A REMARK ON THE LANGLANDS CORRESPONDENCE FOR TORI

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Abstract. For an algebraic torus defined over a local (or global) field F, a celebrated result of R.P. Langlands establishes a natural homomorphism from the group of continuous cohomology classes of the Weil group, valued in the dual torus, onto the space of complex characters of the rational points of the torus (or automorphic characters in the global case). We slightly extend this result by showing that, if we topologize the relevant spaces of continuous homomorphisms and continuous cochains with the compact-open topology, then Langlands's homomorphism is continuous and open. Moreover, we demonstrate that, in both the local and global settings, the subset of unramified characters corresponds to the identity component of the relevant space of characters when viewed in this topological framework. Finally, we compare the group of unramified characters and the Galois (co)invariants of the dual torus.

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1. Introduction

Let T be an algebraic torus defined over a field F (all fields in this article will either be a local or a global field) and splitting over a finite Galois extension E/F, with Galois group $\Gamma_{E/F}$. Denote by $X = \mathrm{Hom}(T_E, \mathbb{G}_m)$ the character lattice, by $\widehat{X} = \mathrm{Hom}(X, \mathbb{Z})$ the cocharacter lattice and by $\widehat{T} = \mathrm{Hom}(\widehat{X}, \mathbb{C}^*)$ the complex dual torus of T. Whenever F is global, we let $\mathbb{A}_F$ denote its ring of adèles.

In [La], Langlands described the space of continuous characters $T(F) \to \mathbb{C}^*$ (local case) and of automorphic characters $T(\mathbb{A}_F)/T(F) \to \mathbb{C}^*$ (global case). Let

$$C_E = \begin{cases} E^* & \text{if } E \text{ local,} \\ \mathbb{A}_E^*/E^* & \text{if } E \text{ global,} \end{cases} \tag{1}$$

and denote by $W_{E/F}$ the relative Weil group, which is defined via the extension

$$1 \longrightarrow C_E \longrightarrow W_{E/F} \xrightarrow{p} \Gamma_{E/F} \longrightarrow 1 \tag{2}$$

of $\Gamma_{E/F}$ by C_E. In what follows, we shall denote the image of the map $W_{E/F} \xrightarrow{p} \Gamma_{E/F}$ by $p(\omega) = \overline{\omega}$. Langlands showed that there is a finite-to-one map from the group $H^1_c(W_{E/F}, \widehat{T})$, of continuous first-cohomology classes of $W_{E/F}$ with values in $\widehat{T}$, onto the relevant space of continuous characters. For F local, the correspondence is one-to-one and for F global the correspondence is one-to-one, modulo local equivalence $\sim_{l.e.}$ (see Definition 8, below).

The aim of this article is to show that, when suitably topologizing the cohomology groups and the spaces of characters, Langlands's canonical map is continuous, open and that the space of unramified characters, X_T (see Definitions 15 and 16), is the identity component of the relevant space of characters. Our first main result is the following. Given our torus T defined over F, let

$$\mathsf{X}_{\mathbb{C}}(T, F) = \begin{cases} \mathrm{Hom}_c(T(F), \mathbb{C}^*) & \text{if } F \text{ local,} \\ \mathrm{Hom}_c(T(\mathbb{A}_F)/T(F), \mathbb{C}^*) & \text{if } F \text{ global.} \end{cases} \tag{3}$$

In both cases $\mathsf{X}_{\mathbb{C}}(T, F)$ is topologized with the compact-open topology, in which case it becomes a metrizable space [Ar, Section 8].

Theorem 1. *For F local or global, $X_T = \mathsf{X}_{\mathbb{C}}(T, F)^\circ$. Furthermore, the identity component $\mathsf{X}_{\mathbb{C}}(T, F)^\circ$ is open in $\mathsf{X}_{\mathbb{C}}(T, F)$.*

The proof of this theorem is given in Corollary 23, below. We now turn our attention to the cohomological side of Langlands's correspondence. Using a formalism of C. Moore [Mo1, Mo2, Mo3, Mo4] we consider the theory of measurable cohomology groups $\underline{H}^*(W_{E/F}, \widehat{T})$ (see Section 5), which provides useful interaction between topological and homological algebra features. We then show that $H^1_c(W_{E/F}, \widehat{T})$ has naturally the structure of a complete metric space (see Theorem 27) and prove the following result.

Theorem 2. *For F local or global, the identity component $H^1_c(W_{E/F}, \widehat{T})^\circ$ is an open subgroup of $H^1_c(W_{E/F}, \widehat{T})$. Furthermore, the canonical surjective homomorphism*

$$H^1_c(W_{E/F}, \widehat{T}) \to \mathsf{X}_{\mathbb{C}}(T, F)$$

established by Langlands is continuous, open and maps $H^1_c(W_{E/F}, \widehat{T})^\circ$ onto X_T.

We shall discuss Langlands's map in the next section (see Theorem 6) and provide arguments to prove Theorem 2 in Propositions 22, 31 and Theorem 37, below. We also prove the following result.

Theorem 3. *If F is a number field or an Archimedean local field, then X_T is topologically isomorphic to* $\mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}}$. *If F is a local non-Archimedean field then X_T is topologically isomorphic to $\widehat{T}_{\Gamma_{E/F}}$ and if F is a function field, then there are finite epimorphisms*

$$(\widehat{T}^{\Gamma_{E/F}})^\circ \to X_T \to \widehat{T}_{\Gamma_{E/F}}, \tag{4}$$

where $(\widehat{T}^{\Gamma_{E/F}})^\circ$ denotes the identity component of the $\Gamma_{E/F}$-invariants of $\widehat{T}$.

1.1. **Notations.** Throughout this paper, the symbol $\mathbb{T}$ will denote the unit circle in $\mathbb{C}$. Given two topological groups G, H we shall write $G \cong H$ to mean that G is (algebraically) isomorphic to H and $G \approx H$ to mean that G is topologically (and algebraically) isomorphic to H. Also, we shall denote by G° the identity component of the topological group G.

2. Langlands's Parametrization

We start by recalling Langlands's parametrization (see also the work of Labesse [**?**]). We retain the notations regarding the extension E/F as above. There is a well-known [Bo, 8.12] equivalence of categories

$$\left\{ \begin{array}{c} \text{Algebraic tori } T/F \\ \text{that split over } E. \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Lattices } X \text{ with} \\ \Gamma_{E/F}\text{-action.} \end{array} \right\}$$

which is given by sending $T \mapsto X = \mathrm{Hom}(T_E, \mathbb{G}_m)$. Under this correspondence, we have $T(E) = \mathrm{Hom}(X, E^*)$ and $T(F) = \mathrm{Hom}_{\Gamma_{E/F}}(X, E^*)$. Denote by $\widehat{X} := \mathrm{Hom}(X, \mathbb{Z})$ and $\widehat{T} := \mathrm{Hom}(\widehat{X}, \mathbb{C}^*)$. We define an action of the Weil group $W_{E/F}$ on $\widehat{T}$ by means of the map $p : W_{E/F} \to \Gamma_{E/F}$; in particular, C_E acts trivially.

We shall write ${}^{\mathsf{L}}T := \widehat{T} \rtimes \Gamma_{E/F}$ and we say that a homomorphism $\varphi : W_{E/F} \to {}^{\mathsf{L}}T$ is an *L-homomorphism* if $\mathrm{pr}_2 \varphi = p$. We recall [La] that the space of Langlands parameters of T, denoted $\Phi(T)$, is defined as the group of all continuous L-homomorphisms $\varphi : W_{E/F} \to {}^{\mathsf{L}}T$, modulo $\widehat{T}$ conjugation.

Definition 4. For a locally compact group $\mathcal{G}$ endowed with a continuous action on $\widehat{T}$, let $Z^1_c(\mathcal{G},\widehat{T})$ denote the (abelian) group of continuous 1-cocycles of $\mathcal{G}$ with values in $\widehat{T}$ and let $B^1_c(\mathcal{G},\widehat{T})$ denote the subgroup of continuous 1-coboundaries. We topologize these spaces with the compact-open topology, and we let the quotient $H^1_c(\mathcal{G},\widehat{T})$ denote the group of continuous 1-cohomology classes, endowed with the quotient topology.

As is well-known [La, p. 232] we have the following result.

Proposition 5 ([La])**.** *The space $\Phi(T)$ is isomorphic to $H^1_c(W_{E/F},\widehat{T})$, as abelian groups.*

Theorem 6 ([La])**.** *Let F be local or global. There are canonical homomorphisms*

$$H^1_c(W_{E/F},\widehat{T}) \xrightarrow{\Lambda} \mathrm{Hom}_c(\mathrm{Hom}_{\Gamma_{E/F}}(X,C_E),\mathbb{C}^*) \xrightarrow{\rho} \mathsf{X}_{\mathbb{C}}(T,F),$$

where Λ is an isomorphism and ρ surjective. If F is local, then ρ is also an isomorphism. If F is global, then the kernel of ρ is finite and consists of the cohomology classes which are locally trivial.

We now describe the locally trivial classes. For a local or global field F, let $(W_F,\varphi,\{r_E\})$ be the absolute Weil group of F (see [Ta2]). Here, the index E varies over all finite extensions of F inside a fixed separable closure $\bar{F}/F$, φ is a continuous map with dense image $W_F\to\Gamma_F$ (where Γ_F is the absolute Galois group), and $r_E : C_E \to W_E^{\mathrm{ab}}$ is the reciprocity isomorphism, where for each finite extension E/F we have $W_E=\varphi^{-1}(\Gamma_E)$.

When F is global, let $\mathrm{Pl}(F)$ denote the set of places of F. From the local-global relationship [Ta2, (1.6.1)], for each $v\in\mathrm{Pl}(F)$, the inclusion $i_v:\bar{F}\to\bar{F}_v$ induces a unique (up to inner automorphisms of W_F) continuous map $\theta_v : W_{F_v}\to W_F$ such that $\varphi\theta_v=i_v\varphi_v$, where we also denote by $i_v:\Gamma_{F_v}\to\Gamma_F$ the induced map between the absolute Galois groups of F_v and F. We have the following well-known result.

