# SIMULTANEOUS UNIFORMIZATION OF CHORD-ARC CURVES AND BMO TEICHMULLER SPACE

KATSUHIKO MATSUZAKI

Abstract. By using a method of simultaneous uniformization, we parametrize the space of chord-arc curves within the product of BMO Teichmüller spaces. This method provides a biholomorphic correspondence between such embeddings  $\gamma$  onto chord-arc curves  $\Gamma$  and  $\log \gamma'$  in the Banach space of BMO functions. In our previous work, through this correspondence  $\Lambda$ , we have clarified the arguments of Teichmüller space theory for chord-arc curves, solved the problem of the discontinuity of the Riemann mapping parametrization of chord-arc curves, and simplified the proof for a real-analytic diffeomorphism onto the space of reparametrizations by strongly quasisymmetric homeomorphisms. In this paper, we first review these results by renewing the arguments in an effort to minimize reliance on other results. Then, we present a new application of simultaneous uniformization to the Cauchy transform of BMO functions on a chord-arc curve. We show that the Cauchy transform can be described by the derivative of the biholomorphic map  $\Lambda$  and hence depends holomorphically on the variation of chord-arc curves. Finally, we organize and demonstrate the corresponding results for the VMO Teichmüller space.

## 1. INTRODUCTION

The universal Teichmüller space is a space that encompasses all Teichmüller spaces formulated by quasiconformal mappings. It can be viewed as the space of all normalized quasisymmetric homeomorphisms on the real line. Depending on the regularity of these mappings, the subspaces contained within the universal Teichmüller space split into two directions. The Teichmüller space of a compact hyperbolic surface is typically the space of all totally singular quasisymmetric homeomorphisms, while many Teichmüller spaces associated with function spaces consist of (locally) absolutely continuous quasisymmetric homeomorphisms.

<sup>2020</sup> Mathematics Subject Classification. Primary 30C62, 30H35, 30E20; Secondary 42A50, 42A45, 26A46.

Key words and phrases. BMO Teichmüller space, simultaneous uniformization, chord-arc curve, quasiconformal, pre-Schwarzian derivative Bers embedding, BMOA, strongly quasisymmetric, Carleson measure,  $A_{\infty}$ -weight, composition operator, conformal welding, Szegö projection, Hilbert transform, Cauchy integral, VMO.

Research supported by supported by Japan Society for the Promotion of Science (KAKENHI 23K25775 and 23K17656).

These subspaces  $T_X$  of the universal Teichmüller space T can be defined by various means, such as by quasisymmetric homeomorphisms, by Beltrami coefficients as the complex dilatations of quasiconformal mappings, by the (pre-)Schwarzian derivatives of quasiconformally extendable conformal mappings on the half-plane H, or by quasicircles which are the images of the real line  $\mathbb R$  under quasiconformal self-homeomorphisms of the whole plane C. The first step in studying the structure of a given Teichmüller space  $T_X$  is to establish the correspondence between these different representations. Furthermore, if  $T_X$ is defined by a family of absolutely continuous quasisymmetric homeomorphisms  $h$ , the logarithm of their derivatives  $\log h'$  form a function space, which can be used to provide  $T_X$  with more interesting analytic structures. We call generically such  $T_X$  absolutely continuous Teichmüller spaces.

To investigate a family of quasicircles, we introduce a method called simultaneous uniformization by Bers, which has been used in the theory of deformation of quasi-Fuchsian groups and associated hyperbolic 3-manifolds. Especially, complex analytic aspects of Thurston's theory are built upon this foundation. We apply this method for the analysis of families of curves Γ. This allows us to coordinate them in the direct product of their corresponding Teichmüller spaces  $T_X$ . Furthermore, we can prove a biholomorphic relationship between the product of  $T_X$  and a domain in the function space to which  $\log \gamma'$  of a quasisymmetric embedding  $\gamma : \mathbb{R} \to \Gamma \subset \mathbb{C}$  belongs.

Specifically, we focus on the BMO Teichmüller space  $T_B$  in this paper. This is because  $T_B$  leads other absolutely continuous Teichmüller spaces  $T_X$  in the sense that  $T_B$  includes  $T_X$  in many cases and the research on  $T_B$  has advanced significantly. Additionally, within the BMO Teichmüller space, we consider a region defined by certain rectifiable quasicircles called chord-arc curves and investigate problems related to the structure of this family of curves using Teichmüller space theory. These curves possess a weaker regularity among non-fractal curves, and there have been many studies in real analysis concerning function spaces defined on them.

A locally rectifiable Jordan curve passing through infinity is called a chord-arc curve if the length of any "arc" along the curve between any two points is uniformly bounded by a constant multiple of the length of the line segment "chord" connecting the two points. If we replace the length of the "arc" with its diameter, it characterizes a quasicircle. A quasicircle  $\Gamma$  passing through infinity is also characterized as the image of  $\mathbb R$  under a quasiconformal self-homeomorphism of C. The set of all such quasicircles, up to affine translations, can be identified with the universal Teichmüller space  $T$ . Similarly, chord-arc curves are the images of  $\mathbb R$  under bi-Lipschitz self-homeomorphisms of  $\mathbb C$ .

A quasisymmetric homeomorphism  $h : \mathbb{R} \to \mathbb{R}$  is the extension of a quasiconformal selfhomeomorphism of  $\mathbb H$ . All the normalized quasisymmetric self-homeomorphisms form a group denoted by QS and it can be identified with the universal Teichmüller space  $T$ . If h is locally absolutely continuous and its derivative h ′ belongs to the class of Muckenhoupt's  $A_{\infty}$ -weights, then h is called strongly quasisymmetric. The set of all normalized strongly quasisymmetric self-homeomorphisms is a subgroup of QS denoted by SQS.

BMO functions are an important class of functions that appear and play an essential role in many problems of real analysis. Let  $BMO(\mathbb{R})$  be the space of all complex-valued BMO functions on  $\mathbb{R}$ , and BMO<sup>\*</sup>( $\mathbb{R}$ ) the subset consisting of  $\phi \in BMO(\mathbb{R})$  for which  $|e^{\phi}|$ is an  $A_{\infty}$ -weight. This forms a convex open subset of the complex Banach space BMO( $\mathbb{R}$ ). For any  $h \in SQS$ , it can be understood that  $\log h'$  is in Re BMO<sup>\*</sup>( $\mathbb{R}$ ), the subset consisting of real-valued functions. In fact, the correspondence  $SQS \to \text{Re}\,\text{BMO}^*(\mathbb{R})$  is bijective.

The BMO Teichmüller space  $T_B$  in T possesses several characterizations as described above. As a space of quasisymmetric homeomorphisms, it coincides with SQS. The corresponding space of Beltrami coefficients on the half-plane  $\mathbb H$  is denoted by  $M_B(\mathbb H)$ , for which  $T_B$  is defined by the quotient under the Teichmüller equivalence. More explicitly, an element in  $M_B(\mathbb{H})$  is defined by the Carleson measure condition for a Beltrami coefficient.

We can apply the method of simultaneous uniformization to chord-arc curves. For any  $\mu^+ \in M_B(\mathbb{H}^+)$  and  $\mu^- \in M_B(\mathbb{H}^-)$ , where  $\mathbb{H}^{\pm}$  are the upper and lower half-planes, we consider the normalized quasiconformal self-homeomorphism  $G(\mu^+, \mu^-)$  having the prescribed complex dilatations on  $\mathbb{H}^{\pm}$ . A BMO embedding  $\gamma : \mathbb{R} \to \mathbb{C}$  is given by  $\gamma = G(\mu^+, \mu^-)|_{\mathbb{R}}$  and it is well-defined by a pair of Teichmüller classes  $([\mu^+], [\mu^-])$ . The space of all BMO embeddings is identified with the product  $T_B^+ \times T_B^-$  of the BMO Teichmüller spaces. We call these pairs the Bers coordinates. Among the Bers coordinates of BMO embeddings  $\gamma : \mathbb{R} \to \mathbb{C}$ , the subset whose image  $\Gamma = \gamma(\mathbb{R})$  is a chord-arc curve is defined to be  $\tilde{T}_C \subset T_B^+ \times T_B^-$ . Any  $\gamma = \gamma([\mu^+], [\mu^-])$  for  $([\mu^+], [\mu^-]) \in \tilde{T}_C$  is locally absolutely continuous, and  $\log \gamma'$  belongs to BMO<sup>\*</sup>( $\mathbb{R}$ ).

The following theorem is a fundamental assertion in studying chord-arc curves from the viewpoint of Teichmüller spaces. This has been proved in  $[45]$ , and in this paper, we review its proof with a view towards studying the Cauchy transform on a chord-arc curve. Additionally, along the way of its proof, we include several different arguments from the existing ones.

**Theorem 1.1.**  $\tilde{T}_C$  is an open subset of  $T_B^+ \times T_B^-$ , and the map  $\Lambda([\mu^+], [\mu^-]) = \log \gamma'$ defined for  $\gamma = \gamma([\mu^+], [\mu^-])$  and  $([\mu^+], [\mu^-]) \in T_C$  is a biholomorphic homeomorphism onto an open subset of  $\text{BMO}^*(\mathbb{R})$  containing  $\text{Re BMO}^*(\mathbb{R})$ .

The importance of this result lies in the fact that the holomorphic dependence of BMO functions can be transformed into a complex-analytic structure of the Teichmüller space. Chord-arc curves have been studied from the perspective of harmonic analysis as plane curves, but research from Teichmüller space theory through such simultaneous uniformization has not been done so far. Hereinafter, we will see that such a perspective can provide concise proofs for some discussions related to chord-arc curves and clarify them.

In the above theorem, we also see that the derivative  $d_{\left(\lceil \mu^+\rceil, \lceil \mu^-\rceil\right)}\Lambda$  of the biholomorphic homeomorphism  $\Lambda$  at  $([\mu^+], [\mu^-]) \in T_C$  induces an isomorphism between the tangent spaces. Hence, according to the direct sum decomposition of the tangent space associated with  $T_B^+ \times T_B^-$ ,  $d_{([\mu^+],[\mu^-])}\Lambda$  induces the topological direct sum decomposition of BMO(R)

and the bounded projections to these factors. We denote these projections in  $BMO(\mathbb{R})$  by  $P^+_{([\mu^+],[\mu^-])}$  and  $P^-_{([\mu^+],[\mu^-])}$ . It holds  $P^+_{([\mu^+],[\mu^-])}\phi + P^-_{([\mu^+],[\mu^-])}\phi = \phi$  for every  $\phi \in \text{BMO}(\mathbb{R})$ . Let  $\Gamma$  be a chord-arc curve, and  $\Omega^+$  and  $\Omega^-$  the complementary domains of  $\mathbb C$  divided

by Γ. The Cauchy integrals of a BMO function  $\psi$  on Γ are defined by

$$
(P_{\Gamma}^{\pm}\psi)(\zeta) = \frac{-1}{2\pi i} \int_{\Gamma} \left(\frac{\psi(z)}{\zeta - z} - \frac{\psi(z)}{\zeta_0^{\pm} - z}\right) dz \qquad (\zeta \in \Omega^{\pm})
$$

for some fixed  $\zeta_0^{\pm} \in \Omega^{\pm}$ , where the line integrals over  $\Gamma$  are taken in the positive directions with respect to  $\Omega^{\pm}$ . These are holomorphic functions on  $\Omega^{\pm}$ , and have non-tangential limits almost everywhere on Γ. These boundary functions on Γ are called the Cauchy projections of  $\psi$  on Γ, denoted by the same notations  $P_{\Gamma}^{\pm}\psi$ . The Plemelj formula implies that  $P_{\Gamma}^+ \psi + P_{\Gamma}^- \psi = \psi$ .

We compare the Cauchy projections  $P_{\Gamma}^{\pm}$  with the conjugates of the projections  $P_{([\mu^+],[\mu^-])}^{\pm}$ under  $\gamma$ . Applied to a BMO function  $\psi$  on  $\Gamma$ , they induce the holomorphic functions on  $\Omega^{\pm}$  having the same jump  $\psi$  (or the same sum  $\psi$  depending on the choice of directions of the line integral) across Γ. In these circumstances, general arguments deduce that they coincide with each other. Thus, the boundedness on the BMO function space and the holomorphic dependence on  $\gamma$  possessed by  $P^{\pm}_{([\mu^+],[\mu^-])}$  can be imported into the properties of the Cauchy projections  $P_{\Gamma}^{\pm}$ .

**Theorem 1.2.** Let  $\gamma = \gamma([\mu^+], [\mu^-])$  be a BMO embedding for  $([\mu^+], [\mu^-]) \in T_C$  with a chord-arc curve  $\Gamma = \gamma(\mathbb{R})$  as its image. Then, the Cauchy projections  $P_{\Gamma}^{\pm}$  are the conjugates by  $\gamma$  of the projections  $P^{\pm}_{([\mu^+],[\mu^-])}$  associated with the topological direct sum decomposition of BMO( $\mathbb{R}$ ). Moreover,  $P_{([\mu^+],[\mu^-])}^{\pm}$  depend holomorphically on  $([\mu^+],[\mu^-]) \in$  $T_C$  as bounded linear operators acting on BMO(R).

The Cauchy transform of a BMO function on a chord-arc curve  $\Gamma$  is defined by the singular integral

$$
(\mathcal{H}_{\Gamma}\psi)(\xi) = \text{p.v.} \frac{1}{\pi} \int_{\Gamma} \left( \frac{\psi(z)}{\xi - z} - \frac{\psi(z)}{\zeta_0^{\pm} - z} \right) dz \quad (\xi \in \Gamma).
$$

By the Plemelj formula,  $\mathcal{H}_{\Gamma}$  can be represented by the Cauchy projections  $P_{\Gamma}^{\pm}$ . The Cauchy transform of BMO functions on a chord-arc curve is an important subject in real analysis. This originates in Calderón's work. Simultaneous uniformization makes it possible to investigate it in the framework of complex-analytic Teichmüller space theory.

**Corollary 1.3.** The conjugate of the Cauchy transform  $\mathcal{H}_{\Gamma}$  under  $\gamma$  represented by

$$
\mathcal{H}_{([{\mu}^+],[{\mu}^-])} = -i(P^+_{([{\mu}^+],[{\mu}^-])} - P^-_{([{\mu}^+],[{\mu}^-])})
$$

depend holomorphically on  $([\mu^+], [\mu^-]) \in \widetilde{T}_C$  as the bounded linear operators on BMO( $\mathbb{R}$ ).

This result should be useful for considering and simplifying several problems on function spaces on chord-arc curves. We demonstrate an application.

Finally, we touch on the theory of the little subspace of  $T_B$ , the VMO Teichmüller space  $T_V$ . This corresponds to the closed subspace VMO(R) of BMO(R), which consists of BMO functions whose norm equalities satisfy a canonical vanishing property. Results for various little subspaces defined by their vanishing conditions are often stated in a way that mirrors those for the original Teichmüller spaces. However, in the case of  $T_V$ , it possesses preferable properties that  $T_B$  does not, such as a topological group structure and a global section for the Teichmüller projection. Moreover, subtle differences arise when considering  $T_V$  defined on S versus on R. Emphasizing these points, we provide a concise exposition of this theory and present new observations as well.

In this paper, we only deal with the BMO Teichmüller space and chord-arc curves, but there is another important class for which our arguments are very effective. That is the integrable Teichmüller space and the Weil–Petersson curves. A separate paper  $[25]$ develops parallel arguments to those in this paper.

## 2. BMO TEICHMÜLLER SPACE AND BMOA

Let  $M(\mathbb{H})$  denote the open unit ball of the Banach space  $L^{\infty}(\mathbb{H})$  of all essentially bounded measurable functions on the half-plane  $\mathbb{H}$ . An element in  $M(\mathbb{H})$  is called a Beltrami coefficient. The universal Teichmüller space  $T$  is the set of all Teichmüller equivalence classes  $[\mu]$  of Beltrami coefficients  $\mu$  in  $M(\mathbb{H})$ . Here,  $\mu_1$  and  $\mu_2$  in  $M(\mathbb{H})$  are equivalent if  $h(\mu_1) = h(\mu_2)$  on R, where  $h(\mu) = H(\mu)|_{\mathbb{R}}$  is the boundary extension of the quasiconformal self-homeomorphism  $H = H(\mu)$  of H, called a *quasisymmetric homeomorphism*, such that its complex dilatation  $\partial H/\partial H$  is  $\mu \in M(\mathbb{H})$  and  $h(\mu)$  satisfies the normalization condition keeping the points 0, 1 and  $\infty$  fixed.

We denote the quotient projection by  $\pi : M(\mathbb{H}) \to T$ , which is called the Teichmüller projection. Thus, we can identify  $T$  with the set QS of all normalized quasisymmetric homeomorphisms  $h(\mu)$  for  $\mu \in M(\mathbb{H})$ . The topology of T is defined as the quotient topology induced from  $M(\mathbb{H})$  by  $\pi$ . The universal Teichmüller space possesses the group structure under the identification  $T \cong QS$ . The composition  $h(\mu) \circ h(\nu)$  in QS is denoted by  $[\mu] * [\nu]$  in T and the inverse  $h(\mu)^{-1}$  is denoted by  $[\mu]^{-1}$ . For every  $[\nu] \in T$ , the right translation  $r_{[\nu]} : T \to T$  on the group T is defined by  $[\mu] \mapsto [\mu] * [\nu]$ .

Let  $F^{\mu}$  denote the normalized  $(0, 1 \text{ and } \infty \text{ are fixed})$  quasiconformal self-homeomorphism of C whose complex dilatation is  $\mu \in M(\mathbb{H}^+)$  on the upper half-plane  $\mathbb{H}^+$  and 0 on the lower half-plane  $\mathbb{H}^-$ . For  $\mu_1$  and  $\mu_2$  in  $M(\mathbb{H}^+)$ , we see that  $\pi(\mu_1) = \pi(\mu_2)$  if and only if  $F^{\mu_1}|_{\mathbb{H}^-} = F^{\mu_2}|_{\mathbb{H}^-}.$ 

We define the following spaces of holomorphic functions  $\Psi$  and  $\Phi$  on  $\mathbb H$  as follows:

$$
A(\mathbb{H}) = \{ \Psi \mid \|\Psi\|_{A} = \sup_{z \in \mathbb{H}} |\text{Im } z|^{2} |\Psi(z)| < \infty \};
$$
  

$$
B(\mathbb{H}) = \{ \Phi \mid \|\Phi\|_{B} = \sup_{z \in \mathbb{H}} |\text{Im } z| |\Phi'(z)| < \infty \}.
$$

Here,  $A(\mathbb{H})$  is a complex Banach space with the hyperbolic  $L^{\infty}$ -norm  $\|\cdot\|_A$ , and  $B(\mathbb{H})$  is the Bloch space with the semi-norm  $\lVert \cdot \rVert_B$ . By ignoring the difference in constant functions, we regard  $B(\mathbb{H})$  as a complex Banach space with the norm  $\|\cdot\|_B$ .

The pre-Schwarzian derivative map  $L : M(\mathbb{H}^+) \to B(\mathbb{H}^-)$  is defined by the correspondence  $\mu \mapsto \log(F^{\mu}|_{\mathbb{H}^{-}})'$  and the Schwarzian derivative map  $S : M(\mathbb{H}^{+}) \to A(\mathbb{H}^{-})$  is defined by  $S(\mu) = L(\mu)'' - (L(\mu)')^2/2$ , where  $B(\mathbb{H})$  and  $A(\mathbb{H})$  serve appropriate spaces as the targets of L and S, respectively. Let  $D(\Phi) = \Phi'' - (\Phi')^2/2$  for  $\Phi \in B(\mathbb{H}^-)$ , which satisfies  $S = D \circ L$ . Then, D restricted to  $L(M(\mathbb{H}^+))$  is a holomorphic bijection onto  $S(M(\mathbb{H}^+)).$ 

It is proved that  $S$  is a holomorphic split submersion onto the bounded contractible domain  $S(M(\mathbb{H}^+))$  in  $A(\mathbb{H}^-)$ . Since  $S = D \circ L$ , we see that  $D: L(M(\mathbb{H}^+)) \to S(M(\mathbb{H}^+))$ is a biholomorphic homeomorphism and  $L$  is also a holomorphic split submersion onto the bounded contractible domain  $L(M(\mathbb{H}^+))$  in  $B(\mathbb{H}^-)$ . Moreover, these maps induce well-defined injections  $\alpha : T \to A(\mathbb{H}^-)$  such that  $\alpha \circ \pi = S$  and  $\beta : T \to B(\mathbb{H}^-)$  such that  $\beta \circ \pi = L$ . We call  $\alpha$  the Bers embedding and  $\beta$  the pre-Bers embedding. By the facts that S and L are split submersions, we see that  $\alpha$  and  $\beta$  are homeomorphisms onto the bounded contractible domains  $\alpha(T) = S(M(\mathbb{H}^+))$  in  $A(\mathbb{H}^-)$  and  $\beta(T) = L(M(\mathbb{H}^+))$  in  $B(\mathbb{H}^-)$ , respectively.

