## **STEIN SPACES AND STEIN ALGEBRAS**

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ABSTRACT. We prove that the category of Stein spaces and holomorphic maps is anti-equivalent to the category of Stein algebras and C-algebra morphisms. This removes a finite dimensionality hypothesis from a theorem of Forster.

#### **INTRODUCTION**

Complex spaces are a generalization of complex manifolds allowing singularities, and as such are the basic objects of study in complex-analytic geometry. Formally, they are defined to be C-ringed spaces that are locally isomorphic to model spaces defined by the vanishing of finitely many holomorphic functions in a domain of  $\mathbb{C}^N$ for some  $N \geq 0$  (see [\[GR84,](#page-5-0) 1, §1.5]). We assume that they are second-countable, but not necessarily reduced or finite-dimensional.

A complex space *S* is said to be *Stein* if  $H^k(S, \mathcal{F}) = 0$  for all coherent sheaves  $\mathcal F$ on *S* and all  $k > 0$  (see [\[GR79\]](#page-5-1)). Stein spaces are the complex-analytic analogues of affine algebraic varieties. For instance, the Stein spaces of finite embedding dimension are exactly those complex spaces that may be realized as closed complex subspaces of  $\mathbb{C}^N$  for some  $N \geq 0$  (see [\[Nar60,](#page-5-2) Theorem 6]).

If *S* is a complex space, the C-algebra  $\mathcal{O}(S)$  of holomorphic functions on *S* carries a canonical Fréchet topology (see [\[GR79,](#page-5-1) V, §6]). A topological C-algebra of the form O(*S*) for some Stein space *S* is called a *Stein algebra*.

In algebraic geometry, the anti-equivalence of categories between affine varieties over C and C-algebras of finite type is a basic tool to study affine algebraic varieties. Our main theorem is a counterpart of this result in complex-analytic geometry.

<span id="page-0-0"></span>**Theorem 0.1** (Theorem [3.3\)](#page-4-0)**.** *The contravariant functor*



*given by*  $S \mapsto \mathcal{O}(S)$  *is an anti-equivalence of categories.* 

Very significant particular cases of Theorem [0.1](#page-0-0) were previously known. First, Forster has shown in [\[For67,](#page-4-1) Satz 1] that Theorem [0.1](#page-0-0) holds if one replaces the right-hand side of [\(3.1\)](#page-4-2) by the category of Stein algebras and continuous C-algebra morphisms. From this point of view, our contribution is an automatic continuity result for morphisms of Stein algebras (see Theorem [3.2](#page-4-3) below).

Second, Forster has proven this automatic continuity result in restriction to finite-dimensional Stein spaces (see [\[For66,](#page-4-4) Theorem 5]). In particular, Theorem [0.1](#page-0-0) was already known in restriction to finite-dimensional Stein spaces and their associated Stein algebras. Forster's theorem was later generalized by Markoe [\[Mar73\]](#page-5-3) and Ephraim [\[Eph78,](#page-4-5) Theorem 2.3] who made weaker finite dimensionality assumptions. Our contribution is to remove these finite dimensionality hypotheses altogether. This problem was raised by Forster in [\[For66,](#page-4-4) Remark p.162].

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Our strategy to prove Theorem [0.1](#page-0-0) is to reduce to the finite-dimensional case treated by Forster by means of the next theorem.

<span id="page-1-0"></span>**Theorem 0.2** (Theorem [2.1\)](#page-2-0)**.** *Let S be a Stein space. Then there exists a holomorphic map*  $f : S \to \mathbb{C}^2$  *all of whose fibers are finite-dimensional.* 

Our proof of Theorem [0.2](#page-1-0) is an application of Oka theory. It uses in a crucial way new examples of Oka manifolds constructed by Forstnerič and Wold [\[FW24\]](#page-5-4) (based on and extending earlier work of Kusakabe [\[Kus21,](#page-5-5) [Kus24\]](#page-5-6)), as well as an extension theorem for holomorphic maps from Stein spaces to Oka manifolds due to Forstnerič [\[For05,](#page-4-6) [For17\]](#page-5-7).

