ON THE VARIATION OF THE MILNOR NUMBER OF FOLIATIONS UNDER BLOW-UPS OF NON-ISOLATED SINGULARITIES

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Abstract. Let $\mathcal{F}$ be a one-dimensional holomorphic foliation on $\mathbb{P}^n$ such that $\mathbf{W} \subset \mathrm{Sing}(\mathcal{F})$, where $\mathbf{W}$ is a smooth complete intersection variety. We determine and compute the variation of the Milnor number $\mu(\mathcal{F}, \mathbf{W})$ under blowups, which depends on the vanishing order of the pullback foliation along the exceptional divisor, as well as on numerical and topological invariants of $\mathbf{W}$. This represents a higher-dimensional version of Van den Essen's formula for projective foliations. As an application, in $\mathbb{P}^3$, we provide a Seidenberg type theorem for non-isolated singularities, without assuming that they are absolutely isolated. That is, the foliation has a birational model on a smooth manifold which is generically log canonical.

1. Introduction

In his celebrated work [3], Bott provided a method to compute residues of global holomorphic vector fields along non-degenerate and non-isolated singularities. In general, determining the residue for degenerate singularities in the case of meromorphic vector fields is challenging, except for isolated singularities, as shown by Baum and Bott in [1].

F. Bracci and T. Suwa established in [4] that Baum-Bott indices continuously vary under smooth deformations of holomorphic foliations, and in particular, such residues/indices can be computed via deformation. Therefore, following Bracci and Suwa, we can, in particular, define and compute the Milnor number of a one-dimensional foliation along a smooth subvariety of high codimension contained in the singular set as follows. Consider a one-dimensional holomorphic foliation $\mathcal{F}_0$ on a complex manifold $\mathbf{M}_0$ induced by a global section of $v_0 \in H^0(X, T\mathbf{M}_0 \otimes L)$, for some fixed line bundle L, such that its singular locus $\mathrm{Sing}(\mathcal{F}_0)$ contains a smooth subvariety $\mathbf{W}_0$ of codimension $d \geq 2$. Now, let $\mathcal{F}_t$ be a generic holomorphic deformation of $\mathcal{F}_0$, for $t \in D(0, \epsilon)$, with ϵ sufficiently small such that $\mathcal{F}_t$ is induced by a section of $v_t \in H^0(X, T\mathbf{M}_0 \otimes L)$, $\lim\limits_{t\to 0} v_t = v_0$ and $\mathrm{Sing}(\mathcal{F}_t) = \{p_1^t, \ldots, p_{m_t}^t\}$, where each p_j^t is an isolated closed point. Then *Milnor number* $\mu(\mathcal{F}_0, \mathbf{W}_0)$ of $\mathcal{F}_0$ along $\mathbf{W}_0$ is given by

$$\mu(\mathcal{F}_0, \mathbf{W}_0) = \sum_{\lim\limits_{t\to 0} p_j^t \in \mathbf{W}_0} \mu(\mathcal{F}_t, p_j^t),$$

where $\mu(\mathcal{F}_t, p_j^t)$ is the usual Milnor number for isolated singularities. In [10], we computed the Milnor number for the case where $\mathbf{W}_0$ is a non-dicritical component. Here, we extend that result to the case where $\mathbf{W}_0$ is a dicritical component of $\mathrm{Sing}(\mathcal{F}_0)$.

It is natural to ask how the Milnor number varies under certain maps that modify the foliation. In [11], the authors show that $\mu(\mathcal{F}_0, \mathbf{W}_0)$ on a three-dimensional manifold $\mathbf{M}_0$ remains invariant under topological equivalences C^1. In this work, our focus is on determining the *Milnor number* $\mu(\mathcal{F}_0, \mathbf{W}_0)$ and computing its variation under *blow-ups* for foliations on projective spaces.

In order to present our first results we need to fix the following notation: $\mathbf{W}_0 := Z(f_1, \ldots, f_d)$ will be a smooth complete intersection variety on $\mathbf{M}_0 = \mathbb{P}^n$ where f_j is a reduced polynomial with $k_j = \deg(f_j)$ for $j = 1, \ldots, d$. Let $\mathcal{T}_{\mathbf{W}_0}$ and $\mathcal{N} := \mathcal{N}_{\mathbf{W}_0}$ be tangent and normal bundles of $\mathbf{W}_0$ in $\mathbf{M}_0$ and with their

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total Chern classes $c(\mathcal{T}_{|W_0}) = \sum \tau_i^{(d)}\mathbf{h}^i$ and $c(\mathcal{N}) = \sum \sigma_i^{(d)}\mathbf{h}^i$, respectively, where $\mathbf{h}$ is the hyperplane class of $\mathbb{P}^n$. Consider

$$\mathcal{W}_\delta^{(d)} := \mathcal{W}_\delta^{(d)}(k_1,\ldots,k_d) = \sum_{i_1+\ldots+i_d=\delta} k_1^{i_1}\ldots k_d^{i_d},$$

the complete symmetric function of degree δ in d variables at the multi-indices $(k_1,\ldots,k_d)$. Now, let $\pi_1 : \mathbf{M}_1 \to \mathbf{M}_0$ be the blowup of $\mathbf{M}_0 = \mathbb{P}^n$ along $\mathbf{W}_0$, with the exceptional divisor $\mathbf{E}_1 = \pi_1^{-1}(\mathbf{W}_0)$. The kernel $\nu(\mathcal{F}_0,\mathbf{W}_0,\varphi_a)$ is defined as follows

$$\nu(\mathcal{F}_0,W_0,\varphi_a) = -\deg(\mathbf{W}_0)\sum_{|a|=0}^{n-d}\sum_{m=0}^{n-d-|a|}(-1)^{\delta_{|a|}^m}\frac{\varphi_a^{(m)}(\ell)}{m!}(k-1)^m\sigma_{a_1}^{(d)}\tau_{a_2}^{(d)}\mathcal{W}_{\delta_{|a|}^m}^{(d)},$$

where $k=\deg(\mathcal{F}_0)$, $a=(a_1,a_2)\in\mathbb{Z}^2$, $|a|=a_1+a_2\geq 0$, $\delta_{|a|}^m := n-d-|a|-m$; and ℓ is given by

$$\ell = \begin{cases} m_{\mathbf{E}_1}(\pi_1^*\mathcal{F}_0), & \text{if } \mathbf{W}_0 \text{ is non-dicritical} \\ m_{\mathbf{E}_1}(\pi_1^*\mathcal{F}_0)-1, & \text{if } \mathbf{W}_0 \text{ is dicritical,}\end{cases}$$

where $m_{\mathbf{E}_1}(\pi_1^*\mathcal{F}_0)$ denotes the vanishing order of the pullback foliation $\pi_1^*\mathcal{F}_0$ at $\mathbf{E}_1$ and the function $\varphi_a(x) := x^{n-d-a_2}(1+x)^{d-a_1}$ with $\varphi_a^{(m)}(x) = \dfrac{d^m}{dx^m}\varphi_a(x)$.

Theorem 1.1. *Let $\mathcal{F}_0$ be a holomorphic foliation by curves on $\mathbb{P}^n$, with $n\geq 3$, of degree k. Suppose that the singular set of $\mathcal{F}_0$ is the disjoint union of a smooth scheme-theoretic complete intersection subvariety $\mathbf{W}_0$ of pure codimension $d\geq 2$, and closed points $p_1,\ldots,p_s$. Consider the blow-up $\pi_1 : \mathbf{M}_1 \to \mathbf{M}_0$ centered on $\mathbf{W}_0$ being $\mathbf{E}_1 = \pi_1^{-1}(\mathbf{W}_0)$ the exceptional divisor and $\mathcal{F}_1$ the strict transform foliation obtained from $\mathcal{F}_0$ via π_1. Then*

(a) $\displaystyle\sum_{i=1}^{s}\mu(\mathcal{F}_0,p_i) = \sum_{i=0}^{n}k^i + \nu(\mathcal{F}_0,W_0,\varphi_a) - N(\mathcal{F}_0,\mathcal{A}_{\mathbf{W}_0})$,

(b) $\mu(\mathcal{F}_0,\mathbf{W}_0) = -\nu(\mathcal{F}_0,\mathbf{W}_0,\varphi_a) + N(\mathcal{F}_0,\mathcal{A}_{\mathbf{W}_0}) \geq -\nu(\mathcal{F}_0,\mathbf{W}_0,\varphi_a)$,

(c) $\mu(\mathcal{F}_1,\bigcup_i \mathbf{W}_i^{(1)}) = \mu(\mathcal{F}_0,\mathbf{W}_0) + \nu(\mathcal{F}_0,W_0,\vartheta_a)$

where $N(\mathcal{F}_0,\mathcal{A}_{\mathbf{W}_0})$ is the number of embedding closed points associated to $\mathbf{W}_0$, counted with multiplicities, $\mathbf{W}_i^{(1)} \subset \mathbf{E}_1$ is each connected component of $\mathrm{Sing}(\mathcal{F}_1)$ *and*

$$\vartheta_a(x) = \varphi_a(x) + x^{n-d-a_2-1}\big(1-(1+x)^{d-a_1}\big).$$

This is a higher-dimensional version of *Van den Essen's formula*; see [25, Theorem 1.3] and [17, Appendix]. Moreover, Theorem 1.1 provides a lower bound for the Milnor number $\mu(\mathcal{F}_0,\mathbf{W}_0)$ and calculates its variation under blow-up. Remarkably, Item (c) remains valid even when $\mathbf{W}_0$ is a closed point.

In the next part of this paper, we explore a holomorphic foliation $\mathcal{F}_0$ defined on $\mathbb{P}^3$, where its singular set contains a smooth regular curve $\mathbf{W}_0$. We start with $\mathbf{M}_0 = \mathbb{P}^3$ and consider a sequence of blow-ups $\pi_j : \mathbf{M}_j \to \mathbf{M}_{j-1}$ centered, for each $j\geq 1$, along a component $\mathbf{W}_{j-1}\subset \mathrm{Sing}(\mathcal{F}_{j-1})$, where $\mathcal{F}_j$ is the strict transform obtained from $\mathcal{F}_{j-1}$ under π_j, and $\mathbf{E}_j$ is the exceptional divisor. In short, we denote this sequence by $\{\pi_j,\mathbf{M}_j,\mathbf{W}_j,\mathcal{F}_j,\mathbf{E}_j\}$. Generally, when we restrict the singular set of $\mathcal{F}_j$ to the exceptional divisor $\mathbf{E}_j$, it comprises new curves and potentially isolated closed points. However, these new curves can be categorized into two main types: those that are homeomorphic to $\mathbf{W}_0$ and those that are homeomorphic to $\mathbb{P}^1$. As a result, we present the following theorem, which efficiently determines $\mu(\mathcal{F}_i,\mathbf{W}_i)$ under natural hypotheses for $i\geq 0$.

Theorem 1.2. *Let $\mathcal{F}_0$ be a one-dimensional holomorphic foliation defined on $\mathbb{P}^3$ such that $\mathbf{W}_0\subset$* $\mathrm{Sing}(\mathcal{F}_0)$ *where $\mathbf{W}_0$ is a smooth curve of degree* $\deg(\mathbf{W}_0)$ *and Euler characteristic $\chi(\mathbf{W}_0)$.*

If there is a blow-up sequence $\pi_j,\mathbf{M}_j,\mathbf{W}_j,\mathcal{F}_j,\mathbf{E}_j$ where $\mathbf{W}_j$ is homeomorphic to $\mathbf{W}_{j-1}$ and $\pi_j(\mathbf{W}_j) = \mathbf{W}_{j-1}$ for $j\geq 1$, then

(a)

$$\mu(\mathcal{F}_j,\mathbf{W}_j) =\big(\ell_{j+1}+1\big)\Bigg((\ell_{j+1}+1)\chi(\mathbf{W}_0)+(3\ell_{j+1}+1)(k-1)\deg(\mathbf{W}_0)+\frac{\ell_{j+1}^2}{2^j}\Lambda_0^{(3)}+ + (3\ell_{j+1}+1)\Lambda_0^{(3)}\sum_{i=1}^{j}\frac{\ell_i}{2^i}\Bigg)+N(\mathcal{F}_j,\mathcal{A}_{\mathbf{W}_j})$$

where $\Lambda_0^{(3)} := \chi(\mathbf{W}_0)-4\deg(\mathbf{W}_0)$, $\ell_i = m_{\mathbf{E}_i}\big(\pi_i^*\mathcal{F}_{i-1}\big)$ *for* $i=1,\dots,j+1$ *and* $N(\mathcal{F}_j,\mathcal{A}_{\mathbf{W}_j})$ *the number of embedding closed points associated with* $\mathbf{W}_j$*, counted with multiplicities.*

(b)

$$\mu(\mathcal{F}_{j+1},\bigcup_i \mathbf{W}_i^{(j+1)}) = \mu(\mathcal{F}_j,\mathbf{W}_j)+\big(1-\ell_{j+1}-\ell_{j+1}^2\big)\chi(\mathbf{W}_0)+\frac{\ell_{j+1}}{2^j}\big(1-\ell_{j+1}^2\big)\Lambda_0^{(3)}- - (3\ell_{j+1}^2+2\ell_{j+1}-1)\Bigg((k-1)\deg(\mathbf{W}_0)+\Lambda_0^{(3)}\sum_{i=1}^{j}\frac{\ell_1}{2^i}\Bigg)$$

where $\mathbf{W}_i^{(j+1)} \subset \mathbf{E}_{j+1}$ *is each connected component of* $\mathrm{Sing}(\mathcal{F}_{j+1})$.

Here we assume that $\sum_{i=\alpha}^{\beta} a_i = 0$ *for* $\alpha<\beta$.

The map $\pi_j : \mathbf{M}_j \setminus \mathbf{E}_j \to \mathbf{M}_{j-1} \setminus \mathbf{W}_{j-1}$ is a biholomorphism which implies that the sequence $N(\mathcal{F}_j,\mathcal{A}_{\mathbf{W}_j})$ is non-increasing. Since $\mu(\mathcal{F}_j,\mathbf{W}_j)$ is a natural number, the order $m_{\mathbf{W}_i}(\mathcal{F}_i)$ (the order of vanishing of the foliation $\mathcal{F}_i$ along the curve $\mathbf{W}_i$) typically increases by one during a blow-up sequence. This increase is expected to be $m_{\mathbf{W}_0}(\mathcal{F}_0)+1$. By Theorem 1.2, we conclude that the order of annulment $\ell_i = m_{\mathbf{E}_i}(\pi_i^*\mathcal{F}_{i-1})$ will be zero for sufficiently large i. This implies that after a certain point in the sequence of blow-ups, the foliation no longer increases in complexity, and the process stabilizes. Consequently, this result offers a new approach to extending Seidenberg's Theorem for foliations with non-isolated singularities. A well-known fact is that for $n=2$, the resolution Theorem of Seidenberg [22] asserts that all the singularities of $\mathcal{F}_i$ are elementary for i large enough. This means that if $p\in\mathrm{Sing}(\mathcal{F}_i)$ then $\mathcal{F}_i$ is locally generated by a vector field X_k having a linear part with eigenvalues λ_1 and λ_2 where $\lambda_1/\lambda_2 \notin \mathbb{Q}_+$ or at least one eigenvalues is non-zero.

As defined in [6], the component $\mathbf{W}_0$ will be called an *absolutely isolated singularity* if all the components $\mathbf{W}_i$, which appear in this process, have the same dimension as $\mathbf{W}_0$. In [6], the desingularization theorem is proven when $\mathbf{W}_0$ is a non-dicritical absolutely isolated singularity. The non-dicritical condition is removed in [2]. However, it is unfortunate that a general birational desingularization theorem is not possible, as shown in [8], where F. Sanz and F. Sancho presented an example of a vector field X that *cannot be* desingularized by any sequence of blowups. For further details, refer to Example 5.9 and Proposition 5.5. For complex 3-folds, Panazzolo [20] and McQuillan–Panazzolo [18] provide a non-birational desingularization (after performing smoothed weighted blow-ups) and Cano in [7] proposes a desingularization approach that permits the use of formal, non-algebraic blow-up centers. Our next result states that we can proceed with a birational desingularization such that, for generic points of the curves in the desingularized model, the singularities are elementary. Furthermore, in the sense of [5, Proposition 2.20], this birational model is generically log canonical.

Theorem 1.3. *Let* $\mathcal{F}_0$ *be a one-dimensional holomorphic foliation defined on* $\mathbf{M}_0=\mathbb{P}^3$ *such that* $\mathbf{W}_0\subset\mathrm{Sing}(\mathcal{F}_0)$ *where* $\mathbf{W}_0$ *is a smooth curve.*

If there is a blow-up sequence $\{\pi_i,\mathbf{M}_i,\mathbf{W}_i,\mathcal{F}_i,\mathbf{E}_i\}$ *such that* $\mathbf{W}_i$ *is a homeomorphic curve to* $\mathbf{W}_{i-1}$ *and* $\pi_j(\mathbf{W}_j)=\mathbf{W}_{i-1}$*, then there is a natural number* k *such that* $\mathbf{W}_i$ *is an elementary component of* $\mathcal{F}_i$ *for* $i\geq k$*, for almost all points of* $\mathbf{W}_i$*. In particular,* $\mathcal{F}_i$ *is generically log canonical along* $\mathbf{W}_i$*. for all* $i\geq k$*.*

In [21], F. Sancho de Salas presented a similar theorem, but with a key difference: the case where $\mathbf{W}_0$ has *codimension two* and is also a *absolutely isolated* component. Theorem 1.3 generalizes the

Sancho de Salas' result for foliations on $\mathbb{P}^3$, as *different dimensional singularities* may emerge during the desingularization process; see example 4.7. Despite the challenges posed, we establish that if such a sequence exists, then, starting from a certain index, the singular components become elementary, *regardless of whether they are absolutely isolated or not.* An additional complication we encounter is that determining whether a singularity is absolutely isolated is a highly challenging task. For further details, see Examples 4.7 and 4.5. Rebelo and Reis in [24] show the transcendental nature of foliations that cannot be resolved by birational maps. They demonstrate that such foliations have a birational model with a formal separatrix passing through one of their singular points. In the example of Sanz and Sancho, the foliations have strictly formal separatrices. We also observe in Proposition 5.10 that a foliation which cannot be birationally desingularized does not admit any formal first integrals along a curve of its singular set. From Theorem 1.3, we can conclude that after a finite number of blow-ups along homeomorphic curves to $\mathbf{W}_0$, the foliation $\mathcal{F}_k$ in the open set U_k is described by the following vector field.

$$X_k = x_1\left(\lambda_1^{(k)} + P_1(x)\right)\frac{\partial}{\partial x_1} + \left(x_1 r_1(x_3) + x_2\lambda_2^{(k)} + P_2(x)\right)\frac{\partial}{\partial x_2} + P_3(x)\frac{\partial}{\partial x_3}$$

where $\lambda_i^{(k)} = \lambda_i^{(k)}(x_3)$ for $i = 1, 2$, the singular component $\mathbf{W}_k$ is defined as $x_1 = x_2 = 0$ and X_k having a linear part with eigenvalues $\lambda_1^{(k)}$ and $\lambda_2^{(k)}$ where $\lambda_1^{(k)}/\lambda_2^{(k)} \notin \mathbb{Q}_+$, for almost all points of $\mathbf{W}_k$, or at least one of these eigenvalues is not identically zero.

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2. Preliminaries

2.1. **Holomorphic foliations with non-isolated singularities.** Let $\mathcal{F}_0$ be a one-dimensional holomorphic foliation on $\mathbb{P}^n$, with $n \geq 3$, such that its singular set contains a smooth subvariety $\mathbf{W}_0$ of pure codimension $d \geq 2$. Given that $\mathbf{W}_0$ is smooth, for each closed point $p \in \mathbf{W}_0$ there exists an open set U_0 of p and a coordinate system $z \in \mathbb{C}^n$ such that $\mathbf{W}_0 \cap U$ can be defined as $\{z_1 = \ldots = z_d = 0\}$. Therefore, in U_0 the foliation $\mathcal{F}_0$ is described by the following vector field

$$X_0 = P_1(z)\frac{\partial}{\partial z_1} + \ldots + P_n(z)\frac{\partial}{\partial z_n} \tag{1}$$

which we can write the local sections as

$$P_i(z) = \sum_{|a|=m_i} z_1^{a_1}\cdots z_d^{a_d} P_{i,a}(z) \tag{2}$$

where $a := (a_1, \ldots, a_d) \in \mathbb{Z}^d$ with $|a| := a_1 + \ldots + a_d$, $a_i \geq 0$ and for each i at least one among the $P_{i,a}(z)$ does not vanish at $\{z_1 = \ldots = z_d = 0\}$. The natural number m_i will be called the multiplicity of P_i along $\mathbf{W}_0$ and denoted by $m_{\mathbf{W}_0}(P_i)$.

Definition 2.1. *The multiplicity of $\mathcal{F}_0$ along $\mathbf{W}_0$ is defined as follows*

$$m_{\mathbf{W}_0}(\mathcal{F}_0) = \min\{m_1, \ldots, m_n\}.$$

Lemma 2.2. *Let $\mathcal{F}_0$ be a holomorphic foliation by curves, defined in the neighborhood of p by the vector field*

$$X_p = \sum_{j=1}^n P_j(z)\frac{\partial}{\partial z_j}$$

as in (1) with $m_i = m_{W_0}(P_i)$. Then, by a linear change of coordinates, $\mathcal{F}_0$ may be described by the vector field

$$Y_p = \sum_{j=1}^n Q_j(w)\frac{\partial}{\partial w_j}$$

with

$$m_{W_0}(Q_j) = \begin{cases} m'_{\mathbf{W}_0}(\mathcal{F}_0), & \text{for} \quad j = 1, \ldots, d \\ m_{\mathbf{W}_0}(\mathcal{F}_0), & \text{for} \quad j = d+1, \ldots, n \end{cases}$$

where $m'_{\mathbf{W}_0}(\mathcal{F}_0) = \min\{m_1, \ldots, m_d\}$.

Proof. In fact, it is enough to consider the linear transformation $w = Az$ where $A = (a_{ij}) \in GL(n, \mathbb{C})$ with $a_{ij} = 0$ for $i = 1, \ldots, d$ and $j = d+1, \ldots, n$. In this way, $B = (b_{ij}) = A^{-1}$ has the same properties, that is, the subspace W given by $\{z_1 = \ldots = z_d = 0\}$ is an invariant set under this transformation. Adjusting the coefficients a_{ij}, if necessary, we can admit that

$$\dot{w}_i = \begin{cases} \displaystyle\sum_{j=1}^{d} a_{ij}\dot{z}_j = \sum_{j=1}^{d} a_{ij}P_j(Bw) = Q_i(w), & \text{for} \quad i = 1, \ldots, d \\ \displaystyle\sum_{j=1}^{n} a_{ij}\dot{z}_j = \sum_{j=1}^{n} a_{ij}P_j(Bw) = Q_i(w), & \text{for} \quad i = d+1, \ldots, n \end{cases}$$

having the required properties. □

Without loss of generality, we can assume that the vector field in (1) is such that

$$m_{\mathbf{W}_0}(P_j) = \begin{cases} m_1, & \text{for } j = 1, \ldots, d \\ m_n, & \text{for } j = d+1, \ldots, n. \end{cases}$$

with $m_1 \geq m_n = \mathrm{m}_{\mathbf{W}_0}(\mathcal{F})$.

From now on, wee will consider the $d \times d$ complex matrix

$$\mathbf{A}_{X_0} = \left[\frac{\partial P_i}{\partial z_j}\right]_{1 \leq i,j \leq d} \tag{3}$$

Definition 2.3. *The component* $\mathbf{W}_0$ *will be referred to as locally elementary of* $\mathcal{F}_0$ *if the matrix* $\mathbf{A}_{X_0}$ *has at least one nonzero eigenvalue for almost all points* $z \in \mathbf{W}_0 \cap U$, *where* U *is an open set. On the other hand, if all the eigenvalues of* $\mathbf{A}_{X_0}|_{\mathbf{W}_0}$ *are identically zero, then* $\mathbf{W}_0$ *will be called a non-elementary component of* $\mathrm{Sing}(\mathcal{F}_0)$.

Clearly, if $m_{\mathbf{W}_0}(\mathcal{F}_0) \geq 2$ then $\mathbf{A}_{X_0} \equiv 0$ for all $z \in \mathbf{W}_0 \cap U_0$. Moreover, the linear part of the vector field X_0 restricted to $\mathbf{W}_0$ has at most d identically zero eigenvalues.