We can refer to a textbook [\[22\]](#page-34-2) for most of the aforementioned facts on the universal Teichmüller space. Concerning the pre-Schwarzian derivative map, see [\[38\]](#page-34-3).

Our subject turns to the BMO Teichmüller space  $T_B$ , which lies in T as introduced by Astala and Zinsmeister [\[3\]](#page-33-0). In general, we say that a measure  $\lambda$  on  $\mathbb H$  is a *Carleson* measure if

$$
\|\lambda\|_c = \sup_{I \subset \mathbb{R}} \frac{\lambda(I \times (0, |I|))}{|I|} < \infty,
$$

where the supremum is taken over all bounded intervals I in R. For  $\mu \in L^{\infty}(\mathbb{H})$ , we consider an absolutely continuous measure  $\lambda_{\mu} = |\mu(z)|^2 dx dy/y$  and let  $||\mu||_c = ||\lambda_{\mu}||_c^{1/2}$ . Then, we provide a stronger norm  $\|\mu\|_{\infty} + \|\mu\|_{c}$  for  $\mu$ . Let  $L_B(\mathbb{H})$  denote the Banach space consisting of all elements  $\mu \in L^{\infty}(\mathbb{H})$  with  $\|\mu\|_{\infty} + \|\mu\|_{c} < \infty$ , namely,  $\lambda_{\mu}$  is a Carleson measure on H. Moreover, we define the corresponding space of Beltrami coefficients as  $M_B(\mathbb{H}) = M(\mathbb{H}) \cap L_B(\mathbb{H}).$ 

**Definition 2.1.** The BMO Teichmüller space  $T_B \subset T$  is defined as  $\pi(M_B(\mathbb{H}))$ , equipped with the quotient topology from  $M_B(\mathbb{H})$  by  $\pi$ .

We introduce the space of holomorphic functions

$$
A_B(\mathbb{H}) = \{ \Psi \in A(\mathbb{H}) \mid ||\lambda_{\Psi}^{(2)}||_c^{1/2} < \infty \},
$$

where  $\lambda_{\Psi}^{(2)} = |\Psi(z)|^2 y^3 dx dy$  is a Carleson measure on H. This is a complex Banach space with norm  $\|\Psi\|_{A_B} = \|\lambda_{\Psi}^{(2)}\|_{c}^{1/2}$ . Similarly,

$$
BMOA(\mathbb{H}) = \{ \Phi \in B(\mathbb{H}) \mid ||\lambda_{\Phi}^{(1)}||_{c}^{1/2} < \infty \},
$$

where  $\lambda_{\Phi}^{(1)} = |\Phi'(z)|^2 y dx dy$  is a Carleson measure on H. This space modulo constants is a complex Banach space with norm  $\|\Phi\|_{\text{BMOA}} = \| \lambda_{\Phi}^{(1)} \|$  $\frac{(1)}{\Phi}$   $\| \frac{1}{2}$ .

The following result is proved by Shen and Wei [\[36,](#page-34-4) Theorem 5.1] extending [\[3,](#page-33-0) Theorem 1].

<span id="page-6-0"></span>**Proposition 2.1.** The Schwarzian derivative map S is a holomorphic map on  $M_B(\mathbb{H}^+)$ into  $A_B(\mathbb{H}^-)$ . Moreover, for each point  $\Psi$  in  $S(M_B(\mathbb{H}^+))$ , there exists a neighborhood  $V_{\Psi}$ of  $\Psi$  in  $A_B(\mathbb{H}^-)$  and a holomorphic map  $\sigma : V_{\Psi} \to M_B(\mathbb{H}^+)$  such that  $S \circ \sigma$  is the identity on  $V_{\Psi}$ .

This fact also leads a claim on the pre-Schwarzian derivative map L on  $M_B(\mathbb{H}^+)$ , and we have that  $L(M_B(\mathbb{H}^+)) \subset \text{BMOA}(\mathbb{H}^-)$  and  $L : M_B(\mathbb{H}^+) \to \text{BMOA}(\mathbb{H}^-)$  is holomorphic. Moreover, we also see that D restricted to  $L(M_B(\mathbb{H}^+))$  is a holomorphic bijection onto  $S(M_B(\mathbb{H}^+))$ . Hence, likewise to the case of the universal Teichmüller space, the pre-Schwarzian derivative map L satisfies the same property on  $M_B(\mathbb{H}^+)$  as in Proposition [2.1.](#page-6-0) These arguments are given in [\[36,](#page-34-4) Section 6].

<span id="page-6-1"></span>**Proposition 2.2.** The pre-Schwarzian derivative map L is a holomorphic map on  $M_B(\mathbb{H}^+)$ into BMOA( $\mathbb{H}^-$ ), and at each point in the image  $L(M_B(\mathbb{H}^+))$ , there exists a local holomorphic right inverse of L.

Propositions [2.1](#page-6-0) and [2.2](#page-6-1) also imply that  $S(M_B(\mathbb{H}^+))$  and  $L(M_B(\mathbb{H}^+))$  are open subsets in  $A_B(\mathbb{H}^-)$  and BMOA( $\mathbb{H}^-$ ) respectively, and  $D: L(M_B(\mathbb{H}^+)) \to S(M_B(\mathbb{H}^+))$  is a biholomorphic homeomorphism.

Under these properties of S and L on  $M_B(\mathbb{H}^+)$ , the Bers embedding  $\alpha$  and the pre-Bers embedding  $\beta$  of the BMO Teichmüller space  $T_B$  can be established in the same way as in the case of T. These maps induce complex Banach structures to  $T_B$  which are biholomorphically equivalent, and hence  $\alpha$  and  $\beta$  are biholomorphic homeomorphisms.

<span id="page-6-2"></span>**Theorem 2.3.** (1) The Bers embedding  $\alpha : T_B \to S(M_B(\mathbb{H}^+)) \subset A_B(\mathbb{H}^-)$  is a homeomorphism onto the image. (2) The pre-Bers embedding  $\beta : T_B \to L(M_B(\mathbb{H}^+)) \subset BMOA(\mathbb{H}^-)$ is a homeomorphism onto the image.

Next, we focus on the relationship between BMOA and BMO. A locally integrable complex-valued function  $\phi$  on  $\mathbb R$  is of *bounded mean oscillation* (BMO) if

$$
\|\phi\|_{\text{BMO}} = \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_I |\phi(x) - \phi_I| dx < \infty,
$$

where the supremum is taken over all bounded intervals I on R and  $\phi_I$  denotes the integral mean of  $\phi$  over I. The set of all complex-valued BMO functions on R is denoted by BMO( $\mathbb{R}$ ). This is regarded as a Banach space with norm  $\|\cdot\|_{\text{BMO}}$  by ignoring the difference in complex constant functions.

The *John–Nirenberg inequality* for BMO functions (see [\[15,](#page-33-1) VI.2]) asserts that there exist two universal positive constants  $C_0$  and  $C_{JN}$  such that for any complex-valued BMO

function  $\phi$ , any bounded interval I of R, and any  $\lambda > 0$ , it holds that

$$
\frac{1}{|I|} |\{t \in I : |\phi(t) - \phi_I| \ge \lambda\}| \le C_0 \exp\left(\frac{-C_{JN}\lambda}{\|\phi\|_{\text{BMO}}}\right). \tag{1}
$$

Concerning the boundary extension of  $\Phi \in BMOA(\mathbb{H})$  to  $\mathbb{R}$ , we note that  $\Phi$  has nontangential limits almost everywhere on R and the Poisson integral of this boundary function reproduces  $\Phi$ . This links the BMO properties of  $\Phi$  on H and on R. The following theorem is well known, which can be seen from [\[49,](#page-35-0) Theorems 9.17 and 9.19].

<span id="page-7-1"></span>**Theorem 2.4.** Let  $E(\Phi)$  be the boundary extension of  $\Phi \in \text{BMOA}(\mathbb{H})$  defined by the nontangential limits on  $\mathbb{R}$ . Then,  $E(\Phi) \in \text{BMO}(\mathbb{R})$ , and the trace operator  $E : \text{BMOA}(\mathbb{H}) \to$  $BMO(\mathbb{R})$  is a Banach isomorphism onto the image.

By considering the trace operators  $E^+$  and  $E^-$  for the half-planes  $\mathbb{H}^+$  and  $\mathbb{H}^-$ , we obtain the closed subspaces  $E^+(BMOA(\mathbb{H}^+))$  and  $E^-(BMOA(\mathbb{H}^-))$  in BMO( $\mathbb{R}$ ). Functions in  $E^+(BMOA(\mathbb{H}^+))$  and  $E^-(BMOA(\mathbb{H}^-))$  correspond by complex conjugation. By the identification under the Banach isomorphism  $E^{\pm}$ : BMOA( $\mathbb{H}^{\pm}$ )  $\rightarrow$  BMO( $\mathbb{R}$ ), we may regard BMOA( $\mathbb{H}^{\pm}$ ) as closed subspaces of BMO( $\mathbb{R}$ ).

Conversely, the projection from BMO(R) to BMOA(H)  $\cong$  E(BMOA(H)) associated with  $E$  is specifically provided by using the following map.

**Definition 2.2.** For  $\phi \in BMO(\mathbb{R})$ , we define the singular integral

$$
\mathcal{H}(\phi)(x) = \text{p.v.} \frac{1}{\pi} \int_{-\infty}^{\infty} \phi(t) \left( \frac{1}{x - t} + \frac{t}{1 + t^2} \right) dt
$$

to be a linear operator on  $BMO(\mathbb{R})$  called the *Hilbert transform*.

It is well known that H gives a Banach automorphism of BMO(R) satisfying  $H \circ H = -I$ (see [\[15,](#page-33-1) Chapter VI]). Let  $P^{\pm} = \frac{1}{2}$  $\frac{1}{2}(I \pm i\mathcal{H})$ , which we call the *Riesz projections*. We can apply the Riesz projections  $P^{\pm}$  to BMO( $\mathbb{R}$ ) as bounded linear operators. We note that  $P^+ + P^- = I$  and  $P^+ \circ P^- = P^- \circ P^+ = O$  by the definition of  $P^{\pm}$  and the property  $\mathcal{H} \circ \mathcal{H} = -I$ . Moreover, the images of  $P^{\pm}$  coincide with  $E^{\pm}(\text{BMOA}(\mathbb{H}^{\pm}))$ , which are the closed subspaces of  $BMO(\mathbb{R})$  consisting of all elements that extend to holomorphic functions on  $\mathbb{H}^{\pm}$  by the Poisson integral.

<span id="page-7-0"></span>**Theorem 2.5.** The Riesz projections  $P^{\pm}$  in BMO(R) are bounded linear projections onto the closed subspaces  $E^{\pm}(\text{BMOA}(\mathbb{H}^{\pm}))$ . They yield the topological direct sum decomposition

$$
BMO(\mathbb{R}) = E^+(\text{BMOA}(\mathbb{H}^+)) \oplus E^-(\text{BMOA}(\mathbb{H}^-)).
$$

Holomorphic functions of the upper and the lower half-planes  $\mathbb{H}^{\pm}$  defined by the Cauchy integrals of  $\phi \in BMO(\mathbb{R}),$ 

$$
\frac{-1}{2\pi i} \int_{\mathbb{R}} \phi(t) \left( \frac{1}{z-t} + \frac{t}{1+t^2} \right) dt \quad (z \in \mathbb{H}^{\pm}),
$$

are called the Szegö projections of  $\phi$ . Here, the integration over R is taken in the increasing direction  $\int_{-\infty}^{\infty}$  when  $z \in \mathbb{H}^+$  and in the decreasing direction  $\int_{\infty}^{-\infty}$  when  $z \in \mathbb{H}^-$ .

We see that the Szegö projections give the bounded linear maps  $BMO(\mathbb{R}) \to BMO(A(\mathbb{H}^{\pm}))$ whose composition with the trace operators  $E^{\pm}$ : BMOA( $\mathbb{H}^{\pm}$ )  $\rightarrow$  BMO( $\mathbb{R}$ ) coincide with the Riesz projections  $P^{\pm}$  (by the Plemelj formula in the special case). In the sequel, we do not distinguish them, denote both of them by  $P^{\pm}$ , and call the Szegö projections. Moreover, we regard BMOA( $\mathbb{H}^{\pm}$ ) as the subspaces of BMO( $\mathbb{R}$ ) by omitting  $E^{\pm}$  and represent the topological direct sum decomposition of  $BMO(\mathbb{R})$  in Theorem [2.5](#page-7-0) by

$$
BMO(\mathbb{R}) = BMOA(\mathbb{H}^+) \oplus BMOA(\mathbb{H}^-). \tag{2}
$$

## 3. Strongly quasisymmetric homeomorphisms

The universal Teichmüller space  $T$  is identified with the set QS of all normalized quasisymmetric homeomorphisms. A quasisymmetric homeomorphism  $h$  can be characterized by the doubling property for the pull-back of the Lebesgue measure on  $\mathbb R$  by h. The BMO Teichmüller space  $T_B$  is identified with the subset of quasisymmetric homeomorphisms  $h(\mu) = H(\mu)|_{\mathbb{R}}$  for  $\mu \in M_B(\mathbb{H})$ . We consider intrinsic characterization of these quasisymmetric homeomorphisms.

<span id="page-8-0"></span>**Definition 3.1.** A quasisymmetric homeomorphism  $h : \mathbb{R} \to \mathbb{R}$  is called *strongly qua*sisymmetric if there are positive constants K and  $\alpha$  such that

$$
\frac{|h(E)|}{|h(I)|} \le K \left(\frac{|E|}{|I|}\right)^{\alpha} \tag{3}
$$

for any bounded interval  $I \subset \mathbb{R}$  and for any measurable subset  $E \subset I$ .

We denote the set of all normalized strongly quasisymmetric homeomorphisms by SQS. As QS is a group under the composition, SQS is a subgroup of QS by definition [\(3.1\)](#page-8-0). We also see that  $h \in SQS$  is locally absolutely continuous, and hence it can be represented as  $h(x) = \int_0^x h'(t)dt.$ 

<span id="page-8-1"></span>**Theorem 3.1.** Let  $h(\mu)$  be a normalized quasisymmetric homeomorphism of  $\mathbb{R}$ . Then,  $[\mu]$  belongs to  $T_B$  if and only if  $h(\mu)$  is strongly quasisymmetric.

The "only-if" part of this theorem follows from [\[13,](#page-33-2) Theorem 2.3] and the "if" part follows from [\[13,](#page-33-2) Theorem 4.2]. Later, we will give a different proof for Theorem [3.1](#page-8-1) in Theorems [4.7](#page-13-0) and [6.5.](#page-18-0) Yet other proofs through other equivalent conditions are summarized in [\[36,](#page-34-4) Theorem A].

Here, we show the way of extending a strongly quasisymmetric homeomorphism of  $\mathbb R$  to a quasiconformal self-homeomorphism of H whose complex dilatation induces a Carleson measure. This is introduced by Fefferman, Kenig and Pipher [\[13\]](#page-33-2). There is a detailed exposition in [\[41,](#page-34-5) Theorem 3.4].

Let  $\phi(x) = \frac{1}{\sqrt{x}}$  $\frac{1}{\pi}e^{-x^2}$  and  $\psi(x) = \phi'(x) = -2x\phi(x)$ . We extend a strongly quasisymmetric homeomorphism  $h : \mathbb{R} \to \mathbb{R}$  to  $\mathbb{H}$  by setting a real-analytic diffeomorphism  $H : \mathbb{H} \to \mathbb{C}$ 

by

$$
H(x, y) = U(x, y) + iV(x, y);
$$
  
 
$$
U(x, y) = (h * \phi_y)(x), V(x, y) = (h * \psi_y)(x),
$$

where  $\varphi_y(x) = y^{-1} \varphi(y^{-1}x)$  for  $x \in \mathbb{R}$  and  $y > 0$ , and  $*$  is the convolution. We call this extension the variant of the *Beurling–Ahlfors extension* by the heat kernel. The original extension in [\[4\]](#page-33-3) uses the kernels  $\phi(x) = \frac{1}{2} 1_{[-1,1]}(x)$  and  $\psi(x) = \frac{1}{2} 1_{[-1,0]}(x) - \frac{1}{2}$  $\frac{1}{2}1_{[0,1]}(x).$ 

The quasiconformal extension theorem can be summarized as follows. The latter statement is in  $|43$ , Proposition 3.2.

<span id="page-9-0"></span>**Theorem 3.2.** For  $h \in SQS$ , the map H given by the variant of the Beurling–Ahlfors extension by the heat kernel is a quasiconformal real-analytic self-diffeomorphism of  $\mathbb H$ whose complex dilatation belongs to  $M_B(\mathbb{H})$ . Moreover, H is bi-Lipschitz with respect to the hyperbolic metric.

We note that in the case where the BMO norm of  $\log h'$  is sufficiently small for  $h \in SQS$ , Semmes [\[34,](#page-34-7) Proposition 4.2] used a modified Beurling–Ahlfors extension H by compactly supported kernels  $\phi$  and  $\psi$  to prove the same properties as in Theorem [3.2.](#page-9-0) By dividing the weight h' into small pieces and composing the resulting maps, the assumption on the small BMO norm can be removed to obtain a quasiconformal extension of the same properties.

A locally integrable non-negative measurable function  $\omega \geq 0$  on R is called a weight. We say that  $\omega$  is an  $A_{\infty}$ -weight if it satisfies the reverse Jensen inequality, namely, there exists a constant  $C_{\infty} \geq 1$  such that

$$
\frac{1}{|I|} \int_{I} \omega(x) dx \le C_{\infty} \exp\left(\frac{1}{|I|} \int_{I} \log \omega(x) dx\right)
$$
 (4)

for every bounded interval  $I \subset \mathbb{R}$ . On the contrary,  $\omega$  is defined to be an  $A_{\infty}$ -weight if  $h(x) = \int_0^x \omega(t)dt$  is a strongly quasisymmetric homeomorphism of R by Coifman and Fefferman [\[8,](#page-33-4) Theorem III]. In other words,  $h'$  is an  $A_{\infty}$ -weight if  $h \in SQS$ . It is known that these definitions are equivalent (see [\[18\]](#page-33-5)). Moreover, the constants K and  $\alpha$  for the strong quasisymmetry in [\(3.1\)](#page-8-0) can be estimated in terms of  $C_{\infty}$ . Concerning the relationship with  $A_p$ -weight  $(p > 1)$  of Muckenhoupt [\[28\]](#page-34-8), see [\[14,](#page-33-6) Section IV.2] and [\[15,](#page-33-1) Section VI.6]. In particular,  $\omega$  is an  $A_{\infty}$ -weight if and only if it is an  $A_p$ -weight for all sufficiently large p.

We also define  $\omega$  to be an  $A_1$ -weight if there exists a constant  $C_1 \geq 1$  such that

$$
\frac{1}{|I|} \int_I \omega(x) dx \le C_1 \operatorname{ess\,inf}_{x \in I} \omega(x)
$$

for any bounded interval  $I \subset \mathbb{R}$ . By the Jones factorization theorem (see [\[14,](#page-33-6) Corollary IV.5.3]),  $\omega$  is an  $A_{\infty}$ -weight if and only if there are  $A_1$ -weights  $\omega_0$ ,  $\omega_1$  and  $p > 1$  such that  $\omega = \omega_0 \omega_1^{1-p}.$ 

the real subspace of BMO(R) consisting of all real-valued BMO functions. In particular,  $\log h' \in \text{Re BMO}(\mathbb{R})$  for  $h \in \text{SQS}$ . Conversely, we know the following fact (see [\[14,](#page-33-6) p.409] and [\[15,](#page-33-1) Lemma VI.6.5]).