We note that Theorem [0.2](#page-1-0) is optimal in the sense that there may not exist a holomorphic map  $f : S \to \mathbb{C}$  with finite-dimensional fibers (see Proposition [2.3\)](#page-3-0). An earlier version of this article, relying on the Oka manifolds constructed by Kusakabe [\[Kus24,](#page-5-6) Theorem 1.6], only produced such a map with values in  $\mathbb{C}^3$ . We are grateful to Franc Forstnerič for drawing our attention to the article [\[FW24\]](#page-5-4), thereby allowing us to prove Theorem [0.2](#page-1-0) in the form stated above.

The results of Oka theory that we need are gathered in Section [1.](#page-1-1) These tools are used to prove Theorem [0.2](#page-1-0) in Section [2.](#page-2-1) In Section [3,](#page-3-1) we deduce Theorem [0.1](#page-0-0) from Theorem [0.2](#page-1-0) and from Forster's works [\[For66,](#page-4-4) [For67\]](#page-4-1).

# 1. Tools from Oka theory

<span id="page-1-1"></span>We recall that a complex manifold *Y* is said to be *Oka* if for all convex compact subsets  $K \subset \mathbb{C}^N$  and all open neighborhoods  $\Omega$  of  $K$  in  $\mathbb{C}^N$ , any holomorphic map  $\Omega \to Y$  can be approximated uniformly on *K* by holomorphic maps  $\mathbb{C}^N \to Y$ (see [\[For09,](#page-4-7) Definition 1.2]).

We now introduce the Oka manifolds of interest to us. For  $r \in \mathbb{R}$ , define

$$
Y_r := \{ (z_1, z_2) \in \mathbb{C}^2 \mid \text{Im}(z_2) < |z_1|^2 + \text{Re}(z_2)^2 + r \}.
$$

The next proposition is a particular case of a theorem of Forstnerič and Wold [\[FW24,](#page-5-4) Corollary 1.5] (pointed out in [\[FW24,](#page-5-4) (1.2)]).

<span id="page-1-3"></span>**Proposition 1.1.** *For*  $r \in \mathbb{R}$ *, the complex manifold*  $Y_r$  *is Oka.* 

<span id="page-1-2"></span>The following easy lemma implies in particular that  $Y_r$  is contractible.

**Lemma 1.2.** *Fix*  $r \in \mathbb{R}$ *. There is a homotopy*  $(h_t)_{t \in [0,1]} : \mathbb{C}^2 \to \mathbb{C}^2$  *inducing strong deformation retractions of both*  $\mathbb{C}^2$  *and*  $Y_r$  *onto*  $\{(z_1, z_2) \in \mathbb{C}^2 \mid \text{Im}(z_2) \le r - 1\}.$ 

*Proof.* The homotopy  $(h_t)_{t \in [0,1]}$  defined by

$$
h_t(z_1, z_2) = (z_1, z_2 - it(\text{Im}(z_2) - r + 1)) \quad \text{if } \text{Im}(z_2) \ge r - 1
$$
  

$$
h_t(z_1, z_2) = (z_1, z_2) \quad \text{if } \text{Im}(z_2) \le r - 1
$$

has the required properties.

We will make use of the Oka property and of the contractibility of *Y<sup>r</sup>* through the next extension result, which is an application of theorems of Forstnerič (see [\[For05,](#page-4-6) Theorem 1.1] and the more general [\[For17,](#page-5-7) Theorem 5.4.4]).

<span id="page-1-4"></span>**Proposition 1.3.** *Fix*  $r \in \mathbb{R}$ *. Let S be a reduced Stein space and let S*<sup>*'*</sup> *be a (possibly nonreduced)* closed complex subspace of *S*. Let  $f' : S' \to Y_r$  be a holomorphic map. *Then there exists a holomorphic map*  $f : S \to Y_r$  *with*  $f|_{S'} = f'$ .

