$, we found the following.

- For $L = 10^3$, the errors for the interpolation was about 10^{-5} for (1), and about 10^{-6} for (3).
- For $L = 10^3$, the errors for the interpolation was about 10^{-5} for (1), and about 10^{-6} for (3).
- For $L = 10^4$, the errors for the interpolation was about 10^{-5} for (1), and about 10^{-8} for (3).
- For $L = 10^5$, the errors for the interpolation was about 10^{-6} for (1), and about 10^{-11} for (3). More precisely, the error for interpolation (1) was $1.577106079952983464 \cdot 10^{-6}$, and the error for interpolation (3) was $8.7279850013999293878 \cdot 10^{-11}$.

These experiments show that formula (3) can replace formula (1) for band-limited vanishing signals, and can be used for effectively for non-vanishing signals as well.

4 Background: spectral representation for non-vanishing signals

In this section, we outline some results being used in the proof for the main Theorem 2.2.

4.1 Some notations and definitions for spaces of functions

We denote by $\overline{z}$ the complex conjugation. We denote by $*$ the convolution

$$(h * x)(t) := \int_{-\infty}^{\infty} h(t-s)x(s)ds, \quad t \in \mathbf{R}.$$

For a Banach space $\mathcal{X}$, we denote by $\mathcal{X}^*$ its dual.

For $r \in [1,\infty)$, we denote by ℓ_r the set of all processes (signals) $x : \mathbb{Z} \to \mathbf{C}$, such that $\|x\|_{\ell_r} := \left(\sum_{t=-\infty}^{\infty} |x(t)|^r\right)^{1/r} < +\infty$. We denote by ℓ_∞ the set of all processes (signals) $x : \mathbb{Z} \to \mathbf{C}$, such that $\|x\|_{\ell_\infty} := \sup_{t\in\mathbb{Z}} |x(t)| < +\infty$.

We denote by $W_2^1(\mathbf{R})$ the Sobolev space of functions $f : \mathbf{R} \to \mathbf{C}$ that belong to $L_2\mathbf{R}$ together with the distributional derivatives up to the first order.

Clearly, the embeddings $W_2^1(\mathbf{R}) \subset C(\mathbf{R})$ and $C(\mathbf{R})^* \subset W_2^1(\mathbf{R})^*$ are continuous.

Let $\overset{p}{W_2^1}(-\pi,\pi)$ denote the Sobolev space of functions $f : [-\pi,\pi] \to \mathbf{C}$ that belong to $L_2(-\pi,\pi)$ together with the distributional derivatives up to the first order, and such that $f(-\pi) = f(\pi)$.

Let $\mathcal{C}$ be the space of functions $f \in C([-\pi,\pi])$ with the finite norm $\|f\|_{\mathcal{C}} := \sum_{k\in\mathbb{Z}} |\widehat{f}_k|$, where $\widehat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\omega s} f(s)ds$ are the Fourier coefficients of f. In other words, $\mathcal{C}$ is the space of absolutely convergent Fourier series on $[-\pi,\pi]$. By the choice of its norm, this is a separable Banach space that is isomorphic to ℓ_1.

Lemma 4.1 *i. The embedding* $\overset{p}{W_2^1}(-\pi,\pi) \subset \mathcal{C}$ *is continuous.*

ii. *If $f \in \mathcal{C}$ and $g \in \mathcal{C}$, then $h = fg \in \mathcal{C}$, and $\|h\|_{\mathcal{C}} \le \|f\|_{\mathcal{C}} \|g\|_{\mathcal{C}}$.*

iii. *For $f \in \mathcal{C}$, define $g_f(\omega, m) := e^{im\omega} f(\omega)$, where $m \in \mathbb{Z}$, $\omega \in \mathbf{R}$. Then $g_f(\cdot, t) \in \mathcal{C}$ and $\|g_f(\cdot, t\|_{\mathcal{C}} = \|f\|_{\mathcal{C}}$.*

Let $\mathcal{A}$ be the space of continuous functions $f \in C(\mathbf{R})$ with the finite norm $\|f\|_{\mathcal{A}} := \int_{\mathbf{R}} |\widehat{f}(\omega)| d\omega$, where $\widehat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega s} f(s) ds$ is the Fourier transform of f. By the choice of this norm, this is a separable Banach space that is isomorphic to $L_1(\mathbf{R})$.