<span id="page-10-1"></span>**Proposition 3.3.** Suppose that a weight  $\omega \geq 0$  satisfies  $\log \omega \in \text{Re BMO}(\mathbb{R})$ . If the BMO norm  $\|\log \omega\|_{\text{BMO}}$  is less than the constant  $C_{JN}$  of [\(2\)](#page-6-2), then  $\omega$  is an  $A_{\infty}$ -weight.

There is an example of  $\phi \in \text{Re BMO}(\mathbb{R})$  such that  $e^{\phi}$  is not an  $A_{\infty}$ -weight:  $\phi(x) =$ log(1/|x|). Let BMO<sup>\*</sup>(R) denote the proper subset of BMO(R) consisting of all BMO functions  $\phi$  such that  $e^{\text{Re}\phi} = |e^{\phi}|$ | is an  $A_{\infty}$ -weight. Moreover, let  $\text{Re BMO}^*(\mathbb{R}) =$  $\text{Re BMO}(\mathbb{R}) \cap \text{BMO}^*(\mathbb{R})$ . We have the following claim.

<span id="page-10-0"></span>**Proposition 3.4.** ReBMO<sup>\*</sup>( $\mathbb{R}$ ) is a convex open subset of the real Banach subspace  $\text{Re BMO}(\mathbb{R})$ . Hence, so is  $\text{BMO}^*(\mathbb{R})$  of the complex Banach space BMO $(\mathbb{R})$ .

*Proof.* For the convexity of Re BMO<sup>\*</sup>( $\mathbb{R}$ ), we have to show that if  $\omega$  and  $\tilde{\omega}$  are  $A_{\infty}$ -weights, then  $\omega^s \tilde{\omega}^t$  is also an  $A_\infty$ -weight for  $s, t \geq 0$  with  $s + t = 1$ . By the Jones factorization, for any sufficiently large  $p > 1$ , we have the decomposition  $\omega = \omega_0 \omega_1^{1-p}$  and  $\tilde{\omega} = \tilde{\omega}_0 \tilde{\omega}_1^{1-p}$ by these  $A_1$ -weights. Then,

$$
\omega^s \tilde{\omega}^t = \omega_0^s \omega_1^{s(1-p)} \tilde{\omega}_0^t \tilde{\omega}_1^{t(1-p)} = (\omega_0^s \tilde{\omega}_0^t) (\omega_1^s \tilde{\omega}_1^t)^{(1-p)}.
$$
\n
$$
\tag{5}
$$

We can verify that  $\omega_0^s \tilde{\omega}_0^t$  and  $\omega_1^s \tilde{\omega}_1^t$  are  $A_1$ -weights:

$$
\frac{1}{|I|} \int_{I} \omega_0(x)^s \tilde{\omega}_0(x)^t dx \le \left(\frac{1}{|I|} \int_{I} \omega_0(x) dx\right)^s \left(\frac{1}{|I|} \int_{I} \tilde{\omega}_0(x) dx\right)^t
$$
  

$$
\lesssim \left(\operatorname*{ess\,inf}_{x \in I} \omega_0(x)\right)^s \left(\operatorname*{ess\,inf}_{x \in I} \tilde{\omega}_0(x)\right)^t \le \operatorname*{ess\,inf}_{x \in I} \omega_0(x)^s \tilde{\omega}_0(x)^t
$$

for any bounded interval  $I \subset \mathbb{R}$ . Hence,  $\omega_0^s \tilde{\omega}_0^t$  is an  $A_1$ -weight. The same estimate holds true for  $\omega_1^s \tilde{\omega}_1^t$ , which is also an  $A_1$ -weight. Again by the factorization in [\(3\)](#page-10-0), we see that  $\omega^s \tilde{\omega}^t$  is an  $A_p$ -weight, and hence  $A_\infty$ -weight.

As another property of  $A_{\infty}$ -weights, we know that if  $\omega$  is an  $A_p$ -weight, then there is some  $\varepsilon > 0$  such that  $\omega^r$  is an  $A_p$ -weight for every  $r \in [0, 1+\varepsilon)$  (see [\[14,](#page-33-6) Theorem IV.2.7]). Combining these properties with the fact in Proposition [3.3](#page-10-1) that the open ball centered at the origin of ReBMO( $\mathbb{R}$ ) with radius  $C_{JN}$  is contained in ReBMO<sup>\*</sup>( $\mathbb{R}$ ), we can prove that  $\text{Re BMO}^*(\mathbb{R})$  is open. Indeed,  $\text{Re BMO}^*(\mathbb{R})$  is the union of open cones spanned by the  $C_{JN}$ -neighborhood of the origin having any points of Re BMO<sup>\*</sup>( $\mathbb{R}$ ) as their vertices.

Because  $\text{BMO}^*(\mathbb{R}) = \text{Re BMO}^*(\mathbb{R}) \oplus i \text{Re BMO}(\mathbb{R})$ , we also see that  $\text{BMO}^*(\mathbb{R})$  is convex and open.

A strongly quasisymmetric homeomorphism  $h : \mathbb{R} \to \mathbb{R}$  can be also characterized by the composition operator on the Banach space  $BMO(\mathbb{R})$ . The pre-composition of h to  $\phi \in BMO(\mathbb{R})$  gives a change of the variables, and we denote this linear operator on  $BMO(\mathbb{R})$  by  $C_h$ . Its boundedness is proved by Jones [\[20\]](#page-33-7) as follows. Concerning the dependence of the constants, see [\[17,](#page-33-8) Example 2.3].

<span id="page-11-2"></span>**Theorem 3.5.** An increasing homeomorphism h of  $\mathbb{R}$  onto itself belongs to SQS if and only if the composition operator  $C_h : \phi \mapsto \phi \circ h$  gives an automorphism of BMO(R), that is,  $C_h$  and  $C_h^{-1}$  are bounded linear operators. Moreover, the operator norm satisfies an estimate

$$
||C_h|| \asymp ||C_h^{-1}|| \lesssim \frac{K}{\alpha}
$$

in terms of the constants K and  $\alpha$  for the strong quasisymmetry of h in [\(3.1\)](#page-8-0).

## 4. Chord-arc curves and conformal welding

The universal Teichmüller space  $T$  is also identified with the set of all normalized quasicircles. As the corresponding characterization for the BMO Teichmüller space  $T_B$ , a certain geometric condition is obtained by Bishop and Jones [\[6,](#page-33-9) Theorem 4], which is preserved under a bi-Lipschitz self-homeomorphism of C. This is a sort of localization of chord-arc condition, and a chord-arc curve defined below satisfies this condition. The subset consisting of all chord-arc curves occupies a certain portion of  $T_B$ .

**Definition 4.1.** A Jordan curve  $\Gamma$  in  $\mathbb{C}$  passing through  $\infty$  is called a *chord-arc curve* if Γ is locally rectifiable and there exists a constant  $κ ≥ 1$  such that the length of the arc between any two points  $z_1, z_2 \in \Gamma$  is bounded by  $\kappa |z_1 - z_2|$ . In other words, the arc length parametrization of  $\Gamma$  yields a bi-Lipschitz embedding of  $\mathbb R$  into  $\mathbb C$ .

A Jordan curve  $\Gamma$  in C passing through  $\infty$  is a quasicircle if  $\Gamma$  is the image of R under a quasiconformal self-homeomorphism of C. This is known to be equivalent to satisfying a weaker condition than in the above definition by replacing the length of the arc between  $z_1$  and  $z_2$  with its diameter even though  $\Gamma$  is not necessarily locally rectifiable (see [\[1,](#page-33-10) Theorems IV.4, 5]). In particular, a chord-arc curve is a quasicircle. The corresponding characterization of a chord-arc curve by the image of  $\mathbb R$  was shown in [\[19,](#page-33-11) Proposition 1.13], [\[30,](#page-34-9) Theorems 7.9, 7.10], and [\[39,](#page-34-10) Theorem] as follows.

<span id="page-11-0"></span>**Proposition 4.1.** A Jordan curve  $\Gamma$  passing through  $\infty$  is a chord-arc curve if and only if  $\Gamma$  is the image of  $\mathbb R$  under a bi-Lipschitz self-homeomorphism of  $\mathbb C$  with respect to the Euclidean metric. In fact, any bi-Lipschitz embedding  $\gamma : \mathbb{R} \to \mathbb{C}$  extends to a bi-Lipschitz self-homeomorphism of C.

First, we note a basic property of the boundary extension of a conformal homeomorphism of H determined by  $\mu \in M_B(\mathbb{H})$  in general. Let  $F^{\mu}$  be the normalized quasiconformal self-homeomorphism of  $\mathbb C$  that is conformal on  $\mathbb H^-$  and has the complex dilatation  $\mu$  on  $\mathbb{H}^+$ . By Proposition [2.2,](#page-6-1) the condition  $\mu \in M_B(\mathbb{H}^+)$  implies  $\log(F^{\mu}|_{\mathbb{H}^-})' \in \text{BMOA}(\mathbb{H}^-)$ . Moreover, the converse is also true (see Theorem [4.9\)](#page-14-0).

<span id="page-11-1"></span>**Lemma 4.2.** If  $\Phi = \log(F^{\mu}|_{\mathbb{H}^{-}})'$  belongs to BMOA( $\mathbb{H}^{-}$ ), then  $f = F^{\mu}|_{\mathbb{R}}$  has its derivative with  $f'(x) \neq 0$  almost everywhere on  $\mathbb{R}$ , and  $\log f'$  coincides with  $E(\Phi) \in \text{BMO}(\mathbb{R})$ .

*Proof.* By Theorem [2.4,](#page-7-1) the boundary extension  $\phi = E(\Phi)$  is in BMO( $\mathbb{R}$ ), and in particular,  $\phi(x)$  is finite almost everywhere on R. Since  $f(\mathbb{R})$  is a quasicircle, it is known that

 $f'(x)$  exists and coincides with the angular derivative of  $F^{\mu}$  at x almost everywhere on R (see [\[30,](#page-34-9) Theorem 5.5]). Hence,  $\log f' = \phi$ .  $\prime = \phi$ .

In this setting of quasicircle, the Lavrentiev theorem in particular gives a condition under which the image of  $\mathbb R$  by  $F^{\mu}$  is a chord-arc curve. See [\[19,](#page-33-11) Theorem 4.2] and [\[30,](#page-34-9) Theorem 7.11]. We note that every function in BMOA(H−) can be represented by the Poisson integral of its boundary extension on R by Theorem [2.5.](#page-7-0)

<span id="page-12-0"></span>**Theorem 4.3.** For a quasiconformal self-homeomorphism  $F^{\mu}$  of  $\mathbb C$  with  $\mu \in M(\mathbb H^+)$ ,  $\Gamma =$  $F^{\mu}(\mathbb{R})$  is a chord-arc curve if and only if  $\log(F^{\mu}|_{\mathbb{H}}-)'$  belongs to BMOA( $\mathbb{H}^-$ ),  $f = F^{\mu}|_{\mathbb{R}}$  is locally absolutely continuous, and  $|f'|$  is an  $A_{\infty}$ -weight. Namely, the equivalent condition is that  $\log f' \in \text{BMO}^*(\mathbb{R})$ .

In fact, any chord-arc curve falls in this situation as the following claim asserts. Overall expositions around these arguments are in [\[27\]](#page-34-11).

<span id="page-12-2"></span>Corollary 4.4. Every chord-arc curve passing through  $\infty$  is the image of R under some  $F^{\mu}$  with  $\log f' \in \text{BMO}^*(\mathbb{R})$  for  $f = F^{\mu}|_{\mathbb{R}}$ .

*Proof.* By Proposition [4.1,](#page-11-0)  $\Gamma$  is the image of  $\mathbb R$  under a quasiconformal self-homeomorphism  $F^{\mu}$  of C. We may assume that  $F^{\mu}$  is conformal on H by pre-composing a quasiconformal self-homeomorphism of  $\mathbb C$  preserving  $\mathbb R$ . Then, Theorem [4.3](#page-12-0) yields the assertion.  $\Box$ 

**Definition 4.2.** A subset of  $T_B$  consisting of all elements  $[\mu]$  such that  $F^{\mu}(\mathbb{R})$  is a chordarc curve is denoted by  $T_C$ .

There exists some  $[\mu] \in T_B$  that is not contained in  $T_C$ . In fact, there are examples of  $F^{\mu}$  such that  $F^{\mu}(\mathbb{R})$  are not locally rectifiable (see [\[3,](#page-33-0) Theorem 6], [\[5,](#page-33-12) Theorem 1.1] and [\[32,](#page-34-12) Theorem]).

<span id="page-12-3"></span>**Proposition 4.5.**  $T_C$  is a proper open subset of  $T_B$  containing the origin.

*Proof.* We consider the composition of the pre-Bers embedding  $\beta: T_B \to \text{BMOA}(\mathbb{H})$  and the trace operator  $E : BMOA(\mathbb{H}) \to BMO(\mathbb{R})$ . Since  $E \circ \beta : T_B \to BMO(\mathbb{R})$  is continuous, BMO<sup>\*</sup>(R) is open in BMO(R) by Proposition [3.4,](#page-10-0) and  $T_C = (E \circ \beta)^{-1}(\text{BMO}^*(\mathbb{R}))$  by Theorem [4.3,](#page-12-0) we see that  $T_C$  is an open subset of  $T_B$  containing the origin.

**Remark 4.1.** Whether  $T_C$  is connected or not is an open problem. Since the inverse image  $E^{-1}(\text{BMO}^*(\mathbb{R}))$  is also a convex open subset of BMOA( $\mathbb{H}$ ), the shape of the image of  $T_B$  under the pre-Bers embedding  $\beta$  comes into question.

<span id="page-12-1"></span>The following properties for  $T_C$  are easily obtained. See [\[26,](#page-34-13) Theorem 4].

**Proposition 4.6.** (1) Every element  $[\mu] \in T_B$  can be obtained by a finite composition  $[\mu] = [\mu_1] * \cdots * [\mu_n]$  of elements  $[\mu_i] \in T_C$   $(i = 1, ..., n)$ . (2) If  $[\mu] \in T_C$  then  $[\mu]^{-1} \in T_C$ .

In general, a quasisymmetric homeomorphism  $h : \mathbb{R} \to \mathbb{R}$  can be expressed as the discrepancy between the boundary values  $f_1 = F_1|_{\mathbb{R}}$  and  $f_2 = F_2|_{\mathbb{R}}$  of two conformal

homeomorphisms  $F_1 : \mathbb{H}^- \to \Omega^-$  and  $F_2 : \mathbb{H}^+ \to \Omega^+$ , where  $\Omega^-$  and  $\Omega^+$  are the complementary domains in  $\mathbb C$  with  $\partial \Omega^- = \partial \Omega^+$ . This expression  $h = f_2^{-1} \circ f_1$  is called *conformal* welding. This allows us to see that the property of h is determined by that of  $f_1$  and  $f_2$ .

We determine the class of  $h \in \text{QS}$  whose quasiconformal extension  $H(\mu)$  to  $\mathbb H$  is given by  $\mu \in M_B(\mathbb{H})$ . The following theorem corresponds to the "only if" part of Theorem [3.1.](#page-8-1) Representing a quasisymmetric homeomorphism  $h(\mu) = H(\mu)|_{\mathbb{R}}$  with  $\mu \in M_B(\mathbb{H})$  by conformal welding, we will give an alternative proof for it.

<span id="page-13-0"></span>**Theorem 4.7.** If  $\mu \in M_B(\mathbb{H})$ , then  $h(\mu) \in SQS$ . In other words,  $h = h(\mu)$  is locally absolutely continuous and  $\phi = \log h'$  belongs to BMO<sup>\*</sup>(R).

The following method of conformal welding works only when  $[\mu]$  belongs to  $T_C$ , that is to say, when the conformal welding is done along a chord-arc curve  $F^{\mu}(\mathbb{R})$ . For the general case, we decompose  $[\mu]$  into finitely many such elements, and apply this to each of them.

<span id="page-13-1"></span>**Lemma 4.8.** If  $F^{\mu}(\mathbb{R})$  is a chord-arc curve, then  $h(\mu)$  is strongly quasisymmetric.

*Proof.* Let  $F_1 = F^{\mu}$  as before and let  $F_2 = F_{\bar{\mu}^{-1}}$  be the normalized quasiconformal selfhomeomorphism of  $\mathbb C$  that is conformal on  $\mathbb{H}^+$  and has complex dilatation  $\bar{\mu}^{-1}$  on  $\mathbb{H}^-$ . Here,  $\bar{\mu}$  is the reflection of  $\mu$  with respect to R defined by  $\bar{\mu}(\bar{z}) = \mu(z)$ . We note that the normalization of  $F_1$  and  $F_2$  requires  $F_1(\mathbb{R}) = F_2(\mathbb{R})$ . Let  $f_1 = F_1|_{\mathbb{R}}$  and  $f_2 = F_2|_{\mathbb{R}}$ . Then, we have that  $f_2 \circ h = f_1$ .

Theorem [4.3](#page-12-0) asserts that  $f_1$  and  $f_2$  are locally absolutely continuous and log  $f'_1$  and log  $f'_2$ belong to BMO<sup>\*</sup>( $\mathbb{R}$ ). Namely, |f'<sub>1</sub>| and |f'<sub>2</sub>| are  $A_{\infty}$ -weights. Moreover,  $f'_{2}(x) \neq 0$  (a.e.) by Lemma [4.2,](#page-11-1) and this implies that  $h$  is also locally absolutely continuous. Then, taking the absolute value of the derivative for the conformal welding  $f_2 \circ h = f_1$ , we have  $|f'_2| \circ h \cdot h' = |f'_1|.$ 

Let  $\tilde{f}_1(x) = \int_0^x |f_1'(t)|dt$  and  $\tilde{f}_2(x) = \int_0^x |f_2'(t)|dt$ , which belong to SQS. Moreover,

$$
\tilde{f}_1(x) = \int_0^x |f'_2| \circ h(t) \cdot h'(t) dt = \int_0^{h(x)} |f'_2(\tau)| d\tau = \tilde{f}_2 \circ h(x).
$$

Hence,  $h = \tilde{f}_2^{-1} \circ \tilde{f}_1$ , which is also a strongly quasisymmetric homeomorphism.

*Proof of Theorem [4.7.](#page-13-0)* We represent  $h = h(\mu)$  for  $\mu \in M_B(\mathbb{H}^+)$  by conformal welding as  $h = (F_{\bar{\mu}^{-1}})^{-1} \circ F^{\mu}|\mathbb{R}$ . If  $F^{\mu}(\mathbb{R})$  is a chord-arc curve, then Lemma [4.8](#page-13-1) implies that  $h \in SQS$ . In the general case, we decompose  $[\mu] \in T_B$  into a finite number of elements in  $T_C$  by Proposition [4.6;](#page-12-1)  $[\mu] = [\mu_1] * \cdots * [\mu_n]$  for  $[\mu_i] \in T_C$   $(i = 1, \ldots, n)$ . By the above argument, each  $h_i = h(\mu_i)$  is in SQS. Hence, we see that  $h = h_1 \circ \cdots \circ h_n$  is strongly quasisymmetric.

Finally, we mention the characterizations for a Beltrami coefficient  $\mu$  to be in  $M_B(\mathbb{H})$ in terms of the Schwarzian and the pre-Schwarzian derivative maps S and L.

<span id="page-14-0"></span>**Theorem 4.9.** Suppose that  $F^{\mu}(\mathbb{R})$  is a chord-arc curve for  $\mu \in M(\mathbb{H}^+)$ . Then, the following conditions are equivalent: (a)  $\mu \in M_B(\mathbb{H}^+);$  (b)  $S(\mu) \in A_B(\mathbb{H}^-);$  (c)  $L(\mu) \in$  $BMOA(\mathbb{H}^-)$ .

*Proof.* The implications  $(a) \Rightarrow (b) \Rightarrow (c)$  are along the same line as Propositions [2.1](#page-6-0) and [2.2.](#page-6-1) The crucial step is to prove  $(c) \Rightarrow (a)$ . If  $F^{\mu}(\mathbb{R})$  is a chord-arc curve, Lemma [4.8](#page-13-1) shows that  $h(\mu)$  is strongly quasisymmetric. Then, by "if" part of Theorem [3.1](#page-8-1) (=Theorem [6.5\)](#page-18-0), we conclude that  $[\mu] \in T_B$ , that is,  $\mu$  belongs to  $M_B(\mathbb{H}^+)$ .