We have the following proposition

Proposition 2.4. *The definitions (2.1) and (2.3) are independent of the chosen coordinate system.*

Proof. In fact, it is enough to consider the biholomorphism $\Phi : U \to \mathbb{C}^n$ given by $w = \Phi(z) = (\Phi_1(z), \ldots, \Phi_n(z))$ such that $\Phi_i(z) \equiv 0$ for all $z \in \mathbf{W}_0 \cap U$, for $i = 1, \ldots, d$. Let us admit that $\mathcal{F}_0$ is described in an other coordinate system by the following vector field

$$Y_0(w) = Q_1(w)\frac{\partial}{\partial w_1} + \ldots + Q_n(w)\frac{\partial}{\partial w_n}$$

where each $Q_i(w)$ vanishing along $\{w_1 = \ldots = w_d = 0\}$. Let $\mathbf{B}_{Y_0}$ be the $d \times d$ complex matrix given by

$$\mathbf{B}_{Y_0} = \left[\frac{\partial Q_i}{\partial w_j}\right]_{1 \leq i,j \leq d}.$$

Since Φ is a local biholomorphism, we have that $det(D\Psi|_{\mathbf{W}_0}) \neq 0$ and by consequence $w_i = \displaystyle\sum_{j=1}^{d} z_j\phi_{i,j}(z)$ for $i = 1, \ldots, d$ and the $d \times d$ complex matrix

$$\mathbf{C} = \left[\phi_{i,j}\right]_{1 \leq i,j \leq d}$$

is not singular for all $z \in \mathbf{W}_0 \cap U$. In the same way, $z = \Psi(w) = \Phi^{-1}(w)$ and $z_i = \sum_{j=1}^{d} w_j \psi_{i,j}(w)$ for some functions $\psi_{i,j}$ and also for $i = 1, \ldots, d$. Thus, $X_0 = \Phi^*(Y_0)$ and $\mathbf{B}_{Y_0} = \mathbf{C} \cdot \mathbf{A}_{X_0} \cdot \mathbf{C}^{-1}$, which concluded that the definition (2.3) is independent of the coordinate system chosen. But,

$$P_i \circ \Psi(w) = \sum_{|a|=m_i} z_1^{a_1} \cdots z_d^{a_d} P_{i,a}(z)|_{z=\Psi(w)} = \sum_{|a|=m_i} w_1^{a_1} \cdots w_d^{a_d} \widetilde{P}_{i,a}(w)$$

with some $\widetilde{P}_{i,a}(w)|_{\mathbf{W}_0} \not\equiv 0$ which results

$$Q_i(w) = \begin{cases} \displaystyle\sum_{j=1}^{d} \phi_{i,j} \circ \Psi(w) \cdot P_j \circ \Psi(w), & 1 \le i \le d \\ \displaystyle\sum_{j=1}^{n} \frac{\partial \Phi_i}{\partial z_j} \circ \Psi(w) \cdot P_j \circ \Psi(w), & d < i \le n. \end{cases}$$

Let $q_i := \mathrm{m}_{\mathbf{W}_0}(Q_i)$ for all $i = 1, \ldots, n$. Now, let us suppose by absurd that $\mathrm{m}_{\mathbf{W}_0}(\mathcal{F}_0) \neq \min\{q_1, \ldots, q_n\}$. Then, applying the same arguments for the vector field Y_0 with $z = \Psi(w)$ we will get $\min\{m_1, \ldots, m_n\} \neq \mathrm{m}_{\mathbf{W}_0}(\mathcal{F}_0)$. □

Let $\pi_1 : U_1 \to U_0$ be the blowup of an open set U_0 along at $\mathbf{W}_0 \cap U_0$, with $\mathbf{E}_1$ is the exceptional divisor and $\mathcal{F}_1$ is the strict transform of the foliation on U_1. In order to simplify the notation, we will denote $I_n = \{1, 2, \ldots, n\}$ and $J_d = \{2, \ldots, d\}$. In the chart $\big((U_1)_1, \sigma_1(u)\big)$, with coordinates $u = (u_i) \in \mathbb{C}^n$ (in the similar manner in the others charts $\Big(\big(U_1\big)_j, \sigma_j(v)\Big)$ such that

$$z = \sigma_1(u) = (u_1, u_1u_2, \ldots, u_1u_d, u_{d+1}, \ldots, u_n) = (z_1, \ldots, z_d, z_{d+1}, \ldots, z_n) \tag{4}$$

and the pull-back foliation $\pi_1^*\mathcal{F}_0$ is described by the following vector field

$$\mathcal{D}_{\pi_1^*\mathcal{F}_0} = \sum_{i \in I_n \setminus J_d} u_1^{m_i} \big(Q_i(u) + u_1 \widetilde{P}_i(u)\big) \frac{\partial}{\partial u_i} + u_1^{m_1 - 1} \sum_{i \in J_d} \big(G_i(u) + u_1 \widetilde{P}_i(u)\big) \frac{\partial}{\partial u_i} \tag{5}$$

with

$$Q_i(u) = \sum_{|a|=m_i} u_2^{a_2} \cdots u_d^{a_d} p_{i,a}(u_{d+1}, \ldots, u_n), \quad G_i(u) = Q_i(u) - u_i Q_1(u),$$

where $p_{i,a}(u_{d+1}, \ldots, u_n) = P_{i,a}(0, \ldots, 0, u_{d+1}, \ldots, u_n)$ for certain functions $\widetilde{P}_i$. For more details, see [10] or [14].

As usual, we will say that $\mathbf{W}_0$ is a *non-dicritical* component if the exceptional divisor $\mathbf{E}_1$ is invariant by $\mathcal{F}_1$, otherwise $\mathbf{W}_0$ is a *dicritical* component.

After a division of (5) by an adequate power of u_1, we obtain the expressions of the vector field X_1 that generates the induced foliation $\mathcal{F}_1$.

In the situation where $\mathbf{W}_0$ is a non-dicritical component of $\mathrm{Sing}(\mathcal{F}_0)$, is described in the following three cases.

Case (i) : $m_n + 1 = m_1$ and $G_j \not\equiv 0$ for some $j \in I_d$.In this situation, equation (5) is divided by $u_1^{m_1-1}$,

$$X_1 = \sum_{i \in I_n \setminus J_d} u_1^{m_i - m_1 + 1} \Big(Q_i(u) + u_1 \widetilde{P}_i(u)\Big) \frac{\partial}{\partial u_i} + \sum_{i \in J_d}^{d} \Big(G_i(u) + u_1 \widetilde{P}_i(u)\Big) \frac{\partial}{\partial u_i}. \tag{6}$$

This case has been widely studied in [9], [10], [12] and [13].

Case (ii) : $m_n + 1 < m_1$. As in the case before, dividing (5) by $u_1^{m_n}$ we get

$$\begin{aligned} X_1 =& u_1^{m_1-m_n}\Big(Q_1(u) + u_1\widetilde{P_1}(u)\Big)\frac{\partial}{\partial u_1} + u_1^{m_1-m_n-1}\sum_{i\in J_d}\Big(G_i(u) + u_1\widetilde{P}_i(u)\Big)\frac{\partial}{\partial u_2} \\ &+ \sum_{i=d+1}^{n}\Big(Q_i(u) + u_1\widetilde{P}_i(u)\Big)\frac{\partial}{\partial u_i} \end{aligned} \tag{7}$$

In this case, the leaves of $\mathcal{F}_1$ contained in $\mathbf{E}_1$ can be locally described as follows

$$\Big\{\varphi(t,p) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t), \varphi(0,p) = p = (p_i)_i, t \in D(0,\epsilon)\Big\} \tag{8}$$

with $\varphi_1(t) \equiv 0$ and $\varphi_i(t) \equiv p_i$ are constant functions for all $i \in J_d$. In other words, these leaves are generically transverse to fibers $\pi_1^{-1}(q)$, $q \in \mathbf{W}_0$.

By other side, the singular set of $\mathcal{F}_1$ when it is restricted to $\mathbf{E}_1$ is given by the following $n-d$ equations

$$Q_i(0, u_2, \ldots, u_n) = 0, \quad \text{for} \quad i = d+1, \ldots, n \tag{9}$$

which can be generically solved for $n-d$ unknown variables $u_{d+1}, \ldots, u_n$ as follows

$$\mathbf{W}_i = \{u \in \mathbf{E}_1 | u_k = \psi_k(u_2, \ldots, u_d), k > d\}. \tag{10}$$

Thus, each singular component $\mathbf{W}_i$ has dimension equal to $d-1$ and is homeomorphic to $\mathbb{P}^{d-1} = \big\{[u_1 : u_2 : \ldots : u_d]\big\}$. Note that $\mathbf{W}_i$ can be given by the graph of the function $\Psi : \mathbb{P}^{d-1} \to \mathbf{E}_1$ locally defined as

$$\Psi(u_1, u_2, \ldots, u_d) = (u_1, u_2, \ldots, u_d, \psi_{d+1}(u_2, \ldots, u_d), \ldots, \psi_n(u_2, \ldots, u_d)). \tag{11}$$

with $u_1 = 0$ in this chart.

Case (iii) : $m_n = m_1$ and $G_{i_0} \not\equiv 0$ for some $i_0 \in I_d$. We may divide (5) by $u_1^{m_1-1}$. As a consequence, we obtain

$$X_1 = u_1 \sum_{i\in I_n\setminus I_d}\Big(Q_i(u) + u_1\widetilde{P}_i(u)\Big)\frac{\partial}{\partial u_i} + \sum_{i\in J_d}\Big(G_i(u) + u_1\widetilde{P}_i(u)\Big)\frac{\partial}{\partial u_i}. \tag{12}$$

Unlike the previous case, the leaves of $\mathcal{F}_1$ restricted to the exceptional divisor are described below

$$\Big\{\varphi(t,p) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)), \varphi(0,p) = p = (p_i)_i, t \in D(0,\epsilon\Big\} \tag{13}$$

such that $\varphi_1(t,0) \equiv 0$ and $\varphi_i(t,p_i) \equiv p_i$ are constant functions for $i > d$. More precisely, the leaves of $\mathcal{F}_1$ in $\mathbf{E}_1$ are tangent to the fibers $\pi_1^{-1}(q)$, $q \in \mathbf{W}_0$. But in this situation, the singular set of $\mathcal{F}_1$ restricted to $\mathbf{E}_1$ is given by the following $d-1$ equations

$$Q_i(0, u_2, \ldots, u_n) - u_i Q_1(0, u_2, \ldots, u_n) = 0, \quad \text{for} \quad i = 2, \ldots, d \tag{14}$$

which can be generically solved for $d-1$ unknown variables $u_2, \ldots, u_d$ as follows

$$\mathbf{W}_i = \{u \in \mathbf{E}_1 | u_k = \psi_k(u_{d+1}, \ldots, u_n), \quad k = 2, \ldots d\}. \tag{15}$$

Thus, each singular component $\mathbf{W}_i$ has dimension equal to $d-n$ and is homeomorphic to $\mathbf{W}_0$ with $\pi_1(\mathbf{W}_i) = \mathbf{W}_0$. Note that $\mathbf{W}_i$ can be given by the graph of the function $\Psi : \mathbf{W}_0 \to \mathbf{E}_1$ locally defined as

$$\Psi(u_{d+1}, \ldots, u_n) = (0, \psi_2(u_{d+1}, \ldots, u_n), \ldots, \psi_d(u_{d+1}, \ldots, u_n), u_{d+1}, \ldots, u_n). \tag{16}$$

since $\mathbf{W}_0$ is locally defined as $z_1 = \ldots = z_d = 0$ and $u_i = z_i$ for $i > d$.

In the case where $\mathbf{W}_0$ is a dicritical component, it is only described by the following condition.

Case (iv) : $m_n = m_1$ and $G_j \equiv 0$ for $j \in I_d$. Then, after the division of (5) by $u_1^{m_1}$ we get

$$X_1 = \sum_{i\in I_n}\Big(Q_i(u) + u_1\widetilde{P}_i(u)\Big)\frac{\partial}{\partial u_i} + \sum_{i\in J_d}\widetilde{P}_i(u)\frac{\partial}{\partial u_i}. \tag{17}$$

In this situation, the exceptional divisor $\mathbf{E}_1$ is not an invariant set of $\mathcal{F}_1$. With the notation given in this section, the number $\ell := m_{\mathbf{E}_1}(\pi_1^*\mathcal{F}_0)$ will be called the order of annulment of $\pi_1^*\mathcal{F}_0$ at $\mathbf{E}_1$ is defined as follows.

$$\ell = \begin{cases} \min\{m_1 - 1, m_n\}, & \text{if } \mathbf{W}_0 \text{ is non-dicritical} \\ m_1, & \text{if } \mathbf{W}_0 \text{ is dicritical.} \end{cases} \tag{18}$$

In particular, if $m_1 = m_n$ then

$$\ell = \begin{cases} \mathrm{m}_{\mathbf{W}_0}(\mathcal{F}_0) - 1, & \text{if } \mathbf{W}_0 \text{ is non-dicritical} \\ \mathrm{m}_{\mathbf{W}_0}(\mathcal{F}_0), & \text{if } \mathbf{W}_0 \text{ is dicritical.} \end{cases} \tag{19}$$

Definition 2.5. *Let $\mathbf{W}_0$ be a non-dicritical component of $\mathrm{Sing}(\mathcal{F}_0)$. We will say that $\mathbf{W}_0$ is of type I, II and III if $m_n + 1 = m_1$, $m_n + 1 < m_1$ and $m_n = m_1$, respectively.*

Definition 2.6. *The foliation $\mathcal{F}_0$ will be called special along $\mathbf{W}_0$ if the induced foliation $\mathcal{F}_1$ has the exceptional divisor $\mathbf{E}_1 = \pi_1^{-1}(W_0)$ as an invariant set, and $\mathrm{Sing}(\mathcal{F}_1)$ meets $\mathbf{E}_1$ at isolated singularities at most.*

Remark 2.7. If $\mathcal{F}_0$ is special along $\mathbf{W}_0$ then necessary $\mathbf{W}_0$ is of type I and $G_j \not\equiv 0$ for all $j \in I_d$.

As in [10], we will denote by $\mathcal{N}(\mathcal{F}_0, Z)$ the singularity number of $\mathcal{F}_0$ in Z, counted with multiplicities, where Z is a smooth subvariety of $\mathbb{P}^n$ of arbitrary dimension which is an invvariant set of $\mathcal{F}_1$.

2.2. **Chern classes.** Let us consder a blow-up sequence $\pi_j : \mathbf{M}_j \to \mathbf{M}_{j-1}$ along a smooth curve $\mathbf{W}_{j-1}$, with exceptional divisor $\mathbf{E}_j$ such that $\mathbf{W}_j \subset \mathbf{E}_j$ for all $j \geq 1$. Furthermore, we will admit that $\mathbf{W}_j$ is homeomorphic to $\mathbf{W}_{j-1}$ and $\pi_j(\mathbf{W}_j) = \mathbf{W}_{j-1}$. Set $\mathcal{N}_j := \mathcal{N}_{\mathbf{W}_j/\mathbf{M}_j}$ the normal bundle of $\mathbf{W}_j$ in $\mathbf{M}_j$ and $\rho_j := \pi_j|_{\mathbf{E}_j}$. Since $\mathbf{E}_j \cong \mathbb{P}(\mathcal{N}_{j-1})$, recall that $A(\mathbf{E}_j)$ is generated as an $A(\mathbf{W}_{j-1})$-algebra by the Chern class

$$\zeta_j := c_1(\mathcal{O}_{\mathcal{N}_{j-1}}(-1))$$

with the single relation

$$\sum_{i=0}^{n-1} (-1)^i \zeta_j^{n-1-i} \cdot \rho_j^* c_i(\mathcal{N}_{j-1}) = 0. \tag{20}$$

The normal bundle $\mathcal{N}_{\mathbf{E}_j/\mathbf{M}_j}$ agrees with the tautological bundle $\mathcal{O}_{\mathcal{N}_{j-1}}(-1)$, and hence

$$\zeta_j = c_1(\mathcal{N}_{\mathbf{E}_j/\mathbf{M}_j}) = [\mathbf{E}_j]. \tag{21}$$

If $\iota_j : \mathbf{E}_j \hookrightarrow \mathbf{M}_j$ is the inclusion map, we also get

$$\iota_{j_*}(\zeta_j^i) = (-1)^i [\mathbf{E}_j]^{i+1}. \tag{22}$$

Given that

$$\int_{\mathbf{E}_j} \rho^* c_i(\mathcal{N}_{j-1}) \zeta_j^{n-i-1} = (-1)^{n-i-1} \int_{\mathbf{W}_{j-1}} c_i(\mathcal{N}_{j-1}) = 0$$

for $i \geq 2$, we have

$$\begin{aligned} \int_{E_j} \zeta_j^{n-1} &= \int_{E_j} \rho^* c_1(\mathcal{N}_{j-1}) \zeta_j^{n-2} = (-1)^n \int_{\mathbf{W}_0} c_1(\mathcal{N}_{j-1}) \\ &= (-1)^n \int_{\mathbf{W}_{j-1}} c_1(\mathcal{T}_{M_{j-1}} \otimes \mathcal{O}_{\mathbf{W}_{j-1}}) - c_1(\mathcal{T}_{\mathbf{W}_{j-1}}) := \Lambda_j^{(n)} \end{aligned} \tag{23}$$

In particular, for $\mathbf{M}_0 = \mathbb{P}^n$ we have that

$$\Lambda_0^{(n)} = \chi(\mathbf{W}_0) - (n+1)\deg(\mathbf{W}_0), \tag{24}$$

where $\chi(\mathbf{W}_0)$ is the Euler characterist of $\mathbf{W}_0$. By other side, from Porteous' Theorem (see [23]), it holds that

$$c_1(\mathcal{T}_{\mathbf{M}_j}) = \pi_j^* c_1(\mathcal{T}_{\mathbf{M}_{j-1}}) - (n-2)\mathbf{E}_j. \tag{25}$$

In particular, for $n = 3$, from Whitney formula, we have that

$$c_1(\mathcal{T}_{\mathbf{W}_j}) + c_1(\mathcal{N}_j)\Big|_{\mathbf{W}_j} = c_1(\mathcal{T}_{M_j})\Big|_{\mathbf{W}_j} = \Big(\pi_j^* c_1(\mathcal{T}_{\mathbf{M}_{j-1}}) - \mathbf{E}_j\Big)\Big|_{\mathbf{W}_j} \tag{26}$$

which results for $j = 1$ that

$$\int_{\mathbf{E}_2} \zeta_2^2 = -\int_{\mathbf{W}_1} c_1(\mathcal{N}_1) = -\int_{\mathbf{W}_1} \Big(\pi_1^* c_1(\mathcal{T}_{\mathbf{M}_0}) - c_1(\mathcal{T}_{\mathbf{W}_1}) - \mathbf{E}_1\Big). \tag{27}$$

Therefore, given that $\mathbf{W}_1$ is homeomorphic to $\mathbf{W}_0$ and $\pi_1(\mathbf{W}_1) = \mathbf{W}_0$ then it is not hard to see that

$$\int_{\mathbf{W}_1} \mathbf{E}_1 = -\frac{\Lambda_0^{(3)}}{2} \tag{28}$$

since $\chi(\mathbb{P}^1) = 2$. Thus, the equations (27) and (28) make us to conclude that

$$\int_{\mathbf{E}_2} \zeta_2^2 = \frac{\chi(\mathbf{W}_0) - 4\deg(\mathbf{W}_0)}{2} = \frac{\Lambda_0^{(3)}}{2}. \tag{29}$$

Now, we will consider a finite induction hypothesis as follows

$$\int_{\mathbf{E}_j} \zeta_j^2 = \frac{\Lambda_0^{(3)}}{2^{j-1}}, \quad \int_{\mathbf{W}_j} \mathbf{E}_j = -\frac{\Lambda_0^{(3)}}{2^j} \tag{30}$$

for $j = 1, \dots, k$. So, from (26), we obtain that

$$\begin{aligned} \int_{\mathbf{E}_{k+1}} \zeta_{k+1}^2 &= -\int_{\mathbf{W}_k} c_1(\mathcal{N}_k)) = \\ &= \int_{\mathbf{W}_k} \big(c_1(\mathcal{T}_{\mathbf{W}_k}) - \pi_k^* c_1(\mathcal{T}_{M_{k-1}}) + \mathbf{E}_k\big) \\ &= \int_{\mathbf{W}_{k-1}} c_1(\mathcal{T}_{\mathbf{W}_k}) - \int_{\mathbf{W}_{k-1}} c_1(\mathcal{T}_{\mathbf{M}_{k-1}}) + \int_{\mathbf{W}_k} \mathbf{E}_k \\ &= -\int_{\mathbf{W}_{k-1}} c_1(\mathcal{N}_{k-1})) + \int_{\mathbf{W}_k} \mathbf{E}_k. \end{aligned} \tag{31}$$

Therefore,

$$\int_{\mathbf{E}_{k+1}} \zeta_{k+1}^2 = \int_{\mathbf{E}_k} \zeta_k^2 + \int_{\mathbf{W}_k} \mathbf{E}_k = \frac{\Lambda_0^{(3)}}{2^{k-1}} - \frac{\Lambda_0^{(3)}}{2^k} = \frac{\Lambda_0^{(3)}}{2^k}. \tag{32}$$

From this fact, we get

$$\int_{\mathbf{W}_{k+1}} \mathbf{E}_{k+1} = -\frac{\Lambda_0^{(3)}}{2^{k+1}}. \tag{33}$$

Theorem 2.8. *Let $\mathcal{F}_0$ be a holomorphic foliation by curves on $\mathbf{M}_0 = \mathbb{P}^n$ such that*

$$\mathrm{Sing}(\mathcal{F}_0) = \mathbf{W}_0 \cup \{p_1, \dots, p_r\}$$

where each p_i is a closed point and $\mathbf{W}_0 = Z(f_1, \dots, f_d)$ is a codimension-d smooth variety with $k_j = \deg(f_j)$. Let $\pi_1 : \mathbf{M}_1 \to \mathbf{M}_0$ be the blow-up of $\mathbb{P}^n$ centered along $\mathbf{W}_0$, $\mathbf{E}_1 = \pi_1^{-1}(\mathbf{W}_0)$and $\mathcal{F}_1$ the induced foliation by $\mathcal{F}_0$ via π_1.

(a) *If $\mathcal{F}_0$ is special along $\mathbf{W}_0$ then*

$$\mathcal{N}(\mathcal{F}_1, \mathbf{E}_1) = -\nu(\mathcal{F}_0, \mathbf{W}_0, \psi_a)$$

where

$$\psi_a(x) = \big((1+x)^{d-a_1} - 1\big)x^{n-d-a_2-1}.$$

(b) *In addition, if all the singularities of $\mathcal{F}_1$ are isolated closed points then*

$$\mathcal{N}(\mathcal{F}_1, \mathbf{M}_1) = \sum_{i=0}^{n} k^i + \nu(\mathcal{F}_0, \mathbf{W}_0, \varphi_a - \psi_a).$$

Proof. To proof the item (a) see the proof of theorem 3.2 in [10]. More precisely, see equations (24), (37), (40) and (41) in [10]. To proof the item (b), it is enough to observe the equation (37) in [10] which is written below

$$\mathcal{N}(\mathcal{F}_1, \mathbf{M}_1) = \sum_{i=0}^{n} k^i + \deg(\mathbf{W}_0) \sum_{|a|=0}^{n-d} \sum_{j=|a|}^{n} \sum_{m=0}^{n-d-|a|} (-1)^{\delta_{|a|}^m} \binom{n-j}{m} \Gamma_a^j \ell^{n-j-m} (k-1)^m \sigma_{a_1}^{(d)} \tau_{a_2}^{(d)} \mathcal{W}_{\delta_{|a|}^m}^{(d)}$$

where

$$\Gamma_a^j = \binom{d-a_1}{j-|a|-1} - \binom{d-a_1}{j-|a|}. \tag{34}$$

and ℓ given by (18). Here we are assuming that $\binom{p}{q} = 0$ if $p < q$ or $q < 0$. However,

$$\sum_{j=|a|+1}^{n} \binom{n-j}{m} \binom{d-a_1}{j-|a|-1} \ell^{n-j-m} = \frac{\psi_a^{(m)}(\ell)}{m!},$$

$$\sum_{j=|a|}^{n} \binom{n-j}{m} \binom{d-a_1}{j-|a|} \ell^{n-j-m} = \frac{\varphi_a^{(m)}(\ell)}{m!}$$

which yields

$$\mathcal{N}(\mathcal{F}_1, \mathbf{M}_1) = \sum_{i=0}^{n} k^i + \nu(\mathcal{F}_0, \mathbf{W}_0, \varphi_a - \psi_a).$$

□

3. Fundamental Lemma of deformation

In this section, we will present a generalization for the Fundamental Lemma of deformation given in [10]. Essentially, we will treat the case in which the singular set of $\mathcal{F}_0$ contains a dicritical component.

Lemma 3.1. *Let $\mathcal{F}_0$ be a one-dimensional holomorphic foliation on $\mathbf{M}_0 = \mathbb{P}^n$, $n \geq 3$, of degree k. Suppose that*

$$Sing(\mathcal{F}_0) = \mathbf{W}_0 \cup \{p_1, \dots, p_s\},$$

where $\mathbf{W}_0 = Z(f_1, \dots, f_d)$ is a smooth complete intersection of $\mathbb{P}^n$ and p_j are isolated points disjoint to $\mathbf{W}_0$. As before, $\pi_1 : \mathbf{M}_1 \to \mathbf{M}_0$ is the blow-up of $\mathbb{P}^n$ along $\mathbf{W}_0$, with an exceptional divisor $\mathbf{E}_1$. Then, there exists a one-parameter family of one-dimensional holomorphic foliations on $\mathbb{P}^n$, denoted by $\{\mathcal{F}_t\}_{t \in D}$ where $D = D(0, \epsilon) = \{t \in \mathbb{C} : |t| < \epsilon\}$ such that

(i) $\lim_{t \to 0} \mathcal{F}_t = \mathcal{F}_0$;
(ii) $\deg(\mathcal{F}_t) = \deg(\mathcal{F})$ *for all* $t \in D$;
(iii) *If* $m_{\mathbf{W}_0}(\mathcal{F}_0) = 1$ *then* $\mathbf{W}_0$ *is an invariant set of* $\mathcal{F}_t$ *for any* $t \in D \setminus \{0\}$.
(iv) *If* $m_{\mathbf{W}_0}(\mathcal{F}_0) \geq 2$ *then* $\mathrm{Sing}(\mathcal{F}_t) = \mathbf{W}_0 \cup \{p_1^t, \dots, p_{s_t}^t\}$ *and* $\mathcal{F}_t$ *is special along* $\mathbf{W}_0$ *for any* $t \in D \setminus \{0\}$;
(v) *For any* $t \in D \setminus \{0\}$, *the order of annulment of* $\pi_1^*(\mathcal{F}_t)$ *at* $\mathbf{E}_1$ *is*

$$m_{\mathbf{E}_1}(\pi_1^* \mathcal{F}_t) = \begin{cases} m_{\mathbf{E}_1}(\pi_1^* \mathcal{F}_0) & \text{if } \mathbf{W}_0 \text{ is non-dicritical} \\ m_{\mathbf{E}_1}(\pi_1^* \mathcal{F}_0) - 1 & \text{if } \mathbf{W}_0 \text{ is dicritical.} \end{cases}$$

Proof. Consider that foliation $\mathcal{F}_0$ is described by the polynomial vector field $X_0 = \sum_{i=1}^{n} P_i(z) \frac{\partial}{\partial z_i}$ in some open affine set $U_j \subset \mathbb{P}^n$. Given that $\mathbf{W}_0$ is smooth variety, by a reordering of the variables, if necessary, we can admit that the $d \times d$ matrix

$$\mathbf{M} = \left[\frac{\partial f_i}{\partial z_j} \right]_{1 \leq i,j \leq d}$$

is not singular in some open set U such that $\mathbf{W}_0 \cap U \neq \emptyset$. Therefore, $F : U \subseteq U_j \to V \subseteq \mathbb{C}^n$ defined as

$$w = F(z) = (f_1(z), \ldots, f_d(z), z_{d+1}, \ldots, z_n)$$

is a local biholomorphism. Furthermore, the image $F(\mathbf{W}_0 \cap U) = \widetilde{\mathbf{W}_0}$ is defined as $w_1 = \ldots = w_d = 0$. Let $\mathcal{G}_0 = F_*(\mathcal{F}_0)$ be the push-forward foliation defined in V which is described by the vector field

$$X_{\mathcal{G}_0} = Q_1(w) \frac{\partial}{\partial w_1} + \ldots + Q_n(w) \frac{\partial}{\partial w_n}$$

where

$$Q_i(w) = \sum_{|a|=m_i} w_1^{a_1} \cdots w_d^{a_d} Q_{i,a}(w) \tag{35}$$

with at least one $Q_{i,a}(z)$ not vanishing at $\widetilde{\mathbf{W}_0}$. Thus,

$$Q_i \circ F(z) = \begin{cases} \sum_{j=1}^{n} \frac{\partial f_i}{\partial z_j} \cdot P_j(z), & i = 1, \ldots, d \\ P_i(z), & i = d+1, \ldots, n \end{cases} \tag{36}$$

Solving this system and applying the factor $\det(\mathbf{M})$ for normalizing, we can admit that

$$P_i(z) = \begin{cases} \det(A_i(z)), & i = 1, \ldots, d \\ \det(\mathbf{M}) \cdot Q_i \circ F(z), & i = d+1, \ldots, n \end{cases} \tag{37}$$

where A_i is obtained replacing the i-th column of DF by the vector column $(Q_1 \circ F(z), \ldots, Q_n \circ F(z))$.

