Commutation of transfer and Aubert-Zelevinski involution for metaplectic groups

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Abstract

A result of K. Hiraga says endoscopic transfer is compatible with Aubert-Zelevinski involution. In this short note, we generalize Hiraga's result to metaplectic group setting.

1 Introduction

The theory of endoscopy was developed to understand the internal structure of Lpackets and it is foundational to the stabilization of the trace formula. The formulation of endoscopy was generalized by W-W. Li [Li11] to metaplectic groups and had lead to the recent stabilization of trace formula for metaplectic groups [Li21].

An important property of endoscopic transfer, proved by Hiraga [Hir04] for ordinary endoscopy and by B. Xu [Xu17a, Appendix] for twisted endoscopy, is that Aubert-Zelevinski involution is compatible with endoscopic transfer. This property plays a role in Arthur's endoscopic classification [Art13] when one want to study non-tempered representations.

The goal of this short note is to prove endoscopic transfer for metaplectic groups is also compatible with Aubert-Zelevinski involution:

Theorem 1.1. Let \tilde{G} be a metaplectic group and let $G^!$ be an endoscopic datum of \tilde{G} . Then for any stable virtual character Θ of $G^!$ we have

$$D_{\tilde{G}} \circ \mathcal{T}_{G^!,\tilde{G}}^{\vee}(\Theta) = \mathcal{T}_{G^!,\tilde{G}}^{\vee} \circ D_{G^!}(\Theta)$$

where D stands for the Aubert-Zelevinski involution and $\mathcal{T}_{G^!,\tilde{G}}^{\vee}$ stands for endoscopic transfer for distributions.

The proof is based on the idea of Hiraga. However, there are some new features in metaplectic group case which makes it (at least to the author) not clear whether Hiraga's result can be applied word for word. For example, when we considering relation of $\mathcal{T}_{G',\tilde{G}}^{\vee}$ with parabolic induction and Jacquet module, we need to include certain twists. Besides, some steps of Hiraga's proof uses combinatorial properties of Weyl groups, these are replaced by arguments using explicit partitions in this note.

Organization The structure of this article is as follows. In Section 2, we recall facts about endoscopy for \tilde{G} established by Li. In Section 3, we discuss compatibility of endoscopic transfer with Jacquet functors. In the last section, we prove Theorem 1.1.

Preliminaries 2

In this section, we fix our notations and recall facts about metaplectic groups and the theory of endoscopy for them following [Li11, Li19, Li21]. We will work over a non-Archimedean local field F such that char(F) = 0. All algebraic groups H are defined over F and we use the same notation for their group of F-points.

2.1Metaplectic groups

Let Sp(2n) be the symplectic group associated to a 2n-dimensional symplectic vector space. After fixing a Borel subgroup and a maximal torus inside, all the standard Levi subgroups of $\operatorname{Sp}(2n)$ takes the form $M = \prod_{i=1}^{k} \operatorname{GL}(n_i) \times \operatorname{Sp}(2m)$ where $n_i \in \mathbb{Z}_{>0}, m \in \mathbb{Z}_{\geq 0}$ and $\sum_{i=1}^{k} n_i + m = n$. The set of standard Levi subgroup is in bijection with $(\{n_i\}_i, m)$ of the above form.

Let SO(2n+1) be the split odd orthogonal group over F. Then after fixing a Borel pair, all standard Levi subgroup takes the form $\prod_{i=1}^{k} \operatorname{GL}(n_i) \times \operatorname{SO}(2m+1)$. We fix a non-trivial additive character $\psi: F \to \mathbb{C}^{\times}$, then the Weil representation give

rise to a central extension

$$1 \to \mathbb{C}^{\times} \to \overline{\operatorname{Sp}}_{\psi}(2n) \to \operatorname{Sp}(2n) \to 1.$$

