

Lie–Hamilton systems on Riemannian and Lorentzian spaces from conformal transformations and some of their applications

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Abstract

We propose a generalization of two classes of Lie–Hamilton systems on the Euclidean plane to two-dimensional curved spaces, leading to novel Lie–Hamilton systems on Riemannian spaces (flat 2-torus, product of hyperbolic lines, sphere and hyperbolic plane), pseudo-Riemannian spaces (anti-de Sitter, de Sitter, and Minkowski spacetimes), as well as to semi-Riemannian spaces (Newtonian or non-relativistic spacetimes). The vector fields, Hamiltonian functions, symplectic form and constants of the motion of the Euclidean classes are recovered by a contraction process. The construction is based on the structure of certain subalgebras of the so-called conformal algebras of the two-dimensional Cayley–Klein spaces. These curved Lie–Hamilton classes allow us to generalize naturally the Riccati, Kummer–Schwarz and Ermakov equations on the Euclidean plane to curved spaces, covering both the Riemannian and Lorentzian possibilities, and where the curvature can be considered as an integrable deformation parameter of the initial Euclidean system.

Keywords: Cayley–Klein geometries; Lie systems; nonlinear superposition rules; constants of motion; Ermakov equation; Riccati equations; Kummer–Schwarz equations

MSC 2020 codes: 34A26 (Primary), 17B66, 70G45, 34A34 (Secondary)

PACS 2010 codes: 02.20.Sv, 02.30.Hq, 02.40.Dr, 02.40.Ky

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1 Introduction

A *Lie system* is a time-dependent system of first-order ordinary differential equations (ODEs in short) in normal form that admits a *superposition rule*, i.e., a time-independent map through which the general solution of the system is expressed in terms of a generic finite family of particular solutions and some constants related to the initial conditions [1–5]. The fundamental Lie–Scheffers theorem states that a Lie system is equivalent to a time-dependent vector field that can be seen as a curve taking values in a finite-dimensional Lie algebra of vector fields, a so-called *Vessiot–Guldberg Lie algebra* (VG Lie algebra in short) [6].

A *Lie–Hamilton system* (LH system in short) is a Lie system whose VG Lie algebra is formed by Hamiltonian vector fields with respect to a certain symplectic structure. These enriched Lie systems have been widely studied and classified under local diffeomorphisms on the real plane (see [7, 8] and references therein). One of the most remarkable properties of LH systems is that they allow an algorithmic computation of time-independent constants of the motion through the so-called *coalgebra formalism*, hence eventually simplifying the deduction of a superposition rule [9].

In order to explicitly state our main objectives, let us consider the so-called *Ermakov equation* [10] on the Euclidean line $\mathbf{E}^1 := \mathbb{R}$ (see also [11–14] and references therein), given by

$$\frac{d^2u}{dt^2} = -\Omega^2(t)u + \frac{c}{u^3}, \quad c \in \mathbb{R},$$

where $\Omega(t)$ is any t -dependent real or pure imaginary function. Recall that this equation is also known as the Milne–Pinney equation [15, 16] and that it is equivalent to the following first-order system of ODEs on the Euclidean plane $\mathbf{E}^2 := \mathbb{R}^2$:

$$\frac{du}{dt} = v, \quad \frac{dv}{dt} = -\Omega^2(t)u + \frac{c}{u^3}. \quad (1.1)$$

Hence this system is associated with the t -dependent vector field $\mathbf{X} : \mathbb{R} \times \mathbf{E}^2 \rightarrow T\mathbf{E}^2$ given by

$$\mathbf{X} = \mathbf{X}_3 + \Omega^2(t)\mathbf{X}_1, \quad (1.2)$$

where the vector fields

$$\mathbf{X}_1 = -u \frac{\partial}{\partial v}, \quad \mathbf{X}_2 = \frac{1}{2} \left(v \frac{\partial}{\partial v} - u \frac{\partial}{\partial u} \right), \quad \mathbf{X}_3 = v \frac{\partial}{\partial u} + \frac{c}{u^3} \frac{\partial}{\partial v} \quad (1.3)$$

fulfil the commutation relations

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_1, \quad [\mathbf{X}_1, \mathbf{X}_3] = 2\mathbf{X}_2, \quad [\mathbf{X}_2, \mathbf{X}_3] = \mathbf{X}_3. \quad (1.4)$$

Therefore they span a VG Lie algebra V^X of \mathbf{X} (1.2) isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \simeq \mathfrak{so}(2, 1)$ for any value of the constant c [7–9]. Hence, the t -dependent vector field \mathbf{X} (or its associated equation (1.1)) is a Lie system (see [5] and references therein).

In addition, the vector fields (1.3) are Hamiltonian vector fields relative to the canonical symplectic form

$$\omega = du \wedge dv \quad (1.5)$$

on \mathbf{E}^2 , as they satisfy the invariance condition

$$\mathcal{L}_{\mathbf{X}_i} \omega = 0, \quad 1 \leq i \leq 3. \quad (1.6)$$

In order to clarify the terminology used throughout the paper, we observe that, as we always deal with vector fields and a symplectic form ω , the condition on the vanishing of the Lie derivative with respect to ω implies that the vector field is locally Hamiltonian (also called symplectic in the literature, see for instance [17]). In the particular case of the Ermakov system, as the domain of the latter is contractible, it can be inferred that the Hamiltonian functions are globally defined.

The associated Hamiltonian functions of the vector fields (1.3) are then determined by the inner product

$$\iota_{\mathbf{X}_i}\omega = dh_i, \quad 1 \leq i \leq 3, \quad (1.7)$$

turning out to be

$$h_1 = \frac{1}{2}u^2, \quad h_2 = -\frac{1}{2}uv, \quad h_3 = \frac{1}{2}\left(v^2 + \frac{c}{u^2}\right). \quad (1.8)$$

The Poisson bracket $\{\cdot, \cdot\}_\omega$, induced by ω , of the above functions leads to

$$\{h_1, h_2\}_\omega = -h_1, \quad \{h_1, h_3\}_\omega = -2h_2, \quad \{h_2, h_3\}_\omega = -h_3, \quad (1.9)$$

so that they span a Lie–Hamilton algebra (LH algebra in short) $\mathcal{H}_\omega \simeq \mathfrak{sl}(2, \mathbb{R})$. The Casimir element \mathcal{C} , that Poisson commutes with h_i , reads

$$\mathcal{C} = h_1h_3 - h_2^2, \quad (1.10)$$

which under the functional realization (1.8) gives the constant

$$\mathcal{C} = c/4. \quad (1.11)$$

Consequently, the t -dependent vector field \mathbf{X} (1.2) not only determines a Lie system but also a LH system [5].

It is worth noting that if we identify (u, v) with the usual canonical variables (q, p) of $T^*\mathbb{R}$ in classical mechanics, the Ermakov system (1.1) can also be obtained from the Hamilton equations determined by the t -dependent Hamiltonian given by (see (1.2))

$$h = h_3 + \Omega^2(t)h_1 = \frac{1}{2}p^2 + \frac{1}{2}\Omega^2(t)q^2 + \frac{c}{2q^2}.$$

In this form, h can be interpreted as the one-dimensional (1D) t -dependent counterpart [7–9] of the Smorodinsky–Winternitz oscillator [18], with time-dependent frequency $\Omega(t)$, unit mass and with a Rosochatius or Winternitz potential depending on the constant c ; note that the latter is in fact a centrifugal barrier whenever $c > 0$ (see [19] and references therein).

Consequently, although the value of the parameter c does not interfere in the algebraic part of the LH structure associated with the Ermakov equation (1.1), the sign of c (1.11) does matter. This leads to three types of submanifolds determined by the surfaces with constant value of the Casimir \mathcal{C} for the Poisson structure of $\mathfrak{sl}(2, \mathbb{R})$ [20]. This result agrees perfectly with the classification of LH systems on the Euclidean plane [7, 8]. Explicitly, there are three different classes of non-diffeomorphic $\mathfrak{sl}(2, \mathbb{R})$ -LH systems on \mathbb{E}^2 , which are distinguished among them by the value of c : class P_2 for $c > 0$; class I_4 for $c < 0$; and class I_5 for $c = 0$. This means that there does not exist any local diffeomorphism mapping one class into another, so that any LH system related to a VG Lie algebra of Hamiltonian vector fields isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ must be, up to a local change of coordinates, of the form (1.1) for a positive, zero or negative value of c .

The resulting (Hamiltonian) vector fields corresponding to one representative from each of these three classes are displayed in Table 1 in Cartesian coordinates (x, y) , along with its associated

Table 1: Classes P_2 , I_4 and I_5 of LH systems on the plane [7, 8]. The VG Lie algebra spanned by the vector fields \mathbf{X}_i ($1 \leq i \leq 3$) is isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \simeq \mathfrak{so}(2, 1)$ in the three cases, while the Hamiltonian functions h_i span a LH algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ with respect to the symplectic form ω .

Class	Vector fields \mathbf{X}_i	Hamiltonian functions h_i	ω	Domain	Casimir
P_2	$\partial_x, x\partial_x + y\partial_y, (x^2 - y^2)\partial_x + 2xy\partial_y$	$-\frac{1}{y}, -\frac{x}{y}, -\frac{x^2 + y^2}{y}$	$\frac{dx \wedge dy}{y^2}$	$\mathbb{R}_{y \neq 0}^2$	1
I_4	$\partial_x + \partial_y, x\partial_x + y\partial_y, x^2\partial_x + y^2\partial_y$	$\frac{1}{x-y}, \frac{x+y}{2(x-y)}, \frac{xy}{x-y}$	$\frac{dx \wedge dy}{(x-y)^2}$	$\mathbb{R}_{x \neq y}^2$	$-\frac{1}{4}$
I_5	$\partial_x, x\partial_x + \frac{1}{2}y\partial_y, x^2\partial_x + xy\partial_y$	$-\frac{1}{2y^2}, -\frac{x}{2y^2}, -\frac{x^2}{2y^2}$	$\frac{dx \wedge dy}{y^3}$	$\mathbb{R}_{y \neq 0}^2$	0

symplectic form and the constant value of the Casimir (1.10); note that these variables are different from (u, v) such that ω now becomes a non-canonical symplectic form. The vector fields \mathbf{X}_i obey the commutation relations (1.4), while the Hamiltonian functions h_i fulfil (1.9) for the three classes. Recall that, in addition to the Ermakov or Milne–Pinney equation, it has been shown in [7, 8] that $\mathfrak{sl}(2, \mathbb{R})$ -LH systems in Table 1 also comprise several types of Riccati equations [21–31] (see also references therein) and second-order Kummer–Schwarz equations [32–34].

The striking point now is that the vector fields of class P_2 expressed exactly in the form of Table 1 admit a natural geometric interpretation as conformal symmetries of \mathbf{E}^2 . In particular, they span a Lie subalgebra of the conformal Euclidean algebra $\mathfrak{so}(3, 1)$ such that the vector field \mathbf{X}_1 corresponds to the generator of translations along the axis x , \mathbf{X}_2 gives rise to dilations and \mathbf{X}_3 provides specific conformal transformations again related to the axis x . Likewise, the vector fields of class I_4 in Table 1 can be directly interpreted as conformal transformations of the space $\mathbf{E}^1 \times \mathbf{E}^1$: \mathbf{X}_1 as the translation, \mathbf{X}_2 as the dilation and \mathbf{X}_3 as the conformal transformation. However, the remaining class I_5 does not allow a conformal interpretation.

These ideas naturally suggest to construct the ‘curved’ counterparts of the classes P_2 and I_4 by making use of the known conformal symmetries of 2D spaces of constant curvature [35, 36]. Our procedure will make use of a Cayley–Klein (CK) setting that requires to deal with two explicit real (graded) contraction parameters κ_1 and κ_2 . The former corresponds to the constant (Gaussian) curvature of the space, while the latter determines the signature, so the nine 2D CK spaces are collectively denoted by $\mathbf{S}_{[\kappa_1], \kappa_2}^2$. The three 1D CK spaces $\mathbf{S}_{[\kappa]}^1$ are obtained as suitable submanifolds of some 2D CK spaces. In this way, we will obtain unified expressions that generalize those for P_2 and I_4 presented in Table 1 to the curved spaces $\mathbf{S}_{[\kappa_1], \kappa_2}^2$ and $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$, respectively. In particular, our novel results will cover LH systems on the three classical Riemannian spaces (sphere, Euclidean and hyperbolic), the three Lorentzian (Minkowski and (anti-)de Sitter) and the three Newtonian (Galilei and Newton–Hooke) spacetimes of constant curvature, as well as on the flat 2-torus and the product of two hyperbolic lines. We recall that a similar approach was already developed in [37] for the class P_1 of LH systems, but in that case it was based on the isometries of 2D CK spaces. Similarly, examples of Lie systems admitting a VG Lie algebra of conformal vector fields can also be found in [8, 31, 38] and, for the same reasons pointed out in [37], these results cannot be applied to the new Lie systems we propose in this work.

In addition, in the framework of Lie systems on homogeneous spaces it is worth recalling that in [3, 39] an ‘almost’ complete classification of superposition rules for *complex* Lie systems with primitive transitive VG Lie algebras of vector fields on homogeneous spaces has been achieved. However, we stress that there are not many results for Lie systems endowed with *real* VG Lie

algebras on homogeneous spaces, which constitute a much more difficult task (cf. [3, 39]). In this respect, some results on spheres can be found in [40].

Consequently, the construction developed in [37] constitutes, in fact, the first known example of LH systems on curved Riemannian, Lorenzian and Newtonian spaces, and a superposition rule for all these nine curved systems was obtained in a very geometrical way through the trigonometry of the 2D CK spaces [41]. Nevertheless, no relevant applications were found, as the Euclidean P_1 -LH class does not possess any remarkable system apart from the complex Bernoulli equation [8]. This last point was the main motivation of our novel conformal-based approach, as the Euclidean P_2 and I_4 -LH classes comprise several notable applications, as pointed out before.

This paper is structured as follows. After explicitly setting the conformal-based motivation of our work, in Section 2 we review the conformal symmetries of the 2D CK spaces $\mathbf{S}_{[\kappa_1], \kappa_2}^2$ introduced in [35], from which the conformal symmetries of the 1D CK spaces $\mathbf{S}_{[\kappa]}^1$ are deduced. In Section 3 we construct the curved counterpart of the Euclidean I_4 -LH class (see Table 1) on the spaces $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$, which we call the *curved I_4 -LH class*. Their t -independent constants of the motion are computed in Subsection 3.1, and a superposition rule for this curved class is derived in Subsection 3.2. In Section 4, some relevant applications of the curved I_4 -LH class are given, generalizing to $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$ well-known LH systems on the Euclidean plane: curved coupled Riccati equations, a curved split-complex Riccati equation, a curved diffusion Riccati system, a curved Kummer–Schwarz equation and a curved Ermakov equation. The same ansatz is applied in Section 5, where we construct the *curved P_2 -LH class* on the 2D CK spaces $\mathbf{S}_{[\kappa_1], \kappa_2}^2$, from which the Euclidean P_2 -LH class (see Table 1) is recovered after contraction. Their t -independent constants of motion are computed in Subsection 5.1, and a superposition rule on those 2D CK spaces $\mathbf{S}_{[\kappa_1], \kappa_2}^2$ with some $\kappa_i = 0$ is obtained in Subsection 5.2, as the computations in the general case are very cumbersome. In Section 6 we give remarkable applications of the curved P_2 -LH class: a curved complex Riccati equation, a curved Kummer–Schwarz equation and a curved Ermakov equation. Finally, in Section 7, some conclusions and open problems are drawn. In particular, the novel interpretation of the curvature as an integrable deformation parameter of LH systems is addressed to.

2 Conformal symmetries of Cayley–Klein spaces revisited

Let us consider the three-dimensional real Lie algebra $\mathfrak{so}(3)$ with commutation relations

$$[J_{12}, P_1] = P_2, \quad [J_{12}, P_2] = -P_1, \quad [P_1, P_2] = J_{12}$$

over a basis $\{P_1, P_2, J_{12}\}$ and with Casimir given by

$$C = P_1^2 + P_2^2 + J_{12}^2.$$

The automorphisms $\Theta_0, \Theta_{01} : \mathfrak{so}(3) \rightarrow \mathfrak{so}(3)$ defined by

$$\begin{aligned} \Theta_0(P_1) &= -P_1, & \Theta_0(P_2) &= -P_2, & \Theta_0(J_{12}) &= J_{12}, \\ \Theta_{01}(P_1) &= P_1, & \Theta_{01}(P_2) &= -P_2, & \Theta_{01}(J_{12}) &= -J_{12} \end{aligned} \quad (2.1)$$

generate a $\mathbb{Z}_2 \times \mathbb{Z}_2$ group of commuting involutive automorphisms of $\mathfrak{so}(3)$, inducing the following $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading of $\mathfrak{so}(3)$:

$$\mathfrak{so}(3) = E_{(1,0)} \oplus E_{(0,1)} \oplus E_{(1,1)},$$

where $E_{(1,0)} := \langle P_1 \rangle$, $E_{(0,1)} := \langle J_{12} \rangle$, $E_{(1,1)} := \langle P_2 \rangle$ and $E_{(0,0)} := \{0\}$ satisfy $[E_{(\alpha_1, \alpha_2)}, E_{(\beta_1, \beta_2)}] = E_{(\alpha_1 + \beta_1, \alpha_2 + \beta_2)}$ for every $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{Z}_2$. A particular solution of the set of $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded

contractions from $\mathfrak{so}(3)$ leads to a two-parametric family of real Lie algebras, denoted by $\mathfrak{so}_{\kappa_1, \kappa_2}(3)$ and given by [42]

$$[J_{12}, P_1] = P_2, \quad [J_{12}, P_2] = -\kappa_2 P_1, \quad [P_1, P_2] = \kappa_1 J_{12}, \quad (2.2)$$

where κ_1 and κ_2 are two real graded contraction parameters that can be reduced to either $+1$, 0 or -1 through a rescaling of the Lie algebra generators. Recall that the vanishment of any κ_i is equivalent to applying an *Inönü–Wigner contraction* [42, 43]. The corresponding Casimir reads

$$C_\kappa = \kappa_2 P_1^2 + P_2^2 + \kappa_1 J_{12}^2, \quad \kappa := (\kappa_1, \kappa_2).$$

The family $\mathfrak{so}_{\kappa_1, \kappa_2}(3)$ contains nine specific Lie algebras (some of them isomorphic). Simple Lie algebras arise when both $\kappa_i \neq 0$ ($\mathfrak{so}(3)$ for positive values and $\mathfrak{so}(2, 1) \simeq \mathfrak{sl}(2, \mathbb{R})$ otherwise) as well as non-simple ones when one $\kappa_i = 0$ (the inhomogenous $\mathfrak{iso}(1, 1)$, $\mathfrak{iso}(2)$ and $\mathfrak{iiso}(1)$, where $\mathfrak{iso}(1) \equiv \mathbb{R}$). The remarkable fact is that $\mathfrak{so}_{\kappa_1, \kappa_2}(3)$ contains all the Lie algebras of the motion groups of the so-called 2D *Cayley–Klein (CK) homogeneous spaces* [44–46] and thus $\mathfrak{so}_{\kappa_1, \kappa_2}(3)$ is called a *CK algebra*.

Let us now exhibit the connection between the CK algebras $\mathfrak{so}_{\kappa_1, \kappa_2}(3)$ and the 2D CK homogeneous spaces explicitly. The automorphism Θ_0 (2.1) gives rise to the Cartan decomposition

$$\mathfrak{so}_{\kappa_1, \kappa_2}(3) = \mathfrak{h}_0 \oplus \mathfrak{p}_0, \quad \mathfrak{h}_0 := \langle J_{12} \rangle, \quad \mathfrak{p}_0 := \langle P_1, P_2 \rangle. \quad (2.3)$$

Consider now the faithful representation of the CK algebra $\Gamma : \mathfrak{so}_{\kappa_1, \kappa_2}(3) \rightarrow \text{End}(\mathbb{R}^3)$ given by

$$\Gamma(J_{12}) = -\kappa_2 E_{12} + E_{21}, \quad \Gamma(P_1) = -\kappa_1 E_{01} + E_{10}, \quad \Gamma(P_2) = -\kappa_1 \kappa_2 E_{02} + E_{20}, \quad (2.4)$$

where E_{ij} denotes the 3×3 elementary matrix with a single non-zero entry 1 at row i and column j ($0 \leq i, j \leq 2$). This representation establishes an isomorphism of Lie algebras between the CK algebra $\mathfrak{so}_{\kappa_1, \kappa_2}(3)$ and the Lie algebra of real 3×3 matrices M satisfying

$$M^T \mathbb{I}_\kappa + \mathbb{I}_\kappa M = 0, \quad \mathbb{I}_\kappa := \text{diag}(1, \kappa_1, \kappa_1 \kappa_2). \quad (2.5)$$

The elements of the representation Γ generated by matrix exponentiation are referred to as the *CK Lie group* $\text{SO}_{\kappa_1, \kappa_2}(3)$. In particular, the following one-parameter subgroups of the CK Lie group $\text{SO}_{\kappa_1, \kappa_2}(3)$ are obtained from (2.4):

$$\begin{aligned} \exp(\alpha \Gamma(P_1)) &= \begin{pmatrix} C_{\kappa_1}(\alpha) & -\kappa_1 S_{\kappa_1}(\alpha) & 0 \\ S_{\kappa_1}(\alpha) & C_{\kappa_1}(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \exp(\gamma \Gamma(J_{12})) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{\kappa_2}(\gamma) & -\kappa_2 S_{\kappa_2}(\gamma) \\ 0 & S_{\kappa_2}(\gamma) & C_{\kappa_2}(\gamma) \end{pmatrix}, \\ \exp(\beta \Gamma(P_2)) &= \begin{pmatrix} C_{\kappa_1 \kappa_2}(\beta) & 0 & -\kappa_1 \kappa_2 S_{\kappa_1 \kappa_2}(\beta) \\ 0 & 1 & 0 \\ S_{\kappa_1 \kappa_2}(\beta) & 0 & C_{\kappa_1 \kappa_2}(\beta) \end{pmatrix}, \quad \alpha, \beta, \gamma \in \mathbb{R}, \end{aligned} \quad (2.6)$$

where we have introduced the so-called κ -dependent *cosine* and *sine* functions defined by (see [35, 41, 46] for details)

$$\begin{aligned} C_\kappa(u) &:= \sum_{l=0}^{\infty} (-\kappa)^l \frac{u^{2l}}{(2l)!} = \begin{cases} \cos \sqrt{\kappa} u & \kappa > 0, \\ 1 & \kappa = 0, \\ \cosh \sqrt{-\kappa} u & \kappa < 0. \end{cases} \\ S_\kappa(u) &:= \sum_{l=0}^{\infty} (-\kappa)^l \frac{u^{2l+1}}{(2l+1)!} = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} u & \kappa > 0, \\ u & \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh \sqrt{-\kappa} u & \kappa < 0. \end{cases} \end{aligned} \quad (2.7)$$

The above κ -dependent trigonometric functions give rise to the κ -*tangent* and the κ -*versed sine* (or *versine*), which are defined as

$$\mathbb{T}_\kappa(u) := \frac{\mathbb{S}_\kappa(u)}{\mathbb{C}_\kappa(u)}, \quad \mathbb{V}_\kappa(u) := \frac{1}{\kappa}(1 - \mathbb{C}_\kappa(u)). \quad (2.8)$$

These κ -functions cover the usual circular ($\kappa > 0$) as well as the hyperbolic ($\kappa < 0$) trigonometric functions, while the parabolic or Galilean ones ($\kappa = 0$) are obtained from the contraction $\kappa \rightarrow 0$ as

$$\mathbb{C}_0(u) = 1, \quad \mathbb{S}_0(u) = \mathbb{T}_0(u) = u, \quad \mathbb{V}_0(u) = u^2/2. \quad (2.9)$$

Their main relations, necessary for further computations, are summarized in the Appendix.

