

INTERACTION SPACES: TOWARDS A UNIVERSAL MATHEMATICAL THEORY OF COMPLEX SYSTEMS

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ABSTRACT. We present the first steps of *interaction spaces theory*, a universal mathematical theory of complex systems which is able to embed cellular automata, agent based models, master equation based models, stochastic or deterministic, continuous or discrete dynamical systems, networked dynamical models, artificial neural networks and genetic algorithms in a single notion. Therefore, interaction spaces represent a common mathematical language that can be used to describe several complex systems modeling frameworks. This is the first step to start a mathematical theory of complex systems. Every notion is introduced both using an intuitive description by listing lots of examples, and using a modern mathematical language.

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1. INTRODUCTION: WHY DO WE NEED A MATHEMATICAL THEORY OF COMPLEX SYSTEMS?

Throughout the history of science, several disciplines have considerably gained from a sound mathematical foundation: quantum mechanics, continuum mechanics, thermodynamics, medicine, biology, information science, economics, social sciences, and urban studies, to name but a few. Indeed, the contribution of mathematics to many disciplines can be considered a general process that occurs when the solution of problems requires the strongest notion of rational truth corroborated by a meaningful validation.

At present, different modeling methods are adopted to study complex systems (CS): among the most used, we can cite, e.g., cellular automata (CA), see e.g. [41], agent based models (ABM), e.g. [58], master equation based models, [49, 26], networked dynamical systems, [45], artificial neural networks, [28, 40, 42], and genetic algorithms, [5, 43]. However, there is no universal mathematical theory of CS, i.e. a theory sufficiently powerful to range over all these systems, from ABM to systems described by some type of differential equations, and, at the same time, to produce meaningful general mathematical results applicable to large classes of systems. The problem is well-known and discussed in literature: see e.g. [29, 46, 9, 11, 12, 6, 19, 8, 50, 30, 17], where you can find both opinions in favour or against the possibility of such a theory.

In this article, we introduce a new mathematical structure, called *interaction space* (IS), having the property to include (i.e. to faithfully embed preserving their original mathematical structure) in a single notion all the previously listed modeling frameworks. In our opinion, such a founding mathematical theory could provide great impact from the perspectives of a common language, precise definitions and general results which would hence be applicable to all these settings (see [25, 17, 34] for very similar viewpoints).

Other aims we have in mind are the following:

- A common mathematical language can be useful to precisely formulate problems like bifurcations, phase transitions and critical phenomena, pattern formation theory, ergodic theory, study of ABM as dynamical systems, etc. (see e.g. [12, 52, 20, 46] for similar problems).
- With our results on Markovian IS and power law for complex adaptive IS, we show the possibility to prove general results applicable to large classes of CS, see [21, 22].
- The description of the dynamics of non-Markovian IS with a system of mean derivative equations, represents a new important general mathematical result. This demonstrates that using a modern mathematical setting, powerful mathematical tools can be used to solve open problems, see [22].

- IS theory represents a proposal for a sound mathematical definition of ABM. This definition would open the possibility to start a mathematical study of a large family of these models (see e.g. [34] for a mathematical approach to the dynamics of some types of ABM).
- In an IS we also have a language of cause-effect relations, where elementary modeling-dependent cause-effect relations between interacting entities can be composed into more complex cause-effect graphs. Using a suitable language of multicategory theory, these cause-effect relations can be used both to model hierarchies of complex systems and new general methods of artificial intelligence, see e.g. [47, 23]. See also [3, 18, 7] for a similar point of view.

In other words, a mathematical theory of CS aims to link phenomenological studies (e.g. estimates of power laws) to a modern mathematical theory, so that to make a step further obtaining more general, clear and widely applicable results.

1.1. Other mathematical theories of complex systems. As far as we know, only the following approaches claim to be mathematical theories of at least suitable classes of complex systems:

- (i) *Kinetic theory for active particles*, see [33, 2, 10, 4] and references therein. This approach is used to describe the dynamics of a large number of interacting entities in living systems which are distributed over a network. Usually, entities are homogeneously distributed within each node and the model provides a mesoscopic description, i.e. through the probability distribution over the microscopic states. The mathematical methods are near to those of statistical mechanics and game theory.
- (ii) *Memory evolutive systems*, see e.g. [13, 14, 32] and references therein. This theory is mainly proposed as a possible foundation of biology. Deeply based on category theory, it makes extensive use of limits and colimits of diagrams to model evolving hierarchical category of living systems. Because of its abstract approach, the scope of memory evolutive systems is probably very general. In spite of this abstractness, it captures essential aspects of biological organization and hence it could lead to concrete hypotheses which are capable of being tested.
- (iii) *Universal dynamics*, see [39, 35, 36, 37, 38, 53]. This approach is also based on category theory, and claims to be a universal theory for every complex system. The basic structure is elementary and given by a category with a selected family of arrows, called fundamental. On the other hand, only local Markovian dynamics in discrete time is considered because the dynamics depends only on a finite number of past times. A notion of locality and of neighborhood is defined using composition of fundamental arrows. In particular, we underscore its applications in information science in [53, 37].
- (iv) *Networks and networked dynamical systems*, see e.g. [45, 44]. Even if this theory does not usually claim to be a universal mathematical theory of CS, frequently it is one of the most effectively used point of view on CS. For example, there is no general definition of CS nor of complex adaptive system within this theory, see e.g. [31, 16] for arguments supporting the idea that network theory is insufficient to model several interesting CS. Moreover, the only network structure of a given model of a CS does not uniquely determine this model, i.e. classical models of CS cannot be identified with their network.

As we will see in Sec. 4.6, we could say that IS theory can be considered as a more general and abstract version of this theory, even if it is actually more near to hypergraphs or multicategories/operads, see e.g. [16, 31, 3, 18]. Indeed, in Sec. 4.6 we also prove that every networked dynamical system can be faithfully embedded as IS.

All these theories, even when they claim to be universal, do not show clear relations with the most used modeling approaches for CS. For this reason, one cannot state that their theorems can be applied to a large family of these models. On the contrary, they present some limitations, like the mesoscopic, or discrete or Markovian dynamics. Finally, in our opinion, the abstract approach used both in (ii) and (iii) sometimes represents an impediment in their spreading in the scientific community of CS modeling and in their practical implementation as a computational tool.

In the present work, we see that IS theory includes all classical models of CS and have a clear cause-effect structure. This allows us in [21] to introduce a meaningful *mathematical* notion of complex adaptive system by formalizing informal ideas frequently used in modeling of CS.

It is important to note that the universality of IS theory allows one to be sure that sufficiently general mathematical results have a satisfactorily range of applications for a diverse range of different modeling frameworks of CS. For theorems already going in this direction, see [22, 21]. Note that this does not force anyone to switch to IS from his favorite CS setting, but it only establishes a general common mathematical language for CS.

2. INTUITIVE DESCRIPTION OF INTERACTION SPACES AND THEIR DYNAMICS

We first describe a generic IS by using only an intuitive approach and giving several examples, exactly like agent based models (ABM) are frequently presented. Secondly, we present a mathematical approach, clearly explaining why this mathematics corresponds to the related intuitive description.

IS theory aims at modeling complex systems enclosed in the following general frame:

2.1. Interacting entities and their state. The system is made by *interacting entities* $e \in E$ described by dynamical *state variables* $x_e(t)$ for $t_{st} \leq t \leq t_{end} \leq +\infty$. Intuitively, an interacting entity is everything able to send or receive propagator signals (of any type) to interact with other interacting entities. In general, we think state variables as vectors made of several components. In case of stochastic dynamics, we can think at the function of time $x_e(-)$ as a sample path followed by the state of the entity e for some random elementary event ω , which encloses all the stochastic events from which this dynamics depends on.

Examples of interacting entities are: agents of an ABM, a vehicle, a traffic light or the stretch of road between two following cars, advertisements in a street, goods exchanged in a market, a whole population of individuals sharing common features and interacting with other entities, words in a text, cells of a CA (even if in this case the propagator signals are not considered in the CA model), etc.

2.2. Interactions. These interacting entities are involved in interactions $i \in I$, each one of a given type α , that can be described as a causally directed elementary process in which a set of *agent* entities a_1, \dots, a_n modify the state of a *patient*

entity p through a *propagator* entity r . We distinguish between the type α of the interaction, which is usually a label useful to classify different interactions, and the interaction $i = (a_1, \dots, a_n, r, \alpha, p)$ that includes all these information. The propagator r can be thought of as a signal-entity activated by agents, and carrying the cause-effect relation sent by agents a_1, \dots, a_n to the patient p . We also think that a subspace of the state space of the propagator r works as a *resource space* R_i for the changing of the state of the patient p ; we will see later why this is important to define CAS.

The general form of an interaction i is hence:

$$i : a_1, \dots, a_n \text{ have an interaction } \alpha \text{ with } p \text{ through } r \quad (2.1)$$

which will be also indicated with the notation

$$i : a_1, \dots, a_n \xrightarrow{r, \alpha} p \quad (2.2)$$

or with a diagram as in Fig. 2.1. This is a sort of *primitive cause-effect relation* (i.e. it depends on the constructed model of the considered CS), and our interest lies more on the possible *cause-effect graphs* that can be built up by concatenating these elementary relations. In other words, agents a_1, \dots, a_n represent the sites of information storage, and the communication topology of information flows within a system is explicitly given by propagators in cause-effect interactions such as (2.2). See also Sec. 3.4.1 for polyadic interactions, as well as [16, 31, 3, 18] and references therein for similar viewpoints.

Examples: a physical interaction between one particle p_1 sending a signal s to another particle p_2 , $i = (p_1, s, \text{sendSignal}, p_2)$; or a firm (agent) sending an advertisement (propagator) and hence changing the state of several people (patients); a suitable set of goods in a market (agents) sending a signal (propagator) that carries information useful for buyers (patients); a biological entity (agents) sending a chemical signal (propagator) to another entity (patients) having receptors able to recognize that signal; in a given text, an adjective a_1 specifies a name a_2 hence changing its state as a patient $p = a_2$, and the propagator r can measure the amount of information specified by the adjective a_1 ; an object in an object oriented program sending a message to another object; a single neuron has multiple dendrites a_1, \dots, a_n (inputs from other neurons), and sends electrical and chemical signals r of type α to its unique axon p . In urban models, agents can be individuals acting in the urban space (e.g. as builders or residents), patients can be lots of terrain, propagator signals can be volumes and surfaces produced for different uses so that the state space of propagators is linked to the available surface and volume at disposal, depending on the master plan (which represents the space of resources; see [55, 56, 1]). Note that we can have more interactions acting on the same patient, such as in the case of a car and a pedestrian simultaneously approaching another pedestrian. We also want to have a sufficient freedom in setting an IS as a model of a complex system, so that, if needed, we can consider interacting entities as mathematical idealized entities: for example, think at a collision between two balls of steel b_1 and b_2 , and the possibility to set as propagator the subbody of the Cartesian product $b_1 \times b_2$ actually involved in the collision. We can also be interested in considering as ideally infinite the speed of this propagator in case of elastic collision, so that the aforementioned subbody is ideally given by the single point of contact.