**Remark 4.2.** The above theorem is still valid without the assumption that  $F^{\mu}(\mathbb{R})$  is a chord-arc curve. This is proved in [\[3,](#page-33-0) Theorem 4]. The step  $(c) \Rightarrow (a)$  relies on the geometric characterization of a curve  $F^{\mu}(\mathbb{R})$  for  $\log(F^{\mu}|_{\mathbb{H}^{-}})' \in \text{BMOA}(\mathbb{H}^{-})$  given by [\[6,](#page-33-9) Theorem 4]. As a consequence, it follows that

$$
S(M_B(\mathbb{H}^+)) = S(M(\mathbb{H}^+)) \cap A_B(\mathbb{H}^-), \quad L(M_B(\mathbb{H}^+)) = L(M(\mathbb{H}^+)) \cap BMOA(\mathbb{H}^-).
$$

## 5. BMO Embeddings in Bers Coordinates

We generalize strongly quasisymmetric homeomorphisms  $h : \mathbb{R} \to \mathbb{R}$  to BMO embeddings  $\gamma : \mathbb{R} \to \mathbb{C}$ , and consider those whose images are chord-arc curves. Then, we use the BMO Teichmüller space  $T_B$  to coordinate these embeddings. We remark here that a BMO embedding is a mapping  $\gamma$  of R, that is, the image  $\Gamma = \gamma(\mathbb{R})$  together with its parametrization, whereas a chord-arc curve refers to the image  $\Gamma$  itself of a certain special BMO embedding  $\gamma$ .

**Definition 5.1.** A topological embedding  $\gamma : \mathbb{R} \to \mathbb{C}$  passing through  $\infty$  is called a *BMO* embedding if there is a quasiconformal self-homeomorphism G of  $\mathbb C$  with  $G|_{\mathbb R} = \gamma$  whose complex dilatation  $\mu = \overline{\partial}G/\partial G$  satisfies  $\mu|_{\mathbb{H}^+} \in M_B(\mathbb{H}^+)$  and  $\mu|_{\mathbb{H}^-} \in M_B(\mathbb{H}^-)$ .

We first consider the derivative of a BMO embedding  $\gamma : \mathbb{R} \to \mathbb{C}$  though  $\gamma$  is not necessarily absolutely continuous. The following claim is proved in a more general setting in [\[27,](#page-34-11) Theorem 6.2]. See [\[45,](#page-34-0) Proposition 3.3] for the proof along quasiconformal theory. Later in Theorem [5.3,](#page-15-0) we show this in the special case where the image of  $\gamma$  is a chord-arc curve.

<span id="page-14-1"></span>**Proposition 5.1.** A BMO embedding  $\gamma : \mathbb{R} \to \mathbb{C}$  has its derivative  $\gamma'$  almost everywhere on  $\overline{\mathbb{R}}$  and  $\log \gamma'$  belongs to BMO( $\mathbb{R}$ ).

We consider parametrization of all BMO embeddings. The simultaneous uniformization due to Bers works for it. In general, a quasisymmetric embedding  $\gamma : \mathbb{R} \to \Gamma \subset \mathbb{C}$  onto a quasicircle  $\Gamma$  passing through  $\infty$  is induced by a quasiconformal self-homeomorphism  $G(\mu^+,\mu^-)$  of  $\overline{\mathbb{C}}$  for  $\mu^+ \in M(\mathbb{H}^+)$  and  $\mu^- \in M(\mathbb{H}^-)$ . We assume that  $G(\mu^+,\mu^-)$  is normalized so that it fixes 0, 1, and  $\infty$ . We see in the following proposition that such an embedding  $\gamma = G(\mu^+, \mu^-)|_{\mathbb{R}}$  is determined by the Teichmüller equivalence class pair  $([\mu^+], [\mu^-])$ . The proof is the same as that in [\[45,](#page-34-0) Proposition 4.1].

<span id="page-15-1"></span>**Proposition 5.2.** For  $\mu^+, \nu^+ \in M(\mathbb{H}^+)$  and  $\mu^-, \nu^- \in M(\mathbb{H}^-)$ ,  $G(\mu^+, \mu^-)|_{\mathbb{R}} = G(\nu^+, \nu^-)|_{\mathbb{R}}$ if and only if  $[\mu^+] = [\nu^+]$  and  $[\mu^-] = [\nu^-]$  in T.

Thus, the space of all such normalized quasisymmetric embeddings can be identified with the product  $T^+ \times T^-$  of the universal Teichmüller space for  $T^+ = \pi(M(\mathbb{H}^+))$  and  $T^{-} = \pi(M(\mathbb{H}^{-}))$ . We refer to this as the Bers coordinates. We can introduce these coordinates for the BMO embeddings as well. Let  $T_B^+ = \pi(M_B(\mathbb{H}^+))$  and  $T_B^- = \pi(M_B(\mathbb{H}^-))$ . Any BMO embedding is represented by  $\gamma([\mu^+], [\mu^-]) = G(\mu^+, \mu^-)|_{\mathbb{R}}$  for  $([\mu^+], [\mu^-]) \in$  $T_B^+ \times T_B^-$ , and thus  $T_B^+ \times T_B^-$  becomes the Bers coordinates of the space of normalized BMO embeddings.

Let  $\bar{\mu} \in M_B(\mathbb{H}^-)$  denote the reflection  $\mu(\bar{z})$  of a Beltrami coefficient  $\mu(z)$  for  $z \in \mathbb{H}^+$ . Then,  $G(\mu, \bar{\mu})$  is nothing but the normalized quasiconformal homeomorphism  $H(\mu): \mathbb{C} \to$ © preserving ℝ. The *axis of symmetry* of the product space  $T_B^+ \times T_B^-$  is defined as

$$
\text{Sym}(T_B^+ \times T_B^-) = \{([\mu], [\bar{\mu}]) \mid [\mu] \in T_B^+\}.
$$

For  $[\mu] \in T_B$ , the corresponding  $h = h(\mu) \in SQS$  is expressed as  $h = G(\mu, \bar{\mu})|_{\mathbb{R}} =$  $\gamma([\mu],[\bar{\mu}])$ . The canonical map  $\iota : T_B \to \text{Sym}(T_B^+ \times T_B^-) \subset T_B^+ \times T_B^-$  defined by  $[\mu] \mapsto ([\mu], [\bar{\mu}])$  is a real-analytic embedding, and hence  $Sym(T_B^+ \times T_B^-)$  is a real-analytic submanifold of  $T_B^+ \times T_B^-$ .

For every  $[\nu] \in T_B$ , the right translation  $r_{[\nu]} : T_B \to T_B$  of the group structure of  $T_B \cong$  SQS is defined by  $[\mu] \mapsto [\mu] * [\nu]$  for every  $[\mu] \in T_B$ . It is known that  $r_{[\nu]}$  is a biholomorphic automorphism of  $T_B$  (see [\[36,](#page-34-4) Remark 5.1] and [\[45,](#page-34-0) Lemma 4.2]). This can be extended to  $T_B^+ \times T_B^-$  as the parallel translation

$$
R_{[\nu]}([\mu^+],[\mu^-]) = (r_{[\nu]}([\mu^+]),r_{[\bar{\nu}]}([\mu^-])) = ([\mu^+]*[\nu],[\mu^-]*[\bar{\nu}]),
$$

which is a biholomorphic automorphism of  $T_B^+ \times T_B^-$  that preserves Sym $(T_B^+ \times T_B^-)$ .

We see that any chord-arc curve  $\Gamma$  passing through  $\infty$  is the image of some BMO embedding by Corollary [4.4](#page-12-2) and Theorem [4.9.](#page-14-0) We have defined  $T_C$  as the subset of T<sub>B</sub> consisting of all  $[\mu] \in T_B$  such that  $F^{\mu}(\mathbb{R})$  is a chord-arc curve, which is an open subset of  $T_B$  by Proposition [4.5.](#page-12-3) Let  $\widetilde{T}_C$  be the subset of  $T_B^+ \times T_B^-$  consisting of the Bers coordinates of BMO embeddings  $\gamma : \mathbb{R} \to \mathbb{C}$  whose images are chord-arc curves. By definition,  $T_C \times \{[0]\}, \{[0]\} \times T_C$ , and  $Sym(T_B^+ \times T_B^-)$  are all contained in  $T_C$ . We will see later that  $T_C$  is an open subset of  $T_B^+ \times T_B^-$ .

In Theorem [4.7,](#page-13-0) by using the method of conformal welding, we have obtained that any element  $h = \gamma([\mu], [\bar{\mu}])$  with  $[\mu] \in T_B$  is locally absolutely continuous and log h' belongs to Re BMO<sup>∗</sup> (R). The characterization of a chord-arc curve as in Theorem [4.3](#page-12-0) can be generalized as follows by using the inverse method of conformal welding.

<span id="page-15-0"></span>**Theorem 5.3.** Let  $\gamma = \gamma([\mu^+], [\mu^-])$  be a BMO embedding with  $([\mu^+], [\mu^-]) \in T_B^+ \times T_B^-$ . Then, the image of  $\gamma : \mathbb{R} \to \mathbb{C}$  is a chord-arc curve, that is  $([\mu^+], [\mu^-]) \in \widetilde{T}_C$ , if and only if  $\gamma$  is locally absolutely continuous and  $\log \gamma'$  belongs to BMO<sup>\*</sup>( $\mathbb{R}$ ).

*Proof.* We represent  $\gamma$  as the following composition:

$$
\gamma([\mu^+], [\mu^-]) = \gamma([0], [\mu^-] * [\overline{\mu^+}]^{-1}) \circ \gamma([\mu^+], [\overline{\mu^+}]).
$$

Here,  $f = \gamma([0], [\mu^-] * [\overline{\mu^+}]^{-1})$  is given as  $F_{\nu}|_{\mathbb{R}}$  for the normalized quasiconformal selfhomeomorphism  $F_{\nu}$  of  $\mathbb C$  that is conformal on  $\mathbb{H}^+$  and has complex dilatation  $\nu$  on  $\mathbb{H}^$ with  $[\nu] = [\mu^-] * [\overline{\mu^+}]^{-1}$ , and  $h = \gamma([\mu^+], [\overline{\mu^+}])$  is given as  $H(\mu^+)|_{\mathbb{R}}$  for the normalized quasiconformal self-homeomorphism  $H(\mu^+)$  of  $\mathbb C$  with the indicated complex dilatation. We have  $[\nu] \in T_B$  since  $T_B \cong \text{SQS}$  is a group. Hence, f is a BMO embedding and the image of f coincides with that of  $\gamma$ .

Suppose that the image of  $\gamma$  is a chord-arc curve. Then, f is locally absolutely continuous and  $\log f' \in \text{BMO}^*(\mathbb{R})$  by Theorem [4.3.](#page-12-0) In addition,  $h \in \text{SQS}$  by Theorem [4.7.](#page-13-0) Therefore,  $\gamma = f \circ h$  is locally absolutely continuous and satisfies  $|\gamma'| = |f'| \circ h \cdot h'$ . Let  $\tilde{f}(x) = \int_0^x |f'(t)| dt$ , which is in SQS. Then,

$$
\tilde{\gamma}(x) = \int_0^x |\gamma'(t)| dt = \int_0^x |f'| \circ h(t) \cdot h'(t) dt = \int_0^{h(x)} |f'(\tau)| d\tau = \tilde{f} \circ h(x)
$$

is also in SQS. This implies that  $|\gamma'|$  is an  $A_{\infty}$ -weight, and hence  $\log \gamma' \in \text{BMO}^*(\mathbb{R})$ .

Conversely, suppose that  $\gamma$  is locally absolutely continuous and log  $\gamma'$  is in BMO<sup>\*</sup>(R). Then, f is also locally absolutely continuous. Under the above definition, we consider  $\tilde{\gamma}$ and  $\tilde{f}$ . Since  $|\gamma'|$  is an  $A_{\infty}$ -weight,  $\tilde{\gamma}$  is in SQS, and hence so is  $\tilde{f}$ . Thus,  $|f'|$  is an  $A_{\infty}$ weight and  $\log f'$  is in BMO<sup>\*</sup>( $\mathbb{R}$ ). By Theorem [4.3](#page-12-0) again, the image of f is a chord-arc curve. This implies that the image of  $\gamma$  is also a chord-arc curve.

In particular, if a BMO embedding  $\gamma$  satisfies  $|\gamma'| = 1$ , which means that  $\gamma$  is parametrized by its arc-length, then  $\gamma(\mathbb{R})$  is a chord-arc curve.

## 6. Holomorphy to the BMO space

For a BMO embedding  $\gamma : \mathbb{R} \to \mathbb{C}$  with  $\gamma = \gamma([\mu^+], [\mu^-])$  for  $([\mu^+], [\mu^-]) \in T_B^+ \times T_B^-$ , we have  $\log \gamma' \in \text{BMO}(\mathbb{R})$  by Proposition [5.1.](#page-14-1) By this correspondence, we define a map

$$
\Lambda: T_B^+ \times T_B^- \to \text{BMO}(\mathbb{R}).
$$

<span id="page-16-0"></span>We first prove that this correspondence is holomorphic in the Bers coordinates.

**Theorem 6.1.** The map  $\Lambda: T_B^+ \times T_B^- \to \text{BMO}(\mathbb{R})$  is holomorphic.

*Proof.* By the Hartogs theorem for Banach spaces (see [\[7,](#page-33-13)  $\S14.27$ ]), to see that  $\Lambda$  is holomorphic, it suffices to show that  $\Lambda$  is separately holomorphic. Namely, we fix, say  $[\mu_0^+]$ , and prove that  $\Lambda([\mu_0^+],[\mu^-])$  is holomorphic on  $[\mu^-]$ . The other case is treated in the same way.

Let  $C_{h_0}$  be the composition operator acting on BMO(R) induced by  $h_0 = h(\mu_0^+) \in$ SQS. We define the affine translation  $Q_{h_0}(\phi)$  of  $\phi \in \text{BMO}(\mathbb{R})$  by  $C_{h_0}(\phi) + \log h'_0$ . Then,  $\Lambda \circ R_{\lbrack \mu_0^+ \rbrack} = Q_{h_0} \circ \Lambda$  holds. This relation yields a useful representation

$$
\Lambda([\mu_0^+], \cdot) = Q_{h_0} \circ \Lambda([0], r_{\bar{[\mu_0^+]}}^{-1}(\cdot)). \tag{6}
$$

Here,  $\Lambda([0], \cdot)$  becomes the trace operator  $E^+$  in Theorem [2.4](#page-7-1) by composing the pre-Bers embedding  $\beta^- : T_B^- \to \text{BMOA}(\mathbb{H}^+)$ . Namely,

$$
\Lambda([0],[\mu]) = E^+(\beta^-([\mu])) \quad ([\mu] \in T_B^-).
$$

Since  $E^+$  is a bounded linear operator and  $r_{\lceil \vec{\mu}_0^+ \rceil}^{-1}$  is holomorphic, we conclude that  $\Lambda([\mu_0^+], \cdot)$ is holomorphic.  $\Box$ 

In the above proof, we see that the affine translation  $Q_{h_0}$  of BMO(R) preserves the image  $\Lambda(T_C)$  invariant because  $R_{\lbrack \mu_0^+ \rbrack}$  preserves  $T_C$  invariant.

**Proposition 6.2.**  $\widetilde{T}_C = \Lambda^{-1}(\text{BMO}^*(\mathbb{R}))$ , and  $\widetilde{T}_C$  is an open subset of  $T_B^+ \times T_B^-$ .

*Proof.* The first assertion follows from Theorem [5.3.](#page-15-0) Since  $\Lambda$  is continuous by Theorem [6.1](#page-16-0) and BMO<sup>\*</sup>( $\mathbb{R}$ ) is open by Proposition [3.4,](#page-10-0) we obtain the second assertion.  $\Box$ 

In the sequel, we restrict  $\Lambda$  to  $\tilde{T}_C$  and define this map as

$$
\Lambda: \widetilde{T}_C \to \text{BMO}^*(\mathbb{R})
$$

<span id="page-17-0"></span>using the same notation  $\Lambda$ .

**Proposition 6.3.** The map  $\Lambda : \widetilde{T}_C \to \text{BMO}^*(\mathbb{R})$  is a holomorphic injection.

*Proof.* Holomorphy follows from Theorem [6.1.](#page-16-0) Let  $\Lambda([\mu_1^+], [\mu_1^-]) = \log \gamma_1'$ ,  $\Lambda([\mu_2^+], [\mu_2^-]) =$  $\log \gamma'_2$ , and suppose that  $\log \gamma'_1 = \log \gamma'_2$ . Since  $\gamma_1 = \gamma([\mu_1^+], [\mu_1^-])$  and  $\gamma_2 = \gamma([\mu_2^+], [\mu_2^-])$ are locally absolutely continuous, we have  $\gamma_1 = \gamma_2$  by the normalization. This implies that  $[\mu_1^+] = [\mu_2^+]$  and  $[\mu_1^-] = [\mu_2^-]$  by Proposition [5.2,](#page-15-1) and hence  $\Lambda$  is injective.  $\Box$ 

We denote the tangent space of  $T_B$  at  $[\mu]$  by  $\mathscr{T}_{[\mu]}T_B$ . The tangent space of  $T_B^+ \times T_B^-$  at  $([\mu^+], [\mu^-])$  is represented by the direct sum

$$
\mathscr{T}_{([\mu^+],[\mu^-])}(T^+_B \times T^-_B) = \mathscr{T}_{[\mu^+]}T^+_B \oplus \mathscr{T}_{[\mu^-]}T^-_B.
$$

By the identification  $T_B^+$  $B_B^+ \cong \beta(T_B^+)$  $\mathcal{L}_B^{\uparrow +}$ )  $\subset$  BMOA( $\mathbb{H}^-$ ) and  $T_B^- \cong \beta(T_B^-) \subset$  BMOA( $\mathbb{H}^+$ ) under the pre-Bers embedding by Theorem [2.3,](#page-6-2) we may assume that  $\mathscr{T}_{\mu+1}T_B^+$  $\mathcal{L}_B^+ \cong \text{BMOA}(\mathbb{H}^-)$  and  $\mathscr{T}_{[\mu^-]}T_B^- \cong \text{BMOA}(\mathbb{H}^+)$ . Then, the derivative  $d_{([\mu^+],[\mu^-])}\Lambda$  of  $\Lambda$  at  $([\mu^+],[\mu^-])$  is regarded as the linear mapping

$$
d_{([\mu^+],[\mu^-])}\Lambda: {\rm BMOA}(\mathbb{H}^-)\oplus {\rm BMOA}(\mathbb{H}^+)\to {\rm BMO}(\mathbb{R})={\rm BMOA}(\mathbb{H}^-)\oplus {\rm BMOA}(\mathbb{H}^+)
$$

by taking the direct sum decomposition [\(2\)](#page-7-0) into account.

The derivative  $d_{([0],[0])}$   $\Lambda$  at the origin can be easily understood. By checking that the restriction of  $\Lambda$  to  $T_B^+$  $T_B^+$  and  $T_B^-$  coincides with

$$
\Lambda|_{T^+_B\times \{[0]\}}=\beta^+:T^+_B\rightarrow \operatorname{BMOA}(\mathbb{H}^-),\quad \Lambda|_{\{[0]\}\times T^-_B}=\beta^-:T^-_B\rightarrow \operatorname{BMOA}(\mathbb{H}^+),
$$

we see that the derivative  $d_{([0],[0])}\Lambda$  is the identity map of BMOA( $\mathbb{H}^-$ )  $\oplus$  BMOA( $\mathbb{H}^+$ ). This implies the following claim by the inverse mapping theorem (see [\[7,](#page-33-13) §7.18]).

<span id="page-18-1"></span>**Proposition 6.4.** The inverse map  $\Lambda^{-1}$  is holomorphic in some neighborhood U of 0 in BMO<sup>\*</sup>( $\mathbb{R}$ ) with  $U \subset \Lambda(\widetilde{T}_C)$ .

Theorem [4.7](#page-13-0) implies that  $\Lambda([\mu],[\bar{\mu}]) \in \text{Re BMO}^*(\mathbb{R})$  for every  $([\mu],[\bar{\mu}]) \in \text{Sym}(T_B^+\times T_B^-)$ . The converse of this claim also holds.