Consider now the Lie subgroup $H_0 := \text{SO}_{\kappa_2}(2)$ of $\text{SO}_{\kappa_1, \kappa_2}(3)$ associated with the Lie subalgebra $\mathfrak{h}_0 \subset \mathfrak{so}_{\kappa_1, \kappa_2}(3)$ (2.3). The *CK family of 2D symmetrical homogeneous spaces* is defined by the quotient

$$\mathbf{S}_{[\kappa_1], \kappa_2}^2 := \text{SO}_{\kappa_1, \kappa_2}(3)/\text{SO}_{\kappa_2}(2). \quad (2.10)$$

It can be shown that the graded contraction parameter κ_1 becomes the (Gaussian) *curvature* of the space, while κ_2 determines the *signature* of the metric through $\text{diag}(1, \kappa_2)$.

2.1 Coordinate systems on the Cayley–Klein spaces

The matrix representation (2.6) of $\text{SO}_{\kappa_1, \kappa_2}(3)$ shows that every element $g \in \text{SO}_{\kappa_1, \kappa_2}(3)$ satisfies $g^T \mathbb{I}_\kappa g = \mathbb{I}_\kappa$, so we can consider the Lie group action of $\text{SO}_{\kappa_1, \kappa_2}(3)$ on \mathbb{R}^3 as isometries of \mathbb{I}_κ (2.5). This action is not transitive, as it preserves the quadratic form induced by \mathbb{I}_κ . Also, $\text{SO}_{\kappa_2}(2) = \langle \exp(\gamma \Gamma(J_{12})) \rangle$ is the isotropy group of the point $O := (1, 0, 0)$, which is thus taken as the *origin* in the space $\mathbf{S}_{[\kappa_1], \kappa_2}^2$. Nevertheless, the action becomes transitive when we restrict ourselves to the orbit of the point O ; namely, the connected component of the submanifold

$$\Sigma_\kappa := \left\{ v := (x^0, x^1, x^2) \in \mathbb{R}^3 : \mathbb{I}_\kappa(v, v) = (x^0)^2 + \kappa_1(x^1)^2 + \kappa_1\kappa_2(x^2)^2 = 1 \right\} \quad (2.11)$$

containing the origin O , thus allowing us to identify the space $\mathbf{S}_{[\kappa_1], \kappa_2}^2$ (2.10) with the latter orbit. The coordinates (x^0, x^1, x^2) on \mathbb{R}^3 satisfying the constraint (2.11) on Σ_κ are called *ambient* or *Weierstrass* coordinates. In terms of these variables, the metric ds_κ^2 of $\mathbf{S}_{[\kappa_1], \kappa_2}^2$ comes from the flat ambient metric in \mathbb{R}^3 divided by the curvature κ_1 and restricted to Σ_κ :

$$ds_\kappa^2 = \frac{1}{\kappa_1} \left((dx^0)^2 + \kappa_1(dx^1)^2 + \kappa_1\kappa_2(dx^2)^2 \right) \Big|_{\Sigma_\kappa} = \frac{\kappa_1(x^1 dx^1 + \kappa_2 x^2 dx^2)^2}{1 - \kappa_1(x^1)^2 - \kappa_1\kappa_2(x^2)^2} + (dx^1)^2 + \kappa_2(dx^2)^2. \quad (2.12)$$

Therefore, the Lie group $\text{SO}_{\kappa_1, \kappa_2}(3)$ is the isometry group of the space $\mathbf{S}_{[\kappa_1], \kappa_2}^2$ in such a manner that J_{12} is a rotation generator, while P_1 and P_2 are translations moving the origin O along two geodesics l_1 and l_2 orthogonal at O , respectively (see Figure 1). We next introduce three relevant kinds of geodesic coordinate systems of a point $Q := (x^0, x^1, x^2)$ in $\mathbf{S}_{[\kappa_1], \kappa_2}^2$ (2.10): *geodesic parallel* coordinates of type I (x, y) and type II (x', y') , and *geodesic polar* coordinates (r, ϕ) . These are defined through the action of the one-parametric groups (2.6) on the origin O as follows [35]:

$$Q = \exp(x\Gamma(P_1))(\exp(y\Gamma(P_2))O) = \exp(y'\Gamma(P_2)) \exp(x'\Gamma(P_1))O = \exp(\phi\Gamma(J_{12})) \exp(r\Gamma(P_1))O,$$

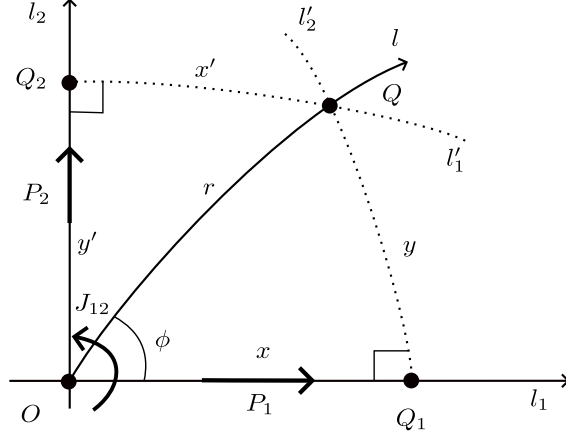


Figure 1: Isometry infinitesimal generators $\{J_{12}, P_1, P_2\}$ and geodesic coordinates (x, y) , (x', y') and (r, ϕ) (2.13) of a point $Q = (x^0, x^1, x^2)$ on the 2D CK space $\mathbf{S}_{[\kappa_1], \kappa_2}^2$ (2.10).

obtaining

$$\begin{aligned}
x^0 &= C_{\kappa_1}(x) C_{\kappa_1 \kappa_2}(y) = C_{\kappa_1}(x') C_{\kappa_1, \kappa_2}(y') = C_{\kappa_1}(r), \\
x^1 &= S_{\kappa_1}(x) C_{\kappa_1 \kappa_2}(y) = S_{\kappa_1}(x') = S_{\kappa_1}(r) C_{\kappa_2}(\phi), \\
x^2 &= S_{\kappa_1 \kappa_2}(y) = C_{\kappa_1}(x') S_{\kappa_1 \kappa_2}(y') = S_{\kappa_1}(r) S_{\kappa_2}(\phi).
\end{aligned} \tag{2.13}$$

In these coordinates, the metric (2.12) adopts the form

$$ds_{\kappa}^2 = C_{\kappa_1 \kappa_2}^2(y) dx^2 + \kappa_2 dy^2 = dx'^2 + \kappa_2 C_{\kappa_1}^2(x') dy'^2 = dr^2 + \kappa_2 S_{\kappa_1}^2(r) d\phi^2. \tag{2.14}$$

As Figure 1 shows, the variable r is the distance between the origin O and the point Q measured along the geodesic l joining both points, while ϕ is the angle formed by the geodesics l and l_1 . Consider the intersection point Q_1 of l_1 with its orthogonal geodesic l'_2 through Q , so x is the geodesic distance between O and Q_1 measured along l_1 and y is the geodesic distance between Q_1 and Q measured along l'_2 . Similarly, if Q_2 is the intersection point of l_2 with its orthogonal geodesic l'_1 , then x' is the geodesic distance between Q and Q_2 measured along l'_1 and y' is the geodesic distance between O and Q_2 measured along l_2 . Note that $(x', y') \neq (x, y)$ if the curvature $\kappa_1 \neq 0$, while on the flat Euclidean plane $\mathbf{E}^2 \equiv \mathbf{S}_{[0], +}^2$ we recover the usual Cartesian coordinates $(x, y) = (x', y')$ and the polar ones (r, ϕ) .

Summing up, depending on the values of the parameters κ_i , the CK family $\mathbf{S}_{[\kappa_1], \kappa_2}^2$ (2.10) comprises *nine* specific 2D symmetrical homogeneous spaces, which are classified into three types according to the value of the signature parameter κ_2 [35], which are displayed by rows in Figure 2:

- *Riemannian spaces* for $\kappa_2 > 0$. In this case the isotropy subgroup $\text{SO}_{\kappa_2}(2) \equiv \text{SO}(2)$ and we find: the *sphere* \mathbf{S}^2 ($\kappa_1 > 0$); the *Euclidean plane* \mathbf{E}^2 ($\kappa_1 = 0$) with $x^0 = +1$; and the *hyperbolic space* \mathbf{H}^2 ($\kappa_1 < 0$) as the part with $x^0 \geq 1$ of a two-sheeted hyperboloid.
- *Pseudo-Riemannian spaces* or *Lorentzian spacetimes* for $\kappa_2 < 0$. Now $\text{SO}_{\kappa_2}(2) \equiv \text{SO}(1, 1)$ is the Lorentz subgroup and these spaces admit a kinematical interpretation as $(1 + 1)\text{D}$ spacetimes. Thus we obtain: the *co-hyperbolic* or *anti-de Sitter space* \mathbf{AdS}^{1+1} ($\kappa_1 > 0$); the *Minkowskian plane* \mathbf{M}^{1+1} ($\kappa_1 = 0$); and the *doubly-hyperbolic* or *de Sitter space* \mathbf{dS}^{1+1} ($\kappa_1 < 0$). In all these cases, the generators J_{12}, P_1 and P_2 of $\mathfrak{so}_{\kappa_1, \kappa_2}(3)$ correspond to the infinitesimal

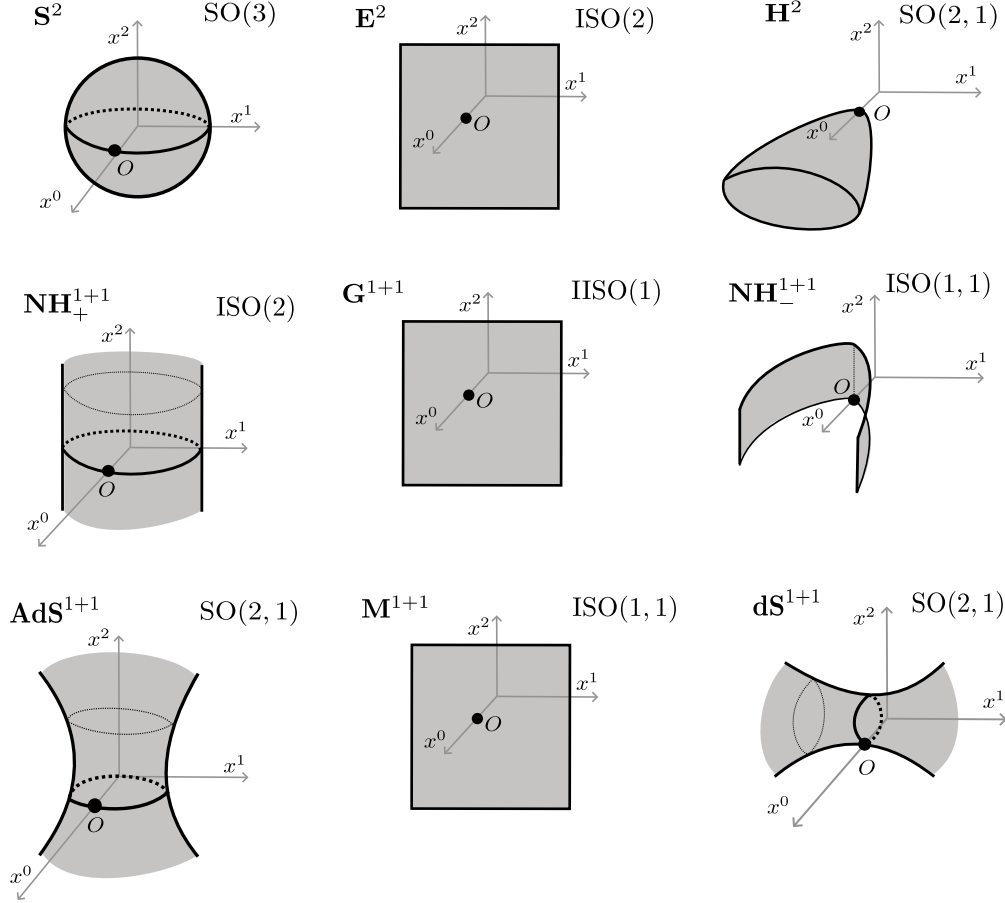


Figure 2: The nine 2D CK spaces $\mathbf{S}_{[\kappa_1, \kappa_2]}^2$ (2.10) in Weierstrass coordinates (x^0, x^1, x^2) (2.11) together with their correspondent 2D CK motion group $\text{SO}_{\kappa_1, \kappa_2}(3)$ according to the ‘normalized’ values of the contraction parameters $\kappa_i \in \{-1, 0, 1\}$.

generators of boosts, time translations and spatial translations, respectively. Moreover, from a physical viewpoint the κ_i parameters are related to the cosmological constant Λ and the speed of light c through

$$\kappa_1 = -\Lambda, \quad \kappa_2 = -1/c^2. \quad (2.15)$$

The geodesic parallel coordinates of type I (x, y) are just the time t and spatial y ones.

- *Semi-Riemannian spaces or Newtonian spacetimes* for $\kappa_2 = 0$. Here $\text{SO}_{\kappa_2}(2) \equiv \text{ISO}(1) \equiv \mathbb{R}$ and the three spaces can be interpreted as non-relativistic spacetimes with $c = \infty$ (2.15). These are: the co-Euclidean or *oscillating Newton–Hooke (NH) space* \mathbf{NH}_+^{1+1} ($\kappa_1 > 0$); the *Galilean plane* \mathbf{G}^{1+1} ($\kappa_1 = 0$); and the co-Minkowskian or *expanding NH space* \mathbf{NH}_-^{1+1} ($\kappa_1 < 0$). The metric (2.14) is degenerate and its kernel gives rise to an integrable foliation of $\mathbf{S}_{[\kappa_1], 0}^2$ which is invariant under the action of the CK group $\text{SO}_{\kappa_1, 0}(3)$. A subsidiary metric $ds_{\kappa}^{\prime 2}$ appears being well-defined when restricted to each leaf of the foliation. In geodesic parallel coordinates of type I (x, y) the metrics read

$$ds_{\kappa}^2 = dx^2, \quad ds_{\kappa}^{\prime 2} = dy^2 \quad \text{on } x = \text{constant}. \quad (2.16)$$

The main metric ds_{κ}^2 provides ‘absolute-time’ t , the leaves of the invariant foliation are the ‘absolute-space’ at $t = t_0$ and $ds_{\kappa}^{\prime 2}$ is the subsidiary spatial metric defined on each leaf.

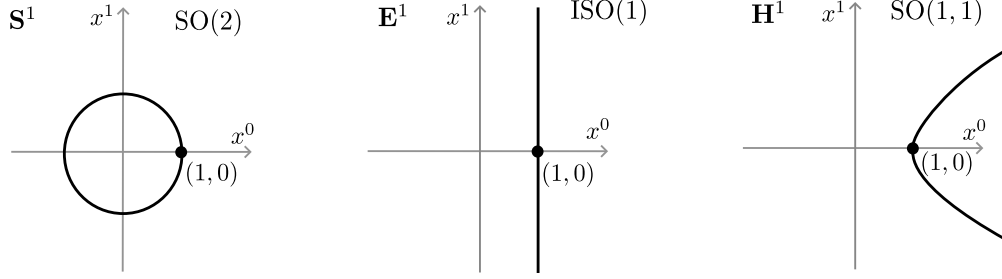


Figure 3: The three 1D CK spaces $\mathbf{S}_{[\kappa]}^2$ in Weierstrass coordinates (x^0, x^1) (2.18) together with their corresponding 1D CK motion group $\text{SO}_{\kappa}(2)$ according to the ‘normalized’ values of the contraction parameter $\kappa \in \{-1, 0, 1\}$.

Consider now the projection

$$\mathbb{R}^3 \ni (x^0, x^1, x^2) \mapsto (x^0, x^1) \in \mathbb{R}^2, \quad (2.17)$$

so the image of the submanifold Σ_{κ} (2.11) under this mapping is

$$\Sigma_{\kappa} := \left\{ (x^0, x^1) \in \mathbb{R}^2 : (x^0)^2 + \kappa(x^1)^2 = 1 \right\}, \quad (2.18)$$

where we have set $\kappa := \kappa_1$. The image of the origin O of $\mathbf{S}_{[\kappa_1], \kappa_2}^2$ under the projection is the point $(1, 0)$, so we call 1D CK space $\mathbf{S}_{[\kappa]}^1$ the connected component of Σ_{κ} containing the point $(1, 0)$. For $\kappa > 0$ we obtain the 1D sphere \mathbf{S}^1 , while the case $\kappa < 0$ gives rise to the hyperbolic line \mathbf{H}^1 , which is the branch of the hyperbola $(x^0)^2 - (x^1)^2 = 1$ with $x^0 \geq 1$. The Euclidean line \mathbf{E}^1 , identified with the line $x^0 = 1$ in \mathbb{R}^2 , appears after the contraction $\kappa = 0$ as depicted in Figure 3.

Analogously to the 2D case, the coordinates (x^0, x^1) on \mathbb{R}^2 satisfying the constraint (2.18) are called *ambient* or *Weierstrass* coordinates. The single coordinate system (x) on $\mathbf{S}_{[\kappa]}^1$ is given by

$$x^0 = C_{\kappa}(x), \quad x^1 = S_{\kappa}(x), \quad (2.19)$$

and it can be obtained from the coordinates (2.13) of $\mathbf{S}_{[\kappa_1], \kappa_2}^2$ by setting $y = 0$, $\kappa_1 \equiv \kappa$ and $\kappa_2 \equiv 0$. Thus, the 1D CK space $\mathbf{S}_{[\kappa]}^1$ is canonically identified with the submanifold $y = 0$ of the 2D CK space $\mathbf{S}_{[\kappa], 0}^2$. Consequently, the (Riemannian) metric ds_{κ}^2 of $\mathbf{S}_{[\kappa]}^1$ reads from (2.14) as

$$ds_{\kappa}^2 = dx^2. \quad (2.20)$$

Note that in these spaces the parameter κ cannot be interpreted as the Gaussian curvature, as the three 1D spaces are flat. Nevertheless, there are some differences better described after the value of κ . The associated 1D CK groups $\text{SO}_{\kappa}(2)$ are given by

$$\text{SO}_{\kappa}(2) = \left\{ \begin{pmatrix} C_{\kappa}(\alpha) & -\kappa S_{\kappa}(\alpha) \\ S_{\kappa}(\alpha) & C_{\kappa}(\alpha) \end{pmatrix} : \alpha \in \mathbb{R} \right\},$$

which correspond to the projection (2.17) to \mathbb{R}^2 of the one-parameter subgroup (2.6) of $\text{SO}_{\kappa, 0}(3)$ generated by P_1 . If $\mathbb{I}_{\kappa} := \text{diag}(1, \kappa)$ denotes the quadratic form inducing the submanifold Σ_{κ} (2.18), then every element $g \in \text{SO}_{\kappa}(2)$ satisfies $g^T \mathbb{I}_{\kappa} g = \mathbb{I}_{\kappa}$, showing that $\text{SO}_{\kappa}(2)$ is a group of isometries of the 1D CK space $\mathbf{S}_{[\kappa]}^1$. After the values $\kappa = +1$, $\kappa = 0$ and $\kappa = -1$ we recover from $\text{SO}_{\kappa}(2)$ the well-known groups of isometries $\text{SO}(2)$, $\text{ISO}(1) \equiv \mathbb{R}$ and $\text{SO}(1, 1)$ of \mathbf{S}^1 , \mathbf{E}^1 and \mathbf{H}^1 , respectively.