Proposition 7. *For F global, there is a bijection* $\mathrm{Inf} : H^1_c(W_{E/F},\widehat{T})\cong H^1_c(W_F,\widehat{T})$*. Furthermore, there is a natural map*

$$\prod_{v\in\mathrm{Pl}(F)}\mathrm{Res}_v : H^1_c(W_F,\widehat{T}) \to \prod_{v\in\mathrm{Pl}(F)} H^1_c(W_{F_v},\widehat{T})$$

to the product of the cohomology groups of each local absolute Weil group.

Definition 8. We say that a Langlands parameter $\varphi\in\Phi(T)\cong H^1_c(W_{E/F},\widehat{T})$ is *locally trivial* if its image in $\prod_{v\in\mathrm{Pl}(F)} H^1_c(W_{F_v},\widehat{T})$ is trivial.

3. Remarks on the compact-open topology

In this paper we will deal a lot with the compact-open topology on the space of continuous functions between two topological spaces, so we gather in this section some relevant information about this topology, for convenience of the reader.

Let A, B be topological spaces. Given any $K \subset A$ compact and $W \subset B$ open, we let $V(K,W) = \{f : A \to B \text{ continuous } \mid f(K) \subset W\}$. The compact-open topology on the space $\mathscr{C}(A,B)$ of continuous functions between A and B is the topology generated by the sub-basis $\{V(K,W) \mid K \subset A \text{ compact and } W \subset B \text{ open}\}$.

Furthermore, recall that a topological space A is called hemicompact if there is a sequence $K_1, K_2, \ldots$ of compact subsets of A such that $A = \cup_{n=1}^{\infty} K_n$ and any compact subset K of A is contained in a finite union $K_{n_1} \cup \cdots \cup K_{n_p}$ of compacts sets in that sequence. The multiplicative groups C_F (with F local or global) and finite extensions of them are examples of hemicompact topological spaces.

Proposition 9. *The compact-open topology on $\mathscr{C}(A,B)$ satisfy the following properties.*

(a) If B is a metric space, then a sequence $(f_n)_{n\in\mathbb{Z}_{\geq 0}}$ in $\mathscr{C}(A,B)$ converges in the compact-open topology to a continuous function f if, and only if, f_n converges uniformly to $f \in \mathscr{C}(A,B)$ on any compact subset K of A.

(b) If B is a metric space, then $\mathscr{C}(A,B)$ is metrizable if and only if A is hemicompact.

(c) If B is a metric space and A is hemicompact, then the compact-open topology is entirely determined by uniform convergence on compact sets.

(d) If A, B are topological groups with A hemicompact and B a metric space, then the space of continuous homomorphisms $\mathrm{Hom_c}(A,B)$ is a closed subspace of $\mathscr{C}(A,B)$.

(e) If A, B_1, B_2 are topological groups with A hemicompact and B_1, B_2 metric spaces, then

$$\mathrm{Hom}_c(A, B_1 \times B_2) \approx \mathrm{Hom}_c(A, B_1) \times \mathrm{Hom}_c(A, B_2).$$

(f) If A_1, A_2, B are topological groups with A_1, A_2 hemicompact and B an abelian group with a metric topology, then

$$\mathrm{Hom}_c(A_1 \times A_2, B) \approx \mathrm{Hom}_c(A_1, B) \times \mathrm{Hom}_c(A_2, B).$$

Proof. Items (a), (b) are proved in [Ar] and (c) is a consequence of (b). Item (d) follows, as the homomorphism property is a closed condition. For (e), arguing with sequences (or otherwise) it is straightforward to check that the topological isomorphism is realized by $\Phi : \mathrm{Hom}_c(A, B_1 \times B_2) \to \mathrm{Hom}_c(A, B_1) \times \mathrm{Hom}_c(A, B_2)$ defined by

$$\Phi(f) = (\pi_1 f, \pi_2 f),$$

(where π_1, π_2 denote the projection on the relevant factor) whose continuous inverse is given by $\Phi^{-1}(g,h)(a) = (g(a), h(a))$ for all $a \in A$. For (f) one checks that the topological isomorphism is realized by $\Psi : \mathrm{Hom}_c(A_1 \times A_2, B) \to \mathrm{Hom}_c(A_1, B) \times \mathrm{Hom}_c(A_2, B)$ with

$$\Psi(f) = (f\iota_1, f\iota_2)$$

(where ι_1 denote the inclusion $a \mapsto (a,e)$ and similarly for ι_2) whose continuous inverse is given by $\Psi^{-1}(g,h)(a_1,a_2) = g(a_1)h(a_2)$, for all $(a_1,a_2) \in A_1 \times A_2$. $\square$

4. Unramified characters

In this section we will recall the definitions of unramified characters of a torus and establish some facts about them. We start by recalling the relevant absolute values in the groups we are interested in.

4.1. **Absolute values.** Suppose first that F is a local field endowed with a normalized absolute value $|\cdot|_F : F \to \mathbb{R}_{\geq 0}$ with respect to which F is a complete topological field (and locally compact). When F is non-Archimedean, we let κ_F denote its residue field, $q_F = \#\kappa_F$ and let $v_F : F \to \mathbb{Z}$ denote its discrete valuation. If F is Archimedean (i.e. F is either $\mathbb{R}$ or $\mathbb{C}$) let $|\alpha|$ denote the usual absolute value in these fields. In each case, the normalized absolute value satisfies

$$|\alpha|_F = \begin{cases} |\alpha|, & \text{if } F = \mathbb{R}, \\ |\alpha|^2, & \text{if } F = \mathbb{C}, \\ q_F^{-v_F(\alpha)}, & \text{if } F \text{ is non-Archimedean.} \end{cases} \tag{5}$$

In the case when F is a global field with ring of adèles $\mathbb{A}_F$, for each $v \in \mathrm{Pl}(F)$ we let $|\cdot|_v$ be the normalized absolute value of the local completion F_v of F, as in (5). Here, any subscript referring to a local field F_v will be only denoted by v. If $\alpha = (a_v)_v \in \mathbb{A}_F$, let $|\alpha|_F = \prod_v |a_v|_v$ denote the adelic absolute value, which restricts to a homomorphism $|\cdot|_F : \mathbb{A}_F^* \to \mathbb{R}_+$ and induces a homomorphism $C_F \to \mathbb{R}_+$.

Definition 10. For F both local or global, the absolute value homomorphism[1] is denoted by $|\cdot|_F : C_F \to \mathbb{R}_+$. We define C_F^1 and V_F to be the kernel and the image of $|\cdot|_F$.

We now summarize the properties of these topological groups.

Proposition 11. *The following assertions hold true.*

(a) C_F is isomorphic to $V_F \times C_F^1$ both topologically and algebraically.
(b) C_F^1 is a compact subgroup of C_F.
(c) When F is a number field or a local Archimedean field, we have $V_F = \mathbb{R}_+$.
(d) When F is a global function field or a local non-Archimedean field, we have $V_F = q_F^{\mathbb{Z}}$, where q_F is the cardinality of the residue field in the local case and q_F is the cardinality of the field of constants in the global case.

[1]The notation $|\cdot|_F$ for the adelic absolute value in the global case is not usual, but it is convenient given the unified notation for the multiplicative groups C_F.

Proof. This is rather well-known. We refer to [Ta1, (1.4.6)] for the algebraic part of (a). In the global case, the continuous section splitting the sequence topologically is canonically defined in the number field case [We, Corollary 2, p. 75] and not canonical in the function field case. We refer to [RV, Theorem 5-15] for (b) and [RV, Theorem 5-14] for (c) and (d) (see also [NSW, Capter VII, §2]). □

Remark 12. We note that under the finite Galois extension E/F, the norm $|\cdot|_E$ is $\Gamma_{E/F}$-invariant. The (topological and algebraic) splitting $C_E = C_E^1 \times V_E$ of Proposition 11(a) is not, in general, a splitting as $\Gamma_{E/F}$-modules. For example, in the local non-Archimedean case, a $\Gamma_{E/F}$-splitting exists if and only if E/F is an unramified extension.

Finally, if F is a local or global, recall (see [Ta2]) that the absolute Weil group W_F comes equipped with an absolute value extending the normalized absolute value $|\cdot|_F$ on $C_F \cong W_F^{\mathrm{ab}} = W_F/W_F^{\mathrm{c}}$ (here, W_F^{c} means the closure of the subgroup generated by all commutators in W_F). Since for any extension E/F we have $W_E^{\mathrm{c}} \subseteq W_F^{\mathrm{c}}$, the absolute value on W_F descends to an absolute value on $W_{E/F} \cong W_F/W_E^{\mathrm{c}}$, denoted $|\cdot|_{E/F}$. We let $W_{E/F}^1$ denote the kernel. Clearly, the image of $|\cdot|_{E/F}$ is V_F, the same as the image of $|\cdot|_F$.

4.2. **Unramified characters of multiplicative groups.** Given a field F, local or global, we say F is of $\mathbb{Z}$- or $\mathbb{R}$-type if, respectively, we have $V_F \cong \mathbb{Z}$ or $V_F \cong \mathbb{R}$.

Definition 13. An *unramified character* of C_F is a continuous homomorphism $C_F \to \mathbb{C}^*$ which is trivial on C_F^1.

Proposition 14 ([Ta1])**.** *Any unramified character of C_F is of the type $\chi : \alpha \mapsto |\alpha|_F^s$ for some $s \in \mathbb{C}$. If F is of $\mathbb{R}$-type, then s is uniquely defined from χ. If F is of $\mathbb{Z}$-type, s is uniquely defined modulo $2\pi i/\log(q_F)$.*

We now recall how to extend this notion of unramified characters of the multiplicative groups C_F to any torus.