*Proof.* Since *S* is Stein, the restriction map  $\mathcal{O}(S) \to \mathcal{O}(S')$  is onto. It follows that there exists a holomorphic map  $f_1: S \to \mathbb{C}^2$  such that  $f_1|_{S'} = f'$ .

Define  $U := f_1^{-1}(Y_r)$ . It is an open neighborhood of  $S'$  in  $S$ . Let  $Z \subset U$  be a closed neighborhood of  $S'$  in  $U$ . By the Tietze–Urysohn extension theorem, there exists a continuous map  $\tau : S \to [0,1]$  which is equal to 0 on *Z* and to 1 on  $S \setminus U$ .

Define a continuous map  $f_2 : S \to Y_r$  by the formula  $f_2(s) = h_{\tau(s)}(f_1(s)),$ where  $(h_t)_{t \in [0,1]}$  is the homotopy given by Lemma [1.2.](#page-1-2) Since  $f_2$  is equal to  $f_1$  on  $U$ , it is holomorphic in a neighborhood of  $S'$  and satisfies  $f_2|_{S'} = f'$ .

As *Y<sup>r</sup>* is Oka by Proposition [1.1,](#page-1-3) it now follows from the jet interpolation part of [\[For17,](#page-5-7) Theorem 5.4.4] (applied with  $\pi$  equal to be the first projection map  $S \times Y \to S$  and with *S* equal to the ideal sheaf of *S*<sup> $\prime$ </sup> in *S*) that  $f_2$  is homotopic to a holomorphic map  $f : S \to Y_r$  with  $f|_{S'} = f_2|_{S'}$ , and hence  $f|_{S'} = f'$ . This completes the proof of the proposition.

## 2. Holomorphic maps with finite-dimensional fibers

<span id="page-2-1"></span><span id="page-2-0"></span>The next theorem is the key to our main results.

**Theorem 2.1.** *Let S be a Stein space. Then there exists a holomorphic map*  $f: S \to \mathbb{C}^2$  *all of whose fibers are finite-dimensional.* 

*Proof.* Let  $S^{\text{red}}$  be the reduction of *S*. Since *S* is Stein, the restriction map  $\mathcal{O}(S) \to \mathcal{O}(S^{\text{red}})$  is onto, and we may assume that *S* is reduced.

Let  $(S_k)_{0 \leq k \leq n}$  with  $n \in \mathbb{N} \cup \{+\infty\}$  be the irreducible components of *S*, viewed as reduced closed complex subspaces of *S*. Let Θ be the collection of all reduced and irreducible closed complex subspaces of *S* that may be obtained as irreducible components of an intersection of finitely many of the  $S_k$ . The set  $\Theta$  is at most countable, and any compact subset of *S* meets at most finitely many elements of Θ.

For  $d \geq 0$ , we let  $\Theta_d \subset \Theta$  be the set of all *d*-dimensional elements of  $\Theta$ . Let  $(Z_{d,j})_{0\leq j\leq m(d)}$  with  $m(d) \in \mathbb{N} \cup \{+\infty\}$  be an enumeration of the elements of  $\Theta_d$ . We henceforth identify  $\Theta$  with the set of all pairs  $(d, j)$  with  $d \geq 0$  and  $0 \leq j \leq m(d)$  and endow it with the lexicographical order. It is a well-ordered set. For all  $(d, j) \in \Theta$ , we view  $W_{d,j} := \bigcup_{(d',j') \leq (d,j)} Z_{d',j'}$  and  $W'_{d,j} := \bigcup_{(d',j') < (d,j)} Z_{d',j'}$ as reduced closed complex subspaces of *S*. Finally, for  $(d, j) \in \Theta$ , we let  $r(d, j)$  be the biggest integer  $k \geq 1$  such that  $Z_{d,j} \subset S_k$ .

We will now construct holomorphic functions  $f_{d,j}: W_{d,j} \to \mathbb{C}^2$  for all  $(d,j) \in \Theta$ with the property that  $f_{d,j}|_{W_{d',j'}} = f_{d',j'}$  and  $f_{d,j}(Z_{d',j'}) \subset Y_{r(d',j')}$  whenever  $(d', j') \leq (d, j)$ . The construction is by induction on the pair  $(d, j) \in \Theta$  (which is legitimate since  $\Theta$  is well-ordered).