2.2 Conformal Cayley–Klein algebras

Infinitesimal conformal symmetries of the nine 2D CK spaces $\mathbf{S}_{[\kappa_1], \kappa_2}^2$ were obtained in [35] by looking for one-parameter subgroups of cycle-preserving transformations (cycles are lines with constant geodesic curvature), while their corresponding conformal Lie groups and compactification were constructed in [36]. This global approach requires to consider the three sets of geodesic coordinates (2.13) and then to express all the conformal Lie generators in a common set. The resulting conformal symmetries span a 6D real Lie algebra $\mathfrak{conf}_{\kappa_1, \kappa_2}$, known as the *conformal algebra* of $\mathbf{S}_{[\kappa_1], \kappa_2}^2$. Let us summarize the main results necessary for this work.

We consider the basis $\{P_i, J_{12}, G_i, D : 1 \leq i \leq 2\}$ corresponding to the generators of translations along the basic geodesic l_i , rotations around the origin (see Figure 1), specific conformal transformations related to the geodesic l_i and dilations, respectively. The commutation relations of $\mathfrak{conf}_{\kappa_1, \kappa_2}$ over such a basis are given by

$$\begin{aligned}
[J_{12}, P_1] &= P_2, & [J_{12}, P_2] &= -\kappa_2 P_1, & [P_1, P_2] &= \kappa_1 J_{12}, \\
[J_{12}, G_1] &= G_2, & [J_{12}, G_2] &= -\kappa_2 G_1, & [G_1, G_2] &= 0, \\
[D, P_i] &= P_i + \kappa_1 G_i, & [D, G_i] &= -G_i, & [D, J_{12}] &= 0, \\
[P_1, G_1] &= D, & [P_2, G_2] &= \kappa_2 D, & & \\
[P_1, G_2] &= -J_{12}, & [P_2, G_1] &= J_{12}. & &
\end{aligned} \tag{2.21}$$

Recall that for the six kinematical cases with $\kappa_2 \leq 0$ (see Figure 2), P_1 is the time-like translation generator, P_2 is the space-like one and J_{12} is just the boost. Hence, G_1 and G_2 can be regarded as time- and space-like specific conformal transformations.

As some relevant Lie subalgebras of $\mathfrak{conf}_{\kappa_1, \kappa_2}$, observe that: (i) the isometries $\{P_1, P_2, J_{12}\}$ span the CK algebra $\mathfrak{so}_{\kappa_1, \kappa_2}(3)$ (2.2) as expected; (ii) the set $\{P_1, P_2, J_{12}, D\}$ (isometries plus dilations) only closes on a subalgebra in the flat spaces with $\kappa_1 = 0$, the so-called Weyl subalgebra; (iii) the set $\{G_1, G_2, J_{12}\}$ spans a subalgebra isomorphic to the inhomogenous $\mathfrak{so}_{0, \kappa_2}(3) \simeq \mathfrak{iso}_{\kappa_2}(2)$; (iv) $\{P_1, G_1, D\}$ provides a subalgebra isomorphic to $\mathfrak{so}(2, 1)$; and (v) $\{P_2, G_2, D\}$ again leads to a subalgebra isomorphic to $\mathfrak{so}(2, 1)$ whenever $\kappa_2 \neq 0$, but to Poincaré $\mathfrak{iso}(1, 1)$ for $\kappa_2 = 0$.

The conformal CK algebra $\mathfrak{conf}_{\kappa_1, \kappa_2}$ (2.21) covers nine specific conformal Lie algebras which are isomorphic to three cases according to the value of the signature parameter κ_2 (regardless of the curvature value κ_1):

- $\mathfrak{so}(3, 1)$, isomorphic to the 3D hyperbolic algebra or to the $(2 + 1)$ D de Sitter algebra, for the three 2D Riemannian spaces with $\kappa_2 > 0$.
- $\mathfrak{iso}(2, 1)$, isomorphic to the $(2 + 1)$ D Poincaré algebra, for the three $(1 + 1)$ D non-relativistic or Newtonian spacetimes with $\kappa_2 = 0$.
- $\mathfrak{so}(2, 2)$, isomorphic to the $(2 + 1)$ D anti-de Sitter algebra, for the three $(2 + 1)$ D relativistic or Lorentzian spacetimes with $\kappa_2 < 0$.

A differential realization of $\mathfrak{conf}_{\kappa_1, \kappa_2}$ (2.21) can be derived in any of the three sets of geodesic coordinates (2.13). Hereafter we will consider geodesic parallel coordinates of type I (x, y) that

give rise to the following vector fields [35]

$$\begin{aligned}
\mathbf{P}_1 &= -\frac{\partial}{\partial x}, & \mathbf{P}_2 &= -\kappa_1 \kappa_2 S_{\kappa_1}(x) T_{\kappa_1 \kappa_2}(y) \frac{\partial}{\partial x} - C_{\kappa_1}(x) \frac{\partial}{\partial y}, \\
\mathbf{J}_{12} &= \kappa_2 C_{\kappa_1}(x) T_{\kappa_1 \kappa_2}(y) \frac{\partial}{\partial x} - S_{\kappa_1}(x) \frac{\partial}{\partial y}, \\
\mathbf{G}_1 &= \frac{1}{C_{\kappa_1 \kappa_2}(y)} (V_{\kappa_1}(x) - \kappa_2 V_{\kappa_1 \kappa_2}(y)) \frac{\partial}{\partial x} + S_{\kappa_1}(x) S_{\kappa_1 \kappa_2}(y) \frac{\partial}{\partial y}, \\
\mathbf{G}_2 &= \kappa_2 S_{\kappa_1}(x) T_{\kappa_1 \kappa_2}(y) \frac{\partial}{\partial x} - (V_{\kappa_1}(x) - \kappa_2 V_{\kappa_1 \kappa_2}(y)) \frac{\partial}{\partial y}, \\
\mathbf{D} &= -\frac{S_{\kappa_1}(x)}{C_{\kappa_1 \kappa_2}(y)} \frac{\partial}{\partial x} - C_{\kappa_1}(x) S_{\kappa_1 \kappa_2}(y) \frac{\partial}{\partial y},
\end{aligned} \tag{2.22}$$

which are always well-defined for any value of κ_1 and κ_2 . The relations (2.13) allow one to express these vector fields in the remaining two sets of geodesic coordinates.

It is worth observing that in the six kinematical spaces with $\kappa_2 \leq 0$, the expressions (2.22) will be well adapted to construct ‘time-like’ curved LH systems, since the time translation generator is simply $\mathbf{P}_1 = -\partial_x$, which is our aim here. Nevertheless, it would also be possible to deduce ‘space-like’ curved LH systems, but for this it would be necessary to choose geodesic parallel coordinates of type II (x', y') such that the spatial translation generator becomes $\mathbf{P}_2 = -\partial_{y'}$. Note that the dilation vector field takes the simplest form in geodesic polar coordinates: $\mathbf{D} = -S_{\kappa_1}(r) \partial_r$.

In addition, it can be easily shown that the vector fields (2.22) satisfy the conformal Killing equations for the CK metric g_1 (2.14), that is, the Lie derivative must be $\mathcal{L}_{\mathbf{X}} g_1 = \mu_{\mathbf{X}} g_1$ where $\mu_{\mathbf{X}}$ is the conformal factor of the vector field \mathbf{X} . Explicitly, the conformal factors in the geodesic (x, y) and ambient (x^0, x^1, x^2) coordinates are found to be

$$\begin{aligned}
\mu_{\mathbf{P}_1} = \mu_{\mathbf{P}_2} = \mu_{\mathbf{J}_{12}} &= 0, & \mu_{\mathbf{D}} &= -2 C_{\kappa_1}(x) C_{\kappa_1 \kappa_2}(y) \equiv -2x^0, \\
\mu_{\mathbf{G}_1} &= 2 S_{\kappa_1}(x) C_{\kappa_1 \kappa_2}(y) \equiv 2x^1, & \mu_{\mathbf{G}_2} &= 2 \kappa_2 S_{\kappa_1 \kappa_2}(y) \equiv 2\kappa_2 x^2.
\end{aligned} \tag{2.23}$$

In order to illustrate the ‘intrinsic’ contraction-scheme of the CK approach, let us write the resulting flat contraction $\kappa_1 = 0$ of the above vector fields. By taking into account (2.9) we directly obtain that the conformal vector fields (2.22) for the three spaces $\mathbf{S}_{[0], \kappa_2}^2$ are

$$\begin{aligned}
\mathbf{P}_1 &= -\frac{\partial}{\partial x}, & \mathbf{P}_2 &= -\frac{\partial}{\partial y}, & \mathbf{J}_{12} &= \kappa_2 y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}, & \mathbf{D} &= -x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, \\
\mathbf{G}_1 &= \frac{1}{2} (x^2 - \kappa_2 y^2) \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, & \mathbf{G}_2 &= \kappa_2 xy \frac{\partial}{\partial x} - \frac{1}{2} (x^2 - \kappa_2 y^2) \frac{\partial}{\partial y},
\end{aligned} \tag{2.24}$$

corresponding to the Euclidean $\mathbf{E}^2 \equiv \mathbf{S}_{[0], +}^2$, Galilean $\mathbf{G}^{1+1} \equiv \mathbf{S}_{[0], 0}^2$ and Minkowskian $\mathbf{M}^{1+1} \equiv \mathbf{S}_{[0], -}^2$ spaces. The conformal factors (2.23) reduce to $\mu_{\mathbf{D}} = -2$, $\mu_{\mathbf{G}_1} = 2x$ and $\mu_{\mathbf{G}_2} = 2\kappa_2 y$.

As a byproduct of the above results, we remark that the conformal algebra \mathfrak{conf}_{κ} of the 1D CK space $\mathbf{S}_{[\kappa]}^1$ (2.19) (see Figure 3) can be obtained in straightforward manner from $\mathfrak{conf}_{\kappa_1, \kappa_2}$ by the restriction of its vector fields (2.22) to the submanifold $y = 0$ of $\mathbf{S}_{[\kappa], 0}^2$, such that $\kappa \equiv \kappa_1$ and $\kappa_2 = 0$. Therefore, \mathfrak{conf}_{κ} turns out to be a 3D Lie algebra spanned by the vector fields $\{\mathbf{P}_1, \mathbf{G}_1, \mathbf{D}\}$ given by

$$\mathbf{P}_1 = -\frac{\partial}{\partial x}, \quad \mathbf{G}_1 = V_{\kappa}(x) \frac{\partial}{\partial x}, \quad \mathbf{D} = -S_{\kappa}(x) \frac{\partial}{\partial x}, \tag{2.25}$$

whose geometrical meaning is preserved, since they are still a translation, a specific conformal transformation and a dilation on $\mathbf{S}_{[\kappa]}^1$, respectively. When $\kappa = 0$, they reduce to

$$\mathbf{P}_1 = -\frac{\partial}{\partial x}, \quad \mathbf{G}_1 = \frac{1}{2}x^2\frac{\partial}{\partial x}, \quad \mathbf{D} = -x\frac{\partial}{\partial x}. \quad (2.26)$$

Finally, the Lie brackets for (2.25) are

$$[\mathbf{D}, \mathbf{P}_1] = \mathbf{P}_1 + \kappa\mathbf{G}_1, \quad [\mathbf{D}, \mathbf{G}_1] = -\mathbf{G}_1, \quad [\mathbf{P}_1, \mathbf{G}_1] = \mathbf{D}, \quad (2.27)$$

so $\mathfrak{conf}_\kappa \simeq \mathfrak{so}(2, 1) \simeq \mathfrak{sl}(2, \mathbb{R})$ regardless of the value of κ , as expected.

3 The class \mathbf{I}_4 of Lie–Hamilton systems on curved spaces

We now have all the geometrical ingredients to construct new LH systems on 2D spaces from their conformal symmetries. Let us start with the class \mathbf{I}_4 of LH systems on the Euclidean plane \mathbf{E}^2 spanned by the vector fields given in Table 1, that is,

$$\mathbf{X}_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad \mathbf{X}_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad \mathbf{X}_3 = x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}, \quad (3.1)$$

which obey the commutation relations (1.4) of $\mathfrak{so}(2, 1) \simeq \mathfrak{sl}(2, \mathbb{R})$. According to the previous section, it emerges that such vector fields are just the sum of two conformal realizations (2.26) on the Euclidean line $\mathbf{E}^1 \equiv \mathbf{S}_{[0]}^1$ and, therefore, defined on $\mathbf{E}^1 \times \mathbf{E}^1$ with coordinates (x, y) . If we restrict ourselves to the first coordinate x , we find that the relationships between \mathbf{X}_i (3.1) and the Euclidean conformal vector fields (2.26) read: $\mathbf{X}_1 = -\mathbf{P}_1$, $\mathbf{X}_2 = -\mathbf{D}$ and $\mathbf{X}_3 = 2\mathbf{G}_1$.

Consequently, a natural generalization of the Euclidean class \mathbf{I}_4 (3.1) to curved spaces is provided by the sum of two realizations of $\mathfrak{so}(2, 1)$ on the 1D CK space $\mathbf{S}_{[\kappa]}^1$, obtained by replacing the conformal symmetries (2.26) of $\mathbf{E}^1 \equiv \mathbf{S}_{[0]}^1$ by their curved counterparts (2.25) of $\mathbf{S}_{[\kappa]}^1$. This yields the following vector fields on $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$:

$$\mathbf{X}_{\kappa,1} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad \mathbf{X}_{\kappa,2} = S_\kappa(x)\frac{\partial}{\partial x} + S_\kappa(y)\frac{\partial}{\partial y}, \quad \mathbf{X}_{\kappa,3} = 2V_\kappa(x)\frac{\partial}{\partial x} + 2V_\kappa(y)\frac{\partial}{\partial y}, \quad (3.2)$$

and their commutation relations are given by (see (2.27))

$$[\mathbf{X}_{\kappa,1}, \mathbf{X}_{\kappa,2}] = \mathbf{X}_{\kappa,1} - \frac{1}{2}\kappa\mathbf{X}_{\kappa,3}, \quad [\mathbf{X}_{\kappa,1}, \mathbf{X}_{\kappa,3}] = 2\mathbf{X}_{\kappa,2}, \quad [\mathbf{X}_{\kappa,2}, \mathbf{X}_{\kappa,3}] = \mathbf{X}_{\kappa,3}. \quad (3.3)$$

The t -dependent vector field

$$\mathbf{X}_\kappa := b_1(t)\mathbf{X}_{\kappa,1} + b_2(t)\mathbf{X}_{\kappa,2} + b_3(t)\mathbf{X}_{\kappa,3}, \quad (3.4)$$

where $b_i(t)$ ($1 \leq i \leq 3$) are arbitrary t -dependent functions, is associated with the first-order system of ODEs on $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$ given by

$$\frac{dx}{dt} = b_1(t) + b_2(t)S_\kappa(x) + 2b_3(t)V_\kappa(x), \quad \frac{dy}{dt} = b_1(t) + b_2(t)S_\kappa(y) + 2b_3(t)V_\kappa(y). \quad (3.5)$$

Notice that the contraction $\kappa = 0$ leads to the system

$$\frac{dx}{dt} = b_1(t) + b_2(t)x + b_3(t)x^2, \quad \frac{dy}{dt} = b_1(t) + b_2(t)y + b_3(t)y^2, \quad (3.6)$$

known as coupled Riccati equations, considered in [21], and which is a particular case of the systems of Riccati equations studied in [9]. Thus we call the system (3.5), the *curved coupled Riccati equations*.

From (3.3) we see that \mathbf{X}_κ is a Lie system with VG Lie algebra $V^{X_\kappa} \simeq \mathfrak{so}(2, 1)$ spanned by the vector fields (3.2). Furthermore, they are Hamiltonian vector fields relative to the symplectic form

$$\omega_\kappa = \frac{1}{4\mathbf{S}_\kappa^2(\frac{1}{2}(x-y))} dx \wedge dy, \quad (3.7)$$

so fulfilling (1.6), and their associated Hamiltonian functions, determined by the condition (1.7), turn out to be

$$h_{\kappa,1} = \frac{1}{2\mathbf{T}_\kappa(\frac{1}{2}(x-y))}, \quad h_{\kappa,2} = \frac{\mathbf{S}_\kappa(\frac{1}{2}(x+y))}{2\mathbf{S}_\kappa(\frac{1}{2}(x-y))}, \quad h_{\kappa,3} = \frac{2\mathbf{S}_\kappa(\frac{1}{2}x)\mathbf{S}_\kappa(\frac{1}{2}y)}{\mathbf{S}_\kappa(\frac{1}{2}(x-y))}. \quad (3.8)$$

With respect to the Poisson bracket $\{\cdot, \cdot\}_{\omega_\kappa}$ induced by the symplectic structure (3.7), the brackets of the Hamiltonian functions (3.8) are

$$\{h_{\kappa,1}, h_{\kappa,2}\}_{\omega_\kappa} = -h_{\kappa,1} + \frac{1}{2}\kappa h_{\kappa,3}, \quad \{h_{\kappa,1}, h_{\kappa,3}\}_{\omega_\kappa} = -2h_{\kappa,2}, \quad \{h_{\kappa,2}, h_{\kappa,3}\}_{\omega_\kappa} = -h_{\kappa,3}, \quad (3.9)$$

showing that \mathbf{X}_κ is a LH system on $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$ with LH algebra $\mathcal{H}_{\omega_\kappa} \simeq \mathfrak{so}(2, 1)$ spanned by the Hamiltonian functions (3.8).

Observe that the product manifold $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$ is naturally endowed with the product Riemannian metric $dx^2 + dy^2$, (which is the product metric (2.20)) showing that the 2D space $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$ is flat. In particular, for $\kappa > 0$ we obtain the flat 2-torus $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$, while the case $\kappa < 0$ gives rise to the product of two hyperbolic lines $\mathbf{H}^1 \times \mathbf{H}^1$. The Euclidean plane \mathbf{E}^2 , corresponding to $\kappa = 0$, is the only 2D CK space which is a product of 1D CK spaces.

Hence, we have obtained a LH class on the space $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$ which generalizes the Euclidean class I_4 to $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$, as the vector fields, symplectic form and Hamiltonian functions of the Euclidean I_4 -LH class, shown in Table 1, are recovered after the contraction $\kappa = 0$ from (3.2), (3.7) and (3.8), respectively. We call such a LH class on $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$ the *curved I_4 -LH class*, which is explicitly described in Table 2 for each of the three spaces $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$. Note that the domain of the coordinates (x, y) requires the condition $x \neq y$ in the three spaces, ensuring a well-defined symplectic form and Hamiltonian functions.

It is also worth emphasizing that the contraction parameter κ can alternatively be interpreted as a perturbation parameter of the Euclidean class I_4 . In particular, if κ is considered to have a small value, we can take a power series expansion in κ of the system (3.5). For instance, by truncating at the first-order in κ , we obtain the following approximation:

$$\frac{dx}{dt} = b_1(t) + b_2(t)x + b_3(t)x^2 - \frac{1}{12}\kappa \left(2b_2(t)x^3 + b_3(t)x^4 \right) + o[\kappa^2], \quad (3.10)$$

(and similarly for the second equation in y) which can be interpreted as a fourth-order perturbation of the Riccati equation (3.6). In this sense, the introduction of κ can be regarded as the opposite process to the contraction procedure, i.e. a classical deformation. This idea is quite similar to Poisson–Hopf deformations of LH systems, for which the quantum deformation parameter can always be thought of as a perturbation (see, e.g., [47, eq. (4.19)] for an example).

Table 2: The curved I_4 -LH class on the three flat spaces $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$ according to the ‘normalized’ values $\kappa \in \{-1, 0, 1\}$. For each space it is shown, in coordinates (x, y) , its Riemannian metric, domain of the variables, VG Lie algebra V^{X_κ} (3.3) with vector fields $\mathbf{X}_{\kappa,i}$ (3.2), LH algebra $\mathcal{H}_{\omega_\kappa}$ (3.9) with Hamiltonian functions $h_{\kappa,i}$ (3.8) and symplectic form ω_κ (3.7). For the sake of clarity, we drop the index κ .

