

NON-AUTONOMOUS SOLITON HIERARCHIES

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ABSTRACT. A formalism of systematic construction of integrable non-autonomous deformations of soliton hierarchies is presented. The theory is formulated as an initial value problem (IVP) for an associated Frobenius integrability condition on a Lie algebra. It is showed that this IVP has a formal solution and within the framewrok of two particular subalgebras of the hereditary Lie algebra the explicit forms of this formal solution are presented. Finally, this formalism is applied to Kortevveg-de Vries, dispersive water waves and Ablowitz-Kaup-Newell-Segur soliton hierarchies.

1. INTRODUCTION

In this article we present a systematic method of deforming of commuting hierarchies of *autonomous* evolutionary flows, i.e. systems of evolutionary PDE's of the form

$$u_{t_n} = K_n[u], \quad n = 1, 2, \dots, \quad (1.1)$$

(where $u = u(x) = (u_1(x), \dots, u_N(x))^T$ and where each $K_n[u]$ is some vector field depending on u and a finite number of its x -derivatives, but not explicetely on times (i.e. evolution parameters t_i) and such that

$$[K_m, K_n] = 0, \quad m, n = 1, 2, \dots, \quad (1.2)$$

to the *non-autonomous* hierarchies of evolutionary flows

$$u_{t_n} = \mathbb{K}_n([u], x, t_1, \dots, t_n) \quad n = 1, 2, \dots, \quad (1.3)$$

that satisfy the Frobenius integrability condition

$$\frac{\partial \mathbb{K}_n}{\partial t_m} - \frac{\partial \mathbb{K}_m}{\partial t_n} + [\mathbb{K}_m, \mathbb{K}_n] = 0, \quad m, n = 1, 2, \dots \quad (1.4)$$

In (1.1) and (1.3) K_n and \mathbb{K}_n are vector fields, depending on u and a finite number of its x -derivatives on some infinite-dimensional functional manifold and K_n do not depend explicitly on times t_i . Note that in (1.3) we assume *triangular* dependence of vector fields \mathbb{K}_n on times t_i for $1 \leq i \leq n$, which in result simplifies the Frobenius condition (1.4) to

$$\frac{\partial \mathbb{K}_n}{\partial t_m} + [\mathbb{K}_m, \mathbb{K}_n] = 0, \quad m < n. \quad (1.5)$$

The condition (1.2) (which is nothing else than the Frobenius integrability condition for the autonomous system (1.1)) means that the system (1.1) has a common, multi-time solution through each initial condition $u(x, 0, 0, \dots) = u_0(x)$. Likewise, the Frobenius condition (1.4) means that the system (1.3) has a common, multi-time solution through each initial condition $u(x, 0, 0, \dots) = u_0(x)$. If these compatibility conditions are not met it makes no sense to consider the systems in (1.1) or the systems in (1.3) as hierarchies; they are simply not compatible.

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In order to highlight the main algebraic ingredients of our construction we first (Section 2) formulate our theory in a more general framework, as an initial-value problem (IVP) for \mathcal{A} -valued functions \mathbb{K}_n satisfying (1.5), where \mathcal{A} is a non-abelian Lie algebra. In Theorem 3 and Corollary 4 we show that this IVP has a formal solution. Next, we present solutions of the IVP for particular subalgebras of the hereditary Lie algebra [1, 2, 3] (Section 3) and then we apply these results to soliton hierarchies (Section 4). In Section 4 we also find the zero-curvature representations for the non-autonomous hierarchies (1.3) from zero-curvature representations of the corresponding autonomous hierarchies (1.1). We illustrate our method on three examples: KdV hierarchy (Section 5), dispersive water wave (DWW) hierarchy in the framework of [4, 5] (Section 5) and Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy [6] (Section 7).

We believe that the results presented in this article are important as the majority of research in the theory of integrable PDE's focuses on autonomous systems of type (1.1). To our best knowledge, the non-autonomous deformations (1.3) of soliton hierarchies have not been previously studied. The usual approach to non-autonomous soliton equations is to modify a single chosen soliton equation by assuming some time-dependence of one or more of its coefficients, see for example [7, 8] or [9].

This article was inspired by our results presented in [10, 11, 12], where we have considered polynomial in times deformations of autonomous Liouville integrable finite dimensional systems, i.e. systems of the form

$$\begin{aligned} \frac{dx}{dt_i} &= \pi dh_i, & h_i &= h_i(x), & i &= 1, \dots, n, \\ \{h_i, h_j\}_\pi &= 0, & i, j &= 1, \dots, n, \end{aligned}$$

on some $2n$ -dimensional manifold equipped with a Poisson bivector π , to non-autonomous Frobenius integrable systems

$$\begin{aligned} \frac{dx}{dt_i} &= \pi dH_i, & H_i &= H_i(x, t_1, \dots, t_n), & i &= 1, \dots, n, \\ \{H_i, H_j\}_\pi + \frac{\partial H_i}{\partial t_j} - \frac{\partial H_j}{\partial t_i} &= 0, & i, j &= 1, \dots, n, \end{aligned}$$

on the same manifold; here x denotes points on this manifold. Another inspiration for this work was the article [13] in which the author constructed non-autonomous KdV hierarchies from Painlevé systems that were obtained as deformations of Stäckel separable systems. This construction, however, was dependent on the dimension n of the underlying Stäckel systems; increasing n led to a completely different finite hierarchy. This drawback is not present in the theory we develop in this article.

2. FROBENIUS INTEGRABILITY CONDITION IN LIE ALGEBRAS

In this section we present a *formal* solution to the Frobenius integrability condition (1.5) formulated as an initial-value problem (IVP) for finite or infinite sets $\{\mathbb{K}_0, \mathbb{K}_1, \dots\}$ of elements \mathbb{K}_n that belong to a non-abelian Lie algebra \mathcal{A} and such that each element \mathbb{K}_n depends on (at most) $n + 1$ real evolutionary parameters (times) t_i , so that

$$\mathbb{K}_n = \mathbb{K}_n(t_0, \dots, t_n). \quad (2.1)$$

Thus, all elements \mathbb{K}_n will be some \mathcal{A} -valued functions of a finite number of real parameters t_i . The word *formal* means in this context that we do not consider any convergence issues that may arise in the formulas presented in this section. However, in the next sections we will apply our theory to soliton hierarchies and then all the expressions appearing in this section will become convergent (in fact, finite); the \mathcal{A} -valued functions \mathbb{K}_n will then become Frobenius integrable deformations of soliton hierarchies.

Definition 1. We say that the set (finite or not) $\{\mathbb{K}_0, \mathbb{K}_1, \dots\}$ of \mathcal{A} -valued functions $\mathbb{K}_i = \mathbb{K}_i(t_0, \dots, t_i)$ satisfies the Frobenius integrability condition (in a triangular form) if

$$\frac{\partial \mathbb{K}_j}{\partial t_i} + \text{ad}_{\mathbb{K}_i} \mathbb{K}_j = 0, \quad 0 \leq i < j. \quad (2.2)$$

Here and in what follows $\text{ad}_{\mathbb{K}_i}$ is the adjoint action in the Lie algebra \mathcal{A} so that $\text{ad}_{\mathbb{K}_i} \mathbb{K}_j \equiv [\mathbb{K}_i, \mathbb{K}_j]$. Note that the *triangular* dependence of \mathbb{K}_n on t_i given in (2.1) means that (2.2) is the actual complete Frobenius condition

$$\frac{\partial \mathbb{K}_j}{\partial t_i} - \frac{\partial \mathbb{K}_i}{\partial t_j} + [\mathbb{K}_i, \mathbb{K}_j] = 0, \quad 0 \leq i < j,$$

simply because (2.1) implies that $\frac{\partial \mathbb{K}_i}{\partial t_j} = 0$ for $i < j$.

We begin our exposition by presenting a well-known fact.

Proposition 2. *Consider the following linear initial value problem*

$$\frac{d\mathbf{v}}{dt} + \mathbb{A}\mathbf{v} = 0, \quad \mathbf{v}(0) = \mathbf{v}_0, \quad (2.3)$$

where $\mathbb{A} = \mathbb{A}(t)$ is some time-dependent linear operator acting in \mathcal{A} , $\mathbf{v} = \mathbf{v}(t)$ is an \mathcal{A} -valued function and where $\mathbf{v}_0 \in \mathcal{A}$. Then the formal solution of this problem is

$$\begin{aligned} \mathbf{v}(t) &= \mathbf{v}_0 - \partial_t^{-1} \mathbb{A} \mathbf{v}_0 + \partial_t^{-1} \mathbb{A} \partial_t^{-1} \mathbb{A} \mathbf{v}_0 - \partial_t^{-1} \mathbb{A} \partial_t^{-1} \mathbb{A} \partial_t^{-1} \mathbb{A} \mathbf{v}_0 + \dots \\ &\equiv (1 + \partial_t^{-1} \mathbb{A})^{-1} \mathbf{v}_0, \end{aligned}$$

where

$$\partial_t^{-1} = \int_0^t dt'$$

is the formal linear operator of definite integration with respect to the time variable t so that $\partial_t^{-1} \mathbb{A}$ is a t -dependent operator on \mathcal{A} .

Proof. Indeed, given that for any linear operator \mathbb{B} in \mathcal{A} we have, formally,

$$(1 + \mathbb{B})^{-1} \equiv 1 - \mathbb{B} + \mathbb{B}^2 - \mathbb{B}^3 + \dots = 1 - \mathbb{B} (1 + \mathbb{B})^{-1}, \quad (2.4)$$

then, taking $\mathbb{B} = \partial_t^{-1} \mathbb{A}$, we obtain

$$\begin{aligned} \frac{d\mathbf{v}(t)}{dt} &= \partial_t (1 + \partial_t^{-1} \mathbb{A})^{-1} \mathbf{v}_0 = \partial_t (\mathbf{v}_0 - \partial_t^{-1} \mathbb{A} (1 + \partial_t^{-1} \mathbb{A})^{-1} \mathbf{v}_0) \\ &= -\mathbb{A} (1 + \partial_t^{-1} \mathbb{A})^{-1} \mathbf{v}_0 = -\mathbb{A} \mathbf{v}(t). \end{aligned}$$

□

In the case \mathbb{A} does not depend on t we obtain the well known solution of the IVP (2.3) in the form of the exponential of \mathbb{A} :

$$\frac{d\mathbb{A}}{dt} = 0 \quad \Rightarrow \quad \mathbf{v}(t) = (1 + \partial_t^{-1} \mathbb{A})^{-1} \mathbf{v}_0 = \exp(-t\mathbb{A}) \mathbf{v}_0. \quad (2.5)$$

We will now generalize this result to the case of a system of linear time-dependent equations. As notified above, we will work with \mathcal{A} -valued functions of some parameters t_i , which we will often call *times*.

Theorem 3. *Suppose that n \mathcal{A} -valued functions $\mathbb{K}_0, \dots, \mathbb{K}_{n-1}$ satisfy the following conditions:*

$$\mathbb{K}_i = \mathbb{K}_i(t_0, \dots, t_i), \quad i = 0, 1, \dots, n-1, \quad (2.6a)$$

$$\frac{\partial \mathbb{K}_j}{\partial t_i} + \text{ad}_{\mathbb{K}_i} \mathbb{K}_j = 0, \quad 0 \leq i < j < n. \quad (2.6b)$$

Then the initial value problem

$$\frac{\partial \mathbb{K}}{\partial t_i} + \text{ad}_{\mathbb{K}_i} \mathbb{K} = 0, \quad i = 0, \dots, n-1, \quad (2.7a)$$

$$\mathbb{K}(0, \dots, 0) = \bar{\mathbb{K}} \in \mathcal{A}. \quad (2.7b)$$

for the \mathcal{A} -valued function $\mathbb{K} = \mathbb{K}(t_0, \dots, t_{n-1})$ has the formal solution

$$\mathbb{K}(t_0, \dots, t_{n-1}) = (1 + \partial_{t_{n-1}}^{-1} \text{ad}_{\mathbb{K}_{n-1}})^{-1} \cdots (1 + \partial_{t_1}^{-1} \text{ad}_{\mathbb{K}_1})^{-1} (1 + \partial_{t_0}^{-1} \text{ad}_{\mathbb{K}_0})^{-1} \bar{\mathbb{K}}, \quad (2.8)$$

where

$$\partial_{t_i}^{-1} f(t_i) \equiv \int_0^{t_i} f(t) dt.$$

Assuming that $\bar{\mathbb{K}}$ in (2.7b) depends on an additional evolution parameter t_n we arrive at the following corollary.

Corollary 4. *Suppose that n \mathcal{A} -valued functions $\mathbb{K}_0, \dots, \mathbb{K}_{n-1}$ satisfy the conditions (2.6). Suppose also that an \mathcal{A} -valued function $\mathbb{K}_n = \mathbb{K}_n(t_0, \dots, t_n)$ satisfies the following initial value problem*

$$\frac{\partial \mathbb{K}_n}{\partial t_i} + \text{ad}_{\mathbb{K}_i} \mathbb{K}_n = 0, \quad 0 \leq i < n, \quad (2.9a)$$

$$\mathbb{K}_n(\underbrace{0, \dots, 0}_n, t_n) = \bar{\mathbb{K}}_n(t_n), \quad (2.9b)$$

where $\bar{\mathbb{K}}_n(t_n)$ is an \mathcal{A} -valued function of t_n . Then, the IVP (2.9) has the unique (formal) solution

$$\mathbb{K}_n(t_0, \dots, t_n) = (1 + \partial_{t_{n-1}}^{-1} \text{ad}_{\mathbb{K}_{n-1}})^{-1} \cdots (1 + \partial_{t_1}^{-1} \text{ad}_{\mathbb{K}_1})^{-1} (1 + \partial_{t_0}^{-1} \text{ad}_{\mathbb{K}_0})^{-1} \bar{\mathbb{K}}_n(t_n). \quad (2.10)$$

For the proof of Theorem 3, see Appendix A. Let us now comment this theorem and the corollary that follows. The conditions (2.6) means that the \mathcal{A} -valued functions $\mathbb{K}_0, \dots, \mathbb{K}_{n-1}$ satisfy the Frobenius integrability condition (2.2). Further in the article the \mathcal{A} -valued functions \mathbb{K}_i are vector fields on some finite or infinite-dimensional manifold \mathcal{M} , so that $\mathbb{K}_i = \mathbb{K}_i(t_0, \dots, t_i, u)$ with $u \in \mathcal{M}$, and the Frobenius condition (2.2) in turn implies that the corresponding non-autonomous dynamical systems

$$\frac{du}{dt_i} = \mathbb{K}_i(t_0, \dots, t_i, u), \quad i = 0, \dots, n-1,$$

posses (at least locally) a common, multi-time solution $u = u(t_0, \dots, t_{n-1}; u_0)$ through each point u_0 of the manifold \mathcal{M} . Theorem 3 yields us then a tool to add one more vector field \mathbb{K} to the set $\{\mathbb{K}_0, \dots, \mathbb{K}_{n-1}\}$ so that the set $\{\mathbb{K}_0, \dots, \mathbb{K}_{n-1}, \mathbb{K}_n \equiv \mathbb{K}\}$ still satisfies the Frobenius integrability condition (2.2). If now the initial condition $\bar{\mathbb{K}}$ for (2.7) depends additionally on a new evolution parameter t_n then the solution (2.10) in Corollary 4 provides us with the set $\{\mathbb{K}_0, \dots, \mathbb{K}_{n-1}, \mathbb{K}_n \equiv \mathbb{K}\}$ of non-autonomous vector fields, depending now on one more evolution parameter t_n , and such that they satisfy the Frobenius condition (2.2) for $0 \leq i < j \leq n$.

