## A NOTE OF CHARACTERISTIC CLASS FOR SINGULAR VARIETIES

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ABSTRACT. In this work we study characteristic classes of possibly singular varieties embedded as a closed subvariety of a nonsingular variety. In special, we express the Schwartz-MacPherson class in terms of the  $\mu$ -class and Chern class of the sheaves of logarithmic and multi-logarithmic differential forms. As an application we show an expression for Euler characteristic of a complement of a singular variety.

Keywords: Characteristic class; Logarithmic sheaf; Milnor number; Chern class.

# 1. INTRODUCTION

Let X be a possibly singular variety in a nonsingular variety M. When X is regular, there is a well know notion of the characteristic class of X, the Chern class, defined in many ways under its tangent bundle, see for instance, [16, 18]. For the original definition see [14]. On the other hand, if X is singular one has different ways to generalize this characteristic class of X.

The first extension of Chern classes for a possibly singular variety is the Schwartz-MacPherson class, it was done independently by M.-H. Schwartz in 1965, see [12, 26] as an element on the cohomology group and by R. MacPherson in 1974 in [22] as an element of the homology group.

Another important class with which we going to work in this paper is the Fulton class, denoted by  $c_F(X)$ , see [16]. This class is defined over a scheme X that can be embedded in a nonsingular variety M and it is proved that it is independent of the choice of embedded. An advantage of this class is that it can be defined over arbitrary fields and in a completely algebraic fashion. On the other hand, a disadvantage it is does not satisfy at first sight nice functorial properties.

The Milnor number was initially defined by Milnor in 1968, see [23], to a hypersurface with isolated singularities. After, independently in 1971 Hamm [17] and Lê in 1974 [21] extended this index for local complete intersections still with isolated singular points. More recently, in 1988, see [24], Parusinski extended the notion of Milnor number to nonisolated singularities. P. Aluffi in 1995, see [6], defined another important class for the singular set of a hypersurface, the  $\mu$ -class, which in the case of isolated singularities coincides with the Minor number.

In [9] Aluffi et al., proved a formula relating the Mather class, Schwartz-MacPherson class, and the class of the virtual tangent bundle of a hypersurface in a nonsingular variety, under certain assumptions in the singular locus of a hypersurface X.

<sup>2020</sup> Mathematics Subject Classification. Primary 32S65, 37F75; secondary 14F05.

Using the computer, Aluffi in [8] developed an algorithm computing to calculate some characteristic classes. The program computed the push-forward to  $\mathbb{P}^n$  of the Schwartz-MacPherson class, and the Fulton class, and showed as is well known, that the Euler characteristic equals the degree of the 0-dimension component of the Schwartz-MacPherson.

In general, it is very hard to calculate the Chern class to singular variety, and there are few results in this direction. On the other hand, as suggested by J.-P. Brasselet, P. Aluffi in [5] (see Theorem 1.1 below) shows an expression to the Schwartz-MacPherson class by using the sheaf of logarithmic differential forms.

The guiding theme of this paper is to express a way of to calculate the Schwartz-MacPherson class in terms of the sheaf of logarithmic and multi-logarithmic differential forms with poles in certain varieties, when this sheaf is not locally free. For this, we use the comparison between the Schwartz-MacPherson and Fulton classes for hypersurface due P. Aluffi, see Theorem 2.1. When the variety X is a local complete intersection, the difference between these classes is called Milnor class, see ([10], Definition 1, p. 46).

Let X be an algebraic variety over an algebraically closed field of characteristic zero. Assume that X is embedded as a closed subvariety of a nonsingular variety M, by  $i: X \to M$ .

**Theorem 1.1** ([5], Theorem 1). Let X be as above. Let  $\pi : \tilde{M} \to M$  be a birational map with  $\tilde{M}$  a nonsingular variety, such that  $X' = (\pi^{-1}(X))_{red}$  is a divisor with smooth components and normal crossing in  $\tilde{M}$ , and  $\pi|_{\tilde{M}-X'}$  is an isomorphism. Then

$$i_{*}c_{SM}(X) = c(TM) \cap [M] - \pi_{*}\left(c(\Omega^{1}_{\tilde{M}}(\log X^{'})^{\vee}) \cap [\tilde{M}]\right) \in A_{*}M.$$

Now, follows our main result:

**Theorem 1.2.** Let  $i : X \longrightarrow M$  be an embedding of a closed subvariety X in a complex algebraic nonsingular variety M. Let  $\pi : \tilde{M} \longrightarrow M$  be a proper birational map with  $\tilde{M}$  a nonsingular variety, such that  $(\pi^{-1}(X))_{red} = \tilde{X}$  is a hypersurface with its singular scheme denoted by  $\tilde{Y}$  and  $\pi|_{\tilde{M}\setminus\tilde{X}}$  is an isomorphism. We assume that  $\operatorname{codim}_{\tilde{M}}(\tilde{Y}) \geq 3$ . Then

(1) 
$$i_*c_{SM}(X) = c(TM) \cap [M] - \pi_* \Big( c(\Omega^1_{\tilde{M}}(\log \tilde{X})^{\vee}) \cap [\tilde{M}] \Big) + \pi_* j_* c(\mathcal{L})^{\dim \tilde{X}} \cap \Big( \mu_{\mathcal{L}}(\tilde{Y})^{\vee} \otimes_{\tilde{M}} \mathcal{L} \Big),$$

where  $\mu_{\mathcal{L}}(\tilde{Y})$  denotes the  $\mu$ -class of  $\tilde{Y}$  with respect to  $\mathcal{L} = \mathcal{O}_{\tilde{M}}(\tilde{X})$ .

We observe P. Aluffi, in the previous theorem, assumes that X' is a normal crossing divisor. This hypothesis is natural, and it is guaranteed for an embedded resolution of singularities in characteristic zero. However, it is possible to obtain a normal crossing divisor with a finite sequence of the blowups. We note that in some cases the amount of blowups can be very large. In our main result, we changed the normal crossing hypothesis in divisor by a divisor whose its singular set has codimension greater than or equal to three.

These hypotheses are satisfied at the case listed below, called Nash Construct for foliations. Let  $\mathcal{F}$  be a holomorphic foliation of dimension k on a n-dimensional manifold M. We consider, for each  $x \in M$  the following vector space

$$F(x) = \{v(x); v \in \mathcal{F}_x\},\$$

where  $\mathcal{F}_x$  denotes the stalk of the sheaf  $\mathcal{F}$  at x. We note that  $\dim F(x) \leq k$  and the equality is when  $x \in M \setminus \operatorname{Sing}(\mathcal{F})$ . Then, we define a section

$$s: M \setminus \operatorname{Sing}(\mathcal{F}) \longrightarrow G(k, n),$$

gives by s(x) = F(x), where G(k, n) denotes de Grassmannian bundle of k-planes in TM. We define  $M^{\nu}$  as the closure of the Im(s) in G(k, n) and call it of the Nash modification of Mwith respect to  $\mathcal{F}$ , see [11, 28] for more details. In this context, we have that  $\pi : M^{\nu} \to M$  is a proper birational map, which is induced by the projection map of the bundle G(k, n), since  $\pi$  is an isomorphism from  $M^{\nu} \setminus \pi^{-1}(\operatorname{Sing}(\mathcal{F}))$  to  $M \setminus \operatorname{Sing}(\mathcal{F})$ . In a special case, Sertoz in [28] was studied this construction with the hypothesis that  $M^{\nu}$  is a manifold, that occurs when the coherent sheaf  $\mathcal{F}$  is "nice", namely, either it is gives by complex actions of reductive groups or it has locally free tangent sheaf and its singular set is smooth, see ([28], Corollary 1.2 p.230).

**Example 1.3.** Let  $M = \mathbb{P}^n$  be a complex projective space and  $\mathcal{F}$  be an one-dimensional holomorphic foliation on  $\mathbb{P}^n$  with isolated singularities. Consider  $\operatorname{Sing}(\mathcal{F}) = X$ . Thus by ([28], Corollary 1.2 p.230) one has  $M^{\nu} \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$  is a nonsingular variety of dimension n. And  $\pi^{-1}(\operatorname{Sing}(\mathcal{F})) = \operatorname{Sing}(\mathcal{F}) \times \mathbb{P}^{n-1} = \tilde{X}$  with  $\tilde{Y} = \operatorname{Sing}(\tilde{X}) = \emptyset$ . So, applying the Theorem 1.2, we have:

(2) 
$$i_*c_{SM}(X) = c(T\mathbb{P}^n) \cap [\mathbb{P}^n] - \pi_*\Big(c(\Omega^1_{M^\nu}(\log \mathbb{P}^{n-1})^\vee) \cap [M^\nu]\Big).$$

In particular, taking the degrees, we have

(3) 
$$\chi(X) = \chi(\mathbb{P}^n) - \int \pi_* \left( c(\Omega^1_{M^\nu}(\log(\tilde{X})^\vee) \cap [M^\nu] \right)$$

or

$$\chi(\mathbb{P}^n \setminus X) = \int \pi_* \left( c(\Omega^1_{M^\nu}(\log(\tilde{X})^\vee) \cap [M^\nu] \right) = \chi(M^\nu \setminus \tilde{X}).$$

The last equality follows from Corollary 1.2, see ([13] p.491). This result was expected because  $\pi$  is an isomorphism from  $M^{\nu} \setminus \tilde{X}$  to  $\mathbb{P}^n \setminus X$ .