<span id="page-18-0"></span>**Theorem 6.5.** For every  $\phi \in \text{Re BMO}^*(\mathbb{R})$ , there exists  $([\mu],[\bar{\mu}]) \in \text{Sym}(T_B^+ \times T_B^-)$ such that  $\Lambda([\mu], [\bar{\mu}]) = \phi$ . Hence, any strongly quasisymmetric homeomorphism  $h(x) =$  $\int_0^x e^{\phi(t)} dt$  is given as  $h = h(\mu)$  for some  $\mu \in M_B(\mathbb{H}^+)$ .

*Proof.* For each  $s \in [0, 1]$ , let

$$
h_s(x) = \int_0^x e^{s\phi(t)} dt.
$$

Then,  $h_s$  is an increasing and locally absolutely continuous homeomorphism of R with  $h_0 = id$  and  $\log h'_s = s\dot{\phi} \in \text{Re BMO}^*(\mathbb{R})$ . By the Hölder inequality, we see that the constant  $C_{\infty}$  in [\(3\)](#page-9-0) for the  $A_{\infty}$ -weights  $h'_{s}$  for all s are bounded by that for h'. Hence,  $h_s$  are uniformly strongly quasisymmetric homeomorphisms for all  $s \in [0,1]$  in the sense that we can take the constants  $\alpha$  and K in [\(3.1\)](#page-8-0) uniformly, and so are their inverses  $h_s^{-1}$ by [\[8,](#page-33-4) Lemma 5].

We have that

$$
\log(h_{\tilde{s}} \circ h_s^{-1})' = (\log h'_{\tilde{s}} - \log h'_{s}) \circ h_s^{-1} = (\tilde{s} - s)C_{h_s}^{-1}\phi
$$

for any  $s, \tilde{s} \in [0, 1]$ . This belongs to ReBMO(R) and the operator norms  $||C_{h_s}^{-1}||$  are uniformly bounded by Theorem [3.5](#page-11-2) because the constants for strong quasisymmetry of  $h_s^{-1}$  are independent of s. Hence, we can choose a positive integer  $n \geq 1$  such that  $\log(h_{j/n} \circ h_{(j-1)/n}^{-1})'$  belong to the neighborhood U of 0 in BMO<sup>\*</sup>( $\mathbb{R}$ ) taken by Proposition [6.4](#page-18-1) for all  $j = 1, ..., n$ .

By this proposition, there exist  $([\mu_j], [\bar{\mu}_j]) \in \text{Sym}(T_B^+ \times T_B^-)$  such that  $\Lambda([\mu_j], [\bar{\mu}_j]) =$  $\log(h_{j/n} \circ h_{(j-1)/n}^{-1})'$  for all j. Therefore,

$$
h = (h_1 \circ h_{(n-1)/n}^{-1}) \circ \cdots \circ (h_{1/n} \circ h_0^{-1})
$$

is given as  $[\mu] = [\mu_n] * \cdots * [\mu_1]$  in  $T_B$ . Then, we have  $\Lambda([\mu], [\bar{\mu}]) = \log h' = \phi$ , which is the desired assertion.  $\Box$ 

Therefore, combined with Theorem [6.5,](#page-18-0) Theorem [4.7](#page-13-0) is improved to the complete characterization of  $T_B$  in terms of strong quasisymmetry or Re BMO<sup>\*</sup>( $\mathbb{R}$ ). Thus, we obtain the alternative proof of Theorem [3.1,](#page-8-1) independently of the theorems in [\[13\]](#page-33-2).

## 7. Biholomorphic correspondence

We have seen that BMO embeddings  $\gamma$  with chord-arc images are represented in the Bers coordinates, and the map  $\Lambda$  to BMO<sup>\*</sup>( $\mathbb{R}$ ) via log  $\gamma'$  is a holomorphic injection. In fact, this map is a biholomorphic homeomorphism onto its image, as shown in [\[45,](#page-34-0) Theorem 6.1. Here, we restructure the proof to emphasize the surjectivity of the derivative of  $\Lambda$ as the main point.

<span id="page-19-2"></span>**Theorem 7.1.** Let  $\Lambda : \widetilde{T}_C \to \text{BMO}^*(\mathbb{R})$  be the holomorphic injection given by

$$
\Lambda([\mu^+], [\mu^-]) = \log \gamma' \quad (\gamma = \gamma([\mu^+], [\mu^-])).
$$

Then, its image  $\Lambda(\tilde{T}_C)$  is an open subset of BMO<sup>\*</sup>(R) containing ReBMO<sup>\*</sup>(R), and  $\Lambda$  is a biholomorphic homeomorphism onto  $\Lambda(T_C)$ .

In the direct sum decomposition of the tangent space

$$
\mathcal{T}_{([\mu^+],[\mu^-])}\widetilde{T}_C = \mathcal{T}_{[\mu^+]}T_B^+ \oplus \mathcal{T}_{[\mu^-]}T_B^-, \tag{7}
$$

we denote the canonical projections by

$$
J^+ : \mathscr{T}_{([\mu^+],[\mu^-])}(\widetilde{T}_C) \to \mathscr{T}_{[\mu^-]}(T_B^-), \quad J^- : \mathscr{T}_{([\mu^+],[\mu^-])}(\widetilde{T}_C) \to \mathscr{T}_{[\mu^+]}(T_B^+).
$$

We first determine the image of each factor under the derivative  $d_{([\mu^+],[\mu^-])}$ . In the sequel,  $h^{\pm} \in SQS$  always denote the quasisymmetric homeomorphisms given by  $h^{\pm} = h(\mu^{\pm})$ .

<span id="page-19-0"></span>**Lemma 7.2.**  $d_{([\mu^+],[\mu^-])} \Lambda(\mathcal{I}_{[\mu^{\pm}]}T_B^{\pm}) = C_{h^{\mp}}(\text{BMOA}(\mathbb{H}^{\mp})).$ 

Proof. Formula [\(6\)](#page-16-0) shows that

$$
\Lambda([\mu^+], [\mu^-]) = \Lambda \circ R_{[\mu^+]}([0], [\mu^-] * [\bar{\mu}^+]^{-1}) = Q_{h^+} \circ \Lambda([0], r^{-1}_{[\bar{\mu}^+]}([\mu^-])).
$$

By fixing  $[\mu^+]$ , we take the partial derivative of this formula along the direction of  $T_B^-$ . Then,

$$
d_{([\mu^+],[\mu^-])}\Lambda|_{\mathscr{T}_{[\mu^-]}T_B^-}=C_{h^+}\circ d_{([0],[\mu^-]*[\bar\mu^+]^{-1})}\Lambda\circ d_{[\mu^-]}r_{[\bar\mu^+]}^{-1},
$$

where  $d_{([0],[\mu^-]*[\bar{\mu}^+]-1)}\Lambda$  restricted to the tangent subspace  $\mathscr{T}_{[\mu^-]*[\bar{\mu}^+]-1}T_B^- \cong \text{BMOA}(\mathbb{H}^+)$ can be regarded as the identity map. Hence,  $d_{([\mu^+],[\mu^-])} \Lambda(\mathcal{T}_{[\mu^-]}T_B) = C_{h^+}(\text{BMOA}(\mathbb{H}^+)).$ The other equation is similarly proved.

<span id="page-19-1"></span>The surjectivity of the derivative of  $\Lambda$  is reduced to the following condition.

**Lemma 7.3.** If the real subspace i Re BMO( $\mathbb{R}$ ) is contained in Ran  $d_{\left(\lceil \mu^+\rceil,\lceil \mu^-\rceil\right)}$   $\Lambda$ , the image of the derivative of  $\Lambda$  at  $([\mu^+], [\mu^-]) \in T_C$ , then  $d_{([\mu^+], [\mu^-])} \Lambda$  is surjective. Similarly, if the real subspace Re BMO(R) is contained in Ran  $d_{([\mu^+],[\mu^-])}\Lambda$ , then  $d_{([\mu^+],[\mu^-])}\Lambda$  is surjective.

*Proof.* For the former statement, it suffices to show that  $\text{Re BMO}(\mathbb{R}) \subset \text{Ran} d_{([\mu^+],[\mu^-])} \Lambda$ . We note that Lemma [7.2](#page-19-0) implies that  $C_{h^+}(\text{BMOA}(\mathbb{H}^+)) \subset \text{Ran} d_{\left(\lceil \mu^+ \rceil, \lceil \mu^- \rceil\right)} \Lambda$ .

We take any  $\phi \in \text{Re BMO}(\mathbb{R})$ . Since  $C_{h^+}$  maps  $\text{Re BMO}(\mathbb{R})$  isomorphically onto itself,  $C_{h^+}^{-1}(\phi)$  also belongs to ReBMO(R). Its Szegö projection

$$
P^{+}(C_{h^{+}}^{-1}(\phi)) = \frac{1}{2}C_{h^{+}}^{-1}(\phi) + i\frac{1}{2}\mathcal{H} \circ C_{h^{+}}^{-1}(\phi)
$$
\n(8)

is in  $BMOA(\mathbb{H}^+)$ . This implies that

$$
\phi + iC_{h^+} \circ \mathcal{H} \circ C_{h^+}^{-1}(\phi) \in \text{Ran} \, d_{([\mu^+],[\mu^-])} \, \Lambda
$$

by the application of  $2C_{h+}$  to [\(7\)](#page-19-1). Here,  $iC_{h+} \circ \mathcal{H} \circ C_{h+}^{-1}(\phi) \in i \text{Re BMO}(\mathbb{R})$  also belongs to Ran  $d_{([\mu^+],[\mu^-])}\Lambda$  by the assumption. Therefore, we have that  $\phi \in \text{Ran} d_{([\mu^+],[\mu^-])}\Lambda$ .

For the latter statement, it suffices to show that  $i \text{Re BMO}(\mathbb{R}) \subset \text{Ran} d_{(|\mu^+|, |\mu^-|)} \Lambda$ . We take any  $i\phi \in i \text{Re BMO}(\mathbb{R})$ . By [\(7\)](#page-19-1), we have that

$$
iP^{+}(C_{h^{+}}^{-1}(\phi)) = \frac{i}{2}C_{h^{+}}^{-1}(\phi) - \frac{1}{2}\mathcal{H} \circ C_{h^{+}}^{-1}(\phi)
$$

is in  $BMOA(\mathbb{H}^+)$ , and hence

$$
i\phi - C_{h^+} \circ \mathcal{H} \circ C_{h^+}^{-1}(\phi) \in \operatorname{Ran} d_{([\mu^+],[\mu^-])} \Lambda.
$$

Here,  $C_{h^+} \circ \mathcal{H} \circ C_{h^+}^{-1}(\phi) \in \text{Re BMO}(\mathbb{R})$  also belongs to  $\text{Ran} d_{([\mu^+],[\mu^-])} \Lambda$  by the assumption, and thus we have  $i\phi \in \text{Ran} d_{([\mu^+],[\mu^-])} \Lambda$ .

The following is the crucial claim for establishing our argument. If  $\Lambda([\mu_0^+], [\mu_0^-]) = \log \gamma'$ lies in i Re BMO( $\mathbb{R}$ ), then  $\gamma(x) = \int_0^x \gamma'(t) dt$  is the arc-length parametrization of a chordarc curve, since  $|\gamma'(t)| = 1$ . In this context, there are several important studies in real analysis regarding the deformation of chord-arc curves, which are surveyed in [\[31\]](#page-34-14). We extend these results to:

<span id="page-20-0"></span>**Lemma 7.4.** If  $\Lambda([\mu_0^+],[\mu_0^-])$  is in the subspace i Re BMO(R), then i Re BMO(R) is in Ran  $d_{([\mu^+],[\mu^-])}\Lambda$ , the image of the derivative of  $\Lambda$  at  $([\mu^+],[\mu^-])$ .

Proof. This can be seen from the work on chord-arc curves by Semmes [\[34,](#page-34-7) p.254]. In fact, for every  $\phi_0 = \Lambda([\mu_0^+], [\mu_0^-]) \in i \text{Re BMO}(\mathbb{R})$ , there is a neighborhood  $U_0$  of  $\phi_0$  in  $BMO(\mathbb{R})$  and a holomorphic map  $\tau: U_0 \to M_B(\mathbb{H}^+) \times M_B(\mathbb{H}^-)$  such that  $\Lambda \circ (\pi^+ \times \pi^-) \circ \tau$ is the identity on  $U_0$ . To prove this claim, we first see that  $\tau$  is defined to be bounded on some neighborhood  $U_0$  by the quasiconformal extension given in [\[34,](#page-34-7) Proposition 4.13]. Then, by [\[37,](#page-34-15) Formulas (6.7), (6.27)], the complex dilatation  $\tau(\phi)$  for  $\phi \in U_0$  is explicitly represented, and  $\tau(\phi)(z)$  is Gâteaux holomorphic on  $U_0$  for each fixed  $z \in \mathbb{H}^+\cup \mathbb{H}^-$ . Under these conditions, we can conclude that  $\tau$  is holomorphic on  $U_0$  by [\[44,](#page-34-16) Lemma 6.1].  $\Box$ 

*Proof of Theorem [7.1.](#page-19-2)* Proposition [6.3](#page-17-0) asserts that  $\Lambda$  is a holomorphic injection. To show that  $\Lambda$  is biholomorphic, we prove that the derivative  $d_{([{\mu}^+],[{\mu}^-])} \Lambda$  at every  $([\mu^+], [\mu^-]) \in T_C$ is surjective onto  $BMO(\mathbb{R})$ . Then, by the inverse mapping theorem (see [\[7,](#page-33-13) §7.18]), we obtain the required claim.

Let  $\phi = \Lambda([\mu^+], [\mu^-]) \in \text{BMO}^*(\mathbb{R})$ . We can find  $\phi_0 \in i \text{Re BMO}(\mathbb{R}) \cap \Lambda(\widetilde{T}_C)$  and  $[\nu] \in T_B$  such that  $Q_h(\phi_0) = \phi$  with  $h = h(\nu)$ . Indeed, we take  $[\nu] \in T_B$  such that  $\Lambda([\nu], [\bar{\nu}]) = \log h' = \text{Re}\,\phi$  by Theorem [6.5,](#page-18-0) and set  $\phi_0 = iC_h^{-1}(\text{Im}\,\phi)$ . Then,

$$
Q_h(\phi_0) = C_h(\phi_0) + \log h' = i \operatorname{Im} \phi + \operatorname{Re} \phi = \phi.
$$

We consider the derivative of  $\Lambda$  at  $([\mu_0^+], [\mu_0^-]) = R_{[\nu]}^{-1}([\mu^+], [\mu^-])$ , where  $\Lambda([\mu_0^+], [\mu_0^-]) =$  $Q_h^{-1} \circ \Lambda([\mu^+], [\mu^-]) = \phi_0$ . By Lemma [7.4,](#page-20-0) we have  $i \text{Re}\,\text{BMO}(\mathbb{R}) \subset \text{Ran} \, d_{([\mu_0^+],[\mu_0^-])} \Lambda$ . Under this condition, Lemma [7.3](#page-19-1) yields that  $d_{([\mu_0^+],[\mu_0^-])}$   $\Lambda$  is surjective. Since

$$
d_{([\mu^+],[\mu^-])}\Lambda = d_{\phi_0}Q_h \circ d_{([\mu_0^+],[\mu_0^-])}\Lambda \circ d_{([\mu^+],[\mu^-])}R_{[\nu]}^{-1},
$$

this is also surjective.  $\square$ 

The proof of Theorem [7.1](#page-19-2) shows that the derivative

$$
d_{([\mu^+],[\mu^-])}\Lambda:\mathscr{T}_{([\mu^+],[\mu^-])}\widetilde{T}_C\to\mathrm{BMO}(\mathbb{R})
$$

of  $\Lambda$  at every point  $([\mu^+], [\mu^-]) \in T_C$  is a surjective isomorphism. Therefore, in account of Lemma [7.2,](#page-19-0) we obtain the topological direct sum decomposition of BMO( $\mathbb{R}$ ) at  $([\mu^+], [\mu^-])$ as

$$
BMO(\mathbb{R}) = C_{h^-}(BMOA(\mathbb{H}^-)) \oplus C_{h^+}(BMOA(\mathbb{H}^+)).
$$
\n(9)

According to this decomposition, we define the bounded linear projections as

$$
P^{\pm}_{([\mu^+],[\mu^-])}: \mathrm{BMO}(\mathbb{R}) \to C_{h^{\pm}}(\mathrm{BMOA}(\mathbb{H}^{\pm})).
$$

<span id="page-21-0"></span> $\textbf{Lemma 7.5.} \ \ P^{\pm}_{([\mu^+],[\mu^-])}=d_{([\mu^+],[\mu^-])}\,\Lambda \circ J^{\pm} \circ (d_{([\mu^+],[\mu^-])}\,\Lambda)^{-1}.$ 

*Proof.* By Lemma [7.2,](#page-19-0) we see that the derivative  $d_{([\mu^+],[\mu^-])}\Lambda$  preserves each factor of the direct sum decompositions [\(7\)](#page-20-0) and (7). Then, the projections  $J^+$  and  $P^+_{([\mu^+],[\mu^-])}$  to the second factors are conjugated by  $d_{([\mu^+],[\mu^-])}\Lambda$ , and so are the projections  $J^-$  and  $P^-_{([\mu^+],[\mu^-])}$ to the first factors.

Finally, as an application of Theorem [7.1,](#page-19-2) we obtain a result about the real-analytic structure of the BMO Teichmüller space  $T_B$ . We restrict the biholomorphic homeomorphism  $\Lambda$  to the real-analytic submanifold Sym  $(T_B^+ \times T_B^-)$ , as in the setting of Theorem [4.7.](#page-13-0) By composing  $\Lambda$  with the canonical real-analytical embedding  $\iota : T_B \to \text{Sym}(T_B^+ \times T_B^-)$ , we have the following. This has appeared in [\[45,](#page-34-0) Corollary 6.2].

**Corollary 7.6.** The map  $\Lambda \circ \iota : T_B \to \text{Re BMO}^*(\mathbb{R})$  given by  $h \mapsto \log h'$  is a real-analytic diffeomorphism. Hence, the BMO Teichmüller space  $T_B$  is real-analytically equivalent to  $\widetilde{\operatorname{Re}\operatorname{BMO}^*(\mathbb{R})}.$ 

## 8. The Cauchy transform on chord-arc curves

In the course of proving the biholomorphic correspondence  $\Lambda : \widetilde{T}_C \to \text{BMO}^*(\mathbb{R})$  from BMO embeddings with chord-arc image in Bers coordinates to the BMO space, we investigated the derivative  $d_{([\mu^+],[\mu^-])}\Lambda$  at  $([\mu^+],[\mu^-]) \in T_C$ . In this section, we show that the Cauchy transform and the Cauchy projection of BMO functions on the chord-arc curve  $\Gamma = \gamma(\mathbb{R})$  for  $\gamma = \gamma([\mu^+], [\mu^-])$  can be represented by  $d_{([\mu^+], [\mu^-])}\Lambda$ . In particular, this proves the Calderón theorem in real analysis for BMO functions on chord-arc curves.

We first introduce the Cauchy transform and the Cauchy projection by considering the Cauchy integral on a chord-arc curve  $\Gamma = \gamma(\mathbb{R})$ . We define the Banach space of BMO functions on Γ by the push-forward of BMO(R) by  $\gamma$  and identify this pair. Namely,

$$
BMO(\gamma(\mathbb{R})) = \{ \gamma_* \phi = \phi \circ \gamma^{-1} \mid \phi \in BMO(\mathbb{R}) \}
$$

with norm  $\|\gamma_*\phi\|_{\text{BMO}(\gamma)} = \|\phi\|_{\text{BMO}}.$ 

**Remark 8.1.** Usually, a function space on a locally rectifiable curve  $\Gamma$  is defined by using its arc-length parametrization. In our case, for a BMO embedding  $\gamma_0 : \mathbb{R} \to \mathbb{C}$  such that  $\gamma_0$  gives an arc-length parametrization of Γ, we may set BMO( $\gamma_0(\mathbb{R})$ ) by the above definition. However, the difference between  $\gamma$  and  $\gamma_0$  is given by the composition operator  $C_h$  for  $h \in \text{SQS}$  (as explained in the next section in detail), and it can be controlled well. Hence, we adopt the representation  $BMO(\gamma(\mathbb{R}))$  because it simplifies the arguments for considering the dependence of the operators acting on it when the embedding  $\gamma$  varies.