4.3. **Unramifed characters: local setting.** In this section, F is a local field and T is a torus defined over F. If $X = \mathrm{Hom}(T_E, \mathbb{G}_m)$ is the character lattice of T and E/F is a Galois extension splitting T, recall (e.g., from [Bo, Section 8.11]) that $X^{\Gamma_{E/F}}$ is the lattice of rational characters of T. Note that each $\chi \in X^{\Gamma_{E/F}}$ defines a continuous homomorphism $|\chi|_F : T(F) \to V_F \subseteq \mathbb{R}_+$ via $|\chi|_F : t \mapsto |\chi(t)|_F$. Let $T(F)^1 = \cap_\chi \ker(|\chi|_F)$, with $\chi \in X^{\Gamma_{E/F}}$.

Definition 15. A continuous homomorphism $T(F) \to \mathbb{C}^*$ is called an *unramified character* if it is trivial on $T(F)^1$. Denote by $X_{T,F}$ the group of all unramified characters of T. We will omit the reference to F and write only X_T if there is no risk of confusion.

The group $T(F)^1$ plays a crucial role in this theory, and it is important to further characterize it. As we will see below, this group is the maximal compact subgroup of $T(F)$. However, we will delay its study until after we describe the space of complex characters of the group $\mathrm{Hom}_{\Gamma_{E/F}}(X, C_E)$, as this latter group can be treated in a more unified manner for both local and global fields.

4.4. **Unramifed characters: global setting.** In the global case, given a torus T defined over a global field F and splitting over a finite Galois extension E/F, any element $\chi \in X^{\Gamma_{E/F}}$ defines, for each $v \in \mathrm{Pl}(F)$, an algebraic homomorphism $\chi_v : T_v \to F_v^*, t_v \mapsto \chi_v(t_v)$ and a continuous homomorphism $|\chi|_F : T(\mathbb{A}_F) \to V_F \subseteq \mathbb{R}_+$ defined by

$$(t_v)_v \mapsto \prod_v |\chi_v(t_v)|_v.$$

Definition 16. Given T defined over F and splitting over E/F, let $T(\mathbb{A}_F)^1 = \cap_\chi \ker |\chi|_F$ with χ running over $X^{\Gamma_{E/F}}$. An *unramified character* of T is a continuous homomorphism $T(\mathbb{A}_F)/T(F) \to \mathbb{C}^*$ trivial on $T(\mathbb{A}_F)^1$. The space of all such characters is denoted $X_{T,F}$. As before, we shall omit the reference on the field F and write only X_T.

We shall characterize X_T. Following [MW], given the lattice of rational characters $X^{\Gamma_{E/F}}$ of T, consider the real vector spaces $\mathfrak{a}_0^* = \mathbb{R} \otimes X^{\Gamma_{E/F}}$ and $\mathfrak{a}_0 = \mathrm{Hom}(X^{\Gamma_{E/F}}, \mathbb{R})$. Define a homomorphism of abelian groups $\log_T : T(\mathbb{A}_F) \to \mathfrak{a}_0$ characterized by

$$\langle \chi, \log_T(t) \rangle = \log_q(|\chi|_F(t)), \tag{6}$$

for $\chi \in X^{\Gamma_{E/F}}$ and $t \in T(\mathbb{A}_F)$. Here, $q = e$ if F is a number field and, in the function field case, $q = q_F$ is the base of the valuation group V_F. In both cases, we have $\ker(\log_T) = T(\mathbb{A}_F)^1$. Let $L_T := \log_T(T(\mathbb{A}_F))$. Note that $\log_T$ is a continuous surjection from $T(\mathbb{A}_F) \to L_T$, which induces a topological isomorphism between L_T and $T(\mathbb{A}_F)/T(\mathbb{A}_F)^1$. In the number field case we have $L_T = \mathfrak{a}_0$ while in the function field case $L_T \subset \mathfrak{a}_0$ is a sublattice of $\mathrm{Hom}(X^{\Gamma_{E/F}}, \mathbb{Z})$ of finite index. We have a topological identification $X_T \approx \mathrm{Hom}_c(L_T, \mathbb{C}^*)$.

Proposition 17. *There is a continuous and open surjection* $\mathfrak{e} : \mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}} \to X_T$ *which is an isomorphism in the number field case and its kernel is* $\widehat{L}_T = \mathrm{Hom}(L_T, \mathbb{Z})$ *in the function field case.*

Proof. If F is a number field, using $L_T = \mathfrak{a}_0 \cong \mathbb{R} \otimes \mathrm{Hom}(X^{\Gamma_{E/F}}, \mathbb{Z})$ and the tensor-hom adjunction we obtain

$$X_T \cong \mathrm{Hom}_c(L_T, \mathbb{C}^*) \cong \mathrm{Hom}_c(\mathbb{R}, \mathrm{Hom}(\mathrm{Hom}(X^{\Gamma_{E/F}}, \mathbb{Z}), \mathbb{C}^*)) \cong \mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}},$$

where we used that any continuous homomorphism between finite dimensional Lie groups is necessarily smooth. Furthermore, $\mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}} \cong \mathfrak{a}_{\mathbb{C}}^*$. In the function field case, since $L_T \subseteq \mathrm{Hom}(X^{\Gamma_{E/F}}, \mathbb{Z})$ is of finite index, we have $\mathrm{Hom}(L_T, \mathbb{C}) \cong \mathbb{C} \otimes \widehat{L}_T \cong \mathfrak{a}_{\mathbb{C}}^*$. As L_T is free (and discrete), applying $\mathrm{Hom}(L_T, -)$ to the exponential exact sequence yields $0 \to \widehat{L}_T \to \mathfrak{a}_{\mathbb{C}}^* \to X_T \to 1$, finishing the proof. □

Remark 18. When F is local non-Archimedean, we can similarly define $\log_T : T(F) \to \mathfrak{a}_0$ as in (6) and there is a similar sequence $0 \to \widehat{L}_T \to \mathfrak{a}_{\mathbb{C}}^* \to X_T \to 1$ characterizing X_T, with $L_T = \log_T(T(F))$ homeomorphic to $T(F)/T(F)^1$.

4.5. **Compactness and identity components.** In this section, we shall study some topological features of the space $\mathrm{Hom}_c(\mathrm{Hom}_{\Gamma_{E/F}}(X, C_E), \mathbb{C}^*)$. In light of Langlands's result, Theorem 6, the information extracted in this section will provide a proof of Theorem 1. To ease notations, let us make the convention

$$T_{C_F} := \mathrm{Hom}_{\Gamma_{E/F}}(X, C_E). \tag{7}$$

We assume here that the field F is either local or global with finite Galois extension E/F splitting T. The group T_{C_F} is a locally compact topological group with topology determined by that of C_E. Furthermore, if $\chi \in X^{\Gamma_{E/F}}$ is a rational character, then for any $t \in T_{C_F}$, we have $\chi(t) \in C_F$ and hence $|\chi|_F = |\cdot|_F \circ \chi$ is a continuous character of T_{C_F}. Similarly to the definitions in the local and global cases, let us write

$$T_{C_F}^1 = \cap_{\chi \in X^{\Gamma_{E/F}}} \ker(|\chi|_F). \tag{8}$$

Proposition 19. *For F local or global, we have that $T_{C_F}^1$ is the maximal compact subgroup of T_{C_F}.*

Proof. We claim that $T_{C_F}^1 = \mathrm{Hom}_{\Gamma_{E/F}}(X, C_E^1)$. Indeed, note that, by (8), we have $\mathrm{Hom}_{\Gamma_{E/F}}(X, C_E^1) \subseteq T_{C_F}^1$. Conversely, let $t \notin \mathrm{Hom}_{\Gamma_{E/F}}(X, C_E^1)$, that is, $t \in T_{C_F}$ and there exists $x \in X$ with $|x(t)|_E > 1$. We show that $t \notin T_{C_F}^1$. Indeed, for $x \in X$ as above, let

$$\tilde{x} = \sum_{\gamma \in \Gamma_{E/F}} \gamma(x) \in X_{E/F}^{\Gamma}$$

and note that $\tilde{x}(t) = \prod_{\gamma \in \Gamma_{E/F}} \gamma(x(t)) = N_{E/F}(x(t)) \in C_F$, where $N_{E/F}$ is the norm map. It follows that $|\tilde{x}(t)|_F = |N_{E/F}(x(t))|_F = |x(t)|_E > 1$, from which we conclude that $t \notin T_{C_F}^1$. As C_E^1 is compact, then $T_{C_F}^1$ is compact. Now, $T_{C_F}^1$ contains every compact subgroup of T_{C_F}, since $\mathbb{R}_+$ does not admit any non-trivial compact subgroups. This finishes the proof. □

Corollary 20. *If F is a global field, then $T(\mathbb{A}_F)^1/T(F)$ is compact. If F is a local field, then $T(F)^1$ is the maximal compact subgroup of $T(F)$.*

Proof. In the global case we have $T(\mathbb{A}_F)/T(\mathbb{A}_F)^1 = T^1_{C_F} \cap (T(\mathbb{A}_F)/T(F))$, and since $T^1_{C_F}$ is compact and $T(\mathbb{A}_F)/T(F) \subset T_{C_F}$ is closed, then $T^1(A_F)/T(F)$ is compact. If F local, $T_{C_F} = T(F)$ so the claim follows from Proposition 19. □

Now let us consider the continuous restriction map $R : \mathrm{Hom}_c(T_{C_F}, \mathbb{C}^*) \to \mathrm{Hom}_c(T^1_{C_F}, \mathbb{C}^*)$. Its kernel consists of those characters that are trivial on $T^1_{C_F}$ and is naturally identified with $\mathrm{Hom}_c(T_{C_F}/T^1_{C_F}, \mathbb{C}^*)$. Let us write $\ker(R) = \widetilde{X}_T$.

