UNIVERSALITY FOR ROOTS OF DERIVATIVES OF ENTIRE FUNCTIONS VIA FINITE FREE PROBABILITY

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Abstract. A universality conjecture of Farmer and Rhoades [*Trans. Amer. Math. Soc.*, 357(9):3789–3811, 2005] and Farmer [*Adv. Math.*, 411:Paper No. 108781, 14, 2022] asserts that, under some natural conditions, the roots of an entire function should become perfectly spaced in the limit of repeated differentiation. This conjecture is known as Cosine Universality. We establish this conjecture for a class of even entire functions with only real roots which are real on the real line. Along the way, we establish a number of additional universality results for Jensen polynomials of entire functions, including the Hermite Universality conjecture of Farmer [*Adv. Math.*, 411:Paper No. 108781, 14, 2022]. Our proofs are based on finite free probability theory. We establish finite free probability analogs of the law of large numbers, central limit theorem, and Poisson limit theorem for sequences of deterministic polynomials under repeated differentiation, under optimal moment conditions, which are of independent interest.

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A. Campbell partially supported by ERC Advanced Grant "RMTBeyond" No. 101020331.
S. O'Rourke has been partially supported by NSF CAREER grant DMS-2143142.

1. Introduction

Tracking the effects of differential operators on polynomial roots dates back implicitly to at least Rolle's theorem and explicitly to at least Gauss's *electrostatic interpretation* of critical points, which gives a simple proof and intuitive explanation of the Gauss–Lucas theorem [39]. If a degree n polynomial p of a single complex variable has distinct roots $z_1, \dots, z_n \in \mathbb{C}$, then a root $z_* \in \mathbb{C}$ of p' must satisfy

$$\frac{p'(z_*)}{p(z_*)} = \sum_{k=1}^{n} \frac{1}{z_* - z_k} = 0. \tag{1.1}$$

Thus, the roots of p' lie at the equilibrium points of a field created by charges at the roots of p. This electrostatic interpretation, while quite simple, is extremely helpful both as a proof tool and as a heuristic for tracking roots under differentiation. In fact, this heuristic helps explain an observation, first attributed to Riesz [44], that if p has only real roots, then the roots of p' are more evenly spaced than those of p. Any areas of clumping in the roots will have a repulsive effect on the critical points through (1.1).

In some cases this electrostatic interpretation can be extended beyond polynomials to functions with an infinite number of roots, even if the sum in (1.1) does not converge in an absolute sense. For these functions this regularizing effect of differentiation leads to a natural conjecture, due to Farmer and Rhoades [13], with further refinements described by Farmer [12], that the roots of certain entire functions should become perfectly spaced in the limits of repeated differentiation; see Section 1.1 for more details.

In this work, we prove this conjecture for a large class of even entire functions by giving it a probabilistic interpretation through *finite free probability* and the recent body of work [2,7,18,19,41,42,43], connecting polynomial roots under differentiation to operations in random matrix theory and free probability. We believe finite free probability can play a further role in the study of entire functions in the *Laguerre–Pólya* class, i.e., functions f which can be represented as

$$f(z) = C_1 z^m e^{c_1 z - c_2 z^2} \prod_{k=1}^{N} \left(1 + \frac{z}{x_k}\right) e^{-\frac{z}{x_k}},$$

for $C, c_1, x_1, x_2, \dots \in \mathbb{R}$, $c_2 \geq 0$, $m \in \mathbb{N}$, and $N \in \mathbb{N} \cup \{\infty\}$. Functions in the Laguerre–Pólya class are uniform limits of polynomials, and thus enjoy many properties of polynomials, for example their roots interlace with the roots of their derivative. As finite free probability is a newly developing field, we do not assume the reader has any familiarity with it, and provide a concise introduction in Section 3.

Additionally, our main results include a proof of the *Hermite Universality Principle* (Principle 1.3) described in [12] and several finite free probabilistic limit theorems which are of independent interest, giving a generalization of both Griffin, Ono, Rolen, and Zagier [15] and Hoskins and Steinerberger [19]. We additionally introduce, and prove, a *Laguerre Universality Principle* (Principle 1.4) which is closely related to a conjecture of Farmer and Rhoades [13] for even functions.

1.1. **Universal limits of differentiation for the Laguerre–Pólya class.** We now discuss the Cosine Universality conjecture of Farmer and Rhoades [13], see also [6] and [12]. They conjecture that if f is an entire function which is real on the real line with only real zeros and the number of zeros of f in $(0, r)$ and $(-r, 0)$ grows sufficiently nicely, then the zeros of $f^{(n)}$ approach perfect spacing as $n \to \infty$. In fact, they further conjecture that the appearance of perfect spacing arises through cosine, namely that there are real sequences A_n, B_n, D_n with D_n bounded such that

$$\lim_{n\to\infty} A_n e^{B_n z} f^{(n)}(\kappa z + D_n) = \cos(\pi z) \tag{1.2}$$

for a constant $\kappa > 0$. Convergence to cosine has been proven for the Riemann Ξ-function and some generalizations by Ki [22]. Gunns and Hughes [17] proved this conjecture for functions in the extended Selberg class. Pemantle and Subramanian [37] proved this conjecture for a random function whose roots are distributed according to a homogeneous Poisson point process.

Universal attractors of differentiation have also been explored by Berry [6], who expanded the idea to more general functions by moving away from real rooted-ness. Based on the work of Griffin, Ono, Rolen, and Zagier [15], Farmer [12] proposed a potential refinement of Cosine Universality for even entire functions, dubbed in [12] the *Hermite Universality Principle*, where universal attractors can be observed by the appearance of Hermite polynomials as the limits of what Farmer refers to as *even Jensen polynomials*. See Section 1.2 below for details on Jensen polynomials and their universality principles.

Our main tool for proving Cosine and Hermite Universality is *finite free probability theory*. This theory emerged out of the celebrated works of Marcus, Spielman and Srivastava [26, 27, 28, 29, 30] on families of interlacing polynomials, which they used to prove the existence of bipartite Ramanujan graphs of all sizes and degrees, and to solve the Kadison–Singer problem [21].

In addition to Cosine and Hermite Universality, we introduce a third universality principle, which we refer to as *Laguerre Universality*, and demonstrate how these universality principles have natural interpretations as (finite free) probabilistic limit theorems. Specifically, our main results in Section 2.1 prove all three universality principles for even entire functions using only the growth conditions of the roots.

Our probabilistic approach is motivated by the recent connection between free probability and repeated differentiation of polynomials [2, 7, 18, 19, 20, 42, 43]. We prove the related finite free limits theorems for general families of real-rooted polynomials in Section 2.2, which are of independent interest, and apply these results to the universality principles. The statements of these theorems require no background in finite free probability, but interpreting them as probabilistic limit theorems does.

1.2. **Jensen polynomials and the universality principles.** Motivated partially by their role in an equivalent statement of the Riemann Hypothesis, from [38], there has been some renewed interest in the *Jensen polynomials* of an entire function f, which are a sequence of polynomials that uniformly approximate f on compact sets. However, as pointed out by Farmer [12] there are two commonly used Jensen polynomials. We follow the terminology from [12] to distinguish between the two possible choices. In this work we will use finite free probability to study repeated differentiation of Jensen polynomials, which then after taking limits allows us to study repeated differentiation of analytic functions.

We now introduce the classical Jensen polynomials. Consider the series representation of an entire function f:

$$f(z) = \sum_{k=0}^{\infty} \gamma_k \frac{z^k}{k!}. \tag{1.3}$$

Definition 1.1. Let f be as in (1.3). Then, for $d, n \in \mathbb{N}$ the degree d classical Jensen polynomial with shift n of f is defined by

$$C_{d,n}(z) := \sum_{k=0}^{d} \binom{d}{k} \gamma_{k+n} z^k. \tag{1.4}$$

Not only do Jensen polynomials of f give a polynomial approximation of f, but if f is an entire function of order less than 2, then the classical Jensen polynomial with shift n of f approximates the n^{th} derivative of f in the sense that, uniformly on compact subsets we have (see [8] and references therein for some background on Jensen polynomials):

$$\lim_{d\to\infty} C_{d,n}\left(\frac{z}{d}\right) = f^{(n)}(z). \tag{1.5}$$

Furthermore, the shifted Jensen polynomials are easy to track through differentiation. Namely,

$$C_{d,n}(z) = \frac{d!}{(n+d)!} \left(\frac{d}{dz}\right)^n C_{d+n,0}(z). \tag{1.6}$$

In fact, one could take (1.6) as a definition of shifted Jensen polynomials after defining the unshifted version $C_{m,0}$. The Jensen polynomials are useful for studying roots of analytic functions because if f is an entire function of order less than 2, then f has only real zeros if and only if $C_{d,0}$ has only real zeros for any $d \in \mathbb{N}$.

For an even entire function

$$f(z) := \sum_{k=0}^{\infty} \gamma_{2k} \frac{z^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \eta_k \frac{z^{2k}}{k!}, \tag{1.7}$$

there is a more commonly used choice of Jensen polynomials, given by what Farmer refers to as the *even Jensen polynomials* of f.

Definition 1.2. Let f be as in (1.7). Then, for $d, n \in \mathbb{N}$ the degree d even Jensen polynomial with shift n of f is defined by

$$J_{d,n}(z) := \sum_{k=0}^{d} \binom{d}{k} \eta_{k+n} z^k. \tag{1.8}$$

It is straightforward to see that the even Jensen polynomials of f are the classical Jensen polynomials of the positive rooted function $g(z) = f(\sqrt{z})$. Hence, the even Jensen polynomials also satisfy

$$J_{d,n}(z) = \frac{d!}{(n+d)!} \left(\frac{d}{dz}\right)^n J_{d+n,0}(z). \tag{1.9}$$

We note that in contrast to the classical case, the shifted even Jensen polynomials, $J_{d,n}$ no longer have a direct connection to the n^{th} derivative of f. To circumvent this issue we will define another set of polynomials in (1.16) which will converge to $f^{(n)}$. Nevertheless, the shifted even Jensen polynomials are well studied, in particular [38], see also [15], showed that the Riemann Hypothesis is equivalent to the even Jensen polynomials of the Riemann Ξ-function having real zeros for all d.

Motivated by similar results for the Riemann Ξ-function, Farmer [12, Principle 3.3] proposed the following.

Principle 1.3 (Hermite Universality, [12]). *For a large class of functions f (such as those considered in the Cosine Universality conjecture discussed above), there should exist sequences $\mathcal{A}_n, \mathcal{B}_n$, and $\mathcal{C}_n$ such that uniformly on compact subsets*

$$\lim_{n\to\infty} \mathcal{A}_n J_{d,n}\left(\mathcal{C}_n z + \mathcal{B}_n\right) = \mathrm{He}_d(z), \tag{1.10}$$

where He_d is the degree d Hermite polynomial

$$\mathrm{He}_d(z) := \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} \frac{d!(-1)^k}{k!(d-2k)!} \frac{z^{d-2k}}{2^k}. \tag{1.11}$$

Given that Principle 1.3 and the results of [15] are formulated for even functions, we will focus on the even Jensen polynomials and not the classical Jensen polynomials of f. In Theorem 2.5, we verify Principle 1.3 and give explicit formulas for $\mathcal{A}_n, \mathcal{B}_n$, and $\mathcal{C}_n$ in terms of the coefficients $\{\eta_k\}_{k=0}^{\infty}$ from (1.7). However, we do not see a direct connection between Principle 1.3 and Cosine Universality, as taking $n \to \infty$ leads to information about derivatives of g, not f. Additionally, despite the fact that in certain limits, the Hermite polynomials converge to cosine, we do not see how the Hermite polynomials are related to the even Jensen polynomials of cosine. In fact, [10] showed that the Hermite polynomials are not the Jensen polynomials of any function in the Laguerre–Pólya class. However, the Laguerre polynomials, which are related to the Hermite polynomials, see (1.12) and Proposition 1.5, do appear naturally in our approach.

