École polytechnique fédérale de Lausanne

MASTER THESIS

MASTER IN MATHEMATICS

Multiplicity-free tensor products of irreducible modules over simple algebraic groups in positive characteristic

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List of notations

Φ	root system
П	base of Φ
α^{\vee}	coroot of α
X	weight lattice of Φ
X^+	set of dominant weights
\leq	usual partial order on X
ω_i	fundamental dominant weight corresponding to α_i
α_h	highest short root in Φ
ρ	half sum of all positive roots
h	Coxeter number of Φ
s_{lpha}	reflection corresponding to α
W	Weyl group of Φ
w_0	longest element in W
$s_{lpha,mp}$	affine reflection
W_p	affine Weyl group associated to Φ and p
$w \bullet x$	=w(x+ ho)- ho
D	$= X^{+} - \rho$, fundamental domain for the dot action of W on X
\widehat{C}	upper closure of an alcove C
\overline{C}	closure of an alcove C
C_1	fundamental alcove
↑	linkage order on X
g	simple Lie algebra associated to Φ (over \mathbb{C})
$\mathcal{U}(\mathfrak{g})$	universal enveloping algebra of the Lie algebra ${\mathfrak g}$
G	simply connected Chevalley group with root system Φ
X_{α}	root subgroup corresponding to α
T	maximal torus of G
В	Borel subgroup of G
$G_{\mathbb{C}}$	simply connected Chevalley group with root system Φ over $\mathbb C$
$\operatorname{soc} M$	socle of the module M
$\operatorname{rad} M$	radical of the module M
$m_M(\lambda)$	multiplicity of the weight λ in M
S^aM	a-th symmetric power of M
$L(\lambda)$	simple G-module of highest weight λ
$[M:L(\lambda)]$	multiplicity of the composition factor $L(\lambda)$ in M
M^*	dual module of M
M^{τ}	contravariant dual of M
$\Delta(\lambda)$	Weyl module of highest weight λ
$ abla(\lambda)$	cost and ard module of highest weight λ
$T(\lambda)$	indecomposable tilting module of highest weight λ
$\operatorname{ch} M$	character of the module M
$\chi(\lambda)$	Weyl character associated to λ
$ u_p(m)$	p-adic valuation of m
$L_{\mathbb{C}}(\lambda)$	simple $G_{\mathbb{C}}$ -module of highest weight λ

Introduction

Tensor products are of great interest in representation theory. Determining their structure has been the subject of much research. When the category of representations we are working with is semisimple, the central question is to find the decomposition of a tensor product of simple modules into a direct sum of simple modules. In particular, this is the case for representations of simple Lie algebras over the field of complex numbers, and for representations of simple algebraic groups over the same field. In the case of $\mathfrak{sl}_2(\mathbb{C})$, the Clebsch-Gordan formula gives an answer to this question.

Another question that has been the subject of research is to determine whether certain simple modules appear several times in the decomposition of the tensor product, and thus classify tensor products without multiplicity. In the case of simple Lie algebras and simple algebraic groups, this question was resolved in 2003 by Stembridge ([Ste03]). In particular, this classification shows that if the tensor product of two simple modules is multiplicity-free, then the highest weight of one of the two modules is a multiple of a fundamental weight. This fact is no longer true in positive characteristic.

When we move on to a field of positive characteristic, the category of representations of a simple algebraic group is no longer semisimple. Other questions, often complicated, may then arise, such as finding indecomposable direct summands and classifying completely reducible tensor products. The classification of multiplicity-free modules is also of interest. Recently, Gruber showed that a multiplicity-free tensor product of simple modules is necessarily completely reducible, which gives another motivation for this classification.

In this project, we will therefore focus on the classification of multiplicity-free tensor products of simple modules over an algebraically closed field of characteristic p > 0. Our main question is:

Question 1. Given a simply connected simple algebraic group G, for which pairs of simple modules $L(\lambda)$ and $L(\mu)$ is the tensor product $L(\lambda) \otimes L(\mu)$ multiplicity-free ?

We will provide the complete classification in the case of SL_2 and SL_3 , and show a number of important results in the case of Sp_4 . In addition, we will show that, under certain assumptions, being completely reducible implies being multiplicity-free. Using the classification of completely reducible tensor products of simple modules for SL_n over a field of characteristic 2, established by Gruber ([Gru21]), we will answer our question in the case of SL_n for p = 2.

The first part of this project recalls important notions of representation theory that will be used later. In the second part, we recall some results related to tensor products and show that we can restrict our attention to simple modules with *p*-restricted highest weight in order to answer our question. In the third part, we show some connections between multiplicityfreeness over \mathbb{C} and in positive characteristic. In parts 4 to 7, we proceed to the classification of multiplicity-free tensor products in the cases of SL₂, SL₃, Sp₄ (partial classification only), and SL_n for p = 2 respectively. This work could be continued on the one hand by completing the classification for Sp₄, and on the other hand by generalising these results to other simply connected simple algebraic groups.

It should also be noted that we used Magma ([BCP97]) in order to compute the composition factors of certain tensor products and to have concrete examples, which enabled us to have a better understanding of the structure of these tensor products.

We assume that the reader is familiar with the representation theory of semisimple Lie algebras, as well as with the basics of algebraic group theory. We will therefore not repeat the relative notions in the preliminaries, but refer the reader to [Hum00] for the representation theory of semisimple Lie algebras and to [MT11] for the theory of algebraic groups.

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1 Preliminaries

In this section, we introduce all the notions needed to solve our problem and introduce some arguments which will be used several times in the next parts of this project.

1.1 Weights and alcoves

We start by recalling some results about root systems and weights. Then, we will define the notion of an alcove, which has a very important role in the structure of some modules.

Let $X_{\mathbb{R}}$ be a Euclidean space of dimension n with scalar product $(,): X_{\mathbb{R}} \times X_{\mathbb{R}} \to \mathbb{R}$ and let $\Phi \subseteq X_{\mathbb{R}}$ be an irreducible root system. We fix $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ a base of Φ and denote by Φ^+ the set of positive roots with respect to Π . Moreover, we fix p a prime.

For $\alpha \in \Phi$, we define its *coroot* $\alpha^{\vee} := \frac{2\alpha}{(\alpha, \alpha)} \in X_{\mathbb{R}}$. We set

$$X_{\mathbb{R}}^+ := \{ x \in X_{\mathbb{R}} | (x, \alpha^{\vee}) \ge 0 \quad \forall \alpha \in \Phi^+ \},$$

and define the *weight lattice* of Φ to be the set

$$X := \{ \lambda \in X_{\mathbb{R}} | (\lambda, \alpha^{\vee}) \in \mathbb{Z} \quad \forall \alpha \in \Phi \}.$$

Moreover we define the set of *dominant weights* to be the set

$$X^+ := X \cap X^+_{\mathbb{R}} = \{ \lambda \in X | (\lambda, \alpha^{\vee}) \ge 0 \quad \forall \alpha \in \Phi^+ \}.$$

There is a partial order on X given by $\lambda \leq \mu$ if $\lambda - \mu$ is a N-linear combination of simple roots, i.e. $\lambda - \mu = \sum_{\alpha \in \Pi} n_{\alpha} \alpha$ with $n_{\alpha} \in \mathbb{N} \ \forall \alpha \in \Pi$.

Since Π is a basis of $X_{\mathbb{R}}$, it follows that $\{\alpha^{\vee} | \alpha \in \Pi\}$ is a basis of $X_{\mathbb{R}}$. Thus it admits a dual basis with respect to (,) and there exists a set $\{\omega_{\alpha} | \alpha \in \Pi\}$ whose elements satisfy

$$(\omega_{\alpha}, \alpha^{\vee}) = 1$$
 and $(\omega_{\alpha}, \beta^{\vee}) = 0 \quad \forall \beta \in \Pi \setminus \{\alpha\}.$

We observe that the $\omega_{\alpha} \in X^+$ for all $\alpha \in \Pi$. We call the weights ω_{α} with $\alpha \in \Pi$ the fundamental dominant weights. One can easily check that every weight is a \mathbb{Z} -linear combination of the fundamental dominant weights. To simplify the notation, we set $\omega_i := \omega_{\alpha_i}$ for $i \in \{1, \ldots, n\}$. We will use the numeration of simple roots given in [Hum00, 11.4], so the numeration of the fundamental dominant weights will correspond to this labelling of Dynkin diagrams.

Definition 1.1. A dominant weight $\lambda \in X^+$ is called *p*-restricted if $(\lambda, \alpha^{\vee}) < p$ for all $\alpha \in \Pi$.

We denote the highest short root in the root system Φ (with respect to Π) by α_h and the half sum of all positive roots by $\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. This element satisfies $\rho = \sum_{i=1}^n \omega_i$ ([Hum00, 13.3]). The *Coxeter number* of the root system Φ is $h := (\rho, \alpha_h^{\vee}) + 1$.

Remark 1.2. The coroot α_h^{\vee} is the highest root in the dual root system Φ^{\vee} and for any dominant weight $\lambda \in X^+$, we have $(\lambda, \alpha^{\vee}) \leq (\lambda, \alpha_h^{\vee})$ for all $\alpha \in \Phi^+$. More generally, for every $x \in X_{\mathbb{R}}^+$ and $\alpha \in \Phi^+$, we have $(x, \alpha^{\vee}) \leq (x, \alpha_h^{\vee})$. Recall also that $(\alpha, \alpha_h^{\vee}) \geq 0$ for all $\alpha \in \Phi^+$.

Definition 1.3. A set $Y \subseteq X$ is called saturated if for all $\lambda \in Y$, $\alpha \in \Phi$ and i between 0 and (λ, α^{\vee}) , we have $\lambda - i\alpha \in Y$.

For $\alpha \in \Phi$ we define the reflection $s_{\alpha} \in \operatorname{GL}(X_{\mathbb{R}})$ by

$$s_{\alpha}(v) := v - \frac{2(v, \alpha)}{(\alpha, \alpha)} \alpha \quad \text{for } v \in X_{\mathbb{R}}.$$

We also define the Weyl group of Φ by

$$W := \langle s_{\alpha} | \; \alpha \in \Phi \rangle \subseteq \mathrm{GL}(X_{\mathbb{R}}).$$

The group W is generated by $\{s_{\alpha} | \alpha \in \Pi\}$. More precisely, $(W, \{s_{\alpha} | \alpha \in \Pi\})$ is a Coxeter system ([Hum90, 1.5]). It contains a unique longest element which we denote by $w_0 \in W$ ([Hum90, 1.8]).

For $\alpha \in \Phi$ and $m \in \mathbb{Z}$, we define the affine reflection $s_{\alpha,mp} \in AGL(X_{\mathbb{R}})$ by

$$s_{\alpha,mp}(v) := s_{\alpha}(v) + mp\alpha \quad \text{for } v \in X_{\mathbb{R}}.$$

The affine Weyl group associated to Φ and p, denoted by W_p , is the group

 $W_p := \langle s_{\alpha,mp} | \alpha \in \Phi, m \in \mathbb{Z} \rangle \subseteq \operatorname{AGL}(X_{\mathbb{R}}).$

The dot action of W_p on $X_{\mathbb{R}}$ is the group action given by

$$w \bullet x := w(x + \rho) - \rho$$
 for $w \in W_p$ and $x \in X_{\mathbb{R}}$.

For $\alpha \in \Phi$ and $m \in \mathbb{Z}$, we define the *reflection hyperplane* of $s_{\alpha,mp}$ (for the dot action) to be the set

$$H_{\alpha,m} := \{ x \in X_{\mathbb{R}} | (x + \rho, \alpha^{\vee}) = mp \}.$$

Since W is a subgroup of W_p , we can restrict the dot action to W. We set

$$D := \{ \lambda \in X \mid \lambda + \rho \in X^+ \},\$$

which is a fundamental domain for the dot action of W on X. (This follows from the facts that $X^+_{\mathbb{R}}$ is a fundamental domain for the action of W on $X_{\mathbb{R}}$ ([Hum90, 1.12]) and that X is preserved by W.)

Definition 1.4. Let $n = (n_{\alpha})_{\alpha \in \Phi^+} \in \mathbb{Z}^{|\Phi^+|}$. We define

 $C_n := \{ x \in X_{\mathbb{R}} | (n_\alpha - 1)p < (x + \rho, \alpha^{\vee}) < n_\alpha p \quad for \ all \ \alpha \in \Phi^+ \}.$

We say that C_n is an alcove if it is a non-empty set. For C_n an alcove, its upper closure is the set

$$\widehat{C_n} := \{ x \in X_{\mathbb{R}} | (n_\alpha - 1)p < (x + \rho, \alpha^{\vee}) \le n_\alpha p \quad \text{for all } \alpha \in \Phi^+ \},\$$

and its closure is the set

$$\overline{C_n} := \{ x \in X_{\mathbb{R}} | (n_\alpha - 1)p \le (x + \rho, \alpha^{\vee}) \le n_\alpha p \text{ for all } \alpha \in \Phi^+ \}.$$

Alternatively, we can define an alcove to be a connected component of $X_{\mathbb{R}} \setminus \bigcup_{\substack{\alpha \in \Phi \\ m \in \mathbb{Z}}} H_{\alpha,m}$.

Definition 1.5. The fundamental alcove is the alcove

 $C_1 := \{ x \in X_{\mathbb{R}} | \ 0 < (x + \rho, \alpha^{\vee}) < p \quad for \ all \ \alpha \in \Phi^+ \}.$

Remark 1.6. By Remark 1.2, we have

$$C_1 = \{ x \in X_{\mathbb{R}} | (x + \rho, \alpha_h^{\vee})$$

Lemma 1.7. Let $\lambda \in X^+ \cap \widehat{C}_1$ and $\mu \in X^+$ be such that $\mu \leq \lambda$. We have $\mu \in \widehat{C}_1$.

Proof. Let $a_1, \ldots a_n \in \mathbb{N}$ be such that $\mu = \lambda - \sum_{i=1}^n c_i \alpha_i$. By assumption, $\mu \in X^+$, thus $(\mu + \rho, \alpha) \geq 0$ for all $\alpha \in \Pi$. Thus, using Remark 1.6, we only need to show that $(\mu + \rho, \alpha_h^{\vee}) \leq p$. Since $(\lambda + \rho, \alpha_h^{\vee}) \leq p$ and $(\alpha_i, \alpha_h^{\vee}) \geq 0$ for all $\alpha_i \in \Pi$, we have

$$(\mu + \rho, \alpha_h^{\vee}) = (\lambda - \sum_{i=1}^n c_i \alpha_i + \rho, \alpha_h^{\vee}) = (\lambda + \rho, \alpha_h^{\vee}) - \sum_{i=1}^n c_i (\alpha_i, \alpha_h^{\vee}) \le (\lambda + \rho, \alpha_h^{\vee}) \le p,$$
$$\mu \in \widehat{C_1}.$$

so $\mu \in C_1$.

Definition 1.8. An alcove C is p-restricted if there exists a p-restricted dominant weight $\lambda \in X^+$ such that $\lambda \in \widehat{C}$.

Theorem 1.9 ([Hum90, 4.5 and 4.8]). The affine Weyl group W_p acts simply transitively on the set of alcoves. Moreover, $\overline{C_1}$ is a fundamental domain for the dot action of W_p on $X_{\mathbb{R}}$.

Definition 1.10. Let $\lambda, \mu \in X$. The weight λ is linked to μ if $\lambda = \mu$ or if there exist affine reflections $s_{\beta_1,m_1p}, \ldots, s_{\beta_t,m_tp} \in W_p$ such that

$$\lambda \leq s_{\beta_1,m_1p} \bullet \lambda \leq \ldots \leq s_{\beta_t,m_tp} \cdots s_{\beta_1,m_1p} \bullet \lambda = \mu.$$

In this case, we write $\lambda \uparrow \mu$.

Remark 1.11. The relation \uparrow is a partial order on X.

1.2 Chevalley groups and algebraic groups

In this section, we recall some definitions about linear algebraic groups, following [MT11], and we construct Chevalley groups following [Ste16].

1.2.1 Linear algebraic groups

Let G be a linear algebraic group. A Borel subgroup $B \leq G$ is a closed, connected, solvable subgroup of G which is maximal with respect to all these properties. The radical R(G) of G is the maximal closed connected solvable normal subgroup of G. The group G is semisimple if R(G) = 1. A non-trivial semisimple algebraic group G is simple if it has no non-trivial proper closed connected normal subgroups. A representation $\rho : G \to GL(V)$ is rational if ρ is a morphism of algebraic groups.

1.2.2 Chevalley groups

We fix a numeration of the roots $\Phi = \{\alpha_1, \ldots, \alpha_m\}$ such that $\operatorname{ht} \alpha_i \leq \operatorname{ht} \alpha_j$ for all $i \leq j$ (recall that $\Pi = \{\alpha_1, \ldots, \alpha_n\}$). Let \mathfrak{g} be the simple Lie algebra associated to Φ (over \mathbb{C}) with Cartan subalgebra \mathfrak{h} . We denote by $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra and by $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ the *Killing form* on \mathfrak{g} . We fix $\{e_\alpha, h_\beta \mid \alpha \in \Phi, \beta \in \Pi\}$ a Chevalley basis of \mathfrak{g} which satisfies the following properties:

- (1) h_{α} is the coroot of α , i.e. $h_{\alpha} = \frac{2t_{\alpha}}{\kappa(t_{\alpha}, t_{\alpha})}$ where $t_{\alpha} \in \mathfrak{h}$ is the unique element such that $\kappa(t_{\alpha}, h) = \alpha(h)$ for all $h \in \mathfrak{h}$,
- (2) $e_{\alpha} \in \mathfrak{g}_{\alpha}$ for all $\alpha \in \Phi$,
- (3) $[h_{\alpha}h_{\beta}] = 0$ for all $\alpha, \beta \in \Pi$,
- (4) $[h_{\alpha}e_{\beta}] = (\beta, \alpha^{\vee})e_{\alpha},$

- (5) $[e_{\alpha}e_{-\alpha}] = h_{\alpha}$ for all $\alpha \in \Pi$,
- (6) $[e_{\alpha}e_{\beta}] = 0$ if $\alpha + \beta \notin \Phi$ and $\beta \neq -\alpha$,
- (7) if $\beta r\alpha, \dots, \beta + q\alpha$ is the α -string through β , then $[e_{\alpha}e_{\beta}] = N_{\alpha,\beta}e_{\alpha+\beta}$ if $\alpha + \beta \in \Phi$ with $N_{\alpha,\beta} = -N_{-\alpha,-\beta} = \pm (r+1)$.

The existence of this basis is proven in [Hum00, 25.2], [Car89, 4.2.1] and [Ste16, Chapter 1]. For $\alpha \in \Phi^+$, we set $f_\alpha := e_{-\alpha}$. Moreover, for any sequences of non-negative integers $A = (a_1, \ldots, a_m), B = (b_1, \ldots, b_m) \in \mathbb{N}^m, C = (c_1, \ldots, c_n) \in \mathbb{N}^n$, we define

$$E^{A} := \frac{e_{\alpha_{1}}^{a_{1}}}{a_{1}!} \cdots \frac{e_{\alpha_{m}}^{a_{m}}}{a_{m}!} \in \mathcal{U}(\mathfrak{g}),$$
$$F^{B} := \frac{f_{\alpha_{1}}^{b_{1}}}{b_{1}!} \cdots \frac{f_{\alpha_{m}}^{b_{m}}}{b_{m}!} \in \mathcal{U}(\mathfrak{g}),$$
$$H^{C} := \binom{h_{\alpha_{1}}}{c_{1}} \cdots \binom{h_{\alpha_{n}}}{c_{n}} \in \mathcal{U}(\mathfrak{g}),$$

where

$$\binom{h_{\alpha_i}}{c_i} := \frac{h_{\alpha_i}(h_{\alpha_i}-1)\cdots(h_{\alpha_i}-c_i+1)}{c_i!}.$$

Using the PBW Theorem (see [Hum00, 17.3] or [Ste16, Chapter 2]), the set $\{F^B H^C E^A\}$ is a basis of $\mathcal{U}(\mathfrak{g})$. We define $\mathcal{U}(\mathfrak{g})_{\mathbb{Z}}$ to be the subring of $\mathcal{U}(\mathfrak{g})$ generated by $\{\frac{e_a^a}{a!} | \alpha \in \Phi, a \in \mathbb{N}\}$, $\mathcal{U}(\mathfrak{g})_{\mathbb{Z}}^{\pm}$ to be the subring of $\mathcal{U}(\mathfrak{g})$ generated by $\{\frac{e_a^a}{a!} | \alpha \in \Phi^{\pm}, a \in \mathbb{N}\}$, and $\mathcal{U}(\mathfrak{g})_{\mathbb{Z}}^{\circ}$ the subring of $\mathcal{U}(\mathfrak{g})$ generated by $\{\binom{h_\alpha}{c} | \alpha \in \Phi, a \in \mathbb{N}\}$. Then $\{F^B H^C E^A\}$ is a \mathbb{Z} -basis of $\mathcal{U}(\mathfrak{g})_{\mathbb{Z}}$ ([Ste16, Chapter 2]).

Let V be an irreducible finite-dimensional $\mathcal{U}(\mathfrak{g})$ -module with highest weight λ . There exists $v^+ \in V$ a maximal vector for $\mathcal{U}(\mathfrak{g})^\circ \mathcal{U}(\mathfrak{g})^+$ of weight λ . We define $M := \mathcal{U}(\mathfrak{g})_{\mathbb{Z}}^- v^+$, which is a lattice in V. Then M is stable under $\mathcal{U}(\mathfrak{g})_{\mathbb{Z}}$. For an arbitrary field k, we define $\mathcal{U}(\mathfrak{g})_k := \mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{Z}} k$ and $V_k := M \otimes_{\mathbb{Z}} k$, which has thus the structure of a $\mathcal{U}(\mathfrak{g})_k$ -module.

For $\alpha \in \Phi$ and $t \in k$, we define

$$x_{\alpha}(t) := \exp(te_{\alpha}) = \sum_{i=0}^{\infty} t^{i} \frac{e_{\alpha}^{i}}{i!}$$

Since e_{α} acts nilpotently on V_k , the map $x_{\alpha}(t)$ is well-defined and is an automorphism of V_k ([Ste16, Chapter 3]). We call the group

$$G = G(V, k) = \langle x_{\alpha}(t) | \alpha \in \Phi, t \in k \rangle \subseteq \operatorname{GL}(V_k)$$

the Chevalley group associated to V and k. The type of G is the type of the root system Φ .

We fix G = G(V, k) a Chevalley group. For $\alpha \in \Phi$, we define the root subgroup corresponding to α to be

$$X_{\alpha} := \{ x_{\alpha}(t) | \ t \in k \} \le G.$$

We also define

$$U := \langle X_{\alpha} | \alpha \in \Phi^+ \rangle \le G$$
 and $U^- := \langle X_{\alpha} | \alpha \in \Phi^- \rangle \le G$.

Furthermore, for $t \in k^*$, we define

$$w_{\alpha}(t) := x_{\alpha}(t)x_{-\alpha}(-t^{-1})x_{\alpha}(t) \in G \quad \text{and} \quad h_{\alpha}(t) := w_{\alpha}(t)w_{\alpha}(1)^{-1} \in G$$

and set

$$T := \langle h_{\alpha}(t) | \ \alpha \in \Phi, \ t \in k^* \rangle \le G.$$

Finally, we set

$$B := \langle U, T \rangle \le G.$$

From now on, we assume that k is algebraically closed. Then G is a semisimple algebraic group (over k) with maximal torus T and Borel subgroup B ([Ste16, Theorem 6]). We call Φ the root system associated to G.

Let X_G be the lattice of all weights appearing in rational representations of G. Then $X_G \subseteq X$ ([MT11, Section 9.2]), and we say that G is simply connected if $X_G = X$. For each type of root system, there exists a unique simply connected Chevalley group of this type (up to isomorphism). For a root system of type A_n , we have $G = \text{SL}_{n+1}$, for a root system of type B_n , we have $G = \text{Spin}_{2n+1}$, and for a root system of type C_n , we have $G = \text{Sp}_{2n}$ ([Ste16, Chapter 3]).

If G = G(V, k) is simply connected and G(V', k) is another Chevalley group of the same type, there exists a surjective homomorphism $G \to G(V', k)$ ([Ste16, Corollary 5]). In particular, V' has the structure of a G-module.

For the rest of this paper, we fix k an algebraically closed field of characteristic p > 0, Φ a root system with base Π , weight lattice X and set of dominant weights X^+ , G = G(V, k) the simply connected Chevalley group with root system Φ , and $B, T \leq G$ as in the last section. Moreover, we fix W the Weyl group associated to Φ and we set $G_{\mathbb{C}} = G(V, \mathbb{C})$.

1.3 Modules

In this section, we define several notions related to modules for G. In particular, we define Weyl modules and tilting modules. All the modules that we consider are finite-dimensional and correspond to rational representations. Moreover, by module, we always mean G-module.

1.3.1 First definitions and irreducible modules

We start by recalling some basic definitions and the classification of finite-dimensional simple G-modules.

Let M be a G-module. Its *socle*, denoted by soc M, is the sum of all its simple submodules and its *radical*, denoted by rad M, is the intersection of all its maximal submodules. The socle soc M is the largest completely reducible submodule of M. The radical rad M is the smallest submodule of M such that $M/\operatorname{rad} M$ is completely reducible ([Jan03, I 2.14]). A vector $v \in M$ is a *maximal vector* with respect to B if $Bv \subseteq kv$. For $\lambda \in X$, we denote by $m_M(\lambda) := \dim M_{\lambda}$ the multiplicity of the weight λ in M, where M_{λ} is the weight space associated to the weight λ in M. The weight μ is called the *highest weight* of M if every $\nu \in X$ with $m_M(\nu) > 0$ satisfy $\nu \leq \mu$. For $a \in \mathbb{N}$, we define the *a-th symmetric power* of Mto be

$$S^a M := M^{\otimes a} / \langle P - \sigma(P) \rangle$$

where P is a pure tensor and $\sigma(v_1 \otimes \ldots \otimes v_a) = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(a)}$ for $\sigma \in S_a$. By multilinear algebra, if (v_1, \ldots, v_m) is an ordered basis of M, then $\{v_{i_1} \otimes \ldots \otimes v_{i_a}\}_{1 \leq i_1 \leq \ldots \leq i_a \leq m}$ is a basis of $S^a V$.

The irreducible G-modules are classified by their highest weight.

Theorem 1.12 ([Hum75, 31.3]). Let $\lambda \in X^+$ be a dominant weight. Up to isomorphism, there exists a unique irreducible module with highest weight λ which we denote by $L(\lambda)$. This module satisfies $m_{L(\lambda)}(\lambda) = 1$. Moreover, every irreducible module is of the form $L(\nu)$ for some $\nu \in X^+$. **Proposition 1.13** ([Pie12, 2.4]). Let M be a completely reducible module, and N < M be a submodule. Then N and M/N are completely reducible.

Let M be a G-module. A composition series for M is a sequence of submodules

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$$

such that the quotients M_i/M_{i-1} are simple for all $i \in \{1, \ldots, n\}$. For $\nu \in X^+$, we write

$$[M: L(\nu)] := |\{i \in \{1, \dots, n\}| M_i / M_{i-1} \cong L(\nu)\}|.$$

Due to Jordan-Hölder Theorem, the value $[M : L(\nu)]$ does not depend on the choice of the composition series (see for example [Erd18, Theorem 3.11]). An irreducible module $L(\nu)$ is called a *composition factor* of M if it appears in a composition series for M, i.e. if $[M : L(\nu)] > 0$. If M is a G-module with composition series $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$, we write this composition series $[L(\nu_n), \ldots, L(\nu_1)]$ where $L(\nu_i) \cong M_i/M_{i-1}$.

We are now ready to define the central notion of this project.

Definition 1.14. A module M is multiplicity-free if all composition factors appear with multiplicity 1, i.e. $[M : L(\nu)] \leq 1$ for all $\nu \in X^+$. If a module M is not multiplicity-free, we say that M has multiplicity.

Lemma 1.15 ([Tes88, 1.30]). Let $\lambda = \sum_{i=1}^{n} a_i \omega_i \in X^+$ with $0 \le a_i < p$ for all $i \in \{1, ..., n\}$. Then for i = 1, ..., n and $0 \le r \le a_i$, we have

$$m_{L(\lambda)}(\lambda - r\alpha_i) = 1.$$

1.3.2 Duality

Now we define two notions of duality in the category of G-modules.