Proposition 21. *Let F be a local or a global field. Then $\widetilde{X}_T = \mathrm{Hom}_c(T_{C_F}, \mathbb{C}^*)^\circ$. Furthermore, we have that $\widetilde{X}_T$ is an open subgroup of $\mathrm{Hom}_c(T_{C_F}, \mathbb{C}^*)$.*

Proof. As $T^1_{C_F}$ is compact, we have $\mathrm{Hom}_c(T^1_{C_F}, \mathbb{C}^*) = \mathrm{Hom}_c(T^1_{C_F}, \mathbb{T})$ is discrete, as it is well-known that the Pontryagin dual of a compact topological group is discrete [HR, (23.17) Theorem]. Hence, the kernel of R, $\widetilde{X}_T$, is a subgroup which is open and closed, and then must be a connected component, provided it is connected. From [Ma], it suffices to show that $T_{C_F}/T^1_{C_F}$ is torsion-free, which we now show. From the split exact sequence $1 \to C^1_E \to C_E \to V_E \to 1$, we obtain the exact sequence

$$1 \to T^1_{C_F} \to T_{C_F} \to \mathrm{Hom}_{\Gamma_{E/F}}(X, V_E) \to H^1(\Gamma_{E/F}, \mathrm{Hom}(X, C^1_E)).$$

But $\mathrm{Hom}_{\Gamma_{E/F}}(X, V_E) = \mathrm{Hom}((X_{\Gamma_{E/F}})/(X_{\Gamma_{E/F}})_{\mathrm{tor}}, V_E)$ since V_E is trivial for the $\Gamma_{E/F}$-action and torsion-free. It follows that $\mathrm{Hom}_{\Gamma_{E/F}}(X, V_E)$ is torsion-free for both the $\mathbb{Z}$- and the $\mathbb{R}$-cases. Since the quotient $T_{C_F}/T^1_{C_F}$ maps isomorphically to a submodule (the kernel of the last arrow) of a torsion-free space, it is torsion-free. This finishes the proof. □

Proposition 22. *Let F be a global field. Then, the restriction map*

$$\rho : \mathrm{Hom}_c(T_{C_F}, \mathbb{C}^*) \to \mathrm{Hom}_c(T(\mathbb{A}_F)/T(F), \mathbb{C}^*)$$

is continuous, open, and closed, if both spaces are endowed with the compact-open topology.

Proof. When F is global, we have that $T(\mathbb{A}_F)/T(F) \subseteq T_{C_F}$ is a closed inclusion of finite index [La, p. 245]. Hence, $T(\mathbb{A}_F)/T(F)$ is also an open subgroup of T_{C_F} and thus ρ is a continuous and open surjection (see [HR, (24.5) Theorem] for the $\mathbb{T}$-valued case; their proof that this homomorphism is also open can be adapted to the $\mathbb{C}^*$-valued case). As $\ker(\rho)$ is finite, this map is also closed. □

Corollary 23. *If F is a local field, then $X_T = \mathrm{Hom}(T(F), \mathbb{C}^*)^\circ$. If F is a global field, then $X_T = \mathrm{Hom}(T(\mathbb{A}_F)/T(F), \mathbb{C}^*)^\circ$. In both cases, X_T is an open subgroup.*

Proof. If F local, we have $T_{C_F} = T(F)$, and hence $\widetilde{X}_T = X_T$ in this case. If F is global, from Propositions 21 and 22, it follows that

$$X_T = \rho(\widetilde{X}_T) = \mathrm{Hom}_c(T(\mathbb{A}_F)/T(F), \mathbb{C}^*)^\circ,$$

since the closure of the image of the identity component under a continuous and open homomorphism is the identity component of the target space [HR, (7.12) Theorem]. This finishes the proof. □

With the description of $T(F)^1$ at hand, we can be more precise in the description of $T(F)$, in the local case. Given a subtorus S of T we let $X(S)$ denote its character lattice.

Proposition 24. *Let F be a local field. The following assertions hold true:*

(a) *The quotient $T(F)/T(F)^1$ is isomorphic to $S(F)/S(F)^1$, where $S \subset T$ is the largest F-split subtorus of T.*
(b) *If F is a local Archimedean field, then $X_T \approx \mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}}$. If F is a local non-Archimedean field, then $X_T \approx \widehat{T}_{\Gamma_{E/F}}$.*

Proof. Recall [Bo, Proposition 8.15] that $T(F) = T_a(F) \cdot S(F)$ is the almost product of the largest anisotropic and largest split subtori defined over F. Furthermore, in the local case, anisotropic tori are compact (see [Hu, Section 35.3] for the Archimedean case and [Pr] for the non-Archimedean case). So, from Corollary 20, we have $T_a(F) \subset T(F)^1$. Thus, the composition $S(F) \to T(F) \to T(F)/T(F)^1$ is a surjective homomorphism, whose kernel is $S(F) \cap T(F)^1$. It follows that

$$\frac{T(F)}{T(F)^1} \cong \frac{S(F)}{S(F) \cap T(F)^1}.$$

On the other hand, from the finite-index injection $X^{\Gamma_{E/F}} \to X(S)$ (see [MW, p. 6]), it follows that $S(F)^1 \subseteq S(F) \cap T(F)^1$, a closed injection with finite-index. Hence, there is a natural surjective, finite-to-one homomorphism

$$\frac{S(F)}{S(F)^1} \to \frac{S(F)}{S(F) \cap T(F)^1}.$$

But the quotient $S(F)/S(F)^1$ does not admit finite subgroups, since it is either a lattice (in the non-Archimedean case) or a real vector space (in the Archimedean case), from which we conclude these two quotients are isomorphic, proving (a).

As for item (b), it follows from (a) that $T(F)/T(F)^1 = \mathrm{Hom}(X^{\Gamma_{E/F}}, V_F)$. If F is local non-Archimedean, then $X_T \approx \widehat{T}_{\Gamma_{E/F}}$ and if F is Archimedean, then $X_T \approx \mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}}$. □

5. Cohomological side

In order to discuss the continuity of the Langlands's map, we obviously need to topologize the cohomology groups first. We shall use the theory developed by C. Moore in the series of papers [Mo1, Mo2, Mo3, Mo4]. In fact, we shall closely follow the axiomatic treatment of [Mo3, Definition, p. 16]. See also [AM] and [Ra, Section 3] for other accounts on this theory.

5.1. **Measurable cohomology theory.** We briefly recall Moore's construction of his cohomology theory in the next paragraph, for any locally compact group G and polish G-modules A (i.e., a Hausdorff, second countable, abelian topological group whose topology admits a separable complete metric [Mo3, Proposition 1] endowed with a continuous G-action). We are interested in A a lattice, a vector space or a complex torus, so this theory is well-suited.

First, given any σ-finite measurable space $(X, \mathcal{B}, \mu)$ with $(X, \mathcal{B})$ countably generated and A a separable metric space, chose a finite measure ν equivalent to μ and metric ρ on A of finite diameter. Define $I(X, A)$ as the set of equivalence classes of ν-measurable functions $X \to A$ under the almost-everywhere-equality equivalence relation and equipped with the metric

$$\bar{\rho}(\varphi, \psi) = \int_X \rho(\varphi(x), \psi(x)) d\nu(x). \tag{9}$$

The topology on $I(X, A)$ defined by $\bar{\rho}$ depends only on the measure class of ν and the topology of A (see [Mo3, Corollary to Proposition 6]) and not on the particular choices made. Now, let G be a locally compact group with Haar measure μ and A a polish G-module as above. Define the chain complex $(\underline{C}^*(G, A), \underline{\delta}^*)$ where $\underline{C}^n(G, A) = I(G^n, A)$ endowed with the metric $\bar{\rho}$ of (9) and $\underline{\delta}^n$ is the usual algebraic coboundary operator restricted to elements of $\underline{C}^n(G, A)$. The subgroups $\underline{Z}^n(G, A), \underline{B}^n(G, A)$ are given the induced topology. We then let $\underline{H}^n(G, A)$ denote the cohomology groups of the chain complex $(\underline{C}^*(G, A), \underline{\delta}^*)$, topologized with the quotient topology. The following proposition summarizes what we need from this construction. All proofs are found in [Mo3] and [AM].

Proposition 25. *The following assertions hold true:*

1. *The coboundary operators $\underline{\delta}^n$ are continuous for all n.*
2. *If $\underline{B}^n(G, A)$ is a closed subgroup of $\underline{Z}^n(G, A)$, then $\underline{H}^n(G, A)$ is a polish G-module.*
3. *For any locally compact G and polish A we have $\underline{H}^1(G, A) \cong H^1_c(G, A)$ as abelian groups which is a homeomorphism if $H^1_c(G, A)$ is endowed with the quotient topology and $Z^1_c(G, A), B^1_c(G, A)$ are given the compact-open topology.*

(4) *For any short exact sequence* $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ *of polish* G*-modules with* i *a homeomorphism onto its image and* p *continuous and open, all morphisms in the long exact sequence induced by* $\underline{H}^*(G, -)$ *are continuous.*

(5) *Given any* $f : G \to G'$ *continuous homomorphism between locally compact groups,* A, A' *polish* G*- and* G'*-modules and* $\varphi : A' \to A$ *a continuous homomorphism of abelian groups satisfying*

$$\varphi(f(g) \cdot a') = g \cdot \varphi(a'),$$

then the induced morphism $\underline{H}^n(G, A) \to \underline{H}^n(G', A')$ *is continuous.*

(6) *Suppose that the quotient* G/G° *is compact. If* A *is discrete, then* $\underline{H}^p(G, A)$ *is countable and discrete in its quotient topology, for all* $p > 0$*. If* A *is a Euclidean space, then* $\underline{H}^p(G, A)$ *is a Euclidean space in its quotient topology, for all* $p > 0$*.*

(7) *If* G *is compact and* A *is a Euclidean space, then* $\underline{H}^p(G, A) = 0$*, for all* $p > 0$*.*

Proof. Item (1) is [Mo3, Proposition 20]. For item (2), see the discussion on [Mo3, p. 10]. Item (3) is [Mo3, Theorem 3, Corollary 1, Theorem 7]. Item (4) is [Mo3, Proposition 25] and item (5) is, with minor modifications, [Mo3, Proposition 27]. Assertion (6) is in [AM, Theorem D] and the last one in [AM, Theorem A]. □

Note that item (5) of the previous proposition implies that in all homomorphisms between the cohomology groups in the spectral sequence (so in particular in the inflation-restriction exact sequence) are continuous. We shall make extensive use of this property.