When the singular set  $\tilde{Y}$  of  $\tilde{X}$  is supported at a point P, the  $\mu$ -class of  $\tilde{Y}$  is  $m_P[P]$ , where  $m_P$  is the classical Minor number, see ([6], §2). So we get the following.

**Corollary 1.4.** In conditions of the Theorem 1.1 and under the additional hypothesis that  $\tilde{Y}$  is supported in a set of finitely many points, i.e.  $\tilde{Y} = \{x_1, \ldots, x_r\}$ , then

$$i_* c_{SM}(X) = c(TM) \cap [M] - \pi_* c(\Omega^1_{\tilde{M}}(\log \tilde{X})^{\vee}) \cap [\tilde{M}] + \pi_* j_* (-1)^{\dim \tilde{M}} \sum_{i=1}^r m_i[x_i],$$

where  $m_i$  is the Milnor number of  $\tilde{X}$  at  $x_i$ .

*Proof.* Under hypothesis of this corollary one has  $\mu_{\mathcal{L}}(\tilde{Y}) = \sum_{i=1}^{r} m_i[x_i]$ . Now we calculate  $\left(\mu_{\mathcal{L}}(\tilde{Y})^{\vee} \otimes_{\tilde{M}} \mathcal{L}\right)$  by using the definition on ([7], p. 3996),

$$c(\mathcal{L})^{\dim \tilde{X}} \cap \left(\mu_{\mathcal{L}}(\tilde{Y})^{\vee} \otimes_{\tilde{M}} \mathcal{L}\right) = (-1)^{\dim \tilde{M}} \sum_{i=1}^{r} m_{i}[x_{i}].$$

As a consequence of the Theorem 1.1, taking the degree in equation (1), we obtain an expression for the Euler characteristic for the complement  $M \setminus X$ , of the variety X in M.

**Corollary 1.5.** In conditions of Theorem 1.1 with the additional hypothesis that M and  $\tilde{M}$  are complete varieties, then

$$\chi(M \setminus X) = \int_{\tilde{M}} c \left( \Omega^1_{\tilde{M}} (\log \tilde{X})^{\vee} \right) \cap [\tilde{M}] + (-1)^{n+1} \int_{\tilde{M}} \mu_{\mathcal{L}}(\tilde{Y}).$$

**Remark 1.6.** We observe that if  $\tilde{Y}$  is supported in a set of finitely many points, i.e.  $\tilde{Y} = \{x_1, \ldots, x_r\}$ , then we recover the result in ([13], Corollary 1.2 (I), p.495)

$$\chi(\tilde{M} \setminus \tilde{X}) = (-1)^n \int_{\tilde{M}} c \left( \Omega^1_{\tilde{M}}(\log \tilde{X}) \right) \cap [\tilde{M}] + (-1)^{n+1} \sum_{i=1}^r m_i,$$
  
$$\chi(\tilde{M} \setminus \tilde{X}) = \chi(\tilde{M} \setminus \tilde{X}) \text{ and } \int_{\tilde{M}} \mu_c(\tilde{Y}) = \sum_{i=1}^r m_i.$$

since  $\chi(M \setminus X) = \chi(\tilde{M} \setminus \tilde{X})$  and  $\int_{\tilde{M}} \mu_{\mathcal{L}}(\tilde{Y}) = \sum_{i=1} m_i$ .

In the second part of this work we use the sheaves of multi-logarithmic differential forms associated to a complete intersection, see Section 4 and [1, 2] for more details about these sheaves. We prove the following result involved its characteristic classes:

**Theorem 1.7.** Let  $i: X \longrightarrow M$  be an embedding of a closed subvariety X in a complex algebraic nonsingular variety M and  $\pi: \tilde{M} \longrightarrow M$  be a proper birational map with  $\tilde{M}$  a nonsingular variety, such that  $(\pi^{-1}(X))_{red} = C = D_1 \cap D_2$  is a complete intersection of smooth divisors and  $\pi|_{\tilde{M}\setminus C}$  is an isomorphism. We assume that  $\tilde{D} = D_1 \cup D_2$  is a normal crossing divisor. Then

$$i_* c_{SM}(X) = c(TM) \cap [M] - \pi_* \left( c(\Omega^1_{\tilde{M}}(\log C)^{\vee}) \cap [\tilde{M}] \right) + \pi_* j_* c_{SM}(C) - \pi_* j_* c_{SM}(\tilde{D}),$$

where  $\Omega^1_{\tilde{M}}(\log C)$  denotes the sheaf of multi-logarithmic differential 1-forms on  $\tilde{M}$ .