In particular, the definition for $\mathcal{A}$ implies that $Y \in \mathcal{A}$ in Definition 2.1.

It can be noted that there are functions in $C(\mathbf{R})$ that do not belong to $\mathcal{A}$.

Lemma 4.2 *i.* *The embedding $W_2^1(\mathbf{R}) \subset \mathcal{A}$ is continuous.*

ii. *If $f \in \mathcal{A}$ and $g \in \mathcal{A}$, then $h = fg \in \mathcal{A}$, and $\|h\|_{\mathcal{A}} \le \|f\|_{\mathcal{A}} \|g\|_{\mathcal{A}}$.*

iii. *For $f \in \mathcal{A}$, define $g_f(\omega, t) := e^{it\omega} f(\omega)$, where $t, \omega \in \mathbf{R}$. Then $g_f(\cdot, t) \in \mathcal{A}$, $\|g_f(\cdot, t\|_{\mathcal{A}} = \|f\|_{\mathcal{A}}$, and the function $g(\cdot, t)$ is continuous in $\mathcal{A}$ with respect to $t \in \mathbf{R}$.*

In particular, it follows that the embeddings $W_2^1(\mathbf{R}) \subset \mathcal{A} \subset C(\mathbf{R}) \subset L_1(\mathbf{R})$ and $L_1(\mathbf{R})^* \subset C(\mathbf{R})^* \subset \mathcal{A}^* \subset W_2^1(\mathbf{R})^*$ are continuous.

We assume that each $X \in L_1(\mathbf{R})$ represents an element of the dual space $C(\mathbf{R})^*$ such that $\langle X, f \rangle = \int_{-\infty}^{\infty} X(\omega) f(\omega) d\omega$ for $f \in C(\mathbf{R})$. We will use the same notation $\langle \cdot, \cdot \rangle$ for the extension of this bilinear form on $\mathcal{A}^* \times \mathcal{A}$.

4.2 Spectral representation for non-vanishing signals

The space $\mathcal{A}$ and its dual $\mathcal{A}^*$ have been used to define formally a spectral representation for $x \in \ell_\infty$ via $X \in \mathcal{A}^*$ such that $\langle X, f \rangle = \int_{-\infty}^{\infty} x(t) \varphi(t) dt$ for any $f \in \mathcal{A}$, where $\varphi \in L_1(\mathbf{R})$ is the Fourier transfer for f; see, e.g., Chapter VI in [5]. In Chapter III in [3], a similar definition was used for the Fourier transforms for pseudo-measures on $[-\pi, \pi]$ represented as elements of ℓ_∞. However, for the purposes of this paper, we will use a more straightforward definition from [2] based on the following lemma.

Proposition 4.3 *For any $x \in L_\infty(\mathbf{R})$, there exists a weak* limit $X \in \mathcal{A}^*$ of the sequence of functions $X_m(\omega) := \int_{-m}^{m} e^{-i\omega t} x(t) dt$ defined on $\mathbf{R}$ for $m > 0$. This X is such that $\|X\|_{\mathcal{A}^*} = \|x\|_{L_\infty(\mathbf{R})}$.*

It can be noted that, in Proposition 4.3, $X_m \in L_1(\mathbf{R}) \subset C(\mathbf{R})^* \subset \mathcal{A}^*$.

We define a spectral representation of $x \in L_\infty(\mathbf{R})$ via mapping $\mathcal{F} : L_\infty(\mathbf{R}) \to \mathcal{A}^*$ such that $X = \mathcal{F}x$ for $x \in L_\infty(\mathbf{R})$ is the limit in $\mathcal{A}^*$ introduced in Proposition 4.3. By Proposition 4.3, this mapping is linear and continuous.