Definition 1.16. Let M be a G-module. Its dual M^* is the usual dual vector space of M with G-action given by

$$(gf)(m) = f(g^{-1}m)$$
 for $f \in M^*$, $g \in G$, $m \in M$.

Proposition 1.17 ([Jan03, II 1.16]). There exists τ an antiautomorphism of G which satisfies

$$\tau^2 = \mathrm{id}_G, \quad \tau|_T = \mathrm{id}_T \quad and \quad \tau(X_\alpha) = X_{-\alpha} \quad for \ all \ \alpha \in \Phi$$

Moreover, if $G = SL_n(k)$ and $T := \{ diagonal matrices \}$, we can take τ to be the matrix transposition, i.e. $\tau(g) = g^t$ for $g \in G$.

Definition 1.18. Let M be a G-module. Its contravariant dual M^{τ} is the dual vector space M^* with action defined by

$$(gf)(m) = f(\tau(g)m)$$
 for $f \in M^*$, $g \in G$, $m \in M$,

where τ is the antiautomorphism from Proposition 1.17. The module M is called contravariantly self-dual if $M^{\tau} \cong M$.

Remark 1.19. One can easily check that $M^{\tau} \otimes N^{\tau} \cong (M \otimes N)^{\tau}$ for any *G*-modules M, N.

Remark 1.20 ([Jan03, II 2.12]). Irreducible modules are contravariantly self-dual.

Lemma 1.21 ([Gru22, V 4.2]). Let M be a contravariantly self-dual module. If M is multiplicity-free, then M is completely reducible.

1.3.3 Weyl modules

In this section, we will define the so called Weyl modules. In characteristic 0, those modules and the irreducible modules coincide. In positive characteristic, the Weyl modules are no longer irreducible, but are still useful to understand the irreducible modules.

Let $\lambda \in X^+$, V' the $\mathcal{U}(\mathfrak{g})$ -module of highest weight λ and V'_k be as defined in section 1.2. The group G acts naturally on V'_k . We call this G-module the Weyl module of highest weight λ and denote it by $\Delta(\lambda)$. The Weyl module $\Delta(\lambda)$ is generated by a maximal vector for B of weight λ and satisfies the following universal property ([Jan03, II 2.13]):

Lemma 1.22. Let V be a G-module generated by a maximal vector for B of weight $\lambda \in X^+$. There exists a surjective morphism $\Delta(\lambda) \to V$.

Proposition 1.23 ([Hum00, 21.3]). Let $\lambda \in X^+$. The set $\{\nu \in X | m_{\Delta(\lambda)}(\nu) > 0\}$ is saturated with highest weight λ .

To compute weight multiplicities in Weyl modules, we can use Freudenthal's formula, whose proof can be found in [Hum00, 22.3].

Theorem 1.24 (Freudenthal's formula). Let $\lambda \in X^+$ and $\mu \in X$. Then

$$((\lambda + \rho, \lambda + \rho) - (\mu + \rho, \mu + \rho))m_{\Delta(\lambda)}(\mu) = 2\sum_{\alpha \in \Phi^+} \sum_{i=1}^{\infty} m_{\Delta(\lambda)}(\mu + i\alpha)(\mu + i\alpha, \alpha).$$

It can be combined with the following proposition.

Proposition 1.25 ([Cav17, Proposition A]). Let $\lambda = \sum_{i=1}^{n} a_i \omega_i \in X^+$ and $\mu = \lambda - \sum_{i=1}^{n} c_i \alpha_i$ with $c_i \in \mathbb{N}$ for all *i*. Suppose the existence of a non-empty subset $J \subseteq \{1, \ldots, n\}$ such that $c_j \leq a_j$ for all $j \in J$. Let $\lambda' = \lambda - \sum_{j \in J} (a_j - c_j) \omega_j$ and $\mu' = \lambda' - \sum_{i=1}^{n} c_i \alpha_i$. Then

$$m_{\Delta(\lambda)}(\mu) = m_{\Delta(\lambda')}(\mu').$$

For $\lambda \in X^+$, we define the *costandard* module of highest weight λ by

$$\nabla(\lambda) := \Delta(-w_0(\lambda))^*.$$

The module $\nabla(\lambda)$ is also called the *induced module* or *dual Weyl module* in the literature, and satisfies $\nabla(\lambda) \cong \Delta(\lambda)^{\tau}$ ([Jan03, II 2.13]).

Proposition 1.26 ([Jan03, II 2.4 and 2.14]). Let $\lambda \in X^+$. We have $\Delta(\lambda)/\operatorname{rad} \Delta(\lambda) \cong L(\lambda)$ and soc $\nabla(\lambda) \cong L(\lambda)$.

1.3.4 Filtrations and tilting modules

Another class of useful modules are the so-called tilting modules, which we define in this subsection. They will be very useful to show that some tensor products have multiplicity. Before that, we define two special kinds of filtrations. We end this section by stating the classification of indecomposable tilting modules.

Definition 1.27. Let M be a G-module. A Weyl filtration of M is a sequence

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$$

of submodules such that M_i/M_{i-1} is a Weyl module for all $i \in \{1, \ldots, n\}$, i.e. there exist dominant weights $\lambda_1, \ldots, \lambda_n \in X^+$ such that $M_i/M_{i-1} \cong \Delta(\lambda_i)$ for all $i \in \{1, \ldots, n\}$.

Theorem 1.28 ([Mat90, Theorem 1]). Let λ , $\mu \in X^+$ be dominant weights. The tensor product $\Delta(\lambda) \otimes \Delta(\mu)$ admits a Weyl filtration.

Definition 1.29. Let M be a G-module. A good filtration of M is a sequence

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$$

of submodules such that there exist dominant weights $\lambda_1, \ldots, \lambda_n \in X^+$ with $M_i/M_{i-1} \cong \nabla(\lambda_i)$ for all $i \in \{1, \ldots, n\}$.

Remark 1.30. A *G*-module *M* admits a Weyl filtration if and only if its dual M^* admits a good filtration if and only if M^{τ} admits a good filtration.

Definition 1.31. A module M is a tilting module if it admits a Weyl filtration and a good filtration.

An important result is that the tensor product of two tilting modules is again a tilting module. We will use it several times in the next sections without further reference.

Theorem 1.32. Let M, N be two tilting modules. Then $M \otimes N$ is a tilting module.

Proof. This is a direct consequence of [Mat90, Theorem 1].

Like the simple modules, the indecomposable tilting modules are classified by their highest weight.

Proposition 1.33 ([Jan03, II E.6]). Let $\lambda \in X^+$ be a dominant weight. There exists a unique indecomposable tilting module $T(\lambda)$ with highest weight λ and $m_{T(\lambda)}(\lambda) = 1$. Moreover, for every tilting module T, there exist dominant weights $\nu_1, \ldots, \nu_n \in X^+$ such that

$$T \cong \bigoplus_{i=1}^{n} T(\nu_i).$$

Lemma 1.34. Every tilting module is contravariantly self-dual.

Proof. If T is a tilting module, then T^{τ} is a tilting module. By definition, $(-)^{\tau}$ preserves the weights of the representation. Thus, if T is indecomposable with highest weight λ , then so is T^{τ} , and we conclude by uniqueness in Proposition 1.33 that $T(\lambda)^{\tau} \cong T(\lambda)$.

Corollary 1.35. Let $T(\lambda)$ be an indecomposable tilting module. Then $T(\lambda)$ is multiplicity-free if and only if $T(\lambda)$ is irreducible.

Proof. This is a direct consequence of Lemmas 1.34 and 1.21.

Lemma 1.36. Let M be a tilting module. Let $\eta \in X^+$ be such that $L(\eta)$ is a composition factor of M and $T(\eta)$ is not irreducible. Then M has multiplicity.

Proof. Using Proposition 1.33, there exist $\nu_1, \ldots, \nu_s \in X^+$ such that $M \cong \bigoplus_{i=1}^s T(\nu_i)$. There exists $\nu_i \geq \eta$ such that $L(\eta)$ is a composition factor of $T(\nu_i)$. If $\nu_i > \eta$, then $T(\nu_i)$ is not irreducible, hence it has multiplicity by Corollary 1.35, and so M has multiplicity. If $\nu_i = \eta$, we conclude using the assumption that $T(\eta)$ is not irreducible.

Lemma 1.37. Let $\lambda \in X^+$. If $\Delta(\lambda) \cong L(\lambda)$, then $T(\lambda) \cong \nabla(\lambda) \cong L(\lambda)$. Else, $T(\lambda)$ is not irreducible.

Proof. If $\Delta(\lambda)$ is irreducible, then so is $\nabla(\lambda)$. In particular, $L(\lambda) \cong \Delta(\lambda) \cong \nabla(\lambda)$ admits a Weyl filtration and a good filtration. Thus, $L(\lambda)$ is a tilting module, and we conclude by uniqueness in Proposition 1.33.

Otherwise, $\Delta(\lambda)$ appear in the Weyl filtration of $T(\lambda)$, hence $T(\lambda)$ is not irreducible.

1.4 Characters

A lot of information about a G-module M is given by the dimensions of its weight spaces. These informations are encoded in the character of the module, a notion that we define in this section. Later on, we will use those characters to compute the composition factors of tensor products of simple G-modules, and in particular to show that some of them are multiplicity-free.

Lemma 1.38 ([MT11, Lemma 15.3]). Let M be an irreducible G-module, $\lambda \in X$ and $w \in W$. Then

$$m_M(\lambda) = m_M(w\lambda).$$

Definition 1.39. Let M be a G-module. Its character is the formal sum

$$\operatorname{ch} M := \sum_{\lambda \in X} m_M(\lambda) e^{\lambda} \in \mathbb{Z}[X].$$

where $\mathbb{Z}[X]$ has \mathbb{Z} -basis $\{e^{\lambda} | \lambda \in X\}$.

We denote by $\mathbb{Z}[X]^W$ the fixed points in $\mathbb{Z}[X]$ for the natural action of W. By Lemma 1.38, we have $\operatorname{ch} M \in \mathbb{Z}[X]^W$ for every *G*-module *M*.

Remark 1.40. Let M, N be two *G*-modules. We have $ch(M \oplus N) = ch M + ch N$ and $ch(M \otimes N) = ch M \cdot ch N$.

Notation 1.41. We denote the character of the Weyl module of highest weight $\lambda \in X^+$ by

$$\chi(\lambda) := \operatorname{ch} \Delta(\lambda).$$

Theorem 1.42 (Weyl's character formula). For $\lambda \in X^+$, we have

$$\chi(\lambda) = \frac{\sum_{w \in W} \det(w) e^{w(\lambda+\rho)}}{\sum_{w \in W} \det(w) e^{w(\rho)}}$$

Corollary 1.43 (Weyl's degree formula). For $\lambda \in X^+$, we have

$$\dim \Delta(\lambda) = \frac{\prod_{\alpha \in \Phi^+} (\lambda + \rho, \alpha)}{\prod_{\alpha \in \Phi^+} (\rho, \alpha)}$$

Proofs of Weyl's character formula and Weyl's degree formula are given in [Hum00, 24].

Weyl's character formula allows us to extend our definition of character for non-dominant weights.

Definition 1.44. Let $\lambda \in X$. The Weyl character associated to λ is the formal element

$$\chi(\lambda) := \frac{\sum_{w \in W} \det(w) e^{w(\lambda + \rho)}}{\sum_{w \in W} \det(w) e^{w(\rho)}}.$$

Lemma 1.45. For $\lambda \in X$, we have

- (1) $\chi(w \bullet \lambda) = \det(w)\chi(\lambda) \quad \forall w \in W,$
- (2) $\chi(\lambda) = 0 \quad \forall \lambda \in D \setminus X^+.$

Proof. For $g \in W$, we have

$$\chi(g \bullet \lambda) = \frac{\sum_{w \in W} \det(w) e^{w(g \bullet \lambda + \rho)}}{\sum_{w \in W} \det(w) e^{w(\rho)}} = \frac{\sum_{w \in W} \det(w) e^{w(g(\lambda + \rho))}}{\sum_{w \in W} \det(w) e^{w(\rho)}}$$
$$= \det(g) \frac{\sum_{w \in W} \det(wg) e^{wg(\lambda + \rho)}}{\sum_{w \in W} \det(w) e^{w(\rho)}} = \det(g) \frac{\sum_{w \in W} \det(w) e^{w(\lambda + \rho)}}{\sum_{w \in W} \det(w) e^{w(\rho)}}$$
$$= \det(g) \chi(\lambda).$$

Now observe that for $\lambda \in D \setminus X^+$, there exists $\alpha \in \Pi$ such that $s_{\alpha} \cdot \lambda = \lambda$. Therefore, $\chi(\lambda) = -\chi(\lambda)$ so $\chi(\lambda) = 0$.

The following lemma will be useful to compute an explicit decomposition of a product of characters into a sum of irreducible or Weyl characters. It ensures the existence and the uniqueness of such a decomposition. We will use it several times in the next sections without further reference.

Lemma 1.46 ([Jan03, II 5.8]). The set of characters of irreducible modules $\{\operatorname{ch} L(\lambda) \mid \lambda \in X^+\}$ is a \mathbb{Z} -basis of $\mathbb{Z}[X]^W$. Moreover, the set of Weyl characters $\{\chi(\lambda) \mid \lambda \in X^+\}$ is a \mathbb{Z} -basis of $\mathbb{Z}[X]^W$.

Proposition 1.47 ([Ste03, Proposition 2.1]). For $\lambda, \mu \in X^+$, we have

$$\chi(\lambda)\chi(\mu) = \sum_{\nu \in X} m_{\Delta(\lambda)}(\nu) \cdot \chi(\mu + \nu).$$

Corollary 1.48. For $\lambda, \mu \in X^+$, we have

ch
$$L(\lambda) \cdot \chi(\mu) = \sum_{\nu \in X} m_{L(\lambda)}(\nu) \cdot \chi(\mu + \nu).$$

Proof. Since $\operatorname{ch} L(\lambda) \in \mathbb{Z}[X]^W$, there exist $\lambda_1, \ldots, \lambda_n \in X^+$ and $a_1, \ldots, a_n \in \mathbb{Z}$ such that $\operatorname{ch} L(\lambda) = \sum_{i=1}^n a_i \chi(\lambda_i)$ (Lemma 1.46). For $\nu \in X$, we have $m_{L(\lambda)}(\nu) = \sum_{i=1}^n a_i m_{\Delta(\lambda_i)}(\nu)$. Using Proposition 1.47 in the second equality below, we get

$$\operatorname{ch} L(\lambda) \cdot \chi(\mu) = \left(\sum_{i=1}^{n} a_i \chi(\lambda_i)\right) \chi(\mu) = \sum_{i=1}^{n} a_i \sum_{\nu \in X} m_{\Delta(\lambda_i)}(\nu) \cdot \chi(\mu + \nu)$$
$$= \sum_{\nu \in X} \left(\sum_{i=1}^{n} a_i m_{\Delta(\lambda_i)}(\nu)\right) \cdot \chi(\mu + \nu) = \sum_{\nu \in X} m_{L(\lambda)}(\nu) \cdot \chi(\mu + \nu).$$

1.4.1 Jantzen *p*-sum formula

As previously claimed, the Weyl modules are not always irreducible in positive characteristic. Thus, it will be useful to compute their composition factors. An important tool for this computation is the so-called Jantzen *p*-sum formula. We will use it to compute the composition factors of Weyl modules with *p*-restricted highest weight.

Let $m \in \mathbb{N}^*$ be a positive integer. Recall that p is a fixed prime. Let $a, b \in \mathbb{N}$ be such that $p \nmid b$ and $m = p^a b$. The *p*-adic valuation of n is $\nu_p(m) := a$.

Proposition 1.49 (Jantzen *p*-sum formula, [Jan03, II 8.19]). Let $\lambda \in X^+$. There exists a filtration

$$\Delta(\lambda) \supseteq \Delta(\lambda)^1 \supseteq \Delta(\lambda)^2 \supseteq \dots$$

such that

$$\sum_{i>0} \operatorname{ch} \Delta(\lambda)^{i} = \sum_{\alpha \in \Phi^{+}} \sum_{0 < mp < (\lambda + \rho, \alpha^{\vee})} \nu_{p}(mp) \chi(s_{\alpha, mp} \bullet \lambda)$$

and

$$\Delta(\lambda)/\Delta(\lambda)^1 \cong L(\lambda).$$

Notation 1.50. We set

$$\mathrm{JSF}(\lambda) := \sum_{\alpha \in \Phi^+} \sum_{0 < mp < (\lambda + \rho, \alpha^{\vee})} \nu_p(mp) \chi(s_{\alpha, mp} \bullet \lambda).$$

Remark 1.51. Observe that, for $\lambda \in X^+$ a *p*-restricted weight, we have

$$(\lambda + \rho, \alpha^{\vee}) \le p \sum_{i=1}^{n} (\omega_i, \alpha^{\vee}) = p(\rho, \alpha^{\vee}) \le p(\rho, \alpha_h^{\vee}) = p(h-1).$$

Therefore, if $p \ge h - 1$, all the *p*-adic valuations in $JSF(\lambda)$ are equal to 1.

Remark 1.52. Let $\lambda \in X^+$. If there exists $\mu \in X^+$ such that $\text{JSF}(\lambda) = \text{ch } L(\mu)$, then $\Delta(\lambda)$ admits two composition factors, $L(\lambda)$ and $L(\mu)$. Since $\Delta(\lambda)/\operatorname{rad} \Delta(\lambda) \cong L(\lambda)$, it follows that $\Delta(\lambda)$ admits a unique composition series, given by $[L(\lambda), L(\mu)]$.

1.5 Linkage principle

Another useful tool to compute the composition factors of a Weyl module is the Strong Linkage Principle. It allows us to show that Weyl modules with highest weight in the fundamental alcove are irreducible.

Proposition 1.53 (The Strong Linkage Principle, [Jan03, II 6.13]). Let $\lambda, \mu \in X^+$ be dominant weights. If

$$[\Delta(\lambda):L(\mu)] > 0,$$

then $\mu \uparrow \lambda$.

Proposition 1.54 ([Jan03, II 6.24]). Let $\lambda \in X^+$ be a dominant weight. Suppose that $\mu \in X$ is maximal in the set $\{\nu \in X \mid \nu \uparrow \lambda, \nu \neq \lambda\}$ with respect to the ordering \uparrow . If $\mu \in X^+$ and $\mu \notin \{\lambda - p\alpha \mid \alpha \in \Phi^+\}$, then

 $[\Delta(\lambda) : L(\mu)] = 1.$

Lemma 1.55. For every $\lambda \in X^+ \cap \widehat{C_1}$ we have $L(\lambda) \cong \Delta(\lambda)$.

Proof. Let $\mu \in X^+$ be such that $L(\mu)$ is a composition factor of $\Delta(\lambda)$. By the Strong Linkage Principle (Proposition 1.53), we have $\mu \uparrow \lambda$, and in particular, $\mu \leq \lambda$, so $\mu \in \widehat{C_1}$ by Lemma 1.7. Moreover, $\mu \in W_p \bullet \lambda$. Since $\overline{C_1}$ is a fundamental domain for the dot action of W_p (Lemma 1.9), we have $\mu \in \overline{C_1} \cap W_p \bullet \lambda = \{\lambda\}$. We conclude that $\mu = \lambda$, therefore $\Delta(\lambda)$ is irreducible.

Lemma 1.56. Let $\lambda \in \widehat{C_1} \cap X^+$ and $\nu \in X^+$ such that $\nu \leq \lambda$. Then $L(\nu) \cong \Delta(\nu)$.

Proof. This is a direct consequence of Lemmas 1.7 and 1.55.

Lemma 1.57 ([Jan03, II 4.16]). Let λ , $\mu \in X^+$. The tensor product $\Delta(\lambda) \otimes \Delta(\mu)$ admits a submodule isomorphic to $\Delta(\lambda + \mu)$.

1.6 An argument to count multiplicities

In this subsection, we provide an argument to compute the multiplicities of each composition factor of the tensor product of two simple modules. We will use it several times later to show that some tensor products of two simple modules have multiplicity.

Argument 1. Let $\lambda, \mu \in X^+$ be dominant weights, and let $M = L(\lambda) \otimes L(\mu)$. A vector $v \otimes w \in M$ is a weight vector of weight ν if and only if v is a weight vector in $L(\lambda)$ of weight ν_1 , w is a weight vector in $L(\mu)$ of weight ν_2 and $\nu_1 + \nu_2 = \nu$. Therefore,

$$m_M(\nu) = \sum_{\substack{\nu_1, \nu_2 \in X \\ \nu_1 + \nu_2 = \nu}} m_{L(\lambda)}(\nu_1) m_{L(\mu)}(\nu_2).$$

Suppose that $\nu_1, \ldots, \nu_s \in X^+$ is the complete list of the dominant weights corresponding to all composition factors of M (with multiplicity). For every weight $\eta \in X$, we have

$$m_M(\eta) = \sum_{i=1}^s m_{L(\nu_i)}(\eta).$$

We compute the ν_i 's as follows. We set $\nu_1 = \lambda + \mu$. Suppose that we have already ν_1, \ldots, ν_t for t < s. Let $\eta \in X^+$ be such that

$$m_M(\eta) > \sum_{i=1}^t m_{L(\nu_i)}(\eta)$$
 and $m_M(\nu) = \sum_{i=1}^t m_{L(\nu_i)}(\nu)$ for every dominant weight $\nu > \eta$.

It follows that $m_{L(\nu_i)}(\nu) = 0$ for every i > t and every $\nu > \eta$. In particular, $\nu_i \neq \eta$ for every i > t. Moreover, there exists i > t such that $m_{L(\nu_i)}(\eta) > 0$, thus $\nu_i \geq \eta$. Therefore, we deduce that $\nu_i = \eta$ for some i > t, and we can choose $\nu_{t+1} = \eta$.

2 Properties of tensor products

In this section, we establish some properties of tensor products. We state Steinberg's tensor product theorem (see [Jan03, II 3.16] for a proof), which allows us to restrict our attention to tensor products of irreducible modules with p-restricted highest weight in order to answer our question.

2.1 Steinberg's tensor product theorem

Theorem 2.1 ([Spr09, Theorem 9.4.3]). The Frobenius endomorphism $k \to k : c \mapsto c^p$ induces a group endomorphism $F : G \to G$ given by $x_{\alpha}(c) \mapsto x_{\alpha}(c^p)$ for all $\alpha \in \Phi$, $c \in k$ and $F(t) = t^p$ for $t \in T$.

Notation 2.2. Let $\phi : G \to \operatorname{GL}(M)$ be a representation. We denote by $M^{(p^i)}$ the vector space M with G-action corresponding to the representation $\phi \circ F^i : G \to \operatorname{GL}(M)$ where F is the group endomorphism described in Theorem 2.1.

Proposition 2.3 ([MT11, Proposition 16.6]). For every $\lambda \in X^+$, we have an isomorphism of *G*-modules $L(p\lambda) \cong L(\lambda)^{(p)}$.

Theorem 2.4 (Steinberg's tensor product theorem). Let $\lambda = \lambda_0 + p\lambda_1 + \ldots + p^n\lambda_n \in X^+$ be a dominant weight such that λ_i is p-restricted for all $i \in \{0, \ldots, n\}$. We have an isomorphism

$$L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_1)^{(p)} \otimes \cdots \otimes L(\lambda_n)^{(p^n)}.$$

2.2 Reduction to *p*-restricted highest weights

Lemma 2.5 ([Gru21, Lemma 4.12]). Let $\lambda, \mu \in X^+$ be dominant weights. If the G-module $L(\lambda) \otimes L(\mu)$ is multiplicity-free, then it is completely reducible.

Proof. Using Remarks 1.19 and 1.20, we have

$$(L(\lambda) \otimes L(\mu))^{\tau} \cong L(\lambda)^{\tau} \otimes L(\mu)^{\tau} \cong L(\lambda) \otimes L(\mu).$$

We conclude by Lemma 1.21.

Theorem 2.6 ([Gru21, Theorem A]). Let $\lambda, \mu \in X^+$ be p-restricted dominant weights. If the G-module $L(\lambda) \otimes L(\mu)$ is completely reducible, then all its composition factors are p-restricted.

Corollary 2.7. Let $\lambda, \mu \in X^+$ be p-restricted dominant weights. If the G-module $L(\lambda) \otimes L(\mu)$ is multiplicity-free, then all its composition factors are p-restricted.

Proof. This is a direct consequence of Lemma 2.5 and Theorem 2.6.

Corollary 2.8. Let $\lambda, \mu \in X^+$ be *p*-restricted weights. If $\lambda + \mu$ is not *p*-restricted, then $L(\lambda) \otimes L(\mu)$ has multiplicity.

Proof. We know that $L(\lambda + \mu)$ is a composition factor of $L(\lambda) \otimes L(\mu)$. Thus we conclude by Corollary 2.7.

Proposition 2.9. Let $\lambda = \lambda_0 + p\lambda_1 + \ldots + p^n\lambda_n$, $\mu = \mu_0 + p\mu_1 + \ldots + p^n\mu_n \in X^+$ be dominant weights such that λ_i, μ_i are p-restricted for all $i \in \{0, \ldots, n\}$. Then $L(\lambda) \otimes L(\mu)$ is multiplicity-free if and only if $L(\lambda_i) \otimes L(\mu_i)$ is multiplicity-free for all $i \in \{0, \ldots, n\}$.

Proof. First, using Steinberg's tensor product theorem (Theorem 2.4), we have an isomorphism

$$L(\lambda) \otimes L(\mu) \cong \bigotimes_{i=0}^{n} (L(\lambda_i)^{(p^i)} \otimes L(\mu_i)^{(p^i)}) \cong \bigotimes_{i=0}^{n} (L(\lambda_i) \otimes L(\mu_i))^{(p^i)}.$$

Clearly, if there exists $i \in \{0, \ldots, n\}$ such that $L(\lambda_i) \otimes L(\mu_i)$ has multiplicity, then $(L(\lambda_i) \otimes L(\mu_i))^{(p^i)}$ has multiplicity, thus $L(\lambda) \otimes L(\mu)$ has multiplicity.

Now suppose that $L(\lambda_i) \otimes L(\mu_i)$ is multiplicity-free for all $i \in \{0, \ldots, n\}$. By Lemma 2.5 and Corollary 2.7, for each $i \in \{0, \ldots, n\}$, there exist $\nu_i^1, \ldots, \nu_i^{m_i} \in X^+$ distinct and *p*-restricted such that

$$L(\lambda_i) \otimes L(\mu_i) \cong \bigoplus_{j=1}^{m_i} L(\nu_i^j).$$

Therefore, we have

$$L(\lambda) \otimes L(\mu) \cong \bigotimes_{i=0}^{n} \bigoplus_{j=1}^{m_i} L(\nu_i^j)^{(p^i)} \cong \bigoplus_{\vec{j}} \bigotimes_{i=0}^{n} L(\nu_i^{j_i})^{(p^i)},$$

where $\vec{j} = (j_0, \ldots, j_n)$ runs over $\underset{i=0}{\overset{n}{\times}} \{1, \ldots, m_i\}$. Since all the weights ν_i^j are *p*-restricted, we can use Steinberg's tensor product theorem again to get

$$L(\lambda) \otimes L(\mu) \cong \bigoplus_{\vec{j}} L(\sum_{i=0}^{n} p^{i} \nu_{i}^{j_{i}}).$$

By uniqueness of the *p*-adic expansion of a weight, we conclude that $L(\lambda) \otimes L(\mu)$ is multiplicity-free.

By this proposition, in order to classify multiplicity-free tensor products of simple G-modules, we may restrict our attention to the study of the tensor products of simple modules with p-restricted highest weights.

3 Connections between characteristic 0 and positive characteristic

In this section, we show some links between multiplicity-free tensor products in characteristic 0 and in positive characteristic. In particular, we show that in positive characteristic, multiplicity-free tensor products and completely reducible tensor products are closely related. This will allow us to classify multiplicity-free tensor products for p = 2 and G of type A_n (see section 7).

Notation 3.1. For $\lambda \in X^+$, we denote by $L_{\mathbb{C}}(\lambda)$ the irreducible $G_{\mathbb{C}}$ -module of highest weight λ (over \mathbb{C}). Recall that ch $L_{\mathbb{C}}(\lambda) = \chi(\lambda)$.