We consider the one-dimensional holomorphic foliation $\mathcal{G}_t$ defined in V described by the following vector field

$$X_{\mathcal{G}_t} = X_{\mathcal{G}_0} + t \cdot Y(w) \tag{38}$$

where

$$Y(w) = Y_1(w) \frac{\partial}{\partial w_1} + \ldots + Y_n(w) \frac{\partial}{\partial w_n}$$

with

$$Y_i(w) = \sum_{|a|=q_i} w_1^{a_1} \cdots w_d^{a_d} Y_{i,a}(w), \quad q_i = \mathrm{m}_{\widetilde{\mathbf{W}_0}}(Y_i)$$

with at least one $Y_{i,a}(z)$ not vanishing at $F(\mathbf{W}_0 \cap U)$ for all i such that

$$q_1 = q_2 = \ldots = q_d = q_{d+1} + 1 = \ldots = q_n + 1 = 1 + \ell. \tag{39}$$

where

$$\ell = \begin{cases} \mathrm{m}_{\mathbf{E}_1}(\pi_1^* \mathcal{F}_0), & \text{if } \mathbf{W}_0 \text{ non-dicritical} \\ \mathrm{m}_{\mathbf{E}_1}(\pi_1^* \mathcal{F}_0) - 1, & \text{if } \mathbf{W}_0 \text{ dicritical} \end{cases}$$

The one-parameter family of one-dimensional holomorphic foliations $\mathcal{F}_t$ is defined as pull-back of $\mathcal{G}_t$, $\mathcal{F}_t = F^* \mathcal{G}_t$. Therefore, $\mathcal{F}_t$ is described by the vector field X_t as follows

$$X_t = \sum_{i=1}^{n} P_i^t(z) \frac{\partial}{\partial z_i}$$

where $P_i^t(z)$ is obtained from (37) chancing $Q_i \circ F(z))$ by $Q_i \circ F(z)) + tY_i \circ F(z)$. It is immediate that $\lim_{t \to 0} \mathcal{F}_t = \mathcal{F}_0$. The vector field X_t is polynomial since each P_i and F they are also. Then, we can consider $\mathcal{F}_t$ defined in the open affine set U_j by Hartogs Extension Theorem. The proof of (i) is immediate. The functions $Y_{i,a}$ are chosen so that $\deg(\mathcal{F}_t)$ be equal to $\deg(\mathcal{F}_0)$. And for that, some functions Y_i may be constant, null or an affine linear in variables $w_{d+1}, \ldots, w_n$, proving (ii). If $\mathrm{m}_{\mathbf{W}_0}(\mathcal{F}_0) = 1$ then in (39) $q_i = 0$ for $i > d$ which results that $\mathbf{W}_0$ is an invariant set of $\mathcal{F}_t$ for $t \neq 0$. Otherwise, if $\mathrm{m}_{\mathbf{W}_0}(\mathcal{F}_0) \geq 2$, $\mathbf{W}_0 \subset \mathrm{Sing}(\mathcal{F}_t)$ for all $t \in D$. Furthermore, shrinking ϵ, if necessary, we can admit that $\mathrm{Sing}(\mathcal{F}_t)$ contains isolated closed points disjoint from $\mathbf{W}_0$, since F is a local bibolomorphism. By construction, for $\mathcal{F}_t$, with $t \neq 0$, the curve $\mathbf{W}_0$ is of type I, which means that by changing some coefficients of Y we can admit that $\mathcal{F}_t$ is special along $\mathbf{W}_0$ for $t \in D \setminus \{0\}$, proving (iv). Finally, for $t \neq 0$, $\mathrm{m}_{\mathbf{E}_1}(\pi_1^*\mathcal{F}_t) = q_n = \mathrm{mult}_{\mathbf{W}_0}(\mathcal{F}_0) - 1$. Thus, from (19) we get (v). □

3.1. **Embedded closed points.** Lemma 3.1 allows us to determine the embedded closed points associated with $\mathbf{W}_0$ more effectively. In fact, let $\mathcal{F}_t$ be a *special* deformation of $\mathcal{F}_0$ given by the Lemma 3.1 for $t \in D(0, \epsilon)$. Thus, or $\mathbf{W}_0 \subset \mathrm{Sing}(\mathcal{F}_t)$ or $\mathbf{W}_0$ is $\mathcal{F}_t$-invariant for all $t \neq 0$. Whatever, we can assume that all the isolated singularities of $\mathcal{F}_t$ are non-degenerate, so we set

$$A_{\mathbf{W}_0} := \{p_j^t \in \mathrm{Sing}(\mathcal{F}_t) : \lim_{t\to 0} p_j^t \in \mathbf{W}_0, \quad \text{such that} \quad p_j^t \notin \mathbf{W}_0, \forall t \neq 0\}. \tag{40}$$

where each p_j^t is isolated point. We will indicate by $N(\mathcal{F}_0, A_{\mathbf{W}_0})$ the number of elements $p_j^t \in A_{\mathbf{W}_0}$, counted with multiplicities, and such points are called embedding points associated to $\mathbf{W}_0$.

3.2. **Proof of Theorem 1.1.** Let $\mathcal{F}_0$ be a foliation by curves defined on $\mathbb{P}^n$ of degree k such that

$$\mathrm{Sing}(\mathcal{F}_0) = \mathbf{W}_0 \cup \{p_1, \ldots, p_s\},$$

which $\mathbf{W}_0 = Z(f_1, \ldots, f_m)$. By Lemma 3.1, there is a one-parameter family of holomorphic foliations by curves on $\mathbb{P}^n$, given by $\{\mathcal{F}_t\}_{t\in D}$ where $D = D(0, \epsilon)$ satisfies $(i)-(v)$ conditions. Thus, for $\mathrm{m}_{\mathbf{W}_0}(\mathcal{F}_0) \geq 2$, we have

$$\mathrm{Sing}(\mathcal{F}_t) = \mathbf{W}_0 \cup \{p_1^t, \ldots, p_{s_t}^t\}$$

where each p_i^t is a closed point. Let $\pi_1 : \mathbf{M}_1 \to \mathbf{M}_0$ be the blowup of $\mathbf{M}_0 = \mathbb{P}^n$ along $\mathbf{W}_0$ being $\mathbf{E}_1$ and $\widetilde{\mathcal{F}_t}$ the exceptional divisor and the induced foliation in $\mathbf{M}_1$, respectively. Thus,

$$\mathrm{Sing}(\widetilde{\mathcal{F}_t}) = \{\widetilde{p_1^t}, \ldots, \widetilde{p_{r_t}^t}\}.$$

Since $\pi_1 : \mathbf{M}_1 \setminus \mathbf{E}_1 \to \mathbf{M}_0 \setminus \mathbf{W}_0$ is a biholomorphism, we have that

$$\sum_{j=1}^{s} \mu(\mathcal{F}_0, p_j) = \lim_{t\to 0} \sum_{\lim_{t\to 0} \widetilde{p_j^t} \notin \mathbf{E}_1} \mu(\widetilde{\mathcal{F}_t}, \widetilde{p}_j^t).$$

By other hand,

$$\sum_{j=1}^{s} \mu(\mathcal{F}_0, p_j) = \mathcal{N}(\widetilde{\mathcal{F}_t}, \mathbf{M}_1) - \lim_{t\to 0} \sum_{\lim_{t\to 0} \widetilde{p}_j^t \in \mathbf{E}_1} \mu(\widetilde{\mathcal{F}_t}, \widetilde{p}_j^t).$$

Given that $\mathcal{F}_t$ is special along W_0 for $t \neq 0$, from Theorem 2.8, we get

$$\begin{aligned} \sum_{j=1}^{s} \mu(\mathcal{F}_0, p_j) &= \mathcal{N}(\widetilde{\mathcal{F}_t}, \mathbf{M}_1) - \mathcal{N}(\widetilde{\mathcal{F}_t}, \mathbf{E}_1) - N(\mathcal{F}, A_{\mathbf{W}_0}) \\ &= \sum_{i=1}^{n} k^i + \nu(\mathcal{F}_0, \mathbf{W}_0, \varphi_a - \psi_a) + \nu(\mathcal{F}_0, \mathbf{W}_0, \psi_a) - N(\mathcal{F}, A_{\mathbf{W}_0}). \end{aligned}$$

Then,

$$\sum_{j=1}^{s} \mu(\mathcal{F}_0, p_j) = \sum_{i=1}^{n} k^i + \nu(\mathcal{F}_0, \mathbf{W}_0, \varphi_a) - N(\mathcal{F}, A_{\mathbf{W}_0}).$$

If $\ell = \mathrm{m}_{\mathbf{E}_1}(\pi_1^*\mathcal{F}_0) = 0$, then $\mathbf{W}_0$ is $\mathcal{F}_t$-invariant, resulting in

$$\sum_{j=1}^{s} \mu(\mathcal{F}_0, p_j) = \lim_{t\to 0} \sum_{\lim\limits_{t\to 0} p_j^t \notin \mathbf{W}_0} \mu(\widetilde{\mathcal{F}_t}, p_j^t).$$

Therefore,

$$\sum_{j=1}^{s} \mu(\mathcal{F}_0, p_j) = \mathcal{N}(\mathcal{F}_t, \mathbb{P}^n) - \lim_{t\to 0} \sum_{\lim\limits_{t\to 0} p_j^t \in \mathbf{W}_0} \mu(\mathcal{F}_t, p_j^t).$$

Thus,

$$\sum_{j=1}^{s} \mu(\mathcal{F}_0, p_j) = \mathcal{N}(\mathcal{F}_t, \mathbb{P}^n) - \mathcal{N}(\mathcal{F}_t, \mathbf{W}_0) - N(\mathcal{F}, A_{\mathbf{W}_0}).$$

By [10], we get

$$\sum_{j=1}^{s} \mu(\mathcal{F}_0, p_j) = \sum_{i=1}^{n} k^i + \nu(\mathcal{F}_0, \mathbf{W}_0, \varphi_a)|_{\ell=0} - N(\mathcal{F}, A_{\mathbf{W}_0}).$$

It concludes the prove of Items (a) and (b) of Theorem 1.1. Then,

$$\mu(\mathcal{F}_0, \mathbf{W}_0) = -\nu(\mathcal{F}_0, \mathbf{W}_0, \varphi_a) + N(\mathcal{F}_0, A_{\mathbf{W}_0}).$$

Note that $N(\mathcal{F}_0, A_{\mathbf{W}_0})$ is a finite number since $\mathcal{F}_0$ is represented by a polynomial vector field and

$$N(\mathcal{F}_0, A_{\mathbf{W}_0}) \leq \mathcal{N}(\widetilde{\mathcal{F}}_t, \mathbf{M}_1) = \sum_{i=0}^{n} k^i + \nu(\mathcal{F}_0, \mathbf{W}_0, \varphi_a - \psi_a).$$

By the same way,

$$\mu(\mathcal{F}_1, \bigcup_i \mathbf{W}_i^{(1)}) = \mathcal{N}(\mathcal{F}_1, \mathbf{E}_1) + N(\mathcal{F}_1, A_{\mathbf{E}_1}) = -\nu(\mathcal{F}_0, \mathbf{W}_0, \psi_a) + N(\mathcal{F}_1, A_{\mathbf{E}_1})$$

where each $\mathbf{W}_i^{(1)}$ is a connected component of $\mathrm{Sing}(\mathcal{F}_1)$ contained in $\mathbf{E}_1$. But, since $\pi_1|_{\mathbf{M}_1\setminus\mathbf{E}_1} : \mathbf{M}_1\setminus\mathbf{E}_1 \to \mathbf{M}_0 \setminus \mathbf{W}_0$ is a biholomorphism, we get $N(\mathcal{F}_0, A_{\mathbf{W}_0}) = N(\mathcal{F}_1, A_{\mathbf{E}_1})$, i.e.,

$$\mu(\mathcal{F}_1, \bigcup_i \mathbf{W}_i^{(1)}) = \mu(\mathcal{F}_0, \mathbf{W}_0) + \nu(\mathcal{F}_0, \mathbf{W}_0, \varphi_a) - \nu(\mathcal{F}_0, \mathbf{W}_0, \psi_a)$$

But,

$$\nu(\mathcal{F}_0, \mathbf{W}_0, \varphi_a) - \nu(\mathcal{F}_0, \mathbf{W}_0, \psi_a) = \nu(\mathcal{F}_0, \mathbf{W}_0, \varphi_a - \psi_a) = \nu(\mathcal{F}_0, \mathbf{W}_0, \vartheta_a)$$

which concluded the proof of Item (c) of Theorem 1.1.

Remark 3.2. In theorem 1.1, we have in mind that if $\mathbf{W}_i^{(1)} \cap \mathbf{W}_j^{(1)} \neq \emptyset$ then $\mu(\mathcal{F}_1, \mathbf{W}_i^{(1)} \cup \mathbf{W}_j^{(1)}) = \mu(\mathcal{F}_1, \mathbf{W}_i^{(1)}) + \mu(\mathcal{F}_1, \mathbf{W}_j^{(1)}) - N(\mathcal{F}_1, \mathbf{W}_i^{(1)} \cap \mathbf{W}_j^{(1)})$.

Remark 3.3. If $\mathbf{W}_0 = Z(f_1, \ldots, f_n)$ with $\deg(f_j) = 1$ for $j = 1, \ldots, n$ then $\mathbf{W}_0$ is an isolated closed point. Theorem 1.1 assures that

$$\mu(\mathcal{F}_1, \bigcup_i \mathbf{W}_i^{(1)}) = \mu(\mathcal{F}_0, \mathbf{W}_0) + \vartheta_0(\ell)$$

where

$$\vartheta_0(\ell) = (1+\ell)^n + \frac{1-(1+\ell)^n}{\ell} = (1+\ell)^n - \sum_{j=0}^{n-1} (1+\ell)^j.$$

Thus, if $\mathbf{W}_0$ is a non-dicritical component then $\ell = \mathrm{m}_{\mathbf{W}_0}(\mathcal{F}_0) - 1$ which results

$$\mu(\mathcal{F}_1, \bigcup_i \mathbf{W}_i^{(1)}) = \mu(\mathcal{F}_0, \mathbf{W}_0) + (\mathrm{m}_{\mathbf{W}_0}(\mathcal{F}_0))^n - \sum_{j=0}^{n-1} (\mathrm{m}_{\mathbf{W}_0}(\mathcal{F}_0))^j.$$

On the other hand, if $\mathbf{W}_0$ is a dicritical component, then $\ell = \mathrm{m}_{\mathbf{W}_0}(\mathcal{F}_0)$ resulting in the following result.

$$\mu(\mathcal{F}_1, \bigcup_i \mathbf{W}_i^{(1)}) = \mu(\mathcal{F}_0, \mathbf{W}_0) + (1 + \mathrm{m}_{\mathbf{W}_0}(\mathcal{F}_0))^n - \sum_{j=0}^{n-1}(1 + \mathrm{m}_{\mathbf{W}_0}(\mathcal{F}_0))^j.$$

These results agree with [2] for the case which $\mathbf{W}_0$ is absolutely isolated singularity. However, these expressions remain true even if the singular set of $\mathcal{F}_1$ contains components of positive dimension.

Remark 3.4. If $\mathbf{W}_0 = Z(f_1, \ldots, f_{n-1})$ with $\deg(f_j) = k_j$ then $\mathbf{W}_0$ is a smooth curve. Thus, $\tau_1^{(n-1)} = \big(n + 1 - \sum_{j=1}^{n-1} k_j\big)$ and $\sigma_1^{(n-1)} = \sum_{j=1}^{n-1} k_j$, resulting in $\chi(\mathbf{W}_0) = \tau_1^{(n-1)} \cdot \deg(\mathbf{W}_0)$. By Theorem 1.1, we get

$$\mu(\mathcal{F}_1, \bigcup_i \mathbf{W}_i^{(1)}) = \mu(\mathcal{F}_0, \mathbf{W}_0) + \chi(\mathbf{W}_0)\Bigg(\sum_{j=0}^{n-3}(1+\ell)^j - \ell^2(1+\ell)^{n-2}\Bigg) +$$

$$(1+\ell)^{n-2}\deg(\mathbf{W}_0)\Bigg((n - n\ell - 2)(k-1) + (n+1)(\ell^2 - \ell)\Bigg),$$

where

$$\ell = \begin{cases} m_{\mathbf{E}_1}\big(\pi_1^*\mathcal{F}_0\big), & \text{if } \mathbf{W}_0 \text{ is non-dicritical} \\ m_{\mathbf{E}_1}\big(\pi_1^*\mathcal{F}_0\big) - 1, & \text{if } \mathbf{W}_0 \text{ is dicritical} \end{cases}$$

Example 3.5. Let us consider the one-dimensional holomorphic foliation $\mathcal{F}_0$ of degree 3 defined on $\mathbf{M}_0 = \mathbb{P}^3$ described in the open affine set $U_3 = \{[\xi_i] \in \mathbb{P}_3 | \xi_3 \neq 0\}$ by the vector field

$$X = \big(P_2(z) + P_3(z)\big)\frac{\partial}{\partial z_1} + \big(Q_2(z) + Q_3(z)\big)\frac{\partial}{\partial z_2} + R_2(z)\frac{\partial}{\partial z_3}$$

where $P_i(z) = \sum_{j=0}^{i} p_{ij} z_1^{i-j} z_2^j$, $Q_i(z) = \sum_{j=0}^{i} q_{ij} z_1^{i-j} z_2^j$, $R_2(z) = \sum_{j=0}^{2} r_j(z) z_1^{2-j} z_2^j$ with $p_{ij}, q_{ij} \in \mathbb{C}$, $r_j(z) = \alpha_j z_1 + \beta_j z_2 + \gamma_j z_3 + \delta_j$ $z_i = \xi_{i-1}/\xi_3$. We will also assume $z_1 Q_2(z) - z_2 P_2(z) \equiv 0$ which results that $\mathbf{W}_0 = \{[\xi] \in \mathbb{P}_3 | \xi_0 = \xi_1 = 0\}$ is a dicritical component of $\mathrm{Sing}(\mathcal{F}_0)$. Thus, $P_2(z) = p_{20} z^2 + p_{11} z_1 z_2$ and $Q_2(z) = p_{20} z_1 z_2 + p_{11} z_2^2$. Let $p_i(\lambda) = P_i(1, \lambda)$ and $q_i(\lambda) = Q_i(1, \lambda)$ for $i = 2, 3$. We admit that both p_2 and p_3 have no roots in common. Beyond this curve, the singular set of $\mathcal{F}_0$ has 8 more isolated points, counting the multiplicities. Let $\pi_1 : \mathbf{M}_1 \to \mathbf{M}_0$, $\mathbf{E}_1$ and $\mathcal{F}_1$ as before. It is not difficult to see that $m_{\mathbf{E}_1}(\pi_1^*\mathcal{F}_0) = 2$ and $\mathrm{Sing}(\mathcal{F}_1)$ has 12 isolated points, counting the multiplicities, 4 of them in $\mathbf{E}_1$.

Let $\mathcal{F}_t$ be the one-parameter family of holomorphic foliation on $\mathbb{P}^3$ described by the vector field

$$X_t = X + t\Bigg(z_3 A_2(z)\frac{\partial}{\partial z_1} + z_3 B_2(z)\frac{\partial}{\partial z_2} + C_1(z)\frac{\partial}{\partial z_3}\Bigg)$$

where $A_2(z) = a_0 z_1^2 + a_1 z_1 z_2 + a_2 z_2^2$, $B_2(z) = b_0 z_1^2 + b_1 z_1 z_2 + b_2 z_2^2$ and $C_1(z) = z_1 c_0(z) + z_2 c_1(z)$ being each c_i an affine linear function. Varying the coefficients of A_2, B_2 and C_1 if necessary, we can admit that $\mathcal{F}_t$ is special along $\mathbf{W}_0$, $\mathrm{m}_{\mathbf{E}_1}(\pi_1^*\mathcal{F}_t) = 1$ and $\deg(\mathcal{F}_t)$=3 for each $t \neq 0$. Let $\widetilde{\mathcal{F}}_t$ be the induced foliation on $\mathbf{M}_1$ by $\mathcal{F}_t$ via π_1. Therefore, for $t \neq 0$ the singular set of $\mathcal{F}_t$ has 20 isolated closed points, counting the multiplicities since $\mathcal{N}(\widetilde{\mathcal{F}}_t, \mathbf{E}_1) = 10$ and $\mathcal{N}(\widetilde{\mathcal{F}}_t, \mathbf{M}_1) = 30$. See Theorem 2.8. As consequence, $N(\mathcal{F}_0, A_{\mathbf{W}_0}) = 12$ because the foliation $\mathcal{F}_1$ has 8 isolated point outside the exceptional divisor. Therefore, we get

$$\mu(\mathcal{F}_0, \mathbf{W}_0) = -\lim_{t \to 0} \nu(\mathcal{F}_t, \mathbf{W}_0, \varphi_a) + N(\mathcal{F}_0, A_{\mathbf{W}_0}) = 20 + 12 = 32.$$

This result is totally coherent and compatible, since a generic perturbation of $\mathcal{F}_0$ with the same degree of $\mathcal{F}_0$ will produce 40 isolated singularities, which means that the curve $\mathbf{W}_0$ corresponds to 32 isolated singularities. However, if $\mathbf{W}_0$ is a dicritical component of $\mathrm{Sing}(\mathcal{F}_0)$ with $\mathrm{m}_{\mathbf{E}_1}(\pi_1^*\mathcal{F}_0) = 2$ and $\deg(\mathcal{F}_0) = 3$, then $\mu(\mathcal{F}_0, \mathbf{W}_0) \geq 20$.

4. Holomorphic foliations on $\mathbb{P}^3$

In this section, we will only consider a holomorphic foliation by curves $\mathcal{F}_0$ of degree k defined in $\mathbf{M}_0 = \mathbb{P}^3$ such that its singular locus $\mathrm{Sing}(\mathcal{F}_0)$ contains a smooth curve $\mathbf{W}_0$ of degree $\deg(\mathbf{W}_0)$ and Euler characteristic $\chi(\mathbf{W}_0)$.

For each point $p \in \mathbf{W}_0$ there are an open set $U_0 \subset \mathbf{M}_0$ and two polynomials f_1, f_2 such that $p \in U_0$ and $\mathbf{W}_0 \cap U_0$ is defined as $f_1 = f_2 = 0$ since $\mathbf{W}_0$ is a smooth curve which implies that $\mathbf{W}_0$ is a locally intersection complete (lci). Shrinking an open set U_0 or reordering the variables, if necessary, we can admit that $\varphi_0 : U_0 \to V_0 \subset \mathbb{C}^3$ defined as $z = \varphi_0(w) = (f_1(w), f_2(w), w_3)$ is a biholomorphism. Therefore, in V_0 the curve $\mathbf{W}_0$ is is given by $z_1 = z_2 = 0$ and $\mathcal{F}_0$ is described by the following vector field

$$X_0 = P_1(z)\frac{\partial}{\partial z_1} + P_2(z)\frac{\partial}{\partial z_2} + P_3(z)\frac{\partial}{\partial z_3} \tag{41}$$

where each

$$P_i(z) = \sum_{j=m_i} P_{ij}(z), \quad P_{ij}(z) = \sum_{r=0}^{j} z_1^{j-r} z_2^r p_{ijr}(z_3), \quad m_i = \mathrm{m}_{\mathbf{W}_0}(P_i)$$

with $m_3 \leq m_1 = m_2$. At first, At first, we assume that there is a blow-up sequence $\{\pi_i, \mathbf{M}_i, \mathbf{W}_i, \mathcal{F}_i, \mathbf{E}_i\}$ where $\mathbf{W}_i \subset \mathbf{E}_i$ is a homeomorphic curve to $\mathbf{W}_{i-1}$ and $\pi_i(\mathbf{W}_i) = \mathbf{W}_{i-1}$ for all $i \geq 1$. In the chart $\big(U_1, \sigma_1(u)\big)$, the pull-back foliation $\pi_1^*\mathcal{F}_0$ is described by the following vector field

$$X_1 = u_1^{m_1} Q_1(u)\frac{\partial}{\partial u_1} + u_1^{m_1-1}\big(Q_2(u) - u_2 Q_1(u)\big)\frac{\partial}{\partial u_2} + u_1^{m_3} Q_3(u)\frac{\partial}{\partial u_3} \tag{42}$$

where

$$Q_i(u) = \sum_{j=m_i} u_1^{j-m_i} Q_{ij}(u), \quad Q_{ij}(u) = \sum_{r=0}^{j} u_2^r p_{ijr}(u_3).$$

By hypothesis, there is a curve $\mathbf{W}_1 \subset \mathrm{Sing}(\mathcal{F}_1)$ which is locally defined as $u_1 = u_2 - \psi_1(u_3) = 0$ for certain function ψ_1. To continue with this blow-up process, we need to rewrite each Q_{ij} as follows

$$Q_{ij}(u) = \sum_{r=0}^{j} u_2^r p_{ijr}(u_3) = (u_2 - \psi_1(u_3))^{n_{ij}} \widetilde{h}_{ij}(u), \quad 0 \leq n_{ij} \leq j,$$

where $\widetilde{h}_{ij}(u) = \widetilde{h}_{ij}(u_2, u_3) = \displaystyle\sum_{r=0}^{j-n_{ij}} u_2^r q_{ijr}(u_3)$.