2.3. Activation. An interaction $i : a_1, \dots, a_n \xrightarrow{r, \alpha} p$ is occurring only if at least one of the agents a_1, \dots, a_n and its propagator r is *active for that interaction*. Indeed, in the state $x_e(t)$ of each interacting entity e there is always a time dependent state variable $x_e(t)_{1,i} =: ac_i^e(t) \in [0, 1]$ (for simplicity, think at the very common case where $ac_i^e(t) \in \{0, 1\}$ is a Boolean variable) indicating if, with respect to the given interaction i , the entity e is active or not. Intuitively, if the interaction i starts at time t_i^s , then at least one agent a_j must be active with respect to i , i.e. it can be involved in the interaction i , and we have $ac_i^{a_j}(t_i^s) \neq 0$. At the same starting time t_i^s , agents activate the propagator r : $ac_i^r(t_i^s) \neq 0$. The propagator r will take a certain time $t_i^a - t_i^s$ to arrive at the patient p . If no other entity and interaction stops r (in that case $t_i^a = +\infty$), r is still active at the arrival time, $ac_i^r(t_i^a) \neq 0$, and activates the patient: $ac_i^p(t_i^a) \neq 0$. See also Sec. 3.3 and Sec. 3.4 for a more accurate formulation of these conditions.

Active agents can also be interpreted in biological terms as entities sending some kind of chemical signal to patients entities having suitable receptors to recognize it; in this description, propagators are entities carrying the signal. Therefore, agents which are not already active in the initial condition of the system, can pass to an active state as a consequence of an interaction (endogenous or exogenous). Therefore, both from an intuitive and modelling point of view, the syntactic structure $i : a_1, \dots, a_n \xrightarrow{r, \alpha} p$ of an interaction and these activations state variables represent the elementary dynamics of the cause-effect signals that propagate in the system. These signals compose themselves into complex cause-effect graphs, whose study is one of the main interests in modeling CS.

Note that this dynamics of occurrence times and activation functions represents a stronger formalism with respect to the usual cause-effect mathematical formalization, as used e.g. for time series. Indeed, it is well-known that a simple conditioning can fail to localize information, so that Shannon entropy and similar measures are not able to measure information flow, see e.g. [31] and reference therein.

Examples: in the above mentioned example about firm's advertisement, only buyers activated, in some way, for the advertised products will have a state modification; it can also happen that an interaction of higher priority deactivate a buyer with respect to the advertised product; only the biological entities having suitable receptors are active for the corresponding interactions; a computer client is waiting for a signal from a server before restarting a download, so that it can be activated at a stochastic future time or deactivated by another program; a date in a text can activate another specific word, such as one describing a illness; only software objects with a suitable public state variable can receive a message to change that variable; only hungry predators are active for hunting preys, and we can measure in a fuzzy way $0 \leq ac_i^e(t) \leq 1$ their degree of hungriness; a similar fuzzy activation can also be useful in suitable models of Alzheimer disease.

Note that the property of the cause-effect relation (2.2) of being primitive can also be understood in another way: even if agents a_1, \dots, a_n are also intuitively interacting to produce and activate the propagator r , in general we are not interested to model this kind of more elementary interactions between agents (think e.g. at the scattering of two particles or the elastic collision between two balls where we do not model the dynamics during the collision); in this case, this means that in our model there are no interactions of the type $j : a_{k_1}, \dots, a_{k_m} \xrightarrow{s, \beta} a_h$ or

$l : a_{h_1}, \dots, a_{h_i} \xrightarrow{u, \gamma} r$ prior to i . This justifies why i starts at time t_i^s and, *at the same time*, the propagator is activated: $ac_i^r(t_i^s) \neq 0$. In other words, interaction between agents a_1, \dots, a_n happens in a negligible time span with respect to all the other timings happening in the system. On the contrary, if we are interested to model the time used to activate r , we have to also consider interactions of the type $l : a_{h_1}, \dots, a_{h_i} \xrightarrow{u, \gamma} r$ aiming at activating r .

2.4. Occurrence times. Once the propagator r arrives at a time t_i^a , we say that the interaction i is *ongoing*, in the sense that the state of the patient p can start to change. If the interaction i is ongoing at the time t_i^o , this can be instantaneous, i.e. a single time instant $t_i^o = t_i^a$, or continuous, i.e. belonging to an interval $t_i^o \in [t_i^a, t_i^a + \delta_i]$. Therefore, whereas the arrival times t_i^a of an interaction i are always single time instants, both the ongoing times t_i^o and the starting times t_i^s can be discrete (e.g. a discrete dynamical system, such as a CA) or continuous (e.g. when agents continuously send the propagator r to the patient p or its state $x_p(t)$ continuously changes in the interval $[t_i^a, t_i^a + \delta_i]$, e.g. in a continuous dynamical system). Depending on the considered system and on its model, all these times t_i^s , t_i^a and t_i^o can be deterministic or stochastic.

More precisely, we can hence think at them as sample paths $t_i^s = t_i^s(t)$, $t_i^a = t_i^a(t)$ and $t_i^o = t_i^o(t)$ for $t_{st} \leq t \leq t_{end}$, with a suitable model-dependent distribution. We always have $t_i^s(t) \geq t$ (see Sec. 3.3), and $t_i^s(t)$ can be thought of as the first starting time of i after or at the present time t . Of course the interaction i can occur multiple times in $[t_{st}, t_{end}]$, and if $t_i^s(t) = t$ we have that i is starting exactly at t , otherwise that it will start at the time instant $t_i^s(t) > t$. See the precise Def. 6 and Def. 9. Similarly, $t_i^o(t) \geq t$ can be thought of as the first ongoing instant of time of i after or at t . The arrival time $t_i^a(t)$ can actually be defined as the first of the ongoing times $t_i^o(t)$ (see Sec. 3.4). An inequality of the type $t_i^a(t) > t$ means that the propagator r will arrive at a future time $t_i^a(t)$; we interpret $t_i^a(t) = t$ as the arriving at the present time t , and $t_i^a(t) < t$ as the statement that the propagator arrived in the past at $t_i^a(t)$.

Therefore, if $t_i^s(t) = t$, then $t_i^s(t) \leq t_i^a(t)$, i.e. if the interaction i starts at the present time t , then it will arrive in a future time instant $t_i^a(t) \geq t_i^s(t) = t$ (the propagator cannot arrive in a past time instant $t_i^a(t) < t_i^s(t)$).

The distributions of these times t_i^s , t_i^a , t_i^o model the timing of the system, and we can always include the deterministic cases using suitable Dirac delta distributions, i.e. using a trivial probability space.

Clearly, it is because of the universality properties of IS theory that we aim at this generality.

Examples: an interaction where an agent chooses a shop on the basis of its information about quality, prices, and goods availability, occurs at random times with a suitable distribution (e.g. an exponential distribution whose rate reflects the characteristics of the shop) depending both on objective and subjective characteristics; an interaction describing a house leasing occurs at random times depending on several factors, e.g. the rate of birth, of marriage, of immigration, etc; the infection of an organism by a virus depends randomly on the hosts encountered; if this virus is considered as the propagator of the infection interaction, then it will arrive to the possible next organism after a random time depending on its aging; an excited electron (agent) produces a photon (propagator) that, in a time depending on the

media, changes the state of another electron (patient) in a scattering interaction; a word in a text can activate a corresponding mental notion in the reader; the starting of a program randomly depends on the interaction of the user with the program's interface.

2.5. Neighborhood of an interaction. The occurrence of an interaction i and its effects depend on the history of the state of a set of entities $\mathcal{N}_i(t)$ called the *neighborhood of the interaction*. The neighborhood of the interaction i is intuitively defined by all the active entities from which i takes the information it needs to operate, and it can depend on time. The neighborhood of an interaction always includes agent, patient and propagator entities whenever they are active for that interaction.

Examples: if an agent is searching for a new house, only the information collected in some order in its memory will affect its future decisions; only the state of the cells belonging to the neighborhood can influence the future state of a given cell in a CA; a given negation or an adjective in a text can influence only a few near verbs or names; only the (random) objects in the visual field of a pedestrian may influence its goal-oriented path; the information collected in a graphical user interface may influence the possible starting of a given computer program.

2.6. Goods and resources. When an interaction i starts at $t_i^s(t)$, a quantity $\gamma_i(t) := x_r(t)_{2,i}$ (called *good*) is (probabilistically) extracted from the resource subspace R_i of the propagator r . In general, the evolution of the state variables of the patient p depends on the extracted goods $\gamma_i(t)$. In the space R_i we can have a notion of *zero resources* $Z_i \subseteq R_i$, so that if $\gamma_i(t) \in Z_i$, then $\text{ac}_i^p(t) = 0$, i.e. the patient p is not active for i . This implies that the propagator does not arrive at t , i.e. $t_i^a \neq t$ because above we stated that $\text{ac}_i^p(t_i^a) \neq 0$. If, for a given set of interacting entities (a *population*) these resources cannot be zero, then other entities in the population will try to manage this lacking of resources. This is a first very rough explanation why the notions of goods and resources will be used to define CAS, see [21].

Examples: an excited electron (agent) produces a photon (propagator) that changes the state of another electron (patient) in a scattering interaction, and goods are related to the frequency of the photon. A specific adjective in a text sends more goods to a given name than a less specific one. The input currents (propagator) of a neuron are the signals (goods) that will be integrated to produce a suitable changing of the output synapses. A developer decides to build a new house and produces as signal the house's project, hold in the state of a suitable abstract propagator entity. Starting from this project, the state of the building's plot will change in a suitable amount of time, unless the municipal administration blocks the project (resources are emptied). In general, a situation where the resources are exhausted before the finishing of the interaction, is an example where the propagator is deactivated before the ending of the interaction, and hence also the patient will be deactivated.

2.7. Evolution equations. Every model of a CS has corresponding evolution equations satisfied by the state variables $x_p(t)$. These equations can be given by differential equations, possibly stochastic, or discrete ones; they can take into account memory effects (i.e. they are of non-Markov type) or not, and we need a common language for all of them.

Let us consider a patient entity $p \in E$: every model of a CS considers a transition function f_p responsible for the dynamics of the state $x_p(t)$. At the generic present time instant $t \in [t_{\text{st}}, t_{\text{end}}]$, we consider the first arrival time among all the interactions in our system that started at t (i.e. such that $t_i^s(t) = t$):

$$t^1(t) := t^1 := \inf \{t_i^a(t) \mid i \in I, t_i^s(t) = t\}$$

(we read it as “*t first*”). Note that, intuitively, no event occurs in the interval (t, t^1) . If $t^1 = +\infty$ this means that no more interactions occur after t , so we can assume $t^1 < +\infty$.