Clearly, for $x \in L_1(\mathbf{R})$, $\mathcal{F}x$ is the standard Fourier transform, and $\mathcal{G} = \mathcal{F}^{-1}$ is the inverse Fourier transform.

Further, for any $h \in L_1(\mathbf{R})$, define a mapping $\mathcal{M}_h : \mathcal{A}^* \to L_\infty(\mathbf{R})$ such that $y_h = \mathcal{M}_h X$ is defined as

$$y_h(t) = \frac{1}{2\pi}\langle X, H(\cdot)e^{i\cdot t}\rangle \quad \text{for} \quad X \in \mathcal{A}^*, \quad H = \mathcal{F}h, \quad t \in \mathbf{R}.$$

By Lemma 4.1(iii), it follows that $H(\cdot)e^{i\cdot t} \in \mathcal{A}$ for any $t \in \mathbf{R}$, and it is continuous in t in the topology of $\mathcal{A}$.

Remark 4.4 *For the special case where $X \in L_1(\mathbf{R})$, the standard results for Fourier transformations imply for $h \in \mathcal{A}$ that $y_h(t) = \frac{1}{2\pi}\langle Y_h, e^{i\cdot t}\rangle$ for any $t \in \mathbf{R}$, where $Y_h = HX$. In this case, the form $\langle HX, e^{i\cdot t}\rangle$ is well defined since $H \in C(\mathbf{R})$ and hence $HX \in L_1(\mathbf{R})$.*

Clearly, the operator $\mathcal{M}_h : \mathcal{A}^* \to L_\infty(\mathbf{R})$ is linear and continuous for any $h \in \mathcal{A}$. Moreover, $y_h(t)$ is continuous in t, $\mathcal{M}_h(\mathcal{A}^*) \subset C(\mathbf{R})$, and the mapping $\mathcal{M}_h : \mathcal{A}^* \to C(\mathbf{R})$ is continuous.

Lemma 4.5 *i. For any $x \in L_\infty(\mathbf{R})$ and $X = \mathcal{F}x$, we have that $(h * x)(t) = y_h(t)$, where $y_h = \mathcal{M}_h X$.*

ii. For any $X \in \mathcal{A}^*$ and $y = \mathcal{M}_h X$, there exists an unique up to equivalency process $x \in L_\infty(\mathbf{R})$ such that $(h * x)(t) = y_h(t)$ for any $h \in \mathcal{A}$ for all t. For this process, we have that $\|x\|_{L_\infty(\mathbf{R})} \le \|X\|_{\mathcal{A}^*}$, and $\mathcal{F}x = X$.

We define an operator $\mathcal{G} : \mathcal{A}^* \to L_\infty(\mathbf{R})$ such that $x = \mathcal{G}X$ in Lemma 4.5(ii) above.

Theorem 4.6 *The mappings $\mathcal{F} : L_\infty(\mathbf{R}) \to \mathcal{A}^*$ and $\mathcal{G} : \mathcal{A}^* \to L_\infty(\mathbf{R})$ are continuous isometric bijections such that $\mathcal{F} = \mathcal{G}^{-1}$ and $\mathcal{G} = \mathcal{F}^{-1}$.*

4.3 Band-limited signals and spectral representation

The following lemma connects Definition 2.1 with the spectral representation.

Lemma 4.7 *A signal $x \in L_\infty(\mathbf{R})$ has a spectral gap $D \subset \mathbf{R}$ if and only if $\langle \mathcal{F}x, f\rangle = 0$ for any $f \in \mathcal{A}$ such that $f|_{\mathbf{R}\setminus D} \equiv 0$.*

This implies that, for any signal $x \in \mathcal{V}(\Omega)$ and any $f_1, f_2 \in \mathcal{A}$, if $f_1(\omega) = f_2(\omega)$ for all $\omega \in [-\Omega, \Omega]$ then $\langle \mathcal{F}x, f_1\rangle = \langle \mathcal{F}x, f_2\rangle$.

5 Proofs

5.1 Proofs of auxiliary results

The proof for Lemma 4.1 can be found in [1].