If $\mathbf{W}_0$ is of type I then in $(U_1, \sigma_1(u))$, the foliation $\mathcal{F}_1$ is described by the following vector field

$$X_1 = u_1 Q_1(u)\frac{\partial}{\partial u_1} + \big(Q_2(u) - u_2 Q_1(u)\big)\frac{\partial}{\partial u_2} + Q_3(u)\frac{\partial}{\partial u_3} \tag{43}$$

In this situation, the singular set of $\mathcal{F}_1$ restricted to $\mathbf{E}_1$ is defined by the equations

$$u_1 = Q_{2,m_1}(u) - u_2 Q_{1,m_1}(u) = Q_{3,m_3}(u) = 0$$

which may be composed of curves and points. But, in the coordinate system $v = F(u) = (u_1, u_2 - \psi_1(u_3), u_3)$, the vector field (43) is given by

$$X_1 = v_1 R_1 \frac{\partial}{\partial v_1} + \Big(R_2 - \big(v_2 + \psi_1(v_3)\big)R_1 - \psi_1'(v_3)R_3\Big)\frac{\partial}{\partial v_2} + R_3\frac{\partial}{\partial v_3} \tag{44}$$

where

$$R_i(v) = \sum_{j=m_i}^{\infty} v_1^{j-m_i} v_2^{n_{ij}} h_{ij}(v), \quad h_{ij}(v) = \widetilde{h}_{ij}(v_2 + \psi_1(v_3), v_3).$$

Now, if $\mathbf{W}_0$ of type II then in the chart $\big((U_1)_1, \sigma_1(u)\big)$ the foliation $\mathcal{F}_1$ is described by the following vector field as in (7), i.e.,

$$X_1 = u_1^{m_1-m_3} Q_1(u)\frac{\partial}{\partial u_1} + u_1^{m_1-m_3-1}\big(Q_2(u) - u_2Q_1(u)\big)\frac{\partial}{\partial u_2} \ + Q_3(u)\frac{\partial}{\partial u_3}. \tag{45}$$

On the exceptional divisor $\mathbf{E}_1$, the leaves of $\mathcal{F}_1$ are generically defined as $u_1 = u_2 - \alpha = 0$, where α is a scalar, while its singular set as $u_1 = Q_{3,m_3}(u) = 0$. Thus, in this coordinate system $v = (u_1, u_2 - \psi_1(u_3), u_3)$, the vector field in (45) is given by

$$\begin{aligned} X_1 &= v_1^{m_1-m_3} R_1\frac{\partial}{\partial v_1} + \left(v_1^{m_1-m_3-1}\big(R_2 - \big(v_2+\psi_1(v_3)\big)R_1\big) - \psi_1'(v_3)R_3\right)\frac{\partial}{\partial v_2} \\ &+R_3\frac{\partial}{\partial v_3} \end{aligned} \tag{46}$$

with R_i as in (44). By other side, for generically fixed u_2, there are singular points given by $u_1 = 0$ and $u_3 = \psi(u_2)$ for some function ψ. Therefore, by continuity, in the singular set of $\mathcal{F}_1$ there is at least one curve homeomorphic to $\mathbb{P}^1$.

If $\mathbf{W}_0$ is of type III then the foliation $\mathcal{F}_1$ is described by the following vector field as in (12),

$$X_1 = u_1Q_1(u)\frac{\partial}{\partial u_1} + \big(Q_2(u) - u_2Q_1(u)\big)\frac{\partial}{\partial u_2} + u_1Q_3(u)\frac{\partial}{\partial u_3} \tag{47}$$

In this situation, the leaves of $\mathcal{F}_1$ on $\mathbf{E}_1$ are generically given by $u_1 = u_3 - \alpha = 0$, where α is a scalar. Its singular set is defined by $u_1 = Q_{2,m_1}(u) - u_2Q_{1,m_1}(u) = 0$. Moreover, when u_3 is generically fixed, singular points are defined by $u_1 = u_2 - \psi_i(u_3) = 0$, for some function ψ_i. Consequently, within the singular set of $\mathcal{F}_1$, there exist curves that are homeomorphic to $\mathbf{W}_0$. In the coordinate system $v = (u_1, u_2 - \psi_1(u_3), u_3)$, the vector field in Equation (47) can be expressed as follows:

$$X_1 = v_1R_1\frac{\partial}{\partial v_1} + \left(R_2 - \big(v_2+\psi_1(v_3)\big)R_1 - v_1\psi_1'(v_3)R_3\right)\frac{\partial}{\partial v_2} + v_1R_3\frac{\partial}{\partial v_3} \tag{48}$$

where R_i is as defined in Equation (44). However, unlike the previous case, homeomorphic curves to $\mathbb{P}^1$ may not appear unless $Q_2(0, u_2, \alpha) - u_2Q_1(0, u_2, \alpha) \equiv 0$ for some α.

Finally, if $\mathbf{W}_0$ is a dicritical component of $\mathrm{Sing}(\mathcal{F}_0)$ then in the chart $(U_1, \sigma_1(u))$, the foliaion $\mathcal{F}_1$ is described by the following vector field

$$X_1 = Q_1(u)\frac{\partial}{\partial u_1} + G_2\frac{\partial}{\partial u_2} + Q_3(u)\frac{\partial}{\partial u_3} \tag{49}$$

where

$$G_2 = \sum_{j=m_1+1}^{\infty} u_1^{j-m_1-1}\Big(Q_{2j}(u) - u_2Q_{1j}(u)\Big).$$

Thus, $\mathbf{E}_1$ is not invariant set by $\mathcal{F}_1$ and the singular set $\mathrm{Sing}(\mathcal{F}_1)$ restricted to $\mathbf{E}_1$ is given by equations

$$u_1 = Q_{1,m_1}(u) = Q_{2,m_1+1}(u) - u_2Q_{1,m_1+1}(u) = Q_{3,m_3}(u) = 0.$$

Therefore, in the coordinate system $v = (u_1, u_2 - \psi_1(u_3), u_3)$, the vector field (49) can be rewritten as follows

$$X_1 = R_1\frac{\partial}{\partial v_1} + \left(\sum_{j=m_1+1} v_1^{j-m_1-1}G_{2j}(v) - \psi_1'(v_3)R_3\right)\frac{\partial}{\partial v_2} + R_3\frac{\partial}{\partial v_3} \tag{50}$$

where $G_{2j}(v) = v_2^{n_{2j}}h_{2j}(v) - (v_2 + \psi_1(v_3))v_2^{n_{1j}}h_{1j}(v)$ and each R_i is as defined in Equation (44).

Proposition 4.1. *Let $\{\pi_j, \mathbf{M}_j, \mathbf{W}_j, \mathcal{F}_j, \mathbf{E}_j\}$ be a blow-up sequence such that $\mathbf{W}_j$ is homeomorphic to $\mathbf{W}_{j-1}$ with $\pi_j(\mathbf{W}_j) = \mathbf{W}_{j-1}$, where $\mathbf{W}_0$ is a smooth curve and $\mathbf{M}_0 = \mathbb{P}^3$. Then*

$$m_{\mathbf{W}_j}(\mathcal{F}_j) \leq 1 + m_{\mathbf{W}_0}(\mathcal{F}_0), \quad \forall j \geq 1.$$

Proof. In fact, let us consider $\mathcal{F}_i$ is described by the following vector field

$$X_i = \sum_{j=1}^{3} P_j^{(i)}(v)\frac{\partial}{\partial v_j}$$

where X_0 is as in (41) and X_1 as in (44), (46),(48) or (50), depending on the type of $\mathbf{W}_1$. But, whatever the case, $P_3^{(1)}(v) = R_3(v)$ or $P_3^{(1)}(v) = v_1 R_3(v)$. Since

$$R_3(0, v_2, v_3) = v_2^{n_{3,m_3}} h_{3,m_3}(v_2, v_3)$$

with $n_{3,m_3} \leq m_3$ we have that $m_{\mathbf{W}_1}(R_3) \leq m_3 = m_{\mathbf{W}_0}(\mathcal{F}_0)$ which results

$$m_{\mathbf{W}_1}(\mathcal{F}_1) \leq m_{\mathbf{W}_1}(v_1 R_3(v)) = 1 + m_{\mathbf{W}_1}(R_3) \leq 1 + m_{\mathbf{W}_0}(\mathcal{F}_0). \tag{51}$$

Now, in the chart $\big((U_2)_1, v = \sigma_1(t)\big)$, the foliation $\mathcal{F}_2$ is described by the following vector field

$$X_2 = P_1^{(2)}(t)\frac{\partial}{\partial t_1} + P_2^{(2)}(t)\frac{\partial}{\partial t_2} + P_3^{(2)}(t)\frac{\partial}{\partial t_3} \tag{52}$$

where $P_3^{(2)}(t) = R_3^{(2)}(t)$ or $P_3^{(2)}(t) = t_1 R_3^{(2)}(t)$ depending of the type that $\mathbf{W}_1$ is with

$$R_3^{(2)}(t) = \sum_{j=m_i} t_1^{j-m_3+n_{3,j}-\beta_1} t_2^{n_{3,j}} h_{3,j}^{(2)}(t), \quad \beta_1 = m_{\mathbf{W}_1}(R_3), \quad h_{3,j}^{(2)}(t) = h_{3,j}^{(1)}(t_1 t_2, t_3).$$

Let $\mathcal{M} = \{j \geq m_3 | j - m_3 + n_{3,j} - \beta_1 = 0\}$ i.e.; if $j \in \mathcal{M}$ then $n_{3,j} = \beta_1 - (j - m_3) \leq \beta_1$. Again by hypothesis, there is the curve $\mathbf{W}_2$ defined as $t_1 = t_2 - \psi_2(t_3) = 0$ contained in the singular set of $\mathcal{F}_2$. But,

$$R_3^{(2)}(0, t_2, t_3) = \sum_{j \in \mathcal{M}} t_2^{n_{3,j}} h_{3,j}^{(2)}(0, t_3) = (t_2 - \psi_2(t_3))^{n_3} h_3(t_2, t_3).$$

which results

$$m_{\mathbf{W}_2}(R_3^{(2)}) \leq n_3 \leq m_{\mathbf{W}_1}(R_3).$$

Therefore,

$$m_{\mathbf{W}_2}(\mathcal{F}_2) \leq m_{\mathbf{W}_2}(P_3^{(2)}) \leq 1 + m_{\mathbf{W}_1}(R_3) \leq 1 + m_{\mathbf{W}_0}(\mathcal{F}_0). \tag{53}$$

Note that if $m_{\mathbf{W}_1}(P_3^{(2)}) > m_{\mathbf{W}_1}(P_1^{(2)})$ we can use the same arguments for $P_1^{(2)} = t_1 R_1^{(2)}$ instead of $P_3^{(2)}$.

Continuing in this manner, we obtain:

$$m_{\mathbf{W}_i}(\mathcal{F}_i) \leq 1 + m_{\mathbf{W}_0}(\mathcal{F}_0), \quad \forall i \geq 1. \tag{54}$$

□

Theorem 4.2. *Let $\mathcal{F}_0$ be a holomorphic foliation by curves of degree k defined on $\mathbf{M}_0 = \mathbb{P}^3$ such that*

$$\mathbf{W}_0 \subset \mathrm{Sing}\big(\mathcal{F}_0\big)$$

where $\mathbf{W}_0$ is a smooth curve with Euler characteristic $\chi(\mathbf{W}_0)$, degree $\deg(\mathbf{W}_0)$ and non-dicritical component of $\mathrm{Sing}(\mathcal{F}_0)$. Let $\pi_1 : \mathbf{M}_1 \to \mathbf{M}_0$ be the blow-up of $\mathbf{M}_0$ along $\mathbf{W}_0$, with $\mathbf{E}_1 = \pi_1^{-1}(\mathbf{W}_0)$ the exceptional divisor, $\mathcal{F}_1$ the strict transform of $\mathcal{F}_0$ under π_1 and $\ell = m_{\mathbf{E}_1}(\pi_1^*\mathcal{F}_0)$.

(a) *If $\mathbf{W}_0$ is of type II then the singular set of $\mathcal{F}_1$ contains*

$$\chi(\mathbf{W}_0) + (k-1)\deg(\mathbf{W}_0) + \ell\bigg(\chi(\mathbf{W}_0) - 4\deg(\mathbf{W}_0)\bigg)/2$$

homeomorphic curves to $\mathbb{P}^1$, counting the multiplicities.

(b) *If $\mathbf{W}_0$ is of type III then the singular set of $\mathrm{Sing}(\mathcal{F}_1)$ contains*

$$2 + \ell = 1 + m_{\mathbf{W}_0}(\mathcal{F}_0)$$

homeomorphic curves to $\mathbf{W}_0$, counting the multiplicities.

Proof. The proof of Theorem 4.2 is identical to the proof of Theorem 4.7 in [12]. □

Remark 4.3. If $\mathbf{W}_0$ is of type II, then there will necessarily be at least one homeomorphic curve to $\mathbb{P}^1$. Indeed, since $\mathbf{W}_0$ is a smooth curve, it is also a local complete intersection (l.c.i.), which implies that $\mathbf{W}_0$ can be locally defined by two polynomials $f_1 = f_2 = 0$, where $d_j = \deg(f_j)$ with $d_1 \leq d_2$. Let us assume, for the sake of contradiction, that $\mathbf{W}_0$ is of type II, and there is no homeomorphic curve to $\mathbb{P}^1$ contained in $\mathrm{Sing}(\mathcal{F}_1)$. Therefore, $\mathcal{F}_1$ can extend to a foliation on $\mathbf{E}_1$ without singularities on it, and its leaves are homeomorphic to $\mathbf{W}_0$. According to [15], Proposition 2.3, and [12], Theorem 4.6, we have $\chi(\mathbf{W}_0) = 0$, which leads to $k = \deg(\mathcal{F}_0) = 2\ell + 1$, where $\ell = m_{\mathbf{E}_1}(\pi_1^*\mathcal{F}_0)$. However, this is impossible if $d_1 \geq 2$. In fact, if $d_1 \geq 2$, then according to [12], page 315, $\mathcal{F}_0$ is described by the following vector field:

$$X_0 = \bigg(\frac{\partial f_2}{\partial z_2}Q_1(z) - \frac{\partial f_1}{\partial z_2}Q_2(z) + \Delta_{23}Q_3\bigg)\frac{\partial}{\partial z_1} + \bigg(- \frac{\partial f_2}{\partial z_1}Q_1(z) + \frac{\partial f_1}{\partial z_1}Q_2(z) - $$
$$- \Delta_{13}Q_3\bigg)\frac{\partial}{\partial z_2} + \Delta_{12}Q_3(z)\frac{\partial}{\partial z_3}$$

where

$$\Delta_{ij} = \frac{\partial f_1}{\partial z_i}\frac{\partial f_2}{\partial z_j} - \frac{\partial f_1}{\partial z_j}\frac{\partial f_2}{\partial z_i}, \quad Q_i(z) = \sum_{j=0}^{m_i} a_{ij}(z)\big(f_1(z)\big)^{m_i - j}\big(f_2(z)\big)^j.$$

Since $\mathbf{W}_0$ is of type II then $m_3 + 2 \leq m_1 = m_2$. Therefore,

$$deg\bigg(\frac{\partial f_2}{\partial z_i}Q_1(z)\bigg) \geq d_2 - 1 + (\ell + 2)d_1 + j(d_2 - d_1)$$

for $j = 0, \ldots, m_1$ since $a_{1j}(z) \not\equiv 0$. Given that at least one $a_{1j} \not\equiv 0$, we can conclude that

$$deg(\mathcal{F}_0) \geq 2\ell + 5.$$

On the other hand, if $d_1 = 1$ then $\mathbf{W}_0$ is a complete intersection and since $\mathcal{F}_1$ extends to $\mathbf{E}_1$ without singularities we can consider $\mathcal{F}_0$ to be special along $\mathbf{W}_0$. In order to have $\deg(\mathcal{F}_0) = 2\ell + 1$, the unique possibility is $d_2 \leq 2$. See [12] for more details. But it implies $\chi(\mathbf{W}_0) = 2$, which is absurd.

Example 4.4. Let $\mathcal{F}_0$ be a holomorphic foliation by curves of degree k on $\mathbf{M}_0 = \mathbb{P}^3$, induced on the affine open set $U_3 = \{[\xi] \in \mathbb{P}_3 \,|\, \xi_3 \neq 0\}$ by the polynomial vector field

$$X_0 = \sum_{j=0}^{m_1} z_1^j z_2^{m_1 - j} P_{1j}(z)\frac{\partial}{\partial z_1} + \sum_{j=0}^{m_2} z_1^j z_2^{m_2 - j} P_{2j}(z)\frac{\partial}{\partial z_2} + \sum_{j=0}^{m_3} z_1^j z_2^{m_3 - j} P_{3j}(z)\frac{\partial}{\partial z_3}$$

where $z_i = \xi_{i-1}/\xi_3$ with with $p_{ij}(z_3) = P_{ij}(0, 0, z_3) \not\equiv 0$ for some j for each i. Thus, $\mathbf{W}_0 = \{[\xi] \in \mathbb{P}^3 | \xi_0 = \xi_1 = 0\} \subset \mathrm{Sing}(\mathcal{F}_0)$ and $\chi(\mathbf{W}_0) = 2$ and $\deg(\mathbf{W}_0) = 1$. Since $\deg(\mathcal{F}_0) = k$ then there are some i, j such that $\deg(P_{ij}) = k - m_i$. Let $\pi_1 : \mathbf{M}_1 \to \mathbf{M}_0$ be the blowup of $\mathbb{P}^3$ along $\mathbf{W}_0$, $\mathbf{E}_1 = \pi_1^{-1}(\mathbf{W}_0)$ and $\mathcal{F}_1$ be the strict transform foliation on $\mathbf{M}_1$. In the chart $\big((U_1)_1, z = \sigma_1(u)\big)$, we have the relations $\sigma_1(u) = (u_1, u_1u_2, u_3) = z \in \mathbb{C}^3$. Thus, if $\mathbf{W}_0$ is type II then $\mathcal{F}_1$ is generated by the vector field as in (45). For generically fixed u_2 the singular set of $\mathcal{F}_1$ restricted to $\mathbf{E}_1$ is determined by the equation $Q_3(u) = \sum_{j=0}^{m_3} u_2^j p_{3j}(u_3)$. Let $m = \max\{\deg(p_{3j}), j = 0, \ldots, m_3\}$. In this way, for generically fixed u_2 the equation Q_3 has m roots, counted the multiplicities. By continuity, in the singular set of $\mathrm{Sing}(\mathcal{F}_1)$ there are m curves of singularities. But, the fiber $\pi_1^{-1}([0:0:1:0])$ is a curve of singularities with multiplicity exactly equal to $k - m_3 - m + 1$. Consequently, if $\mathbf{W}_0$ is of type II then there are $k - m_3 + 1$ homeomorphic curves to $\mathbb{P}^1$ contained in $\mathrm{Sing}(\mathcal{F}_1)$, counting the multiplicities. Now, if $\mathbf{W}_0$ is of type III then the singular set of $\mathcal{F}_1$, restricted to $\mathbf{E}_1$ is determined by the equations $u_1 = Q_2(u) - u_2Q_1(u) = 0$. It is not difficult to see there are $m_1 + 1$ roots for generically fixed u_3 , counting the multiplicities(in the other chart $\big((U_1)_2, \sigma_2(v)\big)$ if necessary). Again, by continuity, there are $m_1 + 1$ homeomorphic curves to $\mathbf{W}_0$ at $\mathbf{E}_1$.

Example 4.5. Let $\mathcal{F}_0$ be a one-dimensional holomorphic foliation in $\mathbf{M}_0 = \mathbb{P}^3$, such that $\mathbf{W}_0 \subset \mathrm{Sing}(\mathcal{F}_0)$, where $\mathbf{W}_0$ is a smooth curve.

In this example, we will assume that the desingularization process for $\mathbf{W}_0$ does not involve any dicritical curves. Thus, in some affine set, $\mathcal{F}_0$ can be described by a vector field X_0 as in (41). Let $\pi_1 : \mathbf{M}_1 \to \mathbf{M}_0$ be the blow-up of $\mathbf{M}_0$ along $\mathbf{W}_0$.

If $\mathbf{W}_0$ is of type III, then there is at least one curve $\mathbf{W}_1 \subset \mathrm{Sing}(\mathcal{F}_1)$ such that $\mathbf{W}_1$ is homeomorphic to $\mathbf{W}_0$ wth $\pi_1(\mathbf{W}_1) = \mathbf{W}_0$. See Theorem 4.2. Thus, $\mathbf{W}_1$ can be locally defined as $u_1 = u_2 - \psi_1(u_3) = 0$, which results in $\mathcal{F}_1$ is described by a vector field X_1 as in Equation (48). Let $\pi_2 : \mathbf{M}_2 \to \mathbf{M}_1$ be the blowup of $\mathbf{M}_1$ along $\mathbf{W}_1$, which is, by hypothesis, a non-dicritical curve of singularities. Given that the first and third sections of X_1 have v_1 as a factor, it is necessary for the curve $\mathbf{W}_2$, defined as $y_1 = y_2 = 0$, where $\sigma_2(y) = v$, will be contained in $\mathrm{Sing}(\mathcal{F}_2)$. Continuing in this manner, there exists a blow-up sequence $\{\pi_i, \mathbf{M}_i, \mathbf{W}_i, \mathcal{F}_i, \mathbf{E}_i\}$ such that $\mathbf{W}_i$ is homeomorphic to $\mathbf{W}_{i-1}$ and $\pi_i(\mathbf{W}_i) = \mathbf{W}_{i-1}$ for all $i \geq 1$.

If $\mathbf{W}_0$ is of type II, then $\mathcal{F}_1$ is described by a vector field X_1 as in Equation (45). In this situation, there is at least one curve $\mathbf{W}_1 \subset \mathrm{Sing}(\mathcal{F}_1)$ that is homeomorphic to $\mathbb{P}^1$, which can be locally defined as $u_1 = u_3 - \psi_1(u_2) = 0$. Consequently, from (45) the third section of X_1 can be written as follows:

$$Q_3(u) = \sum_{j=m_3}^{\infty} v_1^{j-m_3} v_2^{n_{3,j}} h_{3,j}(v), \quad (v_1, v_2, v_3) = (u_1, u_3 - \psi_1(u_2), u_2).$$

Hence, in this coordinate system (v_1, v_2, v_3), the first and third sections of the vector field X_1 also have v_1 as a factor. Similarly to the previous case, there exists a blow-up sequence $\{\pi_i, \mathbf{M}_i, \mathbf{W}_i, \mathcal{F}_i, \mathbf{E}_i\}$ where $\mathbf{W}_i$ is homeomorphic to $\mathbb{P}^1$ and $\pi_i(\mathbf{W}_i) = \mathbf{W}_{i-1}$, but this time for all $i \geq 2$.

Theorem 4.6. *Let $\mathcal{F}_0$ be a one-dimensional holomorphic foliation defined on $\mathbf{M}_0 = \mathbb{P}^3$ of degree k, and its singular locus contains a smooth curve $\mathbf{W}_0$ of degree $\deg(\mathbf{W}_0)$ and Euler characteristic $\chi(W_0)$.*

We will assume the existence of a finite blow-up sequence $\{\pi_i, \mathbf{M}_i, \mathbf{W}_i, \mathcal{F}_i, \mathbf{E}_i\}_{i=1}^{j-1}$ such that $\mathbf{W}_i \subset \mathbf{E}_i$ is homeomorphic to $\mathbf{W}_{i-1}$ with $\pi_i(\mathbf{W}_i) = \mathbf{W}_{i-1}$ and $\ell_i = m_{\mathbf{E}_i}(\pi_i^* \mathcal{F}_{i-1})$ for $i = 1, \ldots, j$.

(a) If $\mathcal{F}_{j-1}$ is special along $\mathbf{W}_{j-1}$ then

$$\mathcal{N}(\mathcal{F}_j, \mathbf{E}_j) = (\ell_j + 2)\chi(\mathbf{W}_0) + 2(\ell_j + 1)(k-1)\deg(\mathbf{W}_0) + (\ell_j + 1)\Lambda_0^{(3)} \left(\frac{\ell_j}{2^{j-1}} + 2 \sum_{i=1}^{j-1} \frac{\ell_i}{2^i} \right)$$

(b) If $\mathrm{Sing}(\mathcal{F}_j)$ contains only isolated closed points on $\mathbf{M}_j$, then

$$\mathcal{N}(\mathcal{F}_j, \mathbf{M}_j) = \sum_{i=0}^{3} k^i + \chi(\mathbf{W}_0) \sum_{i=1}^{j} \left(1 - \ell_i - \ell_i^2\right) + (k-1)\deg(\mathbf{W}_0) \sum_{i=1}^{j} \left(1 - 2\ell_i - 3\ell_i^2\right)$$
$$+ \Lambda_0^{(3)} \sum_{i=1}^{j} \frac{\ell_i(1 - \ell_i^2)}{2^{i-1}} + \Lambda_0^{(3)} \sum_{r=2}^{j} \sum_{i=1}^{r-1} \left(1 - 2\ell_r - 3\ell_r^2\right) \frac{\ell_i}{2^i}.$$

Here we are also assuming that $\sum_{i=\alpha}^{\beta} a_i = 0$ if $\alpha > \beta$.