It is well known that the derived subgroup of $\overline{\mathrm{Sp}}_{\psi}(2n)$ is an central extension by μ_2 hence we may reduce $\overline{\mathrm{Sp}}_{\psi}(2n)$ to an extension

$$1 \to \mu_8 \to \widetilde{\mathrm{Sp}}(2n) \to \mathrm{Sp}(2n) \to 1.$$

For any subgroup $H \subset \operatorname{Sp}(2n)$, denote the inverse image of H in $\widetilde{\operatorname{Sp}}(2n)$ by \widetilde{H} . One of the advantages of working with extensions by μ_8 , instead of working with the usual μ_2 extensions, is that: for any standard Levi subgroup $M = \prod_{i=1}^{k} \operatorname{GL}(n_i) \times \operatorname{Sp}(2m) \subset \operatorname{Sp}(2n)$, we have an isomorphism (depends on the fixed additive character ψ)

$$\widetilde{M} \simeq \prod_{i=1}^{k} \operatorname{GL}(n_i) \times \widetilde{\operatorname{Sp}}(2m).$$

We will say \tilde{M} is a standard Levi subgroup of $\widetilde{\mathrm{Sp}}(2n)$. A group of the form $\prod_i \mathrm{GL}(n_i) \times$ Sp(2m) will be called a group of metaplectic type.

Let H be a reductive group or a group of meteplectic type. For any standard Levi subgroup $L \subset H$, the normalized parabolic induction (resp. normalized Jacquet module) is denoted by i_L^H (resp. r_L^H). We view them as maps between the spaces of virtual (genuine) characters over relevant groups.

2.2Endoscopy

From now on, we fix an positive integer n and set $G = \operatorname{Sp}(2n)$, $\tilde{G} = \widetilde{\operatorname{Sp}}(2n)$.

Following [Li11] (see also [Li21, Chapter 3] for a summary), an elliptic endoscopic datum of \tilde{G} is a triple $(G^!, n', n'')$, where

$$(n', n'') \in \mathbb{Z}^2_{\geq 0}, n' + n'' = n$$

 $G! := SO(2n' + 1) \times SO(2n'' + 1)$

and $G^!$ is the endoscopic group attached to the datum $(G^!, n', n'')$. If there is no ambiguity, an endoscopic group and endoscopic datum will both be denoted as $G^!$. We would like to mention that (n', n'') and (n'', n') will give rise to unequivalent endoscopic data which is a new feature of endoscopy in metaplectic group case.

For $\tilde{M} \simeq \prod_{i=1}^{k} \operatorname{GL}(n_i) \times \widetilde{\operatorname{Sp}}(2m)$ a group of metaplectic type, an elliptic endoscopic group of \tilde{M} is defined to be a group of the form $\prod_{i=1}^{k} \operatorname{GL}(n_i) \times M^!$ where $M^!$ is an elliptic endoscopic group of $\widetilde{\operatorname{Sp}}(2m)$.

For \tilde{G} , the notion of stable conjugacy is defined in [Li11, Section 5.2]. The set of stable regular semisimple conjugacy classes in \tilde{G} is denoted by $\Sigma_{reg}(\tilde{G})$ and let $\Sigma_{reg}(G^!)$ be the set of stable strongly regular semisimple conjugacy classes in $G^!$. Then a correspondence between semisimple elements

$$\Psi_{G^!,G}: \Sigma_{reg}(G^!) \to \Sigma_{reg}(\tilde{G})$$

is defined as in [Li11, Section 5.1]. This map can be made explicit in terms of eigenvalues. Suppose $\delta = (\delta', \delta'') \in G^!$ and let $((a'_i)_{i=1,\dots,n'}^{\pm 1}, 1)$ (resp. $((a''_i)_{i=1,\dots,n''}^{\pm 1}, 1)$) be the eigenvalues of δ' (resp. δ''). Then the eigenvalues of $\Psi_{G^!,G}(\delta)$ are $(a'_i)_{i=1,\dots,n'}^{\pm 1}, (-a''_i)_{i=1,\dots,n''}^{\pm 1}$.

Let $\Sigma_{G-reg}(G^!) \subset \Sigma_{reg}(G^!)$ be the subset consisting of δ such that $\Psi_{G^!,G}(\delta)$ is regular and let $\Gamma_{reg}(G)$ be the set of regular semisimple conjugacy classes in G. The transfer factor in this setting is a map

$$\Delta_{G^!,G}(-,-):\Sigma_{G-reg}(G^!)\times\Gamma_{reg}(G)\to\mu_8$$

defined in [Li11, Section 5.3].