Thus, Corollary 4 is a useful tool to recursively construct sets of non-autonomous vector fields, *triangularly* depending on t_i and satisfying the Frobenius condition. This corollary will be used in the following sections to produce non-autonomous Frobenius integrable deformations of various soliton hierarchies.

Let us conclude this section with a specification of Theorem 3 to the situation when $\mathbb{K}_i = \mathbb{K}_i(t_0, \dots, t_{i-1})$ only (in such situation we say that neither of \mathbb{K}_i depends on its own evolution parameter t_i). In this particular situation, in accordance with (2.5),

$$(1 + \partial_{t_i}^{-1} \text{ad}_{\mathbb{K}_i})^{-1} = \exp(-t_i \text{ad}_{\mathbb{K}_i}),$$

and thus we have the following remark.

Remark 5. If $\mathbb{K}_i = \mathbb{K}_i(t_0, \dots, t_{i-1})$ for $i = 0, \dots, n-1$ and $\bar{\mathbb{K}}_n(t_n) = \bar{\mathbb{K}}_n$, i.e. if neither of $\bar{\mathbb{K}}_i$ depends on its own evolution parameter t_i , then the solution of the IVP (2.9) takes the form

$$\mathbb{K}_n(t_0, \dots, t_{n-1}) = \exp(-t_{n-1} \operatorname{ad}_{\mathbb{K}_{n-1}}) \cdots \exp(-t_1 \operatorname{ad}_{\mathbb{K}_1}) \exp(-t_0 \operatorname{ad}_{\mathbb{K}_0}) \bar{\mathbb{K}}_n. \quad (2.11)$$

Naturally, in this case

$$\mathbb{K}_n(0, \dots, 0) = \bar{\mathbb{K}}_n.$$

3. FROBENIUS INTEGRABILITY IN HEREDITARY ALGEBRAS

The formulas (2.10) and (2.11) cannot be of any practical use until we have some method to compute the expressions on their right hand sides. To achieve this we will assume that our non-abelian Lie algebra \mathcal{A} is a semi-product of a commutative algebra and the Witt algebra (a centerless Virasoro symmetry algebra), i.e. \mathcal{A} is the so-called hereditary algebra [1, 2, 3]. Specifically, we assume that the hereditary algebra \mathcal{A} is generated by the elements $K_n \in \mathcal{A}$, $n = 1, 2, \dots$, and $\sigma_m \in \mathcal{A}$, $m = -1, 0, 1, \dots$, such that

$$[K_n, K_m] = 0, \quad [\sigma_n, K_m] = (\alpha m + \rho' - 1)K_{n+m}, \quad [\sigma_n, \sigma_m] = \alpha(m - n)\sigma_{n+m},$$

where $\rho', \alpha \in \mathbb{R}$ and $\alpha \neq 0$. This choice is motivated by the fact that hereditary algebras of soliton hierarchies, that we will consider in this article, have this structure. By a simple rescaling of all σ_n we can always set $\alpha = 1$. Thus, in this article we will consider the hereditary algebra \mathcal{A} with the generators K_n and σ_m satisfying the commutation relations

$$\begin{aligned} [K_n, K_m] &= 0, & m, n &= 1, 2, \dots, \\ [\sigma_n, K_m] &= \kappa_m K_{n+m}, & n &= -1, 0, 1, \dots, \quad m = 1, 2, \dots, \\ [\sigma_n, \sigma_m] &= (m - n)\sigma_{n+m}, & m, n &= -1, 0, 1, \dots, \end{aligned} \quad (3.1)$$

(so that $\rho - 1 \equiv \frac{1}{\alpha}(\rho' - 1)$) and where we denote $K_0 \equiv 0$. Here and further on we use the notation

$$\kappa_m \equiv \rho + m - 1. \quad (3.2)$$

We can now choose our initial conditions $\bar{\mathbb{K}}_n(t_n)$ in (2.9b) as some very particular deformations (times-dependent linear combinations) of the above generators of \mathcal{A} . It turns out that formulas (2.10) and (2.11) attain a compact, finite form in two particular cases:

- (1) $\bar{\mathbb{K}}_n(t_n)$ is a \mathcal{A}_{-1} -valued function for all n ,
- (2) $\bar{\mathbb{K}}_n(t_n)$ is a \mathcal{A}_0 -valued function for all n ,

where

$$\mathcal{A}_{-1} := \operatorname{span}\{\sigma_{-1}, K_1, K_2, \dots\} \quad \text{and} \quad \mathcal{A}_0 := \operatorname{span}\{\sigma_0, K_1, K_2, \dots\}$$

are two particular subalgebras of the hereditary algebra \mathcal{A} . They are exceptional in the sense that for any n the sets $\mathcal{A}_\varepsilon^{(n)} \equiv \operatorname{span}\{\sigma_\varepsilon, K_1, K_2, \dots, K_n\}$, where $\varepsilon = -1$ or $\varepsilon = 0$, are finite dimensional subalgebras of \mathcal{A} . On the level of the algebras \mathcal{A}_ε we can interpret the solutions \mathbb{K}_n of IVP (2.9) as leading to construction of new deformed bases of \mathcal{A}_ε satisfying the Frobenius integrability condition (2.2).

3.1. Frobenius integrability in the hereditary subalgebra \mathcal{A}_{-1} . We begin with the first case, i.e. when $\bar{\mathbb{K}}_n(t_n)$ is a \mathcal{A}_{-1} -valued function.

Theorem 6. *Consider the IVP (2.9) with the initial conditions (2.9b) in the form*

$$\mathbb{K}_n(0, \dots, 0, t_n) = \bar{\mathbb{K}}_n(t_n) \equiv \sigma_{-1} + \sum_{i=1}^n \mathbf{a}_{n,i}(t_n) K_i, \quad n = 0, 1, \dots, \quad (3.3)$$

where $\mathbf{a}_{n,i}(t_n)$ are arbitrary differentiable functions (thus $\bar{\mathbb{K}}_n(t_n)$ are \mathcal{A}_{-1} -valued functions). Then, the solution (2.10) of the IVP (2.9) is unique and attains the form

$$\mathbb{K}_n = \sigma_{-1} + \sum_{i=1}^n \mathbf{u}_{n,i}(t_0, \dots, t_n) K_i, \quad (3.4)$$

where

$$\mathbf{u}_{n,1} = -\mathbf{c}_n, \quad \mathbf{u}_{n,i} = \frac{(-1)^i}{[\kappa_i]!} \partial_{t_0}^{i-1} \mathbf{c}_n, \quad i = 2, \dots, n, \quad (3.5)$$

with

$$\begin{aligned} \mathbf{c}_n(t_0, \dots, t_n) \equiv & \sum_{m=2}^{n-1} \sum_{r=2}^m \sum_{s=1}^{r-1} \frac{(-1)^{r-1} [\kappa_r]!}{(r-s-1)!} (\tau_{m-1})^{r-s-1} (\partial_{t_m}^{-1})^s \mathbf{a}_{m,r}(t_m) \\ & + \sum_{r=1}^n \frac{(-1)^r [\kappa_r]!}{(r-1)!} (\tau_{n-1})^{r-1} \mathbf{a}_{n,r}(t_n), \quad n \in \mathbb{N}. \end{aligned} \quad (3.6)$$

Here and in what follows we use the shorthand notations:

$$\tau_m = t_0 + t_1 + \dots + t_m, \quad [\kappa_r]! = \kappa_2 \kappa_3 \cdots \kappa_{r-1} \kappa_r, \quad r > 1, \quad [\kappa_1]! = 1, \quad (3.7)$$

and

$$\partial_{t_m}^{-1} f(t_m) \equiv \int_0^{t_m} f(t) dt.$$

Notice that, the first term in (3.6) disappears for \mathbf{c}_1 and \mathbf{c}_2 . Also notice that, after combining (3.5) with (3.6) and simplifying, one can see that $\mathbf{u}_{n,i}$ are always polynomial in variables κ_m . This means that there is no issue with division by zero if one of (3.2), for some particular ρ , is equal to zero.

Proof. Direct calculation yields

$$\mathbf{c}_n(0, \dots, 0, t_n) = -\mathbf{a}_{n,1}(t_n).$$

Moreover,

$$\begin{aligned} (\mathbf{c}_n)_{t_0} = & \sum_{m=2}^{n-1} \sum_{r=3}^m \sum_{s=1}^{r-2} \frac{(-1)^{r-1} [\kappa_r]!}{(r-s-2)!} (\tau_{m-1})^{r-s-2} (\partial_{t_m}^{-1})^s \mathbf{a}_{m,r}(t_m) \\ & + \sum_{r=2}^n \frac{(-1)^r [\kappa_r]!}{(r-2)!} (\tau_{n-1})^{r-2} \mathbf{a}_{n,r}(t_n), \end{aligned}$$

so that

$$(\partial_{t_0} \mathbf{c}_n)(0, \dots, 0, t_n) = \kappa_2 \mathbf{a}_{n,2}(t_n).$$

Continuing differentiation of \mathbf{c}_n with respect to t_0 we obtain

$$(\partial_{t_0}^{i-1} \mathbf{c}_n)(0, \dots, 0, t_n) = (-1)^i [\kappa_i]! \mathbf{a}_{n,i}(t_n), \quad i = 2, 3, \dots, n.$$

That means that $\mathbb{K}_n(0, \dots, 0, t_n) = \bar{\mathbb{K}}_n(t_n)$ and thus (3.4) satisfies (for each n) the initial conditions (3.3). Further, (3.4) satisfies (2.9a) if and only if

$$(\mathbf{u}_{n,n})_{t_j} = 0, \quad (3.8a)$$

$$(\mathbf{u}_{n,i})_{t_j} + \kappa_i \mathbf{u}_{n,i+1} = 0, \quad j \leq i \leq n-1, \quad (3.8b)$$

$$(\mathbf{u}_{n,i})_{t_j} + \kappa_i (\mathbf{u}_{n,i+1} - \mathbf{u}_{j,i+1}) = 0, \quad 1 \leq i \leq j-1, \quad (3.8c)$$

see Appendix B with $\epsilon = -1$. In Appendix C it is showed that

$$(\mathbf{c}_n)_{t_j} \equiv (\mathbf{c}_n)_{t_0} - (\mathbf{c}_j)_{t_0}, \quad 1 \leq j \leq n-1, \quad (3.9)$$

and hence

$$\partial_{t_0}^{i-1} (\mathbf{c}_n)_{t_j} = \partial_{t_0}^i \mathbf{c}_n - \partial_{t_0}^i \mathbf{c}_j, \quad 1 \leq i \leq n-1,$$

while $\partial_{t_0}^i \mathbf{c}_j = 0$ for $i \geq j$. These properties and the fact that $[\kappa_{i+1}]! \equiv [\kappa_i]! \kappa_{i+1}$ imply that the conditions (3.8) are identically satisfied. So, all \mathbb{K}_n given by (3.4) satisfy the IVP (2.9) with (2.9b) being of the particular form (3.3). Since the conditions (3.8), being linear equations, have a unique solution for any initial conditions, the solutions (3.4) and (2.10) must – for the chosen initial conditions – coincide. \square

The first few non-autonomous vectors (3.4) for the general IVP (3.3) have the form

$$\begin{aligned} \mathbb{K}_0 &= \sigma_{-1}, \\ \mathbb{K}_1 &= \sigma_{-1} + \mathbf{a}_{1,1}(t_1) K_1, \\ \mathbb{K}_2 &= \sigma_{-1} + \left[\mathbf{a}_{2,1}(t_2) - \kappa_2 \tau_1 \mathbf{a}_{2,2}(t_2) \right] K_1 + \mathbf{a}_{2,2}(t_2) K_2, \\ \mathbb{K}_3 &= \sigma_{-1} + \left[\kappa_2 \partial_{t_2}^{-1} \mathbf{a}_{2,2}(t_2) + \mathbf{a}_{3,1}(t_3) - \kappa_2 \tau_2 \mathbf{a}_{3,2}(t_3) + \frac{1}{2} \kappa_2 \kappa_3 \tau_2^2 \mathbf{a}_{3,3}(t_3) \right] K_1 \\ &\quad + \left[\mathbf{a}_{3,2}(t_3) - \kappa_3 \tau_2 \mathbf{a}_{3,3}(t_3) \right] K_2 + \mathbf{a}_{3,3}(t_3) K_3, \\ \mathbb{K}_4 &= \sigma_{-1} + \left[\kappa_2 \partial_{t_2}^{-1} \mathbf{a}_{2,2}(t_2) + \kappa_2 \partial_{t_3}^{-1} \mathbf{a}_{3,2}(t_3) - \kappa_2 \kappa_3 \tau_2 \partial_{t_3}^{-1} \mathbf{a}_{3,3}(t_3) - \kappa_2 \kappa_3 \partial_{t_3}^{-2} \mathbf{a}_{3,3}(t_3) \right. \\ &\quad \left. + \mathbf{a}_{4,1}(t_4) - \kappa_2 \tau_3 \mathbf{a}_{4,2}(t_4) + \frac{1}{2} \kappa_2 \kappa_3 \tau_3^2 \mathbf{a}_{4,3}(t_4) - \frac{1}{6} \kappa_2 \kappa_3 \kappa_4 \tau_3^3 \mathbf{a}_{4,4}(t_4) \right] K_1 \\ &\quad + \left[\kappa_3 \partial_{t_3}^{-1} \mathbf{a}_{3,3}(t_3) + \mathbf{a}_{4,2}(t_4) - \kappa_3 \tau_3 \mathbf{a}_{4,3}(t_4) + \frac{1}{2} \kappa_3 \kappa_4 \tau_3^2 \mathbf{a}_{4,4}(t_4) \right] K_2 \\ &\quad + \left[\mathbf{a}_{4,3}(t_4) - \kappa_4 \tau_3 \mathbf{a}_{4,4}(t_4) \right] K_3 + \mathbf{a}_{4,4}(t_4) K_4, \end{aligned}$$

where κ_m are defined by (3.2) and τ_m by (3.7).