Proof of Lemma 4.7. Let $y \in L_1(\mathbf{R})$, and let $Y = \mathcal{F}y \in L_\infty(\mathbf{R})$. It follows from the definitions that $Y \in \mathcal{A}$. Let $h(t) = \bar{y}(-t)$ and $H = \mathcal{F}h$. We have that $Y = \bar{H}$. Furthermore,

$$\int_{-\infty}^{\infty} x(t)y(t)dt = \int_{-\infty}^{\infty} x(t)\bar{h}(-t)dt = (\bar{h} * x)(0) = \frac{1}{2\pi}\langle X, He^{i\cdot 0}\rangle = \frac{1}{2\pi}\langle X, H\rangle = \frac{1}{2\pi}\langle X, \bar{Y}\rangle.$$

Since $Y(\omega) = 0$ if and only if $\bar{Y}(\omega) = 0$, the lemma statement follows from the definitions. □

The proofs for the remaining statements listed in Section 4.2 can be found in [2].

5.2 Proof of Theorem 2.2

As the first step to prove Theorem 2.2, we need to obtain its more abstract conditional version.

Proposition 5.1 *Let $\Omega \in (0, \pi)$ be given, and let $\Omega_1 \in (\Omega, \pi)$ be selected. Suppose that there exists a function $E : \mathbf{R} \times \mathbf{R} \to \mathbf{C}$ such that the following holds.*

i. $E(t, \omega) = e^{i\omega t}$ for all $t \in \mathbf{R}$ and $\omega \in [-\Omega_1, \Omega_2]$.

ii. For any t, $E(t, \cdot)|_{[-\pi,\pi]} \in \mathcal{C}$ and $\sup_{t\in\mathbf{R}} \|E(t, \cdot)|_{[-\pi,\pi]}\|_{\mathcal{C}} < +\infty$.

iii. For any t, $E(t, \cdot) \in \mathcal{A}$ and $\sup_{t\in\mathbf{R}} \|E(t, \cdot)\|_{\mathcal{A}} < +\infty$.

For $t \in \mathbf{R}$ and $k \in \mathbb{Z}$, let

$$\mathrm{a}_k(t) := \frac{1}{2\pi}\int_{-\pi}^{\pi} E(t, \omega)e^{-i\omega k}d\omega.$$

Then any signal $x \in C(\mathbf{R}) \cap \mathcal{V}(\Omega)$ can be represented as

$$x(t) = \sum_{k\in\mathbb{Z}} \mathrm{a}_k(t)x(k).$$

The corresponding series is absolutely convergent. In addition, if $E(t, \omega) = \overline{E(t, -\omega)}$ for all t and ω, then $\mathrm{a}_k(t) \in \mathbf{R}$.

It can be noted that, under the assumption of Lemma 5.1, we have that

i. $\mathrm{a}_k(k) = 1$ and $\mathrm{a}_k(m) = 0$ for all $t \in \mathbf{R}$ and all integers k and m, $m \neq k$;

ii. $\{\mathrm{a}_k(t)\}_{k\in\mathbb{Z}} \in \ell_1$ for all t.

Proof of Proposition 5.1. Suppose that $E(t,\omega)$ is such as described in Lemma 5.1. Since $E(t,\cdot)\in\mathcal{A}$, we have that $\{\alpha_k(t)\}\in\ell_1$ for all t. Hence

$$E(t,\omega)=\sum_{k\in\mathbb{Z}}\mathrm{a}_k(t)e^{i\omega k}=\sum_{k\in\mathbb{Z}}\mathrm{a}_k(t)E(k,\omega),\quad t\in\mathbf{R},\quad \omega\in[-\Omega_1,\Omega_1],$$

where the series are absolutely convergent for any t, ω. Moreover, the sum

$$E(t,\cdot)=\sum_{k\in\mathbb{Z}}\mathrm{a}_k(t)E(k,\cdot)\tag{4}$$

converges in $\mathcal{A}$ for any $t\in\mathbf{R}$.

It can be reminded that $e^{i\omega t}=\sum_{k\in\mathbb{Z}}\mathrm{a}_k(t)e^{i\omega k}$ for $\omega\in[-\Omega_1,\Omega_1]$, but this does not hold if $|\omega|>\Omega_1$.