The paper is organized as follows. First, in order to make this work as self-contained as possible, we provide some necessary definitions and considerations in Sections 2, 3 and 4. The proofs of our main results appear in Sections 5, 6 and 7.

## Acknowledgements

The authors were partially supported by the FAPEMIG [grant number 38155289/2021] and FAPEMIG RED-00133-21.

#### 2. CHERN CLASSES OF SINGULAR VARIETIES

Let X be a complex algebraic variety and A be a subvariety of X, we denote by  $\mathbb{1}_A$  the characteristic function of A which is constant equal to 1 over A and constant equal to 0 elsewhere. A constructible function on X is an integral linear combination of characteristic functions of closed subvarieties of X. The constructible functions on X forms a group denoted by  $\mathbf{F}(X)$ . It can be made a covariant functor in the following way: for a proper morphism  $f: X \to Y$  the push-forward  $f_*$  is defined by setting

$$f_*(\mathbb{1}_A)(y) = \chi(f^{-1}(y) \cap A),$$

where A is a subvariety of X and extending by linearity.

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It was conjectured by Deligne and Grothendieck in 1969 and proved by R. MacPherson [22] in 1974, that there exists a natural transformation  $c_*$  from the functor **F** to homology, which, on a nonsingular variety V, assigns to the function  $\mathbb{1}_V$  the Poincaré dual of the total Chern class of V. That is,

$$c_*(\mathbb{1}_V) = c(TV) \cap V.$$

So it is natural to consider the class  $c_*(\mathbb{1}_X)$  to arbitrary variety X. This class is denoted by  $c_{SM}(X)$  and called of *Schwartz-MacPherson class* of X. This class is the image, via Alexander duality isomorphism, of the previously class defined by M.-H. Schwartz in 1965 see [12]. Explicitly, MacPherson has proved that for all constructible functions  $\alpha$ ,  $\beta$ , and proper morphism f, the class  $c_*$  satisfy the conditions:

- (i)  $f_*c_*(\alpha) = c_*f_*(\alpha);$
- (ii)  $c_*(\alpha + \beta) = c_*(\alpha) + c_*(\beta);$
- (iii)  $c_*(\mathbb{1}_X) = c(TX) \cap [X]$ , if X is smooth.

Although MacPherson has initially considered complex algebraic varieties, Kennedy [20] indicated how MacPherson's theory can be made completely algebraic, by extending it to varieties over an arbitrary field k of characteristic zero.

Let X be now a scheme which can be embedded as a closed subscheme of a nonsingular variety M. Then the Fulton class of X, denoted by  $c_F(X) \in A_*(X)$ , was defined by W. Fulton 1984, see ([16], Example 4.2.6) by setting

$$c_F(X) = c(TM_{|X}) \cap s(X, M),$$

where s(X, M) denotes the Segre class of X on M. It is independent of the choice of embedding.

In particular case, when X is a divisor, the Segre class of X is given by  $s(X, M) = \frac{|X|}{1+X}$ , where by abuse of notation, we denote by X the first Chern class  $c_1(\mathcal{O}(X))$ . So in this case

(4) 
$$c_F(X) = c(TM_{|X}) \cap \frac{[X]}{1+X}.$$

Another important characteristic class, the  $\mu$ -class, was introduced by P. Aluffi [6] in 1995. Let Y be the singular scheme of a hypersurface X on a smooth variety M, and let  $\mathcal{L} = \mathcal{O}(X)$  be the line bundle associate to X. The  $\mu$ -class of Y with respect to  $\mathcal{L}$  is the class

$$\mu_{\mathcal{L}}(Y) := c(T^*M \otimes \mathcal{L}) \cap s(Y,M)$$

in the Chow group  $A_*(Y)$ .

In particular, when Y is supported at a point P, then  $\mu(Y) = m_P[P]$ , where  $m_P$  is the Milnor number of X at P see ([6], §2). In the following theorem, P. Aluffi has established an interesting relationship between these classes.