**Definition 8.1.** The *Cauchy transform* of  $\psi \in BMO(\gamma(\mathbb{R}))$  on a chord-arc curve  $\Gamma = \gamma(\mathbb{R})$ (oriented by  $\mathbb R$  in the increasing direction) is defined by the singular integral

$$
(\mathcal{H}_{\Gamma}\psi)(\xi) = \text{p.v.} \frac{1}{\pi} \int_{\Gamma} \left( \frac{\psi(z)}{\xi - z} - \frac{\psi(z)}{\zeta_0^{\pm} - z} \right) dz \quad (\xi \in \Gamma),
$$

where  $dz = \gamma'(t)dt$ , and  $\zeta_0^+$  and  $\zeta_0^-$  are arbitrary points in the left and the right domains  $Ω<sup>+</sup>$  and  $Ω<sup>-</sup>$  bounded by Γ, respectively. The Cauchy integrals of  $ψ$  on Γ are defined by

$$
(P_{\Gamma}^{\pm}\psi)(\zeta) = \frac{-1}{2\pi i} \int_{\Gamma} \left(\frac{\psi(z)}{\zeta - z} - \frac{\psi(z)}{\zeta_0^{\pm} - z}\right) dz \quad (\zeta \in \Omega^{\pm}),
$$

which are holomorphic functions on  $\Omega^{\pm}$ . Here, the integration over  $\Gamma$  is taken in the inverse direction when  $\zeta \in \Omega$ <sup>-</sup>.

The point-wise convergence of the Cauchy transform and the Cauchy integrals for  $\psi \in$  $BMO(\gamma(\mathbb{R}))$  are guaranteed by this definition of using the regularized kernel. By the Privalov theorem (see [\[16,](#page-33-14) p.431]), if the Cauchy transform  $(\mathcal{H}_{\Gamma}\psi)(\xi)$  exists a.e. on Γ, then the Cauchy integrals  $(P_{\Gamma}^{\pm}\psi)(\zeta)$  have non-tangential limits a.e. on  $\Gamma$ , and vice versa. The boundary extensions of  $P_{\Gamma}^{\pm} \psi$  to  $\Gamma$  are also denoted by the same symbol and called the *Cauchy projections* of  $\psi$ .

The Plemelj formula for the Riemann–Hilbert problem asserts the following relation between the Cauchy transform and the Cauchy projections. This is a generalization of the relation between the Hilbert transform and the Szegö projections. We remark that the sign of  $P_{\Gamma}^-$  is opposite to the usual one due to the orientation of  $\Gamma$ .

<span id="page-22-0"></span>**Proposition 8.1.** For a function  $\psi \in BMO(\gamma(\mathbb{R}))$  on a chord-arc curve  $\Gamma = \gamma(\mathbb{R})$ , the Cauchy transform  $\mathcal{H}_{\Gamma}$  and the Cauchy projections  $P_{\Gamma}^{\pm}$  satisfy

$$
P_{\Gamma}^{+}\psi = \frac{1}{2}(\psi + i\mathcal{H}_{\Gamma}\psi), \quad P_{\Gamma}^{-}\psi = \frac{1}{2}(\psi - i\mathcal{H}_{\Gamma}\psi).
$$

In other words,

$$
\psi = P_{\Gamma}^{+}\psi + P_{\Gamma}^{-}\psi, \quad i\mathcal{H}_{\Gamma}\psi = P_{\Gamma}^{+}\psi - P_{\Gamma}^{-}\psi
$$

holds.

Next, we consider the space of holomorphic functions to which the Cauchy integrals  $P_{\Gamma}^{\pm}\psi$  belong. Let  $F^{\pm}:\mathbb{H}^{\pm}\to\Omega^{\pm}$  be the normalized Riemann mappings, where  $\Omega^{\pm}$  are the

complementary domains of  $\Gamma$  in  $\mathbb{C}$ . Then, we define the Banach space of BMOA functions on  $\Omega^{\pm}$  by the push-forward of BMOA( $\mathbb{H}^{\pm}$ ) by  $F^{\pm}$  and identify these pairs. Namely,

$$
\text{BMOA}(\Omega^{\pm}) = \{ (F^{\pm})_* \Phi^{\pm} = \Phi^{\pm} \circ (F^{\pm})^{-1} \mid \Phi^{\pm} \in \text{BMOA}(\mathbb{H}^{\pm}) \}
$$

with norm  $||(F^{\pm})_*\Phi^{\pm}||_{\text{BMOA}(\Omega^{\pm})} = ||\Phi^{\pm}||_{\text{BMOA}}.$ 

For  $(F^{\pm})_*\Phi^{\pm} \in \text{BMOA}(\Omega^{\pm})$ , their boundary extensions to  $\Gamma$  are  $E(\Phi^{\pm}) \circ (F^{\pm})^{-1}$ , where the Riemann mappings  $F^{\pm}$  are assumed to extend to homeomorphisms of R onto Γ. We define the boundary extension operators  $E_{\Gamma}^{\pm}$  to  $\Gamma$  by

$$
E_{\Gamma}^{\pm}((F^{\pm})_*\Phi^{\pm}) = E(\Phi^{\pm}) \circ (F^{\pm})^{-1}.
$$

<span id="page-23-0"></span>**Proposition 8.2.** It holds that  $E_{\Gamma}^{\pm}(\text{BMOA}(\Omega^{\pm})) \subset \text{BMO}(\gamma(\mathbb{R}))$ . Moreover, these trace operators  $E_{\Gamma}^{\pm}$  are Banach isomorphisms onto their images, where their operator norms are estimated in terms of the norms of the composition operators  $C_{h^{\pm}}$  for  $h^{\pm} = h(\mu^{\pm}) \in SQS$ , respectively.

*Proof.* The norm on BMOA( $\Omega^{\pm}$ ) is induced from BMOA( $\mathbb{H}^{\pm}$ ) by  $F^{\pm}$  whereas the norm of BMO( $\gamma(\mathbb{R})$ ) is induced from BMO( $\mathbb{R}$ ) by  $\gamma$ . Because  $\gamma = F^{\pm} \circ h^{\pm}$ , the differences between  $E^{\pm}$  and  $E_{\Gamma}^{\pm}$  are caused by the composition operators  $C_{h^{\pm}}$ , respectively.

By this proposition, we see that  $E_{\Gamma}^{\pm}(\text{BMOA}(\Omega^{\pm}))$  are closed subspaces of BMO( $\gamma(\mathbb{R})$ ). Moreover, under the Banach isomorphism  $E_{\Gamma}^{\pm}$ , we may identify  $E_{\Gamma}^{\pm}(\text{BMOA}(\Omega^{\pm}))$  with BMOA $(\Omega^{\pm})$ . Hence, we regard BMOA $(\Omega^{\pm})$  as closed subspaces of BMO $(\gamma(\mathbb{R}))$  without noticing  $E_{\Gamma}^{\pm}$  hereafter.

For  $([\mu^+], [\mu^-]) \in \widetilde{T}_C$ , we have obtained the images of the tangent subspaces  $\mathscr{T}_{[\mu^+]}T^+_B$  $B^+$  and  $\mathscr{T}_{[\mu^-]}T_B^-$  under the surjective derivative  $d_{([\mu^+],[\mu^-])}\Lambda$ , which correspond to the topological direct sum decomposition [\(7\)](#page-20-0). Then, every  $\phi \in BMO(\mathbb{R})$  is uniquely represented by  $\phi = \phi^+ + \phi^-$  for  $\phi^+ \in C_{h^+}(\text{BMOA}(\mathbb{H}^+))$  and  $\phi^- \in C_{h^-}(\text{BMOA}(\mathbb{H}^-))$ . These bounded linear projections are denoted by  $\phi^{\pm} = P^{\pm}_{([\mu^+],[\mu^-])}(\phi)$ .

<span id="page-23-1"></span>**Theorem 8.3.** In the function space BMO( $\gamma(\mathbb{R})$ ) on a chord-arc curve  $\Gamma = \gamma(\mathbb{R})$ , the Cauchy projections  $P_{\Gamma}^{\pm}$  satisfy

$$
P_{\Gamma}^{\pm} = \gamma_{*} \circ P_{([\mu^{+}],[\mu^{-}])}^{\pm} \circ \gamma_{*}^{-1}.
$$

In particular,  $P_{\Gamma}^{\pm}$  maps BMO( $\gamma(\mathbb{R})$ ) onto BMOA( $\Omega^{\pm}$ ), they are bounded linear operators, and their operator norms are estimated in terms of  $||C_{h^{\pm}}||$ .

Proof. Let  $\phi = \gamma_*^{-1}(\psi) = \psi \circ \gamma$  for  $\psi \in BMO(\gamma(\mathbb{R})) = \gamma_*(BMO(\mathbb{R}))$ . Then,  $\phi \in BMO(\mathbb{R})$ and  $P^{\pm}_{([\mu^+],[\mu^-])} \circ \gamma_*^{-1}(\psi) = \phi^{\pm} \in C_{h^{\pm}}(\text{BMOA}(\mathbb{H}^{\pm})).$  Therefore,

$$
\gamma_*\circ P_{([\mu^+],[\mu^-])}^{\pm}\circ\gamma_*^{-1}(\psi)\in {\rm BMOA}(\Omega^{\pm})
$$

since  $\gamma = F^{\pm} \circ h^{\pm}$  and BMOA $(\Omega^{\pm}) = F^{\pm}$ (BMOA $(\mathbb{H}^{\pm})$ ) for the Riemann mappings  $F^{\pm}$ :  $\mathbb{H}^{\pm} \to \Omega^{\pm}$ . By the definition of the projections  $P_{([\mu^+],[\mu^-])}^{\pm}$ , we have

$$
\gamma_* \circ P^+_{([\mu^+],[\mu^-])} \circ \gamma_*^{-1}(\psi) + \gamma_* \circ P^-_{([\mu^+],[\mu^-])} \circ \gamma_*^{-1}(\psi) = \psi.
$$

Let  $\Psi_1$  be a measurable function on  $\mathbb C$  defined by BMOA functions  $\gamma_* \circ P^+_{([\mu^+],[\mu^-])} \circ \gamma_*^{-1}(\psi)$ on  $\Omega^+$  and  $-\gamma_* \circ P^-_{([\mu^+],[\mu^-])} \circ \gamma_*^{-1}(\psi)$  on  $\Omega^-$ . Then,  $\Psi_1$  is locally integrable and is of growth order  $\Psi_1(z) = o(|z|)$  as  $z \to \infty$ .

We can verify that  $-i\bar{\partial}\Psi_1 = \psi dz_\Gamma$  as a distribution according to the argument in [\[33,](#page-34-17) p.204]. Here,  $dz_{\Gamma}$  denotes a continuous linear functional defined by

$$
\langle X, dz_{\Gamma} \rangle = \int_{\Gamma} X(z) dz
$$

for every test function  $X \in C_0^{\infty}(\mathbb{C})$ . Then, we have

$$
\langle X, \psi dz_{\Gamma} \rangle = \int_{\Gamma} X(z) \psi(z) dz = 2i \int_{\mathbb{C}} \bar{\partial}X(z) \Psi_1(z) dx dy = -2i \langle X, \bar{\partial} \Psi_1 \rangle.
$$

The middle equality in the above equations is derived from the Green formula.

To see this more precisely, we choose a simply connected bounded domain  $W \subset \mathbb{C}$  of smooth boundary intersecting  $\Gamma$  and containing the support of X. Moreover, we choose a decreasing sequence of neighborhoods  $U_n$  of  $\Gamma$  with  $\bigcap_{n=1}^{\infty} U_n = \Gamma$  appropriately, and let  $W_n^{\pm} = (W \cap \Omega^{\pm}) \setminus U_n$ . Then, the Green formula implies that

$$
\int_{\partial W_n^\pm} X(z) \Psi_1(z) dz = \int_{W_n^\pm} \bar{\partial} (X(z) \Psi_1(z)) d\bar{z} \wedge dz = 2i \int_{W_n^\pm} \bar{\partial} X(z) \Psi_1(z) dx dy.
$$

Here, the first integrals over  $\partial W_n^{\pm}$  tend to the integrals involving the non-tangential limits of  $\Psi_1$  over  $\Gamma \cap W$  as  $n \to \infty$  because  $\Psi_1$  is made of BMOA( $\Omega^{\pm}$ ) whose restrictions to  $W^{\pm} = W \cap \Omega^{\pm}$  belong to the Hardy space  $H^p(W^{\pm})$ . Then, summing up two equations for  $\pm$  after passing to these limits, we obtain the required equation.

The Cauchy projections  $P_{\Gamma}^{\pm}(\psi)$  satisfy the same properties as  $\gamma_* \circ P_{([\mu^+],[\mu^-])}^{\pm} \circ \gamma_*^{-1}(\psi)$ . They are holomorphic functions on  $\Omega^{\pm}$  such that their boundary extensions satisfy  $P_{\Gamma}^{+}$  $\Gamma_{\Gamma}^{+}(\psi)+$  $P_{\Gamma}(\psi) = \psi$  by Proposition [8.1.](#page-22-0) Let  $\Psi_2$  be a measurable function on  $\mathbb C$  defined by  $\overline{P_{\Gamma}^+}$  $\Gamma_{\Gamma}^{+}(\psi)$  on  $\Omega^{+}$  and  $-P_{\Gamma}^{-}(\psi)$  on  $\Omega^{-}$ . Then,  $\Psi_{2}$  is locally integrable and is of growth order  $\Psi_2(z) = o(|z|)$  as  $z \to \infty$ . These facts are verified in [\[33,](#page-34-17) Lemma 3.2].

Moreover, we have also  $-2i\bar{\partial}\Psi_2 = \psi dz_{\Gamma}$  as a distribution. Indeed, for every test function  $X \in C_0^{\infty}(\mathbb{C})$ , the Pompeiu formula (applied at  $\doteq$ ) implies that

$$
-2i\langle X,\bar{\partial}\Psi_2\rangle = 2i\int_{\mathbb{C}} \bar{\partial}X(\zeta)\Psi_2(\zeta)d\xi d\eta = \frac{-1}{\pi} \int_{\mathbb{C}} \bar{\partial}X(\zeta) \left(\int_{\Gamma} \left(\frac{\psi(z)}{\zeta - z} - \frac{\psi(z)}{\zeta - z}\right) dz\right) d\xi d\eta
$$

$$
= \int_{\Gamma} \left(\frac{-1}{\pi} \int_{\mathbb{C}} \frac{\bar{\partial}X(\zeta)}{\zeta - z} d\xi d\eta\right) \psi(z) dz = \int_{\Gamma} X(z)\psi(z) dz = \langle X, \psi dz_{\Gamma} \rangle.
$$

The exchange of order of the integrations is guaranteed simply by integrability of

$$
|\bar{\partial}X(\zeta)| \left| \frac{\psi(z)}{\zeta - z} - \frac{\psi(z)}{\zeta_0^{\pm} - z} \right|
$$

over  $(\zeta, z) \in \mathbb{C} \times \Gamma$ .

Since we have seen that  $-2i\bar{\partial}\Psi_1 = \psi dz_\Gamma = -2i\bar{\partial}\Psi_2$ ,  $\bar{\partial}$ -derivative of the locally integrable function  $\Psi_1 - \Psi_2$  is 0 on  $\mathbb C$  in the distribution sense. Hence, it is holomorphic on  $\mathbb C$ by the Weyl lemma. The growth order  $\Psi_1(z) - \Psi_2(z) = o(|z|)$   $(z \to \infty)$  forces the entire function  $\Psi_1 - \Psi_2$  to be a constant. Thus, we have  $P_{\Gamma}^{\pm}(\psi) = \gamma_* \circ P_{([\mu^+],[\mu^-])}^{\pm} \circ \gamma_*^{-1}(\psi)$  on  $\Omega^{\pm}$  up to constants.

<span id="page-25-0"></span>**Corollary 8.4.** The Cauchy transform  $\mathcal{H}_{\Gamma}$  on a chord-arc curve  $\Gamma = \gamma(\mathbb{R})$  maps  $\text{BMO}(\gamma(\mathbb{R}))$ to BMO( $\gamma(\mathbb{R})$ ), which is a Banach automorphism of BMO( $\gamma(\mathbb{R})$ ).

*Proof.* Since  $\mathcal{H}_{\Gamma} = -i(P_{\Gamma}^+ - P_{\Gamma}^-)$  by Proposition [8.1,](#page-22-0) the statement follows from Proposi-tion [8.2](#page-23-0) and Theorem [8.3.](#page-23-1)

**Remark 8.2.** The boundedness of the Cauchy transform  $\mathcal{H}_{\Gamma}$  as well as the Cauchy projections  $P_{\Gamma}^{\pm}$  is also verified in [\[23,](#page-34-18) Theorem 1.1] as an application of the corresponding  $L^p$  estimate in [\[12\]](#page-33-15).

Theorem [8.3](#page-23-1) also leads to the Calderón theorem (see [\[9,](#page-33-16) Section 8]) for chord-arc curves.

**Corollary 8.5.** The Cauchy projections  $P_{\Gamma}^{\pm}$  on a chord-arc curve  $\Gamma = \gamma(\mathbb{R})$  are associated with the topological direct sum decomposition

$$
BMO(\gamma(\mathbb{R})) = BMOA(\Omega^+) \oplus BMOA(\Omega^-).
$$

**Remark 8.3.**  $P_{\Gamma}^{\pm} \circ \gamma_* = \gamma_* \circ P_{([\mu^+],[\mu^-])}^{\pm}$  maps BMO( $\mathbb{R}$ ) onto BMOA( $\Omega^{\pm}$ ). When  $[\mu^+] = [0]$ or  $[\mu^-] = [0]$ , this coincides with what is called the Faber operator. See [\[23,](#page-34-18) [24\]](#page-34-19) for related arguments. In our setting, we see not only the boundedness of this operator but also its holomorphic dependence when the embeddings  $\gamma$  vary in the Teichmüller space as is discussed in the next section.

## 9. Holomorphic dependence of the Cauchy transform

<span id="page-25-1"></span>In this section, we consider the variation of the Cauchy transform  $\mathcal{H}_{\Gamma}$  when  $\Gamma = \gamma(\mathbb{R})$ moves according to  $\gamma = \gamma([\mu^+], [\mu^-])$ . To formulate this problem, we take the conjugate of  $\mathcal{H}_{\Gamma}$  so that it acts on BMO( $\mathbb{R}$ ). Namely, for  $([\mu^+] , [\mu^-]) \in \widetilde{T}_C$ , we set

$$
\mathcal{H}_{([\mu^+],[\mu^-])} = \gamma_*^{-1} \circ \mathcal{H}_{\Gamma} \circ \gamma_*,\tag{10}
$$

which is a Banach automorphism of  $BMO(\mathbb{R})$  by Corollary [8.4.](#page-25-0) More explicitly,

$$
\mathcal{H}_{([\mu^+],[\mu^-])}(\phi)(x) = \text{p.v.} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\phi(t)}{\gamma(x) - \gamma(t)} \gamma'(t) dt \quad (x \in \mathbb{R})
$$

for  $\phi \in \text{BMO}(\mathbb{R})$ .