Further, let $\mathcal{A}_\Omega$ be the set of all $h\in\mathcal{A}$ such that $h(t)=0$ if $|t|>\pi-\Omega_1$. Clearly, $(h*e^{i\cdot t})(\omega)=(h*E(t,\cdot))(\omega)$ if $\omega\in[-\Omega,\Omega]$ and $h\in\mathcal{A}_\Omega$. Let $X=\mathcal{F}x$. By Lemma 4.5, by Theorem 4.6, and by the choice of E and X, we have that

$$y_h(t)=(h*x)(t)=\frac{1}{2\pi}\langle X,h*e^{i\cdot t}\rangle=\frac{1}{2\pi}\langle X,h*E(t,\cdot)\rangle$$

for all t and all $h\in\mathcal{A}_\Omega$. Hence

$$x(t)=\frac{1}{2\pi}\langle X,E(t,\cdot)\rangle,\quad t\in\mathbf{R}.$$

In particular,

$$x(k)=\frac{1}{2\pi}\langle X,E(k,\cdot)\rangle,\quad k\in\mathbb{Z}.$$

By (4), it follows that

$$x(t)=\frac{1}{2\pi}\langle X,\sum_{k\in\mathbb{Z}}\mathrm{a}_k(t)E(k,\omega)\rangle=\sum_{k\in\mathbb{Z}}\mathrm{a}_k(t)\frac{1}{2\pi}\langle X,E(k,\omega)\rangle=\sum_{k\in\mathbb{Z}}\mathrm{a}_k(t)x(k).$$

This completes the proof of Lemma 5.1. □

The following step is to find a function E satisfying the assumptions of Proposiotion 5.1.

Up to the end of this paper, we assume that N and g are selected as in Theorem 2.2. In particular, $g(m)=\pi$ for any $m\in\mathbb{Z}$.

Lemma 5.2 *We have that $e^{ig(t)t}=e^{-ig(t)t}=1$ for all $t\in[N,N+1)$. In addition, $e^{ig(t)(t-m)}=e^{-ig(t)(t-m)}=1$ for all $m\in\mathbb{Z}$ and $t\in[N+m,N+m+1)$.*

Proof of Lemma 5.2. Let $t = N + \tau$ and $\tau \in [0, 1)$. We have

$$g(t) = \frac{\pi \lfloor t \rfloor}{t} = \frac{\pi N}{N + \tau}.$$

Clearly, $\Omega_1 \leq g(t) \leq \pi$, $g(t)t = \pi N$ and

$$e^{ig(t)t} = e^{i\pi N} = 1, \quad t \in [N, N + 1).$$

We used here that N is even. Further, let $m \in \mathbb{Z}$. By the choice of g, we have that $g(t+m) = g(t)$, and

$$e^{ig(t)(t-m)} = e^{ig(t-m)(t-m)} = 1, \quad t \in [N + m, N + m + 1).$$

This completes the proof of Lemma 5.2. $\square$

Lemma 5.3 *Let a function* $\widetilde{E} : [N, N + 1] \to \mathbf{C}$ *be defined as*

$$\widetilde{E}(t, \omega) = e^{i\omega t}, \qquad \omega \in [-g(t), g(t)],$$
$$\widetilde{E}(t, \omega) = e^{ig(t)t}, \qquad \omega \notin [-g(t), g(t)].$$

Further, let a function $\xi : [N, N + 1] \times \mathbf{R} \to \mathbf{R}$ *be selected such that* $\xi(t, \cdot) \in W_2^1(\mathbf{R})$ *for any* t, *and* $\xi(t, \omega) = 1$ *for any* $\omega \in [-g(t), g(t)]$. *We define the function* $E_N : \mathbf{R} \times \mathbf{R} \to \mathbf{C}$ *as*

$$E_N(t, \omega) = \widetilde{E}(t, \omega)\xi(t, \omega), \qquad \omega \in \mathbf{R}, \quad t \in [N, N + 1].$$

Further, let $t = m + N + \tau$, *where* $\tau \in [0, 1)$ *and* $m \in \mathbb{Z}$. *Let*

$$E(t, \omega) = E_N(\tau, \omega)e^{i\omega m}, \qquad \omega \in \mathbf{R}, \quad t \in \mathbf{R}.$$

Then the conditions (i)-(iii) of Proposition 5.1 hold for these E, *In addition,* $E(t, \omega) = \overline{E(t, -\omega)}$ *for all* t *and* ω.