Example 7. Suppose that the functions $\mathbf{a}_{n,i}(t_n)$ in the initial conditions (3.3) are given by the simple choice $\mathbf{a}_{n,i}(t_n) = 0$ for $i = 1, \dots, n-1$, and $\mathbf{a}_{n,n}(t_n) = t_n$. Then the initial conditions (3.3) have the form

$$\bar{\mathbb{K}}_0(t_0) \equiv \sigma_{-1}, \quad \bar{\mathbb{K}}_n(t_n) \equiv \sigma_{-1} + t_n K_i, \quad n = 1, 2, \dots, \quad (3.10)$$

and the solution of the IVP (2.9) is given by (3.4) and (3.5) with \mathbf{c}_n in the form

$$\mathbf{c}_n(t_0, \dots, t_n) = \sum_{m=2}^{n-1} \sum_{s=2}^m \frac{(-1)^{m-1} [\kappa_m]!}{(m-s)! s!} (\tau_{m-1})^{m-s} t_m^s + \frac{(-1)^n [\kappa_n]!}{(n-1)!} (\tau_{n-1})^{n-1} t_n. \quad (3.11)$$

Then, using (3.5) with (3.11) we obtain the first \mathbb{K}_n in (3.4) in the explicit form

$$\begin{aligned} \mathbb{K}_0 &= \sigma_{-1}, \\ \mathbb{K}_1 &= \sigma_{-1} + t_1 K_1, \\ \mathbb{K}_2 &= \sigma_{-1} - \kappa_2 (t_0 + t_1) t_2 K_1 + t_2 K_2, \\ \mathbb{K}_3 &= \sigma_{-1} + \frac{1}{2} \kappa_2 [t_2^2 + \kappa_3 (t_0 + t_1 + t_2)^2 t_3] K_1 - \kappa_3 (t_0 + t_1 + t_2) t_3 K_2 + t_3 K_3, \\ \mathbb{K}_4 &= \sigma_{-1} + \frac{1}{2} \kappa_2 \left[t_2^2 - \frac{1}{3} \kappa_3 (3t_0 + 3t_1 + 3t_2 + t_3) t_3^2 - \frac{1}{3} \kappa_3 \kappa_4 (t_0 + t_1 + t_2 + t_3)^3 t_4 \right] K_1 \\ &\quad + \frac{1}{2} \kappa_3 [t_3^2 + \kappa_4 (t_0 + t_1 + t_2 + t_3)^2 t_4] K_2 - \kappa_4 (t_0 + t_1 + t_2 + t_3) t_4 K_3 + t_4 K_4. \end{aligned}$$

3.2. Frobenius integrability in the hereditary subalgebra \mathcal{A}_0 . The second possibility arises when the initial conditions are given by \mathcal{A}_0 -valued functions.

Theorem 8. Consider the IVP (2.9) with the initial conditions (2.9b) in the form

$$\mathbb{K}_n(0, \dots, 0, t_n) = \bar{\mathbb{K}}_n(t_n) \equiv \sigma_0 + \sum_{i=1}^n \mathbf{a}_{n,i}(t_n) K_i, \quad n = 0, 1, 2, \dots, \quad (3.12)$$

where $\mathbf{a}_{n,i}(t_n)$ are arbitrary differentiable functions (thus $\bar{\mathbb{K}}_n(t_n)$ is an \mathcal{A}_0 -valued function). Then, the solution (2.10) is unique and attains the form

$$\mathbb{K}_n = \sigma_0 + \sum_{i=1}^n \mathbf{u}_{n,i}(t_0, \dots, t_n) K_i, \quad (3.13)$$

where

$$\mathbf{u}_{n,i}(t_0, \dots, t_n) = \sum_{r=i}^{n-1} \mathbf{c}_{r,i}(t_r) e^{-\kappa_i \tau_{r-1}} + \mathbf{a}_{n,i}(t_n) e^{-\kappa_i \tau_{n-1}}, \quad (3.14)$$

τ_m are again given by (3.7) and where $\mathbf{c}_{r,i}(t_r)$ are functions that satisfy non-homogeneous linear IVPs

$$\mathbf{c}'_{r,i}(t_r) + \kappa_i \mathbf{c}_{r,i}(t_r) = \kappa_i \mathbf{a}_{r,i}(t_r), \quad \mathbf{c}_{r,i}(0) = 0. \quad (3.15)$$

Note that the solution of (3.15) is

$$\mathbf{c}_{r,i}(t_r) = \kappa_i e^{-\kappa_i t_r} \int_0^{t_r} \mathbf{a}_{r,i}(t) e^{\kappa_i t} dt. \quad (3.16)$$

In particular, for the choice $\mathbf{a}_{r,i}(t_i) = t_i^m$, $m = 0, 1, \dots$,

$$\mathbf{c}_{r,i}(t_r) = m! \left(\sum_{k=0}^m \frac{(-\kappa_i)^{k-m}}{k!} t_r^k - e^{-\kappa_i t_r} \right), \quad \kappa_i \neq 0. \quad (3.17)$$

Proof. Fix $n \in \mathbb{N}$ and assume that the solution \mathbb{K}_r given by (3.13) satisfy the IVP (2.9) for all $r < n$. Clearly, $\mathbb{K}_n(0, \dots, 0, t_n) = \bar{\mathbb{K}}_n(t_n)$ so (3.13) satisfy (for each n) the initial conditions (3.12). Further, (3.13) satisfies (2.9a) if and only if

$$(\mathbf{u}_{n,i})_{t_j} + \kappa_i \mathbf{u}_{n,i} = 0, \quad j+1 \leq i \leq n, \quad (3.18a)$$

$$(\mathbf{u}_{n,i})_{t_j} + \kappa_i (\mathbf{u}_{n,i} - \mathbf{u}_{j,i}) = 0, \quad 1 \leq i \leq j. \quad (3.18b)$$

For proof of (3.18) see Appendix B with $\epsilon = 0$. Note that the equations (3.18) are linear and thus possess solutions for all t_j . Consider (3.14), then for $j+1 \leq i \leq n$ we have

$$\begin{aligned} (\mathbf{u}_{n,i})_{t_j} &= -\kappa_i \sum_{r=i}^{n-1} \mathbf{c}_{r,i}(t_r) e^{-\kappa_i \tau_{r-1}} - \kappa_i \mathbf{a}_{n,i}(t_n) e^{-\kappa_i \tau_{n-1}} \\ &\equiv -\kappa_i \mathbf{u}_{n,i}, \end{aligned}$$

so that (3.18a) is identically true, while for $1 \leq i \leq j$

$$(\mathbf{u}_{n,i})_{t_j} = \mathbf{c}'_{j,i}(t_j) e^{-\kappa_i \tau_{j-1}} - \kappa_i \sum_{r=j+1}^{n-1} \mathbf{c}_{r,i}(t_r) e^{-\kappa_i \tau_{r-1}} - \kappa_i \mathbf{a}_{n,i}(t_n) e^{-\kappa_i \tau_{n-1}}$$

and thus

$$(\mathbf{u}_{n,i})_{t_j} + \kappa_i (\mathbf{u}_{n,i} - \mathbf{u}_{j,i}) = (\mathbf{c}'_{j,i}(t_j) + \kappa_i [\mathbf{c}_{j,i}(t_j) - \mathbf{a}_{j,i}(t_j)]) e^{-\kappa_i \tau_{j-1}},$$

so that (3.18b) is satisfied provided that the differential equations (3.15) hold. As result, all (3.13) satisfy the IVP (2.9) with (2.9b) being of the particular form (3.12) and so, by uniqueness of solutions of (3.18) for any initial conditions, the solutions (3.13) and (2.10) must – for the chosen initial conditions – coincide. \square

The first few non-autonomous vectors (3.13) for the general IVP (3.12) have the form

$$\mathbb{K}_0 = \sigma_0,$$

$$\begin{aligned}
\mathbb{K}_1 &= \sigma_0 + \mathbf{a}_{1,1}(t_1)e^{-\kappa_1\tau_0}K_1, \\
\mathbb{K}_2 &= \sigma_0 + \left[\mathbf{c}_{1,1}(t_1)e^{-\kappa_1\tau_0} + \mathbf{a}_{2,1}(t_2)e^{-\kappa_1\tau_1} \right] K_1 + \mathbf{a}_{2,2}(t_2)e^{-\kappa_2\tau_1}K_2, \\
\mathbb{K}_3 &= \sigma_0 + \left[\mathbf{c}_{1,1}(t_1)e^{-\kappa_1\tau_0} + \mathbf{c}_{2,1}(t_2)e^{-\kappa_1\tau_1} + \mathbf{a}_{3,1}(t_3)e^{-\kappa_1\tau_2} \right] K_1 \\
&\quad + \left[e^{-\kappa_2\tau_1}\mathbf{c}_{2,2}(t_2) + \mathbf{a}_{3,2}(t_3)e^{-\kappa_2\tau_2} \right] K_2 + \mathbf{a}_{3,3}(t_3)e^{-\kappa_3\tau_2}K_3, \\
\mathbb{K}_4 &= \sigma_0 + \left[\mathbf{c}_{1,1}(t_1)e^{-\kappa_1\tau_0} + \mathbf{c}_{2,1}(t_2)e^{-\kappa_1\tau_1} + \mathbf{c}_{3,1}(t_3)e^{-\kappa_1\tau_2} + \mathbf{a}_{4,1}(t_4)e^{-\kappa_1\tau_3} \right] K_1 \\
&\quad + \left[\mathbf{c}_{2,2}(t_2)e^{-\kappa_2\tau_1} + \mathbf{c}_{3,2}(t_3)e^{-\kappa_2\tau_2} + \mathbf{a}_{4,2}(t_4)e^{-\kappa_2\tau_3} \right] K_2 \\
&\quad + \left[\mathbf{c}_{3,3}(t_3)e^{-\kappa_3\tau_2} + \mathbf{a}_{4,3}(t_4)e^{-\kappa_3\tau_3} \right] K_3 + \mathbf{a}_{4,4}(t_4)e^{-\kappa_4\tau_3}K_4,
\end{aligned}$$

where κ_m are defined by (3.2), τ_m by (3.7) and $\mathbf{c}_{r,i}(t_r)$ are given by (3.16).

Example 9. Suppose that the functions $\mathbf{a}_{n,i}(t_n)$ in the initial conditions (3.12) are given by $\mathbf{a}_{n,i}(t_n) = 0$ for $i = 1, \dots, n-1$ and $\mathbf{a}_{n,n}(t_n) = t_n$, so that

$$\bar{\mathbb{K}}_0(t_0) \equiv \sigma_{-1}, \quad \bar{\mathbb{K}}_n(t_n) \equiv \sigma_{-1} + t_n K_i, \quad n = 1, 2, \dots \quad (3.19)$$

In this case, by (3.16) or (3.17)

$$\mathbf{c}_{n,i}(t_n) = 0, \quad i = 1, \dots, n-1, \quad \mathbf{c}_{n,n}(t_n) = t_n + \frac{1}{\kappa_n} (e^{-\kappa_n t_n} - 1), \quad \kappa_n \neq 0. \quad (3.20)$$

Then, if $\kappa_i \neq 0$, the first \mathbb{K}_n in (3.13) have the form

$$\begin{aligned}
\mathbb{K}_0 &= \sigma_0, \\
\mathbb{K}_1 &= \sigma_0 + t_1 e^{-\kappa_1 t_0} K_1, \\
\mathbb{K}_2 &= \sigma_0 + \left[\left(t_1 - \frac{1}{\kappa_1} \right) e^{-\kappa_1 t_0} + \frac{1}{\kappa_1} e^{-\kappa_1 (t_0 + t_1)} \right] K_1 + t_2 e^{-\kappa_2 (t_0 + t_1)} K_2, \\
\mathbb{K}_3 &= \sigma_0 + \left[\left(t_1 - \frac{1}{\kappa_1} \right) e^{-\kappa_1 t_0} + \frac{1}{\kappa_1} e^{-\kappa_1 (t_0 + t_1)} \right] K_1 \\
&\quad + \left[\left(t_2 - \frac{1}{\kappa_2} \right) e^{-\kappa_2 (t_0 + t_1)} + \frac{1}{\kappa_2} e^{-\kappa_2 (t_0 + t_1 + t_2)} \right] K_2 + t_3 e^{-\kappa_3 (t_0 + t_1 + t_2)} K_3, \\
\mathbb{K}_4 &= \sigma_0 + \left[\left(t_1 - \frac{1}{\kappa_1} \right) e^{-\kappa_1 t_0} + \frac{1}{\kappa_1} e^{-\kappa_1 (t_0 + t_1)} \right] K_1 \\
&\quad + \left[\left(t_2 - \frac{1}{\kappa_2} \right) e^{-\kappa_2 (t_0 + t_1)} + \frac{1}{\kappa_2} e^{-\kappa_2 (t_0 + t_1 + t_2)} \right] K_2 \\
&\quad + \left[\left(t_3 - \frac{1}{\kappa_3} \right) e^{-\kappa_3 (t_0 + t_1 + t_2)} + \frac{1}{\kappa_3} e^{-\kappa_3 (t_0 + t_1 + t_2 + t_3)} \right] K_3 + t_4 e^{-\kappa_4 (t_0 + t_1 + t_2 + t_3)} K_4.
\end{aligned}$$

4. NON-AUTONOMOUS SOLITON HIERARCHIES AND THEIR DEFORMED ISOSPECTRAL ZERO-CURVATURE REPRESENTATIONS

From now on we will assume that the algebraic objects like K , \mathbb{K} , or σ are vector fields on some infinite-dimensional manifold \mathcal{M} with the corresponding autonomous evolution equations $u_t = K[u]$ and non-autonomous evolution equations $u_t = \mathbb{K}(x, t, [u])$, where the square bracket denotes the dependence on u and a finite number of derivatives of u w.r.t. x (so $[u]$ denotes jet-coordinates on \mathcal{M}) and where $u = (u_1(x), \dots, u_N(x))^T$ denotes points on the manifold \mathcal{M} .

Consider thus an infinite hierarchy of mutually commuting autonomous evolutionary equations on \mathcal{M} of the form

$$u_{s_n} = K_n[u], \quad n = 1, 2, \dots, \quad (4.1)$$

as well as a hierarchy of non-commuting evolutionary equations on \mathcal{M} :

$$u_{\tau_n} = \sigma_n[u], \quad n = -1, 0, 1, \dots, \quad (4.2)$$

such that the commutation relations (3.1) are valid. The members of the hierarchy (4.2) are called master symmetries for (4.1).

In this section we obtain – under Assumption 10 and Assumption 11 – the Frobenius integrable non-autonomous hierarchies $u_{t_n} = \mathbb{K}_n[u]$, where \mathbb{K}_n is of the form (3.4) or (3.13), from an appropriate deformation of an isospectral zero-curvature representation of (4.1) by a non-standard (see Remark 12) isospectral zero-curve representation of (4.2).

Assumption 10. Suppose that the commuting hierarchy (4.1) can be obtained from the isospectral linear problem

$$\begin{cases} \Psi_x = L\Psi, \\ \Psi_{s_i} = U_i\Psi, \quad i = 1, 2, \dots, \end{cases} \quad (4.3)$$

where $L = L(\lambda, u)$, $U_i = U_i(\lambda, [u])$ are some 2×2 matrices depending on $[u]$ and the auxiliary variable λ , s.t. $\lambda_{s_i} = 0$ for all i .