**Theorem 2.1** (Aluffi [7], Theorem I.5). Let X be a hypersurface in a nonsingular variety M, let Y be its singular scheme, and let  $\mathcal{L} = \mathcal{O}(X)$ . Then

(5) 
$$c_{SM}(X) = c_F(X) + c(\mathcal{L})^{\dim X} \cap (\mu_{\mathcal{L}}(Y)^{\vee} \otimes_M \mathcal{L}).$$

### 3. The sheaf of logarithmic forms

Let M be a complex manifold of dimension n and X a reduced hypersurface on M. We consider  $\Omega_M^q(X)$  the sheaf of differential q-forms on M with at most simple poles along X. We define a *logarithmic* q-form along X on an open subset  $U \subset M$  by a meromorphic q-form  $\omega$  on U, regular on the complement U - X and such that both forms  $\omega$  and  $d\omega$  are in the sheaf  $\Omega_M^q(X)$ .

Logarithmic q-forms along X form a coherent sheaf of  $\mathcal{O}_M$ -modules that we will denote simply by  $\Omega^q_M(\log X)$ . In this case, for any open subset  $U \subset M$  we have

$$\Gamma(U, \Omega_M^q(\log X)) = \{ \omega \in \Gamma(U, \Omega_M^q(X)) : d\omega \in \Gamma(U, \Omega_M^{q+1}(X)) \}.$$

See for example [3], [15] and [27] for more details about the sheaf of logarithmic q-forms along X.

Take  $\Omega^1_M(\log X)$ , the sheaf of logarithmic 1-forms along X. Its dual is the sheaf of logarithmic vector fields along X, denoted by  $T_M(-\log X)$  or  $Der(-\log X)$ . With this notations we have the classical short exact sequence (see [15] §2)

$$0 \longrightarrow T_M(-\log X) \longrightarrow T_M \longrightarrow J_X(X) \longrightarrow 0,$$

where  $J_X$  denotes the Jacobian ideal of X which is defined as the Fitting ideal

$$J_X := F^{n-1}(\Omega^1_X) \subset \mathcal{O}_X.$$

Saito in [27] has showed that in general  $\Omega_M^1(\log X)$  and  $T_M(-\log X)$  are reflexive sheaves. When X is an analytic hypersurface with normal crossing singularities, the sheaves  $\Omega_M^1(\log X)$  and  $T_M(-\log X)$  are locally free. Furthermore, the Poincaré residue map (see [27, Section 2])

Res:  $\Omega^1_{\mathcal{M}}(\log X) \longrightarrow \mathcal{O}_X \cong \bigoplus_{i=1}^N \mathcal{O}_{X_i}$ 

gives us the following exact sequence of sheaves on M:

(6) 
$$0 \longrightarrow \Omega^1_M \longrightarrow \Omega^1_M(\log X) \xrightarrow{\operatorname{Res}} \bigoplus_{i=1}^N \mathcal{O}_{X_i} \longrightarrow 0,$$

where  $\Omega_M^1$  is the sheaf of holomorphic 1-forms on M and  $X_1, \ldots, X_N$  are the irreducible components of X.

Finally, if X is such that  $\operatorname{codim}_M(\operatorname{Sing}(X) \ge 3$ , then there exist the following exact sequence of sheaves on M (see [15]):

(7) 
$$0 \longrightarrow \Omega^1_M \longrightarrow \Omega^1_M(\log X) \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

#### 4. The sheaf of multi-logarithmic forms

In this section we will give some basic definitions on the theory of multi-logarithmic differential forms and the results that we will need in this paper. For more details and properties on this subject see [2] and [25].

Let  $X = X_1 \cup \cdots \cup X_k$  be a decomposition of the reduced hypersurface X in a complex manifold M, where each  $X_i$  is a hypersurface defined by a holomorphic function  $h_i$ , for  $i = 1, \ldots, k$ , on an open subset  $U \subset M$ , and  $C = X_1 \cap \cdots \cap X_k$  is a reduced complete intersection.

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We define a multi-logarithmic q-form along the complete intersection C on an open subset  $U \subset X$  by a meromorphic q-form  $\omega$  on U, regular on U - X and such that  $\omega$  have at most simple poles along C and

$$dh_i \wedge \omega \in \sum_{i=1}^k \Omega^{q+1}(\widehat{X}_i) \text{ for all } i \in \{1, \dots, k\},$$

where  $\widehat{X_i} = X_1 \cup \cdots \cup X_{i-1} \cup X_{i+1} \cup \cdots \cup X_k$ .

We denote by  $\Omega_M^q(\log C)$  the coherent sheaf of germs of multi-logarithmic q-forms along C. A. G. Aleksandrov [2] has proved the following result that characterizes multi-logarithmic forms.