Since more than one interaction can simultaneously act on the patient p during the time interval $[t^1, t^1 + \Delta]$, we consider all of those interactions, i.e. all the interactions whose propagator arrives in this interval

$$I_p(t) := \{i \in I \mid t^1(t) \leq t_i^a(t) \leq t^1(t) + \Delta, \text{pa}(i) = p\}.$$

Here $\Delta \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$ is a model-dependent interval of time representing when the evolution equation defined by f_p is solely responsible for the time change of the state $x_p(t)$ (see also Rem. 1 below for examples and other intuitive interpretations of Δ).

Note that we have to consider the evolution equation only if $I_p(t)$ is not empty because otherwise this would mean that among all the interactions acting on the patient p , no one arrives in the interval $[t^1, t^1 + \Delta]$.

Now, we can take into account all the non-Markovian dependencies by considering the *state of the neighbourhood of p*:

$$\begin{aligned} \text{If } \exists t' \leq t \exists i \in I : \text{pa}(i) = p, t' = t_i^a(t'), \varepsilon \in \mathcal{N}_i(t'), \tau \in [t', t] \quad (2.3) \\ \text{then } \mathfrak{n}_p x(\tau, \varepsilon) := x_\varepsilon(\tau). \end{aligned}$$

Explanation: If in a possible past time $t' \leq t$ the propagator of an interaction $i \in I$ arrived at its patient p (i.e. $t' = t_i^a(t')$), we consider the state $x_\varepsilon(\tau)$ of every interacting entity ε in the neighborhood $\mathcal{N}_i(t')$ for all the following times $\tau \in [t', t]$ (see also Rem. 1 below for examples and other intuitive interpretations of this (possibly) non-Markovian behavior).

The general evolution equation for the patient p can now be stated as follows: There exists an elementary event $\omega \in \Omega_p$ (in a suitable probability space modeling the possible stochastic evolution of p governed by the evolution equation) such that if $t \in [t_{\text{st}}, t_{\text{end}}]$, $t^1(t) < +\infty$ and $I_p(t)$ is not empty, then for all s such that $t^1 \leq s \leq t^1 + \Delta \leq t_{\text{end}}$, we have

$$x_p(s) = f_p(\omega, s, \mathfrak{n}_p x_s), \quad (2.4)$$

where $\mathfrak{n}_p x_s$ denotes the neighborhood function considered only in the interval $[t^1, s]$, i.e.

$$\mathfrak{n}_p x_s : \tau \in [t^1(t), s] \mapsto \mathfrak{n}_p x(\tau, -).$$

Remark 1.

- (a) In several cases (e.g. a discrete dynamical system like a CA, where $\Delta = 1$), this Δ can be thought of as a small interval of time with respect to the speed at which the changing of the state $x_p(t)$ spread out in the system, and no other interactions occur in $(t^1, t^1 + \Delta)$.

- (b) If, in an idealized system, the change of the state $x_p(t)$ spread instantaneously in the whole system, then we have to set $\Delta = 0$ and all the interactions occur instantaneously at t^1 .
- (c) We will see below that in every continuous dynamical system where $x_p(t)$ is described by a differential equation, we can set $\Delta = t_{\text{end}} - t_{\text{st}}$, because the differential equation governs the evolution of p in the entire interval $[t_{\text{st}}, t_{\text{end}}]$. Therefore, in this case, Δ is not small. However, in Sec. 4, we will see that for this IS we have only one interacting entity p and only one interaction i corresponding to the differential equation describing the system.
- (d) Of course, the dependence on past states expressed by (2.3) is a very strong one; however, think for example at the case where ε represents a malignant tumor diagnosis for the patient p at the time $t' = t_i^a(t')$, and all the subsequent ($\tau \in [t', t]$) medical and psychological consequences on p of the state $x_\varepsilon(\tau)$ of the neoplasm ε .
- (e) Since the state $x_p(s)$ includes both the activation $\text{ac}_j^p(s) = x_p(s)_{1,j}$ (see Sec. 2.3) and the goods $\gamma_j(s) = x_p(s)_{2,j}$ (see Sec. 2.6), the evolution equations (2.4) have to also include their dynamics:

$$\begin{aligned} \text{ac}_j^p(s) &= f_p(\omega, s, \mathbf{n}_p x_s)_{1,j} \quad \forall j \in I, \\ \gamma_j(s) &= f_p(\omega, s, \mathbf{n}_p x_s)_{2,j} \quad \forall j \in I : p = \text{pr}(j). \end{aligned}$$

Therefore, these equations also control the cause-effect dynamics represented by activation states, and the dynamics of goods; the latter are important for CAS (see [21]).

- (f) We will see more precisely later how both continuous and discrete dynamical systems can be equivalently described using an equation of the form (2.4). Here, we only mention that an ordinary differential equation (ODE) of the form $x_p'(s) = F(s, x_p(s))$ for all $s \in [t_{\text{st}}, t_{\text{end}}]$ can equivalently be written as $x_p(s) = x_p(t_{\text{st}}) + \int_{t_{\text{st}}}^s F(\tau, x_p(\tau)) \, d\tau =: f_p(s, x_p(-)|_{[t_{\text{st}}, s]})$. On the other hand, if we have $x_p(k+1) = F(k, x_p(k))$ for all $k = 0, \dots, N$ and $x_p(0) = x_0$, then we can define $f_p(s, x_p|_{[0, s]})$ stepwise by

$$f_p(s, x_p|_{[0, s]}) := \begin{cases} F(k, x_p(k)) & \text{if } s = k + 1 \\ F(k - 1, x_p(k - 1)) & \text{if } s \in [k, k + 1) \text{ and } k > 0 \\ x_0 & \text{if } s \in [0, 1), \end{cases} \quad (2.5)$$

to reenter into the language of (2.4). Note that in both cases we consider a dependence only on a suitable restriction of x_p .

- (g) We could explicitly admit that the intervals $\Delta = \Delta_p(t)$ depend both on the patient p and the time t . For example, we can admit that the evolution of $x_p(s)$ is described by an ODE for some p or for certain times t , and by a discrete dynamical system for other p or different times t . However, this would result in more cumbersome notations, and it will never be used in the present paper.
- (h) Note that in the function $\mathbf{n}_p x_s$ we have the dependence from all the interactions $i \in I$ that acted on p in the past. This dependence is expressed through the states $x_\varepsilon(\tau)$ of entities in the neighborhood $\mathcal{N}_i(t')$. If the conditions (2.3) are never satisfied, the evolution function f_p simply does not depend on these states.
- (i) Only taking the closed interval $[t^1, t^1 + \Delta]$ we can consider the case $\Delta = 0$ and the evolution istantaneously occurring at $t = t^1$.

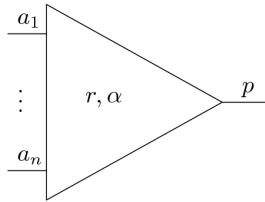


FIGURE 2.1. Representation of an interaction using a diagram.

- (j) It is traditional in physics and mathematics to see that the state variables satisfies some kind of equation attributed to some important scientist. A minimal thinking allows us to say that it is ingenuous to believe that this could happen for all possible complex systems. We are focusing more on a universal mathematical language. CAS and the GEP could play the role of this general law, see [21], but not in the simple form of an equation.

The following examples surely can be described in the previous intuitive formalism: a bouncing billiard ball; a pedestrian between two subsequent interactions with other pedestrians or obstacles; the process of building a house after its starting time and before its end; the internal evolution of a box in a flow chart representing a computer program; the patient p represents a company listed on the stock exchange, and ε represents another company selling the same type of product which experienced a strong decreasing of its shares at time t' . The interacting entity ε represents a neoplasm appearing in a person p at time $t' \leq t$ but still interacting with p at present time t . The last system is clearly non-Markovian, as one can see comparing two different samples paths where $x_\varepsilon(t') = \text{benign}$ or $\bar{x}_\varepsilon(t') = \text{malignant}$.

The terms agent, patient, propagator and members of a neighborhood are collectively named *roles* of entities in an interaction. Of course, interacting entities can play different roles in different interactions and more than one role in the same interaction, e.g. a propagator of i can also be at the same time an agent of the same interaction and a patient of another interaction j which triggers the goods of i . Therefore, if we represent an interaction by means of a graph, like in figure 2.2, and connect two graphs when they share an entity, we obtain a network representing the mentioned causal flows in the system. Note that this informal description already allows for a practical implementation of simulated IS (see e.g. [54]).

The intuitive description above can be summarized by saying: in an interaction, agents activate and the propagator and the goods are sent as a signal to modify the state of the patient; the modification depends on information collected from the neighborhood of that interaction; the starting time and the speed of the signal of the interaction can be stochastic. Occurrence of interactions is causally constrained by logical conditions expressed by the activation of the entities. All the interactions acting on patients cause the evolution of their state during sufficiently small intervals with respect to the spreading of these changes in the system.

Finally note that if we aim to describe “a general CS”, terms such as *interacting entities*, *interactions*, being *active* or not for an interaction, *neighborhood* as the set

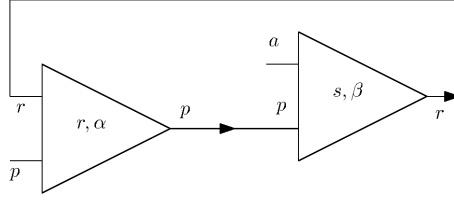


FIGURE 2.2. Graphical representation of two interactions $i : r, p \xrightarrow{r, \alpha} p$ and $j : a, p \xrightarrow{s, \beta} r$, where the interacting entity r is at the same time agent and propagator of the first interaction i and patient of the second interaction j . Using the agent a , we can change the status of r and hence the goods $\gamma_i(t) = x_r(t)_{2,i}$ of the first interaction i .

of all the entities where an interaction takes all the needed information, *occurrence times* and *evolution equations* seems very natural and necessary notions.