The subscript s_i denotes the total derivative with respect to the evolution parameter s_i . The compatibility condition, that is the condition for existence of a common multi-time solution $\Psi(x, s_1, s_2, \dots)$, for the problem (4.3) is

$$(\Psi_x)_{s_i} = (\Psi_{s_i})_x, \quad i = 1, 2, \dots, \quad (4.4a)$$

$$(\Psi_{s_i})_{s_j} = (\Psi_{s_j})_{s_i}, \quad i, j = 1, 2, \dots \quad (4.4b)$$

The condition (4.4a) is equivalent to

$$L_{s_i} = [U_i, L] + (U_i)_x \equiv L' [K_i], \quad i = 1, 2, \dots \quad (4.5)$$

Throughout the whole article $\Omega' [K]$ denotes the directional derivative of the tensor field Ω along the vector field K on \mathcal{M} . The identity in (4.5) is the consequence of Assumption 10, while the condition (4.4b) is equivalent to

$$(U_i)_{s_j} - (U_j)_{s_i} + [U_i, U_j] = 0, \quad i, j = 1, 2, \dots \quad (4.6)$$

Thus, Assumption 10 means that (4.5) is equivalent with the corresponding equation $u_{s_i} = K_i[u]$ in (4.1), i.e. (4.5) is an isospectral zero-curvature representation for (4.1). It also means that $\Psi_{s_i} = \mathcal{L}_{K_i}\Psi$ where \mathcal{L} is the Lie derivative on \mathcal{M} . Then, the equation (4.6) guarantees that all K_i commute, since

$$(\Psi_{s_i})_{s_j} - (\Psi_{s_j})_{s_i} = \mathcal{L}_{K_j}\mathcal{L}_{K_i}\Psi - \mathcal{L}_{K_i}\mathcal{L}_{K_j}\Psi = \mathcal{L}_{[K_j, K_i]}\Psi = \Psi'[[K_j, K_i]] = 0.$$

Note also that (4.6) can be written as

$$U'_i[K_j] - U'_j[K_i] + [U_i, U_j] = 0, \quad i, j = 1, 2, \dots \quad (4.7)$$

Assumption 11. Suppose also that the (non-commuting) hierarchy (4.2) of master symmetries can be obtained from the following *deformed* linear isospectral problem

$$\begin{cases} \Psi_x = L\Psi, \\ \Psi_{\tau_i} = V_i\Psi - \lambda^{i+1}\Psi_\lambda, \quad i = -1, 0, 1, \dots, \end{cases} \quad (4.8)$$

where $L = L(\lambda, u)$ is the same L as in (4.3) while $V_i = V_i(\lambda, [u])$ are some matrices depending on $[u]$ and λ such that $\lambda_{\tau_i} = 0$.

In (4.8) $\Psi_\lambda \equiv \frac{\partial \Psi}{\partial \lambda}$. Obviously, we cannot expect that (4.8) posses a common multi-time solution $\Psi(x, \tau_{-1}, \tau_0, \tau_1, \dots)$. Instead, the Assumption 11 means that (4.8) has, for each i , a solution $\Psi(x, \tau_i)$ so that

$$(\Psi_x)_{\tau_i} = (\Psi_{\tau_i})_x, \quad i = -1, 0, \dots,$$

which is equivalent to

$$L_{\tau_i} = [V_i, L] + (V_i)_x - \lambda^{i+1}L_\lambda \equiv L' [\sigma_i], \quad i = -1, 0, 1, \dots, \quad (4.9)$$

and the identity in (4.9) is the consequence of Assumption 11. This assumption means thus that each equation in (4.9) is equivalent with the corresponding equation $u_{\tau_i} = \sigma_i[u]$ in (4.2). Since the fields σ_i in (4.2) do not commute we clearly cannot expect that $(\Psi_{\tau_i})_{\tau_j} = (\Psi_{\tau_j})_{\tau_i}$. Instead we have

$$(\Psi_{\tau_i})_{\tau_j} - (\Psi_{\tau_j})_{\tau_i} = \Psi'[[\sigma_j, \sigma_i]] = (i-j)\Psi'[\sigma_{i+j}] = (i-j)\Psi_{\tau_{i+j}}, \quad i, j = -1, 0, 1, \dots,$$

which is equivalent to

$$(V_i)_{\tau_j} - (V_j)_{\tau_i} + [V_i, V_j] + \lambda^{j+1}(V_i)_\lambda - \lambda^{i+1}(V_j)_\lambda = (i-j)V_{i+j}, \quad i, j = -1, 0, \dots,$$

that is to

$$V'_i[\sigma_j] - V'_j[\sigma_i] + [V_i, V_j] + \lambda^{j+1}(V_i)_\lambda - \lambda^{i+1}(V_j)_\lambda = (i-j)V_{i+j}, \quad i, j = -1, 0, \dots \quad (4.10)$$

Remark 12. Usually in literature (see for example [3]) one constructs a zero-curvature representation for (4.2), from the *non-isospectral* problem

$$\begin{cases} \Psi_x = L\Psi, \\ \Psi_{\tau_i} = V_i\Psi, \quad i = 1, 2, \dots, \end{cases}$$

with $\lambda_{\tau_i} = \lambda^{i+1}$. The isospectral problem (4.8) is however equivalent (in the sense that it leads to the same zero-curvature equations (4.10)) with the above isospectral problem, while being better adapted to our needs.

We will now construct an isospectral zero-curvature representation of the hierarchies

$$u_{t_n} = \mathbb{K}_n[u],$$

with \mathbb{K}_n given in (3.4) or in (3.13) by combining the isospectral problems (4.3) and (4.8). Consider thus the *deformed* isospectral linear problem

$$\begin{cases} \Psi_x = L\Psi, \\ \Psi_{t_n} = W_n\Psi - \lambda^{\varepsilon+1}\Psi_\lambda, \quad n = 1, 2, \dots, \end{cases} \quad (4.11)$$

with $\varepsilon = -1$ or $\varepsilon = 0$ and where $\lambda_{t_n} = 0$, with $W_n = W_n(\lambda, [u])$ defined as

$$W_n = V_\varepsilon + \sum_{i=1}^n \mathbf{v}_{n,i}(t_0, \dots, t_n)U_i, \quad (4.12)$$

where L, V_ε, U_i are given as above in this section and where $\mathbf{v}_{n,i}$ are so far undetermined functions of evolution parameters t_0, \dots, t_n .

Theorem 13. *The compatibility condition $(\Psi_x)_{t_n} = (\Psi_{t_n})_x$ for (4.11) has the form*

$$L_{t_n} = [W_n, L] + (W_n)_x - \lambda^{\varepsilon+1}L_\lambda \equiv L'[u_{t_n}], \quad n = 1, 2, \dots, \quad (4.13)$$

where

$$u_{t_n} = \mathbb{K}_n \equiv \sigma_\varepsilon + \sum_{i=1}^n \mathbf{v}_{n,i}(t_0, \dots, t_n)K_i[u], \quad n = 1, 2, \dots \quad (4.14)$$

The identity in (4.13) means that (4.13) is equivalent with the corresponding equation $u_{t_n} = \mathbb{K}_n[u]$ in (4.14). Note that so far the vector fields $\mathbb{K}_n[u]$ in (4.14) have nothing in common with \mathbb{K}_n in (3.4) or (3.13).

Proof. Due to the form of W_n we have

$$\begin{aligned} [W_n, L] + (W_n)_x - \lambda^{\varepsilon+1}L_\lambda &= [V_\varepsilon, L] + (V_\varepsilon)_x - \lambda^{\varepsilon+1}L_\lambda + \sum_{i=1}^n \mathbf{v}_{n,i}([U_i, L] + (U_i)_x) \\ &= L'[\sigma_\varepsilon] + \sum_{i=1}^n \mathbf{v}_{n,i}L'[K_i] \equiv L'[\mathbb{K}_n]. \end{aligned}$$

□

So far the functions $\mathbf{v}_{n,i}$ are undetermined. Let us now demand that the compatibility conditions

$$(\Psi_{t_m})_{t_n} = (\Psi_{t_n})_{t_m}, \quad m, n = 1, 2, \dots, \quad (4.15)$$

for (4.11) are satisfied. These conditions are equivalent with the Frobenius integrability conditions (2.2), since

$$\begin{aligned} (\Psi_{t_m})_{t_n} - (\Psi_{t_n})_{t_m} &= (\Psi'[\mathbb{K}_m])_{t_n} - (\Psi'[\mathbb{K}_n])_{t_m} \\ &= \Psi''[\mathbb{K}_n; \mathbb{K}_m] + \Psi' \left[\frac{\partial \mathbb{K}_m}{\partial t_n} + \mathbb{K}'_m[\mathbb{K}_n] \right] - \Psi''[\mathbb{K}_m; \mathbb{K}_n] - \Psi' \left[\frac{\partial \mathbb{K}_n}{\partial t_m} + \mathbb{K}'_n[\mathbb{K}_m] \right] \\ &= \Psi' \left[\frac{\partial \mathbb{K}_m}{\partial t_n} - \frac{\partial \mathbb{K}_n}{\partial t_m} + [\mathbb{K}_n, \mathbb{K}_m] \right], \quad m, n = 1, 2, \dots, \end{aligned}$$

where $\Psi''[\mathbb{K}_n; \mathbb{K}_m] = \Psi''[\mathbb{K}_m; \mathbb{K}_n]$ is the second directional derivative. On the other hand (4.15) hold if and only if

$$(W_n)_{t_m} - (W_m)_{t_n} + [W_n, W_m] + \lambda^{\varepsilon+1}(W_n)_\lambda - \lambda^{\varepsilon+1}(W_m)_\lambda = 0. \quad (4.16)$$

Let us thus investigate the conditions under which (4.16) hold. Assuming $1 \leq m < n$, we have that $(\mathbf{v}_{m,i})_{t_n} = 0$ and then the zero-curvature relations (4.16) reduce to

$$\frac{\partial W_n}{\partial t_m} + W'_n[\mathbb{K}_m] - W'_m[\mathbb{K}_n] + [W_n, W_m] + \lambda^{\varepsilon+1}(W_n)_\lambda - \lambda^{\varepsilon+1}(W_m)_\lambda = 0, \quad 1 \leq m < n. \quad (4.17)$$

We can now prove the following theorem.

Theorem 14. *The zero curvature conditions (4.17) are equivalent with the set of equations on the functions $\mathbf{v}_{n,i}$ that is exactly the same as the set of equations (3.8) and (3.18), respectively for $\varepsilon = -1$ and $\varepsilon = 0$, on the functions $\mathbf{u}_{n,i}$.*

The proof of this theorem can be found in Appendix D. This theorem means that the functions $\mathbf{v}_{n,i}$ and $\mathbf{u}_{n,i}$ pairwise coincide so that the deformed isospectral problem (4.11) leads exactly to the Frobenius integrable hierarchy $u_{t_n} = \mathbb{K}_n[u]$ with \mathbb{K}_n given by (3.4) (for $\varepsilon = -1$) or by (3.13) (for $\varepsilon = 0$). Hence,

$$W_n \equiv V_\varepsilon + \sum_{i=1}^n \mathbf{u}_{n,i}(t_0, \dots, t_n) U_i. \quad (4.18)$$

Remark 15. Thus, we obtain the same non-autonomous Frobenius integrable hierarchies of PDE's starting from the deformed spectral problem (4.11) and starting from the non-autonomous deformations in the case of the subalgebras \mathcal{A}_ε of the hereditary algebra (3.1), that we consider in Section 3.

4.1. Hamiltonian structure of non-autonomous soliton hierarchies. We will now focus on soliton hierarchies. Suppose we have an infinite hierarchy of vector fields K_n on \mathcal{M} that are bi-Hamiltonian with respect to two compatible Poisson structures π_0 and π_1

$$K_n = \pi_0 \delta H_n = \pi_1 \delta H_{n-1}, \quad n = 1, 2, \dots,$$

with π_0 being invertible. Then the operator $N = \pi_1 \pi_0^{-1}$ is an operator with the vanishing Nijenhuis torsion so that for any vector field K on \mathcal{M} we have

$$\mathcal{L}_{NK}N = N\mathcal{L}_KN. \quad (4.19)$$

Any operator satisfying (4.19) is called a hereditary operator. Then,

$$K_n \equiv N^{n-1}K_1, \quad n = 2, 3, \dots, \quad (4.20)$$

and by the hereditary property (4.19)

$$\mathcal{L}_{K_n} N = 0, \quad n = 1, 2, \dots,$$

and

$$[K_n, K_m] = 0, \quad n, m = 1, 2, \dots$$

Define now the infinite sequence of 1-forms

$$\gamma_n \equiv \delta H_n = (N^\dagger)^i \gamma_0, \quad n = 0, 1, \dots, \quad (4.21)$$

where $\gamma_0 = \delta H_0$ and where $N^\dagger = \pi_0^{-1} \pi_1$. By the same hereditary property of N they are all closed and thus there exists an infinite sequence of functionals $H_n = \int h_n dx$ such that $\gamma_n = \delta H_n$. Define now the infinite sequence of Poisson operators

$$\pi_k = N^k \pi_0, \quad k = 2, 3, \dots$$

The operators π_k are pairwise compatible and usually non-local. Then, it follows that the field K_n is $(n+1)$ -Hamiltonian

$$K_n = \pi_0 \delta H_n = \pi_1 \delta H_{n-1} = \dots = \pi_n \delta H_0, \quad n = 1, 2, \dots$$

Consider also a scaling vector field σ_0 such that

$$\begin{cases} \mathcal{L}_{\sigma_0} K_1 = \rho K_1, & \rho \in \mathbf{R}, \\ \mathcal{L}_{\sigma_0} N = N, \end{cases}$$

and define the infinite sequences of vector fields σ_n on \mathcal{M} through

$$\sigma_n = N^n \sigma_0, \quad n = -1, 0, 1, \dots \quad (4.22)$$

Then it can be shown, using the hereditary property (4.19), that the vector fields (4.20) and (4.22) satisfy the commutation relations (3.1).

Finally, let us assume that there exist a vector field σ_{-1} such that $\sigma_0 = N \sigma_{-1}$ and such that it is Hamiltonian with respect to π_0

$$\sigma_{-1} = \pi_0 \delta F.$$

Then, all σ_n are Hamiltonian with respect to the Poisson operator π_{n+1} , that is

$$\sigma_n = \pi_{n+1} \delta F, \quad n = 1, 2, \dots, \quad (4.23)$$

and it immediately follows that every non-autonomous vector field \mathbb{K}_n in (3.4) or in (3.13) is also Hamiltonian (but not bi-Hamiltonian), as

$$\begin{aligned} \mathbb{K}_n &\equiv \sigma_\varepsilon + \sum_{i=1}^n \mathbf{u}_{n,i}(t_0, \dots, t_n) K_i[u] = \\ &= \pi_{\varepsilon+1} \delta \left(F + \sum_{i=1}^n \mathbf{u}_{n,i}(t_0, \dots, t_n) H_{i-\varepsilon-1}[u] \right). \end{aligned}$$

5. NON-AUTONOMOUS KDV HIERARCHY

In this and following sections we apply our theory, developed above, to three well known soliton hierarchies: Kortevæg-de Vries, dispersive water waves and Ablowitz-Kaup-Newell-Segur. The all have the structure exactly as described in subsection 4.1.