Proof of Lemma 5.3. It is easy to see that condition (i) of Proposition 5.1 is satisfied for E.

Further, $\widetilde{E}(t, \cdot)|_{[-\pi,\pi]} \in \overset{p}{W_2^1} (-\pi, \pi)$ for any $t \in [N, N + 1]$, and

$$\sup_{t \in [N, N+1]} \|\widetilde{E}(t, \cdot)|_{[-\pi,\pi]}\|_{\overset{p}{W_2^1}(-\pi,\pi)} < +\infty.$$

Hence $E_N(t, \cdot)|_{[-\pi,\pi]} \in \mathcal{C}$ for any $t \in [N, N + 1)$ and

$$\sup_{t \in [N, N+1]} \|\widetilde{E}(N + \tau, \cdot)|_{[-\pi,\pi]}\|_{\mathcal{C}} < +\infty.$$

Furthermore, $E_N(t,\cdot) \in W_2^1(\mathbf{R})$ for any $t \in [N, N+1]$, and

$$\sup_{t\in[N,N+1]} \|\widetilde{E}(t,\cdot)\|_{W_2^1(\mathbf{R})} < +\infty.$$

Hence condition (ii) of Proposition 5.1 are satisfied for $E_N(t,\omega)$ and for $t \in [N, N+1]$.

Further, by Lemma 4.1(iii), for any $v \in \mathcal{A}$, we have that

$$e^{i\cdot m} v \in \mathcal{A}, \quad \|v\|_{\mathcal{A}} = \|e^{i\cdot m} u\|_{\mathcal{A}}.$$

Similarly, we have for $u \in \mathcal{C}$ that

$$e^{i\cdot m} u \in \mathcal{C}, \quad \|u\|_{\mathcal{C}} = \|e^{i\cdot m} u\|_{\mathcal{C}}.$$

Then conditions (iii) are satisfied for the selected E, and Hence condition (iii) of Proposition 5.1 are satisfied for $E_N(t,\omega)$, for $t \in [N, N+1]$. In addition, $E(t,\omega) = \overline{E(t,-\omega)}$ for all t and ω. This completes the proof of Lemma 5.3. $\square$

It can be noted that $E(t, g(t)) = \bar{E}(t, -g(t)) \neq E(t, -g(t))$ for E selected as in Lemma 5.3 for non-integer $t \notin \{N, N+1\}$.

We will denote by $a_k(t)$ the corresponding coefficients $\mathrm{a}_k(t)$ defined as in Proposition 5.1 with E and g defined by Lemma 5.3.

Proposition 5.4 *Let E be selected as in Lemma 5.3. For $t \in [N, N+1)$, we have that $a_0(t) = 1 - \frac{g(t)}{\pi}$ and*

$$a_k(t) = \frac{t \sin[g(t)k]}{\pi k(k-t)}, \quad k \neq 0.$$

It can be noted that, since $g(t)t = \pi N$ for $t \in [N, N+1)$, we have that

$$a_k(t) = \frac{g(t)t\,\mathrm{sinc}(g(t)k)}{\pi(k-t)} = \frac{N\mathrm{sinc}(g(t)k)}{k-t}, \quad k \neq 0.$$