5.2. **Low-dimensional measurable cohomology groups.** The biggest issue when dealing with the theory of measurable cohomology groups as described by Moore is the statement (2) in Proposition 25: In general, the groups $\underline{H}^r(W_{E/F}, A)$ may not be Hausdorff. There are several partial results to describe when these groups are Hausdorff, and for a discussion on this topic, we refer to [AM, Section 1.4].

Our first task will be to prove that in our context, the cohomology groups $H^1_c(W_{E/F}, A)$ are complete metric spaces when we take A to be $\widehat{T}$, $\mathrm{Lie}(\widehat{T})$ or X. We note that in some possibilities for $W_{E/F}$, this is a consequence of [AM, Theorem D], but we will give an alternative proof that works for all possible cases of $W_{E/F}$. Note also that when T is split over F, the relevant first group of continuous cohomology becomes the space of all continuous functions from C_F to A, which is naturally a metrizable space, when endowed with the compact-open topology. Before we continue, we need a preliminary result.

Lemma 26. *Let* Γ *be a finite group acting linearly on a finite dimensional inner product space* U *with* $U^\Gamma = 0$*. Suppose that* (u_m) *is a sequence on* U *such that* $(\gamma(u_m) - u_m)$ *converges for all* $\gamma \in \Gamma$*. Then,* (u_m) *converges.*

Proof. Since $U^\Gamma = 0$ and $U \cong U^*$, we get $U_\Gamma = 0$ so that $U = \sum_{\gamma\in\Gamma}(\gamma - \mathrm{id})(U)$. Thus, any element $u \in U$ can be written as $u = \sum_\gamma(\gamma^{-1} - \mathrm{id})(u_\gamma)$ for suitable elements $u_\gamma \in U$. Now let $(-,-)$ be a Γ-invariant inner product on U. Then,

$$(u_m, u) = \sum_\gamma (u_m, (\gamma^{-1} - \mathrm{id})(u_\gamma)) = \sum_\gamma (\gamma(u_m) - u_m, u_\gamma)$$

which, from the assumptions, imply (u_m, u) converges and so does (u_m). □

Theorem 27. *For F local or global, we have that $H^1_c(W_{E/F}, \widehat{T})$ is a Polish G-module.*

Proof. From (2) and (3) of Proposition 25, it suffices to show that $\underline{B}^1(W_{E/F}, \widehat{T})$ is closed in $\underline{C}^1(W_{E/F}, \widehat{T})$. To that end, let $(b_m)_{m\in\mathbb{N}}$ be a sequence in $\underline{B}^1(W_{E/F}, \widehat{T})$ converging to some $f \in \underline{C}^1(W_{E/F}, \widehat{T})$. We shall show that $f \in \underline{B}^1(W_{E/F}, \widehat{T})$. Associated to the sequence (b_m) is a sequence (t_m) in $\widehat{T}$ such that $b_m(w) = w(t_m)t_m^{-1}$ for each $w \in W_{E/F}$. Furthermore, for each $m \in \mathbb{N}$ we can choose $\nu_m \in \mathrm{Lie}(\widehat{T})$ such that $\exp(\nu_m) = t_m$. Then,

$$b_m(w) = w(\exp(\nu_m))\exp(-\nu_m) = \exp(w(\nu_m) - \nu_m),$$

since the action of $W_{E/F}$ commutes with the exponential map. We will show that we can modify the sequence (ν_m) without changing (b_m) in such a way that (ν_m) becomes convergent, and hence the limiting cocycle is in fact a coboundary.

From [Mo3, Proposition 6], there exists a measure 0 set $S \subseteq W_{E/F}$ outside which $b_m(w) \to f(w)$ pointwise. Hence, we can choose representatives $\{\gamma\} \subseteq W_{E/F} \setminus S$ for the quotient $W_{E/F}/C_E \cong \Gamma_{E/F}$ such that, for each γ, we have

$$b_m(\gamma) = \exp(\gamma(\nu_m) - \nu_m) \to f(\gamma).$$

Write $\nu_m = \lambda_m + i\mu_m$, where λ_m and μ_m are in the real form $\mathrm{Lie}(\widehat{T})_0 = X \otimes \mathbb{R}$ of $\mathrm{Lie}(\widehat{T}) = X \otimes \mathbb{C}$. First, note that each $\nu_m \in \mathrm{Lie}(\widehat{T})$ is determined up to translations by $(2\pi i)X$. So, we can and will assume that the imaginary parts (μ_m) lie in a bounded region of $\mathrm{Lie}(\widehat{T})_0$. Passing to a subsequence, if needed, we might as well assume (μ_m) is convergent. Then, since $\lambda_m = \nu_m - i\mu_m$, we get

$$\exp(\gamma(\lambda_m) - \lambda_m) = b_m(\gamma)\exp(\gamma(i\mu_m) - i\mu_m)^{-1}$$

and hence $\exp(\gamma(\lambda_m) - \lambda_m)$ converges, since both factors in the right-hand side converge. Using that the exponential map restricted to $\mathrm{Lie}(\widehat{T})_0$ is a homeomorphism onto its image, it then follows that $(\gamma(\lambda_m) - \lambda_m)$ is convergent. Now, if $\nu \in \mathrm{Lie}(\widehat{T})_0^{\Gamma_{E/F}}$, then

$$(\gamma(\lambda_m + \nu) - (\lambda_m + \nu)) = (\gamma(\lambda_m) - \lambda_m),$$

so we can further assume that (λ_m) is a sequence on U, where U is a $\Gamma_{E/F}$-stable complement of $\mathrm{Lie}(\widehat{T})_0^{\Gamma_{E/F}}$. We can now use Lemma 26 to conclude that (λ_m) also converges and, hence, $\lim_m \nu_m = \nu$ for some $\nu \in \mathrm{Lie}(\widehat{T})$. Setting $t = \exp(\nu)$ we obtain that $f(\gamma) = \gamma(t)t^{-1}$ for all γ. Hence, as $W_{E/F}$ can be written as the disjoint union $W_{E/F} = \cup_\gamma \gamma C_E$ and C_E acts trivially on $\widehat{T}$, we get $b_m(w) \to f(w) = w(t)t^{-1}$ for all $w \notin S$, i.e., $f \in \underline{B}^1(W_{E/F}, \widehat{T})$. □

In fact, the proof of Theorem 27 yield similar consequences for X and $\mathrm{Lie}(\widehat{T})$.

Corollary 28. *For F local or global, both $H_c^1(W_{E/F}, X)$ and $H_c^1(W_{E/F}, \mathrm{Lie}(\widehat{T}))$ are Polish G-modules.*

Proof. From Proposition 25(2) and (3), it suffices to show that the respective measurable coboundary group is closed. Let $A = X$ or $A = \mathrm{Lie}(\widehat{T})$. Note that a similar statement as in Lemma 26 holds true if we replace a finite dimensional inner product space by a lattice; the proof would be to embed such lattice in an inner product space as in that lemma. So, if (b_m) is a sequence in $\underline{B}^1(W_{E/F}, A)$ that converges to $f \in \underline{C}^1(W_{E/F}, A)$, then there is an associated sequence (a_m) in A such that $b_m(w) = w(a_m) - a_m$. Just like in the poof of Theorem 27, we can assume that (a_m) is in a $\Gamma_{E/F}$-stable complement of $A^{\Gamma_{E/F}}$ and Lemma 26 (or its lattice version) would imply that (a_m) is convergent and hence $f \in \underline{B}^1(W_{E/F}, \widehat{T})$. □

Proposition 29. *Let F be local or global. Then, $\underline{H}^1(W_{E/F}, X)$ and $\underline{H}^2(W_{E/F}, X)$ are countable and, as topological spaces, discrete.*

Proof. We know from Corollary 28 that $\underline{H}^1(W_{E/F}, X)$ is a complete metric space, so for this group, it suffices to show that this cohomology group is countable. However, using the arguments in the proof of Proposition 7.2 in [AM], it turns out to be a general fact that any countable quotient B/C of Polish spaces with C an analytic subset is necessarily discrete as a topological space, and hence Hausdorff. But, in the $\mathbb{R}$-case, note that C_E is almost connected in the sense that C_E/C_E° is compact; it is profinite in the global case [NSW, Chapter VIII, §2] and trivial in the local case. Hence, $W_{E/F}$ is almost connected, and it follows from [Mo2, Proposition 1.3] that $\underline{H}^r(W_{E/F}, X)$ is countable for $r = 1, 2$. For the $\mathbb{Z}$-case, we use the computations done in [Ra, Proposition] to deduce that $\underline{H}^r(W_{E/F}, X)$ is countable, for $r = 1, 2$. Indeed, first recall that as $\mathbb{Z}$ is free and discrete, then the measurable, continuous and abstract cohomology theories agree (see table in [AM, p. 913]). We thus have $\underline{H}^r(\mathbb{Z}, A) = 0$ for any $\mathbb{Z}$-module A and for all $r > 1$ [Wb, Remark, p.170]. So, from the exact sequence $1 \to W_F^1 \to W_F \to \mathbb{Z} \to 0$, where W_F^1 is compact (and profinite) and W_F is the absolute Weil group of F, we obtain from the inflation-restriction exact sequence that

$$1 \to \underline{H}^1(\mathbb{Z}, X^{W_F^1}) \to \underline{H}^1(W_F, X) \to \underline{H}^1(W_F^1, X)^{\mathrm{Fr}} \to 0.$$

For any W_F-module A, we have $\underline{H}^1(\mathbb{Z}, A) = A/(\mathrm{Fr} - 1)A$ with Fr the image of $1 \in \mathbb{Z}$ in $\mathrm{Aut}(A)$, so $\underline{H}^1(\mathbb{Z}, X^{W_F^1})$ is countable. As W_F^1 is profinite, it is almost connected, so we also have that $\underline{H}^r(W_F^1, X)^{\mathrm{Fr}}$ is countable for $r = 1, 2$ [Mo2, Proposition 1.3], from which $\underline{H}^1(W_F, X)$ is countable. But using again that $\underline{H}^r(\mathbb{Z}, A) = 0$ for $r > 1$, we get the higher inflation-restriction sequence just as in the proof of [Ra, Proposition 6] for $n = 2$:

$$1 \to \underline{H}^1(\mathbb{Z}, \underline{H}^1(W_F^1, X)) \to \underline{H}^2(W_F, X) \to \underline{H}^2(W_F^1, X)^{\mathrm{Fr}} \to 0.$$