**Theorem 4.1** (A. G. Aleksandrov, [2]). Let  $\omega \in \Omega_M^q(X)$ , then  $\omega$  is multi-logarithmic along C if and only if there is a holomorphic function  $g \in \mathcal{O}_M$  which is not identically zero on every irreducible component of the C, a holomorphic differential form  $\xi \in \Omega_M^{q-k}$  and a meromorphic q-form  $\eta \in \sum_{i=i}^k \Omega_M^q(\widehat{X_i})$  such that there exists the following representation

$$g\omega = \frac{dh_1 \wedge \dots \wedge dh_k}{h_1 \dots h_k} \wedge \xi + \eta.$$

For q < k we have the equality (see [25], Remark 2.6):

$$\Omega_M^q(\log C) = \sum_{i=i}^k \Omega_M^q(\widehat{X_i}).$$

Observe that if q = 1 and k = 2 we have

(8) 
$$\Omega^1_M(\log C) = \Omega^1_M(X_1) + \Omega^1_M(X_2)$$

**Proposition 4.2** (see [13]). Let  $X = X_1 \cup X_2$  be a reduced hypersurface on M, where  $X_i$  is a reduced hypersurface, for i = 1, 2, and  $C = X_1 \cap X_2$  is a reduced complete intersection. Then

$$\Omega^1_M(\log X) = \Omega^1_M(\log X_1) + \Omega^1_M(\log X_2).$$

**Lemma 4.3.** Let  $\tilde{D} = D_1 \cup D_2$  be a normal crossing divisor, where  $D_1, D_2$  be smooth hypersurfaces in a complex manifold  $\tilde{M}$ , such that  $C = D_1 \cap D_2$  is a complete intersection. Then

$$c(\Omega^{1}_{\tilde{M}}(\log C)) = c(\Omega^{1}_{\tilde{M}}(\log \tilde{D})).$$

*Proof.* In fact, by the following exact sequence

$$0 \longrightarrow \Omega^1_X \longrightarrow \Omega^1_X(D_1) \oplus \Omega^1_{\tilde{M}}(D_2) \longrightarrow \Omega^1_{\tilde{M}}(D_1) + \Omega^1_X(D_2) \longrightarrow 0$$

one has by Chern class properties,

$$c\Big(\Omega^1_X(D_1)\oplus\Omega^1_X(D_2)\Big)=c\Big(\Omega^1_X(D_1)c\Big(\Omega^1_X(D_2)\Big)=c\Big(\Omega^1_X(D_1)+\Omega^1_X(D_2)\Big)c(\Omega^1_X).$$

Let us consider the exact sequence

$$0 \longrightarrow \Omega^1_X \longrightarrow \Omega^1_X(D_i) \longrightarrow \mathcal{O}_{D_i} \longrightarrow 0.$$

then, one has  $c(\Omega^1_X(D_i)) = c(\Omega^1_X)c(\mathcal{O}_{D_i})$ , where  $c_j(\mathcal{O}_{D_i}) = c_1([D_i])^j$ .

Then  $c\Big(\Omega_X^1(D_1) + \Omega_X^1(D_2)\Big) = c(\mathcal{O}_{D_1})c(\mathcal{O}_{D_2})c(\Omega_X^1)$ . Since  $\Omega_X^1(\log C) = \Omega_X(D_1) + \Omega_X(D_2)$ , we get

$$c(\Omega^1_X(\log C)) = c(\mathcal{O}_{D_1})c(\mathcal{O}_{D_2})c(\Omega_X).$$

On the other hand, since  $\tilde{D} = D_1 \cup D_2$  is a normal crossing divisor, we have the following exact sequence, see sequence (6),

$$0 \longrightarrow \Omega^1_{\tilde{M}} \longrightarrow \Omega^1_{\tilde{M}}(\log(D_1 \cup D_2)) \longrightarrow \mathcal{O}_{D_1} \oplus \mathcal{O}_{D_2} \longrightarrow 0$$

It is follows that  $c(\Omega^1_{\tilde{M}}(\log(D_1 \cup D_2))) = c(\mathcal{O}_{D_1})c(\mathcal{O}_{D_2})c(\Omega_X)$  and we finish the prove.  $\Box$ 

### 5. Proof of Theorem 1.1

*Proof.* Let us consider the exact sequence (7) of reflexive sheaves on  $\tilde{M}$ 

$$0 \longrightarrow \Omega^1_{\tilde{M}} \longrightarrow \Omega^1_{\tilde{M}}(\log \tilde{X}) \longrightarrow \mathcal{O}_{\tilde{X}} \longrightarrow 0$$