2.8. Dynamics of an interaction space. Similarly to an asynchronous CA, the dynamics of a generic IS, is determined by the occurrence times t_i^s , t_i^a and t_i^o (see Sec. 2.4) of all the interactions i , and by the evolution equations (2.4), starting from an initial state of the system:

- (a) The system starts with a given initial value of all the states $x_e(t_{\text{st}})$ for each interacting entity $e \in E$. Note that this includes the initial values of the activation states $\text{ac}_i^p(t_{\text{st}}) = x_p(t_{\text{st}})_{1,i}$ and of the goods $\gamma_i(t_{\text{st}}) = x_p(t_{\text{st}})_{2,i}$ (if $p = \text{pr}(i)$) for any interaction $i \in I$.
- (b) For each interaction $i \in I$, we have to provide the starting time $t_i^s(t_{\text{st}}) \geq t_{\text{st}}$. If the interaction i starts at $\bar{t} := t_i^s(t_{\text{st}})$, we also have to provide the arrival time $t_i^a(\bar{t}) \geq t_i^s(\bar{t})$ and the ongoing time $t_i^o(\bar{t}) \geq t_i^a(\bar{t})$. These are modeling-depending quantities, and frequently they are random variables depending on the state of the neighborhood of i at $t = t_{\text{st}}$, i.e. on the function $n_i x(t_{\text{st}}, \varepsilon) := x_\varepsilon(t_{\text{st}})$ for all $\varepsilon \in \mathcal{N}_i(t_{\text{st}})$. For an arbitrary time $t \in [t_{\text{st}}, t_{\text{end}}]$, these occurrence times can also depend on past time states $n_i x(t', \varepsilon) := x_\varepsilon(t')$ for all $\varepsilon \in \mathcal{N}_i(t')$ and all $t' \in [t_{\text{st}}, t]$.
- (c) We compute the first arrival time t^1 . Assuming, for simplicity, that we have a finite number of interactions, this is given by

$$t^1(t_{\text{st}}) := t^1 := \min \{t_i^a(t_{\text{st}}) \mid i \in I, t_i^s(t) = t\}.$$

If $t^1 = t_{\text{st}}$, this means that at least one propagator instantaneously arrives at $t_i^s(t_{\text{st}}) = t_i^a(t_{\text{st}}) = t_{\text{st}}$. Otherwise, $t^1 > t_{\text{st}}$ and hence all the states remain constant at $x_e(t_{\text{st}})$ for all $t \in [t_{\text{st}}, t^1]$ because only the evolution equations (2.4) can change these states. If $t^1 = t_i^a(t_{\text{st}})$ for some $i \in I$, the occurrence times must coherently satisfy $\bar{t} = t_i^s(\bar{t}) \leq t^1 = t_i^a(\bar{t})$ for some $\bar{t} \geq t_{\text{st}}$ when i actually started ($\bar{t} = t_i^s(\bar{t})$), $\text{ac}_i^{a_j}(\bar{t}) \neq 0 \neq \text{ac}_i^r(\bar{t})$ for some agent a_j and for the propagator r of i , and $\text{ac}_i^r(t^1) \neq 0 \neq \text{ac}_i^p(t^1)$ for the patient p (see Sec. 2.3).

- (d) If $t^1 < +\infty$, then this t^1 is the first time corresponding to the arrival of some propagator, and we can hence update the time as $t = t^1$, in the sense that nothing occurred before. Using the evolution equations (2.4), we can change the state $x_p(s)$ of all the patients of interactions $i \in I_p(t^1)$ whose arrival time $t_i^a \in [t^1, t^1 + \Delta]$.

- (e) Because we have changed the state of these patient entities, we recursively restart from the beginning with the new states $x_e(t)$ at $t = t^1 + \Delta$.
- (f) Note that if an interacting entity is never a patient of at least an interaction i whose propagator actually arrives at some $t = t_i^a(t) \leq t_{\text{end}}$, then its state $x_e(t)$ will never change, because we can never apply the evolution equation (2.4). In this case, the states x_e of these kind of interacting entities work as constant parameters of the system.

We can therefore say that *mathematically solving an IS* means setting the model by deciding interacting entities and interactions, occurrence times, neighborhoods of interactions, initial states $x_e(t_{\text{st}})$ and transition functions f_p , and solve or simulate the evolution equations for the states $x_p(-)$. For some models, the occurrence times or the neighbourhoods can also be considered as unknowns of the study.

More generally, *solving an IS* means:

- 1) Mathematically solving it;
- 2) Validating the obtained results by comparison with independent real world data.
- 3) This comparison is based on a notion of truth which is accepted by a certain community at a certain time.
- 4) This notion have to include understanding and showing frameworks where the model can and where it cannot be applied (falsification).
- 5) Careful checking that the model is applied only in validated settings satisfying all the modeling assumptions.

3. MATHEMATICAL DEFINITION OF IS

A mathematical definition of IS is a necessary step to start a mathematical theory, and hence to prove general theorems in a clear way and using modern and advanced mathematical instruments. We already started this process, e.g. by showing a general master equation for Markov IS (see [22]), proving a systems of mean derivative equations for the description of a general class of non-Markov IS, [22], giving a very comprehensive definition of CAS (see [21]) and proving related power laws, [21].

The usefulness of this mathematical formalization can also be inferred by thinking at the same basic notions of IS, with cause-effect elementary relations represented by interactions (2.2) and activation states. Indeed, these concepts allow one to define the notion of cause-effect graphs occurring in a system, and of hierarchical functors that preserve such relations between pairs of different IS. Therefore, this direction of theoretical development, which we postpone to a subsequent article, finds potential applications in several CS such as the brain and more general intelligent systems, urban systems, the immune system, organisms in biology, social systems, etc. and wherever the intelligibility of a system using cause-effect graphs or a hierarchical description can be helpful, see e.g. [47, 23].

3.1. Interacting entities and interactions. Already in the informal description of IS, it is clear that many components are needed to define an IS: a set of entities, a set of types of interactions, state maps, occurrence times, etc. For this reason, using a nested approach, we introduce four structures that will define the notion of IS. In this way, instead of referring only to the complete notion of IS, we can also focus on only some of these structures and thereby considering more general modeling settings.

Definition 2. A system of entities and interactions $\mathcal{EI} = (E, t_{\text{st}}, t_{\text{end}}, \mathcal{T}, I)$ is given by the following data which satisfy the following conditions:

- (i) A set E , called *the set of interacting entities*.
- (ii) A time interval $[t_{\text{st}}, t_{\text{end}}]$, with $t_{\text{st}} < t_{\text{end}} \leq +\infty$.
- (iii) A finite set \mathcal{T} called *the set of types of interactions*.
- (iv) A set I called *the set of interactions* satisfying the following condition: every interaction $i \in I$ can be written as $i = (a_1, \dots, a_n, r, \alpha, p)$ for some type of interaction $\alpha \in \mathcal{T}$, some entities $a_1, \dots, a_n, r, p \in E$, and where also $n \geq 0$ depends on i .

Remark 3.

- (a) We set $E_i := \{a_1, \dots, a_n, r, p\}$, $\text{ag}(i) := (a_1, \dots, a_n)$, $\text{pa}(i) := p$ and $\text{pr}(i) := r$ to denote all the interacting entities involved in the interaction i , agents, patient and propagator of i , resp. For example, if $i = (a, b, b, \alpha, b) \in I$, where $a, b \in E$ and $\alpha \in \mathcal{T}$, this means (reading backwards) that $\text{pa}(i) = b$, the interaction i is of type α , $\text{pr}(i) = b$, and $\text{ag}(i) = (a, b)$, so that $n = 2$.
- (b) What are naturally thought of as agents or patient can depend on a fixed frame of reference: think e.g. at the interaction of collision between two particles and a frame at rest with respect to one of the two, which can be naturally thought of as the patient of the collision.
- (c) There is no a priori limitation on the cardinality of the set E of interacting entities, even though in several cases it is finite.
- (d) The system is studied in the time interval $[t_{\text{st}}, t_{\text{end}}]$; clearly, if $t_{\text{end}} = +\infty$, we will use the notation $[t_{\text{st}}, t_{\text{end}}] = [t_{\text{st}}, +\infty]$ to mean $[t_{\text{st}}, +\infty)$.
- (e) Generally speaking, the interactions are non Newtonian: they involve more than one agent and they are, in general, not reversible, i.e. there is not an action-reaction principle. For example, it does not seem useful to think as Newtonian the non-colliding interaction of a pedestrian with an obstacle or the interaction of a builder with a house under construction or of an object in an object oriented programming language with another object: even if frequently to each one of these interactions correspond another interaction as answer, in general there is no useful way to say that the intensity (force) of the cause interaction is the opposite of the intensity (force) of the reaction interaction.

3.2. State spaces and activations. As we already intuitively explained in Sec. 2.1, Sec. 2.3 and Sec. 2.6, each interacting entity is described by a state variable x_e , of which activation ac_i^e and goods γ_i are particular cases. Goods γ_i are taken from a subspace $\gamma_i \in R_i$ of the state space called space of resources of an interaction. The main aim of the next definition is to specify, from the mathematical point of view, the entire state space of an interacting entity e , and to underscore that both activation and goods are state variables.

In the following, if X and Y are two sets, Y^X denotes the space of all the functions $f : X \rightarrow Y$ and for the values $f(x) \in Y$ we can also sometime use the notation $f_x \in Y$. For the sake of clarity: if the index set $J = \{j_1, \dots, j_n\}$ is finite, then the product of sets is $\prod_{j \in J} S_j = S_{j_1} \times \dots \times S_{j_n}$.

Definition 4. Let $\mathcal{EI} = (E, t_{\text{st}}, t_{\text{end}}, \mathcal{T}, I)$ be a system of entities and interactions. A system of states $\mathcal{S} = (S, \mathfrak{S}, R, x)$ for \mathcal{EI} is given by the following data which satisfy the following conditions:

- (i) For every interacting entity $e \in E$, a measurable space (S_e, \mathfrak{S}_e) called the *proper state space* of the interacting entity e .

(ST) For each interacting entity $e \in E$ and time $t \in [t_{\text{st}}, t_{\text{end}}]$, a *state function*

$$x_e(t) \in [0, 1]^I \times \prod_{\substack{i \in I \\ e = \text{pr}(i)}} R_i \times S_e =: \bar{S}_e$$

- This means that $x_e(t)$ has three components: the first one $x_e(t)_1 \in [0, 1]^I$ is a function $x_e(t)_1 : I \rightarrow [0, 1]$ and its evaluation at $i \in I$ is denoted by $\text{ac}_i^e(t) := x_e(t)_{1,i} \in [0, 1]$ and called *activation* of e for the interaction i at time t .
- The second component $x_e(t)_2$ is defined only if $e = \text{pr}(i)$ is a propagator of some interaction i (otherwise, it is not defined). Therefore, in general, if $i : a_1, \dots, a_n \xrightarrow{r, \alpha} p$ is an interaction, we set $\gamma_i(t) := x_e(t)_{2,i} \in R_i$, and call R_i the *space of resources* of i . This state variable $\gamma_i(t)$ is called *goods* of i .
- The third component $x_e(t)_3 \in S_e$ lies in the proper state space S_e . Since we use the specific notations $\text{ac}_i^e(t)$ and $\gamma_i(t)$ for the first two components, it is not confusing using simply the classical notation $x_e(t) \in S_e$ for the third one.

Remark 5.

- (a) The property of the proper state space (S_e, \mathfrak{S}_e) of being a measurable space is very weak, from the mathematical point of view, even if usually on a state space there is a richer structure, e.g. $S_e = \mathbb{R}^d$ for some $d > 0$ depending on $e \in E$. Let us note explicitly that the state space is not time dependent.
- (b) We say that the interaction $i : a_1, \dots, a_n \xrightarrow{r, \alpha} p$ has a *notion of zero resources* Z_i (see Sec. 2.6) if
- $\emptyset \neq Z_i \subseteq R_i$;
 - $\forall t \in [t_{\text{st}}, t_{\text{end}}] : \gamma_i(t) \in Z_i \Rightarrow \text{ac}_i^p(t) = 0$, i.e. whenever $\gamma_i(t) \in Z_i$, the patient p is not active for i , i.e. $\text{ac}_i^p(t) = 0$.