The transfer factor satisfies parabolic descent [Li11, Section 5.4]. Suppose $\delta \in G^!$ and $\tilde{\gamma} \in \tilde{G}$ belong to Levi subgroups

$$L = (\prod_{i} \operatorname{GL}(n'_{i}) \times \operatorname{SO}(2m'+1)) \times (\prod_{i} \operatorname{GL}(n''_{i}) \times \operatorname{SO}(2m''+1))$$
$$\tilde{M} = \prod_{i=1}^{k} \operatorname{GL}(n_{i}) \times \widetilde{\operatorname{Sp}}(2m)$$

respectively, where $n'_i + n''_i = n_i$. Then we can write

$$\delta = (\{\delta'_i\}_i, \delta') \times (\{\delta''_i\}_i, \delta'')$$
$$\tilde{\gamma} = (\{\tilde{\gamma}_i\}, \tilde{\gamma}^{\flat})$$

where $\delta'_i \in \operatorname{GL}(n'_i), \, \delta' \in \operatorname{SO}(2m'+1)$ etc. Note that $L^{\flat} := \operatorname{SO}(2m'+1) \times \operatorname{SO}(2m''+1)$ is an elliptic endoscopic group for $\tilde{G}^{\flat} := \operatorname{Sp}(2m)$, the parabolic descent for transfer factor means we have

$$\Delta_{G^{!},G}(\delta,\tilde{\gamma}) = \Delta_{L^{\flat},\tilde{G}^{\flat}}((\delta',\delta''),\tilde{\gamma}^{\flat}).$$

Let $\mathcal{T}_{G^!,\tilde{G}}^{\vee}$ be the transfer map for distributions defined as in [Li21, Section 3.8]. It follows from the results of [Li19] that if Θ is a stable virtual character of $G^!$ then $\mathcal{T}_{G^!,\tilde{G}}^{\vee}(\Theta)$ is a virtual genuine character of \tilde{G} . In this case, we have

$$\mathcal{T}^{\vee}_{G^!,\tilde{G}}(\Theta)(\tilde{\gamma}) = \sum_{\substack{\delta \in \Sigma_{\mathrm{reg}}(G^!) \\ \Psi_{G^!,\tilde{G}}(\delta) = \gamma}} \Delta_{G^!,\tilde{G}}(\delta,\tilde{\gamma}) \Theta(\delta)$$

holds for $\forall \gamma \in \Gamma_{reg}(G)$ and $\tilde{\gamma}$ can be any inverse image of γ .

The formulations above can be extended to the case when \tilde{G} is replace by a group of metaplectic type by treating GL factors and the metaplectic factor separately. Suppose $G^!$ is an endoscopic datum of \tilde{G} then by definition $G^!$ is an elliptic endoscopic datum of some standard Levi \tilde{M} of \tilde{G} . Then we can extend the definition of Ψ, Δ to the pair $(G^!, \tilde{G})$ via the inclusion $\tilde{M} \subset \tilde{G}$ and set

$$\mathcal{T}_{G^!,\tilde{G}}^{\vee}=i_{\tilde{M}}^{\tilde{G}}\circ\mathcal{T}_{G^!,\tilde{M}}^{\vee}$$

all the formulations above extend to this case as well.

3 Jacquet module and endoscopy

In this section, we consider the relation between Jacquet module and endoscopic transfer for metaplectic groups.

Let $G^! = \mathrm{SO}(2n'+1) \times \mathrm{SO}(2n''+1)$ be an elliptic endoscopic datum of \tilde{G} and let $M = \prod_{i=1}^{k} \mathrm{GL}(n_i) \times \mathrm{Sp}(2m)$ be a standard Levi subgroup of G. We define $\mathcal{E}(\tilde{M}, G^!)$ as the set of all sequences $\{(n'_i, n''_i)\}_{i=1,\dots,k}$ satisfying:

- For $\forall 1 \leq i \leq k, n'_i, n''_i \in \mathbb{Z}_{\geq 0}$ and $n'_i + n''_i = n_i$
- $\sum n'_i \le n'$ and $\sum n''_i \le n''$.