Proposition 3.2. Let $\lambda, \mu \in X^+$ be dominant weights such that $\Delta(\lambda) \cong L(\lambda)$ and $\Delta(\mu) \cong L(\mu)$. Suppose that $\Delta(\nu) \cong L(\nu)$ for all dominant weights $\nu \in X^+$ such that $L(\nu)$ is a composition factor of $L(\lambda) \otimes L(\mu)$. Then $L(\lambda) \otimes L(\mu)$ is multiplicity-free if and only if $L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu)$ is multiplicity-free.

Proof. We show that the composition factors in characteristic p and in characteristic 0 are the same. Let $\nu_1, \ldots, \nu_m \in X^+$ be distinct weights and $k_1, \ldots, k_m \in \mathbb{Z}_{>0}$ be such that $\operatorname{ch}(L(\lambda) \otimes L(\mu)) = k_1 \operatorname{ch} L(\nu_1) + \ldots + k_m \operatorname{ch} L(\nu_m)$. By assumption, we have

$$\chi(\lambda)\chi(\mu) = \operatorname{ch} L(\lambda) \operatorname{ch} L(\mu) = \operatorname{ch}(L(\lambda) \otimes L(\mu)) = k_1 \operatorname{ch} L(\nu_1) + \ldots + k_m \operatorname{ch} L(\nu_m)$$
$$= k_1\chi(\nu_1) + \ldots + k_m\chi(\nu_m).$$

By uniqueness of the composition factors and linear independence of the Weyl characters (Lemma 1.46), it follows that $L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu)$ is multiplicity-free if and only if $k_i = 1$ for all $i \in \{1, \ldots, m\}$ if and only if $L(\lambda) \otimes L(\mu)$ is multiplicity-free.

Corollary 3.3. Let $\lambda, \mu \in X^+$ be dominant weights such that $\Delta(\lambda) \cong L(\lambda)$ and $\Delta(\mu) \cong L(\mu)$. If $\Delta(\nu) \cong L(\nu)$ for all dominant weights $\nu \leq \lambda + \mu$, then $L(\lambda) \otimes L(\mu)$ is multiplicity-free if and only if $L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu)$ is multiplicity-free.

Proof. If $L(\nu)$ is a composition factor of $L(\lambda) \otimes L(\nu)$, then ν is dominant and $\nu \leq \lambda + \mu$. Thus we can apply Proposition 3.2

Corollary 3.4. Let $\lambda, \mu \in X^+$ be such that $\lambda + \mu \in \widehat{C_1}$. The module $L(\lambda) \otimes L(\mu)$ is multiplicity-free if and only if $L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu)$ is multiplicity-free.

Proof. By Lemmas 1.7 and 1.55, Corollary 3.3 applies in case $\lambda + \mu \in \widehat{C}_1$.

The next theorem allows us to find the explicit decomposition of some tensor products as a direct sum of irreducible modules. Then we prove a more general version which allows us to conclude that some tensor products of irreducible modules are multiplicity-free without computing the explicit decomposition.

Theorem 3.5. Let $\lambda, \mu \in X^+$ be p-restricted dominant weights such that the following hold:

- (1) $L(\lambda) \cong \Delta(\lambda)$,
- (2) $L(\mu) \cong \Delta(\mu)$,
- (3) $L(\lambda) \otimes L(\mu)$ is completely reducible and
- (4) $L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu)$ is multiplicity-free.

Then $L(\lambda) \otimes L(\mu)$ is multiplicity-free. Moreover, if we have the decomposition

$$L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu) \cong \bigoplus_{i=1}^{m} L_{\mathbb{C}}(\nu_m)$$

for distinct dominant weights $\nu_1, \ldots, \nu_m \in X^+$, then $L(\nu_i) \cong \Delta(\nu_i)$ for all $i \in \{1, \ldots, m\}$ and

$$L(\lambda) \otimes L(\mu) \cong \bigoplus_{i=1}^{m} L(\nu_m)$$

Proof. To simplify the notation, we set $M := L(\lambda) \otimes L(\mu)$.

Since $L(\lambda)$ and $L(\mu)$ are tilting modules, M must be a tilting module. Thus there exist $\nu_1, \ldots, \nu_m \in X^+$ such that $M \cong T(\nu_1) \oplus \ldots \oplus T(\nu_m)$ (Proposition 1.33). Since M is completely reducible, $T(\nu_i)$ must be completely reducible for every $i \in \{1, \ldots, m\}$. Therefore, $T(\nu_i) \cong L(\nu_i) \cong \Delta(\nu_i)$ for every $i \in \{1, \ldots, m\}$, so all dominant weights ν such that $L(\nu)$ is a composition factors of M satisfy $\Delta(\nu) \cong L(\nu)$. Therefore, we can conclude by Proposition 3.2 (and its proof).

Theorem 3.6. Let $\lambda, \mu \in X^+$ be p-restricted dominant weights such that the following hold:

- (1) $L(\lambda) \otimes L(\mu)$ is completely reducible and
- (2) $L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu)$ is multiplicity-free.

Then $L(\lambda) \otimes L(\mu)$ is multiplicity-free.

Proof. We know that we have a surjection $\Delta(\lambda) \to L(\lambda)$ and a surjection $\Delta(\mu) \to L(\mu)$. By right exactness of tensor products, we get a surjection

$$\phi: \Delta(\lambda) \otimes \Delta(\mu) \to L(\lambda) \otimes \Delta(\mu) \to L(\lambda) \otimes L(\mu).$$

Using Theorem 1.28, we fix a Weyl filtration

$$0 = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_m = \Delta(\lambda) \otimes \Delta(\mu).$$

Thus there exist $\nu_1, \ldots, \nu_m \in X^+$ such that $V_i/V_{i-1} \cong \Delta(\nu_i)$ for $i = 1, \ldots, m$. In particular, we have

$$\chi(\lambda)\chi(\mu) = \sum_{i=1}^{m} \chi(\nu_i).$$

Since $L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu)$ is multiplicity-free, we deduce that $\nu_i \neq \nu_j$ for all $i \neq j$.

For $i \in \{1, \ldots, m\}$, we set $W_i := \phi(V_i)$ and we denote by $\phi_i : V_i \to W_i$ the restriction and corestriction of the map ϕ . In particular, ϕ_i is surjective for all $i \in \{1, \ldots, m\}$. By construction, we have a filtration

$$0 = W_0 \subseteq W_1 \subseteq \ldots \subseteq W_m = L(\lambda) \otimes L(\mu)$$

of $L(\lambda) \otimes L(\mu)$.

Now we identify the quotients W_i/W_{i-1} for i = 1, ..., m. Let $\iota_i : V_{i-1} \to V_i$ be the inclusion map. We have the following situation:

Since $\theta_i \circ \phi_i \circ \iota_i = 0$, i.e. $V_{i-1} \subseteq \ker(\theta_i \circ \phi_i)$, there exists a unique map $\psi_i : V_i/V_{i-1} \to W_i/W_{i-1}$ such that the following diagram commutes:

In particular, ψ_i is surjective because θ_i and ϕ_i are surjective. By Proposition 1.13, W_i is completely reducible and so W_i/W_{i-1} is completely reducible. Therefore, rad $\Delta(\nu_i) \subseteq \ker(\psi_i)$, so ψ_i factors as

Therefore, W_i/W_{i-1} is a quotient of $L(\nu_i)$, so either $W_i/W_{i-1} \cong L(\nu_i)$ or $W_i/W_{i-1} = 0$. We deduce that $L(\lambda) \otimes L(\nu)$ is isomorphic to a submodule of $\bigoplus_{i=1}^m L(\nu_i)$, and in particular it is multiplicity-free since all the ν_i 's are distinct.

$4 SL_2$

In this section, we establish the classification of multiplicity-free tensor products of simple $SL_2(k)$ -modules. Recall that $SL_2(k)$ has root system Φ of type A_1 , so $\Phi = \{\alpha, -\alpha\}$. The fundamental weight ω satisfies $\alpha = 2\omega$. Since all weights are integer multiples of ω , we will identify the set of weights with \mathbb{Z} . Under this identification, ω corresponds to 1, the positive root α to 2, and dominant weights are in bijection with \mathbb{N} . Moreover, we have $\rho = \frac{1}{2}\alpha = \omega$ and it corresponds to 1.

Since there exists a unique positive root in Φ , alcoves are in bijection with \mathbb{Z} , with

$$C_n = \{ \lambda \in X_{\mathbb{R}} | (n-1)p < (\lambda + \rho, \alpha^{\vee}) < np \}.$$

Using the identification previously described, we identify $X_{\mathbb{R}}$ with \mathbb{R} . Thus we get

$$C_n = \{ \lambda \in \mathbb{R} \mid (n-1)p < \lambda + 1 < np \}.$$

In particular, there exists a unique *p*-restricted alcove, the fundamental alcove C_1 , and by Lemma 1.55, $L(\lambda) \cong \Delta(\lambda)$ for all *p*-restricted dominant weights $\lambda \in X^+$.

We start by computing the Weyl characters and the decomposition of the product of two such characters.

Lemma 4.1. Let $\lambda \in X^+$. Then

$$\chi(\lambda) = \sum_{i=0}^{\lambda} e^{\lambda - 2i}.$$

Proof. We show this result using Weyl's character formula (Theorem 1.42). We have

$$(e^{1} - e^{-1}) \sum_{i=0}^{\lambda} e^{\lambda - 2i} = \sum_{i=0}^{\lambda} e^{\lambda + 1 - 2i} - \sum_{i=0}^{\lambda} e^{\lambda - 1 - 2i}$$
$$= e^{\lambda + 1} + \sum_{i=1}^{\lambda} e^{\lambda + 1 - 2i} - e^{-\lambda - 1} - \sum_{i=0}^{\lambda - 1} e^{\lambda - 1 - 2i}$$
$$= e^{\lambda + 1} - e^{-\lambda - 1}.$$

Therefore

$$\chi(\lambda) = \frac{e^{\lambda+1} - e^{-\lambda-1}}{e^1 - e^{-1}} = \sum_{i=0}^{\lambda} e^{\lambda-2i}.$$

Proposition 4.2 (Clebsch-Gordan formula). For $\lambda, \mu \in X^+$ with $\lambda \ge \mu$, we have

$$\chi(\lambda)\chi(\mu) = \chi(\lambda + \mu) + \chi(\lambda + \mu - 2) + \ldots + \chi(\lambda - \mu + 2) + \chi(\lambda - \mu).$$

Proof. By Proposition 1.47 and Lemma 4.1, we have

$$\chi(\lambda)\chi(\mu) = \chi(\mu)\chi(\lambda) = \sum_{i=0}^{\mu} \chi(\lambda + \mu - 2i).$$

We are now ready to state the main result of this section.

Proposition 4.3. Let $\lambda, \mu \in X^+$ be p-restricted dominant weights. Then $L(\lambda) \otimes L(\mu)$ is multiplicity-free if and only if $\lambda + \mu$ is p-restricted.

Proof. The "only if" direction is a direct consequence of Corollary 2.8. For the "if" direction, observe that $L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu)$ is multiplicity-free for all $\lambda, \mu \in X^+$ by Proposition 4.2. If $\lambda + \mu < p$, then $\lambda + \mu \in \widehat{C}_1$, and $L(\lambda) \otimes L(\mu)$ is multiplicity-free by Corollary 3.4.

Finally, we state the classification theorem for $SL_2(k)$.

Theorem 4.4. Let $\lambda = \lambda_0 + p\lambda_1 + \ldots + p^n\lambda_n$, $\mu = \mu_0 + p\mu_1 + \ldots + p^n\mu_n \in X^+$ be dominant weights with λ_i, μ_i p-restricted for all $i \in \{0, \ldots, n\}$. Then $L(\lambda) \otimes L(\mu)$ is multiplicity-free if and only if $\lambda_i + \mu_i$ is p-restricted for all $i \in \{0, \ldots, n\}$.

Proof. By Proposition 2.9, $L(\lambda) \otimes L(\mu)$ is multiplicity-free if and only if $L(\lambda_i) \otimes L(\mu_i)$ is multiplicity-free for all $i \in \{0, \ldots, n\}$. We conclude by Proposition 4.3.

5 SL_3

In this section, we establish the classification of multiplicity-free tensor products of simple SL₃-modules with *p*-restricted highest weight. We fix $G = SL_3(k)$ with root system Φ of type A_2 and $\Pi = \{\alpha_1, \alpha_2\}$ a base of Φ . With respect to this base, we have $\rho = \alpha_1 + \alpha_2$. For $\lambda = x\omega_1 + y\omega_2 \in X_{\mathbb{R}}$, we write $\lambda = (x, y)$. In particular, we have $\alpha_1 = (2, -1), \alpha_2 = (-1, 2)$ and $\rho = (1, 1)$.

We will prove the following theorem:

Theorem 5.1. Let $\lambda = (a, b), \mu = (c, d) \in X^+$ be non-zero p-restricted dominant weights. Up to the reordering of λ and μ , the module $L(\lambda) \otimes L(\mu)$ is multiplicity-free if and only if one of the following holds:

- (1) b = d = 0 and a + c < p,
- (2) a = c = 0 and b + d < p,
- (3) b = c = 0 and $a + d or <math>(a, d) \in \{(p 1, 1), (1, p 1)\},\$
- (4) b = 0 and a + c + d ,
- (5) a = 0 and b + c + d ,
- (6) b = 0, c + d = p 1, a + c < p and a < c + 2,
- (7) a = 0, c + d = p 1, b + d
- (8) b = 0, c + d > p 1, a + c < p and a + d < p,
- (9) a = 0, c + d > p 1, b + c
- (10) a + b , <math>c + d = p 1, a + b + c < p and a + b + d < p.

We start by recalling some facts about the structure of those simple modules. Then we consider the relation between characters of simple modules and Weyl characters. Finally, we will establish a sequence of propositions which yield the classification.

5.1 Alcoves

In this subsection, we describe the *p*-restricted alcoves of a root system of type A_2 . There are two such alcoves which we define to be the *fundamental alcove*

$$C_1 := \{ \lambda \in X_{\mathbb{R}} | \ (\lambda + \rho, \alpha_1^{\vee}) > 0, \ (\lambda + \rho, \alpha_2^{\vee}) > 0, \ (\lambda + \rho, (\alpha_1 + \alpha_2)^{\vee})$$

and the second alcove

$$C_2 := \{ \lambda \in X_{\mathbb{R}} | (\lambda + \rho, \alpha_1^{\vee}) < p, (\lambda + \rho, \alpha_2^{\vee}) < p, (\lambda + \rho, (\alpha_1 + \alpha_2)^{\vee}) > p \}.$$

Therefore, all *p*-restricted dominant weights belong to $\widehat{C}_1 \sqcup \widehat{C}_2$. Using the notation previously defined, we get

$$\widehat{C_1} \cap X^+ = \{(a,b) \in \mathbb{N}^2 \mid a+b \le p-2\}$$

and

$$\widehat{C_2} \cap X^+ = \{(a, b) \in \mathbb{N}^2 \mid a < p, \ b < p, \ a + b > p - 2\}.$$

Furthermore, we define the walls

$$\begin{split} F_{1,2} &:= \overline{C_1} \cap \overline{C_2} = \{ (x,y) \in [-1, p-1]^2 | \ x+y = p-2 \}, \\ F_{2,3} &:= \{ \lambda \in \overline{C_2} | \ (\lambda + \rho, \alpha_1^{\vee}) = p \}, \\ F_{2,3'} &:= \{ \lambda \in \overline{C_2} | \ (\lambda + \rho, \alpha_2^{\vee}) = p \}. \end{split}$$

Thus we get

$$F_{1,2} \cap X^+ = \{(a,b) \in \mathbb{N}^2 \mid a+b=p-2\},\$$

$$F_{2,3} \cap X^+ = \{(p-1,b) \in \mathbb{N}^2 \mid b \le p-1\},\$$

$$F_{2,3'} \cap X^+ = \{(a,p-1) \in \mathbb{N}^2 \mid a \le p-1\}$$

Remark 5.2. Following the labelling of the alcoves from [BDM15], we have $F_{2,i} = \overline{C_2} \cap \overline{C_i}$ for i = 3, 3'.

Let us illustrate these alcoves with a picture. The blue arrows form the root system. The black arrows are the fundamental weights, generating the weight lattice (in black). The region containing the dominant weights is coloured in green. The red triangles are the walls of the alcoves.



Figure 1: Alcoves for A_2 and p = 7.

5.2 Structure of Weyl modules and weights in irreducible modules

In this subsection, we compute the composition factors of Weyl modules with p-restricted highest weight and the multiplicity of certain weights in irreducible modules with p-restricted highest weight.

Lemma 5.3. Let $\lambda = (a, b) \in X^+$ be a p-restricted dominant weight.

- If $a + b (i.e. <math>\lambda \in \widehat{C_1}$), then $\Delta(\lambda) \cong L(\lambda)$.
- If a = p 1 or b = p 1 (i.e. $\lambda \in F_{2,3}$ or $\lambda \in F_{2,3'}$), then $\Delta(\lambda) \cong L(\lambda)$.
- Else (i.e. if $\lambda \in C_2$), then $\Delta(\lambda)$ admits exactly two composition factors, $L(\lambda)$ and $L(\lambda (a + b + 2 p)\rho)$.
- In particular, $T(\lambda)$ is irreducible if and only if $\lambda \in \widehat{C_1} \cup F_{2,3} \cup F_{2,3'}$.

Proof. We use the Jantzen *p*-sum formula (Proposition 1.49) to show this result.

- If $\lambda \in \widehat{C}_1$, then $\Delta(\lambda) \cong L(\lambda)$ by Lemma 1.55.
- If $\lambda \in \widehat{C}_2$, then

$$JSF(\lambda) = \chi(s_{\alpha_1+\alpha_2,p} \bullet \lambda) = \chi(\lambda - (a+b+2-p)\rho) = \chi(p-b-2, p-a-2).$$

If a = p - 1 or b = p - 1, then $\text{JSF}(\lambda) = 0$ because $s_{\alpha_1 + \alpha_2, p} \bullet \lambda \in D \setminus X^+$ and $\Delta(\lambda)$ is irreducible. Else, $\Delta(\lambda)$ admits the unique composition series

$$[L(\lambda), L(\lambda - (a+b+2-p)\rho)].$$

The last claim follows directly from Lemma 1.37.

We can also prove Lemma 5.3 using Proposition 1.54, see for example [Sch19, Lemma 2.1.4]. To simplify the notation we define the map

$$\Theta: \begin{array}{ccc} X & \to & \mathbb{Z} \\ (a,b) & \mapsto & a+b+2-p. \end{array}$$

Remark 5.4. By the proof of Lemma 5.3, we have

$$\operatorname{ch} L(\lambda) = \chi(\lambda) - \chi(\lambda - \Theta(\lambda)\rho)$$

for all dominant weights $\lambda \in \widehat{C}_2$.

Lemma 5.5. Let
$$\lambda \in X^+$$
 be p-restricted. Then $\lambda \in \widehat{C_2}$ if and only if $\Theta(\lambda) \ge 1$.

Proof. This is a direct consequence of the definitions of Θ and $\widehat{C_2}$.

Lemma 5.6 ([Tes88, 1.35]). Let $\lambda = (a, b) \in X^+$ be p-restricted with $a \neq 0$ and $b \neq 0$. Then

$$m_{L(\lambda)}(\lambda - \alpha_1 - \alpha_2) = \begin{cases} 1 & \text{if } a + b = p - 1 \\ 2 & \text{otherwise.} \end{cases}$$

Lemma 5.7. Let $\lambda = (a, 0) \in X^+$ with a < p. Then

$$m_{L(\lambda)}(\lambda - i\alpha_1 - j\alpha_2) = \begin{cases} 1 & \text{if } 0 \le j \le i \le a, \\ 0 & \text{else.} \end{cases}$$

Proof. First, observe that for a < p, we have $\Delta(\lambda) \cong L(\lambda)$ by Lemma 5.3. Recall that $\rho = \omega_1 + \omega_2$. Using Weyl's degree formula (Corollary 1.43) with $(\alpha_1, \alpha_1) = 2$, we have

$$\dim L(\lambda) = \frac{(\lambda + \rho, \alpha_1)(\lambda + \rho, \alpha_2)(\lambda + \rho, \alpha_1 + \alpha_2)}{(\rho, \alpha_1)(\rho, \alpha_2)(\rho, \alpha_1 + \alpha_2)} = \frac{(a+1) \cdot 1 \cdot (a+2)}{1 \cdot 1 \cdot 2} = \frac{(a+2)(a+1)}{2}.$$

Observe that $A = \{\lambda - i\alpha_1 - j\alpha_2\}_{0 \le j \le i \le a}$ is saturated with highest weight λ . Thus, by Proposition 1.23, we have $m_{L(\lambda)}(\nu) = m_{\Delta(\lambda)}(\nu) \ge 1$ for all $\nu \in A$. Moreover,

$$|A| = \sum_{i=0}^{a} \sum_{j=0}^{i} 1 = \sum_{i=0}^{a} (i+1) = \sum_{i=1}^{a+1} i = \frac{(a+2)(a+1)}{2} = \dim L(\lambda).$$

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Therefore, $m_{L(\lambda)}(\nu) = 1$ for all $\nu \in A$, as claimed.

5.3 The classification

In this subsection, we prove several propositions which yield the classification of multiplicity-free tensor products of simple $SL_3(k)$ -modules with *p*-restricted highest weight (Theorem 5.1). We start by stating a theorem from Stembridge ([Ste03, Theorem 1.1.A]) which classifies multiplicity-free tensor products of simple $SL_3(\mathbb{C})$ -modules.

Theorem 5.8. Let $(a,b), (c,d) \in X^+$ be dominant weights. Then $L_{\mathbb{C}}(a,b) \otimes L_{\mathbb{C}}(c,d)$ is multiplicity-free if and only if $a \cdot b \cdot c \cdot d = 0$.

Before moving on to the classification, recall that $L(a,b)^* \cong L(b,a)$ for all $a, b \in \mathbb{N}$ and observe that $L(\lambda) \otimes L(\mu)$ is multiplicity-free if and only if $(L(\lambda) \otimes L(\mu))^* \cong L(\lambda)^* \otimes L(\mu)^*$ is multiplicity-free. This allows us to treat several cases simultaneously.

5.3.1 $L(a,0) \otimes L(c,0)$

Proposition 5.9. Let $\lambda = (a, 0), \mu = (c, 0) \in X^+$ be *p*-restricted. Then $L(\lambda) \otimes L(\mu)$ is multiplicity-free if and only if a + c < p.

Proof. The "only if" direction is a direct consequence of Corollary 2.8.

Suppose that a + c < p. By Lemma 5.3, we have $\Delta(\lambda) \cong L(\lambda)$ and $\Delta(\mu) \cong L(\mu)$. If $\nu \leq \lambda + \mu$ is a dominant weight, then either $\nu = \lambda + \mu$ or $\nu \in \widehat{C_1}$. We have $\Delta(\lambda + \mu) \cong L(\lambda + \mu)$ by Lemma 5.3 and $\Delta(\nu) \cong L(\nu)$ for all $\nu \in \widehat{C_1}$ by Lemma 1.56. Thus we can apply Corollary 3.3. By Theorem 5.8, $L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu)$ is multiplicity-free, thus $L(\lambda) \otimes L(\mu)$ is multiplicity-free.

Remark 5.10. By duality, $L(0, b) \otimes L(0, d)$ is multiplicity-free if and only if b + d < p.

5.3.2 $L(a,0) \otimes L(0,d)$

Proposition 5.11. Let $\lambda = (a, 0), \mu = (0, d) \in X^+$ be p-restricted with $a, d \neq 0$. Then $L(\lambda) \otimes L(\mu)$ is multiplicity-free if and only if $a + d , or <math>(a, d) \in \{(p - 1, 1), (1, p - 1)\}$.

Proof. To simplify the notation, we set $M := L(\lambda) \otimes L(\mu)$.

By Lemma 5.3, $L(\lambda)$ and $L(\mu)$ are tilting modules. By Theorem 1.32, M is a tilting module. We know that $L(\lambda + \mu)$ is a composition factor of M. If $\lambda + \mu \in C_2$, then we conclude that M has multiplicity using Lemma 1.36.

If $a + d (i.e. if <math>\lambda + \mu \in \widehat{C_1}$), we apply Corollary 3.4 and Theorem 5.8 to conclude that M is multiplicity-free.

Suppose a = p - 1 and d > 1 (in particular, $\lambda + \mu \in F_{2,3}$). We use Lemmas 1.15 and 5.7 and Argument 1 to compute

$$m_M(\lambda + \mu) = 1,$$
 $m_M(\lambda + \mu - \alpha_1) = 1,$ $m_M(\lambda + \mu - \alpha_2) = 1,$

$$m_M(\lambda + \mu - \alpha_1 - \alpha_2) = 3.$$

Since $m_{L(\lambda+\mu)}(\lambda + \mu - \alpha_1) = m_{L(\lambda+\mu)}(\lambda + \mu - \alpha_2) = 1$ (Lemma 1.15), we conclude that $L(\lambda + \mu - \alpha_1)$ and $L(\lambda + \mu - \alpha_2)$ are not composition factors of M. Moreover, $m_{L(\lambda+\mu)}(\lambda + \mu - \alpha_1 - \alpha_2) = 2$ by Lemma 5.6, thus $L(\lambda + \mu - \alpha_1 - \alpha_2)$ is a composition factor of M. But $\lambda + \mu - \alpha_1 - \alpha_2 = (p - 2, d - 1) \in C_2$, so $T(\lambda + \mu - \alpha_1 - \alpha_2)$ is not irreducible by Lemma 5.3. By Lemma 1.36, we conclude that M has multiplicity. The case a > 1, d = p - 1 is symmetric.

Finally consider the case a = p - 1, d = 1 (in particular, $\lambda + \mu \in F_{2,3}$). Again, we have

$$m_M(\lambda + \mu) = 1,$$
 $m_M(\lambda + \mu - \alpha_1) = 1,$ $m_M(\lambda + \mu - \alpha_2) = 1.$

Therefore, $L(\lambda + \mu)$ is a composition factor, and all other composition factors have highest weight ν with $\nu \leq \lambda + \mu - \alpha_1 - \alpha_2$, $\nu \leq \lambda + \mu - 2\alpha_1$ or $\nu \leq \lambda + \mu - 2\alpha_2$. In particular, $\nu \in \widehat{C}_1$. By Lemma 5.3, this implies that for all dominant weights ν such that $L(\nu)$ is a composition factor of M, we have $L(\nu) \cong \Delta(\nu)$. Using Proposition 3.2 and Theorem 5.8, we conclude that M is multiplicity-free. The case a = 1, d = p - 1 is symmetric

5.3.3 $L(a, 0) \otimes L(c, d)$

Proposition 5.12. Let $\lambda = (a, 0), \mu = (c, d) \in X^+$ be *p*-restricted with $a, c, d \neq 0$ and $a + c + d (i.e. <math>\lambda + \mu \in \widehat{C_1}$). Then $L(\lambda) \otimes L(\mu)$ is multiplicity-free.

Proof. This is a direct consequence of Corollary 3.4 and Theorem 5.8.

Proposition 5.13. Let $\lambda = (a, 0), \mu = (c, d) \in X^+$ be p-restricted with $a, c, d \neq 0$, $a + c + d \geq p - 1$ and $L(\mu) \cong \Delta(\mu)$ (i.e. $\lambda + \mu \notin \widehat{C}_1$ and $\mu \in \widehat{C}_1 \cup F_{2,3} \cup F_{2,3'}$). Then $L(\lambda) \otimes L(\mu)$ has multiplicity.

Proof. To simplify the notation, we set $M := L(\lambda) \otimes L(\mu)$. By assumption $\Delta(\lambda) \cong L(\lambda)$ and $\Delta(\mu) \cong L(\mu)$, in particular these two modules are tilting modules. By Theorem 1.32, M is a tilting module.

If $a + c \ge p$, we conclude directly by Corollary 2.8 that M has multiplicity. So we can restrict our attention to the case a + c < p.

Observe that for d = p - 2, the condition $\Delta(\lambda) \cong L(\lambda)$ forces c = p - 1 (Lemma 5.3), so $a + c \ge p$ and M has multiplicity.