To make the connection between even Jensen polynomials and Cosine Universality more explicit, we propose, and verify in Theorem 2.4, the following universality principle for even Jensen polynomials.

Principle 1.4 (Laguerre Universality). *For a large class of functions f, the unshifted even Jensen polynomials of $f^{(2n)}$ converge, after rescaling, to the generalized Laguerre polynomials*

$$L_d^{(-\frac{1}{2})}(z) = \sum_{k=0}^{d} \frac{(-1)^k \left(d-\frac{1}{2}\right)_{d-k}}{k!(d-k)!} z^k. \tag{1.12}$$

To see why this universality principle is directly connected to Cosine Universality we point out the following proposition whose proof follows from direct computation of coefficients. As we will see in Section 3, the even Jensen polynomials of cosine have an extremely natural position in finite free probability theory.

Proposition 1.5. *Let $\{J_{d,0}\}_{d=0}^{\infty}$ be the even Jensen polynomials of* $\cos(z)$ *of shift* 0 *(so that $J_{d,0}$ is a polynomial of degree d). Then*

$$J_{d,0}(z) = 4^d \frac{(d!)^2}{2d!} L_d^{(-\frac{1}{2})}\left(\frac{z}{4}\right) \tag{1.13}$$

and

$$J_{d,0}(z^2) = (-2)^d \frac{d!}{2d!} \mathrm{He}_{2d}\left(\frac{z}{\sqrt{2}}\right), \tag{1.14}$$

where He_{2d} *and* $L_d^{(-\frac{1}{2})}$ *are defined in* (1.11) *and* (1.12), *respectively.*

For Hermite Universality, (1.9) provides the means for computing the $n \to \infty$ limit of $J_{d,n}$. In order to study the even Jensen polynomials of $f^{(2n)}$, we define the differential operator

$$M := 2\left(D + 2zD^2\right), \tag{1.15}$$

where $D = \frac{d}{dz}$ and z is the multiplication operator. By noting that $g(z) = f(\sqrt{z})$ so that

$$f''(\sqrt{z}) = 4zg''(z) + 2g'(z) = Mg(z),$$

it follows that if $W_{d,n}$ are the unshifted even Jensen polynomials of $f^{(2n)}$, then we have the relationship

$$W_{d,n}(z) = \frac{d!}{(n+d)!} M^n W_{n+d,0}(z), \quad W_{m,0}(z) = J_{m,0}(z). \tag{1.16}$$

We will always use $J_{d,n}$ to denote the degree d even Jensen polynomial of f with shift n and $W_{d,n}$ the degree d even unshifted Jensen polynomial of $f^{(2n)}$. The former corresponds to Hermite Universality and the latter to Laguerre Universality.

We will prove Cosine Universality (Theorem 2.3) using Laguerre Universality (Theorem 2.4). We now present a heuristic argument, that uses the double limits in n and d and for the sake of presentation ignores constants:

$$f^{(2n)}(z) = g_n(z^2) \overset{d \gg 1}{\approx} W_{d,n}\left(\frac{z^2}{d}\right) \overset{n \gg 1}{\approx} d! L_d^{(-\frac{1}{2})}\left(\frac{z^2}{4d}\right) = \mathrm{He}_{2d}\left(\frac{z}{\sqrt{2d}}\right) \overset{d \gg 1}{\approx} \cos(z),$$

where $g_n(z) = f^{(2n)}(\sqrt{z})$. The first $\approx$ follows because $W_{d,n}$ are the classical Jensen polynomials of $g_n(z)$ and the second $\approx$ is Laguerre Universality. The following equality and last $\approx$ are well-known properties of Hermite polynomials. Our proof avoids taking the double limit but morally follows this argument.

We conclude this section with an outline of the remainder of the paper. In Section 2, we formally state our Cosine, Hermite, and Laguerre Universality theorems, as well as our finite free limit theorems for derivatives of polynomials. In Section 3, we present the necessary background on finite free probability. In Section 4, we prove the Cosine, Hermite, and Laguerre Universality theorems, and in Section 5 we prove the finite free limit theorems for derivatives of polynomials. In Section 6 we prove a technical result concerning the coefficients of entire functions. We conclude with examples in Section 7.

2. Main results

We divide our main results into two categories. The first, given in Section 2.1, concerns the Cosine, Hermite, and Laguerre Universality Principles. The latter two of these universality principles are applications of our second category, given in Section 2.2, of main results, which we call finite free limit theorems of differentiation. However, the statements of these limit theorems require no knowledge of finite free or even classical probability. Readers familiar with finite free probability will be able to see why we describe these as a law of large numbers, central limit theorem, and Poisson limit theorem.

2.1. **Results on the universality principles.**

We begin with the exact conditions we wish to assume about the even entire function f.

Assumption 2.1. We say an even entire function f satisfies Assumption 2.1 if f is order less than 2, real on the real line, only has real roots, and

$$\lim_{r\to\infty} \frac{n_+(r)}{h(r)r^\alpha} = 1, \tag{2.1}$$

for some $\alpha \in (0,2)$ and slowly varying[1] positive h, where $n_+(r)$ denotes the number of roots of f (counted with multiplicity) in $[0,r]$.

Slowly varying functions arise naturally in probability theory, and we refer the reader to [40] and references therein for more information on slowly varying functions. For f satisfying Assumption 2.1, we let $a_n > 0$ be any sequence satisfying

$$\lim_{n\to\infty} -a_n \frac{\gamma_{2(n+1)}}{\gamma_{2n}} = 1, \tag{2.2}$$

for the coefficients γ_{2k} defined in (1.7).

We begin with a lemma that gives basic coefficient information directly from the our root density assumption, in particular it can be used to choose a_n in (2.2). We employ the asymptotic notation $O(\cdot)$, $o(\cdot)$, $\lesssim$, $\sim$, etc. under the assumption that some sequence index, such as n or m, tends to infinity. We write $c_n = O(b_n)$ or $c_n \lesssim b_n$ if there exists some constant $C > 0$ such that $|c_n| \le C|b_n|$ for all $n > C$, $c_n = o(b_n)$ if $\frac{c_n}{b_n} \to 0$, and $c_n \sim b_n$ if $\frac{c_n}{b_n} \to 1$.

Lemma 2.2. *Let f be an even entire function, as represented in* (1.7)*, which satisfies Assumption 2.1. Then,*

$$-\frac{\gamma_{2n}}{\gamma_{2(n-1)}} = 4\left(\frac{\alpha}{2}\pi \csc\left(\pi\frac{\alpha}{2}\right)\tilde{h}(n)\right)^{2/\alpha} n^{2-\frac{2}{\alpha}}(1+o(1)), \tag{2.3}$$

and

$$\lim_{n\to\infty} \frac{\gamma_{2(n-2)}\gamma_{2n}}{\gamma_{2(n-1)}^2} = 1, \tag{2.4}$$

for some slowly varying function $\tilde{h}$.

We note that for many choices of the slowly varying function h, $\tilde{h}$ is not too difficult to work out. In fact, for any constant h, $\tilde{h}$ is also a constant. By an inspection of the proofs, we note that in the following Theorems 2.4 and 2.5, our assumption (2.1), on the root density, can be replaced with the existence of the limit (2.4), and in Theorem 2.3 it suffices to assume both the limits exist. In particular, (2.3) is needed to control the tail of the Taylor series of $\frac{1}{\gamma_{2n}} f^{(2n)}(\sqrt{a_n}z)$. The exact constant in the right-hand side of (2.3) is not important and it could be absorbed into the slowly varying function.

We now state our main universality results, the proofs are given in Section 4.

Theorem 2.3 (Cosine Universality for even functions)**.** *Let f be an even entire function, as represented in* (1.7)*, which satisfies Assumption 2.1. Then,*

$$\lim_{n\to\infty} \frac{1}{\gamma_{2n}} f^{(2n)}(\sqrt{a_n}z) = \cos(z), \tag{2.5}$$

uniformly on compact subsets of the complex plane.

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[1] $h : (0,\infty) \to (0,\infty)$ is slowly varying if for any $x \in (0,\infty)$

$$\lim_{r\to\infty} \frac{h(xr)}{h(r)} = 1.$$

Theorem 2.3 will be proven as a corollary of the following theorem.

Theorem 2.4 (Laguerre Universality)**.** *Let f be an even entire function, as represented in* (1.7)*, which satisfies Assumption 2.1, and fix $d \in \mathbb{N}$. Let $W_{d,n}$ be the even Jensen polynomials of $f^{(2n)}$, as in* (1.16)*. Then,*

$$\lim_{n\to\infty} \frac{1}{\gamma_{2n}} W_{d,n}\left(a_n z\right) = 4^d \frac{(d!)^2}{2d!} L_d^{\left(-\frac{1}{2}\right)}\left(\frac{z}{4}\right), \tag{2.6}$$

uniformly on compact subsets of the complex plane. Moreover, the polynomials on the right-hand side of (2.6) *are the even Jensen polynomials of* $\cos(z)$*.*

We additionally prove the Hermite Universality Principle for functions satisfying Assumption 2.1. This result follows from an application of Theorem 2.8, stated below, which shows that Hermite polynomials naturally appear after repeated differentiation of high degree polynomials.

Theorem 2.5 (Hermite Universality)**.** *Let f be an even entire function, as represented in* (1.7)*, which satisfies Assumption 2.1, and let $d \in \mathbb{N}$. Define the centering and normalization sequences*

$$b_n := -2(2(n+d)-1)\frac{\gamma_{2(n+d-1)}}{\gamma_{2(n+d)}}, \tag{2.7}$$

and

$$c_n := 4(2(n+d)-1)^2 \left[\left(\frac{\gamma_{2(n+d)-2}}{\gamma_{2(n+d)}}\right)^2 - \frac{2(n+d)-3}{2(n+d)-1}\frac{\gamma_{2(n+d)-4}}{\gamma_{2(n+d)}}\right]. \tag{2.8}$$

Then, uniformly on compact subsets of the complex plane,

$$\lim_{n\to\infty} \frac{c_n^{-d/2}}{\gamma_{2(n+d)}} J_{d,n}\left(\sqrt{c_n}(z+b_n)\right) = \mathrm{He}_d(z). \tag{2.9}$$

Remark 2.6. The centering and normalization sequences in Theorem 2.5 have quite natural, though not obvious, probabilistic interpretations. If X is a random variable uniformly distributed on the roots of $J_{n+d,0}$, then $b_n = \mathbb{E}X$ and $c_n \sim \frac{\mathrm{Var}(X)}{\sqrt{n+d}}$. Thus, they center and normalize X to have mean 0 and variance approximately $\sqrt{n+d}$.

Farmer [12, Section 5] points out some numerical evidence suggesting that for any fixed d, the rate of convergence to the d-th Hermite polynomial is $1/\sqrt{n}$. We shall see that Theorem 2.5 is a special case of Theorem 2.8, our central limit theorem (CLT) for repeated differentiation. To minimize the assumptions on γ_{2m}, we prove this CLT under the weakest possible assumptions on growth rate of the roots. However, under stronger assumptions our proof can be adapted to get a Berry–Esseen type theorem which would verify Farmer's observation. The analogous rate of convergence for the finite free CLT under the finite free additive convolution appears in [4]. For ease of presentation, we omit the details.