<ul style="list-style-type: none"> • Flat 2-torus $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$ 	<ul style="list-style-type: none"> • Euclidean plane \mathbf{E}^2 	<ul style="list-style-type: none"> • Product of hyperbolic lines $\mathbf{H}^1 \times \mathbf{H}^1$
$ds^2 = dx^2 + dy^2$	$ds^2 = dx^2 + dy^2$	$ds^2 = dx^2 + dy^2$
$x \in (-\pi, \pi], y \in (-\pi, \pi], x \neq y$	$x \in \mathbb{R}, y \in \mathbb{R}, x \neq y$	$x \in \mathbb{R}, y \in \mathbb{R}, x \neq y$
$V^X \simeq \mathfrak{so}(2, 1)$	$V^X \simeq \mathfrak{so}(2, 1)$	$V^X \simeq \mathfrak{so}(2, 1)$
$\mathbf{X}_1 = \partial_x + \partial_y$	$\mathbf{X}_1 = \partial_x + \partial_y$	$\mathbf{X}_1 = \partial_x + \partial_y$
$\mathbf{X}_2 = \sin x \partial_x + \sin y \partial_y$	$\mathbf{X}_2 = x \partial_x + y \partial_y$	$\mathbf{X}_2 = \sinh x \partial_x + \sinh y \partial_y$
$\mathbf{X}_3 = 2((1 - \cos x) \partial_x + (1 - \cos y) \partial_y)$	$\mathbf{X}_3 = x^2 \partial_x + y^2 \partial_y$	$\mathbf{X}_3 = 2((\cosh x - 1) \partial_x + (\cosh y - 1) \partial_y)$
$\mathcal{H}_\omega \simeq \mathfrak{so}(2, 1)$	$\mathcal{H}_\omega \simeq \mathfrak{so}(2, 1)$	$\mathcal{H}_\omega \simeq \mathfrak{so}(2, 1)$
$h_1 = \frac{1}{2 \tan(\frac{1}{2}(x-y))}$	$h_1 = \frac{1}{x-y}$	$h_1 = \frac{1}{2 \tanh(\frac{1}{2}(x-y))}$
$h_2 = \frac{\sin(\frac{1}{2}(x+y))}{2 \sin(\frac{1}{2}(x-y))}$	$h_2 = \frac{x+y}{2(x-y)}$	$h_2 = \frac{\sinh(\frac{1}{2}(x+y))}{2 \sinh(\frac{1}{2}(x-y))}$
$h_3 = \frac{2 \sin(\frac{1}{2}x) \sin(\frac{1}{2}y)}{\sin(\frac{1}{2}(x-y))}$	$h_3 = \frac{xy}{x-y}$	$h_3 = \frac{2 \sinh(\frac{1}{2}x) \sinh(\frac{1}{2}y)}{\sinh(\frac{1}{2}(x-y))}$
$\omega = \frac{1}{4 \sin^2(\frac{1}{2}(x-y))} dx \wedge dy$	$\omega = \frac{1}{(x-y)^2} dx \wedge dy$	$\omega = \frac{1}{4 \sinh^2(\frac{1}{2}(x-y))} dx \wedge dy$

3.1 Constants of the motion of the curved I_4 -class

We now proceed to compute t -independent constants of motion of the LH system \mathbf{X}_κ (3.4) by applying the so-called coalgebra formalism [9, 47]. Let us consider the LH algebra $\mathcal{H}_{\omega_\kappa}$ of the LH system \mathbf{X}_κ expressed in a basis $\{v_1, v_2, v_3\}$ that formally satisfies the same Poisson brackets shown in (3.9):

$$\{v_1, v_2\} = -v_1 + \frac{1}{2}\kappa v_3, \quad \{v_1, v_3\} = -2v_2, \quad \{v_2, v_3\} = -v_3.$$

There exists a non-trivial quadratic Casimir element given by

$$C = v_1 v_3 - v_2^2 - \frac{1}{4}\kappa v_3^2, \quad \{C, \cdot\} = 0. \quad (3.11)$$

As a shorthand notation, let us denote by $\mathcal{M}_{[\kappa]}$ the submanifold of $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$ with $x \neq y$. From (3.8) we construct the following Hamiltonian functions ($1 \leq i \leq 3$):

$$\begin{aligned} h_{\kappa,i}^{(1)} &= h_{\kappa,i}(\mathbf{x}_1) \in C^\infty(\mathcal{M}_{[\kappa]}), \\ h_{\kappa,i}^{(2)} &= h_{\kappa,i}(\mathbf{x}_1) + h_{\kappa,i}(\mathbf{x}_2) \in C^\infty(\mathcal{M}_{[\kappa]} \times \mathcal{M}_{[\kappa]}), \\ h_{\kappa,i}^{(3)} &= h_{\kappa,i}(\mathbf{x}_1) + h_{\kappa,i}(\mathbf{x}_2) + h_{\kappa,i}(\mathbf{x}_3) \in C^\infty(\mathcal{M}_{[\kappa]} \times \mathcal{M}_{[\kappa]} \times \mathcal{M}_{[\kappa]}), \end{aligned} \quad (3.12)$$

where $\mathbf{x}_s = (x_s, y_s)$ denotes the coordinates in the s^{th} -copy of $\mathcal{M}_{[\kappa]}$ within the product manifold. Note that each set of functions of (3.12) so obtained satisfies the Poisson brackets (3.9) with respect to the Poisson bracket induced by taking the product symplectic structure $\omega_\kappa^{[3]}$ of (3.7) to

$\mathcal{M}_{[\kappa]}^3 \equiv \mathcal{M}_{[\kappa]} \times \mathcal{M}_{[\kappa]} \times \mathcal{M}_{[\kappa]}$, that is,

$$\omega_{\kappa}^{[3]} = \frac{1}{4} \sum_{s=1}^3 \frac{1}{S_{\kappa}^2(\frac{1}{2}(x_s - y_s))} dx_s \wedge dy_s.$$

Now, using the Casimir (3.11) we obtain the following constants of motion for the diagonal prolongation $\tilde{\mathbf{X}}_{\kappa}^3$ of \mathbf{X}_{κ} (3.4) to $\mathcal{M}_{[\kappa]}^3$:

$$\begin{aligned} F_{\kappa}^{(1)} &= C(h_{\kappa,1}^{(1)}, h_{\kappa,2}^{(1)}, h_{\kappa,3}^{(1)}) = -\frac{1}{4}, \\ F_{\kappa}^{(2)} &= C(h_{\kappa,1}^{(2)}, h_{\kappa,2}^{(2)}, h_{\kappa,3}^{(2)}) = -\frac{S_{\kappa}(\frac{1}{2}(x_2 - y_1)) S_{\kappa}(\frac{1}{2}(x_1 - y_2))}{S_{\kappa}(\frac{1}{2}(x_1 - y_1)) S_{\kappa}(\frac{1}{2}(x_2 - y_2))}. \end{aligned} \quad (3.13)$$

Observe that $F_{\kappa}^{(1)} < 0$ for any value of κ , so this corresponds to $\mathcal{C} < 0$ (1.11), with $c = -1$ as shown in Table 1 for the Euclidean class I_4 . This fact will distinguish a specific LH system belonging to the *curved I_4 -LH class* from other within the *curved P_2 -LH class*, with $\mathcal{C} > 0$, that we will develop in Section 5.

Note also that the well-known constant of motion of the Euclidean class I_4 [6, 8, 9, 48] is recovered under the contraction $\kappa = 0$:

$$F_0^{(2)} = -\frac{(x_2 - y_1)(x_1 - y_2)}{(x_1 - y_1)(x_2 - y_2)}, \quad (3.14)$$

corresponding to the set of coupled Riccati equations (3.6).

Moreover, $F_{\kappa}^{(2)}$ gives rise to two additional constants of motion through the permutation S_{ij} of the variables $(x_i, y_i) \leftrightarrow (x_j, y_j)$; these are

$$\begin{aligned} F_{\kappa,13}^{(2)} &= S_{13}(F_{\kappa}^{(2)}) = -\frac{S_{\kappa}(\frac{1}{2}(x_2 - y_3)) S_{\kappa}(\frac{1}{2}(x_3 - y_2))}{S_{\kappa}(\frac{1}{2}(x_3 - y_3)) S_{\kappa}(\frac{1}{2}(x_2 - y_2))}, \\ F_{\kappa,23}^{(2)} &= S_{23}(F_{\kappa}^{(2)}) = -\frac{S_{\kappa}(\frac{1}{2}(x_3 - y_1)) S_{\kappa}(\frac{1}{2}(x_1 - y_3))}{S_{\kappa}(\frac{1}{2}(x_1 - y_1)) S_{\kappa}(\frac{1}{2}(x_3 - y_3))}. \end{aligned} \quad (3.15)$$

The functions $h_{\kappa,i}^{(3)}$ (3.12) also allow to obtain another constant of motion which can be expressed in terms of (3.13) and (3.15) as

$$F_{\kappa}^{(3)} = C(h_{\kappa,1}^{(3)}, h_{\kappa,2}^{(3)}, h_{\kappa,3}^{(3)}) = F_{\kappa}^{(2)} + F_{\kappa,13}^{(2)} + F_{\kappa,23}^{(2)} + \frac{3}{4}.$$

3.2 Superposition rules for the curved I_4 -class

So far we have obtained four t -independent constants of motion $\{F_{\kappa}^{(2)}, F_{\kappa,13}^{(2)}, F_{\kappa,23}^{(2)}, F_{\kappa}^{(3)}\}$ for the curved I_4 -LH system \mathbf{X}_{κ} (3.4). To derive a superposition rule in this case, it is necessary to choose two functionally independent constants of motion, say I_1, I_2 , for the diagonal prolongation $\tilde{\mathbf{X}}_{\kappa}^3$ of \mathbf{X}_{κ} to $\mathcal{M}_{[\kappa]}^3$, so verifying the condition

$$\frac{\partial(I_1, I_2)}{\partial(x_1, y_1)} \neq 0. \quad (3.16)$$

This ensures that it is possible to express the general solution $(x_1(t), y_1(t))$ of the LH system \mathbf{X}_κ in terms of two particular solutions $(x_2(t), y_2(t))$ and $(x_3(t), y_3(t))$ and two significative constants, say μ_1 and μ_2 , by solving the equations $I_1 = \mu_1$ and $I_2 = \mu_2$ (see [6, 8] for details).

There are several possible options for the constants I_1 and I_2 . We take $F_\kappa^{(2)}$ and $F_{\kappa,23}^{(2)}$ and then use $F_{\kappa,13}^{(2)}$ in order to simplify the final result, thus we set

$$F_\kappa^{(2)} = -\mu_1, \quad F_{\kappa,23}^{(2)} = -\mu_2, \quad F_{\kappa,13}^{(2)} = -\mu_3. \quad (3.17)$$

For the computations, we rewrite the expression for $F_\kappa^{(2)}$ (3.13) (see the Appendix) as

$$F_\kappa^{(2)} = - \left(\frac{\mathbb{T}_\kappa(\frac{1}{2}x_2) - \mathbb{T}_\kappa(\frac{1}{2}y_1)}{\mathbb{T}_\kappa(\frac{1}{2}x_1) - \mathbb{T}_\kappa(\frac{1}{2}y_1)} \right) \left(\frac{\mathbb{T}_\kappa(\frac{1}{2}x_1) - \mathbb{T}_\kappa(\frac{1}{2}y_2)}{\mathbb{T}_\kappa(\frac{1}{2}x_2) - \mathbb{T}_\kappa(\frac{1}{2}y_2)} \right), \quad (3.18)$$

and similarly for $F_{\kappa,13}^{(2)}$ and $F_{\kappa,23}^{(2)}$ (3.15). Starting from the first equation in (3.17) we find that

$$\mathbb{T}_\kappa(\frac{1}{2}x_1) = \frac{\mu_1 \left(\mathbb{T}_\kappa(\frac{1}{2}x_2) - \mathbb{T}_\kappa(\frac{1}{2}y_2) \right) \mathbb{T}_\kappa(\frac{1}{2}y_1) - \left(\mathbb{T}_\kappa(\frac{1}{2}x_2) - \mathbb{T}_\kappa(\frac{1}{2}y_1) \right) \mathbb{T}_\kappa(\frac{1}{2}y_2)}{\mu_1 \left(\mathbb{T}_\kappa(\frac{1}{2}x_2) - \mathbb{T}_\kappa(\frac{1}{2}y_2) \right) - \left(\mathbb{T}_\kappa(\frac{1}{2}x_2) - \mathbb{T}_\kappa(\frac{1}{2}y_1) \right)}. \quad (3.19)$$

Substituting this result into the second equation in (3.17) and also introducing the third constant μ_3 we arrive at the full superposition principle for y_1 , namely

$$\begin{aligned} \mathbb{T}_\kappa(\frac{1}{2}y_1^\pm) &= \frac{1}{2} \left\{ \left(\mathbb{T}_\kappa(\frac{1}{2}x_2) + \mathbb{T}_\kappa(\frac{1}{2}x_3) \right) \left(\mathbb{T}_\kappa(\frac{1}{2}y_2) - \mathbb{T}_\kappa(\frac{1}{2}y_3) \right) \right. \\ &\quad + \mu_1 \left(\mathbb{T}_\kappa(\frac{1}{2}x_2) - \mathbb{T}_\kappa(\frac{1}{2}y_2) \right) \left(\mathbb{T}_\kappa(\frac{1}{2}x_3) + \mathbb{T}_\kappa(\frac{1}{2}y_3) \right) \\ &\quad \left. - \mu_2 \left(\mathbb{T}_\kappa(\frac{1}{2}x_3) - \mathbb{T}_\kappa(\frac{1}{2}y_3) \right) \left(\mathbb{T}_\kappa(\frac{1}{2}x_2) + \mathbb{T}_\kappa(\frac{1}{2}y_2) \right) \pm \sqrt{\Xi} \right\} \\ &\quad \times \left\{ \mu_1 \left(\mathbb{T}_\kappa(\frac{1}{2}x_2) - \mathbb{T}_\kappa(\frac{1}{2}y_2) \right) - \mu_2 \left(\mathbb{T}_\kappa(\frac{1}{2}x_3) - \mathbb{T}_\kappa(\frac{1}{2}y_3) \right) + \left(\mathbb{T}_\kappa(\frac{1}{2}y_2) - \mathbb{T}_\kappa(\frac{1}{2}y_3) \right) \right\}^{-1} \end{aligned} \quad (3.20)$$

(so with two solutions according to the sign of the square root), where

$$\begin{aligned} \Xi &= \left(\mathbb{T}_\kappa(\frac{1}{2}x_2) - \mathbb{T}_\kappa(\frac{1}{2}x_3) \right)^2 \left(\mathbb{T}_\kappa(\frac{1}{2}y_2) - \mathbb{T}_\kappa(\frac{1}{2}y_3) \right)^2 \\ &\quad + (\mu_1^2 + \mu_2^2 - 2\mu_1\mu_2\mu_3) \left(\mathbb{T}_\kappa(\frac{1}{2}x_2) - \mathbb{T}_\kappa(\frac{1}{2}y_2) \right)^2 \left(\mathbb{T}_\kappa(\frac{1}{2}x_3) - \mathbb{T}_\kappa(\frac{1}{2}y_3) \right)^2 \\ &\quad - 2(\mu_1 + \mu_2 - \mu_1\mu_2) \left(\mathbb{T}_\kappa(\frac{1}{2}x_2) - \mathbb{T}_\kappa(\frac{1}{2}y_2) \right) \left(\mathbb{T}_\kappa(\frac{1}{2}x_3) - \mathbb{T}_\kappa(\frac{1}{2}y_3) \right) \\ &\quad \times \left(\mathbb{T}_\kappa(\frac{1}{2}x_2) - \mathbb{T}_\kappa(\frac{1}{2}x_3) \right) \left(\mathbb{T}_\kappa(\frac{1}{2}y_2) - \mathbb{T}_\kappa(\frac{1}{2}y_3) \right). \end{aligned}$$

Note that the result (3.20) is well-defined whenever the function Ξ is non-negative, which requires some restrictions on the parameters μ_i determined by the specific particular solutions under consideration. Hereafter, this fact will be assumed when a square root appears in a superposition rule (as in Subsection 5.2). Finally, inserting the above expressions into (3.19), we obtain the complete superposition rule for the curved I_4 -LH system \mathbf{X}_κ (3.4) as the general solution $(x_1^\pm(t), y_1^\pm(t))$ with the following dependence

$$x_1^\pm(x_2, y_2, x_3, y_3, \mu_1, \mu_2, \mu_3), \quad y_1^\pm(x_2, y_2, x_3, y_3, \mu_1, \mu_2, \mu_3), \quad \Xi(x_2, y_2, x_3, y_3, \mu_1, \mu_2, \mu_3),$$

such that $\mu_3 = \mu_3(x_2, y_2, x_3, y_3)$ through $F_{\kappa,13}^{(2)} = S_{13}(F_{\kappa}^{(2)}) = -\mu_3$ (3.18).

The superposition rule for the curved I_4 -LH class is rather simplified for the system with $\kappa = 0$, yielding

$$\begin{aligned} y_1^{\pm} &= \frac{(x_2 + x_3)(y_2 - y_3) + \mu_1(x_2 - y_2)(x_3 + y_3) - \mu_2(x_3 - y_3)(x_2 + y_2) \pm \sqrt{\Xi}}{2(\mu_1(x_2 - y_2) - \mu_2(x_3 - y_3) + (y_2 - y_3))}, \\ \Xi &= (x_2 - x_3)^2(y_2 - y_3)^2 + (\mu_1^2 + \mu_2^2 - 2\mu_1\mu_2\mu_3)(x_2 - y_2)^2(x_3 - y_3)^2 \\ &\quad - 2(\mu_1 + \mu_2 - \mu_1\mu_2)(x_2 - y_2)(x_3 - y_3)(x_2 - x_3)(y_2 - y_3), \\ x_1^{\pm} &= \frac{\mu_1(x_2 - y_2)y_1^{\pm} - (x_2 - y_1^{\pm})y_2}{\mu_1(x_2 - y_2) - (x_2 - y_1^{\pm})}, \end{aligned} \quad (3.21)$$

where $\mu_3 = \mu_3(x_2, y_2, x_3, y_3)$ now via $F_{0,13}^{(2)} = S_{13}(F_0^{(2)}) = -\mu_3$ (3.14).

It is worth observing that, to the best of our knowledge, not only the curved LH systems constructed throughout this section together with their constants of the motion and superposition principles constitute new results, but also the superposition rule (3.21) for the Euclidean coupled Riccati equations (3.6), thus completing previous works (see [9] and references therein).

Furthermore, recall that a single Riccati equation, take only the first one in (3.6) with variable x , is a Lie system with VG Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, but it is not a LH system. This system appears in the classification of Lie systems on the plane [7] as the class I_3 for which there does not exist an associated symplectic form, so that the application of the coalgebra formalism to obtain constants of motion and superposition principles is precluded. Nevertheless, the well-known superposition rule for the Riccati equation [6, 48] can easily be derived from the last equation in (3.21). Let us denote by \mathbf{X}^{Ric} the t -dependent vector field for the Riccati equation spanned by the projection $y = 0$ of the conformal vector fields (3.1) with variable $\tilde{x} \in \mathbb{R}$. Under the following identification

$$x_1 \equiv \tilde{x}, \quad y_1 \equiv \tilde{x}_3, \quad x_2 \equiv \tilde{x}_2, \quad y_2 \equiv \tilde{x}_1, \quad (3.22)$$

we retrieve from (3.21) the general solution of the Riccati equation $\tilde{x}(t)$ expressed in terms of three particular solutions $(\tilde{x}_1(t), \tilde{x}_2(t), \tilde{x}_3(t))$ and a single constant μ_1 as

$$\tilde{x} = \frac{\mu_1(\tilde{x}_2 - \tilde{x}_1)\tilde{x}_3 + (\tilde{x}_3 - \tilde{x}_2)\tilde{x}_1}{\mu_1(\tilde{x}_2 - \tilde{x}_1) + (\tilde{x}_3 - \tilde{x}_2)}. \quad (3.23)$$

This result is, in fact, provided by $F_{0,13}^{(2)} = -\mu_1$ (3.14) which is now understood as a constant of motion for the diagonal prolongation $\tilde{\mathbf{X}}^{\text{Ric}}$ of \mathbf{X}^{Ric} to \mathbb{R}^4 .

Likewise, we will call the first equation of (3.5) the *curved Riccati equation*, with variable $x \in \mathbf{S}_{[\kappa]}^1$, whose corresponding t -dependent vector field $\mathbf{X}_{\kappa}^{\text{Ric}}$ is spanned by the projection $y = 0$ of (3.2) (see (2.25)). This is again a Lie system but not a LH one, since no symplectic form on $\mathbf{S}_{[\kappa]}^1$ exists. However, a superposition rule for $\mathbf{X}_{\kappa}^{\text{Ric}}$ is provided by (3.19) under the identification (3.22), determining the general solution $\tilde{x}(t)$:

$$\mathsf{T}_{\kappa}\left(\frac{1}{2}\tilde{x}\right) = \frac{\mu_1\left(\mathsf{T}_{\kappa}\left(\frac{1}{2}\tilde{x}_2\right) - \mathsf{T}_{\kappa}\left(\frac{1}{2}\tilde{x}_1\right)\right)\mathsf{T}_{\kappa}\left(\frac{1}{2}\tilde{x}_3\right) + \left(\mathsf{T}_{\kappa}\left(\frac{1}{2}\tilde{x}_3\right) - \mathsf{T}_{\kappa}\left(\frac{1}{2}\tilde{x}_2\right)\right)\mathsf{T}_{\kappa}\left(\frac{1}{2}\tilde{x}_1\right)}{\mu_1\left(\mathsf{T}_{\kappa}\left(\frac{1}{2}\tilde{x}_2\right) - \mathsf{T}_{\kappa}\left(\frac{1}{2}\tilde{x}_1\right)\right) + \left(\mathsf{T}_{\kappa}\left(\frac{1}{2}\tilde{x}_3\right) - \mathsf{T}_{\kappa}\left(\frac{1}{2}\tilde{x}_2\right)\right)},$$

which is reduced to (3.23) for the contraction $\kappa = 0$. In this curved case, $F_{\kappa}^{(2)} = -\mu_1$ (3.18) is a constant of the motion for the diagonal prolongation $\tilde{\mathbf{X}}_{\kappa}^{\text{Ric}}$ of $\mathbf{X}_{\kappa}^{\text{Ric}}$ to $(\mathbf{S}_{[\kappa]}^1)^4$.

4 Applications of the curved I_4 -class

The Euclidean I_4 -LH class is known to include the following relevant systems [8]: the coupled Riccati equations, the split-complex Riccati equation, a planar diffusion Riccati system, and the Kummer–Schwarz and Ermakov equations for $c < 0$ (1.11). The curved counterpart of the coupled Riccati equations has already been studied, since they appear directly from the conformal vector fields (3.2). Consequently, by making use of the curved I_4 -LH class presented previously, we generalize in this section the remaining particular LH systems to the 2D spaces $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$. In this way, the known Euclidean LH systems are extended to the flat 2-torus $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$ ($\kappa > 0$) and the product of two hyperbolic lines $\mathbf{H}^1 \times \mathbf{H}^1$ ($\kappa < 0$).