Proof of Proposition 5.4. Clearly, $a_k(t) = \mathbb{I}_{k=t}$ for $t \in \mathbb{Z}$, by the choice of $g(k) = \pi$. Further, we have that

$$a_k(t) = \frac{1}{2\pi}(\alpha_k(t) + \beta_k(t)),$$

where

$$\alpha_k(t) = \int_{-g(t)}^{g(t)} e^{-i\omega k} e^{i\omega t} d\omega,$$
$$\beta_k(t) = \int_{-\pi}^{-g(t)} e^{-i\omega k} e^{ig(t)t} d\omega + \int_{g(t)}^{\pi} e^{-i\omega k} e^{ig(t)t} d\omega = e^{ig(t)t} \left(\int_{-\pi}^{-g(t)} e^{-i\omega k} d\omega + \int_{g(t)}^{\pi} e^{-i\omega k} d\omega \right).$$

Assume that $k \neq 0$. In this case,

$$\alpha_k(t) = \int_{-g(t)}^{g(t)} e^{-i\omega k} e^{i\omega t} d\omega = \frac{e^{ig(t)(t-k)} - e^{-ig(t)(t-k)}}{i(t-k)} = \frac{e^{-ig(t)k} - e^{ig(t)k}}{i(t-k)} = -\frac{2\sin(g(t)k)}{t-k},$$

$$\beta_k(t) = e^{ig(t)t} \left(\frac{e^{ig(t)k} - e^{i\pi k}}{-ik} + \frac{e^{-i\pi k} - e^{-ig(t)k}}{-ik} \right) = e^{ig(t)t} \left(\frac{e^{-ig(t)k} - e^{ig(t)k}}{ik} \right)$$
$$= -e^{ig(t)t} \frac{2\sin(g(t)k)}{k},$$

and

$$a_k(t) = \frac{1}{2\pi}(\alpha_k(t) + \beta_k(t)) = -\frac{1}{2\pi} \left(\frac{2\sin(g(t)k)}{t-k} + e^{ig(t)t} \frac{2\sin(g(t)k)}{k} \right).$$

By the choice of even N, we have that $e^{ig(t)t} = e^{i\pi N} = 1$. Hence

$$a_k(t) = \frac{1}{2\pi}(\alpha_k(t) + \beta_k(t)) = -\frac{1}{\pi} \sin(g(t)k) \left(\frac{1}{t-k} + \frac{1}{k} \right) = \frac{1}{\pi} \sin(g(t)k) \frac{t}{k(k-t)}. \qquad (5)$$

Further, assume that $k = 0$. In this case,

$$\alpha_0(t) = \int_{-g(t)}^{g(t)} e^{i\omega t} d\omega = -\frac{e^{ig(t)t} - e^{-ig(t)t}}{it} = -\frac{e^{iN\pi} - e^{-iN\pi}}{it}.$$

Since $t \geq N > 0$, we have that $\alpha_0(t) = 0$. Further, we have $\beta_0(t) = 2e^{ig(t)t}(\pi - g(t)) = 2(\pi - g(t))$. Hence $a_0(t) = 1 - \frac{g(t)}{\pi}$. This completes the proof of Proposition 5.4. $\square$

Lemma 5.5 *Let E be selected as in Lemma 5.3. For any $k, m \in \mathbb{Z}$, we have that*

$$a_k(t+m) = a_{k-m}(t).$$

Proof of Lemma 5.5. Let N be defined as in Lemma 5.3, and let $t = N + \tau$, By the definitions, $E(t,\omega) = e^{iM\omega} E(\tau, t)$, where $M \in \mathbb{Z}$ is such that $\tau = t - M \in [N, N+1)$. Hence $E(t+m,\omega) = e^{im\omega} E(t,\omega)$ and

$$a_k(t+m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{im\omega} E(t,\omega) e^{-i\omega k} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} E(t,\omega) e^{-i\omega(k-m)} d\omega = a_{k-m}(t).$$

This completes the proof of Lemma 5.5. $\square$

Corollary 5.6 *Let E be selected as in Lemma 5.3. Let $a_k(\cdot)$ be defined by (3). Then for any $m \in \mathbb{Z}$, any signal $x \in \mathcal{V}(\Omega) \cap C(R)$ can be represented, for $t \in [N+m, N+m+1)$, as*

$$x(t) = \sum_{k \in \mathbb{Z}} a_{k-m}(t-m) x(k). \qquad (6)$$

The corresponding series is absolutely convergent.