By Proposition [8.1,](#page-22-0) the relationship between the Cauchy transform  $\mathcal{H}_{\Gamma}$  and the Cauchy projections  $P_{\Gamma}^{\pm}$  is given as  $\mathcal{H}_{\Gamma} = -i(P_{\Gamma}^+ - P_{\Gamma}^-)$ . Moreover, Theorem [8.3](#page-23-1) shows  $P_{\Gamma}^{\pm} =$  $\gamma_* \circ P^{\pm}_{([\mu^+],[\mu^-])} \circ \gamma_*^{-1}$ . Then, we have

$$
\mathcal{H}_{([\mu^+],[\mu^-])} = -i(P^+_{([\mu^+],[\mu^-])} - P^-_{([\mu^+],[\mu^-])}). \tag{11}
$$

For  $([0], [0]) \in \tilde{T}_C$ ,  $\mathcal{H}_{([0], [0])}$  coincides with the Hilbert transform  $\mathcal{H}$ . For  $([\mu], [\bar{\mu}]) \in$ Sym  $(T_B^+ \times T_B^-)$ ,  $\mathcal{H}_{([\mu],[\bar{\mu}])}$  is the conjugate of H by the composition operator  $C_h$  for  $h =$  $h(\mu) \in SQS$ . Indeed, [\(9\)](#page-25-1) applied to the case of  $\Gamma = h(\mathbb{R}) = \mathbb{R}$  with  $\gamma([\mu], [\bar{\mu}]) = h$  yields  $\mathcal{H}_{([\mu],[\bar\mu])}=C_h\circ \mathcal{H}\circ C_h^{-1}.$ 

Let  $\mathcal{L}(BMO(\mathbb{R}))$  be the Banach space of all bounded linear operators  $BMO(\mathbb{R}) \rightarrow$  $BMO(\mathbb{R})$  equipped with the operator norm. We consider the map

$$
\eta : \widetilde{T}_C \to \mathcal{L}(\text{BMO}(\mathbb{R}))
$$

<span id="page-26-0"></span>defined by  $([\mu^+], [\mu^-]) \mapsto \mathcal{H}_{([\mu^+],[\mu^-])}.$ 

**Theorem 9.1.**  $\eta : \widetilde{T}_C \to \mathcal{L}(\text{BMO}(\mathbb{R}))$  is holomorphic.

*Proof.* Due to formula [\(9\)](#page-25-1), it suffices to show that  $P^{\pm}_{([\mu^+],[\mu^-])}$  depend holomorphically on  $([\mu^+], [\mu^-]) \in \hat{T}_C$ . By Lemma [7.5,](#page-21-0) we have

$$
P^{\pm}_{([\mu^+],[\mu^-])} = d_{([\mu^+],[\mu^-])} \Lambda \circ J^{\pm} \circ (d_{([\mu^+],[\mu^-])} \Lambda)^{-1} = d_{([\mu^+],[\mu^-])} \Lambda \circ J^{\pm} \circ d_{\Lambda([\mu^+],[\mu^-])} \Lambda^{-1}.
$$

Since  $\Lambda$  is biholomorphic, these operators depend holomorphically on  $([\mu^+] , [\mu^-]) \in T_C$ .  $\Box$ 

A related result can be found in  $[9,$  Théorème 1, though without involving the biholomorphic map  $\Lambda$ , demonstrating in methods of real analysis that  $\eta|_{\text{Sym}(T_B^+\times T_B^-)}$  is realanalytic, in other words, the real-analytic dependence of  $\mathcal{H}_{([\mu],[\bar{\mu}])} = C_h \circ \mathcal{H} \circ C_h^{-1}$  upon  $h \in BMO^*(\mathbb{R})$ . This claim immediately follows from Theorem [9.1.](#page-26-0) In the sequel, as an application of this claim, we consider the theorem of Coifman and Meyer in [\[10,](#page-33-17) Theorem 1]. A survey of this theorem is in [\[31,](#page-34-14) Theorem 5].

We define  $Z = i \text{BMO}(\mathbb{R}) \cap \Lambda(T_C)$  as the real-analytic submanifold of  $\Lambda(T_C)$  consisting of purely imaginary-valued BMO functions which is an open subset of the real Banach subspace  $i$  BMO( $\mathbb{R}$ ) as used in Lemmas [7.3](#page-19-1) and [7.4.](#page-20-0) We do not know whether Z is connected or not. For  $\psi \in Z$ ,

$$
g_0(x) = \int_0^x \exp \psi(t) dt \quad (x \in \mathbb{R})
$$

serves as the *arc-length parameterization* of the chord-arc curve  $g_0(\mathbb{R})$ .

In general, for any BMO embedding with chord-arc image

$$
g(x) = \int_0^x \exp \varphi(t) dt \quad (x \in \mathbb{R})
$$

determined by  $\varphi \in \Lambda(\tilde{T}_C) \subset BMO^*(\mathbb{R})$ , we take a strongly quasisymmetric homeomorphism

$$
h(x) = \int_0^x \exp(\text{Re}\,\varphi(t))dt
$$

and set  $\psi = i \operatorname{Im} \varphi \circ h^{-1} \in Z$ . Then, the arc-length parameter  $g_0$  of the chord-arc curve  $g(\mathbb{R})$  determined by  $\psi$  allows g to be expressed as the *reparametrization* of  $g_0$  by h, that is,  $g = g_0 \circ h$ .

We define  $Y = BMOA(\mathbb{H}^+) \cap \Lambda(\tilde{T}_C)$  as the complex submanifold of  $\Lambda(\tilde{T}_C)$  consisting of BMOA functions on  $\mathbb{H}^+$ . We do not know whether Y is connected or not. For  $\varphi \in Y$ , the corresponding BMO embedding with chord-arc image

$$
f(x) = \int_0^x \exp \varphi(t) dt \quad (x \in \mathbb{R}),
$$

when applied as above, can be expressed as the reparametrization of the arc-length parameter  $g_0$  by a strongly quasisymmetric homeomorphism h so that  $f = g_0 \circ h$ . This correspondence between f and  $g_0$  is bijective, thus defining a mapping from Z to Y. Similarly, the correspondence from the pair  $(f, q_0)$  to their reparametrization h allows the definition of a mapping from Z to Re BMO<sup>\*</sup>( $\mathbb{R}$ ).

To investigate these mappings on  $\tilde{T}_C$  through the biholomorphic homeomorphism  $\Lambda$ , we define

$$
\rho: \widetilde{T}_C \to \{[0]\} \times T_C^-, \quad ([\mu^+], [\mu^-]) \mapsto ([0], [\mu^-] * [\overline{\mu^+}]^{-1});
$$
  

$$
\delta: \widetilde{T}_C \to \text{Sym}(T_B^+ \times T_B^-), \quad ([\mu^+], [\mu^-]) \mapsto ([\mu^+], [\overline{\mu^+}]).
$$

Here,  $\delta$  is the projection to the symmetric axis, which is real-analytic. The unique decomposition

$$
([\mu^+], [\mu^-]) = \rho([\mu^+], [\mu^-]) * \delta([\mu^+], [\mu^-])
$$

corresponds to the decomposition of a quasisymmetric embedding  $g = \gamma([\mu^+], [\mu^-])$  into  $g = f \circ h$  in general, where  $f = \gamma(\rho([\mu^+], [\mu^-]))$  is the boundary extension of the conformal homeomorphism of  $\mathbb{H}^+$  to  $\mathbb{R}$ , and  $h = \gamma(\delta([\mu^+], [\mu^-]))$  is a quasisymmetric homeomorphism of R.

We transform the two maps defined on the submanifold  $Z \subset \Lambda(\widetilde{T}_C)$  into those on  $\widetilde{T}_C$ via Λ. The map  $Z \to Y$  corresponds to

$$
\rho_0 = \rho|_{\Lambda^{-1}(Z)} : \Lambda^{-1}(Z) \to \{[0]\} \times T_C^- = \Lambda^{-1}(Y),
$$

and the map  $Z \to \text{Re BMO}^*(\mathbb{R})$  corresponds to

$$
\delta_0 = \delta|_{\Lambda^{-1}(Z)} : \Lambda^{-1}(Z) \to \text{Sym}(T_B^+ \times T_B^-) = \Lambda^{-1}(\text{Re BMO}^*(\mathbb{R})).
$$

We set  $Z = \Lambda^{-1}(Z)$  and  $Y = \Lambda^{-1}(Y)$ .

We formulate the result on the map  $\delta_0$  shown in [\[10,](#page-33-17) Theorem 1] as follows. The proof follows from what has been demonstrated earlier.

<span id="page-27-0"></span>**Theorem 9.2.**  $\delta_0 : \widetilde{Z} \to \text{Sym}(T_B^+ \times T_B^-)$  is a real-analytic diffeomorphism onto its image.

*Proof.* Since  $\delta_0$  is a real-analytic, it suffices to show that it is injective and its inverse  $\delta_0^{-1}$ is also real-analytic. As before, we consider the conjugate

$$
\Lambda \circ \delta_0 \circ \Lambda^{-1} : Z \to \text{Re BMO}^*(\mathbb{R}).
$$

Let  $g_0(x) = \int_0^x \psi(t)dt$  for  $\psi \in Z$  and  $h(x) = \int_0^x \phi(t)dt$  for  $\phi = \Lambda \circ \delta_0 \circ \Lambda^{-1}(\psi) \in Z$ Re BMO<sup>\*</sup>( $\mathbb{R}$ ). Then,  $f = g_0 \circ h^{-1}$  is a Riemann mapping parametrization of the chord-arc curve and  $\log f' \in Y$ . Taking the logarithm of the derivative of  $g_0 = f \circ h$ , we have

$$
\log g'_0 = \log f' \circ h + \log h'.
$$

Since  $\log g'_0 = \psi$  is purely imaginary and  $\log h' = \phi$  is real, the real and imaginary parts of this equation become

$$
0 = \text{Re}\log f' \circ h + \log h' \quad \text{and} \quad -i\psi = \text{Im}\log f' \circ h. \tag{12}
$$

Moreover, since  $\log f'$  is the boundary extension of the holomorphic function  $\log F'$  for the Riemann mapping F on  $\mathbb{H}^+$ , Relog f' and Im log f' are related by the Hilbert transform  $\mathcal H$  on  $\mathbb R$ :

$$
\operatorname{Im}\log f' = \mathcal{H}(\operatorname{Re}\log f').\tag{13}
$$

Then, the combination of [\(9\)](#page-27-0) and [\(9\)](#page-27-0) yields that

$$
-C_h \circ \mathcal{H} \circ C_h^{-1}(\log h') = -i\psi.
$$
 (14)

This shows that  $\psi = \log g'_0$  is determined by  $\phi = \log h'$  and thus  $\Lambda \circ \delta_0 \circ \Lambda^{-1} : \psi \mapsto \phi$ is injective. Equation [\(9\)](#page-27-0) also gives

$$
\Lambda \circ \delta_0^{-1} \circ \Lambda^{-1}(\phi) = -iC_h \circ \mathcal{H} \circ C_h^{-1}(\phi) = -i\mathcal{H}_{\Lambda^{-1}(\phi)}\phi.
$$

Then, because the Cauchy transform  $\mathcal{H}_{\Lambda^{-1}(\phi)}$  depends real-analytically on  $\phi \in \text{Re BMO}^*(\mathbb{R})$ by Theorem [9.1,](#page-26-0) we see that  $\Lambda \circ \delta_0^{-1} \circ \Lambda^{-1}$  is real-analytic.

Finally, we mention discontinuity of the map  $\rho_0$  in brief. We consider the problem of continuous dependence of parameters of chord-arc curves given by the Riemann mappings. For any chord-arc curve  $\Gamma$ , the normalized Riemann mapping from  $\mathbb{H}^+$  to the domain enclosed by Γ defines a BMO embedding  $\gamma : \mathbb{R} \to \Gamma$ . The set of all such BMO embeddings is identified with  $Y = \{ [0] \} \times T_{\mathcal{Q}}$ , which is a complex analytic submanifold of  $T_{C}$ . The map  $Z \to Y$  and  $\rho_0 : \tilde{Z} \to \tilde{Y}$  give the correspondence of these Riemann mapping parametrizations to the chord-arc curves with arc length parametrizations. The problem of asking whether  $\rho_0$  is continuous or not is essentially raised in [\[21,](#page-34-20) p.303]. It is answered in [\[45,](#page-34-0) Theorem 8.3] as follows.

# <span id="page-28-0"></span>**Theorem 9.3.**  $\rho_0 : \widetilde{Z} \to \widetilde{Y}$  is not continuous.

The proof of this result requires an application of the property that  $T_B \cong SQS$  does not form a topological group (see [\[45,](#page-34-0) Proposition 8.1]). From this property, it follows that  $\rho$ is not continuous on  $\tilde{T}_C$ . However, the discontinuity of  $\rho_0 = \rho|_{\tilde{Z}}$  is a stronger claim than this and is derived from observing the local behavior of  $\delta_0 : \tilde{Z} \to \text{Sym}(T_B^+ \times T_B^-)$  near the origin. The fact that  $\delta_0$  is a homeomorphism onto the image allows for some degree of freedom in choosing elements in  $\tilde{Z}$  via Sym  $(T_B^+ \times T_B^-)$  and enables the construction of a specific sequence of elements demonstrating the discontinuity of  $\rho_0$ .

#### 10. VMO and asymptotically smooth embeddings

The final section is devoted to an exposition on the VMO Teichmüller space. A Carleson measure  $\lambda$  on H is said to be vanishing if

$$
\lim_{|I|\to 0}\frac{\lambda(I\times(0,|I|))}{|I|}=0,
$$

where I is a bounded interval in R. Let  $L_V(\mathbb{H})$  denote the subspace of  $L_B(\mathbb{H})$  consisting of all elements  $\mu$  such that  $\lambda_{\mu} = |\mu(z)|^2 dx dy/y$  is a vanishing Carleson measure on H. We observe that  $L_V(\mathbb{H})$  is closed in  $L_B(\mathbb{H})$ . Moreover, we define the corresponding space of Beltrami coefficients as  $M_V(\mathbb{H}) = M(\mathbb{H}) \cap L_V(\mathbb{H})$ . For  $M_V(\mathbb{H}) \subset M(\mathbb{H})$ , we define the VMO Teichmüller space  $T_V$  as  $\pi(M_V(\mathbb{H}))$ . This is closed in  $T_B$ .

Let  $A_V(\mathbb{H})$  denote the subspace of  $A_B(\mathbb{H})$  consisting of all holomorphic functions  $\Psi$ such that  $\lambda_{\Psi}^{(2)}$  is a vanishing Carleson measure on H. This is a closed subspace of  $A_B(\mathbb{H})$ . Similarly, VMOA( $\mathbb{H}$ ) is defined as the closed subspace of BMOA( $\mathbb{H}$ ) consisting of all holomorphic functions  $\Phi$  such that  $\lambda_{\Phi}^{(1)}$  $\Phi_{\Phi}^{(1)}$  is a vanishing Carleson measure on H.

We apply the Schwarzian and pre-Schwarzian derivative maps  $S$  and  $L$  investigated in Propositions [2.1](#page-6-0) and [2.2](#page-6-1) to  $M_V(\mathbb{H}^+)$ , where  $S: M_B(\mathbb{H}^+) \to A_B(\mathbb{H}^-)$  is a holomorphic map with a local holomorphic right inverse  $\sigma$  at every  $\Psi \in S(M_B(\mathbb{H}^+))$ , and  $L : M_B(\mathbb{H}^+) \to$ BMOA( $\mathbb{H}^-$ ) is a holomorphic map with a holomorphic bijection  $D : L(M_B(\mathbb{H}^+)) \to$  $S(M_B(\mathbb{H}^+))$  satisfying  $D \circ L = S$ . We refer to the results from [\[35,](#page-34-21) Theorems 2.1, 2.2]. In the unit disk case, the corresponding results are in [\[36,](#page-34-4) Sections 5, 6].

**Proposition 10.1.** (1) S maps  $M_V(\mathbb{H}^+)$  into  $A_V(\mathbb{H}^-)$ , and  $\sigma$  maps the local neighborhood in  $S(M_V(\mathbb{H}^+))$  into  $A_V(\mathbb{H}^-)$ . (2) L maps  $M_V(\mathbb{H}^+)$  into VMOA( $\mathbb{H}^-$ ). (3) D maps  $L(M_V(\mathbb{H}^+))$  onto  $S(M_V(\mathbb{H}^+))$ .

These results in particular imply that the Bers embedding  $\alpha$ :  $T_B \rightarrow S(M_B(\mathbb{H}^+))$ maps  $T_V$  onto the domain  $S(M_V(\mathbb{H}^+))$  in  $A_V(\mathbb{H}^-)$ , and that the pre-Bers embedding  $\beta: T_B \to L(M_B(\mathbb{H}^+))$  maps  $T_V$  onto the domain  $L(M_V(\mathbb{H}^+))$  in VMOA( $\mathbb{H}^-$ ). Here, if we apply Proposition [10.2](#page-30-0) below, we have

$$
S(M_V(\mathbb{H}^+)) = S(M_B(\mathbb{H}^+)) \cap A_V(\mathbb{H}^-); \quad L(M_V(\mathbb{H}^+)) = L(M_B(\mathbb{H}^+)) \cap VMOA(\mathbb{H}^-).
$$

Since  $\alpha$  and  $\beta$  are biholomorphic homeomorphisms,  $T_V$  has the complex structure as a closed submanifold of  $T_B$ , which is biholomorphically equivalent to  $L(M_V(\mathbb{H}^+))$  and  $S(M_V(\mathbb{H}^+))$ .

Moreover, it is proved in [\[48,](#page-35-1) Theorem 1.4] that if  $\log F^{\mu} \in \text{VMOA}(\mathbb{H}^-)$  for  $\mu \in M(\mathbb{H}^+),$ then  $\mu \in M_V(\mathbb{H}^+)$ . We note that the strategy of showing this, which is used for  $(c) \Rightarrow (a)$ below, is different from that in Theorem [4.9.](#page-14-0) From this, we obtain characterizations for a Beltrami coefficient  $\mu$  to be in  $M_V(\mathbb{H}^+)$  in terms of S and L.

<span id="page-30-0"></span>**Proposition 10.2.** For a Beltrami coefficient  $\mu \in M(\mathbb{H}^+)$ , the following conditions are equivalent: (a)  $\mu \in M_V(\mathbb{H}^+);$  (b)  $S(\mu) \in A_V(\mathbb{H}^-);$  (c)  $L(\mu) \in VMOA(\mathbb{H}^-)$ . Hence,

$$
S(M_V(\mathbb{H}^+)) = S(M(\mathbb{H}^+)) \cap A_V(\mathbb{H}^-);
$$
  

$$
L(M_V(\mathbb{H}^+)) = L(M(\mathbb{H}^+)) \cap VMOA(\mathbb{H}^-).
$$

A BMO function  $\phi$  on R is said to be of vanishing mean oscillation (VMO) if

$$
\lim_{|I| \to 0} \frac{1}{|I|} \int_I |\phi(x) - \phi_I| dx = 0,
$$

where I is a bounded interval on R. The set of all VMO functions on R is denoted by  $VMO(\mathbb{R})$ . This is a closed subspace of  $BMO(\mathbb{R})$ . The restriction of the trace operator  $E : BMOA(\mathbb{H}) \to BMO(\mathbb{R})$  to VMOA(H) is a Banach isomorphism onto its image in VMO(R). However, the composition operator  $C_h$  for  $h \in SQS$  does not necessarily preserve VMO( $\mathbb{R}$ ). If h is uniformly continuous in addition, then  $C_h$  preserves VMO( $\mathbb{R}$ ) (see  $[44,$  Proposition 3.1]).

A strongly quasisymmetric homeomorphism h is said to be *strongly symmetric* if  $\log h' \in$ VMO(R). Let SS denote the set of all normalized strongly symmetric homeomorphisms. This set is identified with  $T_V$ , but it does not form a subgroup of SQS (see [\[42,](#page-34-22) Corollary 5.6]). For  $h \in SS$ , the map H given by the variant of the Beurling–Ahlfors extension by the heat kernel is a quasiconformal real-analytic self-diffeomorphism of H whose complex dilatation belongs to  $M_V(\mathbb{H})$  (see [\[41,](#page-34-5) Theorem 4.1]).

Because the VMO Teichmüller space defined on  $\mathbb H$  and  $\mathbb R$ , as described above, has several defects alongside its desirable properties, we now shift our focus to a smaller VMO Teichmüller space defined on the unit disk D and the unit circle S. Let  $\Theta(z) = (z - i)/(z + i)$ be the Cayley transformation of the Riemann sphere, which maps R ∪ {∞} onto S, with  $\Theta(\infty) = 1$ . The space VMO(S) of VMO functions on S is defined similarly, and it is a closed subspace of BMO(S). While BMO functions on S and their associated Teichmüller spaces are well studied and equivalent to those defined on  $\mathbb{R}$ —since the Cayley transformation Θ provides an appropriate correspondence between all components of the involved spaces—this equivalence does not hold for VMO functions.

We define  $VcMO(\mathbb{R})$  as the pull-back of  $VMO(\mathbb{S})$  via  $\Theta$ . This coincides with the closure of compactly supported VMO functions on R with respect to the BMO norm, and is sometimes denoted by  $CMO(\mathbb{R})$ . In fact, as noted in [\[11,](#page-33-18) p.639], this is adopted as the definition of VMO functions on  $\mathbb{R}$ . By definition, VcMO $(\mathbb{R})$  is a closed subspace of  $VMO(\mathbb{R}) \subset BMO(\mathbb{R})$ . By considering the relationship between  $VMO(\mathbb{R})$  and other spaces involved in the Teichmüller space  $T_V$ , or by pulling back via the Cayley transformation, we can derive the spaces corresponding to  $VcMO(\mathbb{R})$ . These include the closed subspace  $M_{V_c}(\mathbb{H})$  of Beltrami coefficients, the Banach subspaces VcMOA( $\mathbb{H}$ ) and  $A_{V_c}(\mathbb{H})$  of holomorphic functions, the closed subset SSc of quasisymmetric homeomorphisms of  $\mathbb{R}$ , and the Teichmüller subspace  $T_{V_c}$ .