The procedure requires to obtain an appropriate local diffeomorphism between the specific LH system on $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$ under consideration and the curved I_4 -LH class of Section 3. Recall that, as local diffeomorphisms between open subsets of $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$ are induced from local diffeomorphisms between open subsets of the domain of the parametrization (2.19), the curved analogues of the systems of the Euclidean I_4 -LH class can be constructed by applying local diffeomorphisms of the Euclidean case to the domain of the parametrization (2.19).

Once this is achieved, the corresponding constants of the motion and the superposition principle can be written directly from the results of Subsections 3.1 and 3.2, respectively, although for the sake of brevity we omit these expressions. Nevertheless, we point out that in all applications the first-order constant of motion is $F_\kappa^{(1)} = -\frac{1}{4}$ (3.13), which is a direct consequence of taking a negative value for the constant c (1.11) (see Table 1).

4.1 Curved split-complex Riccati equation

On the Euclidean plane \mathbf{E}^2 with coordinates (u, v) , consider a hypercomplex unit ι which commutes with real numbers and such that $\iota^2 \in \{-1, +1, 0\}$ and define the hypercomplex number $z := u + \iota v$ [44, 45, 49]. The differential equation

$$\frac{dz}{dt} = b_1(t) + b_2(t)z + b_3(t)z^2 \quad (4.1)$$

is known as the *Cayley–Klein Riccati equation* [8, 30], and is equivalent to the following first-order system of differential equations on \mathbf{E}^2 :

$$\frac{du}{dt} = b_1(t) + b_2(t)u + b_3(t)(u^2 + \iota^2 v^2), \quad \frac{dv}{dt} = b_2(t)v + 2b_3(t)uv. \quad (4.2)$$

If $\iota^2 = +1$, we obtain the so-called split-complex numbers $z \in \mathbb{C}'$ and ι is usually referred to as the hyperbolic, double or Clifford unit [44, 49]. In this case, the system (4.2) reads

$$\frac{du}{dt} = b_1(t) + b_2(t)u + b_3(t)(u^2 + v^2), \quad \frac{dv}{dt} = b_2(t)v + 2b_3(t)uv, \quad (4.3)$$

which comes from the so-called *split-complex Riccati equation* (4.1). It is associated with the t -dependent vector field $\mathbf{X} = b_1(t)\mathbf{X}_1 + b_2(t)\mathbf{X}_2 + b_3(t)\mathbf{X}_3$, where

$$\mathbf{X}_1 = \frac{\partial}{\partial u}, \quad \mathbf{X}_2 = u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}, \quad \mathbf{X}_3 = (u^2 + v^2) \frac{\partial}{\partial u} + 2uv \frac{\partial}{\partial v},$$

satisfying the commutation relations (1.4). In addition, these are Hamiltonian vector fields relative to the symplectic form given by

$$\omega = -\frac{1}{2v^2} du \wedge dv,$$

and their associated Hamiltonian functions are

$$h_1 = \frac{1}{2v}, \quad h_2 = \frac{u}{2v}, \quad h_3 = \frac{u^2 - v^2}{2v}.$$

All the above expressions are mapped into those of the Euclidean I_4 -LH class displayed in Table 1 via the change of variables [8]

$$x := u + v, \quad y := u - v, \quad u = \frac{1}{2}(x + y), \quad v = \frac{1}{2}(x - y). \quad (4.4)$$

Observe that the domain for each set of variables is $\mathbb{R}_{x \neq y}^2$ and $\mathbb{R}_{v \neq 0}^2$.

Within the framework established in Section 3, we apply the transformation (4.4) to the curved I_4 -LH class (see Table 2), so now defined on $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$, thus obtaining the t -dependent vector field (3.4) with vector fields, coming from (3.2), given by

$$\begin{aligned} \mathbf{X}_{\kappa,1} &= \frac{\partial}{\partial u}, & \mathbf{X}_{\kappa,2} &= C_\kappa(v) S_\kappa(u) \frac{\partial}{\partial u} + C_\kappa(u) S_\kappa(v) \frac{\partial}{\partial v}, \\ \mathbf{X}_{\kappa,3} &= 2 \left(\frac{1 - C_\kappa(u) C_\kappa(v)}{\kappa} \right) \frac{\partial}{\partial u} + 2 S_\kappa(u) S_\kappa(v) \frac{\partial}{\partial v}. \end{aligned} \quad (4.5)$$

They span a VG Lie algebra $V^{X_\kappa} \simeq \mathfrak{so}(2,1)$, since their commutators are those of (3.3). The corresponding first-order system of ODEs on $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$ turns out to be

$$\begin{aligned} \frac{du}{dt} &= b_1(t) + b_2(t) C_\kappa(v) S_\kappa(u) + 2b_3(t) \left(\frac{1 - C_\kappa(u) C_\kappa(v)}{\kappa} \right), \\ \frac{dv}{dt} &= b_2(t) C_\kappa(u) S_\kappa(v) + 2b_3(t) S_\kappa(u) S_\kappa(v). \end{aligned} \quad (4.6)$$

The vector fields (4.5) are Hamiltonian vector fields relative to the symplectic form

$$\omega_\kappa = -\frac{1}{2S_\kappa^2(v)} du \wedge dv, \quad (4.7)$$

and their associated Hamiltonian functions, fulfilling (1.7), read

$$h_{\kappa,1} = \frac{1}{2T_\kappa(v)}, \quad h_{\kappa,2} = \frac{S_\kappa(u)}{2S_\kappa(v)}, \quad h_{\kappa,3} = \frac{C_\kappa(v) - C_\kappa(u)}{\kappa S_\kappa(v)}. \quad (4.8)$$

They span a LH algebra $\mathcal{H}_{\omega_\kappa} \simeq \mathfrak{so}(2,1)$ with respect to the Poisson bracket $\{\cdot, \cdot\}_{\omega_\kappa}$ induced by the symplectic form (4.7), as their brackets are the same as in (3.9).

The vector fields, the symplectic structure and the Hamiltonian functions of the split-complex Riccati equation (4.3) on \mathbf{E}^2 are recovered under the limit $\kappa \rightarrow 0$ from (4.5), (4.7) and (4.8), respectively. We thus call the system (4.6) the *curved split-complex Riccati equation*.

4.2 A curved diffusion Riccati system

Let us consider the so-called *planar diffusion Riccati system* on \mathbf{E}^2 introduced in [26] and [29] (see [8, Subsection 6.4] and set $c_0 = 1$):

$$\frac{du}{dt} = -b(t) + 2c(t)u + 4a(t)u^2 + a(t)v^4, \quad \frac{dv}{dt} = c(t)v + 4a(t)uv, \quad (4.9)$$

where $a(t), b(t)$ and $c(t)$ are arbitrary t -dependent functions. This system is associated with the t -dependent vector field

$$\mathbf{X} = -2b(t)\mathbf{X}_1 + 2c(t)\mathbf{X}_2 + 2a(t)\mathbf{X}_3, \quad (4.10)$$

where the vector fields

$$\mathbf{X}_1 = \frac{1}{2} \frac{\partial}{\partial u}, \quad \mathbf{X}_2 = u \frac{\partial}{\partial u} + \frac{1}{2} v \frac{\partial}{\partial v}, \quad \mathbf{X}_3 = \frac{1}{2} (4u^2 + v^4) \frac{\partial}{\partial u} + 2uv \frac{\partial}{\partial v},$$

verifying the commutation rules (1.4), are Hamiltonian vector fields with respect to the symplectic form given by [8]

$$\omega = -\frac{2}{v^3} du \wedge dv,$$

with associated Hamiltonian functions

$$h_1 = \frac{1}{2v^2}, \quad h_2 = \frac{u}{v^2}, \quad h_3 = 2\frac{u^2}{v^2} - \frac{1}{2}v^2.$$

The (Hamiltonian) vector fields of the Euclidean I_4 -class in the form shown in Table 1 are reproduced after the change of variables defined by

$$x := 2u + v^2, \quad y := 2u - v^2, \quad u = \frac{1}{4}(x + y), \quad v = \frac{\pm 1}{\sqrt{2}} \sqrt{|x - y|},$$

with domains $\mathbb{R}_{x \neq y}^2$ and $\mathbb{R}_{v \neq 0}^2$.

Then, we apply the above transformation on $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$ to the curved I_4 -LH class, giving rise to the curved t -dependent vector field counterpart \mathbf{X}_κ of (4.10), where the vector fields $\mathbf{X}_{\kappa,i}$ read as

$$\begin{aligned} \mathbf{X}_{\kappa,1} &= \frac{1}{2} \frac{\partial}{\partial u}, & \mathbf{X}_{\kappa,2} &= \frac{1}{2} C_\kappa(v^2) S_\kappa(2u) \frac{\partial}{\partial u} + \frac{1}{2v} C_\kappa(2u) S_\kappa(v^2) \frac{\partial}{\partial v}, \\ \mathbf{X}_{\kappa,3} &= \left(\frac{1 - C_\kappa(2u) C_\kappa(v^2)}{\kappa} \right) \frac{\partial}{\partial u} + \frac{1}{v} S_\kappa(2u) S_\kappa(v^2) \frac{\partial}{\partial y}, \end{aligned} \quad (4.11)$$

closing onto the commutation relations (3.3). The first-order system of ODEs on $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$ associated with \mathbf{X}_κ is given by

$$\begin{aligned} \frac{du}{dt} &= -b(t) + c(t) C_\kappa(v^2) S_\kappa(2u) + 2a(t) \left(\frac{1 - C_\kappa(2u) C_\kappa(v^2)}{\kappa} \right), \\ \frac{dv}{dt} &= c(t) \frac{1}{v} C_\kappa(2u) S_\kappa(v^2) + 2a(t) \frac{1}{v} S_\kappa(2u) S_\kappa(v^2). \end{aligned} \quad (4.12)$$

The vector fields (4.11) turn out to be Hamiltonian vector fields relative to the symplectic form

$$\omega_\kappa = -\frac{2v}{S_\kappa^2(v^2)} du \wedge dv. \quad (4.13)$$

Their associated Hamiltonian functions read

$$h_{\kappa,1} = \frac{1}{2T_{\kappa}(v^2)}, \quad h_{\kappa,2} = \frac{S_{\kappa}(2u)}{2S_{\kappa}(v^2)}, \quad h_{\kappa,3} = \frac{C_{\kappa}(v^2) - C_{\kappa}(2u)}{\kappa S_{\kappa}(v^2)}, \quad (4.14)$$

and fulfil the Poisson brackets (3.9). The vector fields, symplectic form and Hamiltonian functions of the planar diffusion Riccati system on \mathbf{E}^2 (4.9) are recovered from (4.11), (4.13) and (4.14) after the contraction $\kappa \rightarrow 0$. We therefore call the system (4.12) the *curved Riccati diffusion system*.

4.3 Curved Kummer–Schwarz equation for $c < 0$

We now consider the second-order Kummer–Schwarz equation on \mathbb{R} studied in [7–9, 32–34]:

$$\frac{d^2u}{dt^2} = \frac{3}{2u} \left(\frac{du}{dt} \right)^2 - 2cu^3 + 2\eta(t)u,$$

where $\eta(t)$ is an arbitrary t -dependent function and c is a constant. This equation is equivalent to the following first-order system of ODEs on the Euclidean plane \mathbf{E}^2

$$\frac{du}{dt} = v, \quad \frac{dv}{dt} = \frac{3v^2}{2u} - 2cu^3 + 2\eta(t)u, \quad (4.15)$$

which is related to the t -dependent vector field

$$\mathbf{X} = \mathbf{X}_3 + \eta(t)\mathbf{X}_1, \quad (4.16)$$

where the vector fields given by

$$\mathbf{X}_1 = 2u \frac{\partial}{\partial v}, \quad \mathbf{X}_2 = u \frac{\partial}{\partial u} + 2v \frac{\partial}{\partial v}, \quad \mathbf{X}_3 = v \frac{\partial}{\partial u} + \left(\frac{3v^2}{2u} - 2cu^3 \right) \frac{\partial}{\partial v} \quad (4.17)$$

verify the commutation relations (1.4). Whenever $c < 0$, we are dealing with the Euclidean I_4 -class. Let us denote the constant c as

$$c = -\frac{1}{4\lambda^2}, \quad \lambda \in \mathbb{R} - \{0\}.$$

The vector fields (4.17) are Hamiltonian vector fields relative to the symplectic form

$$\omega = \frac{\lambda}{2u^3} du \wedge dv,$$

with associated Hamiltonian functions

$$h_1 = \frac{\lambda}{u}, \quad h_2 = \frac{\lambda v}{2u^2}, \quad h_3 = -\frac{u}{4\lambda} + \frac{\lambda v^2}{4u^3}.$$

The (Hamiltonian) vector fields are mapped into those encompassing the Euclidean I_4 -class (see Table 1) via the change of variables defined by [7]

$$x := \frac{v}{2u} + \frac{u}{2\lambda}, \quad y := \frac{v}{2u} - \frac{u}{2\lambda}, \quad u = \lambda(x - y), \quad v = \lambda(x^2 - y^2),$$

according to the domains $\mathbb{R}_{x \neq y}^2$ and $\mathbb{R}_{u \neq 0}^2$.

Then, we apply the above transformation on $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$ to the curved I_4 -LH class in order to use the results obtained in Section 3. This yields the curved analogue \mathbf{X}_κ of (4.16) where the vector fields

$$\begin{aligned} \mathbf{X}_{\kappa,1} &= 2u \frac{\partial}{\partial v}, \\ \mathbf{X}_{\kappa,2} &= 2\lambda S_\kappa\left(\frac{u}{2\lambda}\right) C_\kappa\left(\frac{v}{2u}\right) \frac{\partial}{\partial u} + 2 \left\{ u C_\kappa\left(\frac{u}{2\lambda}\right) S_\kappa\left(\frac{v}{2u}\right) + \frac{\lambda v}{u} S_\kappa\left(\frac{u}{2\lambda}\right) C_\kappa\left(\frac{v}{2u}\right) \right\} \frac{\partial}{\partial v}, \\ \mathbf{X}_{\kappa,3} &= 4\lambda S_\kappa\left(\frac{u}{2\lambda}\right) S_\kappa\left(\frac{v}{2u}\right) \frac{\partial}{\partial u} + 4 \left\{ u \left(\frac{1 - C_\kappa\left(\frac{u}{2\lambda}\right) C_\kappa\left(\frac{v}{2u}\right)}{\kappa} \right) + \frac{\lambda v}{u} S_\kappa\left(\frac{u}{2\lambda}\right) S_\kappa\left(\frac{v}{2u}\right) \right\} \frac{\partial}{\partial v}, \end{aligned} \quad (4.18)$$

have commutation relations (3.3). The first-order system of ODEs on $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$ associated with \mathbf{X}_κ is given by

$$\begin{aligned} \frac{du}{dt} &= 4\lambda S_\kappa\left(\frac{u}{2\lambda}\right) S_\kappa\left(\frac{v}{2u}\right), \\ \frac{dv}{dt} &= 4u \left(\frac{1 - C_\kappa\left(\frac{u}{2\lambda}\right) C_\kappa\left(\frac{v}{2u}\right)}{\kappa} \right) + 4 \frac{\lambda v}{u} S_\kappa\left(\frac{u}{2\lambda}\right) S_\kappa\left(\frac{v}{2u}\right) + 2\eta(t)u. \end{aligned} \quad (4.19)$$

The vector fields (4.18) are Hamiltonian vector fields relative to the symplectic form

$$\omega_\kappa = \frac{1}{8\lambda u S_\kappa^2\left(\frac{u}{2\lambda}\right)} du \wedge dv, \quad (4.20)$$

and the Hamiltonian functions associated with the vector fields (4.18) read

$$h_{\kappa,1} = \frac{1}{2T_\kappa\left(\frac{u}{2\lambda}\right)}, \quad h_{\kappa,2} = \frac{S_\kappa\left(\frac{v}{2u}\right)}{2S_\kappa\left(\frac{u}{2\lambda}\right)}, \quad h_{\kappa,3} = \frac{C_\kappa\left(\frac{u}{2\lambda}\right) - C_\kappa\left(\frac{v}{2u}\right)}{\kappa S_\kappa\left(\frac{u}{2\lambda}\right)}, \quad (4.21)$$

obeying the Poisson brackets (3.9). The vector fields spanning the VG Lie algebra, symplectic form and Hamiltonian functions of the Kummer–Schwarz equation for $c < 0$ (4.15) on \mathbf{E}^2 are recovered from (4.18), (4.20) and (4.21), respectively, after the contraction $\kappa \rightarrow 0$. Therefore, we call the system (4.19) the *curved Kummer–Schwarz equations* for $c < 0$.

4.4 Curved Ermakov equation for $c < 0$

The Ermakov equation (1.1), outlined in Section 1, with t -dependent vector field \mathbf{X} (1.2) and vector fields (1.3) belongs to the Euclidean I_4 -LH class whenever $c < 0$ [7, 8]. For the computations, we introduce a non-zero real constant λ as

$$c = -\frac{\lambda^4}{4}, \quad \lambda \in \mathbb{R} - \{0\}. \quad (4.22)$$

The change of coordinates on \mathbf{E}^2 defined by

$$x := -\frac{v}{u} + \frac{\lambda^2}{2u^2}, \quad y := -\frac{v}{u} - \frac{\lambda^2}{2u^2}, \quad u = \pm \frac{\lambda}{\sqrt{|x-y|}}, \quad v = \mp \frac{\lambda(x+y)}{2\sqrt{|x-y|}}, \quad (4.23)$$

maps the vector fields (1.3) into the vector fields spanning the Euclidean I_4 -LH class as shown in Table 1. Hence the domains for both set of coordinates are $\mathbb{R}_{x \neq y}^2$ and $\mathbb{R}_{u \neq 0}^2$. Then, we apply the

transformation (4.23) on $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$ to the curved I₄-LH class. This leads to the t -dependent vector field $\mathbf{X}_\kappa = \mathbf{X}_{\kappa,3} + \Omega^2(t)\mathbf{X}_{\kappa,1}$, where the vector fields given by

$$\begin{aligned}\mathbf{X}_{\kappa,1} &= -u \frac{\partial}{\partial v}, \\ \mathbf{X}_{\kappa,2} &= -\frac{u^3}{\lambda^2} \mathrm{C}_\kappa\left(\frac{v}{u}\right) \mathrm{S}_\kappa\left(\frac{\lambda^2}{2u^2}\right) \frac{\partial}{\partial u} + \left\{ u \mathrm{S}_\kappa\left(\frac{v}{u}\right) \mathrm{C}_\kappa\left(\frac{\lambda^2}{2u^2}\right) - \frac{u^2 v}{\lambda^2} \mathrm{C}_\kappa\left(\frac{v}{u}\right) \mathrm{S}_\kappa\left(\frac{\lambda^2}{2u^2}\right) \right\} \frac{\partial}{\partial v}, \\ \mathbf{X}_{\kappa,3} &= \frac{2u^3}{\lambda^2} \mathrm{S}_\kappa\left(\frac{v}{u}\right) \mathrm{S}_\kappa\left(\frac{\lambda^2}{2u^2}\right) \frac{\partial}{\partial u} + \left\{ \frac{2u^2 v}{\lambda^2} \mathrm{S}_\kappa\left(\frac{v}{u}\right) \mathrm{S}_\kappa\left(\frac{\lambda^2}{2u^2}\right) - 2u \left(\frac{1 - \mathrm{C}_\kappa\left(\frac{v}{u}\right) \mathrm{C}_\kappa\left(\frac{\lambda^2}{2u^2}\right)}{\kappa} \right) \right\} \frac{\partial}{\partial v},\end{aligned}\tag{4.24}$$

satisfy the commutation relations (3.3). The first-order system of ODEs on $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$ associated with \mathbf{X}_κ reads

$$\begin{aligned}\frac{du}{dt} &= \frac{2u^3}{\lambda^2} \mathrm{S}_\kappa\left(\frac{v}{u}\right) \mathrm{S}_\kappa\left(\frac{\lambda^2}{2u^2}\right), \\ \frac{dv}{dt} &= \frac{2u^2 v}{\lambda^2} \mathrm{S}_\kappa\left(\frac{v}{u}\right) \mathrm{S}_\kappa\left(\frac{\lambda^2}{2u^2}\right) - 2u \left(\frac{1 - \mathrm{C}_\kappa\left(\frac{v}{u}\right) \mathrm{C}_\kappa\left(\frac{\lambda^2}{2u^2}\right)}{\kappa} \right) - \Omega^2(t)u.\end{aligned}\tag{4.25}$$

Moreover, the vector fields (4.24) are Hamiltonian vector fields relative to the symplectic form

$$\omega_\kappa = \frac{\lambda^4}{4u^4 \mathrm{S}_\kappa^2\left(\frac{\lambda^2}{2u^2}\right)} du \wedge dv.\tag{4.26}$$

Their associated Hamiltonian functions, determined by the condition (1.7), turn out to be

$$h_{\kappa,1} = \frac{\lambda^2}{4 \mathrm{T}_\kappa\left(\frac{\lambda^2}{2u^2}\right)}, \quad h_{\kappa,2} = -\frac{\lambda^2 \mathrm{S}_\kappa\left(\frac{v}{u}\right)}{4 \mathrm{S}_\kappa\left(\frac{\lambda^2}{2u^2}\right)}, \quad h_{\kappa,3} = \lambda^2 \frac{\mathrm{C}_\kappa\left(\frac{\lambda^2}{2u^2}\right) - \mathrm{C}_\kappa\left(\frac{v}{u}\right)}{2\kappa \mathrm{S}_\kappa\left(\frac{\lambda^2}{2u^2}\right)},\tag{4.27}$$

which verify the commutation relations (3.9).