5.1. KdV hierarchy. As a first illustration of our theory, consider the KdV hierarchy. The KdV equation

$$u_t = \frac{1}{4} u_{xxx} + \frac{3}{2} u u_x$$

is a member of the bi-Hamiltonian chain of nonlinear PDE's

$$u_{t_i} = K_i[u] = \pi_0 \delta H_i = \pi_1 \delta H_{i-1}, \quad i = 1, 2, \dots, \quad (5.1)$$

with two compatible Poisson operators

$$\pi_0 = \partial_x, \quad \pi_1 = \frac{1}{4} \partial_x^3 + u \partial_x + \frac{1}{2} u_x.$$

The hierarchy (5.1) is *autonomous* in the sense that none of the vector fields $K_i[u]$ of the hierarchy depends explicitly on the evolution parameters t_j . The KdV hierarchy (5.1) can be generated by the recursion operator and its adjoint

$$N \equiv \pi_1 \pi_0^{-1} = \frac{1}{4} \partial_x^2 + u + \frac{1}{2} u_x \partial_x^{-1}, \quad N^\dagger = \frac{1}{4} \partial_x^2 + u - \frac{1}{2} \partial_x^{-1} u_x,$$

in the sense that (4.20) and (4.21) are valid. In particular, we find that the first vector fields K_n have the form

$$\begin{aligned} K_1 &= u_x, & K_2 &= \frac{1}{4} u_{xxx} + \frac{3}{2} uu_x, \\ K_3 &= \frac{1}{16} u_{5x} + \frac{5}{8} uu_{3x} + \frac{5}{4} u_x u_{xx} + \frac{15}{8} u^2 u_x, \\ K_4 &= \frac{1}{64} u_{7x} + \frac{7}{32} uu_{5x} + \frac{21}{32} u_x u_{4x} + \frac{35}{32} u_{xx} u_{3x} + \frac{35}{32} u_x^3 + \frac{35}{8} uu_x u_{xx} + \frac{35}{32} u^2 u_{3x} + \frac{35}{16} u^3 u_x, \end{aligned}$$

the first conserved one-forms (cosymmetries) $\gamma_n \equiv \delta H_n$ are

$$\begin{aligned} \gamma_0 &= 2, & \gamma_1 &= u, & \gamma_2 &= \frac{1}{4} u_{xx} + \frac{3}{4} u^2, \\ \gamma_3 &= \frac{1}{16} u_{4x} + \frac{5}{8} uu_{xx} + \frac{5}{16} u_x^2 + \frac{5}{8} u^3, \\ \gamma_4 &= \frac{1}{64} u_{6x} + \frac{7}{32} uu_{4x} + \frac{7}{16} u_x u_{3x} + \frac{21}{64} u_{xx}^2 + \frac{35}{32} u^2 u_{xx} + \frac{35}{32} uu_x^2 + \frac{35}{64} u^4, \end{aligned}$$

while the first Hamiltonian densities h_n of the conserved functionals $H_n = \int h_n dx$ are

$$\begin{aligned} h_0 &= 2u, & h_1 &= \frac{1}{2} u^2, & h_2 &= -\frac{1}{8} u_x^2 + \frac{1}{4} u^3, \\ h_3 &= \frac{1}{32} u_{xx}^2 + \frac{5}{32} u^2 u_{xx} + \frac{5}{32} u^4, \\ h_4 &= -\frac{1}{128} u_{3x}^2 + \frac{7}{64} uu_{xx}^2 - \frac{35}{64} u^2 u_x^2 + \frac{7}{64} u^5. \end{aligned}$$

With the KdV hierarchy (5.1) one can also relate the hierarchy of its master symmetries (4.22) with the first few σ_n of the form

$$\begin{aligned} \sigma_{-1} &= 1, & \sigma_0 &= u + \frac{1}{2} xu_x, \\ \sigma_1 &= \frac{1}{2} u_{xx} + \frac{1}{8} xu_{3x} + u^2 + \frac{1}{2} xuu_x + \frac{1}{4} u_x \partial_x^{-1} u. \end{aligned}$$

The master symmetries σ_n are in general non-local. They are Hamiltonian due to (4.23) with the conserved functional

$$F = \int xu dx.$$

The symmetries K_i and master symmetries σ_j of the KdV equation generate the hereditary algebra (3.1) with

$$\rho = \frac{1}{2} \quad \text{so that} \quad \kappa_n = n - \frac{1}{2}. \quad (5.2)$$

The matrix Lax representation (4.5) for the KdV hierarchy is given by

$$L = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}, \quad (5.3)$$

and

$$U_n = \frac{1}{2} \sum_{i=0}^{n-1} \begin{pmatrix} -\frac{1}{2}(\gamma_i)_x & \gamma_i \\ (\lambda - u)\gamma_i - \frac{1}{2}(\gamma_i)_{xx} & \frac{1}{2}(\gamma_i)_x \end{pmatrix} \lambda^{n-i-1}, \quad n = 1, 2, \dots$$

In particular, $U_1 = L$,

$$U_2 = \begin{pmatrix} -\frac{1}{4}u_x & \lambda + \frac{1}{2}u \\ \lambda^2 - \frac{1}{2}u\lambda - \frac{1}{2}u^2 - \frac{1}{4}u_{xx} & \frac{1}{4}u_x \end{pmatrix}$$

and

$$U_3 = \begin{pmatrix} -\frac{1}{4}u_x\lambda - \frac{1}{16}(u_{3x} + 6uu_x) & \lambda^2 + \frac{1}{2}u\lambda + \frac{1}{8}(u_{xx} + 3u^2) \\ \lambda^3 - \frac{1}{2}u\lambda^2 - \frac{1}{8}(u_{xx} + u^2)\lambda - \frac{1}{16}u_{4x} + \frac{1}{2}uu_{xx} + \frac{3}{8}u_x^2 + \frac{3}{8}u^3 & \frac{1}{4}u_x\lambda + \frac{1}{16}(u_{3x} + 6uu_x) \end{pmatrix}.$$

The Lax formulation for the hierarchy of the KdV master symmetries σ_n is given by (4.9) with V_n of the form

$$V_n = \frac{1}{2} \sum_{i=-1}^{n-1} \begin{pmatrix} -\frac{1}{2}\sigma_i & \partial_x^{-1}\sigma_i \\ (\lambda - u)\partial_x^{-1}\sigma_i - \frac{1}{2}(\sigma_i)_x & \frac{1}{2}\sigma_i \end{pmatrix} \lambda^{n-i-1}, \quad n = -1, 0, 1, \dots,$$

so that

$$V_{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad V_0 = \begin{pmatrix} -\frac{1}{4} & \frac{1}{2}x \\ \frac{1}{2}(\lambda - u)x & \frac{1}{4} \end{pmatrix}.$$

5.2. Non-autonomous KdV hierarchy in the case \mathcal{A}_{-1} . We now present the deformed KdV hierarchy $u_{t_n} \in \mathbb{K}_n[u]$ for the case of the hereditary subalgebra \mathcal{A}_{-1} .

For the general initial conditions (3.3) the first members (3.4) of the non-autonomous KdV hierarchy take the form

$$\begin{aligned} u_{t_0} &= 1, \\ u_{t_1} &= 1 + \mathbf{a}_{1,1}(t_1) u_x, \\ u_{t_2} &= 1 + \left[\mathbf{a}_{2,1}(t_2) - \frac{3}{2}(t_0 + t_1) \mathbf{a}_{2,2}(t_2) \right] u_x + \mathbf{a}_{2,2}(t_2) \left[\frac{1}{4}u_{xxx} + \frac{3}{2}uu_x \right], \\ u_{t_3} &= 1 + \left[\frac{3}{2} \partial_{t_2}^{-1} \mathbf{a}_{2,2}(t_2) + \mathbf{a}_{3,1}(t_3) - \frac{3}{2}(t_0 + t_1 + t_2) \mathbf{a}_{3,2}(t_3) + \frac{15}{8}(t_0 + t_1 + t_2)^2 \mathbf{a}_{3,3}(t_3) \right] u_x \\ &\quad + \left[\mathbf{a}_{3,2}(t_3) - \frac{5}{2}(t_0 + t_1 + t_2) \mathbf{a}_{3,3}(t_3) \right] \left[\frac{1}{4}u_{xxx} + \frac{3}{2}uu_x \right] \\ &\quad + \mathbf{a}_{3,3}(t_3) \left[\frac{1}{16}u_{5x} + \frac{5}{8}uu_{3x} + \frac{5}{4}u_x u_{xx} + \frac{15}{8}u^2 u_x \right]. \end{aligned}$$

If we consider the initial conditions given by the choice $\mathbf{a}_{n,i}(t_n) = \delta_{i,n} t_n$, that is (3.10), the deformed vector fields \mathbb{K}_n are given by (3.4) with (3.5) specified by (3.11) with ρ and κ_m as in (5.2). In this case the first few non-autonomous vector fields \mathbb{K}_n are given by formulas in Example 7 and so the first few members of our hierarchy specify to

$$\begin{aligned} u_{t_0} &= 1, \\ u_{t_1} &= 1 + t_1 u_x, \\ u_{t_2} &= 1 - \frac{3}{2}(t_0 + t_1)t_2 u_x + t_2 \left[\frac{1}{4}u_{xxx} + \frac{3}{2}uu_x \right], \\ u_{t_3} &= 1 + \left[\frac{3}{4}t_2^2 + \frac{15}{8}(t_0 + t_1 + t_2)^2 t_3 \right] u_x - \frac{5}{2}(t_0 + t_1 + t_2)t_3 \left[\frac{1}{4}u_{xxx} + \frac{3}{2}uu_x \right] \\ &\quad + t_3 \left[\frac{1}{16}u_{5x} + \frac{5}{8}uu_{3x} + \frac{5}{4}u_x u_{xx} + \frac{15}{8}u^2 u_x \right]. \end{aligned}$$

The Lax representation of the above non-autonomous KdV hierarchy is given by (4.13), with L as in (5.3) and W_n defined by (4.18) with $\varepsilon = -1$, where, in the general case, $\mathbf{u}_{n,i}$ are given by (3.5) and in the case of special initial conditions (3.10) the functions $\mathbf{u}_{n,i}$ are specified by (3.11).

5.3. Non-autonomous KdV hierarchy in the case \mathcal{A}_0 . The deformed KdV hierarchy $u_{t_n} = \mathbb{K}_n[u]$ for the case of the hereditary subalgebra \mathcal{A}_0 and with the general initial conditions (3.12) is given by the vector fields \mathbb{K}_n as in (3.13), thus in particular

$$\begin{aligned} u_{t_0} &= u + \frac{1}{2}xu_x, \\ u_{t_1} &= u + \frac{1}{2}xu_x + \mathbf{a}_{1,1}(t_1)e^{-\frac{1}{2}t_0}u_x, \\ u_{t_2} &= u + \frac{1}{2}xu_x + \left[\mathbf{c}_{1,1}(t_1)e^{-\frac{1}{2}t_0} + \mathbf{a}_{2,1}(t_2)e^{-\frac{1}{2}(t_0+t_1)} \right]u_x + \mathbf{a}_{2,2}(t_2)e^{-\frac{3}{2}(t_0+t_1)} \left[\frac{1}{4}u_{xxx} + \frac{3}{2}uu_x \right], \\ u_{t_3} &= u + \frac{1}{2}xu_x + \left[\mathbf{c}_{1,1}(t_1)e^{-\frac{1}{2}t_0} + \mathbf{c}_{2,1}(t_2)e^{-\frac{1}{2}(t_0+t_1)} + \mathbf{a}_{3,1}(t_3)e^{-\frac{1}{2}(t_0+t_1+t_2)} \right]u_x \\ &\quad + \left[e^{-\frac{3}{2}(t_0+t_1)}\mathbf{c}_{2,2}(t_2) + \mathbf{a}_{3,2}(t_3)e^{-\frac{3}{2}(t_0+t_1+t_2)} \right] \left[\frac{1}{4}u_{xxx} + \frac{3}{2}uu_x \right] \\ &\quad + \mathbf{a}_{3,3}(t_3)e^{-\frac{3}{2}(t_0+t_1+t_2)} \left[\frac{1}{64}u_{7x} + \frac{7}{32}uu_{5x} + \frac{21}{32}u_xu_{4x} + \frac{35}{32}u_{xx}u_{3x} + \frac{35}{32}u_x^3 \right. \\ &\quad \left. + \frac{35}{8}uu_xu_{xx} + \frac{35}{32}u^2u_{3x} + \frac{35}{16}u^3u_x \right]. \end{aligned}$$

If we consider the initial conditions (3.19) the deformed vector fields \mathbb{K}_n are given by (3.13) with (3.14) specified by (3.20) with ρ and κ_m given by (5.2). In this case, as in Example 9, we have

$$\begin{aligned} u_{t_0} &= u + \frac{1}{2}xu_x, \\ u_{t_1} &= u + \frac{1}{2}xu_x + t_1e^{-\frac{1}{2}t_0}u_x, \\ u_{t_2} &= u + \frac{1}{2}xu_x + \left(t_1 + 2e^{-\frac{1}{2}t_1} - 2 \right) e^{-\frac{1}{2}t_0}u_x + t_2e^{-\frac{3}{2}(t_0+t_1)} \left[\frac{1}{4}u_{xxx} + \frac{3}{2}uu_x \right], \\ u_{t_3} &= u + \frac{1}{2}xu_x + \left(t_1 + 2e^{-\frac{1}{2}t_1} - 2 \right) e^{-\frac{1}{2}t_0}u_x + \left(t_2 + \frac{2}{3}e^{-\frac{3}{2}t_2} - \frac{2}{3} \right) e^{-\frac{3}{2}(t_0+t_1)} \left[\frac{1}{4}u_{xxx} + \frac{3}{2}uu_x \right] \\ &\quad + t_3e^{-\frac{3}{2}(t_0+t_1+t_2)} \left[\frac{1}{64}u_{7x} + \frac{7}{32}uu_{5x} + \frac{21}{32}u_xu_{4x} + \frac{35}{32}u_{xx}u_{3x} + \frac{35}{32}u_x^3 + \frac{35}{8}uu_xu_{xx} \right. \\ &\quad \left. + \frac{35}{32}u^2u_{3x} + \frac{35}{16}u^3u_x \right]. \end{aligned}$$

The Lax representation of the above non-autonomous KdV hierarchies is given by (4.13), with L given by (5.3) and with respective W_n defined by (4.18) with $\varepsilon = 0$.