Suppose that d = p - 1. By Lemma 1.15 and Argument 1, we have

 $m_M(\lambda + \mu) = 1,$ $m_M(\lambda + \mu - \alpha_1) = 2.$

By Lemma 1.15, we have $m_{L(\lambda+\mu)}(\lambda+\mu-\alpha_1)=1$, so we deduce that $L(\lambda+\mu-\alpha_1)$ is a composition factor of M. But $\lambda+\mu-\alpha_1=(a+c-2,p)$ is not *p*-restricted, thus M has multiplicity by Corollary 2.7.

Suppose that $a + c and <math>d . By assumption, <math>\lambda + \mu \in C_2$, and $T(\lambda + \mu)$ is not irreducible by Lemma 5.3. Therefore, by Lemma 1.36, M has multiplicity.

Finally, consider the case a + c = p - 1, d . Again, we have

$$m_M(\lambda + \mu) = 1,$$
 $m_M(\lambda + \mu - \alpha_1) = 2,$

and $L(\lambda + \mu - \alpha_1)$ is a composition factor of M. Observe that $\lambda + \mu - \alpha_1 = (p-3, d+1) \in C_2$, so $T(\lambda + \mu - \alpha_1)$ is not irreducible. We conclude by Lemma 1.36 that M has multiplicity. \Box

Proposition 5.14. Let $\lambda = (a, 0), \mu = (c, d) \in X^+$ be p-restricted with $a, c, d \neq 0$. If $d + \min(a, c) \geq p$, then $L(\lambda) \otimes L(\mu)$ has multiplicity.

Proof. To simplify the notation, we set $M := L(\lambda) \otimes L(\mu)$.

If $a + c \ge p$, we conclude directly using Corollary 2.8. Thus we can assume a + c < p. We show inductively that $[M : L(\lambda + \mu - s\alpha_1)] = 1$ for $0 \le s \le p - d \le \min(a, c)$. For $0 \le s \le \min(a, c)$, using Lemma 1.15, we have

$$m_M(\lambda + \mu - s\alpha_1) = \sum_{i=0}^{s} m_{L(\lambda)}(\lambda - i\alpha_1) \cdot m_{L(\mu)}(\mu - (s - i)\alpha_1) = s + 1.$$

Moreover, if $0 \le i < s \le p - d$, then $\lambda + \mu - i\alpha_1 = (a + c - 2i, d + i)$ is *p*-restricted and by Lemma 1.15, we have

$$\sum_{i=0}^{s-1} m_{L(\lambda+\mu-i\alpha_1)}(\lambda+\mu-s\alpha_1) = s.$$

We know that $[M : L(\lambda + \mu)] = 1$, and combining the two previous equations, we conclude inductively that $[M : L(\lambda + \mu - s\alpha_1)] = 1$ for $0 \le s \le p-d$. In particular, $L(\lambda + \mu - (p-d)\alpha_1)$ is a composition factor of M. But $\lambda + \mu - (p-d)\alpha_1 = (a+c-2(p-d), p)$ is not *p*-restricted. We conclude by Corollary 2.7 that M has multiplicity.

Proposition 5.15. Let $\lambda = (a,0), \mu = (c,d) \in X^+$ be p-restricted with $a, c, d \neq 0$, c + d = p - 1 and a + c < p (in particular, $\mu \in \widehat{C_2}$). Then $L(\lambda) \otimes L(\mu)$ is multiplicity-free if and only if a < c + 2.

Proof. We set $M := L(\lambda) \otimes L(\mu)$.

By Remark 5.4, we have $\operatorname{ch} L(\mu) = \chi(\mu) - \chi(\mu - \rho)$ and $\operatorname{ch} L(\lambda) = \chi(\lambda)$. Therefore, by Proposition 1.47, we have

$$\operatorname{ch} M = \chi(\lambda)(\chi(\mu) - \chi(\mu - \rho)) = \sum_{\nu \in X} m_{\Delta(\lambda)}(\nu)(\chi(\mu + \nu) - \chi(\mu + \nu - \rho)).$$

Now we use Lemma 5.7 to rewrite this sum. We get

$$\begin{aligned} \operatorname{ch} M &= \sum_{\nu \in X} m_{\Delta(\lambda)}(\nu)(\chi(\mu + \nu) - \chi(\mu + \nu - \rho)) \\ &= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2} - \rho) \\ &= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2} - \rho) \\ &= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \sum_{i=1}^{a+1} \sum_{j=1}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) \\ &= \sum_{i=0}^{a} \chi(\mu + \lambda - i\alpha_{1}) - \sum_{j=1}^{a+1} \chi(\mu + \lambda - (a + 1)\alpha_{1} - j\alpha_{2}) \\ &= \sum_{i=0}^{a} \chi(\mu + \lambda - i\alpha_{1}) - \sum_{j=0}^{a} \chi(\mu + \lambda - (a + 1)\alpha_{1} - (a + 1 - j)\alpha_{2}) \\ &= \sum_{i=0}^{a} \chi(\mu + \lambda - i\alpha_{1}) - \chi(\mu + \lambda - (a + 1)\alpha_{1} - (a + 1 - i)\alpha_{2}) \end{aligned}$$

$$=\sum_{i=0}^{a}\chi(\mu+\lambda-i\alpha_{1})-\chi(\mu+\lambda-i\alpha_{1}-(a+1-i)\rho)$$
$$=\sum_{i=0}^{a}\chi(\mu+\lambda-i\alpha_{1})-\chi(\mu+\lambda-i\alpha_{1}-\Theta(\mu+\lambda-i\alpha_{1})\rho),$$
(1)

where in the last equality, we used that, for $i \in \{0, ..., a\}$, we have

$$\mu + \lambda - i\alpha_1 = (a + c - 2i, d + i),$$

so $\Theta(\mu + \lambda - i\alpha_1) = a + 1 - i$. Now we dissociate several cases.

First, suppose that $a \leq c$. In this case, for $i \in \{0, \ldots, a\}$ we have $0 \leq a + c - 2i < p$ and $0 < d + i \leq d + c < p$, therefore $\mu + \lambda - i\alpha_1$ is dominant and *p*-restricted. Moreover,

$$\Theta(\mu + \lambda - i\alpha_1) = a + 1 - i \ge 1.$$

By Lemma 5.5, we have $\mu + \lambda - i\alpha_1 \in \widehat{C}_2$ for all $i \in \{0, \ldots, a\}$. Using Remark 5.4, we get

$$\operatorname{ch} M = \sum_{i=0}^{a} \chi(\mu + \lambda - i\alpha_1) - \chi(\mu + \lambda - i\alpha_1 - \Theta(\mu + \lambda - i\alpha_1)\rho)$$
$$= \sum_{i=0}^{a} \operatorname{ch} L(\mu + \lambda - i\alpha_1).$$
(2)

Thus we conclude that M is multiplicity-free.

Suppose that a = c + 1. Using line (1), we have

$$\operatorname{ch} M = \sum_{i=0}^{c} \left(\chi(\mu + \lambda - i\alpha_1) - \chi(\mu + \lambda - i\alpha_1 - \Theta(\mu + \lambda - i\alpha_1)\rho) \right) + \chi(\mu + \lambda - a\alpha_1) - \chi(\mu + \lambda - a\alpha_1 - \Theta(\mu + \lambda - a\alpha_1)\rho).$$

In this case, for $i \in \{0, ..., c\}$ we have $0 \le a + c - 2i < p$ and $0 < d + i \le d + c < p$ so $\mu + \lambda - i\alpha_1$ is dominant and *p*-restricted. Moreover,

$$\Theta(\mu + \lambda - i\alpha_1) = a + 1 - i \ge 2,$$

so by Lemma 5.5, $\mu + \lambda - i\alpha_1 \in \widehat{C}_2$ for all $i \in \{0, \ldots, c\}$. By Remark 5.4, we have

$$\chi(\mu + \lambda - i\alpha_1) - \chi(\mu + \lambda - i\alpha_1 - \Theta(\mu + \lambda - i\alpha_1)\rho) = \operatorname{ch} L(\mu + \lambda - i\alpha_1).$$
(3)

Moreover,

$$\mu + \lambda - a\alpha_1 = (-1, d+a) \in D \setminus X^+,$$

so by Lemma 1.45 we have

$$\chi(\mu + \lambda - a\alpha_1) = 0. \tag{4}$$

Finally,

$$\mu + \lambda - a\alpha_1 - \Theta(\mu + \lambda - a\alpha_1)\rho = \mu + \lambda - a\alpha_1 - \rho = (-2, d + a - 1) = (-2, p - 1).$$

Using again Lemma 1.45, we get

$$\chi(-2, p-1) = -\chi(s_{\alpha_1} \bullet (-2, p-1)) = -\chi(0, p-2) = -\operatorname{ch} L(0, p-2)$$
$$= -\operatorname{ch} L(\lambda + \mu - a\alpha_1 - \alpha_2)$$
(5)

where in the last equality, we used that $\lambda + \mu - a\alpha_1 - \alpha_2 = (0, p - 2) \in X^+$ is *p*-restricted and Lemma 5.3. Combining lines (3), (4) and (5), we conclude that

$$\operatorname{ch} M = \sum_{i=0}^{c} \left(\chi(\mu + \lambda - i\alpha_1) - \chi(\mu + \lambda - i\alpha_1 - \Theta(\mu + \lambda - i\alpha_1)\rho) \right) + \chi(\mu + \lambda - a\alpha_1) - \chi(\mu + \lambda - a\alpha_1 - \Theta(\mu + \lambda - a\alpha_1)\rho) = \operatorname{ch} L(\lambda + \mu - a\alpha_1 - \alpha_2) + \sum_{i=0}^{c} \operatorname{ch} L(\mu + \lambda - i\alpha_1),$$
(6)

and M is multiplicity-free.

Finally, we consider the case $a \ge c+2$. We show that M has multiplicity. Using line (1), we have

$$\operatorname{ch} M = \sum_{i=0}^{c} \left(\chi(\mu + \lambda - i\alpha_1) - \chi(\mu + \lambda - i\alpha_1 - \Theta(\mu + \lambda - i\alpha_1)\rho) \right)$$
(7)

$$+\sum_{i=c+1}^{a} \Big(\chi(\mu+\lambda-i\alpha_1) - \chi(\mu+\lambda-i\alpha_1 - \Theta(\mu+\lambda-i\alpha_1)\rho) \Big).$$
(8)

By the same argument as in the previous case (see line (3)), we write (7) as

$$\sum_{i=0}^{c} \left(\chi(\mu + \lambda - i\alpha_1) - \chi(\mu + \lambda - i\alpha_1 - \Theta(\mu + \lambda - i\alpha_1)\rho) \right) = \sum_{i=0}^{c} \operatorname{ch} L(\mu + \lambda - i\alpha_1).$$
(9)

We rewrite the sum of line (8) by making the expression of the weights explicit. We get

$$\sum_{i=c+1}^{a} \left(\chi(\mu + \lambda - i\alpha_1) - \chi(\mu + \lambda - i\alpha_1 - \Theta(\mu + \lambda - i\alpha_1)\rho) \right)$$

$$= \sum_{i=c+1}^{a} \left(\chi(a + c - 2i, d + i) - \chi(c - i - 1, d + 2i - a - 1) \right)$$

$$= \sum_{i=1}^{a-c} \left(\chi(a - c - 2i, d + c + i) - \chi(-i - 1, d + 2c + 2i - a - 1) \right)$$

$$= \sum_{i=1}^{a-c} \chi(a - c - 2i, p - 1 + i)$$

$$- \sum_{i=1}^{a-c} \chi(-i - 1, d + 2c + 2i - a - 1).$$
(11)

We show that the sum in line (10) is equal to zero. Using Lemma 1.45, we have

$$\sum_{i=1}^{a-c} \chi(a-c-2i, p-1+i)$$

= $-\sum_{i=1}^{a-c} \chi(s_{\alpha_1} \bullet (a-c-2i, p-1+i))$
= $-\sum_{i=1}^{a-c} \chi(-a+c+2i-2, p-i+a-c)$

$$= -\sum_{i=1}^{a-c} \chi(-a+c+2(a-c+1-i)-2, p-(a-c+1-i)+a-c)$$
$$= -\sum_{i=1}^{a-c} \chi(a-c-2i, p-1+i)$$

Therefore,

$$\sum_{i=1}^{a-c} \chi(a-c-2i, p-1+i) = 0.$$
(12)

Now, let us study line (11). Using Lemma 1.45, we have

$$\sum_{i=1}^{a-c} \chi(-i-1, d+2c+2i-a-1) = -\sum_{i=1}^{a-c} \chi(s_{\alpha_1} \bullet (-i-1, d+2c+2i-a-1))$$
$$= -\sum_{i=1}^{a-c} \chi(i-1, d+2c+i-a-1) = -\sum_{i=0}^{a-c-1} \chi(i, p-1+c-a+i).$$

Observe that $(i, p - 1 + c - a + i) \in X^+$ is *p*-restricted for all $i \in \{0, \ldots, a - c - 1\}$. We dissociate the cases a - c even and odd.

If a - c is even, we get

$$\begin{split} &\sum_{i=0}^{a-c-1} \chi(i,p-1+c-a+i) = \sum_{i=0}^{\frac{a-c}{2}-1} \chi(a-c-1-i,p-2-i) + \chi(i,p-1+c-a+i) \\ &= \sum_{i=0}^{\frac{a-c}{2}-1} \chi(a-c-1-i,p-2-i) \\ &\quad + \chi((a-c-1-i,p-2-i) - \Theta(a-c-1-i,p-2-i)\rho). \end{split}$$

For $i \in \{0, ..., \frac{a-c}{2} - 1\}$, we have

$$(a - c - 1 - i) + (p - 2 - i) \ge a - c - 1 + p - (a - c - 2) = p + 1,$$

thus $(a - c - 1 - i, p - 2 - i) \in C_2$. This implies that

$$(a-c-1-i, p-2-i) - \Theta(a-c-1-i, p-2-i)\rho \in C_1,$$

and in particular, by Lemma 5.3 we have

$$\chi((a - c - 1 - i, p - 2 - i) - \Theta(a - c - 1 - i, p - 2 - i)\rho)$$

$$= \operatorname{ch} L((a - c - 1 - i, p - 2 - i)) - \Theta(a - c - 1 - i, p - 2 - i)\rho).$$

Therefore, by Remark 5.4, we have

$$\begin{split} &\sum_{i=0}^{\frac{a-c}{2}-1} \chi(a-c-1-i,p-2-i) + \chi((a-c-1-i,p-2-i) - \Theta(a-c-1-i,p-2-i)\rho) \\ &= \sum_{i=0}^{\frac{a-c}{2}-1} \operatorname{ch} L(a-c-1-i,p-2-i) \\ &\quad + 2\chi((a-c-1-i,p-2-i) - \Theta(a-c-1-i,p-2-i)\rho) \end{split}$$

$$= \sum_{i=0}^{\frac{a-c}{2}-1} \operatorname{ch} L(a-c-1-i, p-2-i) + 2 \operatorname{ch} L((a-c-1-i, p-2-i) - \Theta(a-c-1-i, p-2-i)\rho)$$

$$= \sum_{i=0}^{\frac{a-c}{2}-1} \operatorname{ch} L(a-c-1-i, p-2-i) + 2 \operatorname{ch} L(i, p-1+c-a+i)$$

$$= \sum_{i=1}^{\frac{a-c}{2}} \operatorname{ch} L(a-c-i, p-1-i) + 2 \operatorname{ch} L(i-1, p-2+c-a+i)$$

$$= \sum_{i=1}^{\frac{a-c}{2}} \operatorname{ch} L(\lambda + \mu - c\alpha_1 - i\rho) + 2 \operatorname{ch} L(\lambda + \mu - c\alpha_1 - (a+d-i+2-p)\rho). \quad (13)$$

Similarly, if a - c is odd, we get

$$\begin{split} &\sum_{i=0}^{a-c-1} \chi(i,p-1+c-a+i) \\ &= \sum_{i=0}^{\frac{a-c-1}{2}-1} \Big(\chi(i,p-1+c-a+i) + \chi(a-c-1-i,p-2-i) \Big) \\ &\quad + \chi(\frac{a-c-1}{2},p-1+\frac{c-a-1}{2}). \end{split}$$

Since $(\frac{a-c-1}{2}, p-1+\frac{c-a-1}{2}) \in C_1$, we get, as in the even case,

$$\begin{split} & \frac{a-c-1}{2} \sum_{i=0}^{a-c-1} \left(\chi(i,p-1+c-a+i) + \chi(a-c-1-i,p-2-i) \right) \\ & + \chi(\frac{a-c-1}{2},p-1+\frac{c-a-1}{2}) \\ & = \sum_{i=0}^{\frac{a-c-1}{2}-1} \operatorname{ch} L(a-c-1-i,p-2-i) + 2\operatorname{ch} L(i,p-1+c-a+i) \\ & + \operatorname{ch} L(\frac{a-c-1}{2},p-1+\frac{c-a-1}{2}) \\ & = \sum_{i=1}^{\frac{a-c-1}{2}} \operatorname{ch} L(\lambda+\mu-c\alpha_1-i\rho) + 2\operatorname{ch} L(\lambda+\mu-c\alpha_1-(a+d-i+2-p)\rho) \\ & + \operatorname{ch} L(\lambda+\mu-c\alpha_1-\frac{a-c+1}{2}\rho). \end{split}$$
(14)

Combining lines (9), (12) and (13) (respectively (14)), we find, if a - c is even:

$$\operatorname{ch} M = \sum_{i=0}^{c} \operatorname{ch} L(\mu + \lambda - i\alpha_{1})$$
$$+ \sum_{i=1}^{\frac{a-c}{2}} \operatorname{ch} L(\lambda + \mu - c\alpha_{1} - i\rho) + 2 \operatorname{ch} L(\lambda + \mu - c\alpha_{1} - (a + d - i + 2 - p)\rho)$$

and if a - c is odd:

$$\operatorname{ch} M = \operatorname{ch} L(\lambda + \mu - c\alpha_1 - \frac{a - c + 1}{2}\rho) + \sum_{i=0}^{c} \operatorname{ch} L(\mu + \lambda - i\alpha_1) + \sum_{i=1}^{\frac{a - c - 1}{2}} \operatorname{ch} L(\lambda + \mu - c\alpha_1 - i\rho) + 2 \operatorname{ch} L(\lambda + \mu - c\alpha_1 - (a + d - i + 2 - p)\rho)$$

In both cases, the second sum is non-empty since $a \ge c+2$ by assumption, and we deduce that M has multiplicity.

Proposition 5.16. Let $\lambda = (a,0), \mu = (c,d) \in X^+$ be p-restricted with $a, c, d \neq 0$, c+d > p-1, a+c < p and a+d < p (in particular, $\mu \in C_2$). Then $L(\lambda) \otimes L(\mu)$ is multiplicity-free.

Proof. We set $M := L(\lambda) \otimes L(\mu)$, and $m := \Theta(\mu) = c + d + 2 - p \ge 2$. Observe that our hypotheses imply a < c and a < d.

By Remark 5.4 and Lemma 5.3, we have ch $L(\mu) = \chi(\mu) - \chi(\mu - m\rho)$ and ch $L(\lambda) = \chi(\lambda)$. Therefore, by Proposition 1.47, we have

$$\operatorname{ch} M = \operatorname{ch} L(\lambda) \cdot \operatorname{ch} L(\mu) = \chi(\lambda) \cdot (\chi(\mu) - \chi(\mu - m\rho))$$
$$= \sum_{\nu \in X} m_{\Delta(\lambda)}(\nu)(\chi(\mu + \nu) - \chi(\mu + \nu - m\rho)).$$

We use Lemma 5.7 to rewrite this sum. We get

$$\operatorname{ch} M = \sum_{\nu \in X} m_{\Delta(\lambda)}(\nu)(\chi(\mu + \nu) - \chi(\mu + \nu - m\rho)) = \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2} - m\rho) = \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2} - m\rho) = \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \sum_{i=m}^{a+m} \sum_{j=m}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}).$$
(15)

For $0 \le j \le i \le a$, we have

$$\mu + \lambda - i\alpha_1 - j\alpha_2 = (a + c + j - 2i, d + i - 2j)$$
(16)

with

$$0 < c-a \leq c+a-2i \leq c+a+j-2i \leq c+a-i \leq c+a < p$$

and

$$0 < d-a \le d-j \le d+i-2j \le d+i \le d+a < p.$$

Therefore,

$$\mu + \lambda - i\alpha_1 - j\alpha_2 \in X^+ \text{ is } p\text{-restricted for all } 0 \le j \le i \le a.$$
(17)

Moreover, using line (16), we have

$$\Theta(\lambda + \mu - i\alpha_1 - j\alpha_2) = (a + c + j - 2i) + (d + i - 2j) + 2 - p = m + a - i - j, \quad (18)$$

and if $0 \le j \le i \le \min\{a, m-1\}$, then

$$\Theta(\lambda + \mu - i\alpha_1 - j\alpha_2) = m + a - i - j \ge 1.$$

Using line (17) and Lemma 5.5, we deduce that

$$\lambda + \mu - i\alpha_1 - j\alpha_2 \in \widehat{C}_2 \quad \text{for all } 0 \le j \le i \le \min\{a, m - 1\}.$$
(19)

We dissociate the cases a < m and $a \ge m$. If a < m, using line (15), we have

$$\operatorname{ch} M = \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \sum_{i=m}^{a+m} \sum_{j=m}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2})$$

$$= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \sum_{j=m}^{a+m} \sum_{i=j}^{a+m} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2})$$

$$= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \sum_{i=0}^{a+m} \sum_{j=i}^{a+m} \chi(\mu + \lambda - j\alpha_{1} - i\alpha_{2})$$

$$= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \sum_{i=0}^{a} \sum_{j=i}^{a} \chi(\mu + \lambda - j\alpha_{1} - i\alpha_{2} - m\rho)$$

$$= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - (a - j)\alpha_{1} - (a - i)\alpha_{2} - m\rho)$$

$$= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \chi(\mu + \lambda - (a - j)\alpha_{1} - (a - i)\alpha_{2} - m\rho)$$

$$= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2} - (m + a - i - j)\rho)$$

$$= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2} - \Theta(\mu + \lambda - i\alpha_{1} - j\alpha_{2})\rho),$$

where in the last equality, we use line (18). Since $a \leq m - 1$, we can use line (19) and Remark 5.4 to conclude that

$$\operatorname{ch} M = \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \chi(\mu + \lambda - i\alpha_1 - j\alpha_2 - \Theta(\mu + \lambda - i\alpha_1 - j\alpha_2)\rho)$$
$$= \sum_{i=0}^{a} \sum_{j=0}^{i} \operatorname{ch} L(\mu + \lambda - i\alpha_1 - j\alpha_2).$$
(20)
In particular, ${\cal M}$ is multiplicity-free.

If $a \ge m$, using line (15), we have

$$ch M = \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \sum_{i=m}^{a+m} \sum_{j=m}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2})$$

$$= \sum_{i=0}^{m-1} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) + \sum_{i=m}^{a} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2})$$

$$- \sum_{i=a+1}^{a+m} \sum_{j=m}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2})$$

$$= \sum_{i=0}^{m-1} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) + \sum_{i=m}^{a} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2})$$

$$- \sum_{j=a+1}^{a+m} \sum_{i=j}^{a+m} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \sum_{j=m}^{a} \sum_{i=a+1}^{a+m} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2})$$

$$= \sum_{i=0}^{m-1} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=i}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=i}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - i\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - i\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - i\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - i\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - i\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - i\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - i\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - i\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - i\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - i\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - i\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - i\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - i\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - i\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - i\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - i\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - i\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - i\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - i\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - i\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} -$$

Let us study line (21). We have

$$\begin{split} &\sum_{i=0}^{m-1} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \sum_{i=0}^{m-1} \sum_{j=i}^{m-1} \chi(\mu + \lambda - j\alpha_1 - i\alpha_2 - (a+1)\rho) \\ &= \sum_{i=0}^{m-1} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) \\ &- \sum_{i=0}^{m-1} \sum_{j=0}^{i} \chi(\mu + \lambda - (m-1-j)\alpha_1 - (m-1-i)\alpha_2 - (a+1)\rho) \\ &= \sum_{i=0}^{m-1} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \chi(\mu + \lambda - i\alpha_1 - j\alpha_2 - (a+m-i-j)\rho) \\ &= \sum_{i=0}^{m-1} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \chi(\mu + \lambda - i\alpha_1 - j\alpha_2 - \Theta(\mu + \lambda - i\alpha_1 - j\alpha_2)\rho), \end{split}$$

where in the last equality, we use line (18). Now we can use line (19) and Remark 5.4 to conclude that

$$\sum_{i=0}^{m-1} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \chi(\mu + \lambda - i\alpha_1 - j\alpha_2 - \Theta(\mu + \lambda - i\alpha_1 - j\alpha_2)\rho)$$
$$= \sum_{i=0}^{m-1} \sum_{j=0}^{i} \operatorname{ch} L(\mu + \lambda - i\alpha_1 - j\alpha_2).$$
(23)

Now we study line (22). We have

$$\begin{split} &\sum_{i=m}^{a} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \sum_{i=m}^{a} \sum_{j=0}^{m-1} \chi(\mu + \lambda - (j + a + 1)\alpha_{1} - i\alpha_{2}) \\ &= \sum_{i=m}^{a} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \sum_{i=m}^{a} \sum_{j=0}^{m-1} \chi(\mu + \lambda - ((m - 1 - j) + a + 1)\alpha_{1} - i\alpha_{2}) \\ &= \sum_{i=0}^{a-m} \sum_{j=0}^{m-1} \chi(\mu + \lambda - (i + m)\alpha_{1} - j\alpha_{2}) - \sum_{i=0}^{a-m} \sum_{j=0}^{m-1} \chi(\mu + \lambda - (m + a - j)\alpha_{1} - (i + m)\alpha_{2}) \\ &= \sum_{i=0}^{a-m} \sum_{j=0}^{m-1} \chi(\mu + \lambda - (i + m)\alpha_{1} - j\alpha_{2}) \\ &- \sum_{i=0}^{a-m} \sum_{j=0}^{m-1} \chi(\mu + \lambda - (m + a - j)\alpha_{1} - ((a - m - i) + m)\alpha_{2}) \\ &= \sum_{i=0}^{a-m} \sum_{j=0}^{m-1} \chi(\mu + \lambda - (i + m)\alpha_{1} - j\alpha_{2}) - \sum_{i=0}^{a-m} \sum_{j=0}^{m-1} \chi(\mu + \lambda - (m + a - j)\alpha_{1} - (a - i)\alpha_{2}) \\ &= \sum_{i=0}^{a-m} \sum_{j=0}^{m-1} \chi(\mu + \lambda - (i + m)\alpha_{1} - j\alpha_{2}) - \chi(\mu + \lambda - (m + a - j)\alpha_{1} - (a - i)\alpha_{2}) \\ &= \sum_{i=0}^{a-m} \sum_{j=0}^{m-1} \chi(\mu + \lambda - (i + m)\alpha_{1} - j\alpha_{2}) - \chi(\mu + \lambda - (i + m)\alpha_{1} - j\alpha_{2} - (a - i - j)\rho) \\ &= \sum_{i=0}^{a-m} \sum_{j=0}^{m-1} \chi(\mu + \lambda - (i + m)\alpha_{1} - j\alpha_{2}) - \chi(\mu + \lambda - (i + m)\alpha_{1} - j\alpha_{2}) - \chi(\mu + \lambda - (i + m)\alpha_{1} - j\alpha_{2}) \\ &- \chi(\mu + \lambda - (i + m)\alpha_{1} - j\alpha_{2} - \Theta(\mu + \lambda - (i + m)\alpha_{1} - j\alpha_{2})\rho), \end{split}$$

where in the last equality, we use line (18). For $0 \le i \le a - m$, $0 \le j \le m - 1$, we have

$$\Theta(\mu + \lambda - (i+m)\alpha_1 - j\alpha_2) = a - i - j \ge 1.$$

Therefore, using line (17) and Lemma 5.5, we deduce that $\mu + \lambda - (i+m)\alpha_1 - j\alpha_2 \in \widehat{C}_2$ for all $0 \le i \le a - m$, $0 \le j \le m - 1$. By Remark 5.4, we conclude that

$$\sum_{i=0}^{a-m} \sum_{j=0}^{m-1} \chi(\mu + \lambda - (i+m)\alpha_1 - j\alpha_2) - \chi(\mu + \lambda - (i+m)\alpha_1 - j\alpha_2 - \Theta(\mu + \lambda - (i+m)\alpha_1 - j\alpha_2)\rho)$$
$$= \sum_{i=0}^{a-m} \sum_{j=0}^{m-1} \operatorname{ch} L(\mu + \lambda - (i+m)\alpha_1 - j\alpha_2).$$
(24)

We conclude by combining lines (23) and (24) to obtain

$$\operatorname{ch} M = \sum_{i=0}^{m-1} \sum_{j=0}^{i} \operatorname{ch} L(\mu + \lambda - i\alpha_1 - j\alpha_2) + \sum_{i=0}^{a-m} \sum_{j=0}^{m-1} \operatorname{ch} L(\mu + \lambda - (i+m)\alpha_1 - j\alpha_2)$$
$$= \sum_{i=0}^{m-1} \sum_{j=0}^{i} \operatorname{ch} L(\mu + \lambda - i\alpha_1 - j\alpha_2) + \sum_{i=m}^{a} \sum_{j=0}^{m-1} \operatorname{ch} L(\mu + \lambda - i\alpha_1 - j\alpha_2).$$
(25)

In particular, M is multiplicity-free.