For any $s \in \mathcal{E}(\tilde{M}, G^!)$, we set

$$M_s := (\prod_i \operatorname{GL}(n'_i) \times \prod_i \operatorname{GL}(n''_i)) \times \operatorname{Sp}(2m)$$
$$M_s^! := (\prod_i \operatorname{GL}(n'_i) \times \operatorname{SO}(2m'+1)) \times (\prod_i \operatorname{GL}(n''_i) \times \operatorname{SO}(2m''+1))$$

where $m' = n' - \sum n'_i$ and $m'' = n'' - \sum n''_i$ and we view $M_s^!$ as a standard Levi subgroup of $G^!$ via inclusions

$$\prod_{i} \operatorname{GL}(n'_{i}) \times \operatorname{SO}(2m'+1) \hookrightarrow \operatorname{SO}(2n'+1)$$
$$\prod_{i} \operatorname{GL}(n''_{i}) \times \operatorname{SO}(2m''+1) \hookrightarrow \operatorname{SO}(2n''+1).$$

Here the product respects the order of subscripts and subscripts i with $n'_i = 0$ (resp. $n''_i = 0$) are omitted. Note that M'_s is an elliptic endoscopic group of \tilde{M}_s and M_s is a

standard Levi subgroup of M. Hence we can also view $M_s^!$ as an endoscopic group (not necessarily elliptic) of \tilde{M} .

Let $z_s \in M_s^!$ be the central element whose projections to SO factors, $\operatorname{GL}(n'_i)$ are 1 and projections to $\operatorname{GL}(n''_i)$ are -1. Multiplication by z_s induces a map on distributions over $M_s^!$ which we denote by $[z_s]$. Since z_s has order 2, $[z_s]$ is an involution. With these preparations, we can now state the relation between \mathcal{T}^{\vee} and Jacquet module, which is an analogue of [Hir04, Proposition 5.6].

Theorem 3.1. For any stable virtual character Θ , we have

$$r_{\tilde{M}}^{\tilde{G}} \circ \mathcal{T}_{G^{!},\tilde{G}}^{\vee}(\Theta) = \sum_{s \in \mathcal{E}(M,G^{!})} \mathcal{T}_{M_{s}^{!},\tilde{M}}^{\vee} \circ [z_{s}] \circ r_{M_{s}^{!}}^{G^{!}}(\Theta)$$

Here the right hand side make sense because $r_{M_s^!}^{G^!}$ maps stable virtual characters to stable virtual characters (see eg. [Hir04, Lemma 3.3]).

Lemma 3.2. For $\gamma \in M_{G-reg}$, the following sets are in bijection:

•
$$\{\delta \in \Sigma_{reg}(G^!) | \Psi_{G^!,G}(\delta) = \gamma\}$$

• $\{(s, \delta_s) | s \in \mathcal{E}(M, G^!), \ \delta_s \in \Sigma_{reg}(M_s^!), \Psi_{M_s^!, M}(\delta_s \cdot z_s) = \gamma \}$

Proof. For δ in the first set, let δ' (resp. δ'') be its projection to SO(2n' + 1) (resp. SO(2n'' + 1)). Let γ_i be the projection of γ to $GL(n_i)$. For each $1 \leq i \leq k$, let $2n'_i$ be the number of eigenvalues of δ' which is also an eigenvalue of γ_i . Similarly, we define $2n''_i$ to be the number of eigenvalues of δ'' which is also an eigenvalue of $-\gamma_i$. Then since γ is regular, n'_i, n''_i are well-defined integers and we have $n'_i + n''_i = n_i$. Thus we obtain an element in $\mathcal{E}(M, G^!)$ given by $s = \{(n'_i, n''_i)\}_{i=1,\dots,k}$ and δ can be conjugated to some element $\delta_s \in M^!_s$. By [Li21, Proposition 3.4.9], (s, δ_s) belongs to the second set.

Proof of Theorem 3.1. Let A_M be the center of M and let Δ_G (resp. Δ_M) be the set of simple roots of G (resp. M) for our fixed choice of Borel pair. Define

$$A_M^- := \{ t \in A_M \mid \forall \alpha \in \Delta_G - \Delta_M , |\alpha(t)| < 1 \}.$$