2.2. **Finite free limit theorems of differentiation.** In this section we present the limit theorems for generic sequences of real rooted monic polynomials

$$P_m(z) = \sum_{k=0}^{m} (-1)^k a_{k,m} z^{m-k}. \tag{2.10}$$

Theorems 2.4 and 2.5 will follow by specializing these limit theorems to Jensen polynomials. We define the empirical root measure of the polynomial P_m to be the probability measure

$$\mu_{P_m} := \frac{1}{m} \sum_{z:P_m(z)=0} \delta_z, \tag{2.11}$$

where the roots are counted with multiplicities. We denote the moments and absolute moments of μ_{P_m} by

$$m_j(P_m) := \int_{\mathbb{R}} z^j \, d\mu_{P_m}(z) \quad \text{and} \quad |m|_j(P_m) := \int_{\mathbb{R}} |z|^j \, d\mu_{P_m}(z) \tag{2.12}$$

for $j \in \mathbb{N}$. We present below a finite free law of large numbers, central limit theorem, and Poisson limit theorem in terms of the moments of P_m. The assumptions on moments of μ_{P_m} in these theorems can be translated into assumptions on the coefficients of P_m by using Newton's identities. For positive integers n and j, $(n)_j$ denotes the Pochhammer sequence $(n)_j = \prod_{i=1}^{j}(n-i+1)$.

Theorem 2.7 (Law of large numbers)**.** *Assume $m_1(P_m) \to a \in \mathbb{R}$ and $m_2(P_m) = o(m)$ as $m \to \infty$. Then, for any $d \in \mathbb{N}$,*

$$\lim_{n\to\infty} \frac{1}{(n+d)_n} \left(\frac{d}{dz}\right)^n P_{d+n}(z) = (z-a)^d, \tag{2.13}$$

uniformly on compact subsets of the complex plane.

We define the dilation function on polynomials $\mathfrak{D}_k$ for $k > 0$ by

$$\mathfrak{D}_k p(z) := k^{\deg(p)} p(z/k). \tag{2.14}$$

The roots of $\mathfrak{D}_k p$ are the roots of p multiplied by k, with $\mathfrak{D}_k p$ and p having identical leading degree coefficients.

Theorem 2.8 (Central limit theorem)**.** *Fix $d \in \mathbb{N}$. Assume $m_1(P_m) = o(m^{-1/2})$, $m_2(P_m) \to 1$ and $|m|_{2+\varepsilon}(P_m) = o(m^{\varepsilon/2})$ as $m \to \infty$ for some $\varepsilon > 0$. Then,*

$$\lim_{n\to\infty} \frac{1}{(n+d)_n} \left(\frac{d}{dz}\right)^n \mathfrak{D}_{\sqrt{n+d}} P_{d+n}(z) = \mathrm{He}_d(z), \tag{2.15}$$

uniformly on compact subsets of the complex plane, where He_d is the d-th Hermite polynomial (1.11).

Remark 2.9. As we will demonstrate in Section 5.3, both Theorem 2.7 and Theorem 2.8 are optimal in terms of moments. That is, we will provide counterexamples where the higher moments grow too quickly in the degree and do not converge to $(z-1)^d$ or He_d. It is worth noting that in neither Theorem 2.7 nor Theorem 2.8 do we assume that the empirical measures actually have any limit, merely that the first few moments do not grow too quickly in the degree.

Theorem 2.8 is exactly a deterministic version of the result of Hoskins and Steinerberger [19] for polynomials $p_n(z) = \prod_{i=1}^{n}(z - X_i)$ with the roots, X_i, being independent and identically distributed (iid) random variables. In fact, our result replaces their assumption that all moments of X_i are finite with the assumption that $\mathbb{E}[|X_i|^{2+\varepsilon}]$ is finite. Indeed, under this assumption, by the Law of Large Numbers, almost surely, the $(2+\varepsilon)^{th}$ moment of the empirical root measure, $|m|_{2+\varepsilon}$, is bounded, so Theorem 2.8 can be applied to recover their result.

We give the following corollary for characteristic polynomials of random matrices, which is interesting in its own right.

Corollary 2.10. *Let $\{W_n\}_{n=1}^{\infty}$ be a sequence of $n\times n$ Wigner matrices, i.e., W_n is a Hermitian matrix with mean 0 and variance 1 entries that are independent up to the symmetry condition. We additionally assume the entries have finite moments of all orders*[2]*. If $d\in\mathbb{N}$ and $\Phi_n(z)=\det(z-W_n)$ is the characteristic polynomial of W_n, then almost surely*

$$\lim_{n\to\infty}\frac{1}{(n+d)_n}\left(\frac{d}{dz}\right)^n\Phi_{n+d}(z)=\mathrm{He}_d(z) \tag{2.16}$$

uniformly on compact subsets of the complex plane.

Finally, we present a Poisson limit theorem for polynomials, which we will use to prove the Laguerre Universality Principle for Jensen polynomials.

Theorem 2.11 (Poisson limit theorem)**.** *Assume the polynomials P_m are monic, have only non-negative roots, $\frac{m_1(P_m)}{m}\sim a\in(0,\infty)$ and $m_2(P_m)=o(m^3)$ as $m\to\infty$. Then,*

$$\lim_{n\to\infty}\frac{1}{(2n+2d)_{2n}}M^n\mathfrak{D}_{a^{-1}}P_{n+d}(z)=d!(-1)^dL_d^{(-\frac{1}{2})}(z), \tag{2.17}$$

uniformly on compact subsets of the complex plane, where M is defined in (1.15) *and $L_d^{(-\frac{1}{2})}$ is defined in* (1.12)*.*

Remark 2.12. By a line-by-line modification of the proof, a generalization of Theorem 2.11 holds for any operator $M_{\alpha,t}=t\left((1+\alpha)D+zD^2\right)$ where $t>0$ and $\alpha>-1$ and the limit replaced by $L_d^{(\alpha)}$. However, for notational simplicity, we will just prove Theorem 2.11 and leave the details to the reader.

The following is the Poisson analogue of Corollary 2.10 for positive definite random matrices. We omit the proof for brevity, as it follows directly from Theorem 2.11 and standard random matrix results.

Corollary 2.13. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of $n\times n$ random matrices with iid entries that are mean 0, variance 1, and have finite moments of all orders. If $d\in\mathbb{N}$ and $\Psi_n(z)=\det(z-X_n^*X_n)$ is the characteristic polynomial of $X_n^*X_n$, then almost surely*

$$\lim_{n\to\infty}\frac{1}{(2n+2d)_{2n}}M^n\Psi_{n+d}(z)=d!(-1)^dL_d^{(-\frac{1}{2})}(z), \tag{2.18}$$

uniformly on compact subsets of the complex plane, where M is defined in (1.15) *and $L_d^{(-\frac{1}{2})}$ is defined in* (1.12)*.*

2.3. **A note on complex roots.** We formulated all of our results in terms of real rooted polynomials. However, given the results on the Riemann Ξ-function [15, 22] one may inquire if similar results hold for functions with a small number of complex roots or only under the assumption that the roots are restricted to some strip. This restriction to a strip is motivated by a result of Kim [23] that if f is an entire function of order less than 2 which is real on the real line with all of its roots in a

[2]One could instead assume finite fourth moment, but we choose this assumption for simplicity.

strip containing the real line, then for any $R > 0$ the roots of $f^{(n)}$ in $|z| < R$ are all real for n sufficiently large.

We chose to focus instead on real rooted functions where the theory of finite free probability is much more developed. However, much of what we do does not require real roots. The two main applications of real-rootedness in our arguments are in the proof of Lemma 2.2 and in bounding the higher moments for the empirical root measures of our polynomials. If one assumed conditions on the coefficients of f (similar to [15]) and bounds on all the absolute moments of the polynomials in Section 2.2, then one could adapt our techniques to allow for some complex roots. We leave this direction for future research.

2.4. **Novelties of our results and techniques.** To the best of our knowledge, this work is the first instance of finite free probability being applied to study a function with an infinite number of roots. Given that functions in the Laguerre–Póyla class are well approximated by polynomials with real roots we believe finite free probability should serve as a powerful new tool in the study of such functions. It should be particularly helpful in situations where one suspects only a few root statistics (such as average spacing) govern the behavior (such as limiting spacing of high derivatives).

Previous results in Cosine Universality [17, 22] and Hermite Universality [15, 16, 36] use some property of the function (such as conditions on its Fourier transform or coefficients) which appear to be quiet difficult to prove from some density assumption on the roots. Pemantle and Subramanian [37] do begin only with assumptions on the roots and use properties of the point process and the particularly nice factorization of the function to recover useful coefficient information. Our approach is similar in that we use the nice factorization available for even functions, but our finite free limit theorems require such mild conditions that we are able to consider general collections of deterministic roots.

The results in both [15] and [19] concern polynomials converging to Hermite polynomials under repeated differentiation. However, these two papers share no citations on MathSciNet and neither cites the other. Both of them essentially follow as corollaries of Theorem 2.8 (though [15] make no assumptions on the real-rootedness of the function). The approach in [19] uses the independence of the roots and Newton's identities to apply the classical law of large numbers and central limit theorem to recover the Hermite polynomials in the limit of repeated differentiation. Theorem 2.8 instead follows the intuition from (1.1) that after a large number of derivatives the remaining polynomial should depend on only a few mild empirical statistics of the roots.

3. Background on finite free probability theory

The proofs of our main results are based on finite free probability theory. In this section, we provide a basic introduction to the concepts and results in finite free probability theory that we will need. While many of the concepts in finite free probability are based on those from free probability theory, the reader does not need any background in free probability theory to understand the proofs. We refer the interested reader to [32, 35] for more details on free probability.

In [29], the related field of finite free probability was introduced by defining a convolution on polynomials that gives the expected characteristic polynomial of a random matrix. More precisely, if A and B are n-dimensional Hermitian matrices with characteristic polynomials p and q, respectively, then the finite free additive convolution of p and q is given by:

$$p(z) \boxplus_n q(z) := \mathbb{E}_Q[\chi_{A+QBQ^T}(z)],$$

where the expectation is taken with respect to Q, a Haar-distributed orthonormal matrix, and χ_{A+QBQ^T} is the characteristic polynomial of $A + QBQ^T$. In fact, if

$$p(z) = \sum_{i=0}^{n} z^{n-i}(-1)^i a_i^p \quad \text{and} \quad q(z) = \sum_{i=0}^{n} z^{n-i}(-1)^i a_i^q, \tag{3.1}$$

then $p(z) \boxplus_n q(z)$ can be computed explicitly as

$$p(z) \boxplus_n q(z) = \sum_{k=0}^{n} z^{n-k}(-1)^k \sum_{i+j=k} \frac{(n-i)!(n-j)!}{n!(n-i-j)!} a_i^p a_j^q.$$

It was observed in [29] that $\boxplus_n$ can also computed in terms of differential operators; namely if $\widehat{p}$ and $\widehat{q}$ are such that $\widehat{p}(D)z^n = p(z)$ and $\widehat{q}(D)z^n = q(z)$, where D denotes the differentiation operator, then

$$p(z) \boxplus_n q(z) = \widehat{p}(D)\widehat{q}(D)z^n.$$

Remarkably, this convolution was originally introduced by Walsh [47] in 1922 in a different context, and it enjoys many nice properties; see the discussions in [31, 39] for further details and historical notes.

In [29], the finite free multiplicative convolution is also defined for polynomials p and q given in (3.1) as

$$p(z) \boxtimes_n q(z) := \sum_{k=0}^{n} z^{n-k}(-1)^k \frac{a_k^p a_k^q}{\binom{n}{k}}.$$

This convolution can also be shown to be the expected characteristic polynomial of a random matrix [29]. The finite free multiplicative convolution was also classically studied in [45] in a different context.

The identity polynomial for the finite free multiplicative convolution is $q(z) = (z-1)^n$, which is the characteristic polynomial of the identity matrix. Furthermore, the finite free multiplication of an $n+d$ degree polynomial p with the characteristic polynomial of a projection matrix, $q_{d,n}(z) := z^n(z-1)^d$, can be written in terms of differentiation:

$$p(z) \boxtimes_{n+d} q_{n,n-d}(z) = \frac{1}{(n+d)_n} z^n D^n p(z). \tag{3.2}$$

This connection was used in [2] to connect the derivatives of polynomials and free convolutions, whose connection had already been noted in [43]. More generally, as observed in [33, Section 5.3.4], many differential operators can be implemented by finite free convolutions.