Remark 5.17. By duality, for $\lambda = (0, b), \mu = (c, d) \in X^+$ with 0 < b, c, d < p, the tensor product $L(\lambda) \otimes L(\mu)$ is multiplicity-free if and only if one of the following holds:

- (1) b + c + d ,
- (2) c + d = p 1, b + d < p and b < d + 2 or
- (3) c + d > p 1, b + c < p and b + d < p.

At this step, let us make the following observation, which will be useful in the next proposition.

Corollary 5.18. Let $\lambda = (a, 0), \mu = (c, d) \in X^+$ be p-restricted with $a, c, d \neq 0, c+d \geq p-1$, a + c < p and a + d < p (in particular $\mu \in C_2$). Let $m := \Theta(\mu)$. Then

$$\operatorname{ch}(L(\lambda) \otimes L(\mu)) = \sum_{i=0}^{a} \sum_{j=0}^{i} \operatorname{ch} L(\lambda + \mu - i\alpha_1 - j\alpha_2) - \sum_{i=m}^{a} \sum_{j=m}^{i} \operatorname{ch} L(\lambda + \mu - i\alpha_1 - j\alpha_2).$$

Proof.

- If c + d > p 1 and a < m, then the second double sum is empty and we find the same result as in line (20) (in the proof of Proposition 5.16).
- If c + d > p 1 and $a \ge m$, the result is a consequence of line (25) in the proof of Proposition 5.16 since we have

$$\operatorname{ch}(L(\lambda) \otimes L(\mu)) = \sum_{i=0}^{m-1} \sum_{j=0}^{i} \operatorname{ch} L(\mu + \lambda - i\alpha_1 - j\alpha_2) + \sum_{i=m}^{a} \sum_{j=0}^{m-1} \operatorname{ch} L(\mu + \lambda - i\alpha_1 - j\alpha_2)$$
$$= \sum_{i=0}^{a} \sum_{j=0}^{i} \operatorname{ch} L(\lambda + \mu - i\alpha_1 - j\alpha_2) - \sum_{i=m}^{a} \sum_{j=m}^{i} \operatorname{ch} L(\lambda + \mu - i\alpha_1 - j\alpha_2).$$

• If c + d = p - 1, the condition a + d < p is equivalent to $a \le c$. Moreover, in this case $m = \Theta(\mu) = 1$. Therefore, by line (2) in the proof of Proposition 5.15, we get

$$\operatorname{ch}(L(\lambda) \otimes L(\mu)) = \sum_{i=0}^{a} \operatorname{ch} L(\mu + \lambda - i\alpha_{1})$$
$$= \sum_{i=0}^{a} \sum_{j=0}^{i} \operatorname{ch} L(\lambda + \mu - i\alpha_{1} - j\alpha_{2}) - \sum_{i=1}^{a} \sum_{j=1}^{i} \operatorname{ch} L(\lambda + \mu - i\alpha_{1} - j\alpha_{2})$$
$$= \sum_{i=0}^{a} \sum_{j=0}^{i} \operatorname{ch} L(\lambda + \mu - i\alpha_{1} - j\alpha_{2}) - \sum_{i=m}^{a} \sum_{j=m}^{i} \operatorname{ch} L(\lambda + \mu - i\alpha_{1} - j\alpha_{2})$$

Hence in all cases we are done.

5.3.4 $L(a,b) \otimes L(c,d)$

Lemma 5.19. Let $\lambda = (a,b) \in X^+$ be p-restricted with $a,b \neq 0$ and a+b < p-1 (i.e. $\lambda \in \widehat{C_1}$). Then

$$ch(L(a,0) \otimes L(0,b)) - ch(L(a-1,0) \otimes L(0,b-1)) = ch L(a,b).$$

Proof. First, note that $(a - 1, 0) + (0, b - 1) = \lambda - \rho$. By Lemma 5.3, we have $L(a - 1, 0) \cong \Delta(a - 1, 0), L(a, 0) \cong \Delta(a, 0), L(0, b - 1) \cong \Delta(0, b - 1), L(0, b) \cong \Delta(0, b)$ and $L(\lambda) \cong \Delta(\lambda)$. By Proposition 1.47 and Lemma 5.7, and using Lemma 1.45 in the seventh equality below, we get

$$\begin{split} \operatorname{ch}(L(a,0) \otimes L(0,b)) &- \operatorname{ch}(L(a-1,0) \otimes L(0,b-1)) = \chi(a,0)\chi(0,b) - \chi(a-1,0)\chi(0,b-1) \\ &= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\lambda - i\alpha_{1} - j\alpha_{2}) - \sum_{i=1}^{a-1} \sum_{j=0}^{i} \chi(\lambda - \rho - i\alpha_{1} - j\alpha_{2}) \\ &= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\lambda - i\alpha_{1} - j\alpha_{2}) - \sum_{i=1}^{a} \sum_{j=1}^{i} \chi(\lambda - i\alpha_{1} - j\alpha_{2}) \\ &= \sum_{i=0}^{a} \chi(\lambda - i\alpha_{1}) \\ &= \chi(\lambda) + \sum_{i=1}^{a} \chi(\lambda - i\alpha_{1}) \\ &= \chi(\lambda) + \frac{1}{2} \sum_{i=1}^{a} \chi(\lambda - i\alpha_{1}) + \chi(\lambda - (a+1-i)\alpha_{1}) \\ &= \chi(\lambda) + \frac{1}{2} \sum_{i=1}^{a} \chi(\lambda - i\alpha_{1}) - \chi(s_{\alpha_{1}} \cdot (\lambda - (a+1-i)\alpha_{1})) \\ &= \chi(\lambda) + \frac{1}{2} \sum_{i=1}^{a} \chi(\lambda - i\alpha_{1}) - \chi(\lambda - i\alpha_{1}) \\ &= \chi(\lambda) \\ &= \chi(\lambda) \\ &= \chi(\lambda) \\ &= \chi(\lambda) \end{split}$$

Proposition 5.20. Let $\lambda = (a, b)$, $\mu = (c, d) \in X^+$ be two p-restricted weights with $a, b, c, d \neq 0$. Then $L(\lambda) \otimes L(\mu)$ is multiplicity-free if and only if one of the following holds:

- (1) a + b = p 1, c + d , <math>a + c + d < p and b + c + d < p or
- (2) a + b , <math>c + d = p 1, a + b + c < p and a + b + d < p.

Proof. We set $M := L(\lambda) \otimes L(\mu)$.

If $a + c \ge p$ or $b + d \ge p$, then M has multiplicity by Corollary 2.8. Therefore, we can assume a + c < p and b + d < p. In particular, it cannot happen that $a + b \ge p - 1$ and c + d > p - 1.

Suppose that $a + b \neq p - 1$ and $c + d \neq p - 1$. Using Argument 1, we show that $L(\lambda + \mu - \alpha_1 - \alpha_2)$ has multiplicity at least 2 in M. Using Lemmas 1.15 and 5.6, we have

$$m_M(\lambda + \mu) = 1,$$
 $m_M(\lambda + \mu - \alpha_1) = 2,$ $m_M(\lambda + \mu - \alpha_2) = 2,$

 $m_M(\lambda + \mu - \alpha_1 - \alpha_2) = 6.$

Using Lemmas 1.15 and 5.6 again, we have

$$m_{L(\lambda+\mu)}(\lambda+\mu-\alpha_1) = 1, \qquad \qquad m_{L(\lambda+\mu)}(\lambda+\mu-\alpha_2) = 1.$$

Therefore, $L(\lambda + \mu - \alpha_1)$ and $L(\lambda + \mu - \alpha_2)$ are composition factors of M. Using Lemmas 1.15 and 5.6 a third time, we get

$$m_{L(\lambda+\mu)}(\lambda+\mu-\alpha_1-\alpha_2) \le 2, \qquad m_{L(\lambda+\mu-\alpha_1)}(\lambda+\mu-\alpha_1-\alpha_2) = 1,$$

 $m_{L(\lambda+\mu-\alpha_2)}(\lambda+\mu-\alpha_1-\alpha_2)=1.$

We conclude that $L(\lambda + \mu - \alpha_1 - \alpha_2)$ is a composition factor with multiplicity

$$[M: L(\lambda + \mu - \alpha_1 - \alpha_2)] \ge 6 - 1 - 1 - 2 = 2$$

Therefore, $M = L(\lambda) \otimes L(\mu)$ has multiplicity.

Suppose that a + b = c + d = p - 1. Since we assume a + c < p and b + d < p, we only need to consider the case $\lambda + \mu = (p - 1, p - 1)$. Using the same reasoning as in the previous case, we deduce that $L(\lambda + \mu - \alpha_1)$ is a composition factor of M. But $\lambda + \mu - \alpha_1 = (p - 3, p)$ is not *p*-restricted. Therefore, by Corollary 2.7, M has multiplicity.

Let us consider the case a + b , <math>c + d = p - 1, $a + b + d \ge p$ (the other remaining cases where we claim that M has multiplicity are symmetric). Note that $b + c + d \ge p$, therefore $b + d + \min(a, c) \ge p$. We show inductively that $[M : L(\lambda + \mu - s\alpha_1)] = 1$ for $0 \le s \le p - b - d \le \min(a, c)$. For $0 \le s \le \min(a, c)$, using Lemma 1.15, we have:

$$m_M(\lambda + \mu - s\alpha_1) = \sum_{i=0}^{s} m_{L(\lambda)}(\lambda - i\alpha_1) \cdot m_{L(\mu)}(\mu - (s - i)\alpha_1) = s + 1.$$

Moreover, if i < p-b-d, then $\lambda + \mu - i\alpha_1 = (a+c-2i, b+d+i)$ is p-restricted and we have

$$\sum_{i=0}^{s-1} m_{L(\lambda+\mu-i\alpha_1)}(\lambda+\mu-s\alpha_1) = s.$$

We know that $[M : L(\lambda + \mu)] = 1$, and combining the two previous equations, we conclude inductively that $[M : L(\lambda + \mu - s\alpha_1)] = 1$ for $0 \le s \le p - b - d$. In particular, $L(\lambda + \mu - (p - b - d)\alpha_1)$ is a composition factor of M. But $\lambda + \mu - (p - b - d)\alpha_1 = (a + c - 2(p - b - d), p)$ is not *p*-restricted. We conclude by Corollary 2.7 that M has multiplicity.

Now we consider the cases where we claim that M is multiplicity-free. Up to the reordering of the weights and up to symmetry, we can suppose a + b , <math>c + d = p - 1 and $c \ge d$. Thus the conditions in the statement of the proposition are equivalent to a single one: a + b + c < p. In particular we have $a < d \le c$ and $b < d \le c$. We show that M is multiplicity-free by showing the following equality of characters:

$$\operatorname{ch} M = \sum_{i=0}^{a} \sum_{j=0}^{b} \operatorname{ch} L(\lambda + \mu - i\alpha_1 - j\alpha_2).$$

Let $0 \le i \le a, \ 0 \le j \le b$. We claim that $\lambda + \mu - i\alpha_1 - j\alpha_2 \in \widehat{C_2}$. We have

$$\lambda + \mu - i\alpha_1 - j\alpha_2 = (a + c - 2i + j, b + d - 2j + i)$$

with

$$\begin{aligned} a + c - 2i + j &\leq a + c + b < p, \\ b + d - 2j + i &\leq b + d + a \leq a + b + c < p \quad \text{ and} \\ (a + c - 2i + j) + (b + d - 2j + i) &= a + b + c + d - i - j \geq c + d = p - 1 \end{aligned}$$

Thus $\lambda + \mu - i\alpha_1 - j\alpha_2 \in \widehat{C_2}$ for all $0 \le i \le a, \ 0 \le j \le b$. By Lemma 5.19, we have

 $\operatorname{ch} M = \operatorname{ch} L(a, b) \operatorname{ch} L(c, d)$

$$= (ch(L(a,0) \otimes L(0,b)) - ch(L(a-1,0) \otimes L(0,b-1))) ch L(c,d)$$

= ch(L(a,0) \otimes L(0,b) \otimes L(c,d)) - ch(L(a-1,0) \otimes L(0,b-1) \otimes L(c,d))
= ch L(a,0) ch(L(0,b) \otimes L(c,d)) - ch L(a-1,0) ch(L(0,b-1) \otimes L(c,d)).

Since c + d = p - 1, b + d < p and $b \le d$, we can use line (2) in the proof of Lemma 5.15 to express $ch(L(0,b) \otimes L(c,d))$ and $ch(L(0,b-1) \otimes L(c,d))$. We set $\lambda_b := (0,b)$ and $\lambda_{b-1} := (0,b-1)$. We get

$$\operatorname{ch}(L(0,b)\otimes L(c,d)) = \sum_{k=0}^{b} \operatorname{ch} L(\lambda_{b} + \mu - k\alpha_{2}),$$

and

$$\operatorname{ch}(L(0,b-1)\otimes L(c,d)) = \sum_{k=0}^{b-1} \operatorname{ch} L(\lambda_{b-1} + \mu - k\alpha_2)$$

Observe that $\lambda_b + \mu - k\alpha_2 = (c+k, b+d-2k)$. For $0 \le k \le b$, we have

$$0 < c \le c + k \le c + b < p - 1,$$

$$0 < d - b \le b + d - 2k \le b + d < p - 1 \quad \text{and}$$

$$(c + k) + (b + d - 2k) \ge c + d = p - 1.$$
(26)

Therefore $\lambda_b + \mu - k\alpha_2 \in C_2$ for all $k \in \{0, \dots, b\}$. Moreover, we have

$$a + c + k \le a + b + c$$

$$a + b + d - 2k \le a + b + d \le a + b + c < p.$$
(28)

Similarly, for all $0 \le k \le b-1$, we have $\lambda_{b-1} + \mu - k\alpha_2 = (c+k, b-1+d-2k) \in C_2$ and

$$(c+k) + (b-1+d-2k) \ge p-1,$$
(29)

$$(a-1) + c + k \le a + b + c - 2 < p$$
 and (30)

$$(a-1) + b - 1 + d - 2k \le a + b + d - 2 \le a + b + c - 2 < p.$$
(31)

Let $m := \Theta(\lambda_b + \mu) = b + c + d + 2 - p = b + 1$. Observe that $\Theta(\lambda_b + \mu - k\alpha_2) = m - k$ and $\Theta(\lambda_{b-1} + \mu - k\alpha_2) = m - k - 1$. Lines (26), (27), (28), (29), (30) and (31) allow us to use Corollary 5.18 in the fourth equality below, to get

$$\begin{split} \operatorname{ch} M &= \operatorname{ch} L(a,0) \operatorname{ch} (L(0,b) \otimes L(c,d)) - \operatorname{ch} L(a-1,0) \operatorname{ch} (L(0,b-1) \otimes L(c,d)) \\ &= \operatorname{ch} L(a,0) \left(\sum_{k=0}^{b} \operatorname{ch} L(\lambda_{b} + \mu - k\alpha_{2}) \right) - \operatorname{ch} L(a-1,0) \left(\sum_{k=0}^{b-1} \operatorname{ch} L(\lambda_{b-1} + \mu - k\alpha_{2}) \right) \\ &= \left(\sum_{k=0}^{b} \operatorname{ch} L(a,0) \operatorname{ch} L(\lambda_{b} + \mu - k\alpha_{2}) \right) - \left(\sum_{k=0}^{b-1} \operatorname{ch} L(a-1,0) \operatorname{ch} L(\lambda_{b-1} + \mu - k\alpha_{2}) \right) \\ &= \sum_{k=0}^{b} \sum_{i=0}^{a} \sum_{j=0}^{i} \operatorname{ch} L(\lambda + \mu - i\alpha_{1} - (j+k)\alpha_{2}) \\ &- \sum_{k=0}^{b} \sum_{i=0}^{a} \sum_{j=0}^{i} \operatorname{ch} L(\lambda + \mu - i\alpha_{1} - (j+k)\alpha_{2}) \\ &- \sum_{k=0}^{b-1} \sum_{i=0}^{a-1} \sum_{j=0}^{i} \operatorname{ch} L(\lambda + \mu - \rho - i\alpha_{1} - (j+k)\alpha_{2}) \\ &+ \sum_{k=0}^{b-1} \sum_{i=0}^{a-1} \sum_{j=m-k-1}^{i} \operatorname{ch} L(\lambda + \mu - \rho - i\alpha_{1} - (j+k)\alpha_{2}) \\ &= \sum_{k=0}^{b} \sum_{i=0}^{a} \sum_{j=0}^{i} \operatorname{ch} L(\lambda + \mu - i\alpha_{1} - (j+k)\alpha_{2}) \\ &- \sum_{k=0}^{b} \sum_{i=0}^{a} \sum_{j=0}^{i} \operatorname{ch} L(\lambda + \mu - i\alpha_{1} - (j+k)\alpha_{2}) \\ &- \sum_{k=0}^{b-1} \sum_{i=1}^{a} \sum_{j=i-k}^{i} \operatorname{ch} L(\lambda + \mu - i\alpha_{1} - (j+k)\alpha_{2}) \\ &- \sum_{k=0}^{b-1} \sum_{i=1}^{a} \sum_{j=i-k}^{i} \operatorname{ch} L(\lambda + \mu - i\alpha_{1} - (j+k)\alpha_{2}) \\ &+ \sum_{k=0}^{b-1} \sum_{i=1}^{a} \sum_{j=i-k}^{i} \operatorname{ch} L(\lambda + \mu - i\alpha_{1} - (j+k)\alpha_{2}) \\ &+ \sum_{k=0}^{b-1} \sum_{i=1}^{a} \sum_{j=i-k}^{i} \operatorname{ch} L(\lambda + \mu - i\alpha_{1} - (j+k)\alpha_{2}) \\ &+ \sum_{k=0}^{b-1} \sum_{i=1}^{a} \sum_{j=i-k}^{i} \operatorname{ch} L(\lambda + \mu - i\alpha_{1} - (j+k)\alpha_{2}). \end{split}$$

To simplify the notation, we set $Z(i, j) := \operatorname{ch} L(\lambda + \mu - i\alpha_1 - j\alpha_2)$. Thus we get

$$\operatorname{ch} M = \sum_{k=0}^{b} \sum_{i=0}^{a} \sum_{j=0}^{i} Z(i, j+k) - \sum_{k=0}^{b} \sum_{i=m-k}^{a} \sum_{j=m-k}^{i} Z(i, j+k)$$
$$- \sum_{k=0}^{b-1} \sum_{i=1}^{a} \sum_{j=1}^{i} Z(i, j+k) + \sum_{k=0}^{b-1} \sum_{i=m-k}^{a} \sum_{j=m-k}^{i} Z(i, j+k)$$
$$= \sum_{k=0}^{b-1} Z(0, k) + \sum_{k=0}^{b-1} \sum_{i=1}^{a} Z(i, k) + \sum_{i=0}^{a} \sum_{j=0}^{i} Z(i, j+b) - \sum_{i=m-b}^{a} \sum_{j=m-b}^{i} Z(i, j+b)$$

$$=\sum_{k=0}^{b-1}\sum_{i=0}^{a}Z(i,k) + \sum_{i=0}^{a}Z(i,b) + \sum_{i=1}^{a}\sum_{j=1}^{i}Z(i,j+b) - \sum_{i=m-b}^{a}\sum_{j=m-b}^{i}Z(i,j+b)$$
$$=\sum_{k=0}^{b}\sum_{i=0}^{a}Z(i,k) + \sum_{i=1}^{a}\sum_{j=1}^{i}Z(i,j+b) - \sum_{i=m-b}^{a}\sum_{j=m-b}^{i}Z(i,j+b).$$

Finally, observe that m - b = 1. This allows us to conclude that

$$\operatorname{ch} M = \sum_{k=0}^{b} \sum_{i=0}^{a} Z(i,k) + \sum_{i=1}^{a} \sum_{j=1}^{i} Z(i,j+b) - \sum_{i=1}^{a} \sum_{j=1}^{i} Z(i,j+b)$$
$$= \sum_{k=0}^{b} \sum_{i=0}^{a} Z(i,k)$$
$$= \sum_{k=0}^{b} \sum_{i=0}^{a} \operatorname{ch} L(\lambda + \mu - i\alpha_{1} - k\alpha_{2}).$$
(32)

In particular, $M = L(\lambda) \otimes L(\mu)$ is multiplicity-free.

Now Theorem 5.1 follows from the previous sequence of propositions.

5.4 Decomposition of multiplicity-free tensor products

Since multiplicity-free tensor products of simple modules are completely reducible (Lemma 3.5), we can specify the structure of those modules. For some of the cases, we still need to compute the decomposition in characteristic zero, which we do in the next lemma.

Lemma 5.21. Let $\lambda = (a, 0), \mu = (c, d) \in X^+$. Then

$$\chi(\lambda)\chi(\mu) = \sum_{i=0}^{a} \sum_{j=\max\{0,i-c\}}^{\min\{i,d\}} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2).$$

Moreover, all weights appearing in the sum are dominant.

Proof. By Proposition 1.47 and Lemma 5.7, we get

$$\chi(\lambda)\chi(\mu) = \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2)$$

=
$$\sum_{i=0}^{a} \sum_{j=\max\{0, i-c\}}^{\min\{i,d\}} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2)$$
(33)

$$+\sum_{i=0}^{a}\sum_{j=0}^{\min\{i-c-1,d\}}\chi(\lambda+\mu-i\alpha_{1}-j\alpha_{2})$$
(34)

$$+\sum_{i=0}^{a}\sum_{j=d+1}^{i}\chi(\lambda+\mu-i\alpha_{1}-j\alpha_{2}).$$
(35)

Observe that

 $\lambda + \mu - i\alpha_1 - j\alpha_2 = (a + c - 2i + j, d + i - 2j).$ (36)

In particular, for $i \leq a$ and $j \geq i - c$, we have

$$a + c - 2i + j \ge a - i \ge 0,$$

and for $i \ge 0$ and $j \le \min\{i, d\}$, we have

$$d + i - 2j \ge (d - j) + (i - j) \ge 0.$$

Therefore, all weights appearing in line 33 are dominant.

We show that lines (34) and (35) are equal to zero. We start by line (35). Using Lemma 1.45, and line (36) in the second equality below, we have

$$\sum_{j=d+1}^{i} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2) = -\sum_{j=d+1}^{i} \chi(s_{\alpha_2} \cdot (\lambda + \mu - i\alpha_1 - j\alpha_2))$$

= $-\sum_{j=d+1}^{i} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2 - (d + i - 2j + 1)\alpha_2)$
= $-\sum_{j=d+1}^{i} \chi(\lambda + \mu - i\alpha_1 - (d + i - j + 1)\alpha_2)$
= $-\sum_{j=d+1}^{i} \chi(\lambda + \mu - i\alpha_1 - (d + i - (d + 1 + i - j) + 1)\alpha_2)$
= $-\sum_{j=d+1}^{i} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2).$

Therefore,

$$\sum_{j=d+1}^{i} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2) = 0$$

and

$$\sum_{i=0}^{a} \sum_{j=d+1}^{i} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2) = 0.$$

Now we work on line (34). If i < c + 1, then i - c - 1 < 0. Thus we have

$$\sum_{i=0}^{a} \sum_{j=0}^{\min\{i-c-1,d\}} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2) = \sum_{i=c+1}^{a} \sum_{j=0}^{\min\{i-c-1,d\}} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2)$$
$$= \sum_{j=0}^{\min\{a-c-1,d\}} \sum_{i=c+1+j}^{a} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2).$$
(37)

We fix j and work on the second sum in line (37). Using Lemma 1.45, and line (36) in the second equality below, we get

$$\sum_{i=c+1+j}^{a} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2)$$
$$= -\sum_{i=c+1+j}^{a} \chi(s_{\alpha_1} \bullet (\lambda + \mu - i\alpha_1 - j\alpha_2))$$

$$= -\sum_{i=c+1+j}^{a} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2 - (a+c-2i+j+1)\alpha_1)$$

$$= -\sum_{i=c+1+j}^{a} \chi(\lambda + \mu - (a+c-i+j+1)\alpha_1 - j\alpha_2)$$

$$= -\sum_{i=c+1+j}^{a} \chi(\lambda + \mu - (a+c-(a+c+1+j-i)+j+1)\alpha_1 - j\alpha_2)$$

$$= -\sum_{i=c+1+j}^{a} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2).$$

Therefore,

$$\sum_{i=c+1+j}^{a} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2) = 0$$

 \mathbf{SO}

$$\sum_{j=0}^{\min\{a-c-1,d\}} \sum_{i=c+1+j}^{a} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2) = 0$$

and

$$\sum_{i=0}^{a} \sum_{j=0}^{\min\{i-c-1,d\}} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2) = 0.$$

Remark 5.22. Let (a,b), $(c,d) \in X^+$ be dominant weights. Lemma 5.21 shows that $L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu)$ is multiplicity-free if $a \cdot b \cdot c \cdot d = 0$. Moreover, using Argument 1 like in the proof of Proposition 5.20, one can prove that $[L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu) : L_{\mathbb{C}}(\lambda + \mu - \alpha_1 - \alpha_2)] = 2$ if $a \cdot b \cdot c \cdot d \neq 0$. These two facts provide a proof of Theorem 5.8.

Remark 5.23. The computations in Lemma 5.21 can be done using the Littlewood-Richardson rule, see [LR34, Theorem III] or [FH04, Proposition 15.25] for this specific case.

Corollary 5.24. Let $\lambda = (a, b), \mu = (c, d) \in X^+$ be non-zero and p-restricted such that $L(\lambda) \otimes L(\mu)$ is multiplicity-free. We have the following decompositions:

(1) If b = d = 0, a + c < p and $a \le c$, then

$$L(\lambda) \otimes L(\mu) \cong \bigoplus_{i=0}^{a} L(\mu + \lambda - i\alpha_1).$$

(2) If b = c = 0, $a \le d$ and a + d or <math>(a, d) = (1, p - 1), then

$$L(\lambda) \otimes L(\mu) \cong \bigoplus_{i=0}^{a} L(\mu + \lambda - i\rho).$$

(3) If b = 0 and a + c + d , then

$$L(\lambda) \otimes L(\mu) \cong \bigoplus_{i=0}^{a} \bigoplus_{j=\max\{0,i-c\}}^{\min\{i,d\}} L(\mu + \lambda - i\alpha_1 - j\alpha_2).$$

(4) If b = 0, c + d = p - 1, a + c < p and $a \le c$, then

$$L(\lambda) \otimes L(\mu) \cong \bigoplus_{i=0}^{a} L(\mu + \lambda - i\alpha_1).$$

(5) If b = 0, c + d = p - 1, a + c < p and a = c + 1, then

$$L(\lambda) \otimes L(\mu) \cong L(\lambda + \mu - a\alpha_1 - \alpha_2) \oplus \bigoplus_{i=0}^{c} L(\mu + \lambda - i\alpha_1).$$

(6) If b = 0, c + d > p - 1, a + c < p and a + d < p, then

$$L(\lambda) \otimes L(\mu) \cong \bigoplus_{i=0}^{a} \bigoplus_{j=0}^{\min\{i,\Theta(\mu)-1\}} L(\mu + \lambda - i\alpha_1 - j\alpha_2).$$

(7) If $a \cdot b \cdot c \cdot d \neq 0$, a + b , <math>c + d = p - 1, a + b + c < p and a + b + d < p, then

$$L(\lambda) \otimes L(\mu) \cong \bigoplus_{k=0}^{b} \bigoplus_{i=0}^{a} L(\lambda + \mu - i\alpha_1 - k\alpha_2).$$

Proof.