Assume that the  $f_{d',j'}$  for  $(d',j') < (d,j)$  have been constructed. Since these maps are compatible, they glue to give rise to a holomorphic map  $f'_{d,j}: W'_{d,j} \to \mathbb{C}^2$ . Now  $W_{d,j} = W'_{d,j} \cup Z_{d,j}$ . Define  $V_{d,j} := W'_{d,j} \cap Z_{d,j}$ . It is a possibly nonreduced closed complex subspace of *S*. Note that  $V_{d,j}$  is set-theoretically a union of some of the  $Z_{d',j'}$  with  $(d',j') < (d,j)$ . If  $Z_{d',j'} \subset V_{d,j}$  is one of them, then  $Z_{d',j'} \subset Z_{d,j}$ and hence  $r(d', j') \ge r(d, j)$ . Since  $f_{d', j'}(Z_{d', j'}) \subset Y_{r(d', j')} \subset Y_{r(d, j)}$ , we deduce that  $f'_{d,j}(V_{d,j}) \subset Y_{r(d,j)}$ . Proposition [1.3](#page-1-4) now implies that the holomorphic map  $f'_{d,j}|_{V_{d,j}}: V_{d,j} \to Y_{r(d,j)}$  extends to a holomorphic map  $f''_{d,j}: Z_{d,j} \to Y_{r(d,j)}$ . Since  $f'_{d,j}$  and  $f''_{d,j}$  coincide on  $V_{d,j} = W'_{d,j} \cap Z_{d,j}$ , they glue (by Lemma [2.2](#page-3-2) below) to give rise to a holomorphic map  $f_{d,j}: W_{d,j} \to \mathbb{C}^2$  with the required properties.

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As the  $(f_{d,j})_{(d,j)\in\Theta}$  are compatible, they induce a holomorphic map  $f: S \to \mathbb{C}^2$ . Let us verify that this map has the required property. One has  $f(S_k) \subset Y_k$  for all  $0 \leq k < n$  (as  $S_k$  is one of the  $Z_{d,j}$ ). Since the  $(Y_k)_{k>0}$  form a decreasing family of subsets of  $\mathbb{C}^2$  with empty intersection, we deduce that any point of  $\mathbb{C}^2$  belongs to at most finitely many of the  $f(S_k)$ . In other words, any fiber of f intersects at most finitely many of the  $S_k$ . It follows that all the fibers of f are finite-dimensional.  $\Box$ 

<span id="page-3-2"></span>**Lemma 2.2.** *Let S be a complex space. Let S*<sup>1</sup> *and S*<sup>2</sup> *be closed complex subspaces of S. Set*  $T := S_1 \cap S_2$ *. The following diagram of sheaves on S is exact:* 

<span id="page-3-3"></span>
$$
(2.1) \t\t \t\t \mathcal{O}_S \xrightarrow{f \mapsto (f|_{S_1}, f|_{S_2})} \mathcal{O}_{S_1} \oplus \mathcal{O}_{S_2} \xrightarrow{(g,h) \mapsto g|_T - h|_T} \mathcal{O}_T \to 0.
$$

*If moreover S is reduced and*  $S = S_1 \cup S_2$ *, then the left arrow of [\(2.1\)](#page-3-3) is injective.* 

*Proof.* Fix  $s \in S$ . Write  $A = \mathcal{O}_{S,s}$  and let  $I_1$  (resp.  $I_2$ ) be the ideal of *A* consisting of germs of functions vanishing on  $S_1$  (resp.  $S_2$ ). Then the exactness of [\(2.1\)](#page-3-3) at *s* results from the exactness of  $A \to A/I_1 \oplus A/I_2 \to A/\langle I_1, I_2 \rangle \to 0$ , which is valid for any two ideals  $I_1$  and  $I_2$  of a commutative ring  $A$ .