We conclude, just as for $\underline{H}^1(W_F, X)$, that $\underline{H}^2(W_F, X)$ is countable. From the extension $1 \to W_E^c \to W_F \to W_{E/F} \to 1$ the inflation-restriction exact sequence yields

$$0 \to \underline{H}^1(W_{E/F}, X) \to \underline{H}^1(W_F, X) \to$$
$$\to \underline{H}^1(W_E^c, X)^{W_{E/F}} \to \underline{H}^2(W_{E/F}, X) \to \underline{H}^2(W_F, X).$$

Since W_E acts trivially on X then $\underline{H}^1(W_E^c, X) \cong \mathrm{Hom}_c(W_E^c, X) = 0$ and we conclude that $\underline{H}^r(W_{E/F}, X)$ for $r = 1, 2$ is also countable, finishing the proof. □

Proposition 30. *For F a global or a local field, we have $\underline{H}^1(W_{E/F}, \mathrm{Lie}(\widehat{T})) \approx \mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}}$.*

Proof. Consider the exact sequence $1 \to W_{E/F}^1 \to W_{E/F} \to V_F \to 1$. As $W_{E/F}^1$ is compact and $\mathrm{Lie}(\widehat{T})$ is a Euclidean space, we obtain from [AM, Theorem A] that $\underline{H}^1(W_{E/F}^1, \mathrm{Lie}(\widehat{T})) = 0$. Hence in the $\mathbb{R}$-cases, from the inflation-restriction exact sequence, and using that V_F acts trivially on $\mathrm{Lie}(\widehat{T})^{W_{E/F}^1} = \mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}}$, we obtain a continuous isomorphism of abelian groups $\mathrm{Hom}_c(V_F, \mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}}) \cong \underline{H}^1(W_{E/F}, \mathrm{Lie}(\widehat{T}))$ which is readily seen to be a linear map, hence a homeomorphism, as both spaces are complete metric spaces. In the $\mathbb{Z}$-cases, given that we are working with a linear action of V_F on a finite dimensional vector space, $\underline{H}^1(V_F, \mathrm{Lie}(\widehat{T})^{W_{E/F}^1})$ is naturally homeomorphic to the V_F-coinvariants in $\mathrm{Lie}(\widehat{T})^{W_{E/F}^1}$, hence to the space of invariants $\mathrm{Lie}(\widehat{T})^{W_{E/F}}$. □

5.3. **Long exact sequence in cohomology.** We now apply our knowledge of the low dimensional cohomology groups $\underline{H}^r(W_{E/F}, A)$ with $A = X, A = \mathrm{Lie}(\widehat{T})$ or $A = \widehat{T}$ to some natural constructions in homological algebra. Since $X = \mathrm{Hom}(T_E, \mathbb{G}_m)$ is a free abelian group, applying $\mathrm{Hom}(\widehat{X}, -)$ to the injective resolution $0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 1$ of the trivial module yields the short exact sequence

$$0 \to X \to \mathrm{Lie}(\widehat{T}) \to \widehat{T} \to 1. \tag{10}$$

Using Proposition 25(4), the discreteness of $\underline{H}^2(W_{E/F}, X)$ in Proposition 29 and the homeomorphism of Proposition 30, we get the long exact sequence

$$\begin{aligned} 1 \to X^{\Gamma_{E/F}} \to \mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}} \to \widehat{T}^{\Gamma_{E/F}} \to \\ \to \underline{H}^1(W_{E/F}, X) \to \mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}} \to \underline{H}^1(W_{E/F}, \widehat{T})^\circ \to 1, \end{aligned} \tag{11}$$

where all the arrows are continuous. The first conclusion that we draw from this is the following.

Proposition 31. *If F is a local or a global field, then $H^1_c(W_{E/F}, \widehat{T})^\circ$ is an open subgroup of $H^1_c(W_{E/F}, \widehat{T})$.*

Proof. We use Proposition 25(3) and that $\underline{H}^1(W_{E/F}, \widehat{T})^\circ$ is the inverse image of (the open set) $\{0\} \in \underline{H}^2(W_{E/F}, X)$. □

Theorem 32. *Suppose that $V_F = \mathbb{R}_+$. Then $H^1_c(W_{E/F}, \widehat{T})^\circ \approx \mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}}$.*

Proof. We know that when $V_F = \mathbb{R}_+$, the Weil group $W_{E/F}$ is almost connected. Since X is a discrete $W_{E/F}$-module, we obtain that $\underline{H}^1(W_{E/F}, X)$ is torsion [Mo2, p. 67]. It follows that $\underline{H}^1(W_{E/F}, X) \to \underline{H}^1(W_{E/F}, \mathrm{Lie}(\widehat{T}))$ is the zero map, as $\underline{H}^1(W_{E/F}, \mathrm{Lie}(\widehat{T}))$ is a vector space (see Proposition 30) and hence $\underline{H}^1(W_{E/F}, \mathrm{Lie}(\widehat{T})) \cong \underline{H}^1(W_{E/F}, \widehat{T})^\circ$ is a continuous isomorphism of abelian groups, hence a homeomorphism, as both groups are Euclidean spaces. □

We now turn our attention to the cases when $V_F \cong \mathbb{Z}$. In this case, the Weil group $W_{E/F}$ is not almost connected. Rather, it is locally profinite, thus totally disconnected. We start by noting that the long exact sequence (11) yields the exact sequence

$$0 \to A \to \mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}} \to \underline{H}^1(W_{E/F}, \widehat{T})^\circ \to 1, \tag{12}$$

where $A \subseteq \mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}}$ is the image of the continuous homomorphism $\underline{H}^1(W_{E/F}, X) \to \mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}}$. By Proposition 29 we conclude that A is countable, and since A is also the kernel of the continuous map $\mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}} \to \underline{H}^1(W_{E/F}, \widehat{T})^\circ$ we also see that A is closed. A closed, countable subgroup of a finite dimensional real or complex vector space is necessarily a discrete subgroup. It follows that A is a lattice, and $\underline{H}^1(W_{E/F}, \widehat{T})^\circ$ is a connected abelian complex Lie group with Lie algebra $\mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}}$. We will now study the lattice A in more detail.

Lemma 33. *Let G be a group and D a divisible abelian group. Then, for all $n > 0$ and any left G-module A, we have an isomorphism*

$$H^n(G, \mathrm{Hom}(A, D)) \cong \mathrm{Hom}(H_n(G, A), D).$$

Proof. This is known. It was done, for example, in [BB, Proposition 4.7]. We sketch the proof for convenience. The tensor-hom adjunction yields a natural isomorphism

$$\mathrm{Hom}(M \otimes_{\mathbb{Z}G} A, D) \cong \mathrm{Hom}_G(M, \mathrm{Hom}(A, D))$$

for any right G-module M. By replacing M by a projective resolution $P^\bullet$ of $\mathbb{Z}$ by G-modules, we obtain an isomorphism of chain complexes

$$\mathrm{Hom}(P^\bullet \otimes_{\mathbb{Z}G} A, D) \cong \mathrm{Hom}_G(P^\bullet, \mathrm{Hom}(A, D)).$$

Since D is divisible, the functor $H = \mathrm{Hom}(-, D)$ is an exact contravariant functor and hence it commutes with homology of chain complexes. Taking the homology of the complexes and using the commutativity, we get

$$\begin{aligned}\mathrm{Hom}(H_n(G, A), D) &= H_n(\mathrm{Hom}(P^\bullet \otimes_{\mathbb{Z}G} A, D)) \\ &\cong H_n(\mathrm{Hom}_G(P^\bullet, \mathrm{Hom}(A, D)) \\ &= H^n(G, \mathrm{Hom}(A, D)),\end{aligned}$$

as required. ☐

Proposition 34. *If $V_F \cong \mathbb{Z}$, let A be the lattice of* (12)*. Then, we have an isomorphism $A \cong \mathrm{Hom}_c(\mathrm{Hom}_{\Gamma_{E/F}}(X, C_E), \mathbb{Z})$.*

Proof. Applying the left exact functor $\mathrm{Hom}(\mathrm{Hom}_{\Gamma_{E/F}}(X, C_E), -)$ to the exponential sequence $0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 1$ implies that $\mathrm{Hom}(\mathrm{Hom}_{\Gamma_{E/F}}(X, C_E), \mathbb{Z})$ is isomorphic to the kernel of the map

$$\mathrm{Hom}(\mathrm{Hom}_{\Gamma_{E/F}}(X, C_E), \mathbb{C}) \to \mathrm{Hom}(\mathrm{Hom}_{\Gamma_{E/F}}(X, C_E), \mathbb{C}^*).$$

Using the isomorphism $\mathrm{Hom}_{\Gamma_{E/F}}(X, C_E) \cong H_1(W_{E/F}, \widehat{X})$, which is induced by the restriction map $H_1(W_{E/F}, \widehat{X}) \to H_1(C_E, \widehat{X})$ (as in [La, p. 233]) and the isomorphism $H^1(W_{E/F}, \mathrm{Hom}(\widehat{X}, D)) \cong \mathrm{Hom}(H_1(W_{E/F}, \widehat{X}), D)$ of Lemma 33 (D a divisible abelian group), we get that $\mathrm{Hom}(\mathrm{Hom}_{\Gamma_{E/F}}(X, C_E), \mathbb{Z})$ is the kernel of

$$H^1(W_{E/F}, \mathrm{Hom}(\widehat{X}, \mathbb{C})) \to H^1(W_{E/F}, \mathrm{Hom}(\widehat{X}, \mathbb{C}^*)).$$

Taking continuous classes, it follows that $\mathrm{Hom}_c(\mathrm{Hom}_{\Gamma_{E/F}}(X, C_E), \mathbb{Z})$ is isomorphic to the kernel of $H^1_c(W_{E/F}, \mathrm{Lie}(\widehat{T})) \cong \mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}} \to H^1_c(W_{E/F}, \widehat{T})$ which, by (12), is isomorphic to A, finishing the proof. ☐

Corollary 35. *Suppose that* $V_F \cong \mathbb{Z}$. *Then* $H^1_c(W_{E/F}, \widehat{T})^\circ \approx \mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}}/A$ *is a complex algebraic torus, equipped with a canonical surjective homomorphism* $H^1_c(W_{E/F}, \widehat{T})^\circ \to X_T$ *with finite kernel.*

Proof. For both F local non-Archimedean and a global function field, we can characterize X_T via the exact sequence

$$0 \to \widehat{L}_T \to \mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}} \to X_T \to 1, \tag{13}$$

with L_T the image of the respective $\log_T$ maps, and $\widehat{L}_T$ is the lattice $\mathrm{Hom}(L_T, \mathbb{Z})$. Now, when F is non-Archimedean local, we have $\mathrm{Hom}_{\Gamma_{E/F}}(X, C_E) = T(F)$. From continuity, it follows that $\mathrm{Hom}_c(T(F), \mathbb{Z}) = \mathrm{Hom}_c(T(F)/T(F)^1, \mathbb{Z})$ and hence $A \cong \widehat{L}_T$, using Proposition 34. From (12) and (13) we get $H^1_c(W_{E/F}, \widehat{T})^\circ \cong X_T$.