- (1) This follows from Lemma 5.21 and Proposition 5.9.
- (2) This follows from Lemma 5.21 and Proposition 5.11.
- (3) This follows from Lemma 5.21 and Proposition 5.12.
- (4) This follows from line (2) in the proof of Proposition 5.15.
- (5) This follows from line (6) in the proof of Proposition 5.15.
- (6) This follows from lines (20) and (25) in the proof of Proposition 5.16.
- (7) This follows from line (32) in the proof of Proposition 5.20.

6 Sp₄

In this section, we establish a number of results for the classification of multiplicity-free tensor products of simple Sp₄-modules with *p*-restricted highest weight. We fix $G = \text{Sp}_4(k)$ with root system Φ of type $B_2 = C_2$ and $\Pi = \{\alpha_1, \alpha_2\}$ a base of Φ with $\frac{(\alpha_1, \alpha_1)}{(\alpha_2, \alpha_2)} = 2$. With respect to this base, we have $\rho = \frac{3}{2}\alpha_1 + 2\alpha_2$. We assume $p \ge 5$, so that there exist weights inside the alcoves. Since the Coxeter number of a root system Φ of type B_2 is h = 4, we have in particular $p \ge h$. For $\lambda = x\omega_1 + y\omega_2 \in X$, we write $\lambda = (x, y)$. In particular, we have $\alpha_1 = (2, -2), \ \alpha_2 = (-1, 2)$ and $\rho = (1, 1)$.

In this section, we will also use Euclidean coordinates several times. We fix an orthogonal basis (ϵ_1, ϵ_2) of \mathbb{R}^2 with $\epsilon_1 = \omega_1 = \alpha_1 + \alpha_2$ and $\epsilon_2 = \alpha_2 = 2\omega_2 - \omega_1$. With respect to this basis, we have $\alpha_1 = (1, -1)$, $\alpha_2 = (0, 1)$, $\alpha_1 + \alpha_2 = (1, 0)$, $\alpha_1 + 2\alpha_2 = (1, 1)$ and $\rho = (\frac{3}{2}, \frac{1}{2})$. Since the notation (,) might be confusing, we will always explicitly state when we use Euclidean coordinates. If it is not mentioned, then it means that we use coordinates with respect to the fundamental weights basis. In particular, coordinates with respect to the fundamental weights are used in all statements of propositions in the classification (section 6.3).

6.1 Alcoves

In this subsection, we describe the four *p*-restricted alcoves of a root system of type B_2 . We start by defining a numeration of those alcoves.

Definition 6.1. We set

$$\begin{split} C_1 &:= \{\lambda \in X_{\mathbb{R}} | \ (\lambda + \rho, \alpha_1^{\vee}) > 0, (\lambda + \rho, \alpha_2^{\vee}) > 0, (\lambda + \rho, (\alpha_1 + \alpha_2)^{\vee}) 0, (\lambda + \rho, (\alpha_1 + \alpha_2)^{\vee}) > p, (\lambda + \rho, (\alpha_1 + 2\alpha_2)^{\vee}) p \}, \\ C_4 &:= \{\lambda \in X_{\mathbb{R}} | \ (\lambda + \rho, \alpha_1^{\vee}) < p, (\lambda + \rho, \alpha_2^{\vee}) < p, (\lambda + \rho, (\alpha_1 + \alpha_2)^{\vee}) > 2p \}. \end{split}$$

We called C_1 the fundamental alcove, C_2 the second alcove, C_3 the third alcove and C_4 the fourth alcove.

We also set

$$F_{i,i+1} := \overline{C_i} \cap \overline{C_{i+1}}$$

for i = 1, 2, 3, i.e. $F_{i,i+1}$ is the wall between the alcove C_i and C_{i+1} . Finally, we set

$$\begin{split} F_{3,5} &:= \{\lambda \in \overline{C_3} | \ (\lambda + \rho, \alpha_2^{\vee}) = p\}, \\ F_{4,6} &:= \{\lambda \in \overline{C_4} | \ (\lambda + \rho, \alpha_1^{\vee}) = p\} \ and \\ F_{4,7} &:= \{\lambda \in \overline{C_4} | \ (\lambda + \rho, \alpha_2^{\vee}) = p\}. \end{split}$$

We set $C_5 := s_{\alpha_2,p} \bullet C_3$, $C_6 := s_{\alpha_1,p} \bullet C_4$ and $C_7 := s_{\alpha_2,p} \bullet C_4$. Then $F_{i,j} = \overline{C_i} \cap \overline{C_j}$ for $(i,j) \in \{(3,5), (4,6), (4,7)\}.$

Using coordinates with respect to the fundamental weights, we have $\widehat{C_1} \cap X^+ = \{(a, b) \in \mathbb{N}^2 \mid 2a + b \le p - 3\},\$ $\widehat{C_2} \cap X^+ = \{(a, b) \in \mathbb{N}^2 \mid 2a + b > p - 3, \ a + b \le p - 2\},\$ $\widehat{C_3} \cap X^+ = \{(a, b) \in \mathbb{N}^2 \mid a + b > p - 2, \ 2a + b \le 2p - 3, \ b \le p - 1\},\$ $\widehat{C_4} \cap X^+ = \{(a, b) \in \mathbb{N}^2 \mid 2a + b > 2p - 3, \ a \le p - 1, \ b \le p - 1\}.$ Using Euclidean coordinates, we have

$$\begin{split} X &= \{(a,b) \in (\frac{1}{2}\mathbb{Z})^2 | \ a+b \in \mathbb{Z}\},\\ \widehat{C_1} \cap X^+ &= \{(a,b) \in (\frac{1}{2}\mathbb{N})^2 | \ a+b \in \mathbb{N}, \ a \leq \frac{p-3}{2}, \ b \leq a\},\\ \widehat{C_2} \cap X^+ &= \{(a,b) \in (\frac{1}{2}\mathbb{N})^2 | \ a+b \in \mathbb{N}, \ a > \frac{p-3}{2}, \ a+b \leq p-2\},\\ \widehat{C_3} \cap X^+ &= \{(a,b) \in (\frac{1}{2}\mathbb{N})^2 | \ a+b \in \mathbb{N}, \ a+b > p-2, \ a \leq p-\frac{3}{2}, \ b \leq \frac{p-1}{2}\},\\ \widehat{C_4} \cap X^+ &= \{(a,b) \in (\frac{1}{2}\mathbb{N})^2 | \ a+b \in \mathbb{N}, \ a > p-\frac{3}{2}, \ b \leq \frac{p-1}{2}, \ a-b \leq p-1\}. \end{split}$$

Moreover, a weight $(a,b) \in X$ is dominant if and only if $0 \le b \le a$, and $(a,b) \in D$ if and only if $-\frac{1}{2} \le b \le a+1$.

Let us illustrate these alcoves with a picture. The blue arrows form the root system. The black arrows are the fundamental weights, generating the weight lattice (in black). The region containing the dominant weights is coloured in green. The red triangles are the walls of the alcoves.



Figure 2: Alcoves for B_2 and p = 7.

Remark 6.2. In Euclidean coordinates, we have

- (1) $s_{\alpha_1} \bullet (a, b) = (b 1, a + 1),$
- (2) $s_{\alpha_2} \bullet (a, b) = (a, -b 1),$
- (3) $s_{\alpha_1+\alpha_2,p} \bullet (a,b) = (p-3-a,b),$
- (4) $s_{\alpha_1+2\alpha_2,p} \bullet (a,b) = (a-m,b-m)$ with m = (a+b+2-p).

Lemma 6.3. Let $\lambda = (a, b), \mu = (c, d) \in X$ with $\mu \leq \lambda$. Then $c + d \leq a + b$.

Proof. Let $s, t \in \mathbb{N}$ be such that $\mu = \lambda - s\alpha_1 - t\alpha_2$. We have $\mu = (a - 2s + t, b - 2t + 2s) =: (c, d)$ and $c + d = a + b - t \le a + b$.

6.2 Structure of Weyl modules and weights in irreducible modules

In this subsection, we compute the composition factors of Weyl modules with *p*-restricted highest weight and the multiplicity of certain weights in irreducible modules with *p*-restricted highest weight.

Lemma 6.4. Let $\lambda = (a, b) \in X^+$ be a p-restricted dominant weight.

- If $\lambda \in C_1$, then $\Delta(\lambda) \cong L(\lambda)$.
- If $\lambda \in C_i$ for $i \in \{2, 3, 4\}$, then $\Delta(\lambda)$ admits the unique composition series $[L(\lambda), L(\mu)]$ where $\mu \in C_{i-1}$ is the unique weight linked to λ .
- If $\lambda \in F_{i,j}$ with $(i,j) \neq (4,7)$ then $\Delta(\lambda) \cong L(\lambda)$.
- If $\lambda \in F_{4,7} \setminus F_{4,6}$, then $\Delta(\lambda)$ admits the unique composition series $[L(\lambda), L(\mu)]$ where $\mu \in F_{3,5}$ is the unique weight linked to λ .

In particular, $T(\lambda)$ is irreducible if and only if $\lambda \in \widehat{C_1} \cup F_{2,3} \cup F_{3,4} \cup F_{3,5} \cup F_{4,6}$.

Proof. We prove this lemma using Jantzen *p*-sum formula (Proposition 1.49). For a weight $\lambda \in C_i$, we write λ_j for the unique weight in $(W_p \bullet \lambda) \cap C_j$. We will use Remark 1.52 and Lemma 1.45 several times without further reference.

- If $\lambda \in \widehat{C}_1$ then $\Delta(\lambda) \cong L(\lambda)$ by Lemma 1.55.
- If $\lambda \in C_2$, then

$$JSF(\lambda) = \chi(s_{\alpha_1 + \alpha_2, p} \bullet \lambda) = \chi(\lambda_1) = \operatorname{ch} L(\lambda_1).$$

Therefore, $\chi(\lambda_2) = \operatorname{ch} L(\lambda_2) + \operatorname{ch} L(\lambda_1)$ and $\operatorname{ch} L(\lambda_2) = \chi(\lambda_2) - \chi(\lambda_1)$.

• If $\lambda \in C_3$, then

$$JSF(\lambda) = \chi(s_{\alpha_1 + \alpha_2, p} \bullet \lambda) + \chi(s_{\alpha_1 + 2\alpha_2, p} \bullet \lambda) = -\chi(\lambda_1) + \chi(\lambda_2) = \operatorname{ch} L(\lambda_2).$$

Therefore, $\chi(\lambda_3) = \operatorname{ch} L(\lambda_3) + \operatorname{ch} L(\lambda_2)$ and $\operatorname{ch} L(\lambda_3) = \chi(\lambda_3) - \chi(\lambda_2) + \chi(\lambda_1)$.

• If $\lambda \in C_4$, then

$$JSF(\lambda) = \chi(s_{\alpha_1+\alpha_2,p} \bullet \lambda) + \chi(s_{\alpha_1+\alpha_2,2p} \bullet \lambda) + \chi(s_{\alpha_1+2\alpha_2,p} \bullet \lambda) = \chi(\lambda_1) + \chi(\lambda_3) - \chi(\lambda_2)$$
$$= ch L(\lambda_3).$$

• If $\lambda \in F_{2,3}$, then

$$JSF(\lambda) = \chi(s_{\alpha_1 + \alpha_2, p} \bullet \lambda) = 0$$

because $s_{\alpha_1+\alpha_2,p} \bullet \lambda \in D \setminus X^+$. Thus $\Delta(\lambda)$ is irreducible.

• If $\lambda \in F_{3,4}$, then

$$JSF(\lambda) = \chi(s_{\alpha_1 + \alpha_2, p} \bullet \lambda) + \chi(s_{\alpha_1 + 2\alpha_2, p} \bullet \lambda) = 0$$

because $s_{\alpha_1+2\alpha_2,p} \bullet \lambda \in D \setminus X^+$ and $s_{\alpha_1}s_{\alpha_1+\alpha_2,p} \bullet \lambda \in D \setminus X^+$. Thus $\Delta(\lambda)$ is irreducible.

• If $\lambda \in F_{3,5}$, then

$$JSF(\lambda) = \chi(s_{\alpha_1+\alpha_2,p} \bullet \lambda) + \chi(s_{\alpha_1+2\alpha_2,p} \bullet \lambda) = -\chi(s_{\alpha_1}s_{\alpha_1+\alpha_2,p} \bullet \lambda) + \chi(s_{\alpha_1+2\alpha_2,p} \bullet \lambda) = 0$$

because $s_{\alpha_1}s_{\alpha_1+\alpha_2,p} \bullet \lambda = s_{\alpha_1+2\alpha_2,p} \bullet \lambda$. Thus $\Delta(\lambda)$ is irreducible.

• If $\lambda \in F_{4,6}$, then

$$JSF(\lambda) = \chi(s_{\alpha_1+\alpha_2,p} \bullet \lambda) + \chi(s_{\alpha_1+\alpha_2,2p} \bullet \lambda) + \chi(s_{\alpha_1+2\alpha_2,p} \bullet \lambda)$$
$$= \chi(s_{\alpha_1+\alpha_2,2p} \bullet \lambda) - \chi(s_{\alpha_2}s_{\alpha_1+2\alpha_2,p} \bullet \lambda) = 0$$

where we used that $s_{\alpha_1+\alpha_2}s_{\alpha_1+\alpha_2,p} \bullet \lambda \in D \setminus X^+$ in the second equality and $s_{\alpha_1+\alpha_2,2p} \bullet \lambda = s_{\alpha_2}s_{\alpha_1+2\alpha_2,p} \bullet \lambda$ in the last equality. Thus $\Delta(\lambda)$ is irreducible.

• If $\lambda \in F_{4,7} \setminus F_{4,6}$, then

$$JSF(\lambda) = \chi(s_{\alpha_1 + \alpha_2, p} \bullet \lambda) + \chi(s_{\alpha_1 + \alpha_2, 2p} \bullet \lambda) + \chi(s_{\alpha_1 + 2\alpha_2, p} \bullet \lambda)$$
$$= \chi(s_{\alpha_1}s_{\alpha_1 + \alpha_2}s_{\alpha_1 + \alpha_2, p} \bullet \lambda) + \chi(s_{\alpha_1 + \alpha_2, 2p} \bullet \lambda) - \chi(s_{\alpha_2}s_{\alpha_1 + 2\alpha_2, p} \bullet \lambda)$$
$$= \chi(s_{\alpha_1 + \alpha_2, 2p} \bullet \lambda)$$

where we used that $s_{\alpha_1}s_{\alpha_1+\alpha_2}s_{\alpha_1+\alpha_2,p} \bullet \lambda = s_{\alpha_2}s_{\alpha_1+2\alpha_2,p} \bullet \lambda$ in the last equality. Now observe that $s_{\alpha_1+\alpha_2,2p} \bullet \lambda \in F_{3,5}$, so $\chi(s_{\alpha_1+\alpha_2,2p} \bullet \lambda) = \operatorname{ch} L(s_{\alpha_1+\alpha_2,2p} \bullet \lambda)$ by one of the previous cases and we are done.

The last claim follows directly from Lemma 1.37.

Combining Lemma 6.4 and Remark 6.2, we get the following remark.

Remark 6.5. In this remark, we use Euclidean coordinates. Let $\lambda = (a, b) \in X^+$. We set m := a + b + 2 - p.

- If $\lambda \in \widehat{C_2}$, then $\operatorname{ch} L(\lambda) = \chi(a, b) \chi(p 3 a, b)$.
- If $\lambda \in C_3$, then $\chi(a, b) = \operatorname{ch} L(a, b) + \operatorname{ch} L(a m, b m)$.

Lemma 6.6 ([Tes88, 1.35].). Let $\lambda = (a, b) \in X^+$ be p-restricted with $a \neq 0$ and $b \neq 0$. Then

$$m_{L(\lambda)}(\lambda - \alpha_1 - \alpha_2) = \begin{cases} 1 & \text{if } 2a + b + 2 \equiv 0 \mod p, \\ 2 & \text{otherwise.} \end{cases}$$

$\lambda = (a, b)$	$a \ge 1, b \ge 2$	$a \ge 1, b = 1$	$a = 0, b \ge 2$	$a \ge 1, b = 0$	a = 0, b = 1
$m_{\Delta(\lambda)}(\lambda)$	1	1	1	1	1
$m_{\Delta(\lambda)}(\lambda - \alpha_1)$	1	1	0	1	0
$m_{\Delta(\lambda)}(\lambda - \alpha_2)$	1	1	1	0	1
$m_{\Delta(\lambda)}(\lambda - \alpha_1 - \alpha_2)$	2	2	1	1	1
$m_{\Delta(\lambda)}(\lambda - 2\alpha_2)$	1	0	1	0	0
$m_{\Delta(\lambda)}(\lambda - \alpha_1 - 2\alpha_2)$	3	2	2	1	1

Lemma 6.7. Let $\lambda = (a, b) \in X^+$. We have the following weight multiplicities:

Moreover, for $\lambda = (a, 0) \in X^+$ with $a \ge 2$, we have

$$m_{\Delta(a,0)}(\lambda - \alpha_1 - \alpha_2) = 1, \qquad m_{\Delta(a,0)}(\lambda - \alpha_1 - 2\alpha_2) = 1, \qquad m_{\Delta(a,0)}(\lambda - 2\alpha_1 - \alpha_2) = 1,$$
$$m_{\Delta(a,0)}(\lambda - 2\alpha_1 - 2\alpha_2) = 2.$$

Proof. This follows from Proposition 1.25 and tables of dominant weights ([Bre85]).

Lemma 6.8. Let $\lambda = (a, b) \in X^+$ with $1 \le a < p$ and $2 \le b < p$. Then

$$m_{L(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) = \begin{cases} 2 & \text{if } 2a + b + 2 \equiv 0 \mod p, \\ 2 & \text{if } a + b = p - 1, \\ 3 & \text{otherwise.} \end{cases}$$

Proof.

• If 2a + b + 2 = p, then $\lambda \in C_2$ and by Lemma 6.4, $\Delta(\lambda)$ admits the unique composition series $[L(\lambda), L(\lambda - \alpha_1 - \alpha_2)]$. By Lemma 6.7, we get

$$m_{L(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) = m_{\Delta(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) - m_{\Delta(\lambda - \alpha_1 - \alpha_2)}(\lambda - \alpha_1 - 2\alpha_2) = 3 - 1 = 2.$$

• If 2a + b + 2 = 2p, then $\lambda \in C_4 \cup (F_{4,7} \setminus F_{4,6})$ and by Lemma 6.4, $\Delta(\lambda)$ admits the unique composition series $[L(\lambda), L(\lambda - \alpha_1 - \alpha_2)]$. By Lemmas 6.7 and 1.15, we get

$$m_{L(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) = m_{\Delta(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) - m_{L(\lambda - \alpha_1 - \alpha_2)}(\lambda - \alpha_1 - 2\alpha_2) = 3 - 1 = 2.$$

• If a + b = p - 1, then $\lambda \in C_3$ and by Lemma 6.4, $\Delta(\lambda)$ admits the unique composition series $[L(\lambda), L(\lambda - \alpha_1 - 2\alpha_2)]$. By Lemma 6.7, we get

$$m_{L(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) = m_{\Delta(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) - m_{\Delta(\lambda - \alpha_1 - 2\alpha_2)}(\lambda - \alpha_1 - 2\alpha_2) = 3 - 1 = 2.$$

• In all other cases, either $\Delta(\lambda) \cong L(\lambda)$ or all composition factors of $\Delta(\lambda)$ except $L(\lambda)$ have highest weight $\nu \not\geq \lambda - \alpha_1 - 2\alpha_2$, thus

$$m_{L(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) = m_{\Delta(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) = 3$$

by Lemma 6.7.

Lemma 6.9. Let $\lambda = (a, 1) \in X^+$ with $1 \le a < p$. Then

$$m_{L(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) = \begin{cases} 1 & \text{if } 2a + 3 = p \\ 2 & \text{otherwise.} \end{cases}$$

Proof.

• If 2a + 3 = p, then $\lambda \in C_2$ and by Lemma 6.4, $\Delta(\lambda)$ admits the unique composition series $[L(\lambda), L(\lambda - \alpha_1 - \alpha_2)]$. By Lemma 6.7, we get

$$m_{L(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) = m_{\Delta(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) - m_{\Delta(\lambda - \alpha_1 - \alpha_2)}(\lambda - \alpha_1 - 2\alpha_2) = 2 - 1 = 1.$$

• In all other cases, either $\Delta(\lambda) \cong L(\lambda)$ or all composition factors of $\Delta(\lambda)$ except $L(\lambda)$ have highest weight $\nu \geq \lambda - \alpha_1 - 2\alpha_2$, thus

$$m_{L(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) = m_{\Delta(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) = 2$$

by Lemma 6.7.

Lemma 6.10. Let $2 \le a < p$ and set $W := \Delta(1,0)$. Then $\chi(a,0) = \operatorname{ch} S^{a}(W) - \operatorname{ch} S^{a-2}(W)$ and $\Delta(0,a) \cong S^a(\Delta(0,1))$. In particular $\chi(0,a) = \operatorname{ch} S^a(\Delta(0,1))$.

Proof. For $\chi(a,0)$, see [FH04, §19.5]. For $\Delta(0,a)$, observe that $S^a(\Delta(0,1))$ admits a maximal By Lemma 1.22, there exists a non-zero morphism vector of weight (0, a). $\theta: \Delta(0,a) \to S^a(\Delta(0,1))$. By Lemma 6.4, $\Delta(0,a)$ is irreducible, thus θ is injective. To show that θ is surjective and thus an isomorphism, it is enough to show that $\dim \Delta(0, a)$ = $\dim S^a(\Delta(0,1)).$ Using usual multilinear algebra, we have dim $S^{a}(\Delta(0,1)) = {\binom{a+3}{3}}$. On the other hand, using Weyl's degree formula (Corollary 1.43), with the choice $(\alpha_2, \alpha_2) = 2$ (and hence $(\alpha_1, \alpha_1) = 4$), we have

$$\dim \Delta(0,a) = \frac{(\lambda+\rho,\alpha_1)(\lambda+\rho,\alpha_2)(\lambda+\rho,\alpha_1+\alpha_2)(\lambda+\rho,\alpha_1+2\alpha_2)}{(\rho,\alpha_1)(\rho,\alpha_2)(\rho,\alpha_1+\alpha_2)(\rho,\alpha_1+2\alpha_2)}$$
$$= \frac{2(a+1)(a+3)(2a+4)}{2\cdot 1\cdot 3\cdot 4} = \binom{a+3}{3},$$

hence we are done.

Notation 6.11. Let $\lambda = x\epsilon_1 + y\epsilon_2 \in X$. We define its 1-norm

$$\|\lambda\| := |x| + |y|$$

and its ∞ -norm

$$|\lambda|_{\infty} := \max\{|x|, |y|\}.$$

Lemma 6.12. Let $a \in \mathbb{N}$ and $\lambda \in X$. We have

$$m_{\Delta(a,0)}(\lambda) = \begin{cases} \lfloor \frac{a - \|\lambda\|}{2} \rfloor + 1 & \text{if } \|\lambda\| \in \mathbb{N} \text{ and } \|\lambda\| \le a, \\ 0 & \text{else.} \end{cases}$$

Proof. In this proof, we use Euclidean coordinates. We set $W := \Delta(1,0)$. Let $k \in \mathbb{N}$. We compute $m_{S^kW}(\lambda)$ and then use Lemma 6.10. Let $i := \lfloor \frac{k - \|\lambda\|}{2} \rfloor$. We claim that

$$m_{S^kW}(\lambda) = \begin{cases} \binom{2+i}{2} & \text{if } \|\lambda\| \in \mathbb{N} \text{ and } \|\lambda\| \le k, \\ 0 & \text{else.} \end{cases}$$

We know that dim W = 5 and W admits the five weights $0, \pm \epsilon_1, \pm \epsilon_2$, all of them with multiplicity 1. We fix $(v_{-2}, v_{-1}, v_0, v_1, v_2)$ an ordered basis of W with v_0 a weight vector associated to 0 and $v_{\pm i}$ a weight vector associated to $\pm \epsilon_i$ for i = 1, 2. By multilinear algebra, $S^k W$ admits the basis $\{v_{i_1} \otimes \cdots \otimes v_{i_k}\}_{i_1 \leq i_2 \leq \dots \leq i_k}$ and there exists a natural bijection between

this basis and the set $\{(x, y, z, s, t) \in \mathbb{N}^5 \mid x + y + z + s + t = k\}$ where x is the number of v_{i_j} with $i_j = -2$, y is the number of v_{i_j} with $i_j = -1$, and so on. Moreover, under this bijection, a basis vector associated to (x, y, z, s, t) is a weight vector with weight $(t - x)\epsilon_2 + (s - y)\epsilon_1$. Thus, to compute $m_{S^kW}(\lambda)$, we will count the number of 5-tuples (x, y, z, s, t) with associated weight λ . Let us write $\lambda = f\epsilon_1 + g\epsilon_2$. Then (x, y, z, s, t) is associated to λ if t - x = g and s - y = f. In particular, we need $\|\lambda\| \in \mathbb{N}$ and $\|\lambda\| \leq k$. In this case, we get

$$m_{S^kW}(\lambda) = |\{(x, y, z, s, t) \in \mathbb{N}^5 \mid x + y + z + s + t = k, \ t - x = g, \ s - y = f\}|.$$

Up to symmetry we can assume that both f and g are non-negative. We get

$$\begin{split} m_{S^kW}(\lambda) &= |\{(x, y, z, s, t) \in \mathbb{N}^5 \mid x + y + z + s + t = k, \ t - x = g, \ s - y = f\}| \\ &= |\{(x, y, z) \in \mathbb{N}^3 \mid 2x + g + 2y + f + z = k\}| \\ &= |\{(x, y, z) \in \mathbb{N}^3 \mid 2x + 2y + z = k - \|\lambda\|\}|. \end{split}$$

If $k - \|\lambda\|$ is odd, then so is z, and we have

$$\begin{aligned} |\{(x, y, z) \in \mathbb{N}^3 \mid 2x + 2y + z &= k - \|\lambda\|\}| &= |\{(x, y, z) \in \mathbb{N}^3 \mid 2x + 2y + (z - 1) = k - \|\lambda\| - 1\}|. \\ \text{In this case, let } z' &:= (z - 1)/2. \text{ We get} \\ m_{S^kW}(\lambda) &= |\{(x, y, z') \in \mathbb{N}^3 \mid 2x + 2y + 2z' = k - \|\lambda\| - 1\}| \\ &= |\{(x, y, z') \in \mathbb{N}^3 \mid x + y + z' = (k - \|\lambda\| - 1)/2\}| \end{aligned}$$

$$= |\{(x, y, z) \in \mathbb{N} \mid x + y + z = (k - ||\lambda|| - 1)/2\}$$
$$= \binom{2 + (k - ||\lambda|| - 1)/2}{2}$$
$$= \binom{2 + \lfloor (k - ||\lambda||)/2 \rfloor}{2}$$

where the third equality is a well-known combinatorial result, see for example [MN08, 3.3]. Now suppose that $k - ||\lambda||$ and z are even, and set z' := z/2. Using the same reasoning we get

$$m_{S^{k}W}(\lambda) = |\{(x, y, z') \in \mathbb{N}^{3} | 2x + 2y + 2z' = k - ||\lambda||\}|$$

= $|\{(x, y, z') \in \mathbb{N}^{3} | x + y + z' = (k - ||\lambda||)/2\}|$
= $\binom{2 + (k - ||\lambda||)/2}{2}$
= $\binom{2 + \lfloor (k - ||\lambda||)/2 \rfloor}{2}$.

Hence in both cases we are done with our claim.

Now let $i := \lfloor \frac{a - \|\lambda\|}{2} \rfloor$ and observe that $\lfloor \frac{(a-2) - \|\lambda\|}{2} \rfloor = i - 1$. For $\|\lambda\| \in \mathbb{N}$ and $\|\lambda\| \le a - 2$, by Lemma 6.10, we have

$$m_{\Delta(a,0)}(\lambda) = m_{S^{a}W}(\lambda) - m_{S^{a-2}W}(\lambda) = \binom{2+i}{2} - \binom{1+i}{2} = \frac{(2+i)!}{2!i!} - \frac{(1+i)!}{2!(i-1)!}$$
$$= \frac{(2+i)! - i(1+i)!}{2!i!} = \frac{2(i+1)!}{2!i!} = i + 1 = \lfloor \frac{a - \|\lambda\|}{2} \rfloor + 1.$$

If $\|\lambda\| \in \{a, a-1\}$, then i = 0 and

$$m_{\Delta(a,0)}(\lambda) = m_{S^aW}(\lambda) = \begin{pmatrix} 2\\ 2 \end{pmatrix} = 1.$$

If $\|\lambda\| \notin \mathbb{N}$ or if $\|\lambda\| > a$, we conclude that $m_{\Delta(a,0)}(\lambda) = 0$.