Proof. For $j = 1$, both formulas (a) and (b) align with Theorems 3.1 and 3.3 in [12], respectively. From Porteuos theorem (see [14], page 609) , we obtain

$$\begin{cases} c_1(\mathcal{T}_{\mathbf{M}_i}) = \pi_j^* c_1(\mathcal{T}_{\mathbf{M}_{i-1}}) - \mathbf{E}_j \\ c_2(\mathcal{T}_{\mathbf{M}_i}) = \pi_j^* c_2(\mathcal{T}_{\mathbf{M}_{i-1}}) + \pi_i^* \iota_{i-1}^* [\mathbf{W}_{i-1}] - \pi_j^* c_1(\mathcal{T}_{\mathbf{M}_{i-1}}) \cdot \mathbf{E}_i \\ c_3(\mathcal{T}_{\mathbf{M}_i}) = \pi_j^* c_3(\mathcal{T}_{\mathbf{M}_{i-1}}) - \pi_i^* c_2(\mathcal{N}_{i-1}) \cdot \mathbf{E}_i - \pi_i^* c_1(\mathcal{T}_{\mathbf{M}_{i-1}}) \cdot \mathbf{E}_i^2 + \mathbf{E}_i^3 \end{cases} \tag{55}$$

where $\iota_i : \mathbf{W}_i \to \mathbf{M}_i$ is the inclusion map and $[\mathbf{W}_i] \in H^4(\mathbf{M}_i)$ is the fundamental class of $\mathbf{W}_i$ for $i = 1, \ldots, j$. From [1], it follows that

$$\mathcal{T}_{\mathcal{F}_i} \cong \pi_i^* \left(\mathcal{T}_{\mathcal{F}_{i-1}}\right) \otimes [\mathbf{E}_i]^{\ell_i}$$

which make us conclude

$$c_1(\mathcal{T}_{\mathcal{F}_i}^*) = \pi_j^* c_1(\mathcal{T}_{\mathcal{F}_{i-1}}^*) - \ell_i \cdot \mathbf{E}_i, \quad \text{for} \quad i = 1, \ldots, j. \tag{56}$$

From (30) and (55), we can show by finite induction that

$$\int_{\mathbf{W}_j} c_1(\mathcal{T}_{\mathbf{M}_j}) = \int_{\mathbf{W}_0} c_1(\mathcal{T}_{\mathbf{M}_0}) - \sum_{i=1}^{j} \int_{\mathbf{W}_i} \mathbf{E}_i = 4\deg(\mathbf{W}_0) + \left(1 - \frac{1}{2^j}\right)\Lambda_0^{(3)}, \quad j \geq 0. \tag{57}$$

Similarly, we obtain

$$\int_{\mathbf{W}_j} c_1(\mathcal{T}^*_{\mathcal{F}_j}) = (k-1)\deg(\mathbf{W}_0) + \Lambda_0^{(3)} \sum_{i=1}^{j} \frac{\ell_i}{2^i}, \quad j \geq 0. \tag{58}$$

Using the Baum-Bott's formula, we get that

$$\mathcal{N}(\mathcal{F}_j, \mathbf{E}_j) = \int_{\mathbf{E}_j} c_2(\mathcal{T}_{\mathbf{E}_j} \otimes \mathcal{T}^*_{\mathcal{F}_j})$$

with

$$c_2(\mathcal{T}_{\mathbf{E}_j} \otimes \mathcal{T}^*_{\mathcal{F}_j}) = c_2(\mathcal{T}_{\mathbf{E}_j}) + c_1(\mathcal{T}_{\mathbf{E}_j})c_1(\mathcal{T}^*_{\mathcal{F}_j}) + c_1^2(\mathcal{T}^*_{\mathcal{F}_j}).$$

By hypothesis, we easily get

$$\int_{\mathbf{E}_j} c_2(\mathcal{T}_{\mathbf{E}_j}) = 2\chi(\mathbf{W}_0) \tag{59}$$

From Whitney formula,

$$c_1(\mathcal{T}_{\mathbf{E}_j}) = c_1(\mathcal{T}_{\mathbf{M}_j}) - \mathbf{E}_j = \pi_j^* c_1(\mathcal{T}_{\mathbf{M}_{j-1}}) - 2\mathbf{E}_j$$

which results

$$\int_{\mathbf{E}_j} c_1(\mathcal{T}_{\mathbf{E}_j})c_1(\mathcal{T}^*_{\mathcal{F}_j}) = \ell_j \int_{\mathbf{W}_{j-1}} c_1(\mathcal{T}_{\mathbf{M}_{j-1}}) + 2\int_{\mathbf{W}_{j-1}} c_1(\mathcal{T}^*_{\mathcal{F}_{j-1}}) + 2\ell_j \int_{\mathbf{E}_j} \mathbf{E}_j^2. \tag{60}$$

And finally,

$$\begin{aligned} \int_{\mathbf{E}_j} c_1^2(\mathcal{T}^*_{\mathcal{F}_j}) &= \int_{\mathbf{E}_j} \left(\pi_j^* c_1^2(\mathcal{T}^*_{\mathcal{F}_{j-1}}) - 2\ell_j c_1(\mathcal{T}^*_{\mathcal{F}_{j-1}})\mathbf{E}_j + \ell_j^2 \mathbf{E}_j^2 \right) \\ &= 2\ell_j \int_{\mathbf{W}_{j-1}} c_1(\mathcal{T}^*_{\mathcal{F}_{j-1}}) + \ell_j^2 \int_{\mathbf{E}_j} \mathbf{E}_j^2 \end{aligned} \tag{61}$$

From equations (57), (58), we can add the equations (59), (60) and (61) that after a simple reorganization, we obtain statement (a) of the Theorem. Now, again by Baum-Bott's formula,

$$\mathcal{N}(\mathcal{F}_j, \mathbf{M}_j) = \int_{\mathbf{M}_j} c_3(\mathcal{T}_{\mathbf{M}_j} \otimes \mathcal{T}^*_{\mathcal{F}_j})$$

with

$$c_3(\mathcal{T}_{\mathbf{M}_j} \otimes \mathcal{T}^*_{\mathcal{F}_j}) = c_3(\mathcal{T}_{\mathbf{M}_j}) + c_2(\mathcal{T}_{\mathbf{M}_j})c_1(\mathcal{T}^*_{\mathcal{F}_j}) + c_1(\mathcal{T}_{\mathbf{M}_j})c_1^2(\mathcal{T}^*_{\mathcal{F}_j}) + c_1^3(\mathcal{T}^*_{\mathcal{F}_j}).$$

In order to simplify all these long calculations, we will do four finite inductions as follows. Given that

$$\int_{\mathbf{M}_{j+1}} \pi_{j+1}^* c_2(\mathcal{N}_j) \cdot \mathbf{E}_{j+1} = \int_{\mathbf{E}_{j+1}} \pi_{j+1}^* c_2(\mathcal{N}_j) = -\int_{\mathbf{W}_j} c_2(\mathcal{N}_j) = 0,$$

we get

$$\begin{aligned} \int_{\mathbf{M}_{j+1}} c_3(\mathcal{T}_{\mathbf{M}_{j+1}}) &= \int_{\mathbf{M}_{j+1}} \left(\pi_{j+1}^* c_3(\mathcal{T}_{\mathbf{M}_j}) - \pi_{j+1}^* c_1(\mathcal{T}_{\mathbf{M}_j}) \cdot \mathbf{E}_{j+1}^2 + \mathbf{E}_{j+1}^3 \right) = \\ &= \int_{\mathbf{M}_j} c_3(\mathcal{T}_{\mathbf{M}_j}) + \int_{\mathbf{W}_j} c_1(\mathcal{T}_{\mathbf{M}_j}) + \int_{\mathbf{E}_{j+1}} \mathbf{E}_{j+1}^2 = \\ &= \int_{\mathbf{M}_j} c_3(\mathcal{T}_{\mathbf{M}_j}) + 4\deg(\mathbf{W}_0) + \Lambda_0^{(3)} = \\ &= \int_{\mathbf{M}_j} c_3(\mathcal{T}_{\mathbf{M}_j}) + \chi(\mathbf{W}_0). \end{aligned} \tag{62}$$

Now, the second term,

$$\int_{\mathbf{M}_{j+1}} c_2(\mathcal{T}_{\mathbf{M}_{j+1}})c_1(\mathcal{T}^*_{\mathcal{F}_{j+1}}) = \int_{\mathbf{M}_j} c_2(\mathcal{T}_{\mathbf{M}_j})c_1(\mathcal{T}^*_{\mathcal{F}_j}) + \int_{\mathbf{W}_j} c_1(\mathcal{T}^*_{\mathcal{F}_j}) - \ell_{j+1}\int_{\mathbf{W}_j} c_1(\mathcal{T}_{\mathbf{M}_j}) \tag{63}$$

The third term is given by

$$\int_{\mathbf{M}_{j+1}} c_1(\mathcal{T}_{\mathbf{M}_{j+1}})c_1^2(\mathcal{T}^*_{\mathcal{F}_{j+1}}) = \int_{\mathbf{M}_j} c_1(\mathcal{T}_{\mathbf{M}_j})c_1^2(\mathcal{T}^*_{\mathcal{F}_j}) - \ell_{j+1}^2\int_{\mathbf{W}_j} c_1(\mathcal{T}_{\mathbf{M}_j}) - 2\ell_{j+1}\int_{\mathbf{W}_j} c_1(\mathcal{T}^*_{\mathcal{F}_j}) - \ell_{j+1}^2\int_{\mathbf{E}_j}\mathbf{E}_{j+1}^2 \tag{64}$$

And the last term

$$\int_{\mathbf{M}_{j+1}} c_1^3(\mathcal{T}^*_{\mathcal{F}_{j+1}}) = \int_{\mathbf{M}_j} c_1^3(\mathcal{T}^*_{\mathcal{F}_j}) - 3\ell_{j+1}^2\int_{\mathbf{M}_j} c_1(\mathcal{T}^*_{\mathcal{F}_j}) - \ell_{j+1}^3\int_{\mathbf{E}_j}\mathbf{E}_{j+1}^2. \tag{65}$$

Thus, we will add equations (62), (63), (64) and (65) and so we conclude that

$$\mathcal{N}(\mathcal{F}_{j+1},\mathbf{M}_{j+1}) = \mathcal{N}(\mathcal{F}_j,\mathbf{M}_j) + \chi(\mathbf{W}_0) + (1-2\ell_{j+1}-3\ell_{j+1}^2)\int_{\mathbf{W}_j} c_1(\mathcal{T}^*_{\mathcal{F}_j}) - \ell_{j+1}(1+\ell_{j+1})\int_{\mathbf{W}_j} c_1(\mathcal{T}_{\mathbf{M}_j}) - \ell_{j+1}^2(1+\ell_{j+1})\int_{\mathbf{E}_{j+1}}\mathbf{E}_{j+1}^2 \tag{66}$$

From (57), (58), and admiting that $\mathcal{N}(\mathcal{F}_j,\mathbf{M}_j)$ is given by the statement (b), we directly conclude this theorem. □

4.1. **Proof of Theorem 1.2.** From Lemma (3.1), for each j, there exists a special holomorphic deformation $\mathcal{F}_{jt}$ of $\mathcal{F}_j$ for $0<|t|<\epsilon_j$, where ϵ_j is sufficiently small. Thus, for $t\neq 0$, the deformation $\mathcal{F}_{jt}$ is special along $\mathbf{W}_j$, and $m_{\mathbf{E}_j}(\mathcal{F}_{jt}) = m_{\mathbf{E}_j}(\mathcal{F}_j) = \ell_j$. Furthermore, we can assume $\deg(\pi_j^*\cdots\pi_1^*\mathcal{F}_{jt}) = \deg(\mathcal{F}_0)$. To achieve this, it suffices to consider that $\mathbf{W}_j$ is defined as $u_1 = u_2 - \psi_j(u_3) = 0$, and $\mathcal{F}_j$ is described by the following vector field:

$$X_j = P_1^{(j)}(v)\frac{\partial}{\partial v_1} + P_2^{(j)}(v)\frac{\partial}{\partial v_2} + P_3^{(j)}(v)\frac{\partial}{\partial v_3} \tag{67}$$

where $(v_1,v_2,v_3) = F_j(u) = (u_1, u_2-\psi_j(u_3), u_3)$. Therefore, in this coordinate system, the foliation $\mathcal{F}_{jt}$ is described by the following vector field

$$X_{jt} = X_j + tY_j \tag{68}$$

with the vector field Y_j given as in Lemma (3.1), that is

$$Y_j = \sum_{i=1}^{3} Y_i\frac{\partial}{\partial v_i},\quad Y_i = \sum_{r=0}^{q_i} v_1^{q_i-r}v_2^r a_{ir}(v),\quad q_i = m_{\mathbf{W}_j}(Y_i) \tag{69}$$

where

$$q_1 = q_2 = q_3 + 1 = \ell_j + 1.$$

By Hartog's theorem, the foliation $\mathcal{F}_{jt}$ that is generated by the vector $F_j^*X_{jt}$ can be extend for whole $\mathbf{M}_j$. Furthermore, the coeficients a_{ir} are chosen in order to have $\deg(\pi_j^*\cdots\pi_1^*\mathcal{F}_{jt}) = \deg(\mathcal{F}_0), \forall t$. Varying the coefficients a_{ir}, if necessary, we can admit that $\mathrm{Sing}(\mathcal{F}_{jt})$ is composed of a curve $\mathbf{W}_j$ and some more isolated closed points $p_s^{(j)}$. Let $\widetilde{\mathcal{F}_{jt}}$ be the strict transform of $\mathcal{F}_{jt}$ under π_{j+1}. Therefore, we can determine the Milnor number $\mu(\mathcal{F}_j,\mathbf{W}_j)$ as follows. Since

$$\mathrm{Sing}(\mathcal{F}_{jt}) = \mathbf{W}_j\cup\{p_s^{(j)}, s=1,\dots,s_t\}, \mathrm{Sing}(\widetilde{\mathcal{F}_{jt}}) = \{\widetilde{p}_r^{(j)}, r=1,\dots,r_t\}$$

we get

$$\mu(\mathcal{F}_j,\mathbf{W}_j)=\lim_{t\to 0}\big(\mu(\mathcal{F}_{jt},\mathbf{W}_j)\big)=\lim_{t\to 0}\Bigg(\mathcal{N}(\mathcal{F}_{jt},\mathbf{M}_j)-\sum_{p_s^{(j)}\notin A_{\mathbf{W}_j}}\mu(\mathcal{F}_{jt},p_s^{(j)})\Bigg)$$

$$=\mathcal{N}(\mathcal{F}_j,\mathbf{M}_j)-\lim_{t\to 0}\sum_{p_s^{(j)}\notin A_{\mathbf{W}_j}}\mu(\mathcal{F}_{jt},p_s^{(j)}) \tag{70}$$

On the other hand, since $\pi_{j+1}:\mathbf{M}_{j+1}\setminus\mathbf{E}_{j+1}\to\mathbf{M}_j\setminus\mathbf{W}_j$ is an isomorphism, we get that

$$\lim_{t\to 0}\sum_{p_s^{(j)}\notin A_{\mathbf{W}_j}}\mu(\mathcal{F}_{jt}p_s^{(j)})=\lim_{t\to 0}\sum_{\widetilde{p}_r^{(j)}\notin A_{\mathbf{E}_{j+1}}}\mu(\widetilde{\mathcal{F}_{jt}},\widetilde{p}_r^{(j)})$$

$$=\mathcal{N}(\mathcal{F}_{j+1},\mathbf{M}_{j+1})-\mathcal{N}(\mathcal{F}_{j+1},\mathbf{E}_{j+1})-N(\mathcal{F}_j,\mathcal{A}_{\mathbf{W}_j}) \tag{71}$$

which results

$$\mu(\mathcal{F}_j,\mathbf{W}_j)=\mathcal{N}(\mathcal{F}_j,\mathbf{M}_j)-\mathcal{N}(\mathcal{F}_{j+1},\mathbf{M}_{j+1})+\mathcal{N}(\mathcal{F}_{j+1},\mathbf{E}_{j+1})+N(\mathcal{F}_j,\mathcal{A}_{\mathbf{W}_j}) \tag{72}$$

where necessarily $N(\mathcal{F}_j,\mathcal{A}_{\mathbf{W}_j})\le N(\mathcal{F}_{j-1},\mathcal{A}_{\mathbf{E}_j})=N(\mathcal{F}_{j-1},\mathcal{A}_{\mathbf{W}_{j-1}})$. From (66), we obtain the following

$$\begin{aligned}\mu(\mathcal{F}_j,\mathbf{W}_j)=&\mathcal{N}(\mathcal{F}_{j+1},\mathbf{E}_{j+1})-\chi(\mathbf{W}_0)+(3\ell_{j+1}^2+2\ell_{j+1}-1)\int_{\mathbf{W}_j}c_1(\mathcal{T}^*_{\mathcal{F}_j})+\\ &\ell_{j+1}(1+\ell_{j+1})\int_{\mathbf{W}_j}c_1(\mathcal{T}_{\mathbf{M}_j})+\ell_{j+1}^2(1+\ell_{j+1})\int_{\mathbf{E}_{j+1}}\mathbf{E}_{j+1}^2+N(\mathcal{F}_j,\mathcal{A}_{\mathbf{W}_j})\\ =&(\ell_{j+1}+1)\Big(\chi(\mathbf{W}_0)+(3\ell_{j+1}+1)(k-1)\deg(\mathbf{W}_0)+4\ell_{j+1}\deg(\mathbf{W}_0)\Big)+\\ &+(\ell_{j+1}+1)\Lambda_0^{(3)}\Bigg(\ell_{j+1}+\frac{\ell_{j+1}^2}{2^j}+\big(3\ell_{j+1}+1\big)\sum_{i=1}^{j}\frac{\ell_i}{2^i}\Bigg)+N(\mathcal{F}_j,\mathcal{A}_{\mathbf{W}_j})\\ =&\big(\ell_{j+1}+1\big)\Bigg((\ell_{j+1}+1)\chi(\mathbf{W}_0)+(3\ell_{j+1}+1)(k-1)\deg(\mathbf{W}_0)+\frac{\ell_{j+1}^2}{2^j}\Lambda_0^{(3)}+\\ &(3\ell_{j+1}+1)\Lambda_0^{(3)}\sum_{i=1}^{j}\frac{\ell_i}{2^i}\Bigg)+N(\mathcal{F}_j,\mathcal{A}_{\mathbf{W}_j}).\end{aligned}$$

Notice that for $j=0$, $\mu(\mathcal{F}_0,\mathbf{W}_0)$ agrees with [13]. Now, given that $\pi_{j+1}|_{\mathbf{M}_{j+1}\setminus\mathbf{E}_{j+1}}:\mathbf{M}_{j+1}\setminus\mathbf{E}_{j+1}\to\mathbf{M}_j\setminus\mathbf{W}_j$ is a biholomorphism, we get $N(\mathcal{F}_j,A_{\mathbf{W}_j})=N(\mathcal{F}_{j+1},A_{\mathbf{E}_{j+1}})$ which results

$$\begin{aligned}\mu(\mathcal{F}_{j+1},\bigcup_i\mathbf{W}_i^{(j+1)})&=\mathcal{N}(\mathcal{F}_{j+1},\mathbf{E}_{j+1})+N(\mathcal{F}_{j+1},A_{\mathbf{E}_{j+1}})\\ &=\mu(\mathcal{F}_j,\mathbf{W}_j)+\mathcal{N}(\mathcal{F}_{j+1},\mathbf{M}_{j+1})-\mathcal{N}(\mathcal{F}_j,\mathbf{M}_j).\end{aligned}$$

Again, from From (66), we conclude the Item (b) of Theorem.

Example 4.7. Let $\mathcal{F}_0$ be the one-dimensional holomorphic foliation of degree 4 defined on $\mathbf{M}_0=\mathbb{P}^3$. This foliation is described in the open affine set $U_3=\{[\xi_i]\in\mathbb{P}_3|\xi_3\neq 0\}$ by the following vector field

$$X_0=\big(P_3(z)+P_4(z)\big)\frac{\partial}{\partial z_1}+\big(Q_3(z)+Q_4(z)\big)\frac{\partial}{\partial z_2}+\Bigg(\sum_{i=1}^{4}R_i(z)\Bigg)\frac{\partial}{\partial z_3}, \tag{73}$$

where $z_i=\xi_{i-1}/\xi_3$ and

$$P_i(z)=\sum_{j=0}^{i}z_1^{i-j}z_2^jp_{ij}(z_3),Q_i(z)=\sum_{j=0}^{i}z_1^{i-j}z_2^jq_{ij}(z_3),R_i(z)=\sum_{j=0}^{i}z_1^{i-j}z_2^jr_{ij}(z_3)$$

with $p_{ij},q_{ij},r_{ij}\in\mathbb{C}[z_3]$ are generic polynomials of degree $4-i$. Except for q_{30} and r_{10}, which are identically null.

Thus, $\mathbf{W}_0 = \{[\xi] \in \mathbb{P}^3 | \xi_0 = \xi_1 = 0\} \subset \mathrm{Sing}(\mathcal{F}_0)$ and is of type II. The singular set of $\mathcal{F}_0$ contains another 36 closed points disjunct of $\mathbf{W}_0$. Thus, $\mu(\mathcal{F}_0, \mathbf{W}_0) = 46$, which results in $N(\mathcal{F}_0, \mathcal{A}_{W_0}) = 21$. See Theorem (1.1). Let $\pi_1 : \mathbf{M}_1 \to \mathbf{M}_0$ be the blowup of $\mathbf{M}_0$ along $\mathbf{W}_0$ and $\mathcal{F}_1$ be the strict transform of $\mathcal{F}_0$ under π_1. In the chart $\big((U_3)_1, \sigma_1(u)\big)$, the foliation $\mathcal{F}_1$ is described by the following vector field

$$\begin{aligned} X_1 = &\big(u_1^2 p_{30} + \widetilde{P_1}\big)\frac{\partial}{\partial u_1} + \big(u_1 u_2 (q_{31} - p_{30}) + u_1^2 q_{40} + \widetilde{Q_1}\big)\frac{\partial}{\partial u_2} \\ &+ \left(u_2 r_{11} + u_1 r_{20} + \widetilde{R_1}\right)\frac{\partial}{\partial u_3} \end{aligned} \tag{74}$$

which $\mathbf{W}_1 = \{u \in (U_3)_1 | u_1 = u_2 = 0\} \subset \mathrm{Sing}(\mathcal{F}_1)$ and $\mathrm{m}_{\mathbf{W}_1}(\widetilde{P_1}), \mathrm{m}_{\mathbf{W}_1}(\widetilde{Q_1}) \geq 3$ and $\mathrm{m}_{\mathbf{W}_1}(\widetilde{R_1}) \geq 2$. In addition to $\mathbf{W}_1$, there are 4 other homeomorphic curves to $\mathbb{P}^1$ contained in $\mathbf{E}_1$ of which 3 are given by the roots of $r_{11} = r_{11}(u_3)$ and the fourth is defined as $\pi_1^{(-1)}\big([0:0:1:0]\big)$. Furthermore, $m_{\mathbf{E}_1}(\pi_1^* \mathcal{F}_0) = \ell_1 = 1$.

Thus, $\mathbf{W}_1$ is of type I and is homemorphic to $\mathbf{W}_0$ with $\pi_1(\mathbf{W}_1) = \mathbf{W}_0$. Let $\pi_1 : \mathbf{M}_2 \to \mathbf{M}_1$ be the blow-up of $\mathbf{M}_1$ along $\mathbf{W}_1$ and $\mathcal{F}_2$ be the strict transform of $\mathcal{F}_1$ under π_2 with $m_{\mathbf{E}_2}(\pi_2^* \mathcal{F}_1) = \ell_2 = 1$. But, this time, in $\mathbf{E}_2$ there is no homeomorphic curve to $\mathbf{W}_1$. It is not difficult to see that $\widetilde{\mathcal{F}_2} = \mathcal{F}_2|_{\mathbf{E}_2}$ contains 12 closed points in its singular set. In order to verify Theorem (4.6), we need to make a small perturbation on $\mathcal{F}_1$ because $\mathrm{Sing}(\mathcal{F}_1)$ contains four curves on $\mathbf{E}_1$. Let $\mathcal{F}_{1t}$ be the one-dimensional holomorphic foliation on $\mathbf{M}_1$ which is described in $(U_3)_1$ by following vector field

$$X_{1t} = X_1 + tY_1, \tag{75}$$

where

$$Y_1 = \sum_{j=0}^{2} a_j u_1^{2-j} u_2^j \frac{\partial}{\partial u_1} + \sum_{j=0}^{2} b_j u_1^{2-j} u_2^j \frac{\partial}{\partial u_2} + \big(c_0 u_1 + c_1 u_2\big)\frac{\partial}{\partial u_3}$$

with X_1 given in (74) and $a_i, b_i, c_i \in \mathbb{C}[u_3]$ are generic polynomials of degre $1+i, i$ and $2+i$, respectively.

Therefore, $\mathcal{F}_{1t}$ is a small perturbation of $\mathcal{F}_1$, but now $\mathbf{W}_1$ is a special component of $\mathrm{Sing}(\mathcal{F}_{1t})$, $t \neq 0$. However, in order for there to be another holomorphic family $\mathcal{F}_{0t}$ such that $\mathcal{F}_{1t}$ is the strict transform of $\mathcal{F}_{0t}$ under π_1 then we must need to have $a_2 \equiv 0$ in (75). In fact, let $\mathcal{F}_{0t}$ be the holomorphic family which is described on the affine open set U_3 by the following vector field

$$X_{0t} = X_0 + tY_0, \tag{76}$$

where X_0 as in (73) and

$$Y_0 = (a_0 z_1^3 + a_1 z_1 z_2)\frac{\partial}{\partial z_1} + \left(b_0 z_1^4 + (b_1 + a_0) z_1^2 z_2 + (b_2 + a_1) z_2^2\right)\frac{\partial}{\partial z_2} + (c_0 z_1^2 + c_1 z_2)\frac{\partial}{\partial z_3}.$$

The foliation $\mathcal{F}_{0t}$ also has degree 4 and $\mathbf{W}_0$ is a type I component for $\mathcal{F}_{0t}$, with $t \neq 0$. Furthermore, $m_{\mathbf{E}_1}(\pi_1^* \mathcal{F}_{0t}) = \ell_1 = 1$ and $\mathcal{F}_{1t}$ is the strict transform of $\mathcal{F}_{0t}$ from π_1.