Lemma 3.1 (Lemma 3.24 from [33], see also [29])**.** *If P and Q are polynomials such that $p(z) = P(zD)(z-1)^d$ and $q(z) = Q(zD)(z-1)^d$, then*

$$p(z) \boxtimes_d q(z) = P(zD)Q(zD)(z-1)^d = P(zD)q(z) = Q(zD)p(z)$$

A combinatorial description of finite free convolutions in terms of the posets of partitions is given in [3], which we briefly describe. Before doing so, we will introduce the necessary combinatorial definitions and notations. A partition, $\pi = \{V_1, \ldots, V_r\}$ of $[j] := \{1, \ldots, j\}$ is a collection of pairwise disjoint, non-null, sets V_i

such that $\cup_{i=1}^{r} V_i = [j]$. We refer to V_i as the blocks of π, and denote the number of blocks of π as $|\pi|$. The set of all partitions of $[j]$ is denoted $\mathcal{P}(j)$, and the set of all pair partitions, meaning partitions with $|V_i| = 2$ for all i, is denoted $\mathcal{P}_2(j)$.

The set of partitions can be equipped with the partial order $\preceq$ of reverse refinement, where we define $\pi \preceq \sigma$ if every block of π is completely contained in a block of σ. The minimal element in this ordering is $0_j := \{\{1\},\{2\},\dots,\{j\}\}$ and the maximal element is $1_j = \{\{1,2,\dots,j\}\}$. The supremum of π and σ is denoted $\pi \vee \sigma$. For a partition $\pi = \{V_1,\dots,V_r\}$ and a sequence of numbers $\{c_n\}$, we use

$$c_\pi := \prod_{i=1}^{|\pi|} c_{|V_i|}, \qquad c_{2\pi} = \prod_{i=1}^{|\pi|} c_{2|V_i|}, \tag{3.3}$$

and $(n)_j$ denotes the Pochhammer sequence, $(n)_j = \prod_{i=1}^{j}(n-i+1)$.

The Möbius function (in the set of partitions) is given by

$$\mu(\sigma,\rho) := (-1)^{|\sigma|-|\rho|}(2!)^{r_3}(3!)^{r_4}\dots((n-1)!)^{r_n},$$

where r_i is the number of blocks of ρ that contain exactly i blocks of σ. Note that $\mu(\sigma,\rho)$ is 0 unless $\sigma \preceq \rho$. In particular, we have:

$$\mu(0_n,\rho) = (-1)^{n-|\rho|}\prod_{V\in\rho}(|V|-1)! = (-1)^{n-|\rho|}(2!)^{t_3}(3!)^{t_4}\dots((n-1)!)^{t_n},$$

where t_i is the number of blocks of ρ of size i.

Recall that we associate the empirical root distribution, μ_p, to a polynomial p, given by:

$$\mu_p := \frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i(p)},$$

where $\lambda_1(p),\dots,\lambda_n(p)$ are the roots of p, counted with multiplicity. Furthermore, the moments and absolute moments of this measure are given by:

$$m_j(p) := \frac{1}{n}\sum_{i=1}^{n}\lambda_i(p)^j \quad |m|_j(p) := \frac{1}{n}\sum_{i=1}^{n}|\lambda_i(p)|^j.$$

From the Newton identities, the coefficients of p can be recovered from the moments:

$$a_k^p = \frac{1}{k!}\sum_{\pi\in P(k)} n^{|\pi|}\mu(0_k,\pi)m_\pi(p) \tag{3.4}$$

and

$$m_j(p) = \frac{(-1)^j}{n(j-1)!}\sum_{\pi\in P(j)}(-1)^{|\pi|}N!_\pi(|\pi|-1)!a_\pi^p \tag{3.5}$$

for $j\in[n]$, where $N!_\pi := \prod_{V\in\pi}|V|!$. Here, $m_\pi(p)$ and a_π^p denote the products introduced in (3.3).

In classical probability theory, it is often useful to work with the cumulants of a random variable X, given by the coefficients of the log-moment generating function $K_X(t) = \log(\mathbb{E}[e^{tX}])$ of X. In [3], the analogous finite free cumulants are defined in several ways, the most direct way is

$$\kappa_j^n(p) := \frac{(-n)^j}{n(j-1)!}\sum_{\pi\in P(j)}(-1)^{|\pi|}\frac{N!_\pi a_\pi^p(|\pi|-1)!}{(n)_\pi}, \tag{3.6}$$

where p is given in (3.1). They can also be defined as the coefficients of the (truncated) R-transform, given in [25]. This relationship can be inverted (see [3, Proposition 3.4]) to get:

$$a_k^p = \frac{(n)_k}{n^k k!} \sum_{\pi \in \mathcal{P}(k)} n^{|\pi|} \mu(0_k, \pi) \kappa_\pi^n(p) \tag{3.7}$$

for $k \in [n]$.

An important property of the finite free cumulants is that they linearize the finite free additive convolution, just as the classical cumulants linearize the convolution in classical probability, in the sense that:

$$\kappa_j^n(p \boxplus_n q) = \kappa_j^n(p) + \kappa_j^n(q), \qquad j \in [n].$$

Combining (3.6) and (3.4), or (3.5) and (3.7) gives the following moment-cumulant formulas:

Proposition 3.2 (Theorem 4.2 from [3])**.** *Let p be a monic polynomial of degree n. Then,*

$$\kappa_j^n(p) = \frac{(-n)^{j-1}}{(j-1)!} \sum_{\sigma \in \mathcal{P}(j)} n^{|\sigma|} \mu(0_j, \sigma) m_\sigma(p) \sum_{\pi \geq \sigma} \frac{(-1)^{|\pi|-1}(|\pi|-1)!}{(n)_\pi}, \tag{3.8}$$

for $j = 1, \dots, n$ and

$$m_j(p) = \frac{(-1)^{j-1}}{n^{j+1}(j-1)!} \sum_{\sigma \in \mathcal{P}(j)} n^{|\sigma|} \mu(0_j, \sigma) \kappa_\sigma^n(p) \sum_{\pi \geq \sigma} (-1)^{|\pi|-1}(n)_\pi (|\pi|-1)!, \tag{3.9}$$

for $j \in \mathbb{N}$.

We note that (3.8) gives a crude bound on the cumulants in terms of the absolute moments:

$$|\kappa_j^n(p)| \leq C_j n^{-1} \sum_{\sigma \in \mathcal{P}} n^{|\sigma|} |m|_\sigma(p) \tag{3.10}$$

for some constant $C_j > 0$.

The cumulants and moments of the finite free multiplicative convolution were computed in [2]:

Theorem 3.3 (Theorem 1.1 from [2])**.** *Let p and q be monic polynomials of degree n. Then, the following formulas hold:*

$$\kappa_j^n(p \boxtimes_n q) = \frac{(-1)^{j-1}}{n^{j+1}(j-1)!} \sum_{\substack{\sigma,\tau \in \mathcal{P}(j) \\ \sigma \vee \tau = 1_j}} n^{|\sigma|+|\tau|} \mu(0_j, \sigma) \mu(0_j, \tau) \kappa_\sigma^n(p) \kappa_\tau^n(q), \tag{3.11}$$

and

$$m_j(p \boxtimes_n q) = \frac{(-1)^{j-1}}{n^{j+1}(j-1)!} \sum_{\substack{\sigma,\tau \in \mathcal{P}(j) \\ \sigma \vee \tau = 1_j}} n^{|\sigma|+|\tau|} \mu(0_j, \sigma) \mu(0_j, \tau) \kappa_\sigma^n(p) m_\tau(q). \tag{3.12}$$

We conclude this section with a discussion of limit theorems in finite free probability. In Theorems 1.2 and 1.3 of [2], the leading order and sub-leading in n terms from (3.11) and (3.12) are computed for sequences of polynomials whose moments are bounded in the degree. The leading order and sub-leading terms are given by the corresponding free convolution and its infinitesimal distribution. In Section 5, we consider a similar limit with $q(z) = z^n(z-1)^d$. In contrast to [2], the moments of q decay as n grows with d is fixed, and we must rescale the argument of p to get non-trivial limits. We will then use Lemma 3.1 and (3.2) to prove the main theorems from Section 2.2.

In [25, Section 6] and [3, Section 6] limit theorems are given for a growing number of convolutions of fixed degree polynomials. These limits are analogous to the well known Law of Large Numbers and Central Limit Theorem in classical probability theory, which roughly state that the classical convolution-powers of a measure converge to the mean of the measure or to a Gaussian distribution, depending on their rescaling. Among other similar results, [25] and [3] show that if $\{p_i\}_{i\in\mathbb{N}}$ is a sequence of degree d polynomials each with $m_1(p_i) = 0$ and $m_2(p_i) = 1$, then the n-fold finite free convolution, $n^{-nd/2}p_1(\sqrt{n}z) \boxplus_d \ldots \boxplus_d p_n(\sqrt{n}z)$ converges to the d^{th} Hermite polynomial as $n \to \infty$.

In other words, the Hermite polynomials play the role of the Gaussian distribution in finite free probability. It is notable that all of the (finite free) cumulants after the second vanish for both distributions. The emergence of Hermite polynomials in the finite free central limit theorem and in Theorem 2.8 does not seem to be coincidental. Very recently, in [1, Corollary 3.5], it is shown that the roots of derivatives of polynomials can be computed by fractional finite free convolution powers. So repeated differentiation can be thought of as taking high convolution powers. Using the connection between differentiation and multiplicative convolutions in Lemma 3.1, this connection between differentiation and the fractional finite free convolution is the finite free version of a free probability result of Nica and Speicher [34].

4. Proofs of the theorems in Section 2.1

Throughout this section, we will assume f satisfies Assumption 2.1. Specifically, Lemma 2.2 is available throughout. We will prove Theorem 2.3-2.4 by applying the results of Section 2.2. We begin by specializing the formulas for finite free cumulants, (3.6), to Jensen polynomials:

$$\kappa_j^m(J_{m,0}) = \frac{m^{j-1}}{(j-1)!} \sum_{\pi \in \mathcal{P}(j)} (-1)^{|\pi|}(|\pi|-1)! \frac{(2m)_{2\pi}}{(m)_\pi} \frac{\gamma_{2(m-\pi)}}{\gamma_{2m}^{|\pi|}}. \tag{4.1}$$

In particular, for $j = 1, 2$, we have:

$$\begin{aligned} \kappa_1^m(J_{m,0}) &= -2(2m-1)\frac{\gamma_{2(m-1)}}{\gamma_{2m}}, \\ \kappa_2^m(J_{m,0}) &= 4(2m-1)^2 m \left[\left(\frac{\gamma_{2(m-1)}}{\gamma_{2m}} \right)^2 - \frac{2m-3}{2m-1}\frac{\gamma_{2(m-2)}}{\gamma_{2m}} \right]. \end{aligned} \tag{4.2}$$

Additionally, we can essentially factor out $(\kappa_1^m)^j$ from (4.1) to make the relative growth of the finite free cumulants more transparent:

$$\kappa_j^m(J_{m,0}) = \frac{m^{j-1}2^j(2m-1)^j}{(j-1)!} \left(\frac{\gamma_{2(m-1)}}{\gamma_{2m}} \right)^j \sum_{\pi \in \mathcal{P}(j)} \frac{(-1)^{|\pi|}(|\pi|-1)!}{2^j(2m-1)^j} \frac{(2m)_{2\pi}}{(m)_\pi} \frac{\gamma_{2(m-\pi)}}{\gamma_{2m}^{|\pi|-j}\gamma_{2(m-1)}^j}. \tag{4.3}$$

We will primarily use (2.4) to simplify the summation on the right-hand side of (4.3). Recently, [1] defined an S-transform for finite free probability by using the ratio of the coefficients of a polynomial. In the language of [1], (2.3) and (2.4) can be translated into statements on the finite free S-transforms of the even Jensen polynomials evaluated near 0.