The vector fields (1.3), the symplectic form (1.5) and the Hamiltonian functions (1.8) of the Ermakov equation (1.1) for $c < 0$ on \mathbf{E}^2 are recovered from (4.24), (4.26) and (4.27), respectively, after the contraction $\kappa \rightarrow 0$. We therefore call the system (4.25) the *curved Ermakov equation* for $c < 0$. In addition, such a curved system can alternatively be regarded as the curved Milne–Pinney equation or time-dependent curved Smorodinsky–Winternitz system on $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$. As in the previous applications, the corresponding t -independent constants of the motion and superposition rules can be easily derived from the results of Section 3, now via the change of variables (4.23). Due to their mathematical and physical relevance, let us write, for example, the constants of the motion (3.13) for (4.25):

$$F_\kappa^{(1)} = -\frac{\lambda^4}{16} = \frac{c}{4}, \quad F_\kappa^{(2)} = \frac{\lambda^4}{8} \frac{\mathrm{C}_\kappa\left(\frac{\lambda^2(u_1^2 + u_2^2)}{2u_1^2 u_2^2}\right) - \mathrm{C}_\kappa\left(\frac{u_1 v_2 - u_2 v_1}{u_1 u_2}\right)}{\kappa \mathrm{S}_\kappa\left(\frac{\lambda^2}{2u_1^2}\right) \mathrm{S}_\kappa\left(\frac{\lambda^2}{2u_2^2}\right)}.$$

The contraction $\kappa \rightarrow 0$ leads to the invariant of the Milne–Pinney equation or Smorodinsky–Winternitz system exactly in the form already obtained in [9, 50], namely

$$F_0^{(2)} = \frac{1}{4}(u_1 v_2 - u_2 v_1)^2 - \frac{\lambda^4}{16} \frac{(u_1^2 + u_2^2)^2}{u_1^2 u_2^2} = \frac{1}{4}(u_1 v_2 - u_2 v_1)^2 + \frac{c}{4} \left(\frac{u_1^2}{u_2^2} + \frac{u_2^2}{u_1^2} \right) + \frac{c}{2},$$

which was the cornerstone for deriving a superposition rule in those works. In our case, this would be recovered from (3.21) and the one corresponding to the curved system (4.25) would be provided by (3.20) both through the map (4.23).

Finally, remind, as already commented in Section 3 (see eq. (3.10)), that any curved LH system can be interpreted as a perturbation on the contraction parameter κ of the initial Euclidean system. In particular, for the curved Ermakov system (4.25), we obtain at the first-order in κ the following approximation:

$$\begin{aligned}\frac{du}{dt} &= v + \frac{\kappa}{6} \left(\frac{cv}{u^4} - \frac{v^3}{u^2} \right) + o[\kappa^2], \\ \frac{dv}{dt} &= -\Omega^2(t)u + \frac{c}{u^3} + \frac{\kappa}{12} \left(\frac{c^2}{u^7} - \frac{4cv^2}{u^5} - \frac{v^4}{u^3} \right) + o[\kappa^2],\end{aligned}\tag{4.28}$$

to be compared with (1.1) and where we have introduced the parameter $c < 0$ instead of λ (4.22).

5 The class \mathbf{P}_2 of Lie–Hamilton systems on curved spaces

The class \mathbf{P}_2 of LH systems on the Euclidean plane \mathbf{E}^2 (see Table 1) is spanned by the vector fields

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \quad \mathbf{X}_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad \mathbf{X}_3 = (x^2 - y^2) \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y},\tag{5.1}$$

which give rise to a subalgebra $\mathfrak{so}(2, 1) \simeq \mathfrak{sl}(2, \mathbb{R})$, with commutation rules (1.4), of the conformal algebra $\mathfrak{conf}_{0,+} \simeq \mathfrak{so}(3, 1)$ of $\mathbf{E}^2 \equiv \mathbf{S}_{[0,+]}^2$. Therefore, they have a very concrete geometrical meaning as conformal symmetries. From (2.24) with $\kappa_2 = +1$, it is found that they correspond, in this order, to the generators of translations along the basic geodesic l_1 , dilations and specific conformal transformations related to l_1 on \mathbf{E}^2 , that is, $\mathbf{X}_1 = -\mathbf{P}_1$, $\mathbf{X}_2 = -\mathbf{D}$ and $\mathbf{X}_3 = 2\mathbf{G}_1$.

In order to generalize the Euclidean class \mathbf{P}_2 , let us consider the curved counterparts (2.22) of the vector fields (5.1), thus yielding the following conformal vector fields on the 2D CK spaces $\mathbf{S}_{[\kappa_1], \kappa_2}^2$:

$$\begin{aligned}\mathbf{X}_{\kappa,1} &= \frac{\partial}{\partial x}, & \mathbf{X}_{\kappa,2} &= \frac{S_{\kappa_1}(x)}{C_{\kappa_1\kappa_2}(y)} \frac{\partial}{\partial x} + C_{\kappa_1}(x) S_{\kappa_1\kappa_2}(y) \frac{\partial}{\partial y}, \\ \mathbf{X}_{\kappa,3} &= \frac{2}{C_{\kappa_1\kappa_2}(y)} (V_{\kappa_1}(x) - \kappa_2 V_{\kappa_1\kappa_2}(y)) \frac{\partial}{\partial x} + 2S_{\kappa_1}(x) S_{\kappa_1\kappa_2}(y) \frac{\partial}{\partial y},\end{aligned}\tag{5.2}$$

where we have used the shorthand notation $\kappa = (\kappa_1, \kappa_2)$ introduced in Section 2. Their commutation relations read (see (2.21))

$$[\mathbf{X}_{\kappa,1}, \mathbf{X}_{\kappa,2}] = \mathbf{X}_{\kappa,1} - \frac{1}{2}\kappa_1 \mathbf{X}_{\kappa,3}, \quad [\mathbf{X}_{\kappa,1}, \mathbf{X}_{\kappa,3}] = 2\mathbf{X}_{\kappa,2}, \quad [\mathbf{X}_{\kappa,2}, \mathbf{X}_{\kappa,3}] = \mathbf{X}_{\kappa,3}.\tag{5.3}$$

The t -dependent vector field

$$\mathbf{X}_{\kappa} = b_1(t)\mathbf{X}_{\kappa,1} + b_2(t)\mathbf{X}_{\kappa,2} + b_3(t)\mathbf{X}_{\kappa,3},\tag{5.4}$$

where $b_i(t)$ are arbitrary t -dependent functions, is associated with the following first-order system of ODEs on $\mathbf{S}_{[\kappa_1], \kappa_2}^2$:

$$\begin{aligned}\frac{dx}{dt} &= b_1(t) + b_2(t) \frac{S_{\kappa_1}(x)}{C_{\kappa_1\kappa_2}(y)} + 2b_3(t) \frac{C_{\kappa_1\kappa_2}(y) - C_{\kappa_1}(x)}{\kappa_1 C_{\kappa_1\kappa_2}(y)}, \\ \frac{dy}{dt} &= b_2(t) C_{\kappa_1}(x) S_{\kappa_1\kappa_2}(y) + 2b_3(t) S_{\kappa_1}(x) S_{\kappa_1\kappa_2}(y).\end{aligned}\tag{5.5}$$

Consequently, \mathbf{X}_κ is a Lie system whose VG Lie algebra V^{X_κ} spanned by the vector fields (5.2) is isomorphic to $\mathfrak{so}(2,1)$. In addition, \mathbf{X}_κ is also a LH system since the relation (1.6) provides a symplectic form

$$\omega_\kappa = \frac{C_{\kappa_1\kappa_2}(y)}{S_{\kappa_1\kappa_2}^2(y)} dx \wedge dy, \quad (5.6)$$

turning the vector fields (5.2) into Hamiltonian vector fields via the condition (1.7); these are

$$\begin{aligned} h_{\kappa,1} &= -\frac{1}{S_{\kappa_1\kappa_2}(y)}, & h_{\kappa,2} &= -\frac{S_{\kappa_1}(x)}{T_{\kappa_1\kappa_2}(y)}, \\ h_{\kappa,3} &= \frac{2}{S_{\kappa_1\kappa_2}(y)} (\kappa_1\kappa_2 V_{\kappa_1}(x) V_{\kappa_1\kappa_2}(y) - \kappa_2 V_{\kappa_1\kappa_2}(y) - V_{\kappa_1}(x)) \\ &= 2 \frac{C_{\kappa_1}(x) C_{\kappa_1\kappa_2}(y) - 1}{\kappa_1 S_{\kappa_1\kappa_2}(y)}. \end{aligned} \quad (5.7)$$

Their Poisson brackets $\{\cdot, \cdot\}_{\omega_\kappa}$ induced by ω_κ (5.6) read as

$$\{h_{\kappa,1}, h_{\kappa,2}\}_{\omega_\kappa} = -h_{\kappa,1} + \frac{1}{2}\kappa_1 h_{\kappa,3}, \quad \{h_{\kappa,1}, h_{\kappa,3}\}_{\omega_\kappa} = -2h_{\kappa,2}, \quad \{h_{\kappa,2}, h_{\kappa,3}\}_{\omega_\kappa} = -h_{\kappa,3}, \quad (5.8)$$

showing that \mathbf{X}_κ is a LH system with LH algebra $\mathcal{H}_{\omega_\kappa} \simeq \mathfrak{so}(2,1)$.

Summing up, we call the LH system \mathbf{X}_κ the *curved P_2 -LH class* as it actually constitutes a generalization of the Euclidean P_2 -LH class to the 2D curved spaces $\mathbf{S}_{[\kappa_1],\kappa_2}^2$, recovering the Euclidean LH structure, shown in Table 1, from (5.2), (5.6) and (5.7) under the contraction $\kappa_1 \rightarrow 0$ and setting $\kappa_2 = +1$.

We explicitly display in Table 3 the curved P_2 -LH class on each of the nine 2D CK spaces. In all cases, the VG Lie algebra $V^{X_\kappa} \simeq \mathfrak{so}(2,1)$ (5.3) and the LH algebra $\mathcal{H}_{\omega_\kappa} \simeq \mathfrak{so}(2,1)$ (5.8). The domain of the variables (x, y) guarantees a well-defined symplectic form and Hamiltonian functions.

Recall that the six spaces with $\kappa_2 \leq 0$ can be regarded as $(1+1)$ D kinematical spacetimes considering the relationships (2.15), where x becomes the time coordinate and y the spatial one. Under this interpretation, the vector fields $\mathbf{X}_{\kappa,i}$ (5.2) correspond to the generators of time translations along the time-like geodesic l_1 , dilations and specific conformal transformations also related to l_1 on $\mathbf{S}_{[\kappa_1],\kappa_2}^2$. Therefore, the resulting LH systems shown in Table 3 are of ‘time-like’ type. In this respect, observe that although the ‘main’ time-like metric (2.14) is degenerate in the Newtonian spaces with $\kappa_2 = 0$ ($c = \infty$), with foliation determined by (2.16), our results allow the construction of Newtonian LH systems in a consistent way, as it was already performed considering the isometries on $\mathbf{S}_{[\kappa_1],\kappa_2}^2$ from the Euclidean P_1 -LH class in [37].

Furthermore, we stress that, in this kinematical framework, it would be possible to construct ‘space-like’ LH systems by starting from the generator of spatial translations \mathbf{P}_2 along the geodesic l_2 instead of $\mathbf{P}_1 = \partial_x$ (2.22). In this case, the well adapted variables would be the geodesic parallel coordinates of type II (x', y') (2.13). In particular, for such an analogous ‘space-like’ approach, one should consider the vector fields $\mathbf{X}_{\kappa,1} = -\mathbf{P}_2$, $\mathbf{X}_{\kappa,2} = -\mathbf{D}$ and $\mathbf{X}_{\kappa,3} = 2\mathbf{G}_2$, which in terms of (x', y') are given by [35]

$$\begin{aligned} \mathbf{X}_{\kappa,1} &= \frac{\partial}{\partial y'}, & \mathbf{X}_{\kappa,2} &= S_{\kappa_1}(x') C_{\kappa_1\kappa_2}(y') \frac{\partial}{\partial x'} + \frac{S_{\kappa_1\kappa_2}(y')}{C_{\kappa_1}(x')} \frac{\partial}{\partial y'}, \\ \mathbf{X}_{\kappa,3} &= 2\kappa_2 S_{\kappa_1}(x') S_{\kappa_1\kappa_2}(y') \frac{\partial}{\partial x'} - \frac{2}{C_{\kappa_1}(x')} (V_{\kappa_1}(x') - \kappa_2 V_{\kappa_1\kappa_2}(y')) \frac{\partial}{\partial y'}, \end{aligned}$$

Table 3: The curved P₂-LH class on the nine 2D CK spaces $\mathbf{S}_{[\kappa_1], \kappa_2}^2$ according to the ‘normalized’ values $\kappa_i \in \{1, 0, -1\}$. For each space it is shown, in geodesic parallel coordinates (x, y) (2.13), the metric ds_κ^2 (2.14), domain of the variables, vector fields $\mathbf{X}_{\kappa, i}$ (5.2), Hamiltonian functions $h_{\kappa, i}$ (5.7), and symplectic form ω_κ (5.6). For the sake of clarity, we drop the index $\kappa = (\kappa_1, \kappa_2)$.

<p>• Sphere \mathbf{S}^2 $\mathbf{S}_{[+], +}^2 = \text{SO}(3)/\text{SO}(2)$ $ds^2 = \cos^2 y dx^2 + dy^2$ $x \in (-\pi, \pi], y \in [-\frac{\pi}{2}, \frac{\pi}{2}], y \neq 0$ $\mathbf{X}_1 = \partial_x$ $\mathbf{X}_2 = \frac{\sin x}{\cos y} \partial_x + \cos x \sin y \partial_y$ $\mathbf{X}_3 = 2 \frac{\cos y - \cos x}{\cos y} \partial_x + 2 \sin x \sin y \partial_y$ $h_1 = -\frac{1}{\sin y} \quad h_2 = -\frac{\sin x}{\tan y}$ $h_3 = 2 \frac{\cos x \cos y - 1}{\sin y}$ $\omega = \frac{\cos y}{\sin^2 y} dx \wedge dy$</p>	<p>• Euclidean plane \mathbf{E}^2 $\mathbf{S}_{[0], +}^2 = \text{ISO}(2)/\text{SO}(2)$ $ds^2 = dx^2 + dy^2$ $x \in \mathbb{R}, y \in \mathbb{R}, y \neq 0$ $\mathbf{X}_1 = \partial_x$ $\mathbf{X}_2 = x \partial_x + y \partial_y$ $\mathbf{X}_3 = (x^2 - y^2) \partial_x + 2xy \partial_y$ $h_1 = -\frac{1}{y} \quad h_2 = -\frac{x}{y}$ $h_3 = -\frac{x^2 + y^2}{y}$ $\omega = \frac{1}{y^2} dx \wedge dy$</p>	<p>• Hyperbolic plane \mathbf{H}^2 $\mathbf{S}_{[-], +}^2 = \text{SO}(2, 1)/\text{SO}(2)$ $ds^2 = \cosh^2 y dx^2 + dy^2$ $x \in \mathbb{R}, y \in \mathbb{R}, y \neq 0$ $\mathbf{X}_1 = \partial_x$ $\mathbf{X}_2 = \frac{\sinh x}{\cosh y} \partial_x + \cosh x \sinh y \partial_y$ $\mathbf{X}_3 = 2 \frac{\cosh x - \cosh y}{\cosh y} \partial_x + 2 \sinh x \sinh y \partial_y$ $h_1 = -\frac{1}{\sinh y} \quad h_2 = -\frac{\sinh x}{\tanh y}$ $h_3 = 2 \frac{1 - \cosh x \cosh y}{\sinh y}$ $\omega = \frac{\cosh y}{\sinh^2 y} dx \wedge dy$</p>
<p>• Oscillating NH space \mathbf{NH}_+^{1+1} $\mathbf{S}_{[+], 0}^2 = \text{ISO}(2)/\mathbb{R}$ $ds^2 = dx^2$ $x \in (-\pi, \pi], y \in \mathbb{R}, y \neq 0$ $\mathbf{X}_1 = \partial_x$ $\mathbf{X}_2 = \sin x \partial_x + y \cos x \partial_y$ $\mathbf{X}_3 = 2(1 - \cos x) \partial_x + 2y \sin x \partial_y$ $h_1 = -\frac{1}{y} \quad h_2 = -\frac{\sin x}{y}$ $h_3 = 2 \frac{\cos x - 1}{y}$ $\omega = \frac{1}{y^2} dx \wedge dy$</p>	<p>• Galilean plane \mathbf{G}^{1+1} $\mathbf{S}_{[0], 0}^2 = \text{IISO}(1)/\mathbb{R}$ $ds^2 = dx^2$ $x \in \mathbb{R}, y \in \mathbb{R}, y \neq 0$ $\mathbf{X}_1 = \partial_x$ $\mathbf{X}_2 = x \partial_x + y \partial_y$ $\mathbf{X}_3 = x^2 \partial_x + 2xy \partial_y$ $h_1 = -\frac{1}{y} \quad h_2 = -\frac{x}{y}$ $h_3 = -\frac{x^2}{y}$ $\omega = \frac{1}{y^2} dx \wedge dy$</p>	<p>• Expanding NH space \mathbf{NH}_-^{1+1} $\mathbf{S}_{[-], 0}^2 = \text{ISO}(1, 1)/\mathbb{R}$ $ds^2 = dx^2$ $x \in \mathbb{R}, y \in \mathbb{R}, y \neq 0$ $\mathbf{X}_1 = \partial_x$ $\mathbf{X}_2 = \sinh x \partial_x + y \cosh x \partial_y$ $\mathbf{X}_3 = 2(\cosh x - 1) \partial_x + 2y \sinh x \partial_y$ $h_1 = -\frac{1}{y} \quad h_2 = -\frac{\sinh x}{y}$ $h_3 = 2 \frac{1 - \cosh x}{y}$ $\omega = \frac{1}{y^2} dx \wedge dy$</p>
<p>• Anti-de Sitter space \mathbf{AdS}^{1+1} $\mathbf{S}_{[+], -}^2 = \text{SO}(2, 1)/\text{SO}(1, 1)$ $ds^2 = \cosh^2 y dx^2 - dy^2$ $x \in (-\pi, \pi], y \in \mathbb{R}, y \neq 0$ $\mathbf{X}_1 = \partial_x$ $\mathbf{X}_2 = \frac{\sin x}{\cosh y} \partial_x + \cos x \sinh y \partial_y$ $\mathbf{X}_3 = 2 \frac{\cosh y - \cos x}{\cosh y} \partial_x + 2 \sin x \sinh y \partial_y$ $h_1 = -\frac{1}{\sinh y} \quad h_2 = -\frac{\sin x}{\tanh y}$ $h_3 = 2 \frac{\cos x \cosh y - 1}{\sinh y}$ $\omega = \frac{\cosh y}{\sinh^2 y} dx \wedge dy$</p>	<p>• Minkowskian plane \mathbf{M}^{1+1} $\mathbf{S}_{[0], -}^2 = \text{ISO}(1, 1)/\text{SO}(1, 1)$ $ds^2 = dx^2 - dy^2$ $x \in \mathbb{R}, y \in \mathbb{R}, y \neq 0$ $\mathbf{X}_1 = \partial_x$ $\mathbf{X}_2 = x \partial_x + y \partial_y$ $\mathbf{X}_3 = (x^2 + y^2) \partial_x + 2xy \partial_y$ $h_1 = -\frac{1}{y} \quad h_2 = -\frac{x}{y}$ $h_3 = -\frac{x^2 - y^2}{y}$ $\omega = \frac{1}{y^2} dx \wedge dy$</p>	<p>• De Sitter space \mathbf{dS}^{1+1} $\mathbf{S}_{[-], -}^2 = \text{SO}(2, 1)/\text{SO}(1, 1)$ $ds^2 = \cos^2 y dx^2 - dy^2$ $x \in \mathbb{R}, y \in (-\pi, \pi), y \neq 0$ $\mathbf{X}_1 = \partial_x$ $\mathbf{X}_2 = \frac{\sinh x}{\cos y} \partial_x + \cosh x \sin y \partial_y$ $\mathbf{X}_3 = 2 \frac{\cosh x - \cos y}{\cos y} \partial_x + 2 \sinh x \sin y \partial_y$ $h_1 = -\frac{1}{\sin y} \quad h_2 = -\frac{\sinh x}{\tan y}$ $h_3 = 2 \frac{1 - \cosh x \cos y}{\sin y}$ $\omega = \frac{\cos y}{\sin^2 y} dx \wedge dy$</p>

fulfilling the following commutation relations (see (2.21))

$$[\mathbf{X}_{\kappa,1}, \mathbf{X}_{\kappa,2}] = \mathbf{X}_{\kappa,1} - \frac{1}{2}\kappa_1 \mathbf{X}_{\kappa,3}, \quad [\mathbf{X}_{\kappa,1}, \mathbf{X}_{\kappa,3}] = 2\kappa_2 \mathbf{X}_{\kappa,2}, \quad [\mathbf{X}_{\kappa,2}, \mathbf{X}_{\kappa,3}] = \mathbf{X}_{\kappa,3},$$

and to be compared with (5.3). Hence, they now correspond to the generators of spatial translations along the space-like geodesic l_2 (see Figure 1), dilations and specific conformal transformations related to l_2 on $\mathbf{S}_{[\kappa_1], \kappa_2}^2$. Clearly, such ‘space-like’ LH systems would be diffeomorphic to the previous ‘time-like’ ones only for the three Riemannian spaces with $\kappa_2 > 0$, under an appropriate change of variables, but they would provide new LH systems for the six models of spacetimes.