Remark 5.7 *Corollary 5.6 is due to the particular choice of acceptable E. Possibly, there exist acceptable choices of E such that does not hold.*

Proof of Corollary 5.6. Let $\widetilde{x}(t) := x(t+m)$. It is easy to see that $\widetilde{x} \in \mathcal{V}(\Omega)$. Clearly, $x(t) = \widetilde{x}(t-m)$ for all t. By Proposition 5.4, we have that

$$\widetilde{x}(s) = \sum_{k\in\mathbb{Z}} a_k(s)\widetilde{x}(k), \quad s \in [0,1).$$

Hence, for $t \in [m+N, m+N+1)$,

$$x(t) = \widetilde{x}(t-m) = \sum_{k\in\mathbb{Z}} a_k(t-m)\widetilde{x}(k) = \sum_{k\in\mathbb{Z}} a_k(t-m)x(k+m) = \sum_{k\in\mathbb{Z}} a_{d-m}(t-m)x(d).$$

This completes the proof of Corollary 5.6. □

Proof of Theorem 2.2 follows immediately from Proposition 5.4 and Corollary 5.6. □

Proof Corollary 2.3. By Theorem 2.2, the signal x is uniquely defined by the sequence $\{x(k)\}_{k\in\mathbb{Z}}$. Further, it can be shown that the sequence $\{x(k)\}_{k\in\mathbb{Z}}$ represents a band-limited bounded discrete time signal as defined in Theorem 4 [1]. Then Corollary 2.3 follows from Theorem 4 [1]. □

Concluding remarks

i. Since $|a_k(t)| \sim 1/k^2$ as $|k| \to +\infty$, we have that $\sum_{k\in\mathbb{Z}} |a_k(t)| < +\infty$ for any $t \in \mathbf{R}$. This allows to apply interpolation formula (3) to non-vanishing bounded signals. For these signals, the classical interpolation formula (1) in not applicable, since the coefficients decay as $\sim 1/k$.

ii. It can be seen that selection of N and Ω_1 for interpolation formula (2)-(3) is non-unique. Furthermore, it is possible that there are other potential choices of E in Proposition 5.1, leading to other versions of interpolation formula (2).

iii. The condition that $\Omega \in (0,\pi)$, and that the sampling points are integers, can be removed, as usual, by linear changes of the times scale, i.e., with the replacement of the signal $x(t)$ by signal $x(\mu t)$, with $\mu > 0$. Clearly, less frequent sampling would require $\mu > 1$, and selection of a larger Ω would require $\mu < 1$.

iv. The classical Whittaker-Shannon-Kotelnikov interpolation formula (1) allows spectrum bandwidth $[-\pi,\pi]$. On the other hand, Theorem 2.2 requires that the spectrum bandwidth of x is $[-\Omega,\Omega]$, for $\Omega \in (0,\pi)$. Therefore, the possibility to cover non-vanishing signals is achieved via certain oversampling; this oversampling, however, can be arbitrarily small, since Ω can be arbitrarily close to π.

v. It can be emphasised that the interpolation formula (3) is exact; it is not an approximation. Therefore, for a vanishing signal $x \in L_2(\mathbf{R}) \cap \mathcal{V}(\Omega) \cap C(\mathbf{R})$, both formulae (1) and (3) give the same value. Similarly, for $x \in \mathcal{V}(\Omega_0)$ for $\Omega_0 \in (0, \pi)$, for all possible different choices of $\Omega \in [\Omega_0, \pi)$, $\Omega_1 \in [\Omega, \pi)$, $N = N(\Omega_1)$, and E, the value of the sum (2) is the same. Of course, the values for the corresponding finite truncated sums will be different.

vi. It is known that band-limited signals from $L_2(\mathbf{R})$ are continuous. However, Theorem 2.2 is formulated for signals from $C(\mathbf{R}) \cap \mathcal{V}(\Omega)$ rather than for signals from $\mathcal{V}(\Omega)$, since it is unclear yet if general type band-limited signals $x \in \mathcal{V}(\Omega)$ are continuous. We leave it for the future research.

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