For F a function field, we have $T(\mathbb{A}_F)/T(F) \to \mathrm{Hom}_{\Gamma_{E/F}}(X, C_E)$, a closed inclusion of finite index (see [La, p. 245]) from which we obtain (by Proposition 34) an inclusion $A \hookrightarrow \mathrm{Hom}_c(T(\mathbb{A}_F)/T(F), \mathbb{Z}) = \mathrm{Hom}(T(\mathbb{A}_F)/T(\mathbb{A}_F)^1, \mathbb{Z}) \cong \widehat{L}_T$ with finite cokernel. By (13) this implies the result. □

6. Continuity of Langlands's map

In this section, we shall prove Theorem 3 and also finish the proof of Theorem 2. Recall the notation $T_{C_F} := \mathrm{Hom}_{\Gamma_{E/F}}(X, C_E)$ of (7). As a consequence of Propositions 21 and 31, to study the continuity of Langlands's canonical map

$$H^1_c(W_{E/F}, \widehat{T}) \to \mathrm{Hom}_c(T_{C_F}, \mathbb{C}^*),$$

it will suffice to show continuity when restricting to the identity components of each side. From the exponential sequence $0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 1$ and seeing these abelian groups as trivial T_{C_F}-modules we obtain, from the long exact sequence in cohomology, the exact sequence

$$0 \to \mathrm{Hom}_c(T_{C_F}, \mathbb{Z}) \to \mathrm{Hom}_c(T_{C_F}, \mathbb{C}) \to \mathrm{Hom}_c(T_{C_F}, \mathbb{C}^*) \to \underline{H}^2(T_{C_F}, \mathbb{Z}), \tag{14}$$

where all arrows are continuous. Here, we used that $\underline{H}^1(T_{C_F}, A) \approx \mathrm{Hom}_c(T_{C_F}, A)$, as the actions are trivial (see also Proposition 25(3)) and that $\underline{H}^1(T_{C_F}, \mathbb{Z})$ is discrete [AM, Theorem D], so that the map $\mathbb{C}^* \to \mathrm{Hom}_c(T_{C_F}, \mathbb{Z})$ is the zero map.

Proposition 36. *We have an isomorphism of abelian groups* $\mathrm{Hom}_c(T_{C_F}, \mathbb{C}) \cong \mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}}$ *which is a homeomorphism.*

Proof. We know the splitting $C_E \approx C_E^1 \times V_E$, and since C_E^1 is compact, the continuity coupled with the fact that $\Gamma_{E/F}$ acts trivially on V_E yields that

$$\begin{aligned}\mathrm{Hom}_c(\mathrm{Hom}(X, C_E)^{\Gamma_{E/F}}, \mathbb{C}) &\cong \mathrm{Hom}_c(\mathrm{Hom}(X, V_E)^{\Gamma_{E/F}}, \mathbb{C}) \\ &\cong \mathrm{Hom}_c(\widehat{X}^{\Gamma_{E/F}} \otimes V_E, \mathbb{C}) \\ &\cong \mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}}.\end{aligned}$$

Since $\mathrm{Hom}_c(T_{C_F}, \mathbb{C}) \approx \underline{H}^1(T_{C_F}, \mathbb{C})$ has a natural structure of a Euclidean space [AM, Theorem D], one checks that this isomorphism of abelian groups is linear and thus a homeomorphism. □

Theorem 37. *For F local or global, the natural map*

$$\Lambda : H_c^1(W_{E/F}, \widehat{T}) \to \mathrm{Hom}_c(\mathrm{Hom}_{\Gamma_{E/F}}(X, C_E), \mathbb{C}^*)$$

described by Langlands is a homeomorphism.

Proof. As remarked, to discuss the continuity of the Langlands map, it suffices to restrict to the identity component. Note that in (14), since the space $\underline{H}^2(T_{C_F}, \mathbb{Z})$ is discrete [AM, Theorem D], the last arrow becomes zero, when restricted to the identity components. Furthermore, the continuity of a homomorphism $\mathrm{Hom}_{\Gamma_{E/F}}(X, C_E) \to A$ (with $A = \mathbb{Z}$, $A = \mathbb{C}$ or $A = \mathbb{C}^*$) forces it to factor through $\mathrm{Hom}_{\Gamma_{E/F}}(X, V_E) \to A$. Hence, from the arguments presented thus far, we have the following commutative diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ A_F & \longrightarrow & \mathrm{Hom}_c(\mathrm{Hom}_{\Gamma_{E/F}}(X, V_E), \mathbb{Z}) \\ \downarrow & & \downarrow \\ H_c^1(W_{E/F}, \mathrm{Lie}(\widehat{T})) & \xrightarrow{\mathrm{d}\Lambda} & \mathrm{Hom}_c(\mathrm{Hom}_{\Gamma_{E/F}}(X, V_E), \mathbb{C}) \\ \downarrow & & \downarrow \\ H_c^1(W_{E/F}, \widehat{T})^\circ & \xrightarrow{\Lambda} & \mathrm{Hom}_c(\mathrm{Hom}_{\Gamma_{E/F}}(X, V_E), \mathbb{C}^*)^\circ \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array} \quad . \tag{15}$$

The column in the left is the long exact sequence in cohomology (10) obtained by from the short exact sequence $0 \to X \to \mathrm{Lie}(\widehat{T}) \to \widehat{T} \to 1$, when truncated with respect to the identity component of $H_c^1(W_{E/F}, \widehat{T})$. Here, the module A_F is obtained by modding-out the torsion submodule of $H_c^1(W_{E/F}, X)$. When $V_F \cong \mathbb{R}$

we have $A_F = 0$, while when $V_F \cong \mathbb{Z}$, A_F is the lattice described in Proposition 34. We know that both sides in the middle row carry the structure of a Euclidean vector space. To show that Λ is continuous, it is enough to show that the map $\mathrm{d}\Lambda$ is linear. But note that unwinding the equivalences (and since Langlands's result uses the identifications as in Lemma 33 and Proposition 34) we are tasked with showing that the bijection

$$\mathrm{Hom}_c(V_E, \mathrm{Hom}(\widehat{X}^{\Gamma_{E/F}}, \mathbb{C})) \to \mathrm{Hom}_c(\widehat{X}^{\Gamma_{E/F}} \otimes V_E, \mathbb{C})$$

realized by the tensor-hom adjunction is linear, which is a straightforward computation. We conclude that $d\Lambda$, and hence Λ are homeomorphisms when restricted to the identity components. Since these are open, we are done. $\square$

Coupled with the fact that $\mathrm{Hom}_c(\mathrm{Hom}_{\Gamma_{E/F}}(X, C_E), \mathbb{C}^*) \to \mathrm{Hom}_c(T(\mathbb{A}_F)/T(F), \mathbb{C}^*)$ is open, Theorem 2 is proved.

Remark 38. For F local or global, it is known that $\underline{H}^2(W_F, \widehat{T}) = 0$ (see [Ra]), where W_F is the absolute Weil group of F. Note that from extension $1 \to W_E^c \to W_F \to W_{E/F} \to 1$, the inflation-restriction sequence implies that

$$\underline{H}^1(W_E^c, \widehat{T})^{W_{E/F}} \to \underline{H}^2(W_{E/F}, \widehat{T}) \to \underline{H}^2(W_F, \widehat{T})$$

is exact, and since $\underline{H}^1(W_E^c, \widehat{T}) \cong \mathrm{Hom}_c(W_E^c, \widehat{T}) = 0 = \underline{H}^2(W_F, \widehat{T})$ we get $\underline{H}^2(W_{E/F}, \widehat{T}) = 0$. In its turn, this implies that $\underline{H}^2(W_{E/F}, X) \to \underline{H}^2(W_{E/F}, \mathrm{Lie}(\widehat{T}))$ is surjective. Since the target is a Euclidean space and $\underline{H}^2(W_{E/F}, X)$ is countable and discrete, we conclude $\underline{H}^2(W_{E/F}, \mathrm{Lie}(\widehat{T})) = 0$ as well. It then follows from the isomorphism Λ that

$$\underline{H}^2(W_{E/F}, X) \approx \frac{\underline{H}^1(W_{E/F}, \widehat{T})}{\underline{H}^1(W_{E/F}, \widehat{T})^\circ} \approx \frac{\mathrm{Hom}_c(T_{C_F}, \mathbb{C}^*)}{\mathrm{Hom}_c(T_{C_F}, \mathbb{C}^*)^\circ} \approx \frac{\mathrm{Hom}_c(T_{C_F}, \mathbb{C}^*)}{\mathrm{Hom}_c(T_{C_F}/T_{C_F}^1, \mathbb{C}^*)},$$

from which we conclude that

$$\underline{H}^2(W_{E/F}, X) \approx \mathrm{Hom}_c(T_{C_F}^1, \mathbb{C}^*).$$

When F is local non-Archimedean, we obtain $\underline{H}^2(W_{E/F}, X) \approx \mathrm{Hom}_c(T(F)^1, \mathbb{C}^*)$, recovering an observation made by Schwein in [Sc, Section 5.2].