So we have

(9) 
$$c(\Omega^1_{\tilde{M}}(\log \tilde{X})) = c(\Omega^1_{\tilde{M}})c(\mathcal{O}_{\tilde{X}})$$

Now let us recall the exact sequence, see ([19], p.84).

$$0 \longrightarrow \mathcal{O}(-\tilde{X}) \longrightarrow \mathcal{O}_{\tilde{M}} \longrightarrow \mathcal{O}_{\tilde{X}} \longrightarrow 0$$

So

(10) 
$$c(\mathcal{O}_{\tilde{X}}) = \frac{1}{c(\mathcal{O}(-\tilde{X}))}$$

Then from (9) and (10)

$$c(\Omega^{1}_{\tilde{M}}(\log \tilde{X})) = \frac{c(\Omega^{1}_{\tilde{M}})}{c(\mathcal{O}(-\tilde{X}))}$$

We take the dual in previous expression

$$c(\Omega^1_{\tilde{M}}(\log \tilde{X})^{\vee}) = \frac{c(T\tilde{M})}{c(\mathcal{O}(\tilde{X}))} = \frac{c(T\tilde{M})}{1 + c_1(\mathcal{O}(\tilde{X}))} = \frac{c(T\tilde{M})}{1 + \tilde{X}}$$

Let us consider the following calculation

$$c(T\tilde{M})\left(1-\frac{1}{1+\tilde{X}}\right)\cap[\tilde{M}] = c(T\tilde{M})\left(\frac{\tilde{X}}{1+\tilde{X}}\right)\cap[\tilde{M}]$$
$$= j_*c(T\tilde{M})\cap\left(\frac{[\tilde{X}]}{1+\tilde{X}}\right)$$
$$= j_*c_F(\tilde{X}),$$

where  $j: \tilde{X} \longrightarrow \tilde{M}$  is the inclusion morphism.

On the other hand,

(11)  
$$j_*c_F(\tilde{X}) = c(T\tilde{M})\left(1 - \frac{1}{1 + \tilde{X}}\right) \cap [\tilde{M}] = c(T\tilde{M}) \cap [\tilde{M}] - \frac{c(T\tilde{M})}{1 + \tilde{X}} \cap [\tilde{M}]$$
$$= c(T\tilde{M}) \cap [\tilde{M}] - c(\Omega^1_{\tilde{M}}(\log \tilde{X})^{\vee}) \cap [\tilde{M}].$$

Using the expression in Theorem 2.1 to  $c_F(\tilde{X})$  and applying the homomorphism  $j_*$ , one has

$$j_*c_F(\tilde{X}) = j_*c_{SM}(\tilde{X}) - j_*c(\mathcal{L})^{\dim \tilde{X}} \cap \left(\mu_{\mathcal{L}}(\tilde{Y})^{\vee} \otimes_{\tilde{M}} \mathcal{L}\right).$$

Now, we use the above expression in equation (11)

$$j_* c_{SM}(\tilde{X}) - j_* c(\mathcal{L})^{\dim \tilde{X}} \cap \left( \mu_{\mathcal{L}}(Y)^{\vee} \otimes_{\tilde{M}} \mathcal{L} \right) = c(T\tilde{M}) \cap [\tilde{M}] - c(\Omega^1_{\tilde{M}}(\log \tilde{X})^{\vee}) \cap [\tilde{M}].$$
$$c(\Omega^1_{\tilde{M}}(\log \tilde{X})^{\vee}) \cap [\tilde{M}] - j_* c(\mathcal{L})^{\dim \tilde{X}} \cap \left( \mu_{\mathcal{L}}(\tilde{Y})^{\vee} \otimes_{\tilde{M}} \mathcal{L} \right) = c_{SM}(\tilde{M}) - j_* c_{SM}(\tilde{X})$$

(12) 
$$c(\Omega^{1}_{\tilde{M}}(\log \tilde{X})^{\vee}) \cap [\tilde{M}] - j_{*}c(\mathcal{L})^{\dim \tilde{X}} \cap \left(\mu_{\mathcal{L}}(\tilde{Y})^{\vee} \otimes_{\tilde{M}} \mathcal{L}\right) = c_{*}(\mathbb{1}_{\tilde{M} \setminus \tilde{X}}).$$