Then, as we have already commented in Section 3 (see eq. (3.10)), the contraction parameters $\boldsymbol{\kappa} = (\kappa_1, \kappa_2)$ can be seen as ‘integrable’ perturbation parameters from the initial ‘flat’ system. In the curved P₂-LH class, the most contracted LH system is just the Galilean one, shown in the center of Table 3, leading to following system

$$\frac{dx}{dt} = b_1(t) + b_2(t)x + b_3(t)x^2, \quad \frac{dy}{dt} = b_2(t)y + 2b_3(t)xy, \quad (5.9)$$

which is formed by a Riccati equation in x coupled with another equation in y . From it we can express the curved P₂-LH class (5.5) as a perturbation at the first-order in both (κ_1, κ_2) , namely,

$$\begin{aligned} \frac{dx}{dt} &= b_1(t) + b_2(t)x + b_3(t)x^2 - \frac{1}{12}\kappa_1(2b_2(t)x^3 + b_3(t)x^4) - \kappa_2 b_3(t)y^2 + o[\kappa_1^2, \kappa_2^2, \kappa_1\kappa_2], \\ \frac{dy}{dt} &= b_2(t)y + 2b_3(t)xy - \frac{1}{6}\kappa_1(3b_2(t)x^2y + 2b_3(t)x^3y) + o[\kappa_1^2, \kappa_2^2, \kappa_1\kappa_2]. \end{aligned} \quad (5.10)$$

Other perturbations can be constructed starting from another contracted LH system, that is, from the Euclidean, Minkowkian, and both Newton–Hooke ones.

5.1 Constants of the motion of the curved P₂-class

Analogously to the curved I₄-class (see Section 3.1), we now compute the t -independent constants of the motion of the LH system $\mathbf{X}_{\boldsymbol{\kappa}}$ (5.5). Since the LH algebra $\mathcal{H}_{\omega_{\boldsymbol{\kappa}}}$ (5.8) of the curved P₂-class and the LH algebra of the curved I₄-class (3.9) formally satisfy the same brackets, provided that $\boldsymbol{\kappa} \equiv \kappa_1$ (regardless of the value of κ_2), the Casimir (3.11) also holds for $\mathcal{H}_{\omega_{\boldsymbol{\kappa}}}$. From (5.7) we construct the following Hamiltonian functions ($1 \leq i \leq 3$):

$$\begin{aligned} h_{\boldsymbol{\kappa},i}^{(1)} &= h_{\boldsymbol{\kappa},i}(\mathbf{x}_1) \in C^\infty(\mathbf{S}_{[\kappa_1], \kappa_2}^2), \\ h_{\boldsymbol{\kappa},i}^{(2)} &= h_{\boldsymbol{\kappa},i}(\mathbf{x}_1) + h_{\boldsymbol{\kappa},i}(\mathbf{x}_2) \in C^\infty(\mathbf{S}_{[\kappa_1], \kappa_2}^2 \times \mathbf{S}_{[\kappa_1], \kappa_2}^2), \\ h_{\boldsymbol{\kappa},i}^{(3)} &= h_{\boldsymbol{\kappa},i}(\mathbf{x}_1) + h_{\boldsymbol{\kappa},i}(\mathbf{x}_2) + h_{\boldsymbol{\kappa},i}(\mathbf{x}_3) \in C^\infty(\mathbf{S}_{[\kappa_1], \kappa_2}^2 \times \mathbf{S}_{[\kappa_1], \kappa_2}^2 \times \mathbf{S}_{[\kappa_1], \kappa_2}^2), \end{aligned} \quad (5.11)$$

where $\mathbf{x}_s = (x_s, y_s)$ denotes the coordinates in the s^{th} -copy of $\mathbf{S}_{[\kappa_1], \kappa_2}^2$ within the product manifold. Each set of Hamiltonian functions verifies the Poisson brackets (5.8) with respect to the Poisson bracket induced by taking the product symplectic structure $\omega_{\boldsymbol{\kappa}}^{[3]}$ of (5.6) to $(\mathbf{S}_{[\kappa_1], \kappa_2}^2)^3$:

$$\omega_{\boldsymbol{\kappa}}^{[3]} = \sum_{s=1}^3 \frac{C_{\kappa_1 \kappa_2}(y_s)}{S_{\kappa_1 \kappa_2}^2(y_s)} dx_s \wedge dy_s.$$

Now, using the Casimir (3.11), with $\kappa \equiv \kappa_1$, we obtain the following constants of the motion for the diagonal prolongation $\tilde{\mathbf{X}}_\kappa^3$ of \mathbf{X}_κ to $(\mathbf{S}_{[\kappa_1, \kappa_2]}^2)^3$:

$$\begin{aligned} F_\kappa^{(1)} &= C\left(h_{\kappa,1}^{(1)}, h_{\kappa,2}^{(1)}, h_{\kappa,3}^{(1)}\right) = \kappa_2, \\ F_\kappa^{(2)} &= C\left(h_{\kappa,1}^{(2)}, h_{\kappa,2}^{(2)}, h_{\kappa,3}^{(2)}\right) = 2\left(\kappa_2 + \frac{1 - C_{\kappa_1}(x_1 - x_2) C_{\kappa_1 \kappa_2}(y_1) C_{\kappa_1 \kappa_2}(y_2)}{\kappa_1 S_{\kappa_1 \kappa_2}(y_1) S_{\kappa_1 \kappa_2}(y_2)}\right). \end{aligned} \quad (5.12)$$

We illustrate the CK contraction approach by performing the flat limit $\kappa_1 \rightarrow 0$ and the non-relativistic limit $\kappa_2 \rightarrow 0$ of $F_\kappa^{(2)}$, yielding the constants of the motion for the LH systems shown in the second column and second row in Table 3:

$$F_{(0, \kappa_2)}^{(2)} = \frac{(x_1 - x_2)^2 + \kappa_2(y_1 + y_2)^2}{y_1 y_2}, \quad F_{(\kappa_1, 0)}^{(2)} = 2\left(\frac{1 - C_{\kappa_1}(x_1 - x_2)}{\kappa_1 y_1 y_2}\right). \quad (5.13)$$

Both reduce to the constant of the motion for the (most contracted) Galilean system with $\kappa = (0, 0)$:

$$F_{(0,0)}^{(2)} = \frac{(x_1 - x_2)^2}{y_1 y_2}.$$

Recall that the constant of motion $F_{(0, \kappa_2)}^{(2)}$ leads to the one corresponding to the Euclidean P_2 -class for $\kappa_2 = +1$ obtained in [8].

Furthermore, $F_\kappa^{(2)}$ provides two other constants of the motion by means of the permutation S_{ij} of the variables $(x_i, y_i) \leftrightarrow (x_j, y_j)$:

$$\begin{aligned} F_{\kappa,13}^{(2)} &= S_{13}\left(F_\kappa^{(2)}\right) = 2\left(\kappa_2 + \frac{1 - C_{\kappa_1}(x_3 - x_2) C_{\kappa_1 \kappa_2}(y_2) C_{\kappa_1 \kappa_2}(y_3)}{\kappa_1 S_{\kappa_1 \kappa_2}(y_2) S_{\kappa_1 \kappa_2}(y_3)}\right), \\ F_{\kappa,23}^{(2)} &= S_{23}\left(F_\kappa^{(2)}\right) = 2\left(\kappa_2 + \frac{1 - C_{\kappa_1}(x_1 - x_3) C_{\kappa_1 \kappa_2}(y_1) C_{\kappa_1 \kappa_2}(y_3)}{\kappa_1 S_{\kappa_1 \kappa_2}(y_1) S_{\kappa_1 \kappa_2}(y_3)}\right). \end{aligned} \quad (5.14)$$

Finally, the functions $h_{\kappa,i}^{(3)}$ (5.11) give rise to another constant of the motion that can be written in terms of (5.12) and (5.14) in the form

$$F_\kappa^{(3)} = C\left(h_{\kappa,1}^{(3)}, h_{\kappa,2}^{(3)}, h_{\kappa,3}^{(3)}\right) = F_\kappa^{(2)} + F_{\kappa,13}^{(2)} + F_{\kappa,23}^{(2)} - 3\kappa_2.$$

5.2 Superposition rules for the curved P_2 -class

Similarly to the curved I_4 -class, addressed in Section 3.2, we have deduced four t -independent constants of the motion $\{F_\kappa^{(2)}, F_{\kappa,13}^{(2)}, F_{\kappa,23}^{(2)}, F_\kappa^{(3)}\}$ for the curved P_2 -LH system \mathbf{X}_κ (5.5), so holding for any choice of κ . They allow the derivation of a superposition rule for the diagonal prolongation $\tilde{\mathbf{X}}_\kappa^3$ of \mathbf{X}_κ to $(\mathbf{S}_{[\kappa_1, \kappa_2]}^2)^3$. Again, it is necessary to select two functionally independent constants of the motion, I_1 and I_2 , fulfilling the condition (3.16). In this case, we can set $F_\kappa^{(2)} = \mu_1$ (5.12) and $F_{\kappa,23}^{(2)} = \mu_2$ (5.14), and try to deduce the general solution $(x_1(t), y_1(t))$ of the LH system in terms of two particular solutions $(x_2(t), y_2(t))$ and $(x_3(t), y_3(t))$ along with the constants μ_1 and μ_2 . Nevertheless, the computations are very cumbersome, precluding to express a general superposition rule in an explicit way for an arbitrary $\kappa = (\kappa_1, \kappa_2)$. However, for the contracted cases, with some $\kappa_i = 0$, it is possible to obtain such a rule in an explicit form. Therefore, we shall restrict ourselves to present a superposition principle for the three flat spaces $\mathbf{S}_{[0, \kappa_2]}^2$ and for the three non-relativistic ones $\mathbf{S}_{[\kappa_1, 0]}^2$, separately.

Let us consider the flat LH system $\mathbf{X}_{(0,\kappa_2)}$ (5.5) defined on the Euclidean $\mathbf{E}^2 \equiv \mathbf{S}_{[0,+]}^2$, Galilean $\mathbf{G}^{1+1} \equiv \mathbf{S}_{[0],0}^2$ and Minkowskian $\mathbf{M}^{1+1} \equiv \mathbf{S}_{[0,-]}^2$ planes (displayed in the second column of Table 3). Under the contraction $\kappa_1 \rightarrow 0$, the symplectic form (5.6) and Hamiltonian functions (5.7), determining $\mathbf{X}_{(0,\kappa_2)}$, reduce to

$$\omega_{(0,\kappa_2)} = \frac{1}{y^2} dx \wedge dy, \quad h_{(0,\kappa_2),1} = -\frac{1}{y}, \quad h_{(0,\kappa_2),2} = -\frac{x}{y}, \quad h_{(0,\kappa_2),3} = -\frac{x^2 + \kappa_2 y^2}{y}.$$

Then we take the constant of the motion $F_{(0,\kappa_2)}^{(2)}$ (5.13) together with its permutations $F_{(0,\kappa_2),13}^{(2)}$, $F_{(0,\kappa_2),23}^{(2)}$, and set

$$\begin{aligned} F_{(0,\kappa_2)}^{(2)} &= \frac{(x_1 - x_2)^2 + \kappa_2(y_1 + y_2)^2}{y_1 y_2} = \mu_1, & F_{(0,\kappa_2),23}^{(2)} &= \frac{(x_1 - x_3)^2 + \kappa_2(y_1 + y_3)^2}{y_1 y_3} = \mu_2, \\ F_{(0,\kappa_2),13}^{(2)} &= \frac{(x_2 - x_3)^2 + \kappa_2(y_2 + y_3)^2}{y_2 y_3} = \mu_3. \end{aligned} \quad (5.15)$$

From the first equation we find that

$$x_1^\pm(y_1, x_2, y_2, \mu_1) = x_2 \pm \sqrt{\mu_1 y_1 y_2 - \kappa_2(y_1 + y_2)^2}. \quad (5.16)$$

Substituting this result in the second relation in (5.15), we obtain

$$\begin{aligned} & y_1^\pm(x_2, y_2, x_3, y_3, \mu_1, \mu_2) \\ &= \left\{ y_2 y_3 ((\mu_1 \mu_3 + A)y_2 + (\mu_2 \mu_3 + A)y_3) \pm 2\sqrt{y_2^2 y_3^2 B(\kappa_2(y_2 + y_3)^2 - \mu_3 y_2 y_3)} \right\} \\ & \quad \times \left\{ \mu_1(\mu_1 - 4\kappa_2)y_2^2 + \mu_2(\mu_2 - 4\kappa_2)y_3^2 - 2(\mu_1 \mu_2 + A)y_2 y_3 \right\}^{-1}, \\ & A = -2\kappa_2(\mu_1 + \mu_2 + \mu_3) + 8\kappa_2^2, \\ & B = -\mu_1 \mu_2 \mu_3 + \kappa_2(\mu_1 + \mu_2 + \mu_3)^2 - 8\kappa_2^2(\mu_1 + \mu_2 + \mu_3) + 16\kappa_2^3, \end{aligned}$$

reminding that $\mu_3 = \mu_3(x_2, y_2, x_3, y_3)$ via the third relation in (5.15). Finally, introducing this last expression in (5.16) we arrive at $x_1^\pm(x_2, y_2, x_3, y_3, \mu_1, \mu_2)$, thus completing the superposition principle for the flat LH system $\mathbf{X}_{(0,\kappa_2)}$.

As a second subfamily within the curved P₂-LH system \mathbf{X}_κ (5.5), let us focus on the three non-relativistic systems $\mathbf{X}_{(\kappa_1,0)}$, so with underlying oscillating NH $\mathbf{NH}_+^{1+1} \equiv \mathbf{S}_{[+],0}^2$, Galilean $\mathbf{G}^{1+1} \equiv \mathbf{S}_{[0],0}^2$ (again) and expanding NH $\mathbf{NH}_-^{1+1} \equiv \mathbf{S}_{[-],0}^2$ (1+1)D spacetimes (displayed in the second row of Table 3). Observe that under the non-relativistic limit $\kappa_2 \rightarrow 0$ ($c \rightarrow \infty$), the symplectic form (5.6) and Hamiltonian functions (5.7), defining $\mathbf{X}_{(\kappa_1,0)}$, lead to

$$\omega_{(\kappa_1,0)} = \frac{1}{y^2} dx \wedge dy, \quad h_{(\kappa_1,0),1} = -\frac{1}{y}, \quad h_{(\kappa_1,0),2} = -\frac{S_{\kappa_1}(x)}{y}, \quad h_{(\kappa_1,0),3} = 2 \frac{C_{\kappa_1}(x) - 1}{\kappa_1 y}.$$

We express the constant of the motion $F_{(\kappa_1,0)}^{(2)}$ (5.13) in the form

$$F_{(\kappa_1,0)}^{(2)} = \frac{4S_{\kappa_1}^2\left(\frac{1}{2}(x_1 - x_2)\right)}{y_1 y_2} = \mu_1,$$

so that

$$y_1(x_1, x_2, y_2, \mu_1) = \frac{4S_{\kappa_1}^2\left(\frac{1}{2}(x_1 - x_2)\right)}{\mu_1 y_2}. \quad (5.17)$$

We introduce this result in $F_{(\kappa_1,0),23}^{(2)} = S_{23}\left(F_{(\kappa_1,0)}^{(2)}\right) = \mu_2$ and, after some computations, we get the following equation

$$\mu_1 \frac{y_2}{y_3} \left(\frac{\mathbb{T}_{\kappa_1}\left(\frac{1}{2}x_1\right) \mathbb{C}_{\kappa_1}\left(\frac{1}{2}x_3\right) - \mathbb{S}_{\kappa_1}\left(\frac{1}{2}x_3\right)}{\mathbb{T}_{\kappa_1}\left(\frac{1}{2}x_1\right) \mathbb{C}_{\kappa_1}\left(\frac{1}{2}x_2\right) - \mathbb{S}_{\kappa_1}\left(\frac{1}{2}x_2\right)} \right)^2 = \mu_2.$$

This gives rise to $x_1(x_2, y_2, x_3, y_3, \mu_1, \mu_2)$ in the form

$$\mathbb{T}_{\kappa_1}\left(\frac{1}{2}x_1^\pm\right) = \frac{\sqrt{\mu_1} \mathbb{S}_{\kappa_1}\left(\frac{1}{2}x_3\right) \sqrt{y_2} \pm \sqrt{\mu_2} \mathbb{S}_{\kappa_1}\left(\frac{1}{2}x_2\right) \sqrt{y_3}}{\sqrt{\mu_1} \mathbb{C}_{\kappa_1}\left(\frac{1}{2}x_3\right) \sqrt{y_2} \pm \sqrt{\mu_2} \mathbb{C}_{\kappa_1}\left(\frac{1}{2}x_2\right) \sqrt{y_3}},$$

which, by substitution in (5.17) yields $y_1(x_2, y_2, x_3, y_3, \mu_1, \mu_2)$ completing the full superposition rule for the non-relativistic LH systems $\mathbf{X}_{(\kappa_1,0)}$.

Consequently, we have deduced two equivalent superposition rules for LH systems based on the Galilean $\mathbf{G}^{1+1} \equiv \mathbf{S}_{[0],0}^2$ space, both leading to the following expressions

$$x_1^\pm(x_2, y_2, x_3, y_3, \mu_1, \mu_2) = \frac{\mu_1 x_3 y_2 - \mu_2 x_2 y_3 \pm (x_2 - x_3) \sqrt{\mu_1 \mu_2 y_2 y_3}}{\mu_1 y_2 - \mu_2 y_3},$$

$$y_1^\mp(x_2, y_2, x_3, y_3, \mu_1, \mu_2) = \frac{(x_2 - x_3)^2 (\mu_1 y_2 + \mu_2 y_3 \mp 2 \sqrt{\mu_1 \mu_2 y_2 y_3})}{(\mu_1 y_2 - \mu_2 y_3)^2}.$$

6 Applications of the curved \mathbf{P}_2 -class

The Euclidean \mathbf{P}_2 -LH class is known to include the following relevant systems [8]: the complex Riccati equation, the Kummer–Schwarz and Ermakov equations, both for $c > 0$ (see (1.11) and Table 1). By making use of the results of the previous section on the curved \mathbf{P}_2 -LH class we generalize such Euclidean systems to the nine CK spaces $\mathbf{S}_{[\kappa_1],\kappa_2}^2$.

6.1 Curved complex Riccati equation

The complex Riccati equation on $\mathbf{E}^2 \equiv \mathbf{S}_{[0],+}^2$ corresponds to the case in (4.1) when z is a complex number such that $\iota^2 = -1$. This equation has been widely studied from different points of view (see, e.g., [22–28] and references therein) and leads to the following first-order system on \mathbf{E}^2 :

$$\frac{dx}{dt} = b_1(t) + b_2(t)x + b_3(t)(x^2 - y^2),$$

$$\frac{dy}{dt} = b_2(t)y + 2b_3(t)xy,$$

provided that $z = x + iy$. This system is just recovered from the general t -dependent vector field \mathbf{X}_κ (5.4) with equations (5.5) for the particular case with $\kappa_1 \rightarrow 0$ and $\kappa_2 = +1$. Therefore, we shall call the LH system (5.5) the *curved complex Riccati equation* on $\mathbf{S}_{[\kappa_1],\kappa_2}^2$, being endowed with the VG Lie algebra (5.3), symplectic form (5.6) and LH algebra (5.8).

Observe that for the three flat LH systems $\mathbf{X}_{(0,\kappa_2)}$ (5.5), covering $\mathbf{E}^2 \equiv \mathbf{S}_{[0],+}^2$, $\mathbf{G}^{1+1} \equiv \mathbf{S}_{[0],0}^2$ and $\mathbf{M}^{1+1} \equiv \mathbf{S}_{[0],-}^2$, it is found that

$$\frac{dx}{dt} = b_1(t) + b_2(t)x + b_3(t)(x^2 - \kappa_2 y^2),$$

$$\frac{dy}{dt} = b_2(t)y + 2b_3(t)xy, \tag{6.1}$$

whose superposition rule has been deduced in Section 5.2. Hence, in the Galilei case with $\kappa_2 = 0$ we recover the system (5.9), formed by the usual Riccati equation in x coupled with a second equation in y .

It is worth mentioning that the system (6.1) can be solved without using the superposition rule. Indeed, solving the second equation for x leads to

$$x(t) = \frac{\frac{dy}{dt} - b_2(t)y}{2b_3(t)y}$$

and a second-order ODE in $y(t)$, which, after the change of variables $y(t) = \xi(t)^{-2}$, adopts the form

$$\frac{d^2\xi}{dt^2} = \frac{d}{dt}\log(|b_3(t)|)\frac{d\xi}{dt} - A(t)\xi + \frac{b_3(t)^2\kappa_2}{\xi^3}, \quad (6.2)$$

where

$$A(t) = b_1(t)b_3(t) - \frac{1}{4}b_2(t)^2 + \frac{1}{2}\frac{db_2(t)}{dt} - \frac{b_2(t)\frac{db_3(t)}{dt}}{2b_3(t)}.$$

The ODE (6.2) admits a 3D Lie point symmetry algebra, incidentally isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, implying that it can be linearized via a nonpoint transformation [51, 52]. However, as this transformation also involves the independent variable, it alters the physical significance of the solution.