Then Casselman's character formula, where a proof for covering group setting can be found in [Luo17], implies for any $\gamma \in M_{G-\text{reg}}$, $a \in A_M^-$ and $n \gg 0$ we have

$$r_{\tilde{M}}^{\tilde{G}} \circ \mathcal{T}_{G^{!},\tilde{G}}^{\vee}(\Theta)(a^{n}\tilde{\gamma}) = \mathcal{T}_{G^{!},\tilde{G}}^{\vee}(\Theta)(a^{n}\tilde{\gamma})$$
$$= \sum_{\substack{\delta \in \Sigma_{\mathrm{reg}}(G^{!})\\\Psi_{G^{!},G}(\delta) = a^{n}\gamma}} \Delta_{G^{!},\tilde{G}}(\delta, a^{n}\tilde{\gamma})\Theta(\delta)$$

where $\tilde{\gamma}$ is an inverse image of γ in \tilde{G} . On the other hand, for each $s \in \mathcal{E}(M, G^{!})$, we have

$$\mathcal{T}_{M_{s}^{!},\tilde{M}}^{\vee} \circ [z_{s}] \circ r_{M_{s}^{!}}^{G^{!}}(\Theta)(a^{n}\tilde{\gamma}) = \sum_{\substack{\delta_{s} \in \Sigma_{\mathrm{reg}}(M_{s}^{!})\\\Psi_{M_{s}^{!},M}(\delta_{s}) = a^{n}\gamma}} \Delta_{M_{s}^{!},\tilde{M}}(\delta_{s}, a^{n}\tilde{\gamma})[z_{s}] \circ r_{M_{s}^{!}}^{G^{!}}(\Theta)(\delta_{s})$$
$$= \sum_{\substack{\delta_{s} \in \Sigma_{\mathrm{reg}}(M_{s}^{!})\\\Psi_{M_{s}^{!},M}(\delta_{s} \cdot z_{s}) = a^{n}\gamma}} \Delta_{M_{s}^{!},\tilde{M}}(\delta_{s} \cdot z_{s}, a^{n}\tilde{\gamma})\Theta(\delta_{s})$$

where the second equality is obtained via replacing δ_s by $\delta_s \cdot z_s$. Note that multiplication by z_s only affects GL factors, hence parabolic descent for transfer factors implies

$$\Delta_{M_s^!,\tilde{M}}(\delta_s \cdot z_s, a^n \tilde{\gamma}) = \Delta_{G^!,\tilde{G}}(\delta_s, a^n \tilde{\gamma}).$$

Take summation over $s \in \mathcal{E}(M, G^{!})$, then by Lemma 3.2 we have

$$r_{\tilde{M}}^{\tilde{G}} \circ \mathcal{T}_{G^{!},\tilde{G}}^{\vee}(\Theta)(a^{n}\tilde{\gamma}) = \sum_{s \in \mathcal{E}(M,G^{!})} \mathcal{T}_{M_{s}^{!},\tilde{M}}^{\vee} \circ [z_{s}] \circ r_{M_{s}^{!}}^{G^{!}}(\Theta)(a^{n}\tilde{\gamma})$$

holds for any $\gamma \in M_{G-\text{reg}}$, $a \in A_M^-$ and $n \gg 0$. Hence the virtual characters on two sides are equal.

We end this section by a corollary of Theorem 3.1 about partial Jacquet modules. It will not be used in the following, we include it here mainly for future reference.

Let $M = \operatorname{GL}(d) \times \operatorname{Sp}(2m)$, where m = n - d, be a maximal Levi subgroup of G. For a virtual genuine character Θ of \tilde{G} , we can write

$$r_{\tilde{M}}^{\tilde{G}}(\Theta) = \sum_{i \in I} a_i \Theta_{\pi_i}$$

for some indexing set I, some irreducible genuine representations π_i of \tilde{M} and complex numbers a_i . Each π_i takes the form $\rho_i \boxtimes \pi'_i$ where ρ is an irreducible representation of $\operatorname{GL}(d)$ and π'_i is an irreducible genuine representation of $\widetilde{\operatorname{Sp}}(2m)$.

Let ρ be a unitary irreducible supercuspidal representation of $\operatorname{GL}(d)$. For a real number x, let $|\cdot|^x$ be the character of $\operatorname{GL}(d)$ given by $g \mapsto |\det(g)|^x$. Then the partial Jacquet module with respect to ρ, x is defined as

$$r_{\rho,x}(\Theta) := \sum_{\substack{i \in I \\ \rho_i \simeq \rho | \cdot |^x}} a_i \Theta_{\pi_i}$$

which is a virtual genuine character of $\widetilde{\text{Sp}}(2m)$.