The label (ST) recalls *state function*.

3.3. Clock functions. In the present section, we want to clarify that both the starting time $t \in [t_{\text{st}}, t_{\text{end}}] \mapsto t_i^s(t)$ and ongoing time functions $t \in [t_{\text{st}}, t_{\text{end}}] \mapsto t_i^o(t)$ satisfy similar general properties.

Thinking at the dynamics of an IS presented in Sec. 2.8, we can understand that this dynamics is event based, driven by starting and arrival times of propagators of interactions. This is the concrete notion of time as it naturally plays in an IS. Stating it differently: if we have an interaction sending a propagator at every ticking of a clock, then “Time is what clock shows”, as Einstein is supposed to have said. Conceptually, this is different from the quantity $t \in [t_{\text{st}}, t_{\text{end}}]$, which is only an independent variable to mathematically manage functions such as $t \in [t_{\text{st}}, t_{\text{end}}] \mapsto t_i^s(t) \in [t_{\text{st}}, t_{\text{end}}] \cup \{+\infty\}$.

We start by defining what are the set of time events T we consider in every IS. We can think T as the stochastic values of an exponential distribution representing the intensity of occurrence of a given interaction, see Sec. 3.5.1, or a time interval $[t^1, t^2] \subseteq [t_{\text{st}}, t_{\text{end}}]$ used to model a continuous dynamical system. Each one of these T defines a clock function:

Definition 6. We say that T is a *set of discrete or continuous time events* (we briefly write T *discr./cont.*) if:

- (i) $T \subseteq [t_{\text{st}}, t_{\text{end}}]$ is the disjoint union of single instants t_j for $j \in N \subseteq \mathbb{N}$, or of intervals $[t_k^1, t_k^2]$ for $k \in M \subseteq \mathbb{N}$.
- (ii) Accumulation points of T lie only in its subintervals, i.e.

$$\forall t \in T' \exists k \in M : t \in [t_k^1, t_k^2].$$

We recall that the set of accumulation points is defined by $t \in T'$ if $\forall \delta \in \mathbb{R}_{>0} \exists \bar{t} \in T \cap (t - \delta, t + \delta) : \bar{t} \neq t$, i.e. it is the set T' of all the points that can be arbitrarily approximated using points of T .

Moreover, we say that τ is a *clock function* if

- (iii) $\tau : [t_{\text{st}}, t_{\text{end}}] \longrightarrow [t_{\text{st}}, t_{\text{end}}] \cup \{+\infty\}$.
- (iv) $\exists T$ discr./cont. $\forall t \in [t_{\text{st}}, t_{\text{end}}] : \tau(t) = \inf \{s \geq t \mid s \in T\}$.

We explain the motivations of this definition in the following

Remark 7.

- (a) Written explicitly, condition (i) is $T = \bigcup_{j \in N} \{t_j\} \cup \bigcup_{k \in M} [t_k^1, t_k^2]$, and all these unions are disjoint.
- (b) Condition (ii) excludes situations such as $T = \{1 \pm \frac{1}{n} \mid n \in \mathbb{N}_{>0}\} \cup \{1\}$ where it is not clear whether at $t = 1$ an event occurs instantly or continuously.
- (c) Condition (iv) can be interpreted saying that $\tau(t)$ is the next time event in T after or at t . On the other hand, when we have $\tau(t) = t$, we say that τ is *occurring at t* : e.g. if $t_i^s(t) = t$, we say that the interaction i is starting at t ; if $t_i^s(t) > t$, we say that after t the interaction i will start the first time at the time instant $t_i^s(t)$.
- (d) In figure 3.1, we represented in red the clock function corresponding to the discr./cont. time events T depicted in blue on the y -axis. In this T , at t_{st} , t_1 and t_{end} instantaneous events occur, whereas we have a continuous one in $[t_1^1, t_1^2]$.

We have the following general results, which can be easily proved from the previous definitions.

Theorem 8. *If τ is the clock function defined by the discr./cont. set of events T , then:*

- (i) $\exists \min(T), \max(T)$ and $\tau(t_{\text{st}}) = \min(T) \in T$.
- (ii) If $\max(T) = t_{\text{end}}$, then $\tau(t_{\text{end}}) = t_{\text{end}}$, otherwise $\tau(t) = +\infty$ for all $t \in (\max(T), t_{\text{end}}]$.
- (iii) For all t , we have $\tau(t) \geq t$, and $\tau(t) = t$ if $t \in T$.
- (iv) The function τ is non-decreasing, and hence $\tau(t) = \inf \{\tau(s) \mid s \in [t, t_{\text{end}}]\}$ for all t .
- (v) If $t_2 \in T$, $t_1 < t_2$ and $(t_1, t_2) \cap T = \emptyset$, then $\tau(t) = t_2$ for all $t \in [t_1, t_2]$, i.e. τ is constant and left-continuous in this interval.
- (vi) If $\max(T) = t_{\text{end}}$, then $T = \tau([t_{\text{st}}, t_{\text{end}}])$.
- (vii) If $\max(T) < t_{\text{end}}$, then $T \cup \{+\infty\} = \tau([t_{\text{st}}, t_{\text{end}}])$. Therefore, this and the previous property show that the function τ uniquely determines the set of events T as $T = \tau([t_{\text{st}}, t_{\text{end}}]) \setminus \{+\infty\}$, so that we can equivalently work with T or τ .

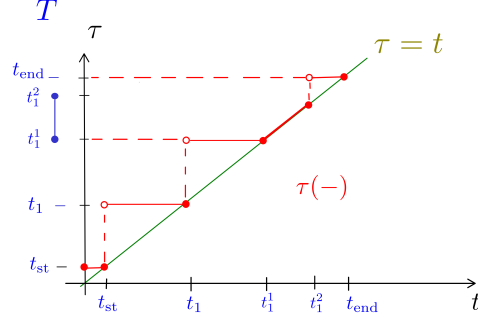


FIGURE 3.1. An example of clock function.

3.4. Data to run an interaction. The previous section gives us the language to formulate the intuitive statements we already introduced in Sec. 2.3, 2.4, 2.5. We also want to show that the arrival time function $t \in [t_{\text{st}}, t_{\text{end}}] \mapsto t_i^{\text{a}}(t)$ can be defined as the minimal value of $t_i^{\text{o}}(t)$.

Definition 9. Let $\mathcal{EI} = (E, t_{\text{st}}, t_{\text{end}}, \mathcal{T}, I)$ be a system of entities and interactions, and let $\mathcal{S} = (S, \mathfrak{S}, R, x)$ be a system of states for \mathcal{EI} . Let $i : a_1, \dots, a_n \xrightarrow{r, \alpha} p$ be an interaction in I , then $\mathcal{D}_i = (t_i^{\text{s}}, t_i^{\text{o}}, \mathcal{N}_i)$ are *data to run i* if:

(CF) t_i^{s} and t_i^{o} are clock functions, called resp. *starting times* and *ongoing times* of i .

Recalling that the set of values $T^{\text{o}} := t_i^{\text{o}}([t_{\text{st}}, t_{\text{end}}]) \setminus \{+\infty\}$ is the set of discr./cont. time events of the ongoing function t_i^{o} , we define the *arrival times* of i as follows:

- $t_i^{\text{a}}(t) := t_j$, if $t_i^{\text{o}}(t) = t_j$ is discrete (i.e. i occurs instantaneously at $t = t_j$).
- $t_i^{\text{a}}(t) := t_k^1$, if $t_i^{\text{o}}(t) \in [t_k^1, t_k^2] \subseteq T^{\text{o}}$ (i.e. i occurs continuously around t).
- $t_i^{\text{a}}(t) := +\infty$, if $t_i^{\text{o}}(t) = +\infty$ (i.e. i never occurs at t or after).

(SA) For all $t \in [t_{\text{st}}, t_{\text{end}}]$, if $t_i^{\text{s}}(t) = t$, i.e. the interaction $i \in I$ is starting at t , then $t_i^{\text{s}}(t) \leq t_i^{\text{a}}(t)$. In other words, if i starts at $t_i^{\text{s}}(t) = t$, then the propagator cannot arrive before this starting time, i.e. $t_i^{\text{s}}(t) > t_i^{\text{a}}(t)$ cannot happen.

We say that $(t_{\text{s}}, t_{\text{a}})$ are *start-arrival events* for i at $t \in [t_{\text{st}}, t_{\text{end}}]$, if $t_{\text{s}} = t_i^{\text{s}}(t) = t \leq t_{\text{a}} = t_i^{\text{a}}(t) < +\infty$. These data have to satisfy the following conditions:

(CE) $\text{ac}_i^{\alpha_j}(t_{\text{s}}) \neq 0$ for some $j = 1, \dots, n$, $\text{ac}_i^r(t_{\text{s}}) \neq 0$, $\text{ac}_i^r(t_{\text{a}}) \neq 0$ and $\text{ac}_i^p(t_{\text{a}}) \neq 0$.

Finally, the *neighborhood* function \mathcal{N}_i satisfies

(NE) $\{e \in E_i \mid \text{ac}_i^e(t) \neq 0\} =: E_i^t \subseteq \mathcal{N}_i(t) \subseteq E_t := \{e \in E_i \mid \exists i \in I : \text{ac}_i^e(t) \neq 0\}$ for all $t \in [t_{\text{st}}, t_{\text{end}}]$, i.e. the neighborhood $\mathcal{N}_i(t)$ always contains the entities of i which are active at t , and it also always contains only active entities.

The labels (CF), (SA), (CE) and (NE) recall *clock functions*, *start-arrival*, *cause-effect* and *neighborhood* respectively.

Let us assume, e.g., that $t_i^{\text{s}}(0) = 0 = t_{\text{st}}$ and $t_i^{\text{o}}(t) = 1$ for $t \in [0, 1]$ and $t_i^{\text{o}}(t) = t$ for $t \in [1, 2]$. We have $t_i^{\text{a}}(t) = 1$ for all $t \in [0, 2]$. We can also have $t_i^{\text{s}}(t) = 3$ for $t > 0$, i.e. after 0 the interaction i will start again at $t = 3$. Note that in this case $t_i^{\text{o}}(1) = 1 < t_i^{\text{s}}(1) = 3$ which simply means that i is ongoing at $t = 1$ and it will start again at $t = 3$. Therefore, in general, the inequality $t_i^{\text{s}}(t) \leq t_i^{\text{o}}(t)$ does not hold. Moreover, $t_i^{\text{a}}(2) = 1 < 2$, therefore the arrival function does not satisfy $t_i^{\text{a}}(t) \geq t$

and hence it is not a clock function. On the other hand, note that at $t = 0$, we have $t_i^s(0) = 0$, i.e. i starts at $t = 0$, and $t_i^a(0) = 1 > t_i^s(0)$. Therefore, also the inequality $t_i^s(t) \leq t_i^a(t)$ in general does not hold (compare this with (SA)).