Changing the coefficients of a_i, b_i, c_i if necessary, we can admit that $\mathbf{W}_1$ is the unique curve of $\mathrm{Sing}(\mathcal{F}_{1t})$ which is also a special component of $\mathrm{Sing}(\mathcal{F}_{1t})$, $t \neq 0$. In addition to $\mathbf{W}_1$, $\mathcal{F}_{1t}$ has four more singularities denoted by $p_i^{(1t)}$ in $\mathbf{E}_1$ and $m_{\mathbf{E}_2}(\pi_2^* \mathcal{F}_{1t}) = \ell_2 = 1$ for all $t \neq 0$. Therefore, from Theorems 1.1 and 1.2, we get

$$\mu(\mathcal{F}_{1t}, \mathbf{W}_1) + \sum_{i=1}^{4} \mu(\mathcal{F}_{1t}, p_i^{(1t)}) = \mu(\mathcal{F}_{0t}, \mathbf{W}_0) - 14 = 14 + N(\mathcal{F}_{0t}, A_{\mathbf{W}_0}) \tag{77}$$

for $t \neq 0$. See also the Remark 3.4. Given $\mu(\mathcal{F}_{1t}, \mathbf{W}_1) = 22$ since $\mathcal{F}_{1t}$ is special along $\mathbf{W}_1$, we conclude

$$N(\mathcal{F}_{0t}, A_{\mathbf{W}_0}) = 8 + \sum_{i=1}^{4} \mu(\mathcal{F}_{1t}, p_i^{(1t)}) = 12. \tag{78}$$

Therefore, for $t \neq 0$, the singular set of $\mathcal{F}_{0t}$ contains 36 + 9 =45 isolated closed points disjunct to $\mathbf{W}_0$. Keeping this notation, let $\mathcal{F}_{2t}$ the strict transform of $\mathcal{F}_{1t}$ from π_2. Thus, it is not difficult to see that

$\mathcal{F}_{2t}$ contains 12 isolated singularities on $\mathbf{E}_2$, counting the multiplicities. Consequently, the singular set of $\mathcal{F}_{2t}$ contains $45+4+12=61$ isolated closed points, counted the muiltiplicities. These facts agree with Theorem 4.6.

5. An application: Seidenberg's Theorem for non-isolated singularities

Lemma 5.1. *Let $\mathcal{F}_0$ be a one-dimensional holomorphic foliation on $\mathbf{M}_0 = \mathbb{P}^3$ of degree k, and its singular locus contains a smooth curve $\mathbf{W}_0$ of degree $\deg(\mathbf{W}_0)$ and Euler characteristic $\chi(\mathbf{W}_0)$. We will assume the existence of a blow-up sequence $\{\pi_j, \mathbf{M}_j, \mathbf{W}_j, \mathcal{F}_j, \mathbf{E}_j\}$.*

If each $\mathbf{W}_j$ is homeomorphic to $\mathbf{W}_{j-1}$ with $\pi_j(\mathbf{W}_j) = \mathbf{W}_{j-1}$ for all $j \geq 1$, then there exists a natural number r such that $m_{\mathbf{W}_r}(\mathcal{F}_r) = 1$ and $\mathbf{W}_r$ is of type III.

Proof. By way of contradiction, let us assume that this theorem is false, i.e., either $m_{\mathbf{W}_j}(\mathcal{F}_j) \geq 2$ or $\mathbf{W}_j$ is not of type III for all $j \geq 1$. From Equation (54), we can make the assumption:

$$1 \leq \ell_j = m_{\mathbf{E}_j}\big(\pi_j^*\mathcal{F}_{j-1}\big) \leq m_{\mathbf{W}_0}(\mathcal{F}_0) + 1, \quad \forall j \geq 1. \tag{79}$$

However, as $\mu(\mathcal{F}_j, \mathbf{W}_j)$ given in Theorem 1.2 is a natural number for all $j \geq 0$, we can infer that the sequence:

$$a_j = (\ell_{j+1}+1)\Lambda_0^{(3)}\left(\frac{\ell_{j+1}^2}{2^j} + \big(3\ell_{j+1}+1\big)\sum_{i=1}^{j}\frac{\ell_i}{2^i}\right) \in \mathbb{N}, \quad \forall j. \tag{80}$$

On the other hand, the sequence of natural numbers ℓ_j is bounded, implying the existence of a subsequence ℓ_{j_i} where each $\ell_{j_i} = \ell$ is a constant for all j_i. Consequently,

$$a_{j_i} = (\ell+1)\Lambda_0^{(3)}\left(\frac{\ell^2}{2^{j_i}} + \big(3\ell+1\big)\sum_{i=1}^{j_i}\frac{\ell_i}{2^i}\right) \in \mathbb{N}, \quad \forall j_i. \tag{81}$$

Therefore,

$$0 < |a_{j_{i_2}} - a_{j_{i_1}}| = (\ell+1)\Big|\Lambda_0^{(3)}\Big|\left(\frac{\ell^2}{2^{j_{i_1}}} + \frac{\ell^2}{2^{j_{i_2}}} + \big(3\ell+1\big)\sum_{r=j_{i_1}}^{j_{i_2}}\frac{\ell_i}{2^i}\right).$$

For sufficiently large j_{i_1} and j_{i_2}, we obtain $0 < |a_{j_{i_2}} - a_{j_{i_1}}| < 1$. However, this is absurd, considering that a_j is a natural number for all j. □

5.1. Maximum number of blowups for the desingularization.

Now, we calculate the maximum number of blowups needed until we reach $\ell_i = m_{\mathbf{E}_i}\big(\pi_i^*\mathcal{F}_{i-1}\big) = 0$. In fact, we will suppose $\ell_j = \ell_1$ for $j = 1, \ldots, N_1+1$. Then, from (80), we get

$$a_j = (\ell_{j+1}+1)\Lambda_0^{(3)}\left(\frac{\ell_{j+1}^2}{2^j} + \big(3\ell_{j+1}+1\big)\sum_{i=1}^{j}\frac{\ell_i}{2^i}\right)$$

is a natural number for $j = 1, \ldots, N_1$. Thus,

$$a_j = (\ell_1+1)\Lambda_0^{(3)}\left(\frac{\ell^2}{2^j} + \big(3\ell_1+1\big)\sum_{i=1}^{j}\frac{\ell_1}{2^i}\right) = (\ell_1+1)\Lambda_0^{(3)}\left(\frac{\ell_1^2}{2^j} + \ell_1\big(3\ell_1+1\big)\big(1-\frac{1}{2^j}\big)\right).$$

Therefore,

$$b_j = \frac{\ell_1(1+\ell_1)(1+2\ell_1)\Lambda_0^{(3)}}{2^j} \in \mathbb{N}, \quad \text{for } j = 1, \ldots, N_1.$$

But, we can consider $\Lambda_0^{(3)} = 2^{\alpha_0}(2\beta_0+1)$ and ℓ_1 or ℓ_1+1 is an even number with $2^{\alpha_1-1} \leq \ell_1, \ell_1+1 \leq 2^{\alpha_1}$ which imply that $N_1 \leq \alpha_0 + \alpha_1$.

In the worth situation, with $N_1 = \alpha_0 + \alpha_1$ blowups, we get ℓ_{j+1} or $\ell_{j+1} + 1$ equal to $2^{\alpha_1 - 1}$ and so we need more $N_2 = \alpha_0 + \alpha_1 - 1$ blowups to get ℓ or $\ell_{j+1} + 1$ equal to $2^{\alpha_1 - 2}$, and so on. Therefore, at most

$$\alpha_0 + \alpha_1 + \alpha_0 + \alpha_1 - 1 + \cdots + \alpha_0 + 1 + \alpha_0 = \frac{(2\alpha_0 + \alpha_1)(\alpha_1 + 1)}{2} \tag{82}$$

will be necessary for we get $\ell_j = m_{\mathbf{E}_j}\left(\pi_j^* \mathcal{F}_{j-1}\right) = 0$.

Example 5.2. Let $\mathcal{F}_0$ be the one-dimensional holomorphic foliation of degree m defined on $\mathbf{M}_0 = \mathbb{P}^3$ which is described in the open affine set $U_3 = \{[\xi_i] \in \mathbb{P}_3 | \xi_3 \neq 0\}$ by the following vector field

$$X_0 = \left(z_1^m - z_1 z_2^2 z_3^{m-2}\right)\frac{\partial}{\partial z_1} + \left(z_1^m - z_2^3 z_3^{m-2}\right)\frac{\partial}{\partial z_2} - z_2^2 z_3^{m-1}\frac{\partial}{\partial z_3}$$

where $z_i = \xi_{i-1}/\xi_3$. We also assume that m is large enough. The singular set of $\mathcal{F}_0$ contains two curves $\mathbf{W}_0 = \{[\xi] \in \mathbb{P}^3 | \xi_0 = \xi_1 = 0\}$ and $\mathbf{W}_0^{(1)} = \{[\xi] \in \mathbb{P}^3 | \xi_0 = \xi_2 = 0$. We will consider the blow-up sequence $\{\pi_j, \mathbf{M}_j, \mathbf{W}_j, \mathcal{F}_j, \mathbf{E}_j\}$ where $\mathbf{W}_j$ is defined in the chart $\left((U_3)_j, \sigma_1(x) = z^{(j-1)}\right)$, with $z^{(0)} = z$, by equations $x_1 = x_2 = 0$. In this chart, the strict transform $\mathcal{F}_j$ from $\mathcal{F}_{j-1}$ via π_j is described by the following vector field

$$X_j = x_1\left(x_1^{m-2j-} - x_2^2 x_3^{m-2}\right)\frac{\partial}{\partial x_1} + \left(x_1^{m-3j}(1 - j x_1^{j-1} x_2) + (j-1)x_2^3 x_3^{m-2}\right)\frac{\partial}{\partial x_2} - x_2^2 x_3^{m-1}\frac{\partial}{\partial x_3}$$

for j such that $m - 3j \geq 0$. Thus, the singular set of $\mathcal{F}_j$ always contains two curves $x_1 = x_2 = 0$ and $x_1 = x_3 = 0$ if $m - 3j \geq 1$. We will determine the Milnor numbers $\mu(\mathcal{F}_j, \mathbf{W}_j)$ for some j. Firstly, for $j = 0$, we can make the holomorphic perturbation $\mathcal{F}_{0t}$ of $\mathcal{F}_0$ which is described the following vector field

$$X_{0t} = X + t\left(\sum_{i=0}^{3} a_i(z_3) z_1^{3-i} z_2^i \frac{\partial}{\partial z_1} + \sum_{i=0}^{3} b_i(z_3) z_1^{3-i} z_2^i \frac{\partial}{\partial z_2} + \sum_{i=0}^{2} c_i(z_3) z_1^{2-i} z_2^i \frac{\partial}{\partial z_3}\right)$$

where a_i, b_i, c_i are generic polynomials of degree $m-3, m-3$ and $m-2$, respectively. Thus, the singular set of $\mathcal{F}_{0t}$ is composed by the curve $\mathbf{W}_0$ and some isolated closed points. It is not difficult to show Theorem 4.6(a) agrees with number of isolated singularities of $\mathcal{F}_{1t}$ on $\mathbf{E}_1$, where $\mathcal{F}_{1t}$ is the strict transform of $\mathcal{F}_{0t}$ from π_1 for $t \neq 0$. From Theorems 1.1 and 1.2, we can determine $\mu(\mathcal{F}_0, \mathbf{W}_0)$. Now, we will determine $\mu(\mathcal{F}_1, \mathbf{W}_1)$. Initially, we will consider the holomorphic perturbations of $\mathcal{F}_1$ as follows

$$X_{1t} = X_1 + t\left(\sum_{i=0}^{3} a_i(x_3) x_1^{3-i} x_2^i \frac{\partial}{\partial x_1} + \sum_{i=0}^{3} b_i(x_3) x_1^{3-i} x_2^i \frac{\partial}{\partial x_2} + \sum_{i=0}^{2} c_i(x_3) x_1^{2-i} x_2^i \frac{\partial}{\partial x_3}\right) \tag{83}$$

where $a_i, b_i, c_i \in \mathbb{C}[z_3]$ of degree $m-5+i, m-6+i$ and $m-4+i$. Thus, $m_{\mathbf{E}_2}(\pi_1^* \mathcal{F}_{1t}) = 2$ for all t. Again, Theorem 4.6(a) agrees with number of isolated singularities of $\mathcal{F}_{2t}$ on $\mathbf{E}_1$, where $\mathcal{F}_{2t}$ is the strict transform of $\mathcal{F}_{1t}$ from π_2 for $t \neq 0$. However, the main problem of these perturbations is that there is no holomorphic foliation $\mathcal{F}_{0t}$ such that $\mathcal{F}_{1t}$ is the strict transform from $\mathcal{F}_{0t}$ via π_1 because a_3 can be not identically null. But, if $a_3 \equiv 0$ then $\mathcal{F}_{1t}$ is not special along $\mathbf{W}_1$. In fact, from (83), there are imbedding closed points associated to $\mathbf{W}_1$, given by $A_i^t(0, 0, x_{3i}^t)$ where x_{3i}^t is a root of $t(b_3(x_3) - x_3^{m-2} = 0$. Thus, in order to make a holomorphic deformations $\mathcal{F}_{1t}$ such that $\mathbf{W}_1$ is special component we need to consider $m_{\mathbf{E}_2}(\pi_2^* \mathcal{F}_{1t}) = 1$. More precisely, we will consider the following deformation $\mathcal{F}_{1t}$ described by vector field

$$X_{1t} = X_1 + t\left(\left(\sum_{i=0}^{2} b_i(x_3) x_1^{2-i} x_2^i + b_3(x_3) x_2^3\right)\frac{\partial}{\partial x_2} + \left(x_1 c_0(x_3) + x_2 c_1(x_3)\right)\frac{\partial}{\partial x_3}\right) \tag{84}$$

where b_i and c_i are generic polynomials of degree $m-5+i$ and $m-3+i$, respectively, except for b_3 which also have degree equal to $m-3$. Now, $\mathcal{F}_{1t}$ is special along $\mathbf{W}_1$ and there exists a one-dimensional holomorphic foliation $\mathcal{F}_{0t}$ on $\mathbf{M}_0$ such that $\mathcal{F}_{1t}$ is the strict transform of $\mathcal{F}_{0t}$ from π_1. In fact, $\mathcal{F}_{0t}$ is described by the following vector field

$$X_{0t} = X_0 + t\left(\left(\sum_{i=0}^{2} b_i(z_3) z_1^{5-2i} z_2^i + b_3(z_3) z_2^3\right)\frac{\partial}{\partial z_2} + \left(z_1^3 c_0(x_3) + z_1 z_2 c_1(z_3)\right)\frac{\partial}{\partial z_3}\right). \tag{85}$$

Let $\mathcal{F}_{2t}$ be strict transform from $\mathcal{F}_{1t}$ via π_2. From Theorem 4.6(a) we have that $\mathcal{N}(\mathcal{F}_{2t}, \mathbf{E}_2) = 4m - 8$ for $t \neq 0$, since $\ell_1 = 2$ and $\ell_1 = 1$. The same idea can be used to calculate $\mu(\mathcal{F}_j, \mathbf{W}_j$ for $j > 1$.

5.2. **Normal forms for non-isolated singularities.** We will focus on the germs of holomorphic foliations $\mathcal{F}_0$ defined in an open set $U_0 \subset \mathbb{C}^3$ such that their singular set contains a smooth curve $\mathbf{W}_0$ of type III with $\mathrm{m}_{\mathbf{W}_0}(\mathcal{F}_0) = 1$. Without loss of generality, we can assume that $\mathbf{W}_0$ is defined as $z_0 = z_1 = 0$. Keeping the notation, we consider the blow-up sequence $\{\pi_i, \mathbf{M}_i, \mathbf{W}_i, \mathcal{F}_i, \mathbf{E}_i\}$.

According to Theorem (4.2), there exist two curves in $\mathbf{E}_1$ that are homeomorphic to $\mathbf{W}_0$, counting multiplicities, since $\mathrm{m}_{\mathbf{E}_1}(\pi_1^*\mathcal{F}_0) = 0$. We will start by considering that $\mathcal{F}_0$ is described by the following vector field:

$$X_0 = \sum_{i=1}^{3} (z_1 p_{i0}(z_3) + z_2 p_{i1}(z_3) + P_i(z)) \frac{\partial}{\partial z_i}, \quad \mathrm{m}_{\mathbf{W}_0}(P_i) \geq 2. \tag{86}$$

By the way, from Equation (3) we have

$$\mathbf{A}_{X_0}|_{\mathbf{W}_0} = \begin{pmatrix} p_{10} & p_{11} \\ p_{20} & p_{21} \end{pmatrix}. \tag{87}$$

Let $\lambda_1 = \lambda_1(z_3)$ and $\lambda_2 = \lambda_2(z_3)$ be the eigenvalues of the matrix (87).Thus, we have the following three cases to consider

(i) $\lambda_1 \cdot \lambda_2 \not\equiv 0$, $\lambda_1 \not\equiv n\lambda_2$ and $\lambda_2 \not\equiv n\lambda_1$, $n \in \mathbb{N}$;
(ii) $\lambda_1 \not\equiv 0$ and $\lambda_2 \equiv 0$;
(iii) $\lambda_1 \equiv \lambda_2 \equiv 0$;

Proposition 5.3. *If the eigenvalues λ_1 and λ_2 of the matrix $\mathbf{A}_{X_0}|_{\mathbf{W}_0}$ given in (87) are non-identically null with $\lambda_1 \not\equiv n\lambda_2$ and $\lambda_2 \not\equiv n\lambda_1$, $\forall n \in \mathbb{N}$ then these conditions are invariant by blowups along curves $\mathbf{W}_i$ homeomorphic to $\mathbf{W}_0$ with $\pi_i(\mathbf{W}_i) = \mathbf{W}_0$.*

Proof. The eigenvalues λ_1 and λ_2 of the matrix (87) are given by

$$\lambda_i = \frac{p_{10} + p_{21}}{2} + (-1)^i \frac{\sqrt{\Delta}}{2} \tag{88}$$

where

$$\Delta = \Delta(z_3) = (p_{10} - p_{21})^2 + 4p_{11}p_{20}.$$

Thus, in the chart $((U_1)_1, \sigma_1(u))$, $\mathcal{F}_1$ is described by the vector field X_1 as follows

$$X_1 = u_1 Q_1(u) \frac{\partial}{\partial u_1} + (Q_2(u) - u_2 Q_1(u)) \frac{\partial}{\partial u_2} + u_1 Q_3(u) \frac{\partial}{\partial u_3} \tag{89}$$

where

$$Q_i(u) = p_{i0} + u_2 p_{i1} + u_1 \widetilde{Q_i}(u), \quad p_{ij} = p_{ij}(u_3), \quad \forall i$$

for some functions $\widetilde{Q_i}$. However, since $\mathbf{W}_0$ is of type III, meaning it is a non-dicritical component, we have:

$$Q_2(0, u_2, u_3) - u_2 Q_1(0, u_2, u_3) = p_{20} + u_2(p_{21} - p_{10}) - u_2^2 p_{11} \not\equiv 0.$$

This implies that there are two curves $\mathbf{W}_i^{(1)}$ that are homeomorphic to $\mathbf{W}_0$, defined as follows: $\mathbf{W}_i^{(1)} = \{u \in (U_1)_1 | u_1 = u_2 - \psi_i(u_3) = 0\}$ where

$$\psi_i(u_3) = \frac{p_{21} - p_{10} + (-1)^i \sqrt{\Delta}}{2p_{11}} \tag{90}$$

for $p_{11} \not\equiv 0$. (If $\lim_{u_3 \to \alpha} \psi_i(u_3) = \infty$ then in the other chart $\big((U_1)_2, \sigma_2(v)\big)$, the curve $\mathbf{W}_i^{(1)}$ is defined as $v_1 - \widetilde{\psi}_i(v_3) = v_2 = 0$ with $\widetilde{\psi}_i(\alpha) = 0$). Let $F_i(u) = (u_1, u_2 - \psi_i(u_3), u_3) = v \in \mathbb{C}^3$ be a local biholomorphism. Now, the push-forward foliation $F_{i_*}(\mathcal{F}_1) = \mathcal{G}_i$ defined in V_i is described by the following vector field

$$Y_1 = \big(v_1 r_{10} + R_1\big) \frac{\partial}{\partial v_1} + \big(v_1 r_{20} + v_2 r_{21} + R_2\big) \frac{\partial}{\partial v_2} + \big(v_1 r_{30} + R_3\big) \frac{\partial}{\partial v_3}$$

where $r_{ij} = r_{ij}(v_3)$ and $\mathrm{m}_{\mathbf{W}_1^{(1)}}(R_i) \geq 2$. But, $r_{10} = p_{10} + \psi_1 p_{11} = \lambda_1$ and $r_{21} = p_{11}(\psi_2 - \psi_1) = \lambda_2 - \lambda_1$. Consequently, the matrix

$$\mathbf{A}_{Y_1}|_{\mathbf{W}_1^{(1)}} = \begin{pmatrix} r_{10} & 0 \\ r_{20} & r_{21} \end{pmatrix} \tag{91}$$

has $\lambda_{11}^{(1)} = \lambda_1$ and $\lambda_{12}^{(1)} = \lambda_2 - \lambda_1$ as eigenvalues which are distinct and non-identically null. Similarly for the other curve $\mathbf{W}_2^{(1)}$ whose matrix $\mathbf{A}_{Y_2}|_{\mathbf{W}_2^{(1)}}$ has $\lambda_{12}^{(1)} = \lambda_2$ and $\lambda_{22}^{(1)} = \lambda_1 - \lambda_2$ as eigenvalues.

If $p_{11} \equiv 0$ then the eigenvalues of (87) are $\lambda_1 = p_{10}$ and $\lambda_2 = p_{21}$. In this way, $\mathbf{W}_1^{(1)} = \{u \in (U_1)_1 | u_1 = u_2 + \dfrac{p_{20}}{p_{21} - p_{10}} = 0\}$ is a curve of singularities with eigenvalues $\lambda_{11}^{(1)} = \lambda_1$ and $\lambda_{12}^{(1)} = \lambda_2 - \lambda_1$ while $\mathbf{W}_2^{(1)} = \{v \in (U_1)_2 | v_1 = v_2 = 0\}$ is the other curve with eigenvalues $\lambda_{21}^{(1)} = \lambda_1 - \lambda_2$ and $\lambda_{22}^{(1)} = \lambda_2$. Note that it is impossible $p_{11} \equiv p_{21} - p_{10} \equiv 0$ since λ_1 and λ_2 are distinct. □

Now, let us consider the second case where $\lambda_1 \not\equiv 0$ and $\lambda_2 \equiv 0$. This leads to $tr(\mathbf{A}_{X_0|\mathbf{W}_0}) = p_{10} + p_{21} = \lambda_1$ and $\det(\mathbf{A}_{X_0|\mathbf{W}_0}) = p_{10}p_{21} - p_{11}p_{20} = 0$. Therefore,

$$\frac{p_{10}(z_3)}{p_{11}(z_3)} = \frac{p_{20}(z_3)}{p_{21}(z_3)} = \varphi(z_3) \implies p_{i0}(z_3) = \varphi(z_3) p_{i1}(z_3).$$

In the chart $((U_1)_1, \sigma_1(u))$, $\mathcal{F}_1$ is described by the vector field X_1 as in (89). But this time, we have $Q_i(u) = p_{i1}(z_3)\big(u_2 + \varphi(u_3)\big) + u_1 \widetilde{Q_i}(u)$ for $i = 1, 2$, and $Q_3(u) = p_{30} + u_2 p_{31} + u_1 \widetilde{Q_3}(u)$. Since

$$Q_2(0, u_2, u_3) - u_2 Q_1(0, u_2, u_3) = \Big(p_{21} - u_2 p_{11}\Big)\Big(u_2 + \varphi(u_3)\Big)$$

there are two homeomorphic curves to $\mathbf{W}_0$ defined when $p_{11} \neq 0$ as follows $\mathbf{W}_1^{(1)} = \big\{u \in (U_1)_1 | u_1 = u_2 + \varphi(u_3) = 0\big\}$ and $\mathbf{W}_2^{(1)} = \big\{u \in (U_1)_1 | u_1 = u_2 - p_{21}/p_{11} = 0\big\}$.

Proposition 5.4. *If the matrix (87) has only one eigenvalue that is not identically zero, denoted by λ_1, then curves $\mathbf{W}_i^{(1)}$ that are homeomorphic to $\mathbf{W}_0$ with $\pi_1(\mathbf{W}_i^{(1)}) = \mathbf{W}_0$ are elementary components of $\mathcal{F}_1$. The eigenvalues of $\mathcal{F}_1$ at $\mathbf{W}_1^{(1)}$ are $\lambda_{11}^{(1)} = \lambda_1$ and $\lambda_{21}^{(1)} \equiv 0$ while the eigenvalues of $\mathcal{F}_1$ at $\mathbf{W}_2^{(1)}$ are $\lambda_{12}^{(1)} = \lambda_1$ and $\lambda_{22}^{(1)} = -\lambda_1$.*

Proof. In fact, let $F_1(u) = (u_2 + \varphi(u_3), u_1, u_3) = (v_1, v_2, v_3)$ be a local biholomorphism. Therefore, the vector field $Y_1 = \big(F_1\big)_* X_1$ can be expressed as follows

$$Y_1 = v_1(v_2 r_{10} + R_1)\frac{\partial}{\partial v_1} + (v_1 r_{20} + v_2 r_{21} + R_2)\frac{\partial}{\partial v_2} + v_1(r_{30} + v_2 r_{31} + R_3)\frac{\partial}{\partial v_3} \tag{92}$$

where $r_{ij} = r_{ij}(v_3)$ and $\mathrm{m}_{\mathbf{W}_1^{(1)}}(R_i) \geq 2$. But, $r_{10} = p_{11}$ and $r_{21} = p_{21} + p_{11}\varphi(v_3) = p_{21} + p_{10} = \lambda_1$. Consequently, we have

$$\mathbf{A}_{Y_1}|_{\mathbf{W}_1^{(1)}} = \begin{pmatrix} 0 & 0 \\ r_{20} & \lambda_1 \end{pmatrix}.$$

Therefore, the matrix $\mathbf{A}_{Y_1}|_{\mathbf{W}_1^{(1)}}$ has only one eigenvalue not identically zero λ_1. In order to analyze the other curve $\mathbf{W}_2^{(1)}$ it is sufficient to consider the local biholomorphism $F_2(u) = (u_1, u_2 - \frac{p_{21}}{p_{11}}, u_3) = (v_1, v_2, v_3)$ defined for $p_{11} \neq 0$. As before, let $Z_1 = \big(F_2\big)_* X_1$, i.e.,

$$Z_1 = v_1(s_{10} + v_2 s_{21} + S_1)\frac{\partial}{\partial v_1} + (v_1 s_{20} + v_2 s_{21} + S_2)\frac{\partial}{\partial v_2} + v_1(s_{30} + v_2 s_{31} + S_3)\frac{\partial}{\partial v_3} \tag{93}$$

where $s_{ij} = s_{ij}(v_3)$ and $\mathrm{m}_{\mathbf{W}_1^{(2)}}(S_i) \geq 2$. But, $s_{10} = -s_{21} = \lambda_1(v_3)$. Therefore, we get

$$\mathbf{A}_{Z_1}|_{\mathbf{W}_2^{(1)}} = \begin{pmatrix} \lambda_1 & 0 \\ s_{20} & -\lambda_1 \end{pmatrix}$$

i.e.; the matrix $\mathbf{A}_{Z_1}|_{\mathbf{W}_2^{(1)}}$ has two distinct eigenvalues denoted by $\lambda_{12}^{(1)} = \lambda_1(v_3)$ and $\lambda_{22}^{(1)} = -\lambda_1(v_3)$ with $\lambda_{12}^{(1)}/\lambda_{22}^{(1)} = -1 \notin Q_+$ for almost all points of $\mathbf{W}_2^{(1)}$. □

Finally, we will consider the third case where λ_1 and λ_2 are identically null, i.e., $tr(\mathbf{A}_{X_0}|_{\mathbf{W}_0}) = det((\mathbf{A}_{X_0}|_{\mathbf{W}_0}) = 0$. Again, we have two distinct situations to consider $p_{10} = 0$ or not. If $p_{10} = 0$ then $p_{21} = 0$ and $p_{11}p_{20} = 0$. Let us only consider the case where $p_{11} = 0$ because the other case is analogous. Therefore, in (86), the multiplicity $\mathrm{m}_{\mathbf{W}_0}(P_1) = m_1 \geq 2$ and

$$\mathbf{A}_{X_0}|_{\mathbf{W}_0} = \begin{pmatrix} 0 & 0 \\ p_{20} & 0 \end{pmatrix} \tag{94}$$

In $\mathbf{E}_1$, there are two types of curves. The first type is given by the roots of p_{20}, which are homeomorphic to $\mathbb{P}^1$. The second type is homeomorphic to $\mathbf{W}_0$.