We would like to apply $r_{\rho,x}$ to $\mathcal{T}_{G^{!},\tilde{G}}^{\vee}(\Theta)$ where Θ is a stable virtual character of $G^{!}$. In the current setting, we have

$$\mathcal{E}(M, G^!) = \{ (d', d'') \mid d', d'' \in \mathbb{Z}_{\geq 0} , d' + d'' = d , d' \leq n', d'' \leq n'' \}$$

and Levi subgroups appear on right hand side of Theorem 3.1 takes the form

$$L = (\operatorname{GL}(d') \times \operatorname{SO}(2m'+1)) \times (\operatorname{GL}(d'') \times \operatorname{SO}(2m''+1))$$

The endoscopic transfer $\mathcal{T}_{L,\tilde{M}}^{\vee}$ map then factors as

$$L \xrightarrow{elliptic} \operatorname{GL}(d') \times \operatorname{GL}(d'') \times \widetilde{\operatorname{Sp}}(2m) \xrightarrow{parabolic} \widetilde{M}$$

Because ρ is a supercuspidal representation, $\mathcal{T}_{M_s^!,\tilde{M}}^{\vee} \circ [z_s] \circ r_{M_s^!}^{G^!}(\Theta)$ has no components isomorphic to $\rho |\cdot|^x \boxtimes \pi'$ unless s = (d, 0) or (0, d).

Now let ϕ be a discrete L-parameter of SO(2n+1). By definition, ϕ is a 2n-dimensional

symplectic representation of the Weil-Deligne group and it can be written as a multiplicity free direct sum

$$\phi = \oplus \phi_i \boxtimes S_{a_i - 1}$$

where each ϕ_i is an irreducible unitary representation of Weil group and S_{a_i-1} is the *a*-dimensional irreducible representation of SL(2, \mathbb{C}). By local Langlands correspondence for GL, each ϕ_i corresponds to an irreducible supercuspidal representation ρ_i . The set of Jordan blocks of ϕ is defined as

$$Jord(\phi) := \{(\rho_i, a_i) | \phi_i \boxtimes S_{a-1} \subset \phi\}.$$

Suppose ϕ factors through $G^!$, then we can write $\phi = (\phi', \phi'')$. We take $(\rho, a) \in Jord(\phi)$ and set $x = \frac{a-1}{2}$. Let $\Theta_{\phi'}$ (resp. $\Theta_{\phi'}$) be the stable virtual character attached to $\phi'(\text{resp.} \phi'')$. If $r_{\rho,x}(\Theta_{\phi'}) \neq 0$ (resp. $r_{\rho,x}(\Theta_{\phi''}) \neq 0$) then (ρ, a) is contained in $Jord(\phi')$ (resp. $Jord(\phi'')$), see eg. [Xu17b, Lemma 7.2]. Since ϕ is discrete, at most one case can happen. In summary, we can obtain the following:

Corollary 3.3. Let $\phi = (\phi', \phi'')$ be a discrete L-parameter of SO(2n + 1) that factors through $G^!$ and let Θ_{ϕ} be the stable virtual character of $G^!$ attached to ϕ . We take a standard Levi $M^! \subset G^!$ defined as

$$M^{!} := \begin{cases} M^{!}_{(d,0)} & \text{if } (\rho,a) \in Jord(\phi') \\ M^{!}_{(0,d)} & \text{if } (\rho,a) \in Jord(\phi''). \end{cases}$$

Write $\tilde{G}_{-} = \widetilde{Sp}(2m)$ and let $G_{-}^{!}$ be the SO part of $M^{!}$ then we have

$$\alpha r_{\rho,x} \circ \mathcal{T}_{G,\tilde{G}}^{\vee}(\Theta_{\phi}) = \mathcal{T}_{G_{-}^{\vee},\tilde{G}_{-}}^{\vee} \circ r_{\rho,x}(\Theta_{\phi}),$$

where α is a sign defined as

$$\alpha = \begin{cases} 1 & \text{if } (\rho, a) \in Jord(\phi') \\ \omega_{\rho}(-1) & \text{if } (\rho, a) \in Jord(\phi'') \end{cases}$$

with ω_{ρ} stands for the central character of ρ .

4 Compatibility

Now we start to prove Theorem 1.1. First we consider the case where $G^!$ is an elliptic endoscopic group of G.