6. NON-AUTONOMOUS DWW HIERARCHY

6.1. Autonomous DWW hierarchy. Let us now apply our theory to the DWW hierarchy. It is a bi-Hamiltonian hierarchy given by

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_n} = K_n = \pi_0\gamma_n = \pi_1\gamma_{n-1}, \quad n = 1, 2, \dots, \quad (6.1)$$

where $\gamma_n \equiv \delta H_n$ are exact one-forms and where

$$\pi_0 = \begin{pmatrix} -\frac{1}{2}v\partial_x - \frac{1}{2}\partial_x v & \partial_x \\ \partial_x & 0 \end{pmatrix}, \quad \pi_1 = \begin{pmatrix} \frac{1}{4}\partial_x^3 + \frac{1}{2}u\partial_x + \frac{1}{2}\partial_x u & 0 \\ 0 & \partial_x \end{pmatrix}$$

are two compatible Poisson operators. This hierarchy is generated by (4.20) and (4.21) with the recursion operator and its adjoint given by

$$N = \pi_1\pi_0^{-1} = \begin{pmatrix} 0 & \frac{1}{4}\partial_x^2 + u + \frac{1}{2}u_x\partial_x^{-1} \\ 1 & v + \frac{1}{2}v_x\partial_x^{-1} \end{pmatrix}, \quad N^\dagger = \begin{pmatrix} 0 & 1 \\ \frac{1}{4}\partial_x^2 + u - \frac{1}{2}\partial_x^{-1}u_x & v - \frac{1}{2}\partial_x^{-1}v_x \end{pmatrix}.$$

In particular, we have the symmetries

$$\begin{aligned} K_1 &= \begin{pmatrix} u_x \\ v_x \end{pmatrix}, & K_2 &= \begin{pmatrix} \frac{1}{4}v_{xxx} + uv_x + \frac{1}{2}vu_x \\ u_x + \frac{3}{2}vv_x \end{pmatrix}, \\ K_3 &= \begin{pmatrix} \frac{3}{8}v^2u_x + \frac{3}{2}uvv_x + \frac{3}{2}uu_x + \frac{1}{4}u_{3x} + \frac{3}{8}vv_{3x} + \frac{9}{8}v_xv_{2x} \\ \frac{3}{2}vu_x + \frac{3}{2}uv_x + \frac{15}{8}v^2v_x + \frac{1}{4}v_{3x} \end{pmatrix}, \end{aligned}$$

cosymmetries

$$\gamma_0 = \begin{pmatrix} 2 \\ v \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} v \\ u + \frac{3}{4}v^2 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} u + \frac{3}{4}v^2 \\ \frac{1}{4}v_{xx} + \frac{3}{2}uv + \frac{5}{8}v^3 \end{pmatrix},$$

and functionals $H_n = \int h_n dx$, where

$$h_0 = 2u + \frac{1}{2}v^2, \quad h_1 = uv + \frac{1}{4}v^3, \quad h_2 = -\frac{1}{8}v_x^2 + \frac{1}{2}u^2 + \frac{3}{4}uv^2 + \frac{5}{32}v^4.$$

With the DWW hierarchy (6.1) one can also relate the hierarchy of its master symmetries (4.22) with the first few σ_n of the form

$$\sigma_{-1} = \begin{pmatrix} -v \\ 2 \end{pmatrix}, \quad \sigma_0 = \begin{pmatrix} 2u + xu_x \\ v + xv_x \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} \frac{3}{4}v_{xx} + \frac{1}{4}xv_{xxx} + uv + xuv_x + \frac{1}{2}xvu_x \\ xu_x + \frac{3}{2}xvv_x + v^2 + 2u \end{pmatrix}.$$

The master symmetries σ_n are Hamiltonian as in (4.23) with

$$F = \int \left(2xu + \frac{1}{2}xv^2 \right) dx.$$

The symmetries K_i and the master symmetries σ_j constitute the hereditary algebra (3.1) with

$$\rho = 1 \quad \text{so that} \quad \kappa_n = n. \quad (6.2)$$

The zero-curvature formulation (4.5) for the DWW hierarchy is given by

$$L = \begin{pmatrix} 0 & 1 \\ \lambda^2 - v\lambda - u & 0 \end{pmatrix} \quad (6.3)$$

and by

$$U_n = \frac{1}{2} \sum_{i=0}^{n-1} \begin{pmatrix} -\frac{1}{2}(\gamma_{i1})_x & \gamma_{i1} \\ (\lambda^2 - v\lambda - u) \gamma_{i1} - \frac{1}{2}(\gamma_{i1})_{xx} & \frac{1}{2}(\gamma_{i1})_x \end{pmatrix} \lambda^{n-i-1}, \quad n = 1, 2, \dots, \quad (6.4)$$

with γ_{i1} denoting the first component of γ_i . In particular, $U_1 = L$ and

$$\begin{aligned} U_2 &= \begin{pmatrix} -\frac{1}{4}v_x & \lambda + \frac{1}{2}v \\ \lambda^3 - \frac{1}{2}v\lambda^2 - (u + \frac{1}{2}v^2) \lambda - \frac{1}{2}uv - \frac{1}{4}v_{2x} & \frac{1}{4}v_x \end{pmatrix}, \\ U_3 &= \begin{pmatrix} -\frac{1}{4}v_x\lambda - \frac{1}{4}u_x - \frac{3}{8}vv_x & \lambda^2 + \frac{1}{2}v\lambda + \frac{3}{8}v^2 + \frac{1}{2}u \\ (U_3)_{21} & \frac{1}{4}v_x\lambda + \frac{1}{4}u_x + \frac{3}{8}vv_x \end{pmatrix}, \end{aligned}$$

where

$$(U_3)_{21} = \lambda^4 - \frac{1}{2}v\lambda^3 - \frac{1}{8}(4u + v^2)\lambda^2 - \frac{1}{8}(8uv + 3v^3 + 2v_{2x})\lambda - \frac{1}{8}(4u^2 + 3uv^2 + 2u_{2x} + 3v_x^2 + 3vv_{2x}).$$

The deformed Lax formulation for the hierarchy of the DWW master symmetries σ_n is given by (4.9) with V_n of the form

$$V_n = \frac{1}{2} \sum_{i=-1}^{n-1} \begin{pmatrix} -\frac{1}{2}\sigma_{i2} & \partial_x^{-1}\sigma_{i2} \\ (\lambda^2 - v\lambda - u) \partial_x^{-1}\sigma_{i2} - \frac{1}{2}(\sigma_{i2})_x & \frac{1}{2}\sigma_{i2} \end{pmatrix} \lambda^{n-i-1}, \quad n = -1, 0, \dots,$$

with σ_{i2} denoting the second component of σ_i , thus

$$V_{-1} = 0, \quad V_0 = \begin{pmatrix} -\frac{1}{2} & x \\ (\lambda^2 - v\lambda - u) x & \frac{1}{2} \end{pmatrix}. \quad (6.5)$$

6.2. Non-autonomous DWW hierarchy in the case \mathcal{A}_{-1} . The first few vector fields of the deformed DWW hierarchy $u_{t_n} = \mathbb{K}_n[u]$ in the case \mathcal{A}_{-1} and with the general initial conditions (3.3) have the form

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix}_{t_0} &= \begin{pmatrix} -v \\ 2 \end{pmatrix}, \\ \begin{pmatrix} u \\ v \end{pmatrix}_{t_1} &= \begin{pmatrix} -v \\ 2 \end{pmatrix} + \mathbf{a}_{1,1}(t_1) \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \\ \begin{pmatrix} u \\ v \end{pmatrix}_{t_2} &= \begin{pmatrix} -v \\ 2 \end{pmatrix} + \left[\mathbf{a}_{2,1}(t_2) - 2(t_0 + t_1) \mathbf{a}_{2,2}(t_2) \right] \begin{pmatrix} u_x \\ v_x \end{pmatrix} + \mathbf{a}_{2,2}(t_2) \begin{pmatrix} \frac{1}{4}v_{xxx} + uv_x + \frac{1}{2}vu_x \\ u_x + \frac{3}{2}vv_x \end{pmatrix}. \end{aligned}$$

If we consider the initial conditions (3.10) the deformed vector fields \mathbb{K}_n are given by (3.4) with (3.5) specified by (3.11) with ρ and κ_n as in (6.2). Explicitly,

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix}_{t_0} &= \begin{pmatrix} -v \\ 2 \end{pmatrix}, \\ \begin{pmatrix} u \\ v \end{pmatrix}_{t_1} &= \begin{pmatrix} -v \\ 2 \end{pmatrix} + t_1 \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \\ \begin{pmatrix} u \\ v \end{pmatrix}_{t_2} &= \begin{pmatrix} -v \\ 2 \end{pmatrix} - 2(t_0 + t_1)t_2 \begin{pmatrix} u_x \\ v_x \end{pmatrix} + t_2 \begin{pmatrix} \frac{1}{4}v_{xxx} + uv_x + \frac{1}{2}vu_x \\ u_x + \frac{3}{2}vv_x \end{pmatrix}. \end{aligned}$$

6.3. Non-autonomous DWW hierarchy in the case \mathcal{A}_0 . The deformed DWW hierarchy $u_{t_n} = \mathbb{K}_n[u]$ in the case \mathcal{A}_0 and with the general initial conditions (3.12) is given by the vector fields (3.13), thus in particular

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix}_{t_0} &= \begin{pmatrix} 2u + xu_x \\ v + xv_x \end{pmatrix}, \\ \begin{pmatrix} u \\ v \end{pmatrix}_{t_1} &= \begin{pmatrix} 2u + xu_x \\ v + xv_x \end{pmatrix} + \mathbf{a}_{1,1}(t_1)e^{-t_0} \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \\ \begin{pmatrix} u \\ v \end{pmatrix}_{t_2} &= \begin{pmatrix} 2u + xu_x \\ v + xv_x \end{pmatrix} + \left[\mathbf{c}_{1,1}(t_1)e^{-t_0} + \mathbf{a}_{2,1}(t_2)e^{-(t_0+t_1)} \right] \begin{pmatrix} u_x \\ v_x \end{pmatrix} \\ &\quad + \mathbf{a}_{2,2}(t_2)e^{-2(t_0+t_1)} \begin{pmatrix} \frac{1}{4}v_{xxx} + uv_x + \frac{1}{2}vu_x \\ u_x + \frac{3}{2}vv_x \end{pmatrix}. \end{aligned}$$

For the initial conditions (3.19) from Example 9 the deformed vector fields \mathbb{K}_n are given by (3.13) with (3.14) specified by (3.20) with ρ and κ_n as in (6.2). In this case

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix}_{t_0} &= \begin{pmatrix} 2u + xu_x \\ v + xv_x \end{pmatrix}, \\ \begin{pmatrix} u \\ v \end{pmatrix}_{t_1} &= \begin{pmatrix} 2u + xu_x \\ v + xv_x \end{pmatrix} + t_1 e^{-t_0} \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \\ \begin{pmatrix} u \\ v \end{pmatrix}_{t_2} &= \begin{pmatrix} 2u + xu_x \\ v + xv_x \end{pmatrix} + (t_1 + (e^{-t_1} - 1))e^{-t_0} \begin{pmatrix} u_x \\ v_x \end{pmatrix} + t_2 e^{-2(t_0+t_1)} \begin{pmatrix} \frac{1}{4}v_{xxx} + uv_x + \frac{1}{2}vu_x \\ u_x + \frac{3}{2}vv_x \end{pmatrix}. \end{aligned}$$

The Lax representation for all the above non-autonomous DWW hierarchies is given by (4.13) with L as in (6.3) and with W_n as in (4.18) with respective $\mathbf{u}_{n,i}$ and $\varepsilon = -1$ or 0 .

7. NON-AUTONOMOUS AKNS HIERARCHY

7.1. Autonomous AKNS hierarchy. In the last example we apply our theory to the AKNS hierarchy. It is a bi-Hamiltonian hierarchy given by

$$\begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = K_n = \pi_0 \gamma_n = \pi_1 \gamma_{n-1}, \quad n = 1, 2, \dots,$$

where $\gamma_n = \delta H_n$ are exact one-forms and where

$$\pi_0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \pi_1 = \begin{pmatrix} -q\partial_x^{-1}q & -\frac{1}{2}\partial_x + q\partial_x^{-1}r \\ -\frac{1}{2}\partial_x + r\partial_x^{-1}q & -r\partial_x^{-1}r \end{pmatrix}$$

are two compatible Poisson operators ($i^2 = -1$ throughout this whole section). This hierarchy is generated by (4.20) and (4.21) with the recursion operator and its adjoint given by

$$N = \pi_1 \pi_0^{-1} = i \begin{pmatrix} \frac{1}{2}\partial_x - q\partial_x^{-1}r & -q\partial_x^{-1}q \\ r\partial_x^{-1}r & -\frac{1}{2}\partial_x + r\partial_x^{-1}q \end{pmatrix}, \quad N^\dagger = -i \begin{pmatrix} \frac{1}{2}\partial_x - r\partial_x^{-1}q & r\partial_x^{-1}r \\ -q\partial_x^{-1}q & -\frac{1}{2}\partial_x + q\partial_x^{-1}r \end{pmatrix}.$$

In particular, we have the symmetries

$$K_1 = i \begin{pmatrix} -2q \\ 2r \end{pmatrix}, \quad K_2 = \begin{pmatrix} qx \\ rx \end{pmatrix}, \quad K_3 = i \begin{pmatrix} \frac{1}{2}qx_x - q^2r \\ -\frac{1}{2}r_{xx} + qr^2 \end{pmatrix}, \quad K_4 = \begin{pmatrix} -\frac{1}{4}q_{xxx} + \frac{3}{2}qrr_x \\ -\frac{1}{4}r_{xxx} + \frac{3}{2}qrr_x \end{pmatrix},$$

cosymmetries

$$\gamma_1 = \begin{pmatrix} -2r \\ -2q \end{pmatrix}, \quad \gamma_2 = i \begin{pmatrix} r_x \\ -q_x \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} \frac{1}{2}r_{xx} - qr^2 \\ \frac{1}{2}q_{xx} - rq^2 \end{pmatrix}, \quad \gamma_4 = i \begin{pmatrix} -\frac{1}{4}r_{xxx} + \frac{3}{2}qrr_x \\ \frac{1}{4}q_{xxx} - \frac{3}{2}qrr_x \end{pmatrix},$$

and respective Hamiltonians $H_n = \int h_n dx$, where

$$h_1 = -2rq, \quad h_2 = -i r q_x, \quad h_3 = -\frac{1}{2}q_x r_x - \frac{1}{2}q^2 r^2, \quad h_4 = i \left(\frac{1}{4}r q_{xxx} + \frac{3}{4}q^2 r r_x \right).$$

With the AKNS hierarchy (6.1) one can relate the hierarchy of its master symmetries (4.22) with σ_{-1} and σ_0 given by

$$\sigma_{-1} = i \begin{pmatrix} -2xq \\ 2xr \end{pmatrix}, \quad \sigma_0 = \begin{pmatrix} (xq)_x \\ (xr)_x \end{pmatrix}.$$

One can directly check that in this case

$$\rho = 0 \quad \text{and} \quad \kappa_n = n - 1 \tag{7.1}$$

in the hereditary algebra (3.1). Moreover, the master symmetries σ_n are Hamiltonian as in (4.23) with

$$F = -2 \int xqr dx.$$

The zero-curvature formulation (4.5) for the AKNS hierarchy is given by [14, 15]

$$L = \begin{pmatrix} -i\lambda & q \\ r & i\lambda \end{pmatrix} \tag{7.2}$$

and by

$$U_n = i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \lambda^{n-1} + \frac{i}{2} \sum_{j=1}^{n-1} \begin{pmatrix} -\partial_x^{-1} [rK_{j1} + qK_{j2}] & K_{j1} \\ -K_{j2} & \partial_x^{-1} [rK_{j1} + qK_{j2}] \end{pmatrix} \lambda^{n-j-1}, \quad n = 1, 2, \dots \tag{7.3}$$