Since we think at the activation state variable $\text{ac}_i^e(t) = x_e(t)_{1,i}$ as a (possible) stochastic path of our CS, from (NE) it follows that also $\mathcal{N}_i(t) \subseteq E_t$ has to be thought of as a (possible) stochastic set.

From the previous definition, we have the following

Theorem 10. *In the previous assumptions, if $t \in [t_{\text{st}}, t_{\text{end}}]$, we have:*

- (i) $t_i^a : [t_{\text{st}}, t_{\text{end}}] \rightarrow [t_{\text{st}}, t_{\text{end}}] \cup \{+\infty\}$ is piecewise constant.
- (ii) $t_i^a(t) \leq t_i^o(t)$.
- (iii) If $i : a_1, \dots, a_n \xrightarrow{r, \alpha} p$ is an interaction in I , and $(t_s, t_a) \geq t$ are start-arrival events for i , then:
 - (i) $\exists j = 1, \dots, n : a_j \in \mathcal{N}_i(t_s)$.
 - (ii) $r \in \mathcal{N}_i(t_s)$ and $r, p \in \mathcal{N}_i(t_a)$.
 - (iii) If i has a notion Z_i of zero resources, then $\gamma_i(t_a) \notin Z_i$.
- (iv) For all $i \in I$ and $t \in [t_{\text{st}}, t_{\text{end}}]$, there exists $t' \in [t_{\text{st}}, t]$ such that $t_i^s(t') = t' \leq t_i^a(t')$, i.e. there exists a time $t' \leq t$ when i started.

3.4.1. *Simultaneous vector interactions.* If two interactions $i : a_1, \dots, a_n \xrightarrow{r, \alpha} p$ and $j : b_1, \dots, b_m \xrightarrow{s, \beta} q$ act on patients p, q and are *simultaneous*, i.e. they have the same occurrence times clock functions $t_i^s(-) = t_j^s(-)$ and $t_i^o(-) = t_j^o(-)$, we can define a *vector interaction* (i, j) by simply considering as agents

$$\text{ag}(i, j) := (a_1, \dots, a_n, b_1, \dots, b_m),$$

as patient $\text{pa}(i, j) := (p, q)$, as propagator $\text{pr}(i, j) := (r, s)$ and as type (α, β) . The activation maps of agents, propagator and patient are defined in a natural way as $\text{ac}_{(i,j)}^e(t) := \text{ac}_i^e(t) \cdot \text{ac}_j^e(t)$. The resource space of (i, j) is the product of the resources of its components $R_{(i,j)} = R_i \times R_j$. Similarly, we can define the state space of the new patient entity $\text{pa}(i, j) = (p, q)$ and the neighborhood. In the particularly interesting case when the two interactions act on the same patient $p = q$, we simply set $\text{pa}(i, j) = p$. At the end, we obtain a cause-effect simultaneous interaction of the form

$$(i, j) : a_1, \dots, a_n, b_1, \dots, b_m \xrightarrow{(r,s), (\alpha, \beta)} p, q.$$

It clearly depends on our modeling aims whether the interaction (i, j) already lies in the set of all the interactions I of our system or if we are more interested in defining a new IS using (i, j) .

The previous construction can be repeated with a finite number i_1, \dots, i_h of simultaneous interactions, so that we can describe arbitrary polyadic cause-effect relations. Therefore, considering simultaneous interactions, we can naturally describe a CS using cause-effect hypergraphs in a polyadic relationships, see e.g. [31, 3, 18].

3.5. **Evolution equations.** We refer to Sec. 2.7 for the motivations of the following definitions:

Definition 11. Let $\mathcal{EI} = (E, t_{\text{st}}, t_{\text{end}}, \mathcal{T}, I)$ be a system of entities and interactions, let $\mathcal{S} = (S, \mathfrak{S}, R, x)$ be a system of states for \mathcal{EI} , and $\mathcal{D}_i = (t_i^s, t_i^o, \mathcal{N}_i)$ the data to run i , for each $i \in I$. Therefore

$$t^1(t) := t^1 := \inf \{t_i^a(t) \mid i \in I, t_i^s(t) = t\} \quad \forall t \in [t_{\text{st}}, t_{\text{end}}],$$

is the *first arrival of all the interactions started at t* . Note that t^1 is the clock function generated by the values of $t_i^a(t)$ for $i \in I$: in fact, t_i^a is piecewise constant (see Thm. 10.(i)), and by Def. 9 of t_i^a , these values are at most countable. Moreover, $t_i^a(t) \geq t = t_i^s(t)$ by condition (SA) of Def. 9.

In this setting, a *system* $\mathcal{EE} = (\Delta, f, \Omega, \mathcal{F}, P)$ for the evolution equations of \mathcal{EI} , \mathcal{S} and $(\mathcal{D}_i)_{i \in I}$ is given by the following data which satisfy the following conditions.

- If the $t^1 < +\infty$ and $p \in E$, we first define

$$I_p(t) := \{i \in I \mid t^1(t) \leq t_i^a(t) \leq t^1(t) + \Delta, \text{pa}(i) = p\} \quad \forall t \in [t_{\text{st}}, t_{\text{end}}],$$

where $\Delta \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$.

- Then, we consider the *state of the neighborhood of $p \in E$* as the function $\mathfrak{n}_p x$ defined by:

$$\text{If } \exists t' \leq t \exists i \in I : \text{pa}(i) = p, t' = t_i^a(t'), \varepsilon \in \mathcal{N}_i(t'), \tau \in [t', t] \quad (3.1)$$

$$\text{then } \mathfrak{n}_p x(\tau, \varepsilon) := x_\varepsilon(\tau).$$

These data have to satisfy the following conditions:

- (i) $(\Omega_p, \mathcal{F}_p, P_p)$ is the *probability space for the evolution of $p \in E$* and the *transition map* $f_p(-, s, \mathfrak{n}_p x_s)_3 : \Omega_p \rightarrow S_p$ is measurable for all $s \in [t_{\text{st}}, t_{\text{end}}]$.
- (EE) There exists $\omega \in \Omega_p$ such that if $t \in [t_{\text{st}}, t_{\text{end}}]$, $t^1 < +\infty$, $I_p(t) \neq \emptyset$ and $t^1(t) \leq s \leq t^1(t) + \Delta \leq t_{\text{end}}$, then

$$x_p(s) = f_p(\omega, s, \mathfrak{n}_p x_s), \quad (3.2)$$

where

$$\mathfrak{n}_p x_s : \tau \in [t^1(t), s] \mapsto \mathfrak{n}_p x(\tau, -).$$

Conditions (i) and (EE) mathematically clarify the intuition about the state variable $x_p(s)$, which results as a (possible) stochastic path of our system. In other words: by running a simulation of the system which follows the algorithm presented in Sec. 2.8, we obtain as outcome a possible value of the state variables $x_p(s)$. The label (EE) recall *evolution equation*.

To further illustrate these concepts, we can consider the following simple examples:

Example 12.

- 1) A person a is throwing a stone p : we can set the propagator $r = p$ as the same patient carrying in its resource state the information of the initial velocity \vec{v}_0 and position x_0 (hence in this example we have $i : a \xrightarrow{p, t} p$ and $\gamma_i(t) = (x_0, \vec{v}_0)$). We can also set $\text{ac}_i^a(t) = \text{ac}_i^p(t) = 1$, and the starting time t_i^s defined by $T^s := \{t_{\text{st}}\}$, so that $t_i^s(t) = t_{\text{st}}$ if $t = t_{\text{st}}$ and $t_i^s(t) = +\infty$ otherwise. The ongoing function t_i^o is defined by $T^o := [t_{\text{st}}, t_{\text{end}}]$ and hence the arrival function is $t_i^a(t) = t_{\text{st}}$ (see also below Sec. 4 for an IS defined by an arbitrary ODE). In the time interval $[t_{\text{st}}, t_{\text{end}}]$ the transition function is of the form $f_p = f_p(s, x_0, \vec{v}_0)$ and gives the deterministic dynamics of the stone. In this case we have a trivial space for stochastic evolution, i.e. $|\Omega_p| = 1$.
- 2) In a more “realistic” modeling of the same system, we can consider the initial condition (x_0, \vec{v}_0) distributed as a 6-dimensional normal distribution, so that we have $S_r = R_j = \mathbb{R}^6$ and $\gamma_j(t)$ is this normal distribution for another initial interaction with $t_j^s = t_j^o = t_j^a = t_{\text{st}}$ and corresponding to this random extraction of the initial condition with normal distributions. The transition function

$f_p(s, t_{\text{st}}, x_0, \vec{v}_0) = x_0 + \vec{v}_0(s - t_{\text{st}}) + \frac{1}{2}\vec{g}(s - t_{\text{st}})^2$ is actually a deterministic function depending on the randomly extracted initial values $(x_0, \vec{v}_0) = (x_0(\omega), \vec{v}_0(\omega))$ (no additionally randomness is introduced after the stone has been thrown). The probability space for the evolution of p is hence again trivial: $|\Omega_p| = 1$.

- 3) Let us consider a pedestrian p receiving at time t_i^s a signal r from a source a , and starting to move in the direction $\gamma_i \in \mathbb{R}^3$, $|\gamma_i| = 1$, with a certain stochastic deviation, both in the direction and in the magnitude of the velocity. We will have $f_p(s, \omega; t_i^s, \gamma_i) = x_0 + \vec{v}(\omega) \cdot (t - t_{\text{st}}i)$, where x_0 is the position of p at time t_i^s and where the expected value of \vec{v} in the space $(\Omega_p, \mathcal{F}_p, P_p)$ is $E(\vec{v}) = v_0 \cdot \gamma_i$; both v_0 and x_0 are taken from the proper state space S_p of the patient p .
- 4) Let $i \in I_p(t)$ be an interaction starting at $t_i^s(t) = t$, arriving at $t_i^a(t) \geq t$ and ongoing in the interval $[t_i^a(t), t_k^2]$. We can have $t^1(t) + \Delta > t_k^2$ if Δ is small because of other interactions simultaneously occurring in the interval $[t^1(t), t^1(t) + \Delta]$, but, at the same time, we want that i continuously acts on p even after $t' := t^1(t) + \Delta$. In order to model this kind of behaviour in the setting of IS, we clearly have to coherently model the starting times so that $t_i^s(t') = t'$, e.g. using a continuous time interval for $t_i^s(-)$.
- 5) Def. 11 states minimal conditions satisfied by a large class of CS. However, it could be also very interesting to consider IS where at time $t^1(t) + \Delta =: t'$ explicitly occur the starting times of feedback interactions $j : p, n_1, \dots, n_k \xrightarrow{r_j, \text{fb}} n_h$, where $\{n_1, \dots, n_k\} \subseteq \mathcal{N}_i(t')$ are entities in the neighborhood of i . Since $t_j^s(t') = t'$ and the transition function f_{n_h} depends also on the state $x_p(t')$ of p , we can say that this is still a particular case of the dynamics described in (e) of Sec. 2.8.