4.1. **Proof of Theorem 2.4.** We will prove Theorem 2.4 by applying Theorem 2.11.

Proof of Theorem 2.4. We begin by showing that after rescaling the argument of $W_{n+d,0} = J_{n+d,0}$ by a_n, the first 2 cumulants satisfy the appropriate assumptions. After rescaling the argument of (4.2) by a_n, we have

$$\kappa_1^{n+d}\left(W_{n+d,0}(a_n z)\right) = -2(2(n+d)-1)\frac{\gamma_{2(n+d-1)}}{\gamma_{2(n+d)}}a_n^{-1}.$$

So by the definition of a_n in (2.2), we have

$$\kappa_1^{n+d}\left(W_{n+d,0}(a_n z)\right)/n \sim 4.$$

Furthermore, rearranging the expression for κ_2 as in (4.3), gives

$$\kappa_2^{n+d}(W_{n+d,0}) = (n+d)\kappa_1^{n+d}(W_{n+d,0})^2\left[1 - \frac{2(n+d)-3}{2(n+d)-1}\frac{\gamma_{2(n+d-2)}\gamma_{2(n+d)}}{\gamma_{2(n+d-1)}^2}\right]$$

Then, by taking $n \to \infty$ and applying Lemma 2.2, we see that $\kappa_2^{n+d}(W_{n+d,0}) = o((n+d)\kappa_1^{n+d}(W_{n+d,0})^2)$. In particular we have that $\kappa_2^{n+d}\left(W_{n+d,0}(a_n z)\right) = o(n^3)$, as required in Theorem 2.11.

Recall the definition of M given in (1.15). In order to apply Theorem 2.11, we first rewrite $\frac{1}{\gamma_{2n}}W_{d,n}(a_n z)$ as the application of M^n to a sequence of monic polynomials:

$$\begin{aligned}\frac{1}{\gamma_{2n}}W_{d,n}(a_n z) &= \frac{1}{\gamma_{2n}}\frac{a_n^{-n}}{(n+d)_n}M^n W_{d+n,0}(a_n z)\\ &= \frac{1}{\gamma_{2n}}\frac{d!}{(2d)!}\frac{(2n+2d)!}{(n+d)!}\frac{a_n^{-n}}{(2n+2d)_{2n}}M^n W_{d+n,0}(a_n z)\\ &= \frac{\gamma_{2(n+d)}}{\gamma_{2n}}\frac{d!}{(2d)!}\frac{a_n^{d}}{(2n+2d)_{2n}}M^n \frac{(2n+2d)!}{(n+d)!\gamma_{2(n+d)}a_n^{n+d}}W_{d+n,0}(a_n z),\end{aligned} \tag{4.4}$$

where in the final line we have multiplied $W_{d+n,0}(a_n z)$ by the appropriate term to make it monic. By the definition of a_n we have

$$\lim_{n\to\infty}\frac{\gamma_{2n+2d}}{\gamma_{2n}}a_n^d = (-1)^d. \tag{4.5}$$

Hence, Theorem 2.4 follows by applying Theorem 2.11, and rescaling the argument by $1/4$ to account for the asymptotic $\kappa_1^{n+d}\left(J_{n+d,0}(a_n z)\right) \sim 4n$. ☐

4.2. **Proof of Theorem 2.5.** To prove Theorem 2.5, we will apply Theorem 2.8 to the polynomials $J_{d+n,0}$.

Proof of Theorem 2.5. As can be seen directly from (4.2) b_n and c_n are exactly the first and second finite free cumulants of $J_{d+n,0}$ respectively. So to apply Theorem 2.8 it suffices to show $|m|_3(J_{d+n,0}) = o(\sqrt{(n+d)c_n^3})$. Furthermore, since the roots of $J_{n+d,0}$ are non-negative, we have $|m|_3(J_{d+n,0}) = m_3(J_{n+d})$.

Notice, from Proposition 3.2 that the leading order of m_3 is in fact $\kappa_3^{n+d}(J_{n+d})$ and it is in fact enough to prove the same bound for the third cumulant. Let $r(m)$ be such that

$$\frac{\gamma_{2(m-2)}\gamma_{2m}}{\gamma_{2(m-1)}^2} = 1 + r(m). \tag{4.6}$$

We will assume $|r(m)| \gg \frac{1}{m}$, otherwise a similar proof to the following is also available. One can check directly from (4.3) that

$$\begin{aligned}\frac{\kappa_3^{n+d}(J_{n+d,0})}{c_n^{3/2}} &= \frac{\sqrt{n+d}}{2}\left[\frac{2+\frac{(2(n+d)-3)(2(n+d)-5)}{(2(n+d)-1)^2}\frac{\gamma_{2(n+d-3)}\gamma_{2(n+d)}^2}{\gamma_{2(n+d-1)}^3}-3\frac{2(n+d)-3}{2(n+d)-1}\frac{\gamma_{2(n+d)}\gamma_{2(n+d-2)}}{\gamma_{2(n+d-1)}^2}}{1-\frac{2(n+d)-3}{2(n+d)-1}\frac{\gamma_{2(n+d)}\gamma_{2(n+d-2)}}{\gamma_{2(n+d-1)}^2}}\right]\\ &\lesssim \sqrt{n+d}\frac{|r(n+d)|^2+o(|r(n+d)|^2)}{[|r(n+d)|+o(|r(n+d)|)]^{3/2}} = o(\sqrt{n+d}).\end{aligned} \tag{4.7}$$

Thus, we can apply Theorem 2.8 to complete the proof of Theorem 2.5. □

4.3. **Proof of Theorem 2.3.** We will prove Theorem 2.3 by showing the power series for rescaled $2n^{th}$ derivative of f converges to the power series of cosine.

Proof of Theorem 2.3. From Proposition 1.5 and Theorem 2.4 we know that the even Jensen polynomials of $\frac{1}{\gamma_{2n}}f^{(2n)}(\sqrt{a_n}z)$ converge to the even Jensen polynomials of $\cos(z)$. Thus, for any fixed $d \in \mathbb{N}$ the first $2d$ coefficients in the series expansion of $\frac{1}{\gamma_{2n}}f^{(2n)}(\sqrt{a_n}z)$ converge to those of $\cos(z)$. We now show the remaining coefficients are negligible. Fix a compact subset of $K \subset \mathbb{C}$ and $\varepsilon > 0$. For $n \in \mathbb{N}$, define

$$\widetilde{f}_n(z) = \frac{1}{\gamma_{2n}}f^{(2n)}(\sqrt{a_n}z) = \sum_{k=0}^{\infty}\frac{\gamma_{2(k+n)}}{\gamma_{2n}(2k)!}a_n^k z^{2k}. \tag{4.8}$$

We let $b_{k,n} := \frac{\gamma_{2(k+n)}}{\gamma_{2n}(2k)!}a_n^k$ so we have:

$$\widetilde{f}_n(z) = \sum_{k=0}^{\infty} b_{k,n}z^{2k}. \tag{4.9}$$

At this point we note that the right-hand side of (2.3) can be written as $\bar{h}(n)n^{2-\frac{2}{\alpha}}(1+o(1))$ for some slowly varying function $\bar{h}$. From the Karamata representation theorem there exists $c, w : (0,\infty) \to [0,\infty)$ such that $c(x) \to c \in (0,\infty)$ as $x \to \infty$, $w(x) \to 0$ as $x \to \infty$, and

$$\bar{h}(x) = c(x)\exp\left(\int_1^x \frac{w(t)}{t}dt\right). \tag{4.10}$$

We can assume without loss of generality that $a_n = \bar{h}(n)^{-1}n^{-2+\frac{2}{\alpha}}$. We additionally assume n is sufficiently large such that the $1+o(1)$ term in the right-hand side of (2.3) is in $(1/2, 2)$, that $\sup_{x\geq 0} c(n+x) \leq 2c$, and that $\sup_{x\geq 0} w(n+x) \leq \frac{1}{2}$.

It follows from (4.10) and the definition of a_n that, for any $k = 0, 1, \ldots,$

$$\begin{aligned}\left|\frac{\gamma_{2(n+k)}}{\gamma_{2n}}a_n^k\right| &= \prod_{i=0}^{k-1}\left|\frac{\gamma_{2(n+i+1)}}{\gamma_{2(n+i)}}a_n\right|\\ &\lesssim 2^{\left|2-\frac{2}{\alpha}\right|k}\prod_{i=0}^{k-1}\frac{\bar{h}(n+i)}{\bar{h}(n)}\left(1+\frac{i}{n}\right)^{2-\frac{2}{\alpha}}\\ &\lesssim 2^{\left|2-\frac{2}{\alpha}\right|k+k}c^k k^{\frac{k}{2}}\prod_{i=0}^{k-1}\left(1+\frac{i}{n}\right)^{2-\frac{2}{\alpha}}\\ &\lesssim 2^{\left|2-\frac{2}{\alpha}\right|k+k}c^k k^{\frac{3k}{2}}.\end{aligned} \tag{4.11}$$

Hence, for n sufficiently large

$$|b_{k,n}| \lesssim \frac{k^{c'k}}{(2k)!}, \tag{4.12}$$

uniformly in $k \in \mathbb{N}$ for some $c' \in (0, 2)$.

Thus, there exists $d \in \mathbb{N}$ such that for n sufficiently large

$$\sum_{k=2d+2}^{\infty} |b_{k,n}| C^{2k} < \varepsilon, \tag{4.13}$$

where C is chosen sufficiently large that $|z| < C$ for all $z \in K$. We then conclude that

$$\begin{aligned} \limsup_{n\to\infty} \sup_{z\in K} |\widetilde{f}_n(z) - \cos(z)| &< \limsup_{n\to\infty} \sum_{k=0}^{2d} \left| b_{k,n} - \frac{(-1)^k}{(2k)!} \right| C^{2k} + \varepsilon \\ &= \varepsilon. \end{aligned} \tag{4.14}$$

As $\varepsilon > 0$ was arbitrary, the proof is complete. □

5. Proofs of the theorems in Section 2.2

We begin by fixing $d \in \mathbb{N}$, and letting

$$q_{d,n}(z) = z^n (z-1)^d. \tag{5.1}$$

We can use (3.2) to write the derivative of P_m from (2.10), in terms of the free convolution with $q_{d,n}$:

$$P_{d,n}(z) := \frac{1}{z^n} \left[P_{d+n}(z) \boxtimes_{n+d} q_{d,n}(z) \right] = \frac{1}{(n+d)_n} \left(\frac{d}{dz} \right)^n P_{n+d}(z). \tag{5.2}$$

It will also be convenient to work with the free finite multiplicative convolution without the z^{-n} prefactor and we define:

$$\widehat{P}_{d,n}(z) = P_{d+n}(z) \boxtimes_{n+d} q_{d,n}(z). \tag{5.3}$$

Our goal is to compute asymptotic (in n) formulas for the finite free cumulants of these polynomials in terms of the finite free cumulants of P_m. For simplicity of presentation we drop the degree dependence in cumulants of P_{d+n} and use the notation $\kappa_j^P \equiv \kappa_j^{d+n}(P_{d+n})$.

5.1. Law of large numbers and the proof of Theorem 2.7.

We begin be computing the moments and cumulants of $\widehat{P}$.