In particular,

$$U_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad U_2 \equiv L, \quad U_2 = \begin{pmatrix} -i\lambda^2 - \frac{i}{2}qr & q\lambda + \frac{i}{2}q_x \\ r\lambda - \frac{i}{2}r_x & i\lambda^2 + \frac{i}{2}qr \end{pmatrix},$$

$$U_3 = \begin{pmatrix} -i\lambda^3 - \frac{i}{2}qr\lambda + \frac{1}{4}rqx - \frac{1}{4}qr_x & q\lambda^2 + \frac{i}{2}q_x\lambda - \frac{1}{4}q_{xx} + \frac{1}{2}q^2r \\ r\lambda^2 - \frac{i}{2}r_x\lambda - \frac{1}{4}r_{xx} + \frac{1}{2}qr^2 & i\lambda^3 + \frac{i}{2}qr\lambda - \frac{1}{4}rqx + \frac{1}{4}qr_x \end{pmatrix}.$$

The deformed Lax formulation for the hierarchy of the master symmetries σ_n is given by (4.9) with V_i of the form

$$V_n = i \begin{pmatrix} -x & 0 \\ 0 & x \end{pmatrix} \lambda^{n+1} + \frac{i}{2} \sum_{j=-1}^{n-1} \begin{pmatrix} -\partial_x^{-1} [r\sigma_{j1} + q\sigma_{j2}] & \sigma_{j1} \\ -\sigma_{j2} & \partial_x^{-1} [r\sigma_{j1} + q\sigma_{j2}] \end{pmatrix} \lambda^{n-j-1},$$

where $n = -1, 0, 1, \dots$, and thus

$$V_{-1} = \begin{pmatrix} -ix & 0 \\ 0 & ix \end{pmatrix}, \quad V_0 = \begin{pmatrix} -ix\lambda & xq \\ xr & ix\lambda \end{pmatrix}. \quad (7.4)$$

7.2. Non-autonomous AKNS hierarchy in the case \mathcal{A}_{-1} . The first four systems of the deformed AKNS hierarchy $u_{t_n} = \mathbb{K}_n[u]$ in the case \mathcal{A}_{-1} and with the general initial conditions (3.3) are

$$\begin{aligned} \begin{pmatrix} q \\ r \end{pmatrix}_{t_0} &= i \begin{pmatrix} -2xq \\ 2xr \end{pmatrix}, \\ \begin{pmatrix} q \\ r \end{pmatrix}_{t_1} &= i \begin{pmatrix} -2xq \\ 2xr \end{pmatrix} + i \mathbf{a}_{1,1}(t_1) \begin{pmatrix} -2q \\ 2r \end{pmatrix}, \\ \begin{pmatrix} q \\ r \end{pmatrix}_{t_2} &= i \begin{pmatrix} -2xq \\ 2xr \end{pmatrix} + i [\mathbf{a}_{2,1}(t_2) - (t_0 + t_1) \mathbf{a}_{2,2}(t_2)] \begin{pmatrix} -2q \\ 2r \end{pmatrix} + \mathbf{a}_{2,2}(t_2) \begin{pmatrix} q_x \\ r_x \end{pmatrix}, \\ \begin{pmatrix} q \\ r \end{pmatrix}_{t_3} &= i \begin{pmatrix} -2xq \\ 2xr \end{pmatrix} + i [\partial_{t_2}^{-1} \mathbf{a}_{2,2}(t_2) + \mathbf{a}_{3,1}(t_3) - \tau_2 \mathbf{a}_{3,2}(t_3) + \tau_2^2 \mathbf{a}_{3,3}(t_3)] \begin{pmatrix} -2q \\ 2r \end{pmatrix} \\ &\quad + [\mathbf{a}_{3,2}(t_3) - 2\tau_2 \mathbf{a}_{3,3}(t_3)] \begin{pmatrix} q_x \\ r_x \end{pmatrix} + i \mathbf{a}_{3,3}(t_3) \begin{pmatrix} \frac{1}{2}q_{xx} - q^2r \\ -\frac{1}{2}r_{xx} + qr^2 \end{pmatrix}, \end{aligned}$$

where $\tau_2 = t_0 + t_1 + t_2$.

If we consider the initial conditions (3.10) the deformed vector fields \mathbb{K}_n are given by (3.4) with (3.5) specified by (3.11) with ρ and κ_n as in (7.1), yielding

$$\begin{aligned} \begin{pmatrix} q \\ r \end{pmatrix}_{t_0} &= i \begin{pmatrix} -2xq \\ 2xr \end{pmatrix}, \\ \begin{pmatrix} q \\ r \end{pmatrix}_{t_1} &= i \begin{pmatrix} -2xq \\ 2xr \end{pmatrix} + i t_1 \begin{pmatrix} -2q \\ 2r \end{pmatrix}, \\ \begin{pmatrix} q \\ r \end{pmatrix}_{t_2} &= i \begin{pmatrix} -2xq \\ 2xr \end{pmatrix} - i(t_0 + t_1) t_2 \begin{pmatrix} -2q \\ 2r \end{pmatrix} + t_2 \begin{pmatrix} q_x \\ r_x \end{pmatrix}, \\ \begin{pmatrix} q \\ r \end{pmatrix}_{t_3} &= i \begin{pmatrix} -2xq \\ 2xr \end{pmatrix} + i \left[\frac{1}{2} t_2^2 + (t_0 + t_1 + t_2)^2 t_3 \right] \begin{pmatrix} -2q \\ 2r \end{pmatrix} - 2(t_0 + t_1 + t_2) t_3 \begin{pmatrix} q_x \\ r_x \end{pmatrix} \\ &\quad + i t_3 \begin{pmatrix} \frac{1}{2}q_{xx} - q^2r \\ -\frac{1}{2}r_{xx} + qr^2 \end{pmatrix}. \end{aligned}$$

7.3. Non-autonomous AKNS hierarchy in the case \mathcal{A}_0 . The deformed AKNS hierarchy $u_{t_n} = \mathbb{K}_n[u]$ in the case \mathcal{A}_0 and with the general initial conditions (3.12) is given by the vector fields (3.13); the first few of them read

$$\begin{pmatrix} q \\ r \end{pmatrix}_{t_0} = \begin{pmatrix} (xq)_x \\ (xr)_x \end{pmatrix},$$

$$\begin{aligned}
\begin{pmatrix} q \\ r \end{pmatrix}_{t_1} &= \begin{pmatrix} (xq)_x \\ (xr)_x \end{pmatrix} + i \mathbf{a}_{1,1}(t_1) \begin{pmatrix} -2q \\ 2r \end{pmatrix}, \\
\begin{pmatrix} q \\ r \end{pmatrix}_{t_2} &= \begin{pmatrix} (xq)_x \\ (xr)_x \end{pmatrix} + i \mathbf{a}_{2,1}(t_2) \begin{pmatrix} -2q \\ 2r \end{pmatrix} + \mathbf{a}_{2,2}(t_2) e^{-(t_0+t_1)} \begin{pmatrix} q_x \\ r_x \end{pmatrix}, \\
\begin{pmatrix} q \\ r \end{pmatrix}_{t_3} &= \begin{pmatrix} (xq)_x \\ (xr)_x \end{pmatrix} + i \mathbf{a}_{3,1}(t_3) \begin{pmatrix} -2q \\ 2r \end{pmatrix} \\
&\quad + \left[e^{-(t_0+t_1)} \mathbf{c}_{2,2}(t_2) + \mathbf{a}_{3,2}(t_3) e^{-\tau_2} \right] \begin{pmatrix} q_x \\ r_x \end{pmatrix} + i \mathbf{a}_{3,3}(t_3) e^{-2\tau_2} \begin{pmatrix} \frac{1}{2} q_{xx} - q^2 r \\ -\frac{1}{2} r_{xx} + qr^2 \end{pmatrix},
\end{aligned}$$

where $\tau_2 = t_0 + t_1 + t_2$. Notice that, in this case $\kappa_1 = 0$ and thus all $\mathbf{c}_{r,1} = 0$ due to (3.16).

For the initial conditions (3.19) the deformed vector fields \mathbb{K}_n are given by (3.13) with (3.14) specified by (3.20) with ρ and κ_n as in (7.1), which yields

$$\begin{aligned}
\begin{pmatrix} q \\ r \end{pmatrix}_{t_0} &= \begin{pmatrix} (xq)_x \\ (xr)_x \end{pmatrix}, \\
\begin{pmatrix} q \\ r \end{pmatrix}_{t_1} &= \begin{pmatrix} (xq)_x \\ (xr)_x \end{pmatrix} + i t_1 \begin{pmatrix} -2q \\ 2r \end{pmatrix}, \\
\begin{pmatrix} q \\ r \end{pmatrix}_{t_2} &= \begin{pmatrix} (xq)_x \\ (xr)_x \end{pmatrix} + i \left[\left(t_1 - \frac{1}{\kappa_1} \right) e^{-\kappa_1 t_0} + \frac{1}{\kappa_1} e^{-\kappa_1 (t_0+t_1)} \right] \begin{pmatrix} -2q \\ 2r \end{pmatrix} + t_2 e^{-\kappa_2 (t_0+t_1)} \begin{pmatrix} q_x \\ r_x \end{pmatrix}, \\
\begin{pmatrix} q \\ r \end{pmatrix}_{t_3} &= \begin{pmatrix} (xq)_x \\ (xr)_x \end{pmatrix} + i \left[\left(t_1 - \frac{1}{\kappa_1} \right) e^{-\kappa_1 t_0} + \frac{1}{\kappa_1} e^{-\kappa_1 (t_0+t_1)} \right] \begin{pmatrix} -2q \\ 2r \end{pmatrix}, \\
&\quad + \left[\left(t_2 - \frac{1}{\kappa_2} \right) e^{-\kappa_2 (t_0+t_1)} + \frac{1}{\kappa_2} e^{-\kappa_2 (t_0+t_1+t_2)} \right] \begin{pmatrix} q_x \\ r_x \end{pmatrix} + i t_3 e^{-\kappa_3 (t_0+t_1+t_2)} \begin{pmatrix} \frac{1}{2} q_{xx} - q^2 r \\ -\frac{1}{2} r_{xx} + qr^2 \end{pmatrix}.
\end{aligned}$$

The Lax representation for all the above non-autonomous AKNS hierarchies are given by (4.13) with L as in (6.3) and with W_n as in (4.18) with respective $\mathbf{u}_{n,i}$ and $\varepsilon = -1$ or 0 .

APPENDIX A.

In this appendix we prove Theorem 3. Consider the following linear integral operators

$$\xi_i = 1 + \partial_{t_i}^{-1} \text{ad}_{\mathbb{K}_i}, \quad i = 0, 1, \dots, n-1$$

acting in \mathcal{A} (notice that $\xi_i = \xi_i(t_0, \dots, t_i)$ due to (2.6a)). Then, by (2.4),

$$\xi_i^{-1} = 1 - \partial_{t_i}^{-1} \text{ad}_{\mathbb{K}_i} \xi_i^{-1}.$$

Therefore

$$\begin{aligned}
\partial_{t_i} \xi_i &= \partial_{t_i} + \text{ad}_{\mathbb{K}_i}, \\
\partial_{t_i} \xi_i^{-1} &= \partial_{t_i} - \text{ad}_{\mathbb{K}_i} \xi_i^{-1},
\end{aligned} \tag{A.1}$$

and

$$\partial_{t_i} \xi_j = \xi_j \partial_{t_i} \quad \text{for } j < i,$$

while

$$\partial_{t_i} \xi_j^{-1} = -\xi_j^{-1} \frac{\partial \xi_j}{\partial t_i} \xi_j^{-1} + \xi_j^{-1} \partial_{t_i} = -\xi_j^{-1} [\xi_j, \text{ad}_{\mathbb{K}_i}] \xi_j^{-1} + \xi_j^{-1} \partial_{t_i} \quad \text{for } j > i.$$

Note that $\partial_{t_i} \xi_j$ and $\frac{\partial \xi_j}{\partial t_i}$ are two different operators, as $\partial_{t_i} \xi_j = \frac{\partial \xi_j}{\partial t_i} + \xi_j \partial_{t_i}$. The formula above is due to the fact that for any time-dependent operator ξ_j

$$\frac{\partial \xi_j^{-1}}{\partial t_i} = -\xi_j^{-1} \frac{\partial \xi_j}{\partial t_i} \xi_j^{-1} \tag{A.2}$$

and the fact that the explicit time dependence of ξ_j on t_i is

$$\begin{aligned} \frac{\partial \xi_j}{\partial t_i} &= \partial_{t_j}^{-1} \frac{\partial \text{ad}_{\mathbb{K}_j}}{\partial t_i} \stackrel{(2.6a)}{=} -\partial_{t_j}^{-1} \text{ad}_{[\mathbb{K}_i, \mathbb{K}_j]} = -\partial_{t_j}^{-1} [\text{ad}_{\mathbb{K}_i}, \text{ad}_{\mathbb{K}_j}] = [\partial_{t_j}^{-1} \text{ad}_{\mathbb{K}_j}, \text{ad}_{\mathbb{K}_i}] \\ &= [\xi_j, \text{ad}_{\mathbb{K}_i}] \quad \text{for } i < j. \end{aligned} \quad (\text{A.3})$$

Notice that $\frac{\partial \xi_j}{\partial t_i} = 0$ for $i > j$.