3.5.1. *Stochastic generation of clock functions.* We already specified in Sec. 2.4 and Sec. 3.3 that the clock functions $t_i^s(t)$, $t_i^o(t)$ can be thought of as sample paths generated by model depending distributions. This can be done using the following procedure:

- (i) For each interaction $i \in I$, we consider the *state of the neighborhood* of i defined as

$$\begin{aligned} \text{If } \exists t' \leq t : t' = t_i^a(t'), \varepsilon \in \mathcal{N}_i(t'), \tau \in [t', t] \\ \text{then } n_i x(\tau, \varepsilon) := x_\varepsilon(\tau). \end{aligned} \quad (3.3)$$

In general, all the probability distributions of the occurrence times depend on this neighborhood function, i.e. on the history of the interaction i .

- (ii) At each time t , we have to decide whether i will start after or at t in a discrete time t_j or a continuous time interval $[t_k^1, t_k^2]$. Even if, in principle, this can also be decided randomly, usually it is a model-related choice.
- (iii) Starting from the previous step and using a model-depending probability distribution $T_i^s(-; n_i x)$ on the space $[t, t_{\text{end}}]$ depending on $n_i x$, we can extract either a sample of the form $t_j \in [t, t_{\text{end}}]$ or a pair

$$(t_k^1, t_k^2) \in \{(t^1, t^2) \in [t, t_{\text{end}}] \mid t^1 < t^2\}$$

(note in both cases the interval $[t, t_{\text{end}}]$). In the first case we set $t_i^s(t) = t_j$, and in the second one we set $t_i^s(t') = t'$ for all $t' \in [t_k^1, t_k^2]$.

- (iv) At each t such that $t_i^s(t) = t$ (i.e. i starts at t), using a model-depending probability distribution $T_i^a(-; n_i x, t_i^s(t))$ on the space $[t_i^s(t), t_{\text{end}}]$, depending also on the previous random value $t_i^s(t) \in [t, t_{\text{end}}]$, we extract the value of

Symbol	Meaning
E	set of interacting entities
$[t_{\text{st}}, t_{\text{end}}]$	time interval
\mathcal{T}	set of types of interactions
I	set of interactions $i = (a_1, \dots, a_n, r, \alpha, p)$
$E_i := \{a_1, \dots, a_n, r, p\}$	interacting entities in the interaction i
$\text{ag}(i) := (a_1, \dots, a_n)$	agents of i
$\text{pa}(i) := p$	patient of i
$\text{pr}(i) := r$	propagator of i

TABLE 1. Summary of Def. 2 of system of entities and interactions $\mathcal{EI} = (E, t_{\text{st}}, t_{\text{end}}, \mathcal{T}, I)$.

Symbol	Meaning	Condition
$\text{ac}_i^e(t)$	activation map	$\text{ac}_i^e(t) \in [0, 1]$
R_i	Resources of i	
$\gamma_i(t)$	goods of the interaction i	$\gamma_i(t) \in R_i$
	state variable $x_e(t)$ and proper state space S_e	$\forall e \in E : x_e(t) \in S_e$

TABLE 2. Summary of Def. 4 of system of state spaces and activation maps $\mathcal{S} = (S, \mathfrak{S}, R, x)$.

$t_i^a(t)$. Clearly, this could depend on the speed of the propagator $r = \text{pr}(i)$. We recall that $t_i^a(-)$ takes only discrete values and is not a clock function, see Thm. 10.

- (v) Finally, using a model-dependent probability distribution $T_i^o(-; n_i x, t_i^a(t))$ on the space $[t_i^a(t), t_{\text{end}}]$, depending also on the previous random value $t_i^a(t) \geq t_i^s(t) = t$, we extract either if the interaction i is instantaneously occurring at $t_j = t_i^a(t)$ or a sample pair $(t_k^1, t_k^2) \in \{(t^1, t^2) \in [t_i^a(t), t_{\text{end}}] \mid t^1 < t^2\}$. In the first case we set $t_i^o(t) = t_j$, whereas in the second one we set $t_i^o(t') = t'$ for all $t' \in [t_k^1, t_k^2]$.
- (vi) Clearly, by considering trivial probability distributions, the previous method also includes the deterministic generation of occurrence times.

3.6. Interaction spaces.

Definition 13. An *interaction space* $\mathfrak{I} = (\mathcal{EI}, \mathcal{SA}, \mathcal{I}, \mathcal{TF})$ is given by considering all the previously defined systems:

- (i) A system of entities and interactions $\mathcal{EI} = (E, t_{\text{st}}, t_{\text{end}}, \mathcal{T}, I)$.
- (ii) A system of state spaces $\mathcal{SA} = (S, \mathfrak{S}, R, x)$ for \mathcal{EI} .
- (iii) Data $\mathcal{D}_i = (t_i^s, t_i^o, \mathcal{N}_i)$ to run each interaction $i \in I$.
- (iv) A system $\mathcal{EE} = (\Delta, f, \Omega, \mathcal{F}, P)$ for the evolution equations of \mathcal{EI} , \mathcal{S} and $(\mathcal{D}_i)_{i \in I}$.

Symbol	Meaning	Condition
$t_i^s(t)$	starting time	(CF)
$t_i^o(t)$	ongoing time	(CF)
$t_i^a(t)$	arrival time	(SA), (CE)
$\mathcal{N}_i(t)$	neighborhood of i at time t	(NE): $E_i^t \subseteq \mathcal{N}_i(t) \subseteq E_t$

TABLE 3. Summary of Def. 9 of data to run an interaction $\mathcal{D} = (t_i^s, t_i^o, \mathcal{N}_i)$.

Symbol	Meaning	Condition
Δ	time of evolution	$\Delta \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$
$(\Omega_p, \mathcal{F}_p, P_p)$	probability space for the evolution of p	
$f_p(\omega, s, \mathfrak{n}_p, x_s)$	transition function of p	(EE)

TABLE 4. Summary of Def. 11 of system for the evolution equations $\mathcal{EE} = (\Delta, f, \Omega, \mathcal{F}, P)$.

After a first look, one can actually recognize that the previous definitions essentially represent the introduction of several mathematical notations, and that the important conditions are only a few, as it is clarified in tables 1, 2, 3 and 4.

4. CLASSICAL MODELS FOR COMPLEX SYSTEMS AS INTERACTION SPACES

We can now explain how classical models for complex system can be embedded as interaction spaces. Of course, these embeddings are injective: e.g. if two CA are equal when viewed as IS, then they are necessarily equal as CA.

Even if in this section we do not always mathematically prove and detail the corresponding embeddings, in our opinion it is clear that the following ways to include these classical models as IS are sufficiently detailed to allow a reconstruction of the initial model from the corresponding IS. We also see that the IS structure allows one to consider several interesting generalizations of these classical models of CS.

4.1. Continuous dynamical systems. Assume that the considered system is described by a system of ODE $x'(s) = F(s, x(s)) \in \mathbb{R}^n$ for $s \in [t_{\text{st}}, t_{\text{end}}]$ starting from a given initial state $x_0 \in \mathbb{R}^n$ at $t = t_{\text{st}}$ (we also recall that any higher order ODE can always be transformed into an equivalent system of first order ODE). We can think at an IS having a single entity p with an initial state $x_p(t_{\text{st}}) = x_0$. At $t \geq t_{\text{st}}$ the dynamics of this IS must be ruled by an evolution equation faithfully corresponding to this ODE.

The following is only one possible way of seeing a dynamical system as an IS, and several other embeddings are possible as well, e.g. because a dynamical system does not have intrinsic notions of activations, goods, neighborhood, etc. However, we will see that these additional notions naturally inspire interesting generalizations.

We therefore set $E = \{p\}$, $t_{\text{st}} < t_{\text{end}} \leq +\infty$, $\mathcal{T} = \{\text{ds}\}$ which means “dynamical system”, $I = \{(p, p, \text{ds}, p)\}$ i.e. $p \xrightarrow{p, \text{ds}} p$, $x_p(t) \in [0, 1]^I \times \{0\}^I \times \mathbb{R}^n$. Since there is only one interaction, in the following we omit the index i . We set trivial activations and goods: $\text{ac}^p(t) := 1$, $\gamma(t) := 0$ for all t . Occurrence times: t^s defined by $T^s := \{t_{\text{st}}\}$, so that $t^s(t) = t_{\text{st}}$ if $t = t_{\text{st}}$ and $t^s(t) = +\infty$ otherwise; t^o defined by $T^o := [t_{\text{st}}, t_{\text{end}}]$, i.e. $t^o(t) = t$ for all t ; Therefore, the arrival time is given by

$t^a(t) = t_{\text{st}}$ for all t . Neighborhood: $\mathcal{N}(t) = \{p\}$. Conditions (CF), (SA), (CE) and (NE) trivially hold. For the evolution equation, we set $\Delta = t_{\text{end}} - t_{\text{st}}$, and we have $t^1(t) = t_{\text{st}}$ if $t = t_{\text{st}}$ and $t^1(t) = +\infty$ otherwise, so that $I_p(t_{\text{st}}) = \{i\}$ and $I_p(t) = \emptyset$ for $t > t_{\text{st}}$ because $t^1(t) = +\infty \leq t^a = t_{\text{st}} \leq t^1(t) + \Delta = +\infty$ is impossible. If (τ, ε) lies in the domain of the neighborhood function $n_p x$ (i.e. if the conditions $p = \text{pa}(i)$, $t' = t_i^a(t')$, $\varepsilon \in \mathcal{N}_i(t')$, $\tau \in [t', t]$ hold for some $t' \leq t$ and $i \in I$ (see (2.3)), then necessarily $t' = t_{\text{st}}$ and $\varepsilon = p$, so that

$$n_p x : \tau \in [t_{\text{st}}, t] \mapsto x_p(\tau) \in \mathbb{R}^n,$$

where we considered only the nontrivial specific state space $S_p := \mathbb{R}^n$. For a deterministic dynamics, we consider a trivial probability space $\Omega_p = \{0\}$. Finally, the assumptions of (EE) are $t^1(t) < +\infty$ (so that $t = t_{\text{st}}$) and $t^1(t) \leq s \leq t^1(t) + \Delta \leq t_{\text{end}}$ (i.e. $t_{\text{st}} \leq s \leq t_{\text{end}}$) and the evolution equation must be $x_p(s) = f_p(s, n_p x_s)$, where the restricted neighborhood function is $n_p x_s : \tau \in [t_{\text{st}}, s] \mapsto x_p(\tau) \in \mathbb{R}^n$, i.e. it is $x_p(-)|_{[t_{\text{st}}, s]}$. As we already anticipated in Rem. 1.(f), if we assume that $x'_p(s) = F(s, x_p(s))$ for all $s \in [t_{\text{st}}, t_{\text{end}}]$, where $F \in \mathcal{C}^0([t_{\text{st}}, t_{\text{end}}] \times U, \mathbb{R}^n)$ is a continuous function and $U \subseteq \mathbb{R}^n$ is an open set, we can define $f_p(s, y) := y(t_{\text{st}}) + \int_{t_{\text{st}}}^s F(\tau, y(\tau)) d\tau$ for all $s \in [t_{\text{st}}, t_{\text{end}}]$ and for all $y \in \mathcal{C}^0([t_{\text{st}}, t_{\text{end}}], U)$ to have that this ODE is satisfied if and only if $x_p(s) = x_p(t_{\text{st}}) + \int_{t_{\text{st}}}^s F(\tau, x_p(\tau)) d\tau = f_p(s, x_p(-)|_{[t_{\text{st}}, s]})$. Moreover, f_p uniquely determine F as $F(t, x) = \frac{d}{ds} f_p(s, x)|_{s=t}$ and this yields the injective embedding from the data (F, x_0) describing the continuous dynamical system to this IS starting from $x_p(t_{\text{st}}) = x_0$.