Lemma 5.1. *Let P_m satisfy the assumptions of Theorem 2.7. For any $j \in \mathbb{N}$*

$$\kappa_j^{n+d}(\widehat{P}_{d,n}) = \left(\kappa_1^P\right)^j \frac{(d+n)^j}{(d+n)_j} \frac{d}{d+n} + o\left(\frac{1}{n}\right), \tag{5.4}$$

and

$$m_j(\widehat{P}_{d,n}) = \left(\kappa_1^P\right)^j \frac{d}{d+n} + o\left(\frac{1}{n}\right) \tag{5.5}$$

as $n \to \infty$.

Proof. From Theorem 3.3 we know

$$\kappa_j(\widehat{P}_{d,n}) = \frac{(-1)^{j-1}}{(n+d)^{j+1}(j-1)!} \sum_{\substack{\sigma,\tau\in\mathcal{P}(j)\\ \sigma\vee\tau=1_j}} (n+d)^{|\sigma|+|\tau|}\mu(0_j,\sigma)\mu(0_j,\tau)\kappa_\sigma^P\kappa_\tau(q_{d,n}) \tag{5.6}$$

and

$$m_j(\widehat{P}_{d,n}) = \frac{(-1)^{j-1}}{(n+d)^{j+1}(j-1)!} \sum_{\substack{\sigma,\tau\in\mathcal{P}(j)\\ \sigma\vee\tau=1_j}} (n+d)^{|\sigma|+|\tau|}\mu(0_j,\sigma)\mu(0_j,\tau)\kappa_\sigma^P m_\tau(q_{d,n}). \tag{5.7}$$

For any j, $m_j(q_{d,n}) = |m|_j(q_{d,n}) = \frac{d}{n+d}$, and thus from (3.10), we also have $|\kappa_j^{d+n}(q_{d,n})| \leq C_j\frac{d}{n+d}$, so any term in the sums in (5.7) or (5.6) is of order at most $(n+d)^{|\sigma|-j-1}|\kappa_\sigma^P|$. Let $r(n) := m_2(P_{n+d})/(n+d)$, by assumption $r(n) > 0$ and converges to 0 as $n\to\infty$. From convexity, for $k\geq 2$, we bound the k^{th} moment by the 2^{nd} moment to get:

$$|m_k|(P_{n+d}) \leq (n+d)^{k-1}r(n)^{k/2}. \tag{5.8}$$

We now bound the κ_σ^P terms in (5.7) and (5.6). For each singleton block of σ we use that $|\kappa_1^P|$ converges to a constant. For the remaining blocks of σ we use the cumulant bound, (3.10), and (5.8) to get

$$|\kappa_i^P| \leq C_i(n+d)^{i-1}r(n)^{i/2},$$

For a general cumulant κ_σ^P, with $\sigma\in\mathcal{P}(j)$, we let ℓ be the number of single elements of σ, we then have

$$|\kappa_\sigma^P| \lesssim (n+d)^{j-\ell-(|\sigma|-\ell)}r(n)^{\frac{j-\ell}{2}} \tag{5.9}$$

rearranging gives:

$$(n+d)^{|\sigma|-j-1}|\kappa_\sigma^P| \lesssim (n+d)^{-1}r(n)^{\frac{j-\ell}{2}}. \tag{5.10}$$

By assumption, $r(n)\to 0$, so the leading order term in (5.7) and (5.6) occurs when $j=\ell$, which is the case only when $\sigma = 0_j$. From the restriction on the sums for σ and τ we thus have that $\tau = 1_j$. The proof is completed by plugging in these choices for σ and τ. □

Proof of Theorem 2.7. We note, as $P_{d,n}$ is simply $\widehat{P}_{d,n}$ with the roots at zero removed we have the simple relation between moments given by

$$m_j(P_{d,n}) = \frac{n+d}{d}m_j(\widehat{P}_{d,n}), \tag{5.11}$$

Thus,

$$m_j(P_{d,n}) = (\kappa_1^P)^j + o(1). \tag{5.12}$$

The moments of $P_{d,n}$ converge to $\{1, a, a^2, \ldots, a^d\}$, which are the moments of the polynomial $(z-a)^d$. As the coefficients of a monic polynomial are a continuous function of the moments of its empirical root measure, we have convergence of the coefficients as a vector in $\mathbb{R}^{d+1}$ equipped with any norm. Let K be some compact subset of $\mathbb{C}$, it is straightforward to show that for any polynomial $p(z)$ of degree at most d that $\sup_{z\in K}|p(z)| \leq C_{d,K}\|p\|_\infty$ where $C_{d,K} > 0$ is some constant that depends only on K and d and $\|p\|_\infty$ is the $\ell^\infty(\mathbb{R}^{d+1})$ norm of the vector of coefficients of p. Thus for polynomials of degree at most d convergence of the coefficients is equivalent to uniform convergence on compact subsets. □

5.2. **Central limit theorem and the proof of Theorem 2.8.** By assumption, $m_1(P_{n+d})$ is smaller than the dilation by $\mathfrak{D}_{\sqrt{n+d}}$ in Theorem 2.8. Thus, by continuity we can assume without loss of generality that $\kappa_1^P \equiv 0$. We saw in Theorem 2.7 that if $\kappa_1^P \sim a$, then the polynomials $P_{d,n}$ satisfy a law of large numbers with no additional scaling required. With $\kappa_1^P = 0$, we will see that the order of moments/cumulants for $\widehat{P}_{d,n}$ and $P_{d,n}$ become quiet different, and will require some rescaling to recover non-degenerate limits. From the perspective of moving from a law of large numbers to a central limit theorem the rescaling by $\sqrt{n+d}$ is natural.

The next lemma follows immediately from (3.10) and convexity, as in (5.8).

Lemma 5.2. *Let P_m satisfy the assumptions of Theorem 2.8. Let $r(n) \to 0$ be positive such that $|m|_{2+\epsilon}(P_{n+d}) \leq r(n)(n+d)^{\varepsilon/2}$ for some $\varepsilon > 0$. Then, for all $j \geq 2+\varepsilon$*

$$|\kappa_j^P| \lesssim |m|_j(P_{n+d}) \leq (n+d)^{\frac{j}{2}-1} r(n)^{\frac{j}{2+\varepsilon}}. \tag{5.13}$$

Lemma 5.3. *Under the assumptions of Theorem 2.8, for any odd $j \in \mathbb{N}$:*

$$m_j(\widehat{P}_{d,n}) = o\left(\frac{1}{n^{j/2+1}}\right), \tag{5.14}$$

and for any even $j \in \mathbb{N}$:

$$m_j(\widehat{P}_{d,n}) = \frac{(-1)^{j/2+1}}{(n+d)^{1+j/2}(j-1)!} \left(\kappa_2^P\right)^{j/2} \sum_{\substack{\tau \in \mathcal{P}(j) \\ \sigma \vee \tau = 1_j \\ \sigma \in \mathcal{P}_2(j)}} d^{|\tau|} \mu(0_j, \tau) + o\left(\frac{1}{n^{1+j/2}}\right) \tag{5.15}$$

as $n \to \infty$.

Proof. Fix d, j. We again consider (5.7). The moments of $q_{d,n}$ are

$$m_k(q_{d,n}) = \frac{d}{n+d}. \tag{5.16}$$

Thus, the order of any term in the summation of (5.7) is $(n+d)^{-j-1+|\sigma|}|\kappa_\sigma^P|$. However, if $|\sigma| > j/2$, then there exists $V \in \sigma$ such that $|V| = 1$, and by the assumption that κ_1^P is zero that term is actually zero.

Let $r(n)$ be as in Lemma 5.2 and $\sigma \in \mathcal{P}(j)$. Let a be the number of blocks of σ of size 2 and let $\ell = |\sigma| - a$. Let $V_1, V_2, \ldots, V_\ell$ be the blocks of σ of size greater than 2. Note that by definition

$$\sum_{k=1}^{\ell} |V_k| = j - 2a. \tag{5.17}$$

Then, the order $(n+d)^{|\sigma|-j-1}|\kappa_\sigma^P|$ is at most

$$\begin{aligned} |(n+d)^{|\sigma|-j-1}\kappa_\sigma^P| &\lesssim (n+d)^{|\sigma|-j-1+\frac{1}{2}(j-2a)-\ell} r(n)^{\frac{j-2a}{2+\varepsilon}} \\ &\lesssim (n+d)^{a-j-1+\frac{1}{2}(j-2a)} r(n)^{\frac{j-2a}{2+\varepsilon}} \\ &= (n+d)^{-\frac{j}{2}-1} r(n)^{\frac{j-2a}{2+\varepsilon}}, \end{aligned} \tag{5.18}$$

which is $o(n^{-1-j/2})$ unless $2a = j$ and hence $\ell = 0$. If j is odd, then $\ell \neq 0$, completing the proof of (5.14). The proof is completed by noting that (5.15) is the restriction of (5.7) to the terms where σ is a pair partition. □

From (5.11), we can immediately use Lemma 5.3 to compute the moments of $P_{d,n}$.

Lemma 5.4. *Under the assumptions of Theorem 2.8*

$$m_j(P_{d,n}) = o\left(\frac{1}{n^{j/2}}\right), \tag{5.19}$$

for any odd $j \in \mathbb{N}$. If $j \in \mathbb{N}$ is even, then

$$\begin{aligned} m_j(P_{d,n}) = \frac{(-1)^{j/2+1}}{d(n+d)^{j/2}(j-1)!} \left(\kappa_2^P\right)^{j/2} \sum_{\sigma \in \mathcal{P}_2(j)} \sum_{\sigma \vee \tau = 1_j} d^{|\tau|} \mu(0_j, \tau) \\ + o\left(\frac{1}{n^{j/2}}\right) \end{aligned} \tag{5.20}$$

as $n \to \infty$.

We will now prove Theorem 2.8 by computing the moments of the d^{th} Hermite polynomial, He_d, and seeing that they agree, to leading order, with Lemma 5.4. In order to do this we need the following technical lemma from [3].

Lemma 5.5 (Lemma 4.5 from [3])**.** *For any $\sigma \in \mathcal{P}(j)$*

$$\sum_{\pi \geq \sigma} \mu(\pi, 1_j)(d)_\pi = \sum_{\sigma \vee \tau = 1_j} d^{|\tau|} \mu(0_j, \tau). \tag{5.21}$$

Proof of Theorem 2.8. He_d is the unique monic degree d real rooted polynomial such that $\kappa_j^d(\mathrm{He}_d) = d\delta_{j2}$, where δ_{j2} is the Kronecker delta, which is one if $j = 2$ and zero otherwise. However, to apply Lemma 5.4 it will be more convenient to consider the moments of He_d, which we compute from the moment-cumulant formula, Proposition 3.2:

$$m_j(\mathrm{He}_d) = \frac{(-1)^{j/2+1}}{d(j-1)!} \sum_{\sigma \in \mathcal{P}_2(j)} \sum_{\pi \geq \sigma} \mu(\pi, 1_j)(d)_\pi. \tag{5.22}$$

Next note that from Lemma 5.4 for any even $j \in \mathbb{N}$

$$m_j(D_{\sqrt{n+d}} P_{d,n}) = \frac{(-1)^{j/2+1}}{d(j-1)!} \left(\kappa_2^P\right)^{j/2} \sum_{\sigma \in \mathcal{P}_2(j)} \sum_{\sigma \vee \tau = 1_j} d^{|\tau|} \mu(0_j, \tau) + o(1). \tag{5.23}$$

Thus, by Lemma 5.5, the $n \to \infty$ limit of (5.23) is (5.22), completing the proof. □

Proof of Corollary 2.10. It follows from well known results in random matrix theory (see for example Theorems 2.3.24 and 2.4.2 in [46]) that almost surely

$$\frac{\kappa_2^{n+d}(\Phi_{n+d})}{n} \sim 1, \quad \kappa_1^{n+p}(\Phi_{n+d}) = o\left(1\right), \text{ and } \quad m_4^{n+d}(\Phi_{n+d}) = O(n^2). \tag{5.24}$$

Hence, shifting and normalizing Φ_{n+d} to have mean 0 and second cumulant 1 is undone by the rescaling in Theorem 2.8. So we may apply Theorem 2.8 directly to Φ_{n+d}. □

5.3. **Optimality of Theorem 2.7 and Theorem 2.8.** The moment assumptions in both Theorems 2.7 and 2.8 are optimal. To see this we consider three counterexamples when the moment conditions are relaxed. Consider the polynomials

$$P_m(z) = z^m - \frac{m^2}{4} z^{m-2}. \tag{5.25}$$

A simple computation shows $|m|_1(J_m) = \frac{1}{2}$, $m_2(J_m) = m$ and

$$\lim_{n\to\infty} \frac{1}{(n+d)_n} \left(\frac{d}{dz}\right)^n P_{d+n}(z) = z^d - \frac{d(d-1)}{4} z^{d-2}.$$

Hence, the second moment assumption in Theorem 2.7 cannot be relaxed in general. A similar counterexample for Theorem 2.8 without the higher order moment bound would be $P_m(z) = z^m - \frac{m}{2} z^{m-2}$. Both limit theorem are sensitive to the polynomials having just a few very large roots. One can use Jensen's inequality to show that having a small number, i.e. bounded in the degree, of maximally large roots, i.e. to maintain first/second moment 1, is the only way polynomials can violate the moment assumptions of Theorems 2.7 and 2.8. One can also check that the classical Jensen polynomials of cosine also violate the moment conditions, and conclusion, of Theorem 2.8.