Using Schwartz-MacPherson class properties one has,

$$\pi_*c_*(\mathbb{1}_{\tilde{M}\setminus\tilde{X}}) = c_*\pi_*(\mathbb{1}_{\tilde{M}\setminus\tilde{X}}) = c_*(\mathbb{1}_{M\setminus X}) = c(TM) \cap [M] - i_*c_{SM}(X).$$

So, applying  $\pi_*$  in the equation (12), we have

$$i_*c_{SM}(X) = c(TM) \cap [M] - \pi_*c(\Omega^1_{\tilde{M}}(\log \tilde{X})^{\vee}) \cap [\tilde{M}] + \pi_*j_*c(\mathcal{L})^{\dim \tilde{X}} \cap \left(\mu_{\mathcal{L}}(\tilde{Y})^{\vee} \otimes_{\tilde{M}} \mathcal{L}\right).$$

# 6. Proof of Corollary 1.5

Proof. We note by Schwartz-MacPherson class properties one has

$$\pi_* c_{SM}(\tilde{M} \setminus \tilde{X}) = c_{SM}(M \setminus X).$$

So, by using the degree properties (see [16], p. 13) we have

$$\int_{\tilde{M}} c_{SM}(\tilde{M} \setminus \tilde{X}) = \int_{M} \pi_* c_{SM}(\tilde{M} \setminus \tilde{X}) = \int_{M} c_{SM}(M \setminus X) = \chi(M \setminus X).$$

Taking the degrees in equation (12)

$$\begin{split} \chi(M \setminus X) &= \int_{\tilde{M}} c(\Omega^{1}_{\tilde{M}}(\log \tilde{X})^{\vee}) \cap [\tilde{M}] - \int_{\tilde{M}} j_{*}c(\mathcal{L})^{\dim \tilde{X}} \cap \left(\mu_{\mathcal{L}}(\tilde{Y})^{\vee} \otimes \mathcal{L}\right) \\ &= \int_{\tilde{M}} c(\Omega^{1}_{\tilde{M}}(\log \tilde{X})^{\vee}) \cap [\tilde{M}] + (-1)^{n+1} \int_{\tilde{M}} \mu_{\mathcal{L}}(\tilde{Y}). \end{split}$$

The last simplification follows from Aluffi (see [7], §4).

### 7. Proof of Theorem 1.7

*Proof.* How  $\tilde{D} = D_1 \cup D_2$  is a normal crossing divisor, we have the short exact sequence (6)

$$0 \longrightarrow \Omega^1_{\tilde{M}} \longrightarrow \Omega^1_{\tilde{M}}(\log(D_1 \cup D_2)) \longrightarrow D_1 \oplus D_2 \longrightarrow 0$$

Following the proof of Theorem 1 in ([5], p.621) we have

$$j_*c_{SM}(\tilde{D}) = c(T\tilde{M}) \cap [\tilde{M}] - c(\Omega^1_{\tilde{M}}(\log \tilde{D})^{\vee}) \cap [\tilde{M}].$$

Now, adding the factor  $(-j_*c_{SM}(C))$  in both sides and applying the morphism  $\pi_*$ 

$$j_*c_{SM}(\tilde{D}) - j_*c_{SM}(C) = c(T\tilde{M}) \cap [\tilde{M}] - c(\Omega^1_{\tilde{M}}(\log \tilde{D})^{\vee}) \cap [\tilde{M}] - j_*c_{SM}(C)$$
$$j_*c_{SM}(\tilde{D}) - j_*c_{SM}(C) = c_*(\mathbb{1}_{\tilde{M}\setminus C}) - c(\Omega^1_{\tilde{M}}(\log \tilde{D})^{\vee}) \cap [\tilde{M}]$$

$$\pi_* j_* c_{SM}(\tilde{D}) - \pi_* j_* c_{SM}(C) = c_*(\mathbb{1}_{M \setminus X}) - \pi_* c(\Omega^1_{\tilde{M}}(\log \tilde{D})^{\vee}) \cap [\tilde{M}].$$

To finish the proof we use the Lemma 4.3

$$\pi_* j_* c_{SM}(\tilde{D}) - \pi_* j_* c_{SM}(C) = c_*(\mathbb{1}_{M \setminus X}) - \pi_* c(\Omega^1_{\tilde{M}}(\log C)^{\vee}) \cap [\tilde{M}].$$

Rearranging it

$$i_*c_{SM}(X) = c(TM) \cap [M] - \pi_* \left( c(\Omega^1_{\tilde{M}}(\log C)^{\vee}) \cap [\tilde{M}] \right) + \pi_* j_* c_{SM}(C) - \pi_* j_* c_{SM}(\tilde{D}),$$

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