For a split reductive group H with a fixed Borel pair, let $\mathcal{L}(H)$ be the set of standard Levi subgroups of H. Recall from [Aub95] that the Aubert-Zelevinski involution is defined as

$$D_H := \sum_{L \in \mathcal{L}(H)} (-1)^{r(L)} i_L^H \circ r_L^H$$

where r(-) stands for semisimple rank. For \tilde{G} , its Aubert-Zelevinski involution is defined similarly as

$$D_{\tilde{G}} := \sum_{M \in \mathcal{L}(G)} (-1)^{r(M)} i_{\tilde{M}}^{\tilde{G}} \circ r_{\tilde{M}}^{\tilde{G}}.$$

Let $G^! = \mathrm{SO}(2n'+1) \times \mathrm{SO}(2n''+1)$ be an elliptic endoscopic group of \tilde{G} . For $L \in \mathcal{L}(G^!)$, define

$$\mathcal{M}(L) := \{ (M, s) | M \in \mathcal{L}(G), s \in \mathcal{E}(M, G^!), L = M_s^! \}$$

By [Li21, Section 3.8], we know

$$\mathcal{T}_{G^!,\tilde{G}}^{\vee} \circ i_{M_s^!}^{G^!} \circ [z_s] = i_{\tilde{M}}^{\tilde{G}} \circ \mathcal{T}_{M_s^!,\tilde{M}}^{\vee}$$

holds for any $M \in \mathcal{L}(G)$, $s \in \mathcal{E}(M, G^{!})$. Then we can compute

$$\begin{split} D_{\tilde{G}} \circ \mathcal{T}_{G^{!},\tilde{G}}^{\vee} &= \sum_{M \in \mathcal{L}(G)} (-1)^{r(M)} i_{\tilde{M}}^{\tilde{G}} \circ r_{\tilde{M}}^{\tilde{G}} \circ \mathcal{T}_{G^{!},\tilde{G}}^{\vee} \\ &= \sum_{M \in \mathcal{L}(G)} (-1)^{r(M)} i_{\tilde{M}}^{\tilde{G}} (\sum_{s \in \mathcal{E}(M,G^{!})} \mathcal{T}_{M_{s}^{!},\tilde{M}}^{\vee} \circ [z_{s}] \circ r_{M_{s}^{!}}^{G^{!}}) \\ &= \sum_{M \in \mathcal{L}(G)} (-1)^{r(M)} \sum_{s \in \mathcal{E}(M,G^{!})} \mathcal{T}_{G^{!},\tilde{G}}^{\vee} \circ i_{M_{s}^{!}}^{G^{!}} \circ r_{M_{s}^{!}}^{G^{!}} \\ &= \mathcal{T}_{G^{!},\tilde{G}}^{\vee} (\sum_{L \in \mathcal{L}(G^{!})} (\sum_{(M,s) \in \mathcal{M}(L)} (-1)^{r(M)} i_{L}^{G^{!}} \circ r_{L}^{G^{!}})) \end{split}$$

Comparing with the definition of $D_{G'}$, we see that it is enough to show

$$\sum_{(M,s)\in\mathcal{M}(L)} (-1)^{r(M)} = (-1)^{r(L)}$$

holds for all $L \in \mathcal{L}(G^!)$. Suppose L takes the form

$$L = (\prod_{i=1}^{k'} \operatorname{GL}(n'_i) \times \operatorname{SO}(2m'+1)) \times (\prod_{i=1}^{k''} \operatorname{GL}(n''_i) \times \operatorname{SO}(2m''+1)).$$

Here, if k' or k'' is 0 then the corresponding product is understood as taken over empty set, similar conventions apply to later discussion. Write $I' = (n'_1, ..., n'_{k'})$ and define I'' similarly.

Lemma 4.1. The set $\mathcal{M}(L)$ is in bijection with the set of triple (k, \bar{I}', \bar{I}'') satisfying:

- $k \in \mathbb{Z}$ and $\max(k', k'') \leq k$
- $\bar{I}' = (\bar{n}'_1, ..., \bar{n}'_k)$ and there exists a subsequence $1 \le i_1 < ... < i_{k'} \le k$ such that

$$\bar{n}'_i = \begin{cases} n'_j & \text{ if } i = i_j \\ 0 & \text{ else} \end{cases}$$

Similar condition holds for \bar{I}'' .