6.2 Curved Kummer–Schwarz equation for $c > 0$

The Kummer–Schwarz equation (4.15) [32–34], with variables (u, v) , t -dependent vector field (4.16) and vector fields (4.17), belongs to the Euclidean P_2 -LH class whenever the constant $c > 0$ [7–9], that we shall express as

$$c = \frac{1}{\lambda^2}, \quad \lambda \in \mathbb{R} - \{0\}.$$

In this case, the change of coordinates on \mathbf{E}^2 defined by

$$x := \frac{v}{2u}, \quad y := \frac{u}{\lambda}, \quad u = \lambda y, \quad v = 2\lambda xy, \quad (6.3)$$

maps the vector fields (4.17) into (5.1) spanning the Euclidean P_2 -LH class with domains $\mathbb{R}_{u \neq 0}^2$ and $\mathbb{R}_{y \neq 0}^2$, respectively. This shows (see Table 1) that the vector fields (4.17) are Hamiltonian vector fields with respect to the symplectic form

$$\omega = -\frac{\lambda}{2u^3} du \wedge dv,$$

with associated Hamiltonian functions

$$h_1 = -\frac{\lambda}{u}, \quad h_2 = -\frac{\lambda v}{2u^2}, \quad h_3 = -\frac{u}{\lambda} - \frac{\lambda v^2}{4u^3}.$$

In order to generalize the Kummer–Schwarz equation (4.15) for $c > 0$ to the nine CK spaces $\mathbf{S}_{[\kappa_1], \kappa_2}^2$ using the results obtained in Section 5, we apply the transformation (6.3) to the curved

P₂-class. This procedure yields the t -dependent vector field $\mathbf{X}_\kappa = \mathbf{X}_{\kappa,3} + \eta(t)\mathbf{X}_{\kappa,1}$ as the curved counterpart of the Euclidean system (4.16), where the vector fields

$$\begin{aligned}\mathbf{X}_{\kappa,1} &= 2u \frac{\partial}{\partial v}, \\ \mathbf{X}_{\kappa,2} &= \lambda C_{\kappa_1} \left(\frac{v}{2u} \right) S_{\kappa_1 \kappa_2} \left(\frac{u}{\lambda} \right) \frac{\partial}{\partial u} + \left\{ 2u \frac{S_{\kappa_1} \left(\frac{v}{2u} \right)}{C_{\kappa_1 \kappa_2} \left(\frac{u}{\lambda} \right)} + \frac{\lambda v}{u} C_{\kappa_1} \left(\frac{v}{2u} \right) S_{\kappa_1 \kappa_2} \left(\frac{u}{\lambda} \right) \right\} \frac{\partial}{\partial v}, \\ \mathbf{X}_{\kappa,3} &= 2\lambda S_{\kappa_1} \left(\frac{v}{2u} \right) S_{\kappa_1 \kappa_2} \left(\frac{u}{\lambda} \right) \frac{\partial}{\partial u} \\ &\quad + \left\{ 4u \left(\frac{C_{\kappa_1 \kappa_2} \left(\frac{u}{\lambda} \right) - C_{\kappa_1} \left(\frac{v}{2u} \right)}{\kappa_1 C_{\kappa_1 \kappa_2} \left(\frac{u}{\lambda} \right)} \right) + \frac{2\lambda v}{u} S_{\kappa_1} \left(\frac{v}{2u} \right) S_{\kappa_1 \kappa_2} \left(\frac{u}{\lambda} \right) \right\} \frac{\partial}{\partial v},\end{aligned}\tag{6.4}$$

obey the commutation relations (5.3). Hence, we obtain the following first-order system of differential equations on $\mathbf{S}_{[\kappa_1], \kappa_2}^2$, so generalizing (4.15):

$$\begin{aligned}\frac{du}{dt} &= 2\lambda S_{\kappa_1} \left(\frac{v}{2u} \right) S_{\kappa_1 \kappa_2} \left(\frac{u}{\lambda} \right), \\ \frac{dv}{dt} &= 4u \left(\frac{C_{\kappa_1 \kappa_2} \left(\frac{u}{\lambda} \right) - C_{\kappa_1} \left(\frac{v}{2u} \right)}{\kappa_1 C_{\kappa_1 \kappa_2} \left(\frac{u}{\lambda} \right)} \right) + \frac{2\lambda v}{u} S_{\kappa_1} \left(\frac{v}{2u} \right) S_{\kappa_1 \kappa_2} \left(\frac{u}{\lambda} \right) + 2\eta(t)u.\end{aligned}\tag{6.5}$$

In addition, the vector fields (6.4) are Hamiltonian vector fields relative to the symplectic form

$$\omega_\kappa = -\frac{C_{\kappa_1 \kappa_2} \left(\frac{u}{\lambda} \right)}{2\lambda u S_{\kappa_1 \kappa_2}^2 \left(\frac{u}{\lambda} \right)} du \wedge dv,\tag{6.6}$$

with associated Hamiltonian functions given by

$$h_{\kappa,1} = -\frac{1}{S_{\kappa_1 \kappa_2} \left(\frac{u}{\lambda} \right)}, \quad h_{\kappa,2} = -\frac{S_{\kappa_1} \left(\frac{v}{2u} \right)}{T_{\kappa_1 \kappa_2} \left(\frac{u}{\lambda} \right)}, \quad h_{\kappa,3} = 2 \frac{C_{\kappa_1} \left(\frac{v}{2u} \right) C_{\kappa_1 \kappa_2} \left(\frac{u}{\lambda} \right) - 1}{\kappa_1 S_{\kappa_1 \kappa_2} \left(\frac{u}{\lambda} \right)}.\tag{6.7}$$

The corresponding Poisson brackets $\{\cdot, \cdot\}_{\omega_\kappa}$, induced by the symplectic form (6.6), are identical to those in (5.8). Observe that, as expected, the system (6.5) can also be obtained from the Hamilton equations of the time-dependent Hamiltonian $h_\kappa = h_{\kappa,3} + \eta(t)h_{\kappa,1}$.

The vector fields spanning the corresponding VG Lie algebra, symplectic form and Hamiltonian functions of the Kummer–Schwarz equation (4.15) for $c > 0$ on \mathbf{E}^2 are recovered from (6.4), (6.6) and (6.7) after the contraction $\kappa_1 \rightarrow 0$ with $\kappa_2 = +1$. Thus, we call the system (6.5) the *curved Kummer–Schwarz equation* for $c > 0$.

6.3 Curved Ermakov equation for $c > 0$

As last application, let us consider the Ermakov equation (1.1) with positive constant $c > 0$ which corresponds to the Euclidean P₂-LH class [7, 8]. If we set

$$c = \lambda^4, \quad \lambda \in \mathbb{R} - \{0\},$$

it can be shown that the change of coordinates on \mathbf{E}^2 given by

$$x := -\frac{v}{u}, \quad y := \pm \frac{\lambda^2}{u^2}, \quad u = \frac{\lambda}{\sqrt{|y|}}, \quad v = -\frac{\lambda x}{\sqrt{|y|}},\tag{6.8}$$

maps the vector fields (1.3) into those determining the Euclidean P₂-LH class (5.1) with domains $\mathbb{R}_{u \neq 0}^2$ and $\mathbb{R}_{y \neq 0}^2$, respectively.

Applying again the transformation (6.8) to the curved P₂-LH class constructed in Section 5, we are led to the following curved analogue

$$\mathbf{X}_\kappa = \mathbf{X}_{\kappa,3} + \Omega^2(t)\mathbf{X}_{\kappa,1}, \quad (6.9)$$

on the nine spaces $\mathbf{S}_{[\kappa_1], \kappa_2}^2$ of the t -dependent Ermakov vector field \mathbf{X} (1.2), where the ‘curved’ vector fields now read

$$\begin{aligned} \mathbf{X}_{\kappa,1} &= -u \frac{\partial}{\partial v}, \\ \mathbf{X}_{\kappa,2} &= -\frac{u^3}{2\lambda^2} C_{\kappa_1} \left(\frac{v}{u} \right) S_{\kappa_1 \kappa_2} \left(\frac{\lambda^2}{u^2} \right) \frac{\partial}{\partial u} + \left\{ \frac{u S_{\kappa_1} \left(\frac{v}{u} \right)}{C_{\kappa_1 \kappa_2} \left(\frac{\lambda^2}{u^2} \right)} - \frac{u^2 v}{2\lambda^2} C_{\kappa_1} \left(\frac{v}{u} \right) S_{\kappa_1 \kappa_2} \left(\frac{\lambda^2}{u^2} \right) \right\} \frac{\partial}{\partial v}, \\ \mathbf{X}_{\kappa,3} &= \frac{u^3}{\lambda^2} S_{\kappa_1} \left(\frac{v}{u} \right) S_{\kappa_1 \kappa_2} \left(\frac{\lambda^2}{u^2} \right) \frac{\partial}{\partial u} \\ &\quad + \left\{ 2u \left(\frac{C_{\kappa_1} \left(\frac{v}{u} \right) - C_{\kappa_1 \kappa_2} \left(\frac{\lambda^2}{u^2} \right)}{\kappa_1 C_{\kappa_1 \kappa_2} \left(\frac{\lambda^2}{u^2} \right)} \right) + \frac{u^2 v}{\lambda^2} S_{\kappa_1} \left(\frac{v}{u} \right) S_{\kappa_1 \kappa_2} \left(\frac{\lambda^2}{u^2} \right) \right\} \frac{\partial}{\partial v}. \end{aligned} \quad (6.10)$$

These satisfy the commutation relations (5.3). Therefore, we obtain the following first-order system of differential equations on $\mathbf{S}_{[\kappa_1], \kappa_2}^2$ associated with \mathbf{X}_κ (6.9):

$$\begin{aligned} \frac{du}{dt} &= \frac{u^3}{\lambda^2} S_{\kappa_1} \left(\frac{v}{u} \right) S_{\kappa_1 \kappa_2} \left(\frac{\lambda^2}{u^2} \right), \\ \frac{dv}{dt} &= 2u \left(\frac{C_{\kappa_1} \left(\frac{v}{u} \right) - C_{\kappa_1 \kappa_2} \left(\frac{\lambda^2}{u^2} \right)}{\kappa_1 C_{\kappa_1 \kappa_2} \left(\frac{\lambda^2}{u^2} \right)} \right) + \frac{u^2 v}{\lambda^2} S_{\kappa_1} \left(\frac{v}{u} \right) S_{\kappa_1 \kappa_2} \left(\frac{\lambda^2}{u^2} \right) - \Omega^2(t)u. \end{aligned} \quad (6.11)$$

Furthermore, the vector fields (6.10) are Hamiltonian vector fields relative to the symplectic form

$$\omega_\kappa = \frac{\lambda^4 C_{\kappa_1 \kappa_2} \left(\frac{\lambda^2}{u^2} \right)}{u^4 S_{\kappa_1 \kappa_2}^2 \left(\frac{\lambda^2}{u^2} \right)} du \wedge dv,$$

such that their associated Hamiltonian functions turn out to be

$$h_{\kappa,1} = \frac{\lambda^2}{2 S_{\kappa_1 \kappa_2} \left(\frac{\lambda^2}{u^2} \right)}, \quad h_{\kappa,2} = -\frac{\lambda^2 S_{\kappa_1} \left(\frac{v}{u} \right)}{2 T_{\kappa_1 \kappa_2} \left(\frac{\lambda^2}{u^2} \right)}, \quad h_{\kappa,3} = \lambda^2 \frac{1 - C_{\kappa_1} \left(\frac{v}{u} \right) C_{\kappa_1 \kappa_2} \left(\frac{\lambda^2}{u^2} \right)}{\kappa_1 S_{\kappa_1 \kappa_2} \left(\frac{\lambda^2}{u^2} \right)},$$

which close on the Poisson brackets (5.8). The differential equations (6.11) can alternatively be obtained as the Hamilton equations of the time-dependent Hamiltonian $h_\kappa = h_{\kappa,3} + \Omega^2(t)h_{\kappa,1}$.

Finally, since the Ermakov equation for $c > 0$ on \mathbf{E}^2 (1.1) is recovered from the above expressions on $\mathbf{S}_{[\kappa_1], \kappa_2}^2$ for the particular case when $\kappa_1 \rightarrow 0$ and $\kappa_2 = +1$, we call the LH system (6.11) the *curved Ermakov equation* for $c > 0$.

7 Conclusions and final remarks

In this work, the isometry-based formalism proposed in [37] has been completed and expanded from a conformal-based point of view. As observed in Section 1, the curved LH systems obtained in [37] do not include any physically relevant applications, as no remarkable system apart from the complex Bernoulli equation belongs to the Euclidean I₂-LH class. Nevertheless, the conformal symmetries of the 1D and 2D CK spaces lead to a natural generalization of the Euclidean classes I₄ and P₂ to curved spaces: the so-called curved I₄ and P₂-LH classes. The former is defined on $\mathbf{S}_{[\kappa]}^1 \times \mathbf{S}_{[\kappa]}^1$, the product of two 1D CK spaces, while the latter is defined on the proper 2D CK spaces $\mathbf{S}_{[\kappa_1], \kappa_2}^2$.

The curved I₄-LH class, obtained in Section 3, yields novel LH systems on two different spaces (see Table 2), being the Euclidean I₄-LH class recovered under the contraction $\kappa \rightarrow 0$. From this, the curved counterparts of systems belonging to the Euclidean I₄-LH class arise in a natural way (see Section 4): curved coupled Riccati equations, a curved split-complex Riccati equation, a curved diffusion Riccati system, a curved Kummer–Schwarz equation and a curved Ermakov equation, being the well-known Euclidean systems retrieved under the contraction $\kappa \rightarrow 0$. In particular, the superposition rule for this curved class produces, after contraction $\kappa \rightarrow 0$, a new superposition rule for the Euclidean coupled Riccati equations (3.6), from which the well-known superposition rule for the Riccati equation is derived after projection. This example suggests the possibility that, although a Lie system on a manifold M is not necessarily a LH one, it may happen that the Lie system on $M \times M$ generated by the sum of the realizations spanning the VG Lie algebra of the system on M is a LH system. In this case, a superposition rule for the system on M can be obtained by projecting a superposition rule for the LH system on $M \times M$, the latter being more feasible to be obtained by means of the coalgebra formalism. Also, as pointed out in the expressions (3.10) and (4.28), we stress that the parameter κ can be considered as an integrable deformation of the initial Euclidean LH system. Following a similar ansatz, the curved P₂-LH class is presented in Section 5, from which new LH systems on the 2D CK spaces emerge (see Table 3), particularly recovering the Euclidean P₂-LH systems under the contraction $\kappa_1 \rightarrow 0$, with $\kappa_2 = +1$. Within this curved LH class, the curved analogues of a complex Riccati equation, of a Kummer–Schwarz equation and of a Ermakov equation are obtained in Section 6. As for the curved P₂-LH class is concerned, the graded contraction parameters κ_1 and κ_2 (corresponding now to the curvature and the signature of the metric (2.14)), can be considered as integrable deformation parameters (see (5.10)). Although the approximate systems obtained by truncation of the series in κ_i are generally not related to a VG algebra, and therefore do not inherit a definite geometrical structure, their solutions may be inferred using either the Lie symmetry method [53] or techniques of perturbation theory [54], taking into account that their solution must provide, in the limit $\kappa_i \rightarrow 0$, the solution of the non-perturbed system.

In this context, some natural open problems emerge, which are expected to be (at least partially) solved by applying the same geometrical formalism. For instance, the higher-dimensional extension of the present work can be carried for the so-called 2ND CK spaces [55], for which the conformal symmetries can be derived following the same geometrical ansatz proposed in [35]. At this point, we must recall that, unlike the 2D case, not all the 2ND CK spaces are symplectic manifolds (the only sphere with a symplectic structure is the 2D one). Thus, it would be interesting to study those 2ND CK spaces admitting a symplectic form turning its conformal symmetries into Hamiltonian vector fields, from which new higher-dimensional (curved) LH classes would emerge. It is also worthy to be mentioned that the so-called rank-two CK spaces seem to possess a natural symplectic structure [55]. Nevertheless, its precise structure has not been found yet. Finally, it

would also be interesting to determine explicit solutions for those curved LH systems based on the book Lie algebra \mathfrak{b}_2 , which constitutes a distinguished subalgebra of $\mathfrak{sl}(2, \mathbb{R})$, as explicit solutions for the \mathfrak{b}_2 -LH systems on the Euclidean plane have been found recently [56].

Work along these various lines is currently in progress.

Acknowledgements

This work has been supported by Agencia Estatal de Investigación (Spain) under the grant PID2023-148373NB-I00 funded by MCIN/AEI/10.13039/501100011033/FEDER, UE. F.J.H. acknowledges support by the Q-CAYLE Project funded by the Regional Government of Castilla y León (Junta de Castilla y León, Spain) and by the Spanish Ministry of Science and Innovation (MCIN) through the European Union funds NextGenerationEU (PRTR C17.I1). O.C. acknowledges a fellowship (grant C15/23) supported by Universidad Complutense de Madrid and Banco Santander. The authors also acknowledge the contribution of RED2022-134301-T funded by MCIN/AEI/10.13039/501100011033 (Spain).

Appendix. Fundamental relations of κ -trigonometric functions

Consider the κ -dependent trigonometric functions: κ -cosine $C_\kappa(u)$, κ -sine $S_\kappa(u)$, κ -tangent $T_\kappa(u)$ and κ -versed sine $V_\kappa(u)$ defined in (2.7) and (2.8). Assume that u and v are two arbitrary real arguments. Observe that in Sections 5 and 6, the label κ corresponds to κ_1 or to the product $\kappa_1\kappa_2$.

Their derivatives are given by

$$\begin{aligned} \frac{d}{du} C_\kappa(u) &= -\kappa S_\kappa(u), & \frac{d}{du} S_\kappa(u) &= C_\kappa(u), \\ \frac{d}{du} T_\kappa(u) &= \frac{1}{C_\kappa^2(u)}, & \frac{d}{du} V_\kappa(u) &= S_\kappa(u). \end{aligned}$$

The main identities for such κ -functions read (see [41] for details):

$$\begin{aligned} C_\kappa^2(u) + \kappa S_\kappa^2(u) &= 1. \\ C_\kappa(2u) &= C_\kappa^2(u) - \kappa S_\kappa^2(u), \\ S_\kappa(2u) &= 2 S_\kappa(u) C_\kappa(u), \\ V_\kappa(2u) &= 2 S_\kappa^2(u) = 4V_\kappa(u) - 2\kappa V_\kappa^2(u), \\ T_\kappa(2u) &= \frac{2 T_\kappa(u)}{1 - \kappa T_\kappa^2(u)}. \\ C_\kappa^2\left(\frac{u}{2}\right) &= \frac{C_\kappa(u) + 1}{2} = 1 - \kappa \frac{V_\kappa(u)}{2}, \\ S_\kappa^2\left(\frac{u}{2}\right) &= \frac{1 - C_\kappa(u)}{2\kappa} = \frac{1}{2} V_\kappa(u), \\ T_\kappa\left(\frac{u}{2}\right) &= \frac{1 - C_\kappa(u)}{\kappa S_\kappa(u)} = \frac{S_\kappa(u)}{C_\kappa(u) + 1} = \frac{V_\kappa(u)}{S_\kappa(u)}. \end{aligned}$$

$$\begin{aligned}
C_\kappa(u) &= \frac{1 - \kappa T_\kappa^2(\frac{u}{2})}{1 + \kappa T_\kappa^2(\frac{u}{2})}, & S_\kappa(u) &= \frac{2 T_\kappa(\frac{u}{2})}{1 + \kappa T_\kappa^2(\frac{u}{2})}, \\
V_\kappa(u) &= \frac{2 T_\kappa^2(\frac{u}{2})}{1 + \kappa T_\kappa^2(\frac{u}{2})}, & T_\kappa(u) &= \frac{2 T_\kappa(\frac{u}{2})}{1 - \kappa T_\kappa^2(\frac{u}{2})}.
\end{aligned}$$

$$\begin{aligned}
C_\kappa(u \pm v) &= C_\kappa(u) C_\kappa(v) \mp \kappa S_\kappa(v) S_\kappa(u), \\
S_\kappa(u \pm v) &= S_\kappa(u) C_\kappa(v) \pm S_\kappa(v) C_\kappa(u), \\
V_\kappa(u \pm v) &= V_\kappa(u) + V_\kappa(v) - \kappa V_\kappa(u) V_\kappa(v) \pm S_\kappa(u) S_\kappa(v), \\
T_\kappa(u \pm v) &= \frac{T_\kappa(u) \pm T_\kappa(v)}{1 \mp \kappa T_\kappa(u) T_\kappa(v)}.
\end{aligned}$$

$$\begin{aligned}
C_\kappa(u + v) + C_\kappa(u - v) &= 2 C_\kappa(u) C_\kappa(v), \\
C_\kappa(u + v) - C_\kappa(u - v) &= -2\kappa S_\kappa(u) S_\kappa(v), \\
S_\kappa(u + v) + S_\kappa(u - v) &= 2 S_\kappa(u) C_\kappa(v), \\
S_\kappa(u + v) - S_\kappa(u - v) &= 2 C_\kappa(u) S_\kappa(v), \\
V_\kappa(u + v) + V_\kappa(u - v) &= 2\{V_\kappa(u) + V_\kappa(v) - \kappa V_\kappa(u) V_\kappa(v)\}, \\
V_\kappa(u + v) - V_\kappa(u - v) &= 2 S_\kappa(u) S_\kappa(v).
\end{aligned}$$

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