To simplify our analysis, we will address the first type of curves in the chart $((U_1)_1, \sigma_1(u))$, while the second type of curves will be handled in the chart $((U_1)_2, \sigma_2(v))$. This allows us to have the following: proposition

Proposition 5.5. *If in the matrix (87) the coefiecients p_{10}, p_{11} and p_{21} are identically null and p_{20} is an affine function with no common root with p_{31} then there exists a blow-up sequence $\{\pi_i, \mathbf{M}_i, \mathbf{W}_i, \mathcal{F}_i, \mathbf{E}_i\}$ such that $\mathbf{W}_i = (\pi_i)^{-1}(q)$, $q \in \mathbf{W}_{i-1}$ and $\mathbf{W}_i$ is a non-elementary component of* $\mathrm{Sing}(\mathcal{F}_i)$ *for all $i \geq 1$.*

Proof. In the chart $((U_1)_1, \sigma_1(u))$, the foliation $\mathcal{F}_1$ is described by the following vector field

$$X_1 = u_1^{m_1} Q_1(u)\frac{\partial}{\partial u_1} + \big(p_{20} + u_1 Q_2\big)\frac{\partial}{\partial u_2} + u_1\big(p_{30} + u_2 p_{31} + u_1 Q_3\big)\frac{\partial}{\partial u_3} \tag{95}$$

where $p_{ij} = p_{ij}(u_3)$ and for certain functions Q_i. Let $\mathbf{W}_1$ be the curve defined by $u_1 = u_3 - \beta = 0$ where β is the root of p_{20}. Therefore, $\mathbf{W}_1 = (\pi_1)^{-1}(0, 0, \beta)$. Without loss of generality, we can assume that $p_{20}(u_3) = \alpha u_3$ with $\alpha \neq 0$, that is, $\beta = 0$. In these coordinates $(w_1, w_2, w_3) = (u_1, u_3, u_2)$ the vector field X_1 in (95) is rewritten as follows

$$\begin{aligned} Y_1 =& w_1^{m_1} R_1(w)\frac{\partial}{\partial w_1} + w_1\big(p_{30}(w_2) + w_3 p_{31}(w_2) + w_1 R_2\big)\frac{\partial}{\partial w_2} \\ &+ \big(\alpha w_2 + w_1 R_3\big)\frac{\partial}{\partial w_3} \end{aligned} \tag{96}$$

But, the second section of Y_1 in (96) is

$$\begin{aligned} w_1\big(p_{30}(w_2) + w_3 p_{31}(w_2) + w_1 R_2\big) &= w_1\big(p_{30}(0) + w_3 p_{31}(0)\big) + \widetilde{R}_2 \\ &= w_1 r_{20}(w_3) + \widetilde{R}_2 \end{aligned} \tag{97}$$

where $r_{20}(w_3) = p_{30}(0) + w_3 p_{31}(0)$ and $\mathrm{m}_{\mathbf{W}_1}(\widetilde{R}_2) \geq 2$. Furthermore, $p_{31}(0) \neq 0$ by hypothesis. So,

$$\mathbf{A}_{Y_1}|_{\mathbf{W}_1} = \begin{pmatrix} 0 & 0 \\ r_{20} & 0 \end{pmatrix} \tag{98}$$

where r_{20} is also an affine function. In this way, the third section of Y_1 is

$$\begin{aligned} \alpha w_2 + w_1 R_3 &= w_1 R_3(0, 0, w_3) + \alpha w_2 + \widetilde{R_3} \\ &= w_1 r_{30} + w_2 r_{31} + \widetilde{R_3} \end{aligned} \tag{99}$$

where $r_{30}(w_3) = R_3(0, 0, w_3)$, $r_{31}(w_3) \equiv \alpha \neq 0$ and $\mathrm{m}_{\mathbf{W}_1}(\widetilde{R_3}) \geq 2$. Consequently, the vector field Y_1 possesses the same properties as X_1 because r_{20} is an affine function without any common root with $r_{31}(w_3) = \alpha \neq 0$. Hence, we can continue the blow-up process indefinitely, leading to $\mathbf{W}_2 = (\pi_2)^{-1}(0, 0, \beta_1)$, where $\beta_1 = -r_{30}(0)/r_{31}(0)$, and so forth. □

Corollary 5.6. *Let us assume in the matrix (87) that p_{10}, p_{11} and p_{21} and identically null and $p_{20}(z_3) = z_3 g(z_3)$. If $g(0)$ and $p_{31}(0)$ are nonzero then there exists a blow-up sequence $\{\pi_i, \mathbf{M}_i, \mathbf{W}_i, \mathcal{F}_i, \mathbf{E}_i\}$ such that $\mathbf{W}_i = (\pi_i)^{-1}(q)$, $q \in \mathbf{W}_{i-1}$ and $\mathbf{W}_i$ is a non-elementary component of* $\mathrm{Sing}(\mathcal{F}_i)$ *for all $i \geq 1$.*

Proof. In the chart $((U_1)_1, \sigma_1(u))$, the foliation $\mathcal{F}_1$ is described by the following vector field

$$X_1 = u_1^{m_1} Q_1(u)\frac{\partial}{\partial u_1} + \big(u_3 g(u_3) + u_1 Q_2(u)\big)\frac{\partial}{\partial u_2} + u_1\big(p_{30} + u_2 p_{31} + u_1 Q_3(u)\big)\frac{\partial}{\partial u_2}$$

where $p_{ij} = p_{ij}(u_3)$ and $Q_i(0, u_2, u_3) = 0$ for $i = 2, 3$. But, in these coordinates $(w_1, w_2, w_3) = (u_1, u_3, u_2)$ the foliation $\mathcal{F}_1$ is described by vector field Y_1 as follows

$$\begin{aligned} Y_1 =& w_1^{m_1} R_1 \frac{\partial}{\partial w_1} + w_1\big(p_{30}(w_2) + w_3 p_{31}(w_2) + w_1 R_2\big)\frac{\partial}{\partial w_2} + \\ &+ \big(w_2 g(w_2) + w_1 R_3\big)\frac{\partial}{\partial w_3}. \end{aligned}$$

But, the second section of Y_1 is

$$\begin{aligned} w_1\big(p_{30}(w_2) + w_3 p_{31}(w_2) + w_1 R_2\big) &= w_1(p_{30}(0) + w_3 p_{31}(0) + \widetilde{R}_2(w) \\ &= w_1 r_{20}(w_3) + \widetilde{R}_2(w) \end{aligned} \tag{100}$$

while its third section is

$$\begin{aligned} w_2 g(w_2) + w_1 R_3(w) &= w_1 R_3(0, 0, w_3) + w_2 g(0) + \widetilde{R}_3(w) \\ &= w_1 r_{30}(w_3) + w_2 r_{31}(w_3) + \widetilde{R}_2(w). \end{aligned} \tag{101}$$

Therefore, $r_{20}(w_3) = p_{30}(0) + w_3 p_{31}(0)$ is an affine function with no commom root with $r_{31}(w_3) = g(0) \neq 0$. Therefore, we can aplly Proposition 5.5 again. □

Proposition 5.7. *If in the matrix (87) the coefiecients p_{10}, p_{11} and p_{21} are identically null and $p_{20} \not\equiv 0$. Then, for any blow-up sequence $\{\pi_i, \mathbf{M}_i, \mathbf{W}_i, \mathcal{F}_i, \mathbf{E}_i)$ such that $\mathbf{W}_i$ is homeomorphic to $\mathbf{W}_{i-1}$ and $\pi_i(\mathbf{W}_i) = \mathbf{W}_{i-1}$ there is a natural number $k \in \mathbb{N}$ such that $\mathbf{W}_i$ is an elementary component of* $\operatorname{Sing}(\mathbf{W}_i)$ *for $i \geq k$.*

Proof. The vector field (86) can be rewritten as follows

$$X_0 = P_1(z)\frac{\partial}{\partial z_1} + \big(z_1 p_{20} + P_2(z)\big)\frac{\partial}{\partial z_2} + \big(z_1 p_{30} + z_2 p_{31} + P_3(z)\big)\frac{\partial}{\partial z_3} \tag{102}$$

where $p_{ij} = p_{ij}(z_3)$ and

$$P_i(z) = \sum_{j=0}^{m_i} z_1^{m_i - j} z_2^j P_{ij}(z) = z_2^{n_i} g_i(z) + z_1 L_i(z), \quad m_i = \mathrm{m}_{\mathbf{W}_0}(P_i) \geq 2$$

with $n_i \geq m_i$, either $g_i(0, 0, z_3) := p_i(z_3) \not\equiv 0$ or $g_i \equiv 0$, $\mathrm{m}_{\mathbf{W}_0}(L_i) \geq m_i - 1$ with $L_i(0, 1, z_3) := q_i(z_3)$ for all i. In addition, if $\mathrm{m}_{\mathbf{W}_0}(L_i) \geq m_i$ then $n_i = m_i$ and $p_i \not\equiv 0$.

From (102), in the chart $((U_1)_2, \sigma_2(v))$ the foliation $\mathcal{F}_1$ is described by the following vector field

$$Y_1 = \big(-v_1^2 p_{20} + R_1 - v_1 R_2\big)\frac{\partial}{\partial v_1} + v_2\big(v_1 p_{20} + R_2\big)\frac{\partial}{\partial v_2} + v_2\big(v_1 p_{30} + p_{31} + R_3\big)\frac{\partial}{\partial v_3} \tag{103}$$

where $p_{ij} = p_{ij}(v_3)$, $R_i = v_2^{n_i - 1} g_i^{(1)}(v) + v_1 v_2^{m_i - 1} L_i^{(1)}(v)$ with $g_i^{(1)}(0, 0, v_3) = p_i(v_3)$ and $L_i^{(1)}(0, 0, v_3) = q_i(v_3)$.

In this chart, the exceptional divisor $\mathbf{E}_1$ is defined by the equation $\{v_2 = 0\}$ and the non-elementary curve $\mathbf{W}_1 = \{v \in (U_1)_2 | v_1 = v_2 = 0\} \subset \operatorname{Sing}(\mathcal{F}_1)$ has multiplicity equal to 2. Besides that $\mathrm{m}_{\mathbf{E}_1}(\pi_1^* \mathcal{F}_0) = 0$. From this point onward, let us focus solely on the fibers $\pi_1^{-1}(0, 0, z_3)$ for which $p_{20}(z_3) \neq 0$. This is because the curves associated with these fibers have already been taken into account in Proposition 5.5.

To continue with our analysis, we need to make some considerations about the possible values of p_{31}. Specifically, if $p_{31} \equiv 0$, then in (102), there must be a function g_i that is not identically zero for some i. Otherwise, the hypersurface $v_1 = 0$ would be entirely contained in the singular set of X_1.

Hence, let us consider the case where $\mathrm{m}_{\mathbf{W}_1}(\mathcal{F}_1) = 1$ which results in $p_{31} \not\equiv 0$ or $\mathrm{m}_{\mathbf{W}_1}(R_1) = 1$.

Therefore, if $p_{31} \not\equiv 0$ and $\mathrm{m}_{\mathbf{W}_1}(R_1) \geq 2$ then $\mathbf{W}_1$ is of type I. In this situation, the singular set of $\mathcal{F}_2$ restricted to the exceptional divisor consists of the elementary curve $\mathbf{W}_2 = \{x \in (U_2)_1 | x_1 = x_2 = 0\}$, with $\sigma_1(x) = v$, and some closed points. The eigenvalues of $\mathcal{F}_2$ at $\mathbf{W}_2$ are $\lambda_{11}^{(2)} = -p_{20}(x_3)$ and $\lambda_{21}^{(2)} = 2p_{20}(x_3)$, i.e. $\lambda_{11}^{(2)}/\lambda_{21}^{(2)} = -1/2 \notin Q_+$ for almost all $x \in \mathbf{W}_2$. More precisely, $\lambda_{11}^{(2)}/\lambda_{21}^{(2)} = -1/2 \notin Q_+$ for all x_3 such that $p_{20}(x_3) \neq 0$.

Now, if $\mathrm{m}_{\mathbf{W}_1}(R_1) = 1$, then it must be the case that $m_1 = n_1 = 2$ and $p_1 \not\equiv 0$, resulting in $\mathbf{W}_1$ being of type III. Consequently, the singular set of $\mathcal{F}_2$, when restricted to $\mathbf{E}_2$, contains a curve that is homeomorphic to $\mathbf{W}_1$, but with multiplicity equal to 2. In fact, in the chart $\big((U_2)_1, \sigma_1(t)\big)$, $\mathcal{F}_2$ can be described by the following vector field:

$$\begin{aligned} X_2 =& t_1\Big(-t_1p_{20} + R_1^{(2)} - t_1R_2^{(2)}\Big)\frac{\partial}{\partial t_1} + t_2\Big(2t_1p_{20} + R_1^{(2)} + 2t_1R_2^{(2)}\Big)\frac{\partial}{\partial t_2} + \\ & + t_1t_2\Big(t_1p_{30} + p_{31} + R_3^{(2)}\Big)\frac{\partial}{\partial t_3} \end{aligned} \tag{104}$$

where

$$R_i^{(2)}(t) = (t_1t_2)^{m_i-1}L_i^{(2)}(t) + t_1^{n_i-2}t_2^{n_i-1}g_i^{(2)}(t), \quad g_i^{(2)}(0,0,t_3) = p_i(t_3).$$

Thus, $\mathbf{W}_2 = \{t \in (U_2)_1 | t_1 = t_2 = 0\} \subset \mathrm{Sing}(\mathcal{F}_2)$. But, in this situation, the singular set of $\mathcal{F}_3$ contains three elementary curves homemorphic to $\mathbf{W}_2$. In fact, on the chart $\big((U_3)_1, \sigma_1(x) = t\big)$, there are two curves $\mathbf{W}_1^{(3)} = \{t \in (U_3)_1 | x_1 = x_2 = 0\}$ and $\mathbf{W}_2^{(3)} = \{t \in (U_3)_1 | x_1 = x_2 - \frac{3p_{20}}{2p_1} = 0\}$ while on the other chart $\big((U_3)_1, \sigma_2(y) = t\big)$, there is the third curve $\mathbf{W}_3^{(3)} = \{y \in (U_3)_2 | y_1 = y_2 = 0\}$. The eigenvalues of $\mathcal{F}_3$ along $\mathbf{W}_1^{(3)}$ are $\lambda_{11}^{(3)} = -p_{20}(x_3)$ and $\lambda_{12}^{(3)} = 3p_{20}(x_3)$, along $\mathbf{W}_2^{(3)}$ are $\lambda_{21}^{(3)} = \frac{p_{20}(x_3)}{2}$ and $\lambda_{22}^{(3)} = -3p_{20}(x_3)$, at $\mathbf{W}_3^{(3)}$ are $\lambda_{31}^{(3)} = 2p_1(x_3) - 2p_{20}(x_3)$ and $\lambda_{32}^{(3)} = 2p_{20}(x_3) - p_1(x_3)$. It is worth noting that the case $\lambda_{31}^{(3)} \equiv \lambda_{32}^{(3)} \equiv 0$ is not possible.

Henceforth, we will exclusively examine the situation where $\mathrm{m}_{\mathbf{W}_1}(\mathcal{F}_1) \geq 2$, which implies that $p_{31} \equiv 0$ and $\mathrm{m}_{\mathbf{W}_1}(R_1) \geq 2$, leading to $\mathbf{W}_1$ being of type III. Furthermore, if $\mathrm{m}_{\mathbf{W}_1}(R_1) = 2$ then $m_1 = 2$ or $n_1 = 3$ while if $\mathrm{m}_{\mathbf{W}_1}(R_2) = 1$ then $m_2 = n_2 = 2$ and $p_2 \not\equiv 0$. Thus, in the chart $\big((U_2)_1, v = \sigma_1(x)\big)$, the foliation $\mathcal{F}_2$ is described by the following vector field

$$\begin{aligned} X_2 =& x_1\big(-p_{20} + R_1^{(2)} - R_2^{(2)}\big)\frac{\partial}{\partial x_1} + x_2\big(2p_{20} - R_1^{(2)} + 2R_2^{(2)}\big)\frac{\partial}{\partial x_2} \\ & + x_1x_2\big(p_{30} + R_3^{(2)}\big)\frac{\partial}{\partial x_3} \end{aligned} \tag{105}$$

where

$$\begin{aligned} R_1^{(2)}(t) =& x_1^{m_1-2}x_2^{m_1-1}L_1^{(2)}(x) + x_1^{n_1-3}x_2^{n_1-1}g_1^{(2)}(x), \\ R_i^{(2)}(x) =& x_1^{m_i-1}x_2^{m_i-1}L_i^{(2)}(x) + x_1^{n_i-2}x_2^{n_i-1}g_i^{(2)}(x), \quad \text{for } i = 2,3 \end{aligned}$$

with $L_i^{(2)}(0,0,x_3) = q_i(x_3)$ and $g_i^{(2)}(0,0,x_3) = p_i(x_3)$.

Here, we assume that $p_1 \equiv 0$ when $n_3 \geq 4$, and similarly $p_2 \equiv q_2 \equiv 0$ when $\mathrm{m}_{\mathbf{W}_1}(R_2) \geq 2$. Thus, the singular set of $\mathcal{F}_2$ is defined as follows

$$x_1 = x_2\Big(2p_{20} + (2p_2 - q_1)x_2 - p_1x_2^2\Big) = 0.$$

Thus, if $p_1 \not\equiv 0$ and $\Delta_1 = (2p_2 - q_1)^2 + 8p_{20}p_1 \not\equiv 0$ then we have 3 curves to consider: $\mathbf{W}_1^{(2)} = \{x \in (U_2)_1 | x_1 = x_2 = 0\}$, $\mathbf{W}_2^{(2)} = \{x \in (U_2)_1 | x_1 = x_2 - \psi_1(x_3) = 0\}$ and $\mathbf{W}_3^{(2)} = \{x \in (U_2)_1 | x_1 = x_2 - \psi_2(x_3) = 0\}$ where

$$\psi_i(x_3) = \frac{2p_2(x_3) - q_1(x_3) + (-1)^i\sqrt{\Delta_1}}{2p_1(x_3)}.$$

The eigenvalues of $\mathcal{F}_2$ at $\mathbf{W}_1^{(2)}$ are $\lambda_{11}^{(2)} = -p_{20}$ and $\lambda_{12}^{(2)} = 2p_{20}$, at $\mathbf{W}_2^{(2)}$ are $\lambda_{21}^{(2)} = p_{20} + \psi_1p_2$ and $\lambda_{22}^{(2)} = \psi_1\sqrt{\Delta_1}$, at $\mathbf{W}_3^{(2)}$ are $\lambda_{31}^{(2)} = p_{20} + \psi_2p_2$ and $\lambda_{32}^{(2)} = -\psi_2\sqrt{\Delta_1}$.

If $\Delta_1 \equiv 0$ and $p_1 \not\equiv 0$ then there are 2 homeomorphic curves to $\mathbf{W}_1$ in the singular set of $\mathcal{F}_2$, namely $\mathbf{W}_1^{(2)} = \{x \in (U_2)_1 | x_1 = x_2 = 0\}$ and $\mathbf{W}_2^{(2)} = \{x \in (U_2)_1 | x_1 = x_2 - \psi_1(x_3) = 0\}$, but this last one has multiplicity equal to 2. The eigenvalues of $\mathcal{F}_2$ are $\lambda_{21}^{(2)} = p_{20} + \psi_1 p_2$ and $\lambda_{22}^{(2)} \equiv 0$.

If $p_1 \equiv 0$ and $2q_2 - q_1 \not\equiv 0$ then there are 3 elementary curves contained in $\mathbf{E}_2$, that is, $\mathbf{W}_1^{(2)} = \{x \in (U_2)_1 | x_1 = x_2 = 0\}$, $\mathbf{W}_2^{(2)} = \{x \in (U_2)_1 | x_1 = x_2 - \frac{2p_{20}}{q_1 - 2p_2} = 0\}$ and $\mathbf{W}_3^{(2)} = \{y \in (U_2)_1 | y_1 = y_2 =\}$ where $\sigma_2(y) = v$. The eigenvalues of $\mathcal{F}_2$ along $\mathbf{W}_2^{(2)}$ are $\lambda_{21}^{(2)} = \frac{q_1 p_{20}}{q_1 - 2p_2}$ and $\lambda_{22}^{(2)} = -2p_{20}$, along $\mathbf{W}_3^{(2)}$ are $\lambda_{31}^{(2)} = q_1 - 2p_2$ and $\lambda_{32}^{(2)} = p_2$.

Now, if $p_1 \equiv 0$ and $2q_2 - q_1 \equiv 0$ then there are 2 elementary curves contained in $\mathbf{E}_2$, the elementary curve $\mathbf{W}_1^{(2)} = \{x \in (U_2)_1 | x_1 = x_2 = 0\}$ and $\mathbf{W}_2^{(2)} = \{y \in (U_2)_1 | y_1 = y_2 = 0\}$, with multiplicity equal to 2. The eigenvalues of $\mathcal{F}_2$ along $\mathbf{W}_2^{(2)}$ are $\lambda_{21}^{(2)} \equiv 0$ and $\lambda_{22}^{(2)} = q_2$. Notice if $q_2 \equiv 0$ then $q_1 \equiv 0$ resulting in $\mathrm{m}_{\mathbf{W}_1}(R_1) > 3$ and $\mathrm{m}_{\mathbf{W}_1}(R_2) > 2$ which is absurd.

Now, we will consider $\mathrm{m}_{\mathbf{W}_1}(R_1) = 3$, $\mathrm{m}_{\mathbf{W}_1}(R_2) = 2$, and $\mathrm{m}_{\mathbf{W}_1}(R_3) = 1$. Hence, $m_3 = n_3 = 2$ and $p_3 \not\equiv 0$. Therefore, the singular set of $\mathcal{F}_2$ has 2 curves homemorphic to W_1, namely, $\mathbf{W}_1^{(2)} = \{x \in (U_2)_1 | x_1 = x_2 = 0\}$ as before and $\mathbf{W}_2^{(2)} = \{y \in (U_2)_2 | y_1 = y_2 = 0\}$ with multilplicity equal to 2. But this curve $\mathbf{W}_2^{(2)}$ is non-elementary and of type I. Thus, let $\mathbf{W}_2 = \mathbf{W}_2^{(2)}$. The singular set of $\mathcal{F}_3$ contains the elementary curve $\mathbf{W}_1^{(3)} = \{t \in (U_3)_1 | t_1 = t_2 = 0\}$ and some isolated closed points. The eigenvalues of $\mathcal{F}_3$ along $\mathbf{W}_1^{(3)}$ are $\lambda_{11}^{(3)} = -2p_{20}$ and $\lambda_{12}^{(3)} = 3p_{20}$.

Finally, if a blow-up sequence $\{\pi_i, \mathbf{M}_i, \mathbf{W}_i, \mathcal{F}_i, \mathbf{E}_i\}$ exists such that $\mathbf{W}_i$ is homeomorphic to $\mathbf{W}_{i-1}$ and $\pi_i(\mathbf{W}_i) = \mathbf{W}_{i-1}$, with $\mathbf{W}_i \subset \mathbf{E}_i$ being a non-elementary curve of $\mathrm{Sing}(\mathcal{F}_i)$ for all i, then, based on our previous observations, the only possibility is $\mathrm{m}_{\mathbf{W}_1}(R_1) \geq 3$, $\mathrm{m}_{\mathbf{W}_1}(R_2) \geq 2$, and $\mathrm{m}_{\mathbf{W}_1}(R_3) \geq 2$, and these conditions must remain invariant under subsequent blow-ups. More precisely, foliation $\mathcal{F}_2$, in chart $\big((U_2)_2, \sigma_2(y)\big)$, is described by the following vector field.

$$
\begin{aligned}
Y_2 = &\big(-2y_1^2 p_{20} + R_1^{(2)} - 2y_1 R_2^{(2)} \big) \frac{\partial}{\partial y_1} + y_2 \big(y_1 p_{20} + R_2^{(2)} \big) \frac{\partial}{\partial y_2} + \\
&+ y_2 \big(y_1 p_{30} + R_3^{(2)} \big) \frac{\partial}{\partial y_3}
\end{aligned}
\tag{106}
$$

where

$$
R_1^{(2)} = y_1 y_2^{m_1 - 2} L_1^{(2)}(y) + y_2^{n_1 - 3} g_1^{(2)}(y), \quad R_i^{(2)} = y_1 y_2^{m_i - 1} L_i^{(2)}(y) + y_2^{n_i - 2} g_i^{(2)}(y)
$$

for $i = 2, 3$, with $L_i^{(2)}(y) = L_i^{(1)} \circ \sigma_2(y)$ and $g_i^{(2)}(y) = g_i^{(1)} \circ \sigma_2(y)$.