5.4. **Poisson limit theorem and the proof of Theorem 2.11.** In this section we prove Theorem 2.11 as corollary of Theorem 2.7. An alternative proof is available by considering $D^{2n} P_{n+d}(z^2)$, applying Theorem 2.8, and using the relationship between Hermite and Laguerre polynomials. However, we choose the below proof as it can be generalized under straightforward modifications to repeated application of the more general operators discussed in Remark 2.12. For clarity of presentation we consider only the operator M.

Proof of Theorem 2.11. We define a new array of polynomials $\{W_{d,n}\}_{d,n=1}^{\infty}$ by

$$W_{d,n}(z) = \frac{1}{(2n+2d)_{2n}} M^n W_{n+d,0}(z), \quad W_{m,0}(z) = P_m(z), \tag{5.26}$$

where M is defined by (1.15).

We then factorize M^n into the product of D^n and a differential operator that only depends on z and D through zD:

$$\frac{1}{(2n+2d)_{2n}} M^n = \frac{(n+d)_n}{(2n+2d)_{2n}} \mathcal{F}_n \frac{D^n}{(n+d)_n}, \tag{5.27}$$

where

$$\mathcal{F}_n = \prod_{k=0}^{n-1} \left(2k + [1 + 2zD]\right), \tag{5.28}$$

which is a polynomial in zD. As an operator on polynomials, $\mathcal{F}_n$ is degree preserving. Then, by Lemma 3.1

$$\begin{aligned}\frac{1}{(2n+2d)_{2n}}M^n P_{n+d}(z) &= \frac{(n+d)_n}{(2n+2d)_{2n}}\mathcal{F}_n \frac{D^n}{(n+d)_n}P_{n+d}(z)\\ &= \frac{D^n}{(n+d)_n}P_{n+d}(z) \boxtimes_d \frac{(n+d)_n}{(2n+2d)_{2n}}\mathcal{F}_n(z-1)^d\\ &= \frac{D^n}{(n+d)_n}n^{-n-d}P_{n+d}(nz) \boxtimes_d \frac{(n+d)_n}{(2n+2d)_{2n}}\mathfrak{D}_n\mathcal{F}_n(z-1)^d.\end{aligned} \tag{5.29}$$

In the last term we simply moved some scaling from P_{n+d} to $(z-1)^d$ using that scalars can be moved across multiplicative convolutions. We know from Theorem 2.7 that

$$\lim_{n\to\infty}\frac{D^n}{(n+d)_n}n^{-n-d}P_{n+d}(nz) = (z-a)^d. \tag{5.30}$$

In the basis $\left\{z^0, z^1, \dots, z^d\right\}$ $\mathcal{F}_n$ is a diagonal matrix:

$$\mathcal{F}_n z^j = 2^n \prod_{k=0}^{n-1}\left(k+\frac{1}{2}+j\right)z^j \tag{5.31}$$

for $0 \le j \le d$. For each $j \in [d]$, we factorize the coefficient as:

$$\prod_{k=0}^{n-1}\left(k+\frac{1}{2}+j\right) = \left(n+j-\frac{1}{2}\right)_j\left(n-\frac{1}{2}\right)_{n-d}\left(d-\frac{1}{2}\right)_{d-j}$$

Hence

$$\begin{aligned}\mathcal{F}_n(z-1)^d &= 2^n\sum_{j=0}^{d}\binom{d}{j}(-1)^{d-j}\left(\left(n+j-\frac{1}{2}\right)_j\left(n-\frac{1}{2}\right)_{n-d}\left(d-\frac{1}{2}\right)_{d-j}\right)z^j\\ &= 2^n d!(-1)^{-d}\left(n-\frac{1}{2}\right)_{n-d}\sum_{j=0}^{d}\frac{\left(d-\frac{1}{2}\right)_{d-j}}{j!(d-j)!}\left(n+j-\frac{1}{2}\right)_j(-1)^j z^j.\end{aligned}$$

We then rescale the argument by n to get:

$$\mathfrak{D}_n\mathcal{F}_n(z-1)^d = 2^n d!(-1)^{-d}\left(n-\frac{1}{2}\right)_{n-d}n^d\sum_{j=0}^{d}\frac{\left(d-\frac{1}{2}\right)_{d-j}}{j!(d-j)!}\frac{\left(n+j-\frac{1}{2}\right)_j}{n^j}(-1)^j z^j. \tag{5.32}$$

Since $\left(n+j-\frac{1}{2}\right)_j$ is a monic degree j polynomial in n, we have that $\frac{\left(n+j-\frac{1}{2}\right)_j}{n^j} \to 1$ as $n\to\infty$. We note that the n-dependent coefficient on the right most side of (5.32) exactly cancels with the scaling term in (5.29):

$$\lim_{n\to\infty}\frac{(n+d)_n}{(2n+2d)_{2n}}2^n n^d\left(n-\frac{1}{2}\right)_{n-d} = 1. \tag{5.33}$$

So $\frac{(n+d)_n}{(2n+2d)_{2n}}\mathfrak{D}_n\mathcal{F}_n(z-1)^d$ converges to the monic polynomial:

$$d!(-1)^{-d}\sum_{j=0}^{d}\frac{(-1)^{-d}\left(d-\frac{1}{2}\right)_{d-j}}{j!(d-j)!}z^j = d!(-1)^{-d}L_d^{\left(-\frac{1}{2}\right)}(z).$$

Hence,

$$\lim_{n\to\infty} W_{d,n}(z) = (z-a)^d \boxtimes_d d!(-1)^{-d} L_d^{(-\frac{1}{2})}(z). \tag{5.34}$$

Normalizing such that $a = 1$ completes the proof. □

6. Proof of Lemma 2.2 and results in complex analysis

In this section we collect some results in complex analysis and prove Lemma 2.2. First, we note that (2.4) follows from (2.3), and hence we focus only on the latter.

We express g as the product and define the series coefficients e_m by

$$g(z) = \prod_{k=1}^{\infty}\left(1 - \frac{z}{x_k^2}\right) = \sum_{k=0}^{\infty}(-1)^k e_k z^k. \tag{6.1}$$

For simplicity, we denote the square of the roots by $r_k = x_k^2$ and let $n(r) = |\{k \in \mathbb{N} : r_k \in [0,r]\}|$. From Assumption 2.1 $n(r) \sim \hat{h}(r)r^{\alpha/2}$ for some positive slowly varying function $\hat{h}$. We define $\rho = \frac{\alpha}{2}$. Before we proceed we note that standard computations yield the following integrals, which we will use in the limits below.

Lemma 6.1. *Let $\rho \in (0,1)$. Then,*

$$\int_0^\infty \frac{u^\rho}{(1+u)^2}\,du = \rho\pi\csc(\pi\rho), \tag{6.2}$$

$$\int_0^\infty \frac{u^\rho}{(1+u)^3}\,du = \frac{1}{2}\rho(1-\rho)\pi\csc(\pi\rho), \tag{6.3}$$

and

$$\int_0^\infty \frac{u^\rho}{u(1+u)}\,du = \pi\csc(\pi\rho). \tag{6.4}$$

6.1. Proof of Lemma 2.2.

We use a saddle point argument similar to that of [37].

Proof. Beginning from Cauchy's integral formula

$$e_k = \frac{1}{2\pi i}\int_\Gamma z^{-k} g(z)\frac{1}{z}dz, \tag{6.5}$$

where Γ is a circle centered at the origin with radius to be chosen later, we define the function $\phi_k(z) = \log g(z) - k\log z$, where log is the logarithm with the branch cut along the negative imaginary axis. Our goal is to find a saddle point σ_k of ϕ_k and a circle Γ centered at the origin passing through σ_k such that

(1) $\phi_k'(\sigma_k) = 0$ and $\phi_k''(\sigma_k) > 0$.
(2) The contribution of the arc of Γ of length $\phi_k''(\sigma_k)^{-1/2}$ centered at σ_k is $\exp(\phi_k(\sigma_k))\sqrt{2\pi/\phi_k''(\sigma_k)}$.
(3) The contributions from the remainder of Γ is small.

The major simplification available compared to [37] is that every sum we will consider converges absolutely and the saddle point will sit exactly on the negative real line. The derivative of ϕ_k is given by

$$\phi_k'(z) = -\frac{k}{z} + \sum_{k=1}^{\infty}\frac{1}{z - r_k} =: -\frac{k}{z} + s(z). \tag{6.6}$$

Thus, we are looking for $z \in \mathbb{R}$ such that

$$zs(z) = k, \tag{6.7}$$

and $-z \gg 1$. Define the measure $\Pi_g = \sum_{k=1}^{\infty} \delta_{r_k}$, where δ_{r_k} is the point mass at r_k, and throughout assume $z = -r$ and large $r > 0$. We will use the asymptotic notation $\sim$ under the assumption that $r \to \infty$ or $k \to \infty$; we distinguish between these two limits with $\sim_r$ and $\sim_k$, respectively. Integration by parts gives

$$\begin{aligned} -s(z) &= \int_0^\infty \frac{1}{r+t} d\Pi_g(t) \\ &= \int_0^\infty r^{-2} \frac{1}{(1+t/r)^2} n(t)dt \\ &= \int_0^\infty r^{-1} \frac{n(ru)}{(1+u)^2} du \\ &\sim_r \left(\int_0^\infty \frac{u^\rho}{(1+u)^2} du \right) r^{-1+\rho} \hat{h}(r). \end{aligned} \tag{6.8}$$

Hence, $-rs(-r) \sim_r \pi\rho \csc(\pi\rho) r^\rho \hat{h}(r)$, and by the intermediate value theorem, (6.7) has at least one solution σ_k on the negative real line such that

$$\sigma_k \sim_k - \left(\frac{k}{c_\rho \hat{h}(-\sigma_k)} \right)^{1/\rho} = - \left(\frac{k}{\pi\rho \csc(\pi\rho) \hat{h}(-\sigma_k)} \right)^{1/\rho}. \tag{6.9}$$

For our purposes the uniqueness of σ_k is immaterial, so we take any choice satisfying (6.9). The higher derivatives of $s^{(j)}$ are

$$s^{(j)}(z) = (-1)^j (j)! \sum_{k=1}^\infty \frac{1}{(z - r_k)^{j+1}}. \tag{6.10}$$