If  $S = S_1 \cup S_2$ , then a holomorphic function in the kernel of the left arrow of [\(2.1\)](#page-3-3) vanishes at all points and hence vanishes if  $S$  is reduced.  $\square$ 

<span id="page-3-0"></span>The next proposition shows the optimality of Theorem [2.1.](#page-2-0)

**Proposition 2.3.** *There exists a Stein space S such that all holomorphic maps*  $f : S \to \mathbb{C}$  *admit an infinite-dimensional fiber.* 

*Proof.* For  $n \ge 1$ , set  $S_n := \mathbb{C}^n$ . Define  $T_n := \{(z_1, \ldots, z_n) \in S_n \mid z_n = 0\}$  and  $T'_n := \{(z_1, \ldots, z_{n+1}) \in S_{n+1} \mid z_n = 0 \text{ and } z_{n+1} = 1\}.$  Let  $\varphi_n : T_n \longrightarrow T'_n$  be the isomorphism given by  $\varphi_n(z_1, \ldots, z_{n-1}, 0) = (z_1, \ldots, z_{n-1}, 0, 1)$ . Let *S* be the complex space obtained from  $\sqcup_{n\geq 1} S_n$  by gluing  $S_n$  and  $S_{n+1}$  transversally along  $T_n$ and  $T'_n$  by means of  $\varphi_n$  (for all  $n \geq 1$ ). The complex space *S* is Stein because so is its normalization  $\sqcup_{n>1} S_n$  (see [\[Nar62,](#page-5-8) Theorem 1]).

Let  $f: S \to \mathbb{C}$  be a holomorphic map. Assume first that  $f|_{S_n}$  is constant for all  $n \gg 0$ . As the subset  $S_n \cap S_{n+1}$  of *S* is nonempty, the value taken by  $f|_{S_n}$  does not depend on  $n \gg 0$ . It follows that f has a (single) infinite-dimensional fiber.

Assume now that the set  $\Sigma := \{n \in \mathbb{N}_{\geq 1} \mid f|_{S_n} \text{ is not constant}\}\$ is infinite. For all  $n \in \Sigma$ , the map  $f|_{S_n} : S_n \to \mathbb{C}$  omits at most one value, by Picard's little theorem. We deduce that at most one complex number is not the image of  $f|_{S_n}$  for all but finitely many  $n \in \Sigma$ . Consequently, all complex numbers except possibly one are in the image of infinitely of the  $f|_{S_n}$ . As the nonempty fibers of  $f|_{S_n}$ have dimension  $\geq n-1$ , we deduce that all the fibers of f except possibly one are infinite-dimensional.  $\hfill \Box$ 

## 3. Morphisms of Stein algebras

<span id="page-3-1"></span>**Theorem 3.1.** *Let S be a Stein space. Let*  $\chi : \mathcal{O}(S) \to \mathbb{C}$  *be a*  $\mathbb{C}$ *-algebra morphism. Then*  $\chi$  *is continuous and there exists*  $s \in S$  *such that*  $\chi(f) = f(s)$  *for all*  $f \in \mathcal{O}(S)$ *.* 

*Proof.* Let  $f: S \to \mathbb{C}^2$  be as in Theorem [2.1.](#page-2-0) Let  $(f_i)_{1 \leq i \leq 2}$  be the components of *f*. Set  $\lambda_i := \chi(f_i) \in \mathbb{C}$ . Let  $T \subset S$  be the closed complex subspace defined by the equations  $\{f_i = \lambda_i\}_{1 \leq i \leq 2}$ . Let  $r_{S,T} : \mathcal{O}(S) \to \mathcal{O}(T)$  be the restriction map, which is continuous by  $|GR79, V, §6.4$  Theorem 6. By [\[Eph78,](#page-4-5) Lemma 1.7], there exists a morphism of C-algebras  $\chi_T : \mathcal{O}(T) \to \mathbb{C}$  such that  $\chi = \chi_T \circ r_{S,T}$ .