Thus, $\mathbf{W}_2 = \{y \in (U_2)_2 | y_1 = y_2 = 0\}$ is a non-elementary and $\mathrm{m}_{\mathbf{W}_2}(\mathcal{F}_2) = 2$. In order for such a sequence to exist we must have $\mathrm{m}_{\mathbf{W}_2}(R_1^{(2)}) \geq 3$, $\mathrm{m}_{\mathbf{W}_2}(R_2^{(2)}) \geq 2$ and $\mathrm{m}_{\mathbf{W}_2}(R_3^{(2)}) \geq 2$ and so on. Therefore, let $\mathbf{W}_k$ be the curve defined in the chart $\big((U)_k, \sigma_2(y^{(k)})\big)$ as follows $\{y^{(k)} \in (U_k)_2 | y_1^{(k)} = y_2^{(k)} = 0\}$, with $\sigma_2(y^{(k)}) = y^{(k-1)}$ for $k \geq 1$. By finite induction, the foliation $\mathcal{F}_k$, in the chart $\big((U_k)_1, \sigma_1(x) = y^{(k-1)}\big)$, is described by the vector field

$$
\begin{aligned}
X_k = &x_1 \Bigg(-(k-1)(p_{20} + S_2^{(k)}) + S_1^{(k)} \Bigg) \frac{\partial}{\partial x_1} + x_2 \Bigg(k(p_{20} + S_2^{(k)}) - S_1^{(k)} \Bigg) \frac{\partial}{\partial x_2^{(k)}} + \\
&+ x_1 x_2 \Bigg(p_{30} + S_3^{(k)} \Bigg) \frac{\partial}{\partial x_3^{(k)}}
\end{aligned}
\tag{107}
$$

where

$$
\begin{aligned}
&S_1^{(k)}(x) = x_1^{\alpha_k} x_2^{\alpha_{k-1}} \tilde{g}_1^{(k)}(x) + x_1^{m_1 - k} (x_2)^{m_1 - k + 1} \tilde{L}_1^{(k)}(x), \quad \alpha_k = n_1 - (2k - 1) \\
&S_i^{(k)}(x) = x_1^{n_i - k} x_2^{n_i - k + 1} \tilde{g}_i^{(k)}(x) + (x_1 x_2)^{m_i - 1} \tilde{L}_i^{(k)}(x), \text{ for } i = 2, 3
\end{aligned}
$$

with $\tilde{L}_i^{(k)}(y) = L_i^{(k-1)}(\sigma_1(x))$ and $\tilde{g}_i^{(k)}(y) = g_i^{(k-1)}(\sigma_1(x))$, while in the chart $\big((U_k)_2, \sigma_2(y) = y^{(k-1)})\big)$, by the vector field

$$Y_k = \Big(- ky_1^2 p_{20} + R_1^{(k)} - ky_1 R_2^{(k)}\Big)\frac{\partial}{\partial y_1} + y_2\Big(y_1 p_{20} + R_2^{(k)}\Big)\frac{\partial}{\partial y_2} + \\ + y_2\Big(y_1 p_{30} + R_3^{(k)}\Big)\frac{\partial}{\partial y_3} \tag{108}$$

where

$$R_1^{(k)}(y) = y_2^{\alpha_k} g_1^{(k)}(y) + y_1 y_2^{m_1-k} L_1^{(k)}(y),$$
$$R_i^{(k)}(y) = y_2^{n_i-k} g_2^{(k)}(y) + y_1 y_2^{m_i-1} L_2^{(k)}(y), \text{ for } i = 2,3$$

with $L_i^{(k)}(y) = L_i^{(k-1)} \circ \sigma_2(y)$ and $g_i^{(k)}(y) = g_i^{(k-1)} \circ \sigma_2(y)$. Furthermore, $g_i^{(k)}(0,0,x_3) = p_i(x_3)$ and $L_i^{(k)}(0,0,x_3) = q_i(x_3)$ for all i.

Since $\mathrm{m}_{\mathbf{W}_{k-1}}(R_1^{(k-1)}) \geq 3$, $\mathrm{m}_{\mathbf{W}_{k-1}}(R_2^{(k-1)}) \geq 2$ and $\mathrm{m}_{\mathbf{W}_{k-1}}(R_3^{(k-1)}) \geq 2$, $\mathcal{F}_k$ is well defined as well as $\alpha_k, n_2 - k, n_3 - k, m_1 - k \geq 0$ with $\mathrm{m}_{\mathbf{E}_i}(\pi_i^* \mathcal{F}_{i-1}) = 1$ for $i = 1, \ldots, k$. By Theorem 4.2 there are three curves $\mathbf{W}_j^{(k)} \subset \mathrm{Sing}(\mathcal{F}_k)$, counting the multiplicities, since $m_{\mathbf{W}_k}(\mathcal{F}_k) = 2$. It is not difficult to see that the curve $\mathbf{W}_1^{(k)} = \{x^{(k)} \in (U_k)_1 | x_1^{(k)} = x_2^{(k)} = 0\}$ is an elementary component of $\mathrm{Sing}(\mathcal{F}_k)$ since $\lambda_{11}^{(k)} = -(k-1)p_{20}(x_3)$ and $\lambda_{21}^{(k)} = kp_{20}(x_3)$, i.e.; $\lambda_{11}^{(k)}/\lambda_{21}^{(k)} = -(k-1)/k \notin Q_+, k \geq 2$ for almost all $x^{(k)} \in \mathbf{W}_1^{(k)}$. Notice that $S_i^{(k)}|_{\mathbf{E}_k} \equiv 0$ for all i resulting in $\mathbf{W}_1^{(k)}$ is the unique homeomorphic curve to $\mathbf{W}_{k-1}$ contained in this chart.

However, considering that at least one of $p_i \not\equiv 0$, it follows that at least one of the three inequalities $\mathrm{m}_{\mathbf{W}_k}(R_1^{(k)}) \leq \min\{m_1 - k + 1, \alpha_k\}$, $\mathrm{m}\mathbf{W}_k(R_2^{(k)}) \leq n_2 - k$, or $\mathrm{m}_{\{W_k}(R_3^{(k)}) \leq n_3 - k$ holds true for all $k > 0$. Consequently, it is impossible for such a sequence to continue indefinitely. This assumption is consistent since it would contradict Lemma 5.1 otherwise. Therefore, for some $k > 0$, the analysis of Equation (108) follows a pattern similar to that of Equation (103), with the substitution of p_{20} and p_2 by kp_{20} and kp_2 in R_1, respectively. It is enough to compare Equations (103) and (108). Thus, the study is reduced to one of the cases that had previously been examined. □

Proposition 5.8. *Let us consider the foliation $\mathcal{F}_0$ described by the vector field (86) with $p_{10} \not\equiv 0$ and the all eigenvalues of (87) are identically null. Then, for any blow-up sequence Let $\{\pi_i, \mathbf{M}_i, \mathbf{W}_i, \mathcal{F}_i, \mathbf{E}_i\}$ be such that $\mathbf{W}_i$ is homeomorphic to $\mathbf{W}_{i-1}$ and $\pi_i(\mathbf{W}_i) = \mathbf{W}_{i-1}$ there is a natural $k \in \mathbb{N}$ such that $\mathbf{W}_i$ are elementary components of $\mathcal{F}_i$ for $i \geq k$.*

.

Proof. From (86), we get

$$X_0 = \sum_{i=1}^{3}\Big(z_1 p_{i0} + z_2 p_{i1} + P_i(z)\Big)\frac{\partial}{\partial z_i}, \tag{109}$$

where

$$P_i(z) = \sum_{j=0}^{m_i} z_1^{m_i-j} z_2^j P_{ij}(z), \quad m_i \geq 2.$$

Under these conditions, since $tr(\mathbf{A}_{X_0}|_{\mathbf{W}_0}) \equiv det(\mathbf{A}_{X_0}|_{\mathbf{W}_0}) equiv 0$ there exists a holomorphic function $\varphi(z_3)$ such that $p_{i0}(z_3) = \varphi(z_3)p_{i1}(z_3)$ for $i = 1, 2$, which results

$$\mathbf{A}_{X_0}|_{\mathbf{W}_0} = \begin{pmatrix} \varphi(z_3)p_{11}(z_3) & p_{11}(z_3) \\ -\varphi^2(z_3)p_{11}(z_3) & -\varphi(z_3)p_{11}(z_3) \end{pmatrix}. \tag{110}$$

In the chart $((U_1)_1, \sigma_1(u))$, the foliation $\mathcal{F}_1$ is described by the following vector field

$$X_1 = u_1\Big((u_2+\varphi(z_3))p_{11} + u_1 Q_1\Big)\frac{\partial}{\partial u_1} - \Big((u_2+\varphi(z_3))^2 p_{11} - u_1(Q_2 - u_2 Q_1)\Big)\frac{\partial}{\partial u_2} + \\ + u_1(p_{30} + u_2 p_{31} + u_1 Q_3)\frac{\partial}{\partial u_3} \tag{111}$$

where

$$Q_i(u) = u_1^{m_i-2}\sum_{j=0}^{m_i} u_2^j P_{ij}(u_1, u_1 u_2, u_3).$$

In this chart, there is the only non-elementary curve $\mathbf{W}_1$ which is defined by Equations $u_1 = u_2 + \varphi(u_3) = 0$ and has multiplicity equal to 2. Just as it was done in the Proposition 5.7, from now on we only consider the fibers $\pi_1^{-1}(0,0,z_3)$ such that $p_{11}(z_3) \neq 0$. In this coordinate system $(v_1, v_2, v_3) = F(u) = (u_1, u_2 + \varphi(u_3), u_3)$, the foliation $\mathcal{F}_1$ is described by the vector field

$$Y_1 = v_1\big(v_2 p_{11} + v_1 R_1\big)\frac{\partial}{\partial v_1} - \Big(p_{11}v_2^2 - v_1\big(G_2 - v_2 R_1 + \varphi'(v_3)(r_{30} + v_2 r_{31})\big)\Big)\frac{\partial}{\partial v_2} + \\ + v_1\big(r_{30} + v_2 r_{31} + v_1 R_3\big)\frac{\partial}{\partial v_3}, \tag{112}$$

where $R_i(v) = Q_i \circ F^{-1}(v)$ and $G_2(v) = R_2(v) + \varphi(v_3)R_1(v) + v_1\varphi'(v_3)R_3(v)$.

Initially, we will consider the case where φ is not a constant function and and $\mathrm{m}_{\mathbf{W}_1}(\mathcal{F}_1) = 1$. Hence, under these conditions, $r_{30} \not\equiv 0$ or $a_0(v_3) := G_2(0,0,v_3) + \varphi'(v_3)r_{30}(v_3) \not\equiv 0$.

Let us consider $a_0 \not\equiv 0$ which results $\mathbf{W}_1$ is of type III. So, in the chart $\big((U_2)_2, v = \sigma_2(t)\big)$, the singular set of $\mathcal{F}_2$ contains the only curve $\mathbf{W}_2 = \{t \in (U_2)_2 | t_1 = t_2 = 0\}$ that is homeomorphic to $\mathbf{W}_1$. The curve $\mathbf{W}_2$ has multiplicity equal to 2 and $\mathrm{m}_{\mathbf{W}_2}(\mathcal{F}_2) = 2$. The singular set of $\mathcal{F}_3$ contains three elementary homeomorphic curves to $\mathbf{W}_2$. In other words, in the chart $(U_3)_1, t = \sigma_1(x))$, there are curves $\mathbf{W}_1^{(3)} = \{x \in (U_3)_1 | x_1 = x_2 = 0\}$, with eingevalues of $\mathcal{F}_3$ at $\mathbf{W}_1^{(3)}$ are $\lambda_{11}^{(3)} = -a_0(x_3)$ and $\lambda_{11}^{(3)} = a_0(x_3)$; and

$$\mathbf{W}_2^{(3)} = \left\{x \in (U_3)_1 | x_1 = x_2 - \frac{2a_0(x_3)}{3p_{11}(x_3)} = 0\right\}$$

with eingevalues of $\mathcal{F}_3$ at $\mathbf{W}_2^{(3)}$ are $\lambda_{21}^{(3)} = a_0(x_3)/3$ and $\lambda_{22}^{(3)} = -2a_0(x_3)$. In the chart $(U_3)_2, t = \sigma_2(y))$, there is the curve $\mathbf{W}_3^{(3)} = \{y \in (U_3)_2 | y_1 = y_2 = 0\}$, with eingevalues of $\mathcal{F}_3$ at $\mathbf{W}_3^{(3)}$ are $\lambda_{31}^{(3)} = 3p_{11}(y_3)$ and $\lambda_{32}^{(3)} = -p_{11}(y_3)$.

Now, we will consider $a_0 \equiv 0$ and $r_{30} \not\equiv 0$ which result $\mathbf{W}_1$ is of type I. Therefore, the singular set of $\mathcal{F}_2$ contains the only curve $\mathbf{W}_2 = \{y \in (U_2)_2 | y_1 = y_2 = 0\}$, $\sigma_2(y) = v$, which is homeomorphic to $\mathbf{W}_1$. The eigenvalues of $\mathcal{F}_2$ at $\mathbf{W}_2$ are $\lambda_{21}^{(2)} = 2p_{11}(y_3)$ and $\lambda_{21}^{(2)} = -p_{11}(y_3)$ whose ratio is a negative rational number for almost every point of $\mathbf{W}_2$.

Thus, we will consider $\mathrm{m}_{\mathbf{W}_1}(\mathcal{F}_1) = 2$ which results $r_{30} \equiv 0$ and $a_0 \equiv 0$ in (112). Consequently, the singular set of $\mathcal{F}_2$ contains three elementary homeomorphic curves to $\mathbf{W}_1$. In the chart $\big((U_2)_1, v = \sigma_1(x)\big)$, the singular set of $\mathcal{F}_2$ is defined by the following Equations

$$x_1 = -2x_2^2 p_{11}(x_3) + x_2 b_1(x_3) + a_1(x_3) = 0 \tag{113}$$

where

$$a_1(x_3) = \frac{\partial G_2}{\partial v_1}(0,0,x_3) \text{ and } b_1(x_3) = \frac{\partial G_2}{\partial v_2}(0,0,x_3) - 2R_1(0,0,v_3) + \varphi'(x_3)r_{31}(x_3).$$

Hence, if $\Delta = b_1^2(x_3) + 8p_{11}(x_3)a_1(x_3) \not\equiv 0$ then in this chart there are two curves which are defined as follows $\mathbf{W}_1^{(2)} = \{x \in (U_2)_1 | x_1 = x_2 - \psi_1(x_3) = 0\}$ and $\mathbf{W}_2^{(2)} = \{x \in (U_2)_1 | x_1 = x_2 - \psi_2(x_3) = 0\}$ where

$$\phi_i(x_3) = \frac{b_1(x_3) - (-1)^i\sqrt{\Delta}}{4p_{11}(x_3)}.$$

The third curve is $\mathbf{W}_3^{(2)} = \{y \in (U_2)_2 | y_1 = y_2 = 0\}$, with $\sigma_2(y) = v$. The eigenvalues of $\mathcal{F}_2$ along $\mathbf{W}_1^{(2)}$ are $\lambda_{11}^{(2)} = \phi_1(x_3)p_{11}(x_3) + R_1(0,0,x_3)$ and $\lambda_{12}^{(2)} = \sqrt{\Delta}$. The eigenvalues of $\mathcal{F}_2$ along $\mathbf{W}_2^{(2)}$ are $\lambda_{21}^{(2)} = \phi_2(x_3)p_{11}(x_3)R_1(0,0,x_3)$ and $\lambda_{22}^{(2)} = \sqrt{\Delta}$. And the eigenvalues of $\mathcal{F}_2$ along $\mathbf{W}_3^{(2)}$ are $\lambda_{31}^{(2)} = 2p_{11}(x_3)$ and $\lambda_{12}^{(2)} = -p_{11}(x_3)$.

If $\Delta \equiv 0$ and $b_1 \not\equiv 0$ in Equation (113) then the curve $\mathbf{W}_1^{(2)}$ has multiplicity equal to 2. Hence, $\mathbf{W}_1^{(2)}$ is an elementar component if $\lambda_{11}^{(2)} = \phi_1(x_3)p_{11}(x_3) + R_1(0,0,x_3) \not\equiv 0$. Otherwise, it is enough to make this change of variables $(t_1,t_2,t_3) = (x_2 - \psi_1(x_3), x_1, x_3$ which results in the vector field (112) being transformed into the vector field (106). The proof then proceeds similarly to that of Proposition (5.7).

In an exact similar way, if φ is not a constant function or $\Delta \equiv b_1 \equiv 0$, the vector field (112) can also be transformed into the vector field (102) by changing the variables (v_1,v_2,v_3) to (t_2,t_1,t_3). So, we finish the proof of the Proposition.

□

5.3. **Proof of Theorem 1.3.**

Proof. Let $\{\pi_i, \mathbf{M}_i, \mathbf{W}_i, \mathcal{F}_i, \mathbf{E}_i\}$ be a blow-up sequence such that $\mathbf{M}_0 = \mathbb{P}^3$ and $\mathbf{W}_i$ is homeomorphic to $\mathbf{W}_{i-1}$ with $\pi_i(\mathbf{W}_i) = \mathbf{W}_{i-1}$ for all $i \geq 1$. From Theorem 1.2, there exists $k_0 \in \mathbb{N}$ such that $m_{\mathbf{E}_i}(\pi_i^*\mathcal{F}_{i-1}) = 0$ for $i \geq k_0$. Hence, $m_{\mathbf{W}_i}(\mathcal{F}_i) = 1$ and $\mathbf{W}_i$ is of type III for some index i. From Propostions 5.3, 5.4, 5.7 and 5.8, we conclude the proof of Theorem 1.3. Thus, $\mathbf{W}_i$ is an elementary component for $i \geq k$, for some natural number k, and for almost all points of $\mathbf{W}_i$. If follows from [5, Proposition 2.20] that $(\mathcal{F}_i) = 1$ is generically log canonical along $\mathbf{W}_i$. □

Example 5.9. *(***F. Sanz and F. Sancho's example***)* Let us consider the holomorphic foliation $\mathcal{F}_0$ defined on $\mathbf{M}_0 = \mathbb{P}^3$ described in the affine open set $U_3 = \{[\xi] \in \mathbb{P}_3, \xi_3 \neq 0\}$ by the following vector field

$$X_0 = z_1^2 \frac{\partial}{\partial z_1} + (-\alpha z_1 z_2 + z_1 z_3)\frac{\partial}{\partial z_2} + (-\lambda z_1 + z_2 - \beta z_1 z_3)\frac{\partial}{\partial z_3},$$

where $z_i = \frac{\xi_{i-1}}{\xi_3}$, $\alpha, \beta \in \mathbb{R}_{\geq 0}$ and $\lambda \in \mathbb{R}_{>0}$. Thus, we have that

$$\mathrm{Sing}(\mathcal{F}_0) = \mathbf{W}_0 \cup \mathbf{W}_1 \cup \{p_1\}$$

where $\mathbf{W}_0 := \{\xi_0 = \xi_1 = 0\}$, $\mathbf{W}_1 : \{\xi_0 = \xi_3 = 0\}$ and $p_1 = [1:0:0:0]$. Let $\pi_1 : \mathbf{M}_1 \to \mathbf{M}_0$ be the blowup of $\mathbb{P}^3$ along $\mathbf{W}_0$ being $\mathbf{E}_1$ and $\mathcal{F}_1$ as in the previous example. Thus, the curve $\mathbf{W}_0$ is type III. The singular set of $\mathcal{F}_1$ contains only one curve which is homeomorphic to $\mathbf{W}_0$ but with multiplicity equal to 2. See Theorem 4.2.

In fact, in the chart $((U_1)_2, z = \sigma_2(v))$, with relations $z = \sigma_2(v) = (v_1 v_2, v_2, v_3)$, the foliation $\mathcal{F}_1$ is described by the following vector field

$$Y_1 = \big(v_2 v_1^2 - v_1^2(-\alpha v_2 + v_3)\big)\frac{\partial}{\partial v_1} + v_2(-\alpha v_1 v_2 + v_1 v_3)\frac{\partial}{\partial v_2} + v_2\big(-v_1(\lambda + \beta v_3) + 1\big)\frac{\partial}{\partial v_3}.$$

It is not difficult to see that the curve defined as $\mathbf{W}_2^{(1)} = \{v \in (U_1)_2 | v_1 = v_2 = 0\}$ is such a curve. In the chart $((U_1)_1, \sigma_1(u))$, the foliation $\mathcal{F}_1$ is described by the vector field

$$X_1 = u_1^2 \frac{\partial}{\partial u_1} + (u_3 - u_1 u_2(1+\alpha))\frac{\partial}{\partial u_2} + u_1\big(-\lambda - \beta v_3 + u_2\big)\frac{\partial}{\partial u_3}.$$

However, the singular set of $\mathcal{F}_1$ contains the curve $\mathbf{W}_1^{(1)} = \{u \in (U_1)_2 | u_1 = u_3 = 0\}$ which is homeomorphic to $\mathbb{P}^1$. F. Sanz and F. Sancho showed that the vector field X_1 is invariant by a blow-up centered at $\mathbf{W}_1^{(1)}$ which results that X_0 cannot be desingularized by blow-ups along such curves. See [8] for more details.

Let $\pi_2 : \mathbf{M}_2 \to \mathbf{M}_1$ be the blowup of $\mathbf{M}_1$ along $\mathbf{W}_2^{(1)}$ being $\mathbf{E}_2$ and $\mathcal{F}_2$ the exceptional divisor and the strict transform foliation, respectively. Thus, the curve $\mathbf{W}_2^{(1)}$ is of type I and $\mathrm{m}_{\mathbf{E}_2}(\pi_2^*\mathcal{F}_1) = 1$. In the chart $((U_2)_1, \sigma_1(w) = v)$, the foliation $\mathcal{F}_2$ is described by the vector field

$$X_2 = w_1\big(-w_3 + (1+\alpha)w_1w_2\big)\frac{\partial}{\partial w_1} + w_2\big(2w_3 - (1+2\alpha)w_1w_2\big)\frac{\partial}{\partial w_2} + w_2\big(1 - w_1(\lambda+\beta)\big)\frac{\partial}{\partial w_3}.$$

Except for $w_3 = 0$, the curve $\mathbf{W}_1^{(2)} = \{w \in (U_2)_1 | w_1 = w_2 = 0\}$ is an elementary curve of $\mathrm{Sing}(\mathcal{F}_2)$ as the eigenvalues of $\mathbf{A}_{X_2}|_{\mathbf{W}_1^{(2)}}$ are $\lambda_{11}^{(2)}(w_3) = -w_3$ and $\lambda_{21}^{(2)}(w_3) = 2w_3$. Furthermore, $\lambda_{11}^{(2)}/\lambda_{21}^{(2)} = -1/2 \notin \mathbb{Q}_+$ for almost all $w \in \mathbf{W}_1^{(1)}$. The exception occurs precisely at the intersection point with the curve $\pi_2^{-1}\big(\mathbf{W}_1^{(1)}\big)$.

5.4. **On the formal first integrals.** The following proposition tells us that a foliation on $(\mathbb{C}^3, 0)$ which cannot be birationally desingularized can admit formal first integrals along a curve of its singular set.

Proposition 5.10. *Let $\mathcal{F}$ be a germ of holomorphic foliation by curves on $(\mathbb{C}^n, 0)$ whose singular set is a pure codimension 2 scheme passing through 0. If $\mathcal{F}$ has a formal integral first $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n-1}, 0)$, then f is convergent. In particular, if $\mathcal{F}$ is a foliation on $(\mathbb{C}^3, 0)$ that cannot be birationally resolved, then $\mathcal{F}$ has no formal first integral along a curve in its singular set.*

Proof. Since the statement is local, consider a point $p \in Z = Sing(\mathcal{F})$, a neighborhood U of p and a germ of the vector field v inducing $\mathcal{F}$ in (U, p). Dualizing the sequence

$$0 \longrightarrow \mathcal{O} \xrightarrow{v} TU \longrightarrow N_{\mathcal{F}} \longrightarrow 0,$$

we obtain

$$0 \to N_{\mathcal{F}}^* \to \Omega_U^1 \xrightarrow{v^\vee} \mathcal{O} \to \mathcal{E}xt^1(N_{\mathcal{F}}, \mathcal{O}) \simeq \mathcal{O}_Z \to 0.$$

Cutting this sequence, we obtain the following short exact sequences

$$0 \to N_{\mathcal{F}}^* \to \Omega_U^1 \xrightarrow{v^\vee} \mathcal{I}_Z \to 0.$$

and

$$0 \to \mathcal{I}_Z \to \mathcal{O} \to \mathcal{E}xt^1(N_{\mathcal{F}}, \mathcal{O}) \simeq \mathcal{O}_Z \to 0.$$

Dualizing the first sequence, we obtain one of the isomorphisms

$$\mathcal{E}xt^p(N_{\mathcal{F}}^*, \mathcal{O}) \simeq \mathcal{E}xt^{p+1}(\mathcal{I}_Z, \mathcal{O})$$

for all $p \geq 1$. Since Z has pure codimension 2, we have that $\mathcal{E}xt^{p+1}(\mathcal{I}_Z, \mathcal{O}) = 0$, for all $p \geq 1$. This shows that $N_{\mathcal{F}}^*$ is locally free, which is equivalent to saying that the foliation is induced by $(n-1)$ holomorphic 1-forms. Therefore, the foliation is given by a decomposable $(n-1)$-form for all $p \in Z$. In fact, it is enough to consider the morphism $N_{\mathcal{F}}^* \simeq \mathcal{O}^{\oplus(n-1)} \to \Omega_U^1$ and take the maximal exterior power of the morphism $N_{\mathcal{F}}^* \simeq \mathcal{O}^{\oplus(n-1)} \to \Omega_U^1$. Now, the result follows from the Malgrange Theorem [16]. Now, suppose that $\mathcal{F}$ is a foliation on $(\mathbb{C}^3, 0)$ with a formal integral first $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$ around a point $p \in Z$. Then the convergence of $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$ implies, in particular, that $\mathcal{F}$ is the intersection of two codimension-one holomorphic foliations. This contradicts [8, Theorem 1]. □

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