4.2. Discrete dynamical systems. If the dynamical system is described by a recursive equation of the form $x(k+1) = F(k, x(k)) \in \mathbb{R}^n$ for all $k = 0, \dots, N$ and $x(0) = x_0 \in \mathbb{R}^n$, we can set the IS as above, changing only the evolution equation as described in (2.5), where we can think at $x_p(-) : \{0, \dots, N\} \rightarrow \mathbb{R}^n$ as an arbitrary function. Therefore, the transition function f_p uniquely determine the values $V_F := \{F(k, y) \in \mathbb{R}^n \mid k \in \{0, \dots, N\}, y \in \mathbb{R}^n\}$ which define all the possible orbits of the given discrete dynamical system. Therefore, this gives an embedding of the data (V_F, x_0) into this IS.

It is natural to think at generalizations of the form:

- Initial interaction with a starting entity $s \in E$ which is responsible for setting the initial condition x_0 .
- Introducing a non-trivial dynamics of goods in a suitable space of resources (e.g. described by another dynamical system) corresponds to coupled dynamical systems.
- We can also consider several levels of non-Markovian dynamical systems taking less trivial occurrence times or neighborhoods. For example, if $t_i^s(t) < t_i^a(t)$ and a previous interaction $j \in I_p(t_i^a(t))$ returns the state $x_p(t')$ back to a previous value $x_p(t' - \tau_D)$ at some $t' < t_i^a(t)$, then we have a delay dynamical system $x'(s) = F(s, x(s - \tau_D))$, see e.g. [15]. We can also consider a nontrivial neighborhood and couple the dynamical system with the past dynamics of another interacting entity.
- Considering a nontrivial probability space for the evolution of p , we can also describe as IS any stochastic dynamical system.

- If the system experiences abrupt changes (i.e. infinite derivatives), like in collisions, we can similarly describe it as an IS by taking as F a generalized smooth function, see e.g. [24].

4.3. Synchronous and asynchronous cellular automata. To embed CA as IS, we clearly set cells with their state space as interacting entities. Depending on the type of cellular automaton, we can have either local or global interactions i , the latter possibly acting on only a subset of cells. Even if in classical CA we can set as always active every cell, in more advanced CA we can think at inserting the activation $ac_i^e(t)$ as a state variable of some cell (see e.g. [55]). Every local interaction has the same type of neighborhood, which corresponds to that of the cell in the CA structure. In every local interaction, agents are all the cells in the neighborhood, and the patient corresponds to the cell on which the interaction acts. Global interactions can be seen as having only one agent that equals the patient on which they act. The dynamics can be synchronous at times $t_i^s(t) = t_i^o(t) = t_i^a(t) = t_{st} + k$ if $t_{st} + k \leq t < t_{st} + k + 1$, $k \in \mathbb{N}$, whereas asynchronous dynamics corresponds to more general occurrence times. Transition functions f_p correspond to the mathematical functions defining the state change of the cellular automaton. In more advanced CA, interactions may also depend on a suitable space of resources and goods (see e.g. [55]), on continuous state space with stochastic, non-Markovian or time-dependent rules. Even if every CA can also be seen as a discrete dynamical system, the setting as IS we are considering here allows one to preserve also the other structures of the automaton, such as neighborhoods and asynchronous dynamics, having in this way an embedding.

4.4. Agent based models. For ABM, we refer to the mathematical definition given in [58]. Although agents naturally correspond to interacting entities of IS, we have to consider that frequently ABM are identified with the corresponding implementation in an (object oriented) programming language, and the corresponding mathematical formalization is not always considered. In that case, the state space of an agent can also include its behavioral rules or methods, implemented as computer codes. Since IS is a mathematical theory, such methods have to be associated to a corresponding mathematical function, but this is clearly always possible because the semantic of every programming language always has a corresponding mathematical theory. More generally, even in case of mathematically formalized ABM, behavioral rules and methods are exactly transition functions in the language of IS. The environment itself, see e.g. [58], has to be considered as an interacting entity. Neighborhoods are defined by all the entities (agents or environment) from which the methods (interactions) take the information they need to operate. Note that, in case another agent is not contained in this neighborhood, it is completely hidden for the interactions of the considered agent. Since an interaction is of the form $i : a_1, \dots, a_n \xrightarrow{r, \alpha} p$, in general interactions are only of local nature, depending on the agents a_1, \dots, a_n, p , on r, α , and on the cause-effect relations with other agents. The dynamics is naturally asynchronous, depending on the signals r sent between agents: these correspond to propagator entities whose speed has to be modeled only in particular cases.

4.5. Master equation based models. In [22], we prove that the dynamics of a Markovian IS is described by a master equation. Any Markov model can be seen as a particular case of one of these Markovian IS. Therefore, this includes several

models used in synergetics, [51, 48, 49, 57]. Usually, additional structures such as propagators, starting, arrival and ongoing times, neighborhoods, etc. are not used in these descriptions.

4.6. Networked dynamical systems. For this kind of model of CS, see e.g. [45]. For all time $t \in [t_{\text{st}}, t_{\text{end}}]$, let $G_t = (V_t, L_t)$ be a graph with set of vertices/nodes V_t and set of edges/links L_t . Every node $e \in V_t$ has a state $x_e(t)$ belonging to a space S_e . This corresponds to an IS where interacting entities are all the nodes of the network plus unordered pairs of vertices (here, for simplicity, we consider only non-directed networks)

$$E := \bigcup_{t \in [t_{\text{st}}, t_{\text{end}}]} V_t \cup \{\{e_1, e_2\} \mid e_1, e_2 \in V_t\}.$$

The network can be easily formalized considering $\{0, 1\}$ -valued interactions between nodes and corresponding to the adjacency matrix G_t : At each time t , we define an interaction $i_l := (l, 1, \text{matr}, l)$ for each pair $l = \{e_1, e_2\} \in E$, with $\text{ag}(i_l) = \text{pa}(i_l) = l$ (and an abstract/trivial propagator), and where at each time t

$$f_l(t) = \begin{cases} 1 & l \in L_t \\ 0 & \text{otherwise} \end{cases}$$

We activate only V_t , i.e.

$$\text{ac}_i^e(t) = 1 \iff e \in V_t. \quad (4.1)$$

Each node interacts only with adjacent nodes (which is hence the neighborhood) and the transition functions correspond to the functions that update the state of each node, and can hence come, e.g., from the solution of suitable differential equations. As it is well known, the update algorithm can be synchronous or asynchronous, and as such it has to be implemented as the times of an IS. Similarly, one can embed as IS networked dynamical systems based on hypergraphs.

The setting (4.1) clearly states that the activation function is trivial in a networked dynamical system. We can hence state that IS allows one to implement a more detailed cause-effect structure using the activation function. Also propagators, and hence the spaces of resources, are not used in this formalization of networked dynamical systems as IS. On the other hand, we could say that IS can be seen as networked (stochastic) dynamical systems over a cause-effect weighted directed network with abstract weights given by propagators.

4.7. Artificial neural networks. As in the previous case, interacting entities are the neurons of the network. The structure of the network and the neighborhoods can be formalized, in the language of IS, using the adjacency matrix as above. The state of each neuron includes the values of the input variables, the bias for each one of these inputs, the activation function, the property of being a start or an end node. State space of propagators include the weights of the links. State space of neurons could also include the property of being active with respect to a change of the inputs or a change of the weights associated to its input links. The most important interactions depend on the type of learning and, in general, have propagators (and hence weights) as patient entities. In case of supervised or reinforcement learning, pairs of examples can be seen as stochastic goods of suitable propagators. We can also have neurons as patients if the learning algorithm changes their activation functions. The dynamics is in general synchronous. If one

is interested in computation times of activation functions, propagators times can also be considered.

4.8. Genetic algorithms. The population of candidate solutions (phenotypes) with their state space (genotype) are the interacting entities. Stochastic interactions are clearly the core of these models. Mutation, crossover, inversion and selection operators can be easily implemented as fitness depending interactions of an IS. The algorithm is synchronous, but asynchronous versions can also be implemented, e.g. by considering more fitted populations as single interacting entities that spread out their genetic code over the entire set of interacting entities. In this case, propagators can be considered, with their times. In this generalization, the introduction of suitable neighborhoods is also relevant.

5. CONCLUSIONS AND FUTURE DEVELOPMENTS

The present paper represents only the first necessary starting point to even imagine a mathematical theory of CS, i.e. the creation of a common universal mathematical language. The universality of IS theory allows one to be sure that sufficiently general mathematical results have a satisfactorily range of applications for a range of different modeling frameworks of CS. For theorems already going in this direction, see [22, 21]. Note that this does not force anyone to switch to IS from his favorite CS setting.

A precise mathematical universal language also provides the necessary setting to try a formalization of concepts such as that of CAS, of hierarchy of CS, of functors preserving cause-effect relations, etc., see [21] for a mathematical definition of CAS by following the idea of Zipf's *principle of least effort*, [59], and [23] for ideas about applications of these notions to a new approach to artificial intelligence.

Note that the embedding results we showed are not related in any way to universal machines: we do not restrict to recursive functions and, first of all, the embeddings are constructed by considering particular cases of IS without mentioning what kind of functions they are able to process.

On the contrary, we already noted that the universality of IS theory also includes several interesting generalization of well-known modeling frameworks for CS. In [55, 56, 54], we already applied this point of view by considering a strong generalization of the notion of CA for the practical motivations of creating validated models of urban growth and vehicular traffic. IS theory actually originated from these practical models, and from the observation that we actually were considering a very general setting applicable to a large class of CS.

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