• For $\forall 1 \leq i \leq k$, at least one of \bar{n}'_i, \bar{n}''_i is non-zero

Proof. Given a triple (k, \bar{I}', \bar{I}'') as above, the corresponding Levi is

$$M = \prod_{i=1}^{k} \operatorname{GL}(\bar{n}'_{i} + \bar{n}''_{i}) \times \operatorname{Sp}(2m)$$

and the sequence $\{(\bar{n}'_i, \bar{n}''_i)\}_{i=1,\dots,k}$ gives an element $s \in \mathcal{E}(M, G^!)$ with $M_s^! = L$.

Conversely, if $M_s^! = L$ where $s = \{(\bar{n}'_i, \bar{n}''_i)\}_{i=1,\dots,k}$ then, by the definition of $M_s^!$, there is a subsequence of $\{\bar{n}'_i\}_i$ (resp. $\{\bar{n}''_i\}_i$) which is equal to I' (resp. I'') and entries of $\{\bar{n}'_i\}_i$ (resp. $\{\bar{n}''_i\}_i$) don't belong to this subsequence equal to 0. Hence $(k, \{(\bar{n}'_i\}, \{\bar{n}''_i\})$) is a triple as above. Clearly the above two maps are inverse to each other.

Note that the latter set of triples depends only on the pair (k', k''), thus we may denote it as $\mathcal{M}(k', k'')$. Note that we have r(L) = r(G) - (k' + k'') and if M is the standard Levi attached to a triple $(k, \bar{I}', \bar{I}'') \in \mathcal{M}(k', k'')$ then we also have r(M) = r(G) - k. Therefore if we set

$$f(k',k'') := \sum_{(k,\bar{I}',\bar{I}'') \in \mathcal{M}(k',k'')} (-1)^k$$

then we have

$$\sum_{(M,s)\in\mathcal{M}(L)} (-1)^{r(M)} = (-1)^{r(G)} f(k',k'').$$

Hence we are reduced to proving the following lemma:

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Lemma 4.2. $f(k',k'') = (-1)^{k'+k''}$

Proof. We can divide $\mathcal{M}(k',k'')$ into three subsets according to the following conditions:

- The first entry of \bar{I}' is non-zero while the first entry of \bar{I}'' is zero.
- The first entry of \bar{I}'' is non-zero while the first entry of \bar{I}' is zero.
- The first entries of \bar{I}', \bar{I}'' are both non-zero.

Then by deleting the first entries of \bar{I}', \bar{I}'' simultaneously, the above subsets are in bijection with $\mathcal{M}(k'-1,k''), \mathcal{M}(k',k''-1), \mathcal{M}(k'-1,k''-1)$ respectively. Thus we have

$$f(k',k'') = -f(k'-1,k'') - f(k',k''-1) - f(k'-1,k''-1)$$

and the lemma follows from induction on k', k''.

Now we move to general case. Let $G^!$ be an (not necessarily elliptic) endoscopic group of \tilde{G} . Then there exists a standard Levi subgroup \tilde{M} such that $G^!$ is an elliptic endoscopic group of \tilde{M} . By [Aub95, Theorem 1.7] we have

$$i_{\tilde{M}}^{\tilde{G}} \circ D_{\tilde{M}} = D_{\tilde{G}} \circ i_{\tilde{M}}^{\tilde{G}}.$$

On the other hand, since $\mathcal{T}_{G^!\tilde{M}}^{\vee}$ is identity on GL factors, the above ellip

$$D_{\tilde{M}} \circ \mathcal{T}_{G^!, \tilde{M}}^{\vee} = \mathcal{T}_{G^!, \tilde{M}}^{\vee} \circ D_{G^!}$$

Therefore we have

$$D_{\tilde{G}} \circ \mathcal{T}_{G^{!},\tilde{G}}^{\vee} = D_{\tilde{G}} \circ i_{\tilde{M}}^{G} \circ \mathcal{T}_{G^{!},\tilde{M}}^{\vee}$$
$$= i_{\tilde{M}}^{\tilde{G}} \circ \mathcal{T}_{G^{!},\tilde{M}}^{\vee} \circ D_{G^{!}}$$
$$= \mathcal{T}_{G^{!},\tilde{G}}^{\vee} \circ D_{G^{!}}$$

which finishs the proof of Theorem 1.1.

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