Following similarly to (6.8)

$$s^{(j)}(-r) \sim_r -(j+1)! \left(\int_0^\infty \frac{u^\rho}{(1+u)^{j+2}} \right) r^{-j-1+\rho} \hat{h}(r). \tag{6.11}$$

We take Γ in (6.5) to be a circle of radius $|\sigma_k|$ centered at the origin. Let $\Gamma_1 = \{z : z = \sigma_k e^{i\theta},\ -k^{-\delta} \le \theta \le k^{-\delta}\}$ be a small arc around σ_k for some fixed $\delta \in (1/3, 1/2)$. Define

$$v_k(\theta) = \phi_k(\sigma_k e^{i\theta}). \tag{6.12}$$

It is straightforward to check using (6.11) that

$$v_k'(0) = 0, \quad v_k''(0) \sim_k k \left[-1 + \frac{2 \int_0^\infty \frac{u^\rho}{(1+u)^3} du}{\int_0^\infty \frac{u^\rho}{(1+u)^2} du} \right], \text{ and } \sup_{|\theta| \le k^{-\delta}} v_k'''(\theta) = O(k). \tag{6.13}$$

From Lemma 6.1, it follows that

$$-1 + \frac{2 \int_0^\infty \frac{u^\rho}{(1+u)^3} du}{\int_0^\infty \frac{u^\rho}{(1+u)^2} du} = -1 + \frac{\pi\rho(1-\rho)\csc(\pi\rho)}{\pi\rho\csc(\pi\rho)} = -\rho. \tag{6.14}$$

and

$$v_k''(0) \sim_k -\rho k. \tag{6.15}$$

Lemma 6.2. *We have*

$$\frac{\int_{\Gamma_1} \frac{g(z)}{z^{k+1}} dz}{g(\sigma_k) \sigma_k^{-k} (\rho k)^{-1/2}} \to i\sqrt{2\pi} \tag{6.16}$$

as $k \to \infty$.

Proof. Following an approach similar to [37], we parametrize Γ_1 in terms of the angle θ and Taylor expand v_k up to third order using (6.13) to see that

$$\int_{\Gamma_1} \frac{g(z)}{z^{k+1}} dz = i \int_{-k^{-\delta}}^{k^{-\delta}} \exp\left(v_k(0) + v_k'(0)\theta + v_k''(0)\frac{\theta^2}{2} \right) d\theta (1 + o(1)), \tag{6.17}$$

where $\exp(v_k(0)) = g(\sigma_k)\sigma_k^{-k}$ and $v_k'(0) = 0$. For the final term we use the substitution $w = \theta\sqrt{-v_k''(0)}$ to see that

$$\int_{-k^{-\delta}}^{k^{-\delta}} \exp\left(v_k''(0)\frac{\theta}{2} \right) d\theta = \frac{1}{\sqrt{-v_k''(0)}} \int_{-\sqrt{-v_k''(0)}k^{-\delta}}^{\sqrt{-v_k''(0)}k^{-\delta}} \exp\left(-\frac{w^2}{2} \right) dw. \tag{6.18}$$

The proof then follows from (6.15). □

Let $\Gamma_2 = \Gamma \setminus \Gamma_1$. For $z \in \Gamma_2$

$$\frac{\mathrm{Re}(z)}{|\sigma_k|} > \cos(\pi \pm k^{-\delta}) = -1 + \frac{k^{-2\delta}}{2} + O\left(k^{-4\delta}\right). \tag{6.19}$$

In the remainder of the proof we aim only to control the relative size of $g(\sigma_k)$. Thus for convenience we use the principle branch, ln, of the logarithm below. Using (6.25)

$$\ln\left| g\left(\sigma_k e^{ik^{-\delta}}\right)\right| - \ln g(\sigma_k) \lesssim -k^{1-2\delta}. \tag{6.20}$$

Thus, $\sup_{z\in\Gamma_2} \frac{|g(z)|}{g(\sigma_k)} = O(\exp(-k^{\varepsilon}))$ for some sufficiently small $\varepsilon > 0$.

It then follows that

$$e_k = g(\sigma_k)\sigma_k^{-k}(2\pi\rho k)^{-1/2}(1 + o(1)). \tag{6.21}$$

Additionally, note that

$$\frac{d}{dz} \ln g(z) = s(z), \tag{6.22}$$

and σ_k is defined to be a solution to

$$s(\sigma_k) = \frac{k}{\sigma_k}. \tag{6.23}$$

One can see that

$$(k+1)\left[\frac{\sigma_k}{\sigma_{k+1}} - 1\right] \geq \ln g(\sigma_k) - \ln g(\sigma_{k+1}) \geq k\left[1 - \frac{\sigma_{k+1}}{\sigma_k}\right]. \tag{6.24}$$

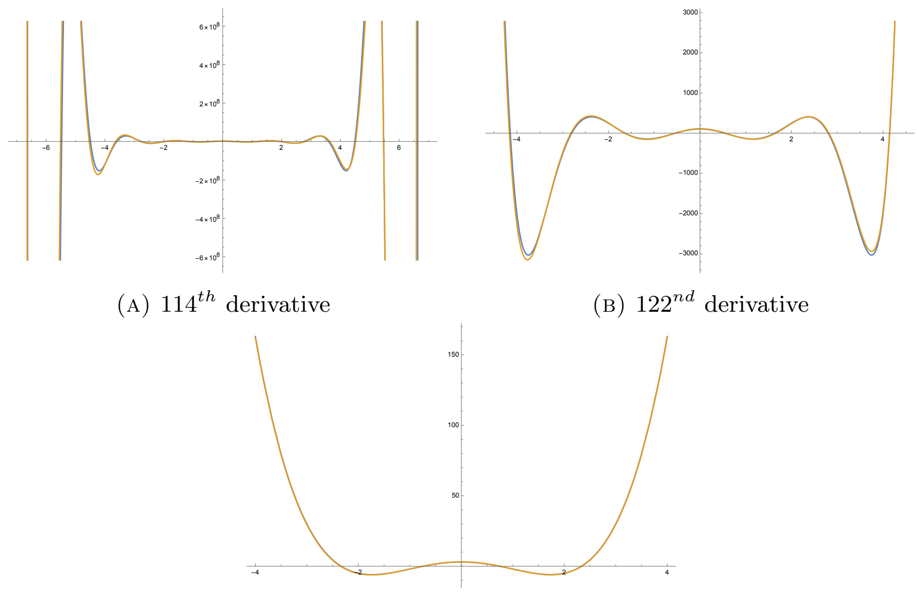
It then follows that

$$\ln g(\sigma_k) - \ln g(\sigma_{k+1}) + k\ln|\sigma_k| - k\ln|\sigma_{k+1}| = o(1). \tag{6.25}$$

We can then apply (6.9), (6.21), and (6.25) to see

$$\begin{aligned} \frac{e_k}{e_{k+1}} &\sim_k \frac{g(\sigma_k)\sigma_k^{-k}(2\pi\rho k)^{-1/2}}{g(\sigma_{k+1})\sigma_{k+1}^{-k-1}(2\pi\rho(k+1))^{-1/2}} \\ &\sim_k \frac{g(\sigma_k)}{g(\sigma_{k+1})}\left(\frac{\sigma_{k+1}}{\sigma_k}\right)^k \sigma_{k+1} \\ &\sim_k \left(\frac{(k+1)}{\pi\rho\csc(\rho\pi)\hat{h}(-\sigma_{k+1})}\right)^{1/\rho}. \end{aligned} \tag{6.26}$$

The proof of (2.3) is then completed by noting $e_k = (-1)^k\gamma_{2k}/(2k)!$ and using properties on the composition of regularly varying functions (see [40, Proposition 0.8])



(A) 114^{th} derivative

(B) 122^{nd} derivative

(C) 126^{nd} derivative

FIGURE 1. Derivatives of p_{130} (blue) compared to the Hermite polynomials (orange).

to rewrite $\hat{h}(\sigma_{k+1})$ as one slowly varying function. As mentioned at the beginning of the section, (2.4) follows from (2.3), so the proof of Lemma 2.2 is completed.

□

7. Examples

In this section, we give some examples of polynomials and entire functions which satisfy the assumptions of our main results. Of course the most important polynomial examples for our purposes are the even Jensen polynomials of functions satisfying Assumption 2.1. As we have already discussed in Section 2.2, random polynomials with iid roots and characteristic polynomials of random matrices serve as examples existing in the literature. We now discuss a few other examples.

We choose the 130 degree polynomial, p_{130}, with roots placed, somewhat arbitrarily at $\big(((-26,26)\cap\mathbb{Z}/2)\cup((-27,27)\cap((\mathbb{Z}/2)^{3/2}+\{3/2\}))\big)\setminus[-2,0]$, where $(\mathbb{Z}/2)^{3/2}$ denotes all numbers of the form $\pm(|k|/2)^{3/2}$ for some $k\in\mathbb{Z}$ and $+$ denotes the Minkowski sum. In Figure 1, we plot the 114^{th}, 122^{nd}, and 126^{th} derivative of p_{130}, after rescaling p_{130} so the its empirical root measure has mean 0 and variance 130, and normalizing the derivative to make it a monic polynomial. We also plot the corresponding Hermite polynomials.

For entire functions, we first follow the lead of [37]. Let $\beta\in(0,1)$ and $N_\beta=\{y_j\}_{j=1}^{\infty}$ be a Poisson point process with intensity measure having density $\beta x^{-(1+\beta)}$ on $(0,\infty)$. We define the entire function g_β as

$$g_\beta(z)=\prod_{k=1}^{\infty}(1-y_k z)\,. \tag{7.1}$$

Then, one can check using standard properties of the Poisson point processes N_β (see for example [9, 24]) that the random entire function $f_\alpha(z) = g_{\alpha/2}\left(z^2\right)$ almost surely satisfies Assumption 2.1 for any $\alpha \in (0, 2)$. Hence, Cosine Universality holds almost surely for f_α, as does Hermite and Laguerre Universality for its Jensen polynomials via Theorems 2.3, 2.4, and 2.5. The case $\alpha = 1$ is exactly an even version of the random entire function considered in [37].

For a deterministic example, we consider the Bessel function of the first kind J_ν. There are multiple definitions for J_ν, and we consider the series definition

$$J_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k k! \Gamma(\nu + k + 1)} z^{2k} \tag{7.2}$$

for some $\nu \geq 0$. For this choice of ν the zeros of J_ν are all real. We remove the (potential) branch point at $z = 0$ and consider the functions

$$\widetilde{J}_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k k! \Gamma(\nu + k + 1)} z^{2k}. \tag{7.3}$$

It is known [11, Chapter 10.21] that if $j_{k,\nu}$ is the k^{th} positive root of $\widetilde{J}_\nu$, then

$$j_{k,\nu} \sim \pi\left(k + \frac{\nu}{2} - \frac{1}{4}\right) \tag{7.4}$$

as $k \to \infty$. Assumption 2.1 follows from (7.4). Thus, Cosine, Hermite, and Laguerre Universality hold for $\widetilde{J}_\nu$ and its even Jensen polynomials via Theorems 2.3, 2.4, and 2.5.

We conclude our examples by mentioning the work of Assiotis [5] where random functions in the Laguerre–Pólya class are expressed as the limit of characteristic polynomials of unitarily invariant random Hermitian matrices. First, many of the examples discussed in [5] could be taken as our choice of function $g(z)$ in the relation $f(z) = g\left(z^2\right)$, and our universality principles would hold almost surely. Second, our work presents a different connection between random matrices and the Laguerre–Pólya class. While [5] considers scaling limits to random functions, we instead use finite free probability (where one averages over unitarily invariant random matrix ensembles) to provide a deterministic application to the Laguerre–Pólya class. While we do not explore possible deeper connections between [5] and our work here, we point out the work of Gorin and Marcus [14] connecting β-ensembles in random matrix theory (which are much of the motivation for [5]) and finite free probability.

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