Corollary 6.13. Let $a \in \mathbb{N}$. We set

$$Y(i,j) := \lfloor \frac{a - |i| - |j|}{2} \rfloor + 1.$$

We have

$$\chi(a,0) = \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} Y(i,j) e^{j\epsilon_1 + i\epsilon_2} + \sum_{i=1}^{a} \sum_{j=i-a}^{a-i} Y(i,j) e^{j\epsilon_1 - i\epsilon_2}.$$

Proof. This is a direct consequence of Lemma 6.12.

Corollary 6.14. Let $a = \frac{p-1}{2}$ and $\lambda \in X$. Then

$$m_{L(a,0)}(\lambda) = \begin{cases} 1 & \text{if } \|\lambda\| \in \mathbb{N} \text{ and } \|\lambda\| \le a, \\ 0 & \text{else.} \end{cases}$$

In particular, we have

ch
$$L(a,0) = \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} e^{j\epsilon_1 + i\epsilon_2} + \sum_{i=1}^{a} \sum_{j=i-a}^{a-i} e^{j\epsilon_1 - i\epsilon_2}.$$

Proof. Observe that $(a, 0) \in C_2$. Moreover, $s_{\alpha_1+\alpha_2,p} \bullet (a, 0) = (a-2, 0)$, so $(a-2, 0) \in C_1$ is the only weight in the first alcove linked to (a, 0). By Lemmas 6.4 and 6.13, we have

$$m_{L(a,0)}(\lambda) = m_{\Delta(a,0)}(\lambda) - m_{\Delta(a-2,0)}(\lambda)$$

$$= \begin{cases} \lfloor \frac{a - \|\lambda\|}{2} \rfloor + 1 - \left(\lfloor \frac{a - 2 - \|\lambda\|}{2} \rfloor + 1\right) & \text{if } \|\lambda\| \in \mathbb{N} \text{ and } \|\lambda\| \le a - 2, \\ \lfloor \frac{a - \|\lambda\|}{2} \rfloor + 1 & \text{if } \|\lambda\| \in \{a, a - 1\}, \\ 0 & \text{else.} \end{cases}$$

Observe that $\lfloor \frac{a - \|\lambda\|}{2} \rfloor + 1 = 1$ for $\|\lambda\| \in \{a, a - 1\}$ and

$$\lfloor \frac{a-\|\lambda\|}{2} \rfloor + 1 - \left(\lfloor \frac{a-2-\|\lambda\|}{2} \rfloor + 1\right) = \lfloor 1 + \frac{a-2-\|\lambda\|}{2} \rfloor - \lfloor \frac{a-2-\|\lambda\|}{2} \rfloor = 1,$$

which allows us to conclude.

Lemma 6.15. Let $0 \le b < p$ and $\lambda \in X$. Then

$$m_{L(0,b)}(\lambda) = m_{\Delta(0,b)}(\lambda) = \begin{cases} \left(\frac{b}{2} - |\lambda|_{\infty}\right) + 1 & \text{if } \frac{b}{2} - |\lambda|_{\infty} \in \mathbb{N}, \\ 0 & \text{else.} \end{cases}$$

Proof. Let $V := \Delta(0,1)$. First, observe that $L(0,b) = \Delta(0,b) = S^b V$ by Lemmas 6.4 and 6.10. We know that dim V = 4 and V admits the four weights $\frac{1}{2}(\pm \epsilon_1 \pm \epsilon_2)$, all of them with multiplicity 1. We fix a basis of weight vectors $(v_{-2}, v_{-1}, v_1, v_2)$ with $v_{\pm 2}$ associated to $\pm \frac{1}{2}(\epsilon_1 + \epsilon_2)$ and $v_{\pm 1}$ to $\pm \frac{1}{2}(\epsilon_1 - \epsilon_2)$. By multilinear algebra, $S^b V$ admits the basis $\{v_{i_1} \otimes \cdots \otimes v_{i_k}\}_{i_1 \leq i_2 \leq \ldots \leq i_k}$ and there exists a natural bijection between this basis and the set $\{(x, y, s, t) \in \mathbb{N}^4 \mid x + y + s + t = b\}$ where x is the number of v_{i_j} with $i_j = -2$, y is

the number of v_{i_j} with $i_j = -1$, and so on. Moreover, under this bijection, a basis vector associated to (x, y, s, t) is a weight vector with weight

$$(t-x)\frac{1}{2}(\epsilon_1+\epsilon_2) + (s-y)\frac{1}{2}(\epsilon_1-\epsilon_2) = \frac{1}{2}(t+s-x-y)\epsilon_1 + (t+y-x-s)\epsilon_2.$$

Thus, to compute $m_{S^bV}(\lambda)$, we will count the number of 4-tuples (x, y, s, t) with associated weight λ . Let us write $\lambda = f \frac{1}{2}(\epsilon_1 + \epsilon_2) + g \frac{1}{2}(\epsilon_1 - \epsilon_2)$. Then (x, y, s, t) is associated to λ if t - x = f and s - y = g. Therefore we get

$$m_{S^{b}V}(\lambda) = |\{(x, y, s, t) \in \mathbb{N}^{4} \mid x + y + s + t = b, \ t - x = f, \ s - y = g\}|.$$

Up to symmetry we can assume that both f and g are non-negative. In particular, this implies that $|\lambda|_{\infty} = \frac{1}{2}(f+g)$, so we get

$$\begin{split} m_{S^{b}V}(\lambda) &= |\{(x, y, s, t) \in \mathbb{N}^{4} \mid x + y + s + t = b, \ t - x = f, \ s - y = g\}| \\ &= |\{(x, y) \in \mathbb{N}^{2} \mid 2x + f + 2y + g = b\}| \\ &= |\{(x, y) \in \mathbb{N}^{2} \mid 2x + 2y = b - 2|\lambda|_{\infty}\}| \\ &= |\{(x, y) \in \mathbb{N}^{2} \mid x + y = \frac{1}{2}b - |\lambda|_{\infty}\}|. \end{split}$$

Clearly, the equality $x + y = \frac{1}{2}b - |\lambda|_{\infty}$ cannot be satisfied for $x, y \in \mathbb{N}$ if $\frac{1}{2}b - |\lambda|_{\infty} \notin \mathbb{N}$. Thus we can restrict our attention to the case $\frac{1}{2}b - |\lambda|_{\infty} \in \mathbb{N}$, and using the combinatorial result ([MN08, 3.3]) again, we get

$$m_{S^{b}V}(\lambda) = |\{(x,y) \in \mathbb{N}^{2} \mid x+y = \frac{1}{2}b - |\lambda|_{\infty}\} = \binom{\frac{1}{2}b - |\lambda|_{\infty} + 1}{1} = \frac{1}{2}b - |\lambda|_{\infty} + 1,$$

thus we are done.

6.3 Classification results

In all statements of this section, unless stated otherwise, we use coordinates with respect to the fundamental weights.

We start by stating a theorem from Stembridge ([Ste03, Theorem 1.1.B]) which classifies multiplicity-free tensor products of simple $\text{Sp}_4(\mathbb{C})$ -modules.

Theorem 6.16. Let $\lambda = (a, b), \mu = (c, d) \in X^+$ be dominant weights. Up to the reordering of λ and μ , $L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu)$ is multiplicity-free if and only if one of the following holds:

- (1) a = 0 and b = 1,
- (2) a = 1 and b = 0,
- (3) a = d = 0,
- (4) a = c = 0,
- (5) b = 0 and d = 1, or
- (6) b = d = 0.

Proposition 6.17. Let $\lambda \in C_2$ and $\mu \in \widehat{C_1}$. If $\lambda + \mu \in C_3$, then $L(\lambda) \otimes L(\mu)$ has multiplicity.

Proof. Let $\lambda' := s_{\alpha_1+\alpha_2,p} \bullet \lambda \in C_1$, so that $\Delta(\lambda)$ admits the unique composition series $[L(\lambda), L(\lambda')]$ and let $\eta := s_{\alpha_1+2\alpha_2,p} \bullet (\lambda + \mu) \in C_2$, so that $\Delta(\lambda + \mu)$ admits the unique composition series $[L(\lambda + \mu), L(\eta)]$ (Lemma 6.4). We have a short exact sequence

$$0 \longrightarrow L(\lambda') \longrightarrow \Delta(\lambda) \longrightarrow L(\lambda) \longrightarrow 0.$$

By Lemma 6.4, we have $L(\mu) \cong \Delta(\mu)$ and $L(\lambda') \cong \Delta(\lambda')$. Taking the tensor product with $L(\mu)$, we get the short exact sequence

$$0 \longrightarrow \Delta(\lambda') \otimes \Delta(\mu) \xrightarrow{\phi} \Delta(\lambda) \otimes \Delta(\mu) \xrightarrow{\psi} L(\lambda) \otimes L(\mu) \longrightarrow 0.$$

By Lemma 1.57, $\Delta(\lambda) \otimes \Delta(\mu)$ admits a submodule isomorphic to $\Delta(\lambda + \mu)$. We will abuse the notation and denote it by $\Delta(\lambda + \mu)$. Thus, we can restrict our exact sequence to

$$0 \longrightarrow \phi^{-1}(\Delta(\lambda + \mu)) \xrightarrow{\phi} \Delta(\lambda + \mu) \xrightarrow{\psi} \psi(\Delta(\lambda + \mu)) \longrightarrow 0.$$

Suppose for contradiction that $L(\lambda) \otimes L(\mu)$ is multiplicity-free. Then in particular it is completely reducible, and $\psi(\Delta(\lambda + \mu))$ is completely reducible (Proposition 1.13). Therefore, rad $\Delta(\lambda + \mu) \subseteq \ker(\psi)$ and $\psi(\Delta(\lambda + \mu)) \cong L(\lambda + \mu)$ or $\psi(\Delta(\lambda + \mu)) = 0$. We claim that the second case is impossible. By exactness, it would imply $\phi^{-1}(\Delta(\lambda + \mu)) \cong \Delta(\lambda + \mu)$, but $\lambda + \mu \nleq \lambda' + \mu$, so it cannot appear as a submodule of $\Delta(\lambda') \otimes \Delta(\mu)$. Thus, $\phi^{-1}(\Delta(\lambda + \mu)) \cong L(\eta)$ and it is a submodule of $\Delta(\lambda') \otimes \Delta(\mu)$.

Using Theorem 1.28, we fix

$$0 = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_m = \Delta(\lambda') \otimes \Delta(\mu)$$

a Weyl filtration. Thus there exist $\nu_1, \ldots, \nu_m \in X^+$ such that $V_i/V_{i-1} \cong \Delta(\nu_i)$ for $i \in \{1, \ldots, m\}$. We set $W_i := V_i \cap \phi^{-1}(\Delta(\lambda + \mu))$. Since $\phi^{-1}(\Delta(\lambda + \mu))$ is irreducible, then $W_i = 0$ or $W_i \cong L(\eta)$. Let j be minimal such that $W_j \cong L(\eta)$ (in particular $W_{j-1} = 0$). We have the following situation



Since ker $\pi_j = V_{j-1}$ and $W_j \cap V_{j-1} = W_{j-1} = 0$, we have an injective map $L(\eta) \to \Delta(\nu_j)$, so $L(\eta)$ is a submodule of $\Delta(\nu_j)$. In particular, $L(\eta)$ is a composition factor of $\Delta(\nu_j)$, so $\eta \uparrow \nu_j$ by the Strong Linkage Principle (Proposition 1.53). Recall at this step that $\eta \uparrow \lambda + \mu$. Now, observe that $\nu_j \leq \lambda' + \mu < \lambda + \mu$, so $\nu_j \neq \lambda + \mu$. By the geometry of alcoves, it follows that $\nu_j \in C_2$, so $\eta = \nu_j$. But $L(\eta)$ is not a submodule of $\Delta(\eta)$ (Lemma 6.4), so we get a contradiction. Therefore, $L(\lambda) \otimes L(\mu)$ has multiplicity.

6.3.1 $L(0,b) \otimes L(0,d)$

Proposition 6.18. Let $\lambda = (0, b), \mu = (0, d) \in X^+$ be p-restricted dominant weights with 0 < b, d < p. Then $L(\lambda) \otimes L(\mu)$ is multiplicity-free if and only if one of the following holds:

- (1) $b+d \leq p-3$ (i.e. $\lambda + \mu \in \widehat{C_1}$) or
- (2) $(b,d) \in \{(1,p-2), (p-2,1)\}.$

Proof. We set $M := L(\lambda) \otimes L(\mu)$. By Lemma 6.4, $L(\lambda)$ and $L(\mu)$ are tilting modules, so M is a tilting module.

If $b + d \ge p$, we conclude directly from Corollary 2.8 that M has multiplicity.

If $b + d \leq p - 3$ (i.e. $\lambda + \mu \in \widehat{C_1}$), we apply Corollary 3.4 and Theorem 6.16 to conclude that M is multiplicity-free.

Suppose that b + d = p - 2. By Lemma 1.15, we have

 $m_{L(\lambda)}(\lambda - \alpha_2) = 1,$ $m_{L(\mu)}(\mu - \alpha_2) = 1,$ $m_{L(\lambda + \mu)}(\lambda + \mu - \alpha_2) = 1.$

Using Argument 1, we have

$$m_M(\lambda + \mu - \alpha_2) = 1 + 1 = 2,$$

and we deduce that $L(\lambda + \mu - \alpha_2)$ is a composition factor of M. Observe that $\lambda + \mu - \alpha_2 = (1, p - 4) \in C_2$, so $T(\lambda + \mu - \alpha_2)$ is not irreducible by Lemma 6.4. We can thus conclude by Lemma 1.36 that M has multiplicity.

If b + d = p - 1 and $b \neq 1, d \neq 1$, we use Argument 1. By Lemma 1.15, we have

$$\begin{split} m_{L(\lambda)}(\lambda - \alpha_2) &= 1, & m_{L(\lambda)}(\lambda - 2\alpha_2) &= 1, \\ m_{L(\mu)}(\mu - \alpha_2) &= 1, & m_{L(\mu)}(\mu - 2\alpha_2) &= 1, \\ m_{L(\lambda + \mu)}(\lambda + \mu - \alpha_2) &= 1, & m_{L(\lambda + \mu)}(\lambda + \mu - 2\alpha_2) &= 1, \\ m_{L(\lambda + \mu - \alpha_2)}(\lambda + \mu - \alpha_2) &= 1, & m_{L(\lambda + \mu - \alpha_2)}(\lambda + \mu - 2\alpha_2) &= 1. \end{split}$$

Therefore, we get

$$m_M(\lambda + \mu - \alpha_2) = 2,$$
 $m_M(\lambda + \mu - 2\alpha_2) = 3.$

We deduce that $L(\lambda + \mu - 2\alpha_2)$ is a composition factor of M. Observe that $\lambda + \mu - 2\alpha_2 = (2, p - 5) \in C_2$, so $T(\lambda + \mu - \alpha_2)$ is not irreducible by Lemma 6.4. As in the previous case, we conclude by Lemma 1.36 that M has multiplicity.

Finally, consider the case b = 1, d = p - 2 (the case b = p - 2, d = 1 is symmetric). By Proposition 1.47 and Lemmas 6.4 and 1.45, we have

$$ch M = \chi(\lambda)\chi(\mu) = \chi(0, p-1) + \chi(1, p-3) + \chi(0, p-3) + \chi(-1, p-1)$$
$$= ch L(0, p-1) + ch L(1, p-3) + ch L(0, p-3).$$

Therefore, M is multiplicity-free.

6.3.2 $L(a,0) \otimes L(c,0)$

Lemma 6.19. Let $a, c \in \mathbb{N}$. We use Euclidean coordinates. For $i, j \in \mathbb{Z}$, we set

$$\delta(a,i,j) := \begin{cases} 1 & \text{if } a-i-j \text{ is even,} \\ 0 & \text{if } a-i-j \text{ is odd.} \end{cases}$$

Then

$$\chi(a\omega_1)\chi(c\omega_1) = \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \delta(a,i,j)\chi(c+j,i).$$

Moreover, if $a \leq c$, then all the weights on the right hand side of the equality are dominant.

Proof. In this proof we use Euclidean coordinates. Like in Corollary 6.13, we set

$$Y(i,j) := \lfloor \frac{a - |i| - |j|}{2} \rfloor + 1.$$

Using Proposition 1.47 and Corollary 6.13 in the first equality below, Lemma 1.45 in the second one and Remark 6.2 in the third one, we get

$$\begin{split} \chi(a\omega_1)\chi(c\omega_1) &= \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} Y(i,j)\chi(c+j,i) + \sum_{i=1}^{a} \sum_{j=i-a}^{a-i} Y(i,j)\chi(c+j,-i) \\ &= \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} Y(i,j)\chi(c+j,i) - \sum_{i=1}^{a} \sum_{j=i-a}^{a-i} Y(i,j)\chi(s_{\alpha_2} \bullet (c+j,-i)) \\ &= \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} Y(i,j)\chi(c+j,i) - \sum_{i=1}^{a} \sum_{j=i-a}^{a-i} Y(i,j)\chi(c+j,i-1) \\ &= \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} Y(i,j)\chi(c+j,i) - \sum_{i=0}^{a-1} \sum_{j=i+1-a}^{a-i-1} Y(i+1,j)\chi(c+j,i) \\ &= \sum_{i=0}^{a-1} \sum_{j=i+1-a}^{a-i-1} (Y(i,j) - Y(i+1,j))\chi(c+j,i) \\ &+ \sum_{i=0}^{a-1} Y(i,a-i)\chi(c+a-i,i) + Y(i,i-a)\chi(c+i-a,i) \\ &+ Y(a,0)\chi(c,a). \end{split}$$

For $i \in \{0, ..., a - 1\}$, we have $Y(i, a - i) = Y(i, i - a) = 1 = \delta(a, i, \pm (a - i))$, and $Y(a, 0) = 1 = \delta(a, a, 0)$. Moreover

$$Y(i,j) - Y(i+1,j) = \left(\lfloor \frac{a-|i|-|j|}{2} \rfloor + 1\right) - \left(\lfloor \frac{a-|i+1|-|j|}{2} \rfloor + 1\right) = \delta(a,i,j).$$

Therefore, we get

$$\chi(a\omega_{1})\chi(c\omega_{1}) = \sum_{i=0}^{a-1} \sum_{j=i+1-a}^{a-i-1} \delta(a,i,j)\chi(c+j,i) + \sum_{i=0}^{a-1} \delta(a,i,a-i)\chi(c+a-i,i) + \delta(a,i,i-a)\chi(c+i-a,i) + \delta(a,a,0)\chi(c,a) = \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \delta(a,i,j)\chi(c+j,i).$$
(38)

Finally, if $0 \le i$, $a \le c$ and $i - a \le j$, then $0 \le i \le i + (c - a) \le c + j$, hence all the weights appearing in line (38) are dominant.

Proposition 6.20. Let $\lambda = (a, 0), \mu = (c, 0) \in X^+$ be two *p*-restricted dominant weights with $0 < a < c, c \geq \frac{p-1}{2}$ and $a + c (i.e. <math>\lambda \in \widehat{C_1}, \mu \in C_2$ and $\lambda + \mu \in \widehat{C_2}$). Then $L(\lambda) \otimes L(\mu)$ is multiplicity-free.

Proof. In this proof we use Euclidean coordinates. We set $M := L(\lambda) \otimes L(\mu)$ and m := 2c - p + 2. Like in Lemma 6.19, for $i, j \in \mathbb{Z}$, we set

$$\delta(a, i, j) := \begin{cases} 1 & \text{if } a - i - j \text{ is even,} \\ 0 & \text{if } a - i - j \text{ is odd.} \end{cases}$$

Note that $\delta(a, i, j) = \delta(a, i, -j)$.

By Remark 6.5 and Proposition 1.47, and using Lemma 6.19 in the second equality below, we have

$$\operatorname{ch} M = \chi(a,0)(\chi(c,0) - \chi(p-3-c,0))$$

$$= \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \delta(a,i,j)\chi(c+j,i) - \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \delta(a,i,j)\chi(p-3-c+j,i)$$

$$= \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \delta(a,i,j)\chi(c+j,i) - \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \delta(a,i,-j)\chi(p-3-c-j,i)$$

$$= \sum_{i=0}^{a} \sum_{j=\max\{i-a,a-i-m\}}^{a-i} \delta(a,i,j)\chi(c+j,i) + \sum_{i=0}^{a} \sum_{j=i-a}^{a-i-m-1} \delta(a,i,j)\chi(c+j,i)$$

$$- \sum_{i=0}^{a} \sum_{j=\max\{i-a,a-i-m\}}^{a-i} \delta(a,i,-j)\chi(p-3-c-j,i)$$

$$= \sum_{i=0}^{a} \sum_{j=\max\{i-a,a-i-m\}}^{a-i} \delta(a,i,j)(\chi(c+j,i)-\chi(p-3-c-j,i))$$

$$= \sum_{i=0}^{a} \sum_{j=\max\{i-a,a-i-m\}}^{a-i} \delta(a,i,j)(\chi(c+j,i)-\chi(p-3-c-j,i))$$

$$(39)$$

$$+\sum_{i=0}^{a} \left(\sum_{j=i-a}^{a-i-m-1} \delta(a,i,j) \chi(c+j,i) - \sum_{j=i-a}^{a-i-m-1} \delta(a,i,j) \chi(p-3-c-j,i) \right)$$
(40)

We show that line (40) is equal to zero. We have

$$\begin{split} \sum_{j=i-a}^{a-i-m-1} \delta(a,i,j)\chi(c+j,i) &- \sum_{j=i-a}^{a-i-m-1} \delta(a,i,j)\chi(p-3-c-j,i) \\ &= \sum_{j=i-a}^{a-i-m-1} \delta(a,i,j)\chi(c+j,i) \\ &- \sum_{j=i-a}^{a-i-m-1} \delta(a,i,-m-1-j)\chi(p-3-c-(-m-1-j),i) \\ &= \sum_{j=i-a}^{a-i-m-1} \delta(a,i,j)\chi(c+j,i) - \sum_{j=i-a}^{a-i-m-1} \delta(a,i,-m-1-j)\chi(c+j,i) \\ &= \sum_{j=i-a}^{a-i-m-1} (\delta(a,i,j) - \delta(a,i,-m-1-j))\chi(c+j,i) = 0, \end{split}$$

where in the last equality, we use that m is odd, thus -m - 1 is even and $\delta(a, i, -m - 1 - j) = \delta(a, i, j)$. Therefore

$$\sum_{i=0}^{a} \left(\sum_{j=i-a}^{a-i-m-1} \delta(a,i,j) \chi(c+j,i) - \sum_{j=i-a}^{a-i-m-1} \delta(a,i,j) \chi(p-3-c-j,i) \right) = \sum_{i=0}^{a} 0 = 0.$$

Now we work on line (39). For $0 \le i \le a$ and $\max\{i-a, a-i-m\} \le j \le a-i$, we claim that $(c+j,i) \in \widehat{C_2}$. Indeed, we have $i \ge 0$ and

$$(c+j)+i \le c+(a-i)+i \le p-2$$

by assumption. Thus it remains to show that $c+j > \frac{p-3}{2}$. If $i-a \ge \frac{p-1}{2} - c$, then

$$c+j \ge c+i-a \ge \frac{p-1}{2} > \frac{p-3}{2}$$

hence we are done. If $i - a < \frac{p-1}{2} - c$, then $a - i > c - \frac{p-1}{2}$ and

$$c+j \ge c+a-i-m > 2c - \frac{p-1}{2} - m = 2c - \frac{p-1}{2} - (2c-p+2) = \frac{p-3}{2}.$$

so we are done. Using Remark 6.5, we get

$$\operatorname{ch} M = \sum_{i=0}^{a} \sum_{j=\max\{i-a,a-i-m\}}^{a-i} \delta(a,i,j) (\chi(c+j,i) - \chi(p-3-c-j,i))$$
$$= \sum_{i=0}^{a} \sum_{j=\max\{i-a,a-i-m\}}^{a-i} \delta(a,i,j) \operatorname{ch} L(c+j,i).$$

We conclude that M is multiplicity-free.

Proposition 6.21. Let $\lambda = (a, 0), \mu = (c, 0) \in X^+$ be p-restricted dominant weights with $0 < a \leq c$. Then $L(\lambda) \otimes L(\mu)$ is multiplicity-free if and only if one of the following holds:

(1) $a + c \leq \frac{p-3}{2}$ (i.e. $\lambda + \mu \in \widehat{C}_1$),

(2)
$$c \geq \frac{p-1}{2}$$
 and $a + c or$

(3)
$$a = c = \frac{p-1}{2}$$
.

Proof. We set $M := L(\lambda) \otimes L(\mu)$.

If $a + c \ge p$, we use Corollary 2.8 to conclude that M has multiplicity.

If $a + c \leq \frac{p-3}{2}$ (i.e. $\lambda + \mu \in \widehat{C_1}$), we apply Corollary 3.4 and Theorem 6.16 to conclude that M is multiplicity-free.

Suppose that $c \leq \frac{p-3}{2}$ and $a + c > \frac{p-3}{2}$ (i.e. $\mu \in \widehat{C_1}$ and $\lambda + \mu \in C_2$). In this case, $L(\lambda)$ and $L(\mu)$ are tilting modules by Lemma 6.4, hence M is a tilting module. Since $T(\lambda + \mu)$ is not irreducible by Lemma 6.4, we conclude that M has multiplicity by Lemma 1.36.

Consider the case a = 1, c = p - 2 (in particular $\mu \in F_{2,3}, \lambda \in \widehat{C_1}$ and $\lambda + \mu \in F_{4,6}$). By Lemma 6.4, we have $L(\lambda)$ and $L(\mu)$ are tilting modules, therefore M is a tilting module. We use Argument 1. By Lemma 1.15, we have

$$m_{L(\lambda)}(\lambda - \alpha_1) = 1,$$
 $m_{L(\mu)}(\mu - \alpha_1) = 1,$ $m_{L(\lambda + \mu)}(\lambda + \mu - \alpha_1) = 1.$

Since $m_M(\lambda + \mu - \alpha_1) = 2$, we deduce that $L(\lambda + \mu - \alpha_1)$ is a composition factor of M. Observe that $\lambda + \mu - \alpha_1 = (p - 3, 2) \in C_3$, so $T(\lambda + \mu - \alpha_1)$ is not irreducible by Lemma 6.4. We conclude by Lemma 1.36 that M has multiplicity.