For these spaces, the equivalent conditions as in Proposition [10.2](#page-30-0) also hold (see [\[36,](#page-34-4) Theorem 4.1]). Furthermore, the aforementioned defects no longer exist. The following

claims are established in the setting of  $\mathcal S$  and  $\mathbb D$ . Claim (1) is found in [\[2,](#page-33-19) p.458], (2) is in [\[40,](#page-34-23) Theorem 4.1], and (3) is verified using a claim in [\[14,](#page-33-6) p.474].

<span id="page-31-0"></span>**Proposition 10.3.** (1) The composition operator  $C_h$  for  $h \in SQS$  preserves VcMO(R). (2)  $T_{Vc}$  ≅ SSc is a closed subgroup of  $T_B$  ≅ SQS. In fact, it is a topological subgroup. (3)  $VcMO(\mathbb{R}) \subset BMO^*(\mathbb{R})$ . Thus,  $F^{\mu}(\mathbb{R})$  is a chord-arc curve for  $\mu \in M_{Vc}(\mathbb{H})$ , and  $T_{Vc}$  is a proper closed subset of  $T_C$ .

Furthermore, the chord-arc curve  $F^{\mu}(\mathbb{R})$  given by  $\mu \in M_{V_c}(\mathbb{H}^+)$  satisfies the following condition on the chord-arc constant  $\kappa \geq 1$ . See [\[29,](#page-34-24) Theorem 2].

**Definition 10.1.** A chord-arc curve  $\Gamma$  in  $\mathbb{C}$  passing through  $\infty$  is called *asymptotically* smooth if the chord-arc constant  $\kappa \geq 1$  tends to 1 uniformly as the spherical distance between two points on  $\Gamma$  tends to 0. Namely, setting  $|\widetilde{z_1z_2}|$  as the length of the arc between any two points  $z_1, z_2 \in \Gamma$  and  $d_{\widehat{\mathcal{C}}}(z_1, z_2)$  as the spherical distance on  $\widehat{\mathbb{C}}$ ,

$$
\lim_{t \to 0} \sup_{d_{\widehat{\mathbb{C}}}(z_1, z_2) < t} \frac{|\widehat{z_1 z_2}|}{|z_1 - z_2|} = 1.
$$

Remark 10.1. We have an equivalent definition for chord-arc curves even if we replace the Euclidean distance with the spherical distance. Moreover, the condition for a Jordan curve Γ in the Riemann sphere  $\widehat{\mathbb{C}}$  to be a chord-arc curve is invariant under Möbius transformations. See [\[27,](#page-34-11) p.877] and [\[39,](#page-34-10) Section 7]. The above definition for asymptotically smoothness is the translation of that for bounded Jordan curves in [\[29\]](#page-34-24) to the unbounded case.

**Proposition 10.4.** For  $\mu \in M(\mathbb{H}^+)$ ,  $F^{\mu}(\mathbb{R})$  is asymptotically smooth if and only if  $\log(F^{\mu}|_{\mathbb{H}^{-}})'$  belongs to VcMOA( $\mathbb{H}^{-}$ ). This is equivalent to the condition  $\mu \in M_{Vc}(\mathbb{H}^{+})$ .

From this fact and the group structure of  $T_{V_c} \cong \text{SSc}$ , we conclude that for a BMO embedding  $\gamma = \gamma([\mu^+], [\mu^-])$  with  $([\mu^+], [\mu^-]) \in T_B^+ \times T_B^-$ , its image  $\gamma(\mathbb{R})$  is asymptotically smooth if and only if  $([\mu^+], [\mu^-]) \in T^+_{Vc} \times T^-_{Vc}$ . We refer to this  $\gamma$  as an asymptotically smooth embedding. In this case, we have  $\log \gamma' \in \text{VcMO}(\mathbb{R})$ . Thus,  $T^+_{Vc} \times T^-_{Vc} \subset \widetilde{T}_C$ .

We now consider the restriction of the biholomorphic map  $\Lambda$  to  $T_{V_c}^+ \times T_{V_c}^-$ . This yields a biholomorphic homeomorphism

$$
\Lambda: T^+_{Vc} \times T^-_{Vc} \to \text{VcMO}(\mathbb{R})
$$

onto its image, which is a connected open subset of  $VcMO(\mathbb{R})$ . In addition,  $\Lambda$  maps  $\text{Sym}(T_{Vc}^+ \times T_{Vc}^-)$  onto ReVcMO(R) as a real-analytic diffeomorphism. Moreover, we can construct a holomorphic right inverse of  $\Lambda$  on some neighborhood of the real subspace  $\text{Re } \text{Vc} \text{MO}(\mathbb{R})$ , as in the following assertion.

**Theorem 10.5.** There exists a neighborhood W of  $\text{Re } V \text{c} \text{MO}(\mathbb{R})$  in  $V \text{c} \text{MO}(\mathbb{R})$  and a holomorphic map  $\Sigma: W \to M_{Vc}(\mathbb{H}^+) \times M_{Vc}(\mathbb{H}^-)$  such that  $(\pi^+ \times \pi^-) \circ W$  is a holomorphic right inverse of  $\Lambda : T^+_{Vc} \times T^-_{Vc} \to \text{VcMO}(\mathbb{R})$  on W.

*Proof.* This follows from the arguments around [\[44,](#page-34-16) Theorem 6.2]. Since  $L^{\infty}(\mathbb{R})$  is dense in VcMO(R), we can define a holomorphic map  $\Sigma$  on some neighborhood W of ReVcMO(R) using the variant of the Beurling–Ahlfors extension as in Theorem [3.2.](#page-9-0) By a proof similar to that of [\[44,](#page-34-16) Proposition 6.3], we see that

$$
\Sigma(W) \subset M_B(\mathbb{H}^+) \times M_B(\mathbb{H}^-); \quad \Sigma(\text{Re} \,\mathrm{VcMO}(\mathbb{R})) \subset M_{Vc}(\mathbb{H}^+) \times M_{Vc}(\mathbb{H}^-),
$$

and in particular,  $\Sigma|_{\text{Re }V\text{cMO}(\mathbb{R})}$  is real-analytic. However, by combining the proof of [\[41,](#page-34-5) Theorem 4.1] with that of [\[44,](#page-34-16) Lemma 5.4], we can actually show that  $\Sigma(W)$  is contained in  $M_{V_c}(\mathbb{H}^+) \times M_{V_c}(\mathbb{H}^-)$ . The results for VMO(R) are able to be applied to VcMO(R) ≅ VMO(S) by lifting functions on S via the universal cover  $\mathbb{R} \to \mathbb{S}$ .

We investigate the correspondence from the arc-length parametrization to the Riemann mapping parametrization for asymptotically smooth curves. Analogous to the general case, we define

$$
Z_c = i BMO(\mathbb{R}) \cap \Lambda(T_{Vc}^+ \times T_{Vc}^-);
$$
  

$$
Y_c = BMOA(\mathbb{H}^+) \cap \Lambda(T_{Vc}^+ \times T_{Vc}^-) \subset VcMOA(\mathbb{H}^+).
$$

The former is a real analytic submanifold, and the latter is a complex analytic submanifold of  $\Lambda(T_{Vc}^+ \times T_{Vc}^-)$ . Moreover, we set  $\widetilde{Z}_c = \Lambda^{-1}(Z_c)$  and  $\widetilde{Y}_c = \Lambda^{-1}(Y_c)$ , and consider the map

$$
\rho: T_{Vc}^+ \times T_{Vc}^- \to \{[0]\} \times T_{Vc}^-,\quad ([\mu^+], [\mu^-]) \mapsto ([0], [\mu^-] * [\mu^+]^{-1}).
$$

Let  $\rho_0 = \rho|_{\widetilde{Z}_c}$ .

By the topological group property of  $T_{V_c}$ , as established in Proposition [10.3,](#page-31-0) we derive the following result in contrast to Theorem [9.3:](#page-28-0)

**Theorem 10.6.**  $\rho_0 : \widetilde{Z}_c \to \widetilde{Y}_c$  is a homeomorphism.

*Proof.* It is straightforward to verify that  $\rho_0$  is bijective. Since  $\rho$  is continuous due to the topological group property of  $T_{V_c}$ , it follows that  $\rho_0$  is also continuous. Hence, it remains to show that  $\rho_0^{-1}$  is continuous. To do this, we consider the conjugation of  $\rho_0^{-1}$  by  $\Lambda$ , that is,

$$
\Lambda \circ \rho_0^{-1} \circ \Lambda^{-1} : Y_c \to Z_c.
$$

For any  $\varphi \in Y_c$ , we have

$$
\Lambda \circ \rho_0^{-1} \circ \Lambda^{-1}(\varphi) = C_h^{-1}(i \operatorname{Im} \varphi),
$$

where  $h(x) = \int_0^x \exp(\text{Re}\,\varphi(t)) dt$ . Since  $h \mapsto h^{-1}$  is continuous in  $T_{Vc} \cong \text{SSc}$ , we conclude that the map

$$
VcMO(\mathbb{R}) \times T_{Vc} \to VcMO(\mathbb{R}), \quad (\phi, h) \mapsto C_h^{-1}(\phi),
$$

<span id="page-32-0"></span>is continuous by Lemma [10.7](#page-32-0) below. Thus,  $\Lambda \circ \rho_0^{-1} \circ \Lambda^{-1}$  is continuous.

**Lemma 10.7.** The map VcMO( $\mathbb{R}$ ) ×  $T_{Vc} \to \text{VcMO}(\mathbb{R})$  defined by  $(\phi, h) \mapsto C_h(\phi)$  is continuous. In particular, a sequence of bounded linear operators  $C_{h_n}$  on  $VcMO(\mathbb{R})$  converges to  $C_h$  strongly as  $h_n \to h$  in  $T_{V_c}$ .

*Proof.* The affine translation  $Q_h(\phi)$  of  $\phi \in VcMO(\mathbb{R})$  is given by  $C_h(\phi) + \log h'$ . Then, we have

$$
\Lambda \circ R_{[\mu]} = Q_h \circ \Lambda
$$

for  $h = h(\mu) \in \text{SSc}$ . From this, we deduce

$$
C_h(\phi) = \Lambda \circ R_{[\mu]} \circ \Lambda^{-1}(\phi) - \log h',
$$

which is a continuous function of  $(\phi, h)$ . The continuity with respect to h follows from the topological group property of  $T_{V_c}$ .

## **REFERENCES**

- <span id="page-33-19"></span><span id="page-33-10"></span>[1] L.V. Ahlfors, Lectures on Quasiconformal Mappings, 2nd ed., Univ. Lect. Ser. 38, Amer. Math. Soc., 2006.
- [2] J.M. Anderson, J. Becker and F.D. Lesley, On the boundary correspondence of asymptotically conformal automorphisms, J. London Math. Soc. 38 (1988), 453–462.
- <span id="page-33-3"></span><span id="page-33-0"></span>[3] K. Astala and M. Zinsmeister, *Teichmüller spaces and BMOA*, Math. Ann. **289** (1991), 613–625.
- [4] A. Beurling and L.V. Ahlfors, The boundary correspondence under quasiconformal mappings, Acta Math. **96** (1956), 125–142.
- <span id="page-33-12"></span>[5] C.J. Bishop, A counterexample in conformal welding concerning Hausdorff dimension, Mich. Math. J. 35 (1988), 151–159.
- <span id="page-33-9"></span>[6] C.J. Bishop and P.W. Jones, *Harmonic measure*,  $L^2$  estimates and the Schwarzian derivative, J. Anal. Math. 62 (1994), 77–113.
- <span id="page-33-13"></span><span id="page-33-4"></span>[7] S.B. Chae, Holomorphy and Calculus in Normed Spaces, Pure and Applied Math. 92, Marcel Dekker, 1985.
- [8] R.R. Coifman and C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals, Studia Math. 51 (1974), 241–250.
- <span id="page-33-16"></span>[9] R.R. Coifman and Y. Meyer, Une généralisation du théorème de Calderón sur l'intégrale de Cauchy, Fourier Analysis (Proc. Sem., El Escorial, 1979), pp.87–116, Asoc. Mat. Española, 1980.
- <span id="page-33-17"></span>[10] R.R. Coifman and Y. Meyer, Lavrentiev's curves and conformal mappings, Institute Mittag-Leffler, Report No.5, 1983.
- <span id="page-33-18"></span>[11] R.R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, Bull. Amer. Math. Soc. 83 (1977), 569–645.
- <span id="page-33-15"></span>[12] G. David, *Opérateurs intégraux singuliers sur certaines courbes du plan complexe*, Ann. Sci. École Norm. Sup. 17 (1984), 157–189.
- <span id="page-33-2"></span>[13] R.A. Fefferman, C.E. Kenig and J. Pipher, The theory of weights and the Dirichlet problems for elliptic equations, Ann. of Math.  $134$  (1991), 65-124.
- <span id="page-33-6"></span>[14] J. García-Cuerve and J.L. Rubio de Francia, Weighted Norm Inequalities and Related Topics, North-Holland Mathematics Studies 116, 1985.
- <span id="page-33-14"></span><span id="page-33-1"></span>[15] J.B. Garnett, Bounded Analytic Functions, Grad. Texts in Math. 236, Springer, 2007.
- <span id="page-33-8"></span>[16] G.M. Goluzin, Geometric Theory of Functions of a Complex Variable, Transl. Math. Mono. 26, AMS. 1969.
- <span id="page-33-5"></span>[17] Y. Gotoh, On composition operators which preserve BMO, Pacific J. Math. 201 (2001), 289–307.
- [18] S.V. Hruščev, A description of weights satisfying the  $A_{\infty}$  condition of Muckenhoupt, Proc. Amer. Math. Soc. 90 (1984), 253–257.
- <span id="page-33-11"></span>[19] D.S. Jerison and C.E. Kenig, *Hardy spaces,*  $A_{\infty}$ , and singular integrals on chord-arc domains, Math. Scand. 50 (1982), 221–247.
- <span id="page-33-7"></span>[20] P.W. Jones, Homeomorphisms of the line which preserve BMO, Ark. Mat. 21 (1983), 229–231.
- <span id="page-34-20"></span><span id="page-34-2"></span>[21] Y. Katznelson, S. Nag, and D.P. Sullivan, On conformal welding homeomorphisms associated to Jordan curves, Ann. Acad. Sci. Fenn. A. I. Math. 15 (1990), 293–306.
- <span id="page-34-18"></span>[22] O. Lehto, Univalent Functions and Teichmüller Spaces, Grad. Texts in Math. 109, Springer, 1987.
- [23] T. Liu and Y. Shen, On the BMO boundedness of the Cauchy integral on a chord-arc curve, Arch. Math. 120 (2023), 203–212.
- <span id="page-34-19"></span><span id="page-34-1"></span>[24] T. Liu and Y. Shen, The Faber operator acting on BMOA, BMO-Teichmüller space and chord-arc curves, Acta. Math. Sinica Eng. Ser. (2024), doi.org/10.1007/s10114-024-2184-4.
- <span id="page-34-13"></span>[25] K. Matsuzaki, *Integrable Teichmüller spaces for analysis on Weil–Petersson curves*, preprint.
- <span id="page-34-11"></span>[26] K. Matsuzaki and H. Wei, Space of chord-arc curves and BMO/VMO Teichmüller space, Ann. Fenn. Math. 48 (2023), 27–42.
- <span id="page-34-8"></span>[27] P. MacManus, *Quasiconformal mappings and Ahlfors–David curves*, Trans. Amer. Math. Soc. **343** (1994), 853–881.
- [28] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc. 165 (1972), 207–226.
- <span id="page-34-24"></span><span id="page-34-9"></span>[29] Ch. Pommerenke, On Univalent Functions, Bloch Functions and VMOA, Math. Ann. 236 (1978), 199–208.
- <span id="page-34-14"></span>[30] Ch. Pommerenke, Boundary Behaviour of Conformal Maps, Springer, 1992.
- [31] S. Semmes, The Cauchy integral, chord-arc curves, and quasiconformal mappings, The Bieberbach conjecture (West Lafayette, Ind., 1985), pp.167–183, Math. Surveys Monogr. 21, Amer. Math. Soc., 1986.
- <span id="page-34-12"></span>[32] S. Semmes, A counterexample in conformal welding concerning chord-arc curves, Ark. Math. 24 (1986), 141–158.
- <span id="page-34-17"></span>[33] S. Semmes, Estimates for  $(\bar{\partial} - \mu \partial)^{-1}$  and Calderón's theorem on the Cauchy integral, Trans. Amer. Math. Soc. 306 (1988), 191–232.
- <span id="page-34-21"></span><span id="page-34-7"></span>[34] S. Semmes, Quasiconformal mappings and chord-arc curves, Trans. Amer. Math. Soc. 306 (1988), 233–263.
- <span id="page-34-4"></span>[35] Y. Shen, VMO–Teichmüller space on the real line, Ann. Fenn. Math. 47 (2022), 57–82.
- <span id="page-34-15"></span>[36] Y. Shen and H. Wei, Universal Teichmüller space and BMO, Adv. Math.  $234$  (2013) 129–148.
- [37] Y. Shen and L. Wu, Weil–Petersson Teichmüller space III: dependence of Riemann mapping for Weil–Petersson curves, Math. Ann. 381 (2021), 875–904.
- <span id="page-34-3"></span>[38] T. Sugawa, *The universal Teichmüller space and related topics*, Proceedings of the international workshop on quasiconformal mappings and their applications, pp. 261–289, Narosa Publishing House, 2007.
- <span id="page-34-10"></span>[39] P. Tukia, Extension of quasisymmetric and Lipschitz embeddings of the real line into the plane, Ann. Acad. Sci. Fenn. Ser. A.I. Math. 6 (1981), 89–94.
- <span id="page-34-23"></span><span id="page-34-5"></span>[40] H. Wei, A note on the BMO-Teichmüller space, J. Math. Anal. Appl.  $435$  (2016), 746–753.
- [41] H. Wei and K. Matsuzaki, Beurling–Ahlfors extension by heat kernel,  $A_{\infty}$ -weights for VMO, and vanishing Carleson measures, Bull. London Math. Soc. 53 (2021), 723–739.
- <span id="page-34-22"></span>[42] H. Wei and K. Matsuzaki, Symmetric and strongly symmetric homeomorphisms on the real line with non-symmetric inversion, Anal. Math. Phys. 11 (2021), 79.
- <span id="page-34-6"></span>[43] H. Wei and K. Matsuzaki, The p-Weil–Petersson Teichm¨uller space and the quasiconformal extension of curves, J. Geom. Anal. **32** (2022), 213.
- <span id="page-34-16"></span>[44] H. Wei and K. Matsuzaki, The VMO-Teichmüller space and the variant of Beurling–Ahlfors extension by heat kernel, Math. Z. 302 (2022), 1739–1760.
- <span id="page-34-0"></span>[45] H. Wei and K. Matsuzaki, BMO embeddings, chord-arc curves, and Riemann mapping parametrization, Adv. Math. **417** (2023) 108933.
- [46] H. Wei and K. Matsuzaki, Strongly symmetric homeomorphisms on the real line with uniform con*tinuity*, Indiana Univ. Math. J. **72** (2023), 1553–1576.

- <span id="page-35-1"></span>[47] H. Wei and K. Matsuzaki, Parametrization of the p-Weil–Petersson curves: holomorphic dependence, J. Geom. Anal. 33 (2023), 292.
- <span id="page-35-0"></span>[48] H. Wei and K. Matsuzaki, Chordal Loewner chains and Teichmüller spaces on the half-plane, [arXiv:2207.02447.](http://arxiv.org/abs/2207.02447)
- [49] K. Zhu, Operator Theory in Function Spaces, Math. Surveys Mono. 138, Amer. Math. Soc., 2007.

Department of Mathematics, School of Education, Waseda University Shinjuku, Tokyo 169-8050, Japan Email address: matsuzak@waseda.jp