Our choice of *f* implies that *T* is a finite-dimensional Stein space. It therefore follows from Forster's theorem [For 66, Theorem 5] that  $\chi_T$  is continuous and hence that so is  $\chi$ . Another theorem of Forster [\[For67,](#page-4-1) Satz 1] then implies that there exists  $s \in S$  such that  $\chi(f) = f(s)$  for all  $f \in \mathcal{O}(S)$ .

# <span id="page-4-3"></span>**Theorem 3.2.** *Any* C*-algebra morphism between Stein algebras is continuous.*

*Proof.* Let *S* and *S'* be two Stein spaces, and let  $\xi$  :  $\mathcal{O}(S') \to \mathcal{O}(S)$  be a C-algebra morphism. Fix a finitely generated maximal ideal  $\mathfrak{m} \subset \mathcal{O}(S)$ . There exists  $s \in S$ such that  $\mathfrak{m} = \{f \in \mathcal{O}(S) \mid f(s) = 0\}$  (see e.g. [\[GR79,](#page-5-1) V, §7.1, statement above Theorem 1]). Evaluation at *s* therefore induces an isomorphism  $\mathcal{O}(S)/\mathfrak{m} \longrightarrow \mathbb{C}$ . We let  $\chi : \mathcal{O}(S) \to \mathbb{C}$  be the induced map.

Apply Theorem [3.3](#page-4-0) to the C-algebra morphism  $\chi \circ \xi : \mathcal{O}(S') \to \mathbb{C}$ . We deduce the existence of  $s' \in S'$  such that  $\chi \circ \xi(f) = f(s')$  for all  $f \in \mathcal{O}(S')$ . It then follows that  $\xi^{-1}(\mathfrak{m}) = \{f \in \mathcal{O}(S') \mid f(s') = 0\}.$  This maximal ideal is closed (by continuity of the evaluation map  $f \mapsto f(s')$ , and hence finitely generated by [\[For67,](#page-4-1) Theorem 2].

Since  $\mathfrak m$  was arbitrary, the continuity of  $\xi$  is now an application of the criterion given in [\[For67,](#page-4-1) Theorem 3].  $\square$ 

<span id="page-4-0"></span>**Theorem 3.3.** *The contravariant functor*

<span id="page-4-2"></span>(3.1) 
$$
\begin{Bmatrix}Stein spaces \\ and holomorphic maps \end{Bmatrix} \rightarrow \begin{Bmatrix}Stein algebras \\ and C-algebra morphisms \end{Bmatrix}
$$

*given by*  $S \mapsto \mathcal{O}(S)$  *is an anti-equivalence of categories.* 

*Proof.* Since C-algebra morphisms of Stein algebras are automatically continuous by Theorem [3.2,](#page-4-3) the theorem is equivalent to [\[For67,](#page-4-1) Satz 1].

We finally record the following consequence of Theorem [3.3](#page-4-0) for later use in [\[Ben\]](#page-4-8). If *S* is a Stein space, we let  $\lambda_S : S \to \text{Spec}(\mathcal{O}(S))$  be the unique morphism of locally ringed spaces such that  $\lambda_S^* : \mathcal{O}(S) \to \mathcal{O}(S)$  is the identity (see [\[SP,](#page-5-9) Lemma 01I1]).

**Proposition 3.4.** *Let X be a complex space and let S be a Stein space. The map*

(3.2) { holomorphic maps 
$$
\rightarrow
$$
  $\rightarrow$   $\rightarrow$   $\rightarrow$  *morphisms of C-locally ringed spaces*  $\rightarrow$  *X*  $\rightarrow$  Spec( $\mathcal{O}(S)$ )

*given by*  $f \mapsto \lambda_S \circ f$  *is a bijection.* 

*Proof.* As the statement is local on *X*, we may assume that *X* is Stein. In this case, the proposition follows from Theorem [3.3](#page-4-0) since the global sections functor induces a bijection between the set of morphisms of C-locally ringed spaces  $X \to \text{Spec}(\mathcal{O}(S))$ and the set of C-algebra morphisms  $\mathcal{O}(S) \to \mathcal{O}(X)$  (see [\[SP,](#page-5-9) Lemma 0111]).

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