Theorem 39. *Suppose that $V_F \cong \mathbb{Z}$. If F is local non-Archimedean, then $(H_c^1(W_{E/F}, \widehat{T}))^\circ$ is isomorphic to X_T. When F is a global function field, then the group $H_c^1(W_{E/F}, \widehat{T})^\circ / \sim_{l.e.}$ is isomorphic to X_T.*

Proof. See Corollary 35 for the proof in the non-Archimedean local cases. For global functions fields, the proof of Corollary 35 yields a commutative diagram (in which the rightmost vertical arrow is a surjection with finite kernel):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & \mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}} & \longrightarrow & H^1_c(W_{E/F},\widehat{T})^\circ & \longrightarrow & 0 \\ & & \big\downarrow & & \big\| & & \big\downarrow & & \\ 0 & \longrightarrow & \widehat{L}_T & \longrightarrow & \mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}} & \longrightarrow & X_T & \longrightarrow & 0 \end{array} \qquad (16)$$

We claim that the kernel of $H^1_c(W_{E/F},\widehat{T})^\circ \to X_T$ consists of locally trivial classes. For each local place $v \in \mathrm{Pl}(F)$, let us denote by $E_{v'} = EF_v$ the induced completion of E and denote by $\Gamma_{v'|v}$ and $W_{v'|v}$ the Galois group and the relative Weil group of the extension $E_{v'}/E_v$. Let also $L_{T,v} \approx T(F_v)/T(F_v)^1$ denote the image of the local $\log_T$ map, and let $X_{T,v}$ denote the space of local unramified characters. Using the isomorphism of $X_{T,v}$ and $H^1_c(W_{v'|v},\widehat{T})^\circ$ of the previous paragraph, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{L}_{T,v} & \longrightarrow & \mathrm{Lie}(\widehat{T})^{\Gamma_{v'|v}} & \longrightarrow & X_{T,v} & \longrightarrow & 0 \\ & & \big\uparrow\sim & & \big\| & & \big\uparrow\sim & & \\ 0 & \longrightarrow & \mathrm{Hom}_c(T(F_v),\mathbb{Z}) & \longrightarrow & \mathrm{Lie}(\widehat{T})^{\Gamma_{v'|v}} & \longrightarrow & H^1_c(W_{v'|v},\widehat{T})^\circ & \longrightarrow & 0 \\ & & \big\uparrow & & \big\uparrow & & \big\uparrow & & \\ 0 & \longrightarrow & A & \longrightarrow & \mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}} & \longrightarrow & H^1_c(W_{E/F},\widehat{T})^\circ & \longrightarrow & 0 \\ & & \big\downarrow & & \big\| & & \big\downarrow & & \\ 0 & \longrightarrow & \widehat{L}_T & \longrightarrow & \mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}} & \longrightarrow & X_T & \longrightarrow & 0 \end{array}.$$

By Theorem 37 the vertical arrows on the top and bottom right-hand side are the local and global Langlands maps respectively, and the map in the middle is the natural assignment from a global to local Langlands parameters. Given a global parameter $\phi \in H^1_c(W_{E/F},\widehat{T})^\circ$, let ϕ_v denote the corresponding local parameters of ϕ, and let $\chi_\phi \in X_T$ denote the corresponding automorphic character of T. Since automorphic characters are completely determined by their local components, since Langlands's maps are compatible with localization to a local place, and since the local Langlands correspondences are bijective, it follows that $\chi_\phi = 1$ if and only if all local parameters ϕ_v are trivial. This finishes the proof. $\square$

6.1. **Explicit Cocycles.** In this last subsection, we exhibit an explicit realization of the cocycles in $Z^1_c(W_{E/F},\widehat{T})^\circ$, and thus the cohomology classes in $H^1_c(W_{E/F},\widehat{T})^\circ$. For F local or global, let us retain the convention of Section 4 and write $\log_q$ to the isomorphism sending V_F to either $\mathbb{R}$ or $\mathbb{Z}$. Given any element $s \in \widehat{T}^{\Gamma_{E/F}}$, note that it determines a continuous function $z_s : W_{E/F} \to \widehat{T}$ given by

$$z_s(\omega) = s^{\log_q |\omega|_{E/F}}, \tag{17}$$

where $\omega \in W_{E/F}$.

Proposition 40. *We have $z_s \in Z^1_c(W_{E/F}, \widehat{T})$, and the assignment $s \mapsto z_s$, with $s \in \widehat{T}^{\Gamma_{E/F}}$, induces continuous a surjection $(\widehat{T}^{\Gamma_{E/F}})^\circ \to H^1_c(W_{E/F}, \widehat{T})^\circ$. Furthermore, if F is a global function field, then we have continuous surjections $H^1_c(W_{E/F}, \widehat{T})^\circ \to X_T \to \widehat{T}_{\Gamma_{E/F}}$.*

Proof. As $s \in \widehat{T}^{\Gamma_{E/F}}$ we have $z_s(\omega_1\omega_2) = z_s(\omega_1)z_s(\omega_2) = z_s(\omega_1)({}^{\overline{\omega_1}}z_s(\omega_2))$, so that $z_s \in Z^1_c(W_{E/F}, \widehat{T})$. Given $s, s' \in \widehat{T}^{\Gamma_{E/F}}$ we compute

$$z_{ss'}(\omega) = (ss')^{\log_q(|\omega|_{E/F})} = z_s(\omega)z_{s'}(\omega)$$

so that we get a homomorphism $\widehat{T}^{\Gamma_{E/F}} \to Z^1_c(W_{E/F}, \widehat{T})$ which is readily seen to be continuous since, with the compact-open topology, $s_n \to s$ implies $z_{s_n} \to z_s$. Composing further with the quotient to $H^1_c(W_{E/F}, \widehat{T})$ and restricting to the identity component, we get the required surjection. Note that for $V_F \cong \mathbb{Z}$ it follows from (12) that restricting to the identity component yields that $(\widehat{T}^{\Gamma_{E/F}})^\circ \to H^1_c(W_{E/F}, \widehat{T})^\circ$ is surjective. On the other hand, when $V_F \cong \mathbb{R}$, we saw in Theorem 32 that in the exact sequence (11), when modding-out torsion and restricting to the identity component, we obtain an isomorphism between $\mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}} \cong H^1_c(W_{E/F}, \widehat{T})^\circ$, so from the lift

$$\begin{array}{ccc} & \nearrow & \mathrm{Lie}(\widehat{T})^{\Gamma_{E/F}} \\ & & \downarrow \\ (\widehat{T}^{\Gamma_{E/F}})^\circ & \longrightarrow & H^1_c(W_{E/F}, \widehat{T})^\circ \end{array}$$

we get the required surjectivity, finishing the proof for the first part.

As for the last statement in the global function field case, the surjectivity of the arrow $H^1_c(W_{E/F}, \widehat{T})^\circ \to X_T$ was discussed in (16). On the other hand, let $S \subset T$ be the largest split subtorus of T. Write $X(S) = \mathrm{Hom}(S, \mathbb{G}_m)$ for the character lattice of S and $\widehat{X}(S) = \mathrm{Hom}(\mathbb{G}_m, S)$ for its cocharacter lattice. As $X^{\Gamma_{E/F}} \subseteq X(S)$ is an inclusion of finite index, it induces an inclusion

$$S(\mathbb{A}_F)/S(\mathbb{A}_F)^1 = L_S \hookrightarrow L_T = T(\mathbb{A}_F)/T(\mathbb{A}_F)^1$$

which is also of finite index. Furthermore, as S is split we have a canonical isomorphism $\widehat{X}(S) = \widehat{X}^{\Gamma_{E/F}} \cong L_S$, from which we obtain an epimorphism

$$\mathrm{Hom}(L_T, \mathbb{C}^*) = X_T \to X_S = \mathrm{Hom}(L_S, \mathbb{C}^*) \cong \widehat{T}_{\Gamma_{E/F}},$$

finishing the proof. □

This concludes the proof of Theorem 3, in the function field case. We finish with a result that shows that the position of $\underline{H}^1(W_{E/F}, \widehat{T})^\circ$ in between connected invariants and the coinvariants of $\widehat{T}$ is sensitive to the value groups V_E and V_F.

Proposition 41. *Suppose that F is a global function field. If $V_E = V_F$ then we have that $H^1_c(W_{E/F}, \widehat{T})^\circ \cong (\widehat{T}^{\Gamma_{E/F}})^\circ$. Otherwise, the surjection $(\widehat{T}^{\Gamma_{E/F}})^\circ \to H^1_c(W_{E/F}, \widehat{T})^\circ$ has a finite kernel.*

Proof. Note that $z_s \in Z^1_c(W_{E/F}, \widehat{T})$ is a coboundary if and only if there is $t \in \widehat{T}$ such that $z_s(\omega) = s^{\log_q |\omega|_{E/F}} = ({}^{\overline{\omega}}t)t^{-1}$ for all $\omega \in W_{E/F}$. This implies that $z_s(\omega) = 1$ for any $\omega = r_E(x)$ with $x \in C_E \hookrightarrow W_{E/F}$, where r_E is the reciprocity map. Hence, if $V_E = V_F$ then z_s is a coboundary implies $s = 1$ and we get $H^1_c(W_{E/F}, \widehat{T})^\circ \cong (\widehat{T}^{\Gamma_{E/F}})^\circ$. If, on the other hand, $V_E \neq V_F$, then there is $d > 1$ such that $V_E = q^{d\mathbb{Z}}$ and $V_F = q^{\mathbb{Z}}$. In this case, z_s a coboundary implies $s \neq 1$ is of finite order, and hence the surjection of Proposition 40 has a finite kernel. □

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