Denote now $\mathbf{t} = (t_0, \dots, t_{n-1})$ and consider (2.8), that is

$$\mathbb{K}(\mathbf{t}) = \xi_{n-1}^{-1} \cdots \xi_1^{-1} \xi_0^{-1} \bar{\mathbb{K}}.$$

We will now show that $\mathbb{K}(\mathbf{t})$ is the solution of the IVP (2.7). Naturally, $\mathbb{K}(\mathbf{0}) = \bar{\mathbb{K}}$. Using (A.2) we obtain

$$\begin{aligned} \frac{\partial \mathbb{K}(\mathbf{t})}{\partial t_i} &= \partial_{t_i} \xi_{n-1}^{-1} \cdots \xi_1^{-1} \xi_0^{-1} \bar{\mathbb{K}} \\ &= -\xi_{n-1}^{-1} \frac{\partial \xi_{n-1}}{\partial t_i} \xi_{n-1}^{-1} \xi_{n-2}^{-1} \cdots \xi_1^{-1} \xi_0^{-1} \bar{\mathbb{K}} + \xi_{n-1}^{-1} \partial_{t_i} \xi_{n-2}^{-1} \cdots \xi_1^{-1} \xi_0^{-1} \bar{\mathbb{K}} = \cdots = \\ &= -\sum_{j=i+1}^{n-1} \xi_{n-1}^{-1} \cdots \xi_j^{-1} \frac{\partial \xi_j}{\partial t_i} \xi_j^{-1} \cdots \xi_{i+1}^{-1} \xi_i^{-1} \cdots \xi_0^{-1} \bar{\mathbb{K}} + \xi_{n-1}^{-1} \cdots \xi_{i+1}^{-1} \partial_{t_i} \xi_i^{-1} \cdots \xi_0^{-1} \bar{\mathbb{K}} \end{aligned}$$

and further by (A.3) and (A.1)

$$\begin{aligned} \frac{\partial \mathbb{K}(\mathbf{t})}{\partial t_i} &= -\sum_{j=i+1}^{n-1} \xi_{n-1}^{-1} \cdots \xi_j^{-1} [\xi_j, \text{ad}_{\mathbb{K}_i}] \xi_j^{-1} \cdots \xi_{i+1}^{-1} \xi_i^{-1} \cdots \xi_0^{-1} \bar{\mathbb{K}} - \xi_{n-1}^{-1} \cdots \xi_{i+1}^{-1} \text{ad}_{\mathbb{K}_i} \xi_i^{-1} \cdots \xi_0^{-1} \bar{\mathbb{K}} \\ &= -\text{ad}_{\mathbb{K}_i} \xi_{n-1}^{-1} \cdots \xi_{i+1}^{-1} \xi_i^{-1} \cdots \xi_0^{-1} \bar{\mathbb{K}} - \sum_{j=i+1}^{n-2} \xi_{n-1}^{-1} \cdots \xi_{j+1}^{-1} \text{ad}_{\mathbb{K}_i} \xi_j^{-1} \cdots \xi_{i+1}^{-1} \xi_i^{-1} \cdots \xi_0^{-1} \bar{\mathbb{K}} \\ &\quad + \sum_{j=i+1}^{n-1} \xi_{n-1}^{-1} \cdots \xi_j^{-1} \text{ad}_{\mathbb{K}_i} \xi_{j-1}^{-1} \cdots \xi_i^{-1} \xi_{i-1}^{-1} \cdots \xi_0^{-1} \bar{\mathbb{K}} - \xi_{n-1}^{-1} \cdots \xi_{i+1}^{-1} \text{ad}_{\mathbb{K}_i} \xi_i^{-1} \cdots \xi_0^{-1} \bar{\mathbb{K}} \\ &= -\text{ad}_{\mathbb{K}_i} \xi_{n-1}^{-1} \cdots \xi_0^{-1} \bar{\mathbb{K}} \equiv -\text{ad}_{\mathbb{K}_i} \mathbb{K}(\mathbf{t}). \end{aligned}$$

APPENDIX B.

We prove here (3.8) and (3.18) for the cases $\varepsilon = -1$ and $\varepsilon = 0$, respectively. In either case we have

$$[\sigma_\varepsilon, K_i] = \kappa_i K_{i+\varepsilon}, \quad i \geq 1, \quad K_0 \equiv 0.$$

Consider (3.13) and (3.4), thus

$$\mathbb{K}_n = \sigma_\varepsilon + \sum_{i=1}^n \mathbf{u}_{n,i}(t_0, \dots, t_n) K_i, \quad \varepsilon = 0, -1.$$

Then for $m < n$:

$$\begin{aligned} \frac{\partial \mathbb{K}_n}{\partial t_m} + [\mathbb{K}_m, \mathbb{K}_n] &= \sum_{i=1}^n (\mathbf{u}_{n,i})_{t_m} K_i + \sum_{i=1}^n \mathbf{u}_{n,i} [\sigma_\varepsilon, K_i] + \sum_{j=1}^m \mathbf{u}_{m,j} [K_j, \sigma_\varepsilon] \\ &= \sum_{i=1}^n (\mathbf{u}_{n,i})_{t_m} K_i + \sum_{i=1-\varepsilon}^n \kappa_i \mathbf{u}_{n,i} K_{i+\varepsilon} - \sum_{i=1-\varepsilon}^m \kappa_i \mathbf{u}_{m,i} K_{i+\varepsilon} \\ &= -\varepsilon (\mathbf{u}_{n,n})_{t_m} K_n + \sum_{i=m+1}^n [(\mathbf{u}_{n,i+\varepsilon})_{t_m} + \kappa_i \mathbf{u}_{n,i}] K_{i+\varepsilon} \\ &\quad + \sum_{i=1-\varepsilon}^m [(\mathbf{u}_{n,i+\varepsilon})_{t_m} + \kappa_i (\mathbf{u}_{n,i} - \mathbf{u}_{m,i})] K_{i+\varepsilon}. \end{aligned}$$

Thus, $\frac{\partial \mathbb{K}_n}{\partial t_m} + [\mathbb{K}_m, \mathbb{K}_n] = 0$ for $m < n$ if and only if

$$\begin{cases} \varepsilon(\mathbf{u}_{n,n})_{t_m} = 0, \\ (\mathbf{u}_{n,i})_{t_m} + \kappa_{i-\varepsilon} \mathbf{u}_{n,i-\varepsilon} = 0, & m+1+\varepsilon \leq i \leq n+\varepsilon, \\ (\mathbf{u}_{n,i})_{t_m} + \kappa_{i-\varepsilon} (\mathbf{u}_{n,i-\varepsilon} - \mathbf{u}_{m,i-\varepsilon}) = 0, & 1 \leq i \leq m+\varepsilon, \end{cases}$$

which yields (3.18) for $\varepsilon = 0$ and (3.8) for $\varepsilon = -1$.

APPENDIX C.

We prove here (3.9). Differentiating (3.6) with respect to t_j yields

$$\begin{aligned} (\mathbf{c}_n)_{t_j} &= \sum_{m=j+1}^{n-1} \sum_{r=3}^m \sum_{s=1}^{r-2} \frac{(-1)^{r-1} [\kappa_r]!}{(r-s-2)!} (\tau_{m-1})^{r-s-2} (\partial_{t_m}^{-1})^s \mathbf{a}_{m,r}(t_m) \\ &\quad + \sum_{r=2}^j \sum_{s=1}^{r-1} \frac{(-1)^{r-1} [\kappa_r]!}{(r-s-1)!} (\tau_{j-1})^{r-s-1} (\partial_{t_j}^{-1})^{s-1} \mathbf{a}_{j,r}(t_j) \\ &\quad + \sum_{r=2}^n \frac{(-1)^r [\kappa_r]!}{(r-2)!} (\tau_{n-1})^{r-2} \mathbf{a}_{n,r}(t_n), \end{aligned}$$

and further

$$\begin{aligned} (\mathbf{c}_n)_{t_j} &= \sum_{m=j}^{n-1} \sum_{r=3}^m \sum_{s=1}^{r-2} \frac{(-1)^{r-1} [\kappa_r]!}{(r-s-2)!} (\tau_{m-1})^{r-s-2} (\partial_{t_m}^{-1})^s \mathbf{a}_{m,r}(t_m) \\ &\quad - \sum_{r=3}^j \sum_{s=1}^{r-2} \frac{(-1)^{r-1} [\kappa_r]!}{(r-s-2)!} (\tau_{j-1})^{r-s-2} (\partial_{t_j}^{-1})^s \mathbf{a}_{j,r}(t_j) \\ &\quad + \sum_{r=2}^j \sum_{s=1}^{r-1} \frac{(-1)^{r-1} [\kappa_r]!}{(r-s-1)!} (\tau_{j-1})^{r-s-1} (\partial_{t_j}^{-1})^{s-1} \mathbf{a}_{j,r}(t_j) \\ &\quad + \sum_{r=2}^n \frac{(-1)^r [\kappa_r]!}{(r-2)!} (\tau_{n-1})^{r-2} \mathbf{a}_{n,r}(t_n). \end{aligned}$$

Thus

$$\begin{aligned} (\mathbf{c}_n)_{t_j} &= \sum_{m=j}^{n-1} \sum_{r=3}^m \sum_{s=1}^{r-2} \frac{(-1)^{r-1} [\kappa_r]!}{(r-s-2)!} (\tau_{m-1})^{r-s-2} (\partial_{t_m}^{-1})^s \mathbf{a}_{m,r}(t_m) \\ &\quad + \left(\sum_{r=2}^j \sum_{s=0}^{r-2} - \sum_{r=3}^j \sum_{s=1}^{r-2} \right) \frac{(-1)^{r-1} [\kappa_r]!}{(r-s-2)!} (\tau_{j-1})^{r-s-2} (\partial_{t_j}^{-1})^s \mathbf{a}_{j,r}(t_j) \\ &\quad + \sum_{r=2}^n \frac{(-1)^r [\kappa_r]!}{(r-2)!} (\tau_{n-1})^{r-2} \mathbf{a}_{n,r}(t_n), \end{aligned}$$

and finally

$$\begin{aligned} (\mathbf{c}_n)_{t_j} &= \left(\sum_{m=2}^{n-1} - \sum_{m=2}^{j-1} \right) \sum_{r=3}^m \sum_{s=1}^{r-2} \frac{(-1)^{r-1} [\kappa_r]!}{(r-s-2)!} (\tau_{m-1})^{r-s-2} (\partial_{t_m}^{-1})^s \mathbf{a}_{m,r}(t_m) \\ &\quad + \left(\sum_{r=2}^n - \sum_{r=2}^j \right) \frac{(-1)^r [\kappa_r]!}{(r-2)!} (\tau_{n-1})^{r-2} \mathbf{a}_{n,r}(t_n) \\ &\equiv (\mathbf{c}_n)_{t_0} - (\mathbf{c}_j)_{t_0}. \end{aligned}$$

APPENDIX D.

We prove here Theorem 14. We start by observing that, due to (3.1)

$$(\Psi_{s_i})_{\tau_j} - (\Psi_{\tau_j})_{s_i} = \Psi'[[\sigma_j, K_i]] = \kappa_i \Psi'[K_{i+j}] = \kappa_i \Psi_{s_{i+j}},$$

which implies

$$(U_i)_{\tau_j} - (V_j)_{s_i} + [U_i, V_j] + \lambda^{j+1}(U_i)_\lambda = \kappa_i U_{i+j},$$

or, equivalently,

$$U'_i[\sigma_j] - V'_j[K_i] + [U_i, V_j] + \lambda^{j+1}(U_i)_\lambda = \kappa_i U_{i+j},$$

so that, in particular

$$U'_i[\sigma_\varepsilon] - V'_\varepsilon[K_i] + [U_i, V_\varepsilon] + \lambda^{\varepsilon+1}(U_i)_\lambda = \kappa_i U_{i+\varepsilon}, \quad \varepsilon = -1 \text{ or } 0. \quad (\text{D.1})$$

Obviously

$$\frac{\partial W_n}{\partial t_m} = \sum_{i=1}^n (\mathbf{v}_{n,i})_{t_m} U_i.$$

Further

$$\begin{aligned} W'_n[\mathbb{K}_m] - W'_m[\mathbb{K}_n] &\stackrel{(4.12)}{=} V'_\varepsilon[\mathbb{K}_m] + \sum_{i=1}^n \mathbf{v}_{n,i} U'_i[\mathbb{K}_m] - V'_\varepsilon[\mathbb{K}_n] - \sum_{j=1}^m \mathbf{v}_{m,j} U'_j[\mathbb{K}_n] \\ &\stackrel{(4.14)}{=} \sum_{j=1}^m \mathbf{v}_{m,j} (V'_\varepsilon[K_j] - U'_j[\sigma_\varepsilon]) - \sum_{i=1}^n \mathbf{v}_{n,i} (V'_\varepsilon[K_i] - U'_i[\sigma_\varepsilon]) + \sum_{i=1}^n \sum_{j=1}^m \mathbf{v}_{n,i} \mathbf{v}_{m,j} (U'_i[K_j] - U'_j[K_i]) \\ &\stackrel{(\text{D.1}), (4.7)}{=} \sum_{j=1}^m \mathbf{v}_{m,j} ([U_j, V_\varepsilon] + \lambda^{\varepsilon+1}(U_j)_\lambda - \kappa_j U_{j+\varepsilon}) \\ &\quad - \sum_{i=1}^n \mathbf{v}_{n,i} ([U_i, V_\varepsilon] + \lambda^{\varepsilon+1}(U_i)_\lambda - \kappa_i U_{i+\varepsilon}) - \sum_{i=1}^n \sum_{j=1}^m \mathbf{v}_{n,i} \mathbf{v}_{m,j} [U_i, U_j], \end{aligned}$$

and

$$[W_n, W_m] \stackrel{(4.12)}{=} \sum_{j=1}^m \mathbf{v}_{m,j} [V_\varepsilon, U_j] + \sum_{i=1}^n \mathbf{v}_{n,i} [U_i, V_\varepsilon] + \sum_{i=1}^n \sum_{j=1}^m \mathbf{v}_{n,i} \mathbf{v}_{m,j} [U_i, U_j],$$

and finally

$$\lambda^{\varepsilon+1}(W_n)_\lambda - \lambda^{\varepsilon+1}(W_m)_\lambda \stackrel{(4.12)}{=} \lambda^{\varepsilon+1} \sum_{i=1}^n \mathbf{v}_{n,i} (U_i)_\lambda - \lambda^{\varepsilon+1} \sum_{j=1}^m \mathbf{v}_{m,j} (U_j)_\lambda.$$

Thus, for $m < n$

$$\begin{aligned} \frac{\partial W_n}{\partial t_m} + W'_n[\mathbb{K}_m] - W'_m[\mathbb{K}_n] + [W_n, W_m] + \lambda^{\varepsilon+1}(W_n)_\lambda - \lambda^{\varepsilon+1}(W_m)_\lambda &= \\ &= \sum_{i=1}^n (\mathbf{v}_{n,i})_{t_m} U_i - \sum_{j=1-\varepsilon}^m \kappa_i \mathbf{v}_{m,i} U_{i+\varepsilon} + \sum_{i=1-\varepsilon}^n \kappa_i \mathbf{v}_{n,i} U_{i+\varepsilon} \\ &= -\varepsilon (\mathbf{v}_{n,n})_{t_m} U_n + \sum_{i=m+1}^n [(\mathbf{v}_{n,i+\varepsilon})_{t_m} + \kappa_i \mathbf{v}_{n,i}] U_{i+\varepsilon} \\ &\quad + \sum_{i=1-\varepsilon}^m [(\mathbf{v}_{n,i+\varepsilon})_{t_m} + \kappa_i (\mathbf{v}_{n,i} - \mathbf{v}_{m,i})] U_{i+\varepsilon} = 0. \end{aligned}$$

Thus, (4.17) holds if and only if

$$\begin{cases} \varepsilon(\mathbf{v}_{n,n})_{t_m} = 0, \\ (\mathbf{v}_{n,i})_{t_m} + \kappa_{i-\varepsilon}\mathbf{v}_{n,i-\varepsilon} = 0, & m+1+\varepsilon \leq i \leq n+\varepsilon \\ (\mathbf{v}_{n,i})_{t_m} + \kappa_{i-\varepsilon}(\mathbf{v}_{n,i-\varepsilon} - \mathbf{v}_{m,i-\varepsilon}) = 0, & 1 \leq i \leq m+\varepsilon \end{cases}$$

which coincides with equations (3.8) for $\varepsilon = -1$ and with (3.18) for $\varepsilon = 0$. Thus, in both cases, $\mathbf{v}_{n,i} \equiv \mathbf{u}_{n,i}$ for all n, i .

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