If a + c = p - 1 and $a \neq 1, \frac{p-1}{2}$, then $\frac{p-1}{2} < c < p - 2$ (i.e. $\lambda \in \widehat{C_1}$ and $\mu \in C_2$) and by Lemma 6.4, we have $L(\lambda) \cong \Delta(\lambda)$. Moreover, $\Delta(\mu)$ admits the unique composition series $[L(\mu), L(\eta)]$ with $\eta = s_{\alpha_1 + \alpha_2, p} \bullet \mu \in C_1$. Since $c \geq \frac{p+1}{2}$, we have in particular $\eta < \mu - 2\alpha_1 - 2\alpha_2$, thus $m_{L(\mu)}(\nu) = m_{\Delta(\mu)}(\nu)$ for all weights $\nu \geq \lambda - 2\alpha_1 - 2\alpha_2$. We use Argument 1 to show that $[M : L(\lambda + \mu - 2\alpha_1 - 2\alpha_2)] = 2$. Using Lemmas 6.7 and 1.15 (note that $a \geq 2$), we have

$$m_M(\lambda + \mu) = 1, \qquad m_M(\lambda + \mu - \alpha_1) = 2,$$

$$m_M(\lambda + \mu - 2\alpha_1) = 3, \qquad m_M(\lambda + \mu - \alpha_2) = m_M(\lambda + \mu - 2\alpha_2) = 0,$$

$$m_M(\lambda + \mu - \alpha_1 - \alpha_2) = 2, \qquad m_M(\lambda + \mu - \alpha_1 - 2\alpha_2) = 2,$$

$$m_M(\lambda + \mu - 2\alpha_1 - \alpha_2) = 4, \qquad m_M(\lambda + \mu - 2\alpha_1 - 2\alpha_2) = 7.$$

We deduce that

 $[M: L(\lambda + \mu)] = [M: L(\lambda + \mu - \alpha_1)] = [M: L(\lambda + \mu - 2\alpha_1)] = 1,$

$$[M : L(\lambda + \mu - \alpha_2)] = [M : L(\lambda + \mu - 2\alpha_2)] = 0.$$

By Lemma 6.4, $L(\lambda + \mu) \cong \Delta(\lambda + \mu)$. Moreover, $\lambda + \mu - \alpha_1 = (p - 3, 2)$. Thus, using Lemmas 1.15 and 6.7, we get

$$[M: L(\lambda+\mu-\alpha_1-\alpha_2)]=0, \qquad \text{and} \qquad [M: L(\lambda+\mu-\alpha_1-2\alpha_2)]=0.$$

Since $2(p-3) + 2 + 2 \neq 0 \mod p$, we have $m_{L(\lambda+\mu-\alpha_1)}(\lambda+\mu-2\alpha_1-\alpha_2) = 2$ by Lemma 6.6. By Lemma 1.15, we have $m_{L(\lambda+\mu-2\alpha_1)}(\lambda+\mu-2\alpha_1-\alpha_2) = 1$. Moreover, by Lemma 6.7, we have $m_{L(\lambda+\mu)}(\lambda+\mu-2\alpha_1-\alpha_2) = 1$. Therefore,

$$[M: L(\lambda + \mu - 2\alpha_1 - \alpha_2)] = 0.$$

Finally observe that (p-3) + 2 = p - 1. Thus, using Lemmas 6.7, 6.8 and 1.15, we have

$$m_{L(\lambda+\mu)}(\lambda+\mu-2\alpha_{1}-2\alpha_{2}) = 2, \qquad m_{L(\lambda+\mu-\alpha_{1})}(\lambda+\mu-2\alpha_{1}-2\alpha_{2}) = 2,$$

$$m_{L(\lambda+\mu-2\alpha_{1})}(\lambda+\mu-2\alpha_{1}-2\alpha_{2}) = 1.$$

We conclude that

$$[M: L(\lambda + \mu - 2\alpha_1 - 2\alpha_2)] = 7 - 2 - 2 - 1 = 2.$$

In particular, M has multiplicity.

If $c \ge \frac{p-1}{2}$ and a + c then M is multiplicity-free by Proposition 6.20.

Finally, suppose that $a = c = \frac{p-1}{2}$ (i.e. $\lambda, \mu \in C_2$ and $\lambda + \mu \in F_{4,6}$). We show that M is multiplicity-free. We have

$$s_{\alpha_1+\alpha_2,p} \bullet \mu = \mu - 2(\alpha_1 + \alpha_2).$$

For the rest of this proof we use Euclidean coordinates. Using Corollary 1.48 and Lemma 6.4 in the first equality below, Corollary 6.14 in the second one and Lemma 1.45 in the third one, we get

$$\begin{split} \operatorname{ch} & M = \sum_{\nu \in X} m_{L(\lambda)}(\nu)(\chi(\mu + \nu) - \chi(\mu - 2(\alpha_1 + \alpha_2) + \nu)) \\ &= \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \chi(a+j,i) - \chi(a-2+j,i) + \sum_{i=1}^{a} \sum_{j=i-a}^{a-i} \chi(a+j,-i) - \chi(a-2+j,-i) \\ &= \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \chi(a+j,i) - \sum_{i=1}^{a} \sum_{j=i-a}^{a-i} \chi(s_{\alpha_2} \cdot (a+j,-i)) \\ &- \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \chi(a-2+j,i) + \sum_{i=1}^{a} \sum_{j=i-a}^{a-i} \chi(s_{\alpha_2} \cdot (a-2+j,-i)) \\ &= \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \chi(a+j,i) - \sum_{i=1}^{a} \sum_{j=i-a}^{a-i} \chi(a+j,i-1) \\ &- \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \chi(a-2+j,i) + \sum_{i=1}^{a} \sum_{j=i-a}^{a-i} \chi(a-2+j,i-1) \\ &= \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \chi(a+j,i) - \sum_{i=0}^{a-i} \sum_{j=i+1-a}^{a-i-1} \chi(a-2+j,i-1) \\ &= \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \chi(a-2+j,i) + \sum_{i=0}^{a-i-1} \sum_{j=i+1-a}^{a-i-1} \chi(a-2+j,i-1) \\ &= \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \chi(a-2+j,i) + \sum_{i=0}^{a-i-1} \sum_{j=i+1-a}^{a-i-1} \chi(a-2+j,i) \\ &= \left(\sum_{i=0}^{a-1} \chi(i,i) + \chi(2a-i,i)\right) + \chi(a,a) \\ &- \left(\sum_{i=0}^{a-1} \chi(i,i) + \chi(2a,0) + \sum_{i=1}^{a-1} \chi(2a-i,i) \\ &- \sum_{i=0}^{a-2} \chi(2a-2-i,i) - \chi(a-1,a-1) - \chi(-2,0) - \sum_{i=1}^{a-1} \chi(i-2,i) - \chi(a-2,a). \end{split}$$

At this step, recall that we use Euclidean coordinates and observe that $(i, i) \in \widehat{C}_1 \cup F_{2,3} \cup F_{3,5}$ for $0 \leq i \leq a = \frac{p-1}{2}$. Thus, by Lemma 6.4, we have $\chi(i, i) = \operatorname{ch} L(i, i)$. Similarly, $(2a, 0) = (p-1, 0) \in F_{4,6}$, hence $\chi(2a, 0) = \operatorname{ch} L(2a, 0)$ by Lemma 6.4. Using those facts and Lemma 1.45, we get

$$\operatorname{ch} M = \sum_{i=0}^{a} \operatorname{ch} L(i,i) + \operatorname{ch} L(2a,0) + \sum_{i=1}^{a-1} \chi(2a-i,i) - \sum_{i=1}^{a-1} \chi(2a-1-i,i-1) + \sum_{i=1}^{a-1} \chi(s_{\alpha_1} \bullet (i-2,i)) + \chi(s_{\alpha_1+\alpha_2} \bullet (-2,0)) - \chi(a-1,a-1) + \chi(s_{\alpha_1} \bullet (a-2,a))$$

$$=\sum_{i=0}^{a} \operatorname{ch} L(i,i) + \operatorname{ch} L(2a,0) + \sum_{i=1}^{a-1} \chi(2a-i,i) - \sum_{i=1}^{a-1} \chi(2a-1-i,i-1) + \sum_{i=1}^{a-1} \chi(i-1,i-1) + \chi(-1,0) - \chi(a-1,a-1) + \chi(a-1,a-1) + \sum_{i=1}^{a-1} \chi(a-1,a-1) + \sum_{i=1}^{a-1} \chi(2a-1,i) + \operatorname{ch} L(2a,0) + \sum_{i=1}^{a-1} \chi(2a-i,i) - \left(\sum_{i=1}^{a-1} \chi(2a-1-i,i-1) - \chi(i-1,i-1)\right),$$

where in the last equality we use that $(-1,0) \in D \setminus X^+$ so $\chi(-1,0) = 0$ by Lemma 1.45. If $i \in \{1, \ldots, a-1\}$, then $(2a-1-i, i-1) \in C_2$. Thus, by Remark 6.2, we have

$$s_{\alpha_1+\alpha_2,p} \bullet (2a-1-i,i-1) = (p-3-(2a-1-i),i-1) = (i-1,i-1),$$

and by Remark 6.5 we get

$$\chi(2a - 1 - i, i - 1) - \chi(i - 1, i - 1) = \operatorname{ch} L(2a - 1 - i, i - 1)$$

Moreover, $(2a - i, i) \in C_3$ and 2a - i + i - (p - 2) = 1, so by Remark 6.5, we have

$$\chi(2a - i, i) = \operatorname{ch} L(2a - i, i) + \operatorname{ch} L(2a - i - 1, i - 1).$$

We can thus conclude that

$$ch M = \sum_{i=0}^{a} ch L(i,i) + ch L(2a,0) + \sum_{i=1}^{a-1} \chi(2a-i,i) - \sum_{i=1}^{a-1} ch L(2a-1-i,i-1)$$
$$= \sum_{i=0}^{a} ch L(i,i) + ch L(2a,0) + \sum_{i=1}^{a-1} ch L(2a-i,i)$$
$$= \sum_{i=0}^{a} ch L(i,i) + \sum_{i=0}^{a-1} ch L(2a-i,i).$$

In particular, M is multiplicity-free.

6.3.3
$$L(a,0) \otimes L(0,d)$$

Lemma 6.22. Let $a, b \in \mathbb{N}$. In Euclidean coordinates, we have

$$\chi(a\omega_1)\chi(b\omega_2) = \sum_{i=0}^{b} \sum_{j=0}^{\min\{i,b-i\}} \chi(a + \frac{b}{2} - i, \frac{b}{2} - j).$$

Moreover, if b < p and a < p, then all weights appearing on the right hand side of the equality belong to $D \cup W \bullet \overline{C_1}$ and the dominant ones are p-restricted.

Proof. In this proof, we use Euclidean coordinates. Let $\lambda = a\omega_1$ and $\mu = b\omega_2$. We set $X(i,j) := m_{\Delta(\mu)}(\frac{b}{2} - i, \frac{b}{2} - j) = \frac{b}{2} - \max\{|\frac{b}{2} - i|, |\frac{b}{2} - j|\} + 1$ (Lemma 6.15). We get

$$\chi(\lambda)\chi(\mu) = \sum_{\nu \in X} m_{\Delta(\mu)}(\nu)\chi(\lambda + \nu) = \sum_{i=0}^{b} \sum_{j=0}^{b} X(i,j)\chi(a + \frac{b}{2} - i, \frac{b}{2} - j).$$

We dissociate the cases b even and b odd. First suppose that b is even. Using Lemma 1.45 in the second equality below and Remark 6.2 in the third one, we get

$$\begin{split} \chi(\lambda)\chi(\mu) &= \sum_{i=0}^{b} \sum_{j=0}^{\frac{b}{2}} X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j) + \sum_{i=0}^{b} \sum_{j=\frac{b}{2}+1}^{b} X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j) \\ &= \sum_{i=0}^{b} \sum_{j=0}^{\frac{b}{2}} X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j) - \sum_{i=0}^{b} \sum_{j=\frac{b}{2}+1}^{b} X(i,j)\chi(s_{\alpha_{2}} \bullet (a+\frac{b}{2}-i,\frac{b}{2}-j)) \\ &= \sum_{i=0}^{b} \sum_{j=0}^{\frac{b}{2}} X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j) - \sum_{i=0}^{b} \sum_{j=\frac{b}{2}+1}^{b} X(i,j)\chi(a+\frac{b}{2}-i,-\frac{b}{2}+j-1) \\ &= \sum_{i=0}^{b} \sum_{j=0}^{\frac{b}{2}} X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j) \\ &- \sum_{i=0}^{b} \sum_{j=1}^{\frac{b}{2}} X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j) \\ &= \sum_{i=0}^{b} \sum_{j=0}^{\frac{b}{2}} X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j) \\ &= \sum_{i=0}^{b} \sum_{j=0}^{\frac{b}{2}} X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j) \\ &= \sum_{i=0}^{b} \sum_{j=0}^{\frac{b}{2}} X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j) - \sum_{i=0}^{b} \sum_{j=1}^{\frac{b}{2}} X(i,b-j+1)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j) \\ &= \sum_{i=0}^{b} X(i,0)\chi(a+\frac{b}{2}-i,\frac{b}{2}) \\ &+ \sum_{i=0}^{b} \sum_{j=1}^{\frac{b}{2}} (X(i,j)-X(i,b-j+1))\chi(a+\frac{b}{2}-i,\frac{b}{2}-j). \end{split}$$

If b is odd, then for any i, we have

$$s_{\alpha_2} \bullet (a + \frac{b}{2} - i, \frac{b}{2} - \frac{b+1}{2}) = s_{\alpha_2} \bullet (a + \frac{b}{2} - i, -\frac{1}{2}) = (a + \frac{b}{2} - i, -\frac{1}{2}),$$

so $\chi(a + \frac{b}{2} - i, \frac{b}{2} - \frac{b+1}{2}) = 0$ by Lemma 1.45. Similarly to the case where b is even, we get

$$\begin{split} \chi(\lambda)\chi(\mu) &= \sum_{i=0}^{b} \sum_{j=0}^{\frac{b-1}{2}} X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j) + \sum_{i=0}^{b} \sum_{j=\frac{b+3}{2}}^{b} X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j) \\ &= \sum_{i=0}^{b} \sum_{j=0}^{\frac{b-1}{2}} X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j) - \sum_{i=0}^{b} \sum_{j=\frac{b+3}{2}}^{b} X(i,j)\chi(s_{\alpha_{2}} \bullet (a+\frac{b}{2}-i,\frac{b}{2}-j)) \\ &= \sum_{i=0}^{b} \sum_{j=0}^{\frac{b-1}{2}} X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j) - \sum_{i=0}^{b} \sum_{j=\frac{b+3}{2}}^{b} X(i,j)\chi(a+\frac{b}{2}-i,-\frac{b}{2}+j-1) \end{split}$$

$$=\sum_{i=0}^{b}\sum_{j=0}^{\frac{b-1}{2}}X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j)$$

$$-\sum_{i=0}^{b}\sum_{j=1}^{\frac{b-1}{2}}X(i,b-j+1)\chi(a+\frac{b}{2}-i,-\frac{b}{2}+(b-j+1)-1)$$

$$=\sum_{i=0}^{b}\sum_{j=0}^{\frac{b-1}{2}}X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j) - \sum_{i=0}^{b}\sum_{j=1}^{\frac{b-1}{2}}X(i,b-j+1)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j)$$

$$=\sum_{i=0}^{b}X(i,0)\chi(a+\frac{b}{2}-i,\frac{b}{2})$$

$$+\sum_{i=0}^{b}\sum_{j=1}^{\frac{b-1}{2}}(X(i,j)-X(i,b-j+1))\chi(a+\frac{b}{2}-i,\frac{b}{2}-j).$$
(42)

At this step, observe that X(i,0) = 1, and set Y(i,j) := X(i,j) - X(i,b-j+1). Using lines (41) and (42), for all b, we have

$$\chi(\lambda)\chi(\mu) = \sum_{i=0}^{b} \chi(a + \frac{b}{2} - i, \frac{b}{2}) + \sum_{i=0}^{b} \sum_{j=1}^{\lfloor \frac{b}{2} \rfloor} Y(i, j)\chi(a + \frac{b}{2} - i, \frac{b}{2} - j)$$

$$= \sum_{i=0}^{b} \chi(a + \frac{b}{2} - i, \frac{b}{2})$$

$$+ \sum_{i=0}^{\lfloor \frac{b}{2} \rfloor} \sum_{j=1}^{\lfloor \frac{b}{2} \rfloor} Y(i, j)\chi(a + \frac{b}{2} - i, \frac{b}{2} - j)$$
(43)

$$+\sum_{i=\lfloor\frac{b}{2}\rfloor+1}^{b}\sum_{j=1}^{\lfloor\frac{b}{2}\rfloor}Y(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j).$$
(44)

We compute the value of Y(i,j) in lines (43) and (44). Recall that Y(i,j) = X(i,j) - X(i,b-j+1) and $X(i,j) = \frac{b}{2} - \max\{|\frac{b}{2} - i|, |\frac{b}{2} - j|\} + 1$. Therefore, for $j \leq \frac{b}{2}$, we have

$$\begin{split} Y(i,j) &= (\frac{b}{2} - \max\{|\frac{b}{2} - i|, |\frac{b}{2} - j|\} + 1) - (\frac{b}{2} - \max\{|\frac{b}{2} - i|, |\frac{b}{2} - (b - j + 1)|\} + 1) \\ &= \max\{|\frac{b}{2} - i|, |j - \frac{b}{2} - 1|\} - \max\{|\frac{b}{2} - i|, |\frac{b}{2} - j|\} \\ &= \max\{|\frac{b}{2} - i|, |\frac{b}{2} - j + 1|\} - \max\{|\frac{b}{2} - i|, |\frac{b}{2} - j|\} \\ &= \max\{|\frac{b}{2} - i|, \frac{b}{2} - j + 1\} - \max\{|\frac{b}{2} - i|, \frac{b}{2} - j\}. \end{split}$$

If $|\frac{b}{2} - i| \ge \frac{b}{2} - j + 1$, then

$$\max\{|\frac{b}{2} - i|, \frac{b}{2} - j + 1\} = \max\{|\frac{b}{2} - i|, \frac{b}{2} - j\} = |\frac{b}{2} - i|$$

so Y(i,j) = 0. If $\left|\frac{b}{2} - i\right| < \frac{b}{2} - j + 1$, then $\left|\frac{b}{2} - i\right| \le \frac{b}{2} - j$ and we have

$$\max\{|\frac{b}{2}-i|, \frac{b}{2}-j+1\} = \frac{b}{2}-j+1 \quad \text{and} \quad \max\{|\frac{b}{2}-i|, \frac{b}{2}-j\} = \frac{b}{2}-j$$

so Y(i, j) = 1. Therefore, we have

$$Y(i,j) = \begin{cases} 0 & \text{if } |\frac{b}{2} - i| \ge \frac{b}{2} - j + 1\\ 1 & \text{if } |\frac{b}{2} - i| < \frac{b}{2} - j + 1. \end{cases}$$
(45)

If $i \leq \frac{b}{2}$, as in line (43), then

$$\left|\frac{b}{2} - i\right| = \frac{b}{2} - i \ge \frac{b}{2} - j + 1 \iff j \ge i + 1.$$
(46)

If $i > \frac{b}{2}$, as in line (44), then

$$\left|\frac{b}{2}-i\right| = i - \frac{b}{2} \ge \frac{b}{2} - j + 1 \iff j \ge b - i + 1.$$
 (47)

Combining lines (45), (46) and (47) with lines (43) and (44), we get

$$\chi(\lambda)\chi(\mu) = \sum_{i=0}^{b} \chi(a + \frac{b}{2} - i, \frac{b}{2}) + \sum_{i=0}^{\lfloor \frac{b}{2} \rfloor} \sum_{j=1}^{i} \chi(a + \frac{b}{2} - i, \frac{b}{2} - j) + \sum_{i=\lfloor \frac{b}{2} \rfloor + 1}^{b} \sum_{j=1}^{b-i} \chi(a + \frac{b}{2} - i, \frac{b}{2} - j) = \sum_{i=0}^{b} \chi(a + \frac{b}{2} - i, \frac{b}{2}) + \sum_{i=0}^{b} \sum_{j=1}^{\min\{i, b-i\}} \chi(a + \frac{b}{2} - i, \frac{b}{2} - j) = \sum_{i=0}^{b} \sum_{j=0}^{\min\{i, b-i\}} \chi(a + \frac{b}{2} - i, \frac{b}{2} - j),$$
(48)

establishing the claim of the first statement.

Finally, assume that a < p and b < p. We do a change of basis to express our weights in coordinates with respect to the fundamental weights. We have

$$(a + \frac{b}{2} - i)\epsilon_1 + (\frac{b}{2} - j)\epsilon_2 = (a - i + j)\omega_1 + (b - 2j)\omega_2$$

For the rest of the proof, we use coordinates with respect to the fundamental weights. Let $0 \le i \le b < p$ and $0 \le j \le \min\{i, b - i\}$. We set $\nu := (a - i + j, b - 2j)$. We have

$$a - i + j \le a < p$$
 and $b - 2j \le b < p$,

so dominant weights appearing in line (48) are *p*-restricted. Moreover, $j \leq \frac{b}{2}$, thus $b-2j \geq 0$. If $(a-i+j) \geq -1$, then $\nu \in D$ and we are done. Assume (a-i+j) < -1. We have

$$s_{\alpha_1} \bullet (a - i + j, b - 2j) = (i - a - j - 2, b + 2a - 2i + 2) =: \nu'.$$

By assumption, we have

$$i - a - j - 2 = -(a - i + j) - 2 > 1 - 2 = -1$$

If $b + 2a - 2i + 2 \ge -1$, then $\nu' \in D$ and

$$2(i - a - j - 2) + (b + 2a - 2i + 2) = b - 2j - 2 \le b - 2 \le p - 3j$$

so $\nu' \in \overline{C_1}$ and we are done.

Finally, suppose b + 2a - 2i + 2 < -1. Then we have

$$s_{\alpha_2} \bullet (i - a - j - 2, b + 2a - 2i + 2) = (b + a - i - j + 1, 2i - b - 2a - 4) =: \eta.$$

By assumption, we have

$$2i - b - 2a - 4 = -(b + 2a - 2i + 2) - 2 > 1 - 2 = -1$$

and

$$b + a - i - j + 1 = (b - i) - j + a + 1 \ge a + 1 > 0,$$

hence $\eta \in D$. Moreover, we have

$$2(b+a-i-j+1) + (2i-b-2a-4) = b - 2j - 2 \le p - 3$$

so $\eta \in \overline{C_1}$. Therefore in all cases we are done.

Proposition 6.23. Let $\lambda = (p - 2, 0), \mu = (0, d) \in X^+$ be p-restricted with d > 0. Then $L(\lambda) \otimes L(\mu)$ is multiplicity-free if and only if d = 1.

Proof. We set $M := L(\lambda) \otimes L(\mu)$. Observe that $\lambda \in F_{2,3}$, so $\Delta(\lambda) \cong L(\lambda)$ by Lemma 6.4.

Suppose that d = 1. Using Corollary 1.48 and the structure of L(0, 1), we have

ch
$$M = \sum_{\nu \in X} m_{L(\mu)}(\nu)\chi(\lambda + \nu)$$

= $\chi(p - 2, 1) + \chi(p - 3, 1) + \chi(p - 2, -1) + \chi(p - 1, -1).$

Observe that $(p-2,1) \in F_{3,4}$ and $(p-3,1) \in F_{2,3}$ so $\chi(p-2,1) = \operatorname{ch} L(p-2,1)$ and $\chi(p-3,1) = \operatorname{ch} L(p-3,1)$ by Lemma 6.4. Moreover, $(p-2,-1), (p-1,-1) \in D \setminus X^+$ so $\chi(p-2,-1) = \chi(p-1,-1) = 0$ by Lemma 1.45. Thus we get

$$\operatorname{ch} M = \operatorname{ch} L(p-2,1) + \operatorname{ch} L(p-3,1).$$

In particular, M is multiplicity-free.

Now suppose that d > 1. In particular, $\lambda + \mu \in C_4 \cup (F_{4,7} \setminus F_{4,6})$. By Lemma 6.4, both $L(\lambda)$ and $L(\mu)$ are tilting modules, so M is a tilting module. Since $L(\lambda + \mu)$ is a composition factor of M but $T(\lambda + \mu)$ is not irreducible, we conclude by Lemma 1.45 that M has multiplicity.

Proposition 6.24. Let $\lambda = (p - 1, 0), \mu = (0, d) \in X^+$ be p-restricted with d > 0. Then $L(\lambda) \otimes L(\mu)$ is multiplicity-free if and only if d = 1.

Proof. We set $M := L(\lambda) \otimes L(\mu)$. Observe that $\lambda \in F_{4,6}$, so $\Delta(\lambda) \cong L(\lambda)$ by Lemma 6.4. Moreover, $\lambda + \mu \in F_{4,6}$.

Suppose that d = 1. Using Corollary 1.48 and the structure of L(0, 1), we get

ch
$$M = \sum_{\nu \in X} m_{L(\mu)}(\nu) \chi(\lambda + \nu)$$

= $\chi(p - 1, 1) + \chi(p - 2, 1) + \chi(p - 1, -1) + \chi(p, -1).$

Observe that $(p-1,1) \in F_{4,6}$ and $(p-2,1) \in F_{3,4}$ so $\chi(p-1,1) = \operatorname{ch} L(p-1,1)$ and $\chi(p-2,1) = \operatorname{ch} L(p-2,1)$ by Lemma 6.4. Moreover, $(p-1,-1), (p,-1) \in D \setminus X^+$ so $\chi(p-1,-1) = \chi(p,-1) = 0$ by Lemma 1.45. Thus we get

$$\operatorname{ch} M = \operatorname{ch} L(p-1,1) + \operatorname{ch} L(p-2,1).$$

In particular, M is multiplicity-free.

Now suppose that d > 1. By Lemma 6.4, both $L(\lambda)$ and $L(\mu)$ are tilting modules, so M is a tilting module. We use Argument 1 to show that $L(\lambda + \mu - \alpha_1 - \alpha_2)$ is a composition factor of M. Using Lemmas 6.12, 6.15, 1.15 and 6.6, we have

$$\begin{split} m_{L(\lambda)}(\lambda - \alpha_1) &= 1, & m_{L(\lambda)}(\lambda - \alpha_1 - \alpha_2) &= 1, \\ m_{L(\mu)}(\mu - \alpha_2) &= 1, & m_{L(\mu)}(\mu - \alpha_1 - \alpha_2) &= 1, \\ m_{L(\lambda + \mu)}(\lambda + \mu - \alpha_1) &= 1, & m_{L(\lambda + \mu)}(\lambda + \mu - \alpha_2) &= 1, \\ m_{L(\lambda + \mu)}(\lambda + \mu - \alpha_1 - \alpha_2) &= 2. \end{split}$$

Therefore $m_M(\lambda + \mu - \alpha_1) = m_M(\lambda + \mu - \alpha_2) = 1$ and $L(\lambda + \mu - \alpha_1)$, $L(\lambda + \mu - \alpha_2)$ are not composition factors of M. Moreover, $m_M(\lambda + \mu - \alpha_1 - \alpha_2) = 3$, thus $L(\lambda + \mu - \alpha_1 - \alpha_2)$ is a composition factor of M. Observe that $\lambda + \mu - \alpha_1 - \alpha_2 = (p - 2, d) \in C_4$. By Lemma 6.4, $T(\lambda + \mu - \alpha_1 - \alpha_2)$ is not irreducible and by Lemma 1.36, M has multiplicity. \square

Proposition 6.25. Let $\lambda = (a, 0), \mu = (0, d) \in X^+$ with $2a + d \leq p - 3$ (i.e. $\lambda + \mu \in \widehat{C_1}$). Then $L(\lambda) \otimes L(\mu)$ is multiplicity-free.

Proof. We apply Corollary 3.4 and Theorem 6.16 to conclude that $L(\lambda) \otimes L(\mu)$ is multiplicity-free.

Proposition 6.26. Let $\lambda = (a, 0), \mu = (0, d) \in X^+$ with $0 < a \leq \frac{p-3}{2}, 0 < d \leq p-3$ and 2a + d > p-3 (i.e. $\lambda, \mu \in \widehat{C_1}$ and $\lambda + \mu \notin \widehat{C_1}$). Then $L(\lambda) \otimes L(\mu)$ is multiplicity-free if and only if (a, d) = (1, p-3).

Proof. We set $M := L(\lambda) \otimes L(\mu)$. Observe that $\lambda + \mu \in C_2 \cup C_3 \cup F_{2,3}$. Moreover, $L(\lambda)$ and $L(\mu)$ are tilting modules, so M is a tilting module.

Suppose that $a + d \neq p - 2$ (i.e. $\lambda + \mu \notin F_{2,3}$, so $\lambda + \mu \in C_2 \cup C_3$). The tilting module $T(\lambda + \mu)$ is thus not irreducible, and since $L(\lambda + \mu)$ is a composition factor of M, we conclude by Lemma 1.36 that M has multiplicity.

Suppose that a + d = p - 2 and $a \neq 1$. Using the same argument as in the proof of Proposition 6.24, we get that $L(\lambda + \mu - \alpha_1 - \alpha_2)$ is a composition factor of M. Observe that $\lambda + \mu - \alpha_1 - \alpha_2 = (a - 1, d) \in C_2$ so $T(\lambda + \mu - \alpha_1 - \alpha_2)$ is not irreducible. We conclude by Lemma 1.36 that M has multiplicity.

Finally, suppose that (a,d) = (1, p - 3). Using Proposition 1.47 and the structure of L(1,0), we get

$$ch M = \sum_{\nu \in X} m_{L(1,0)}(\nu) \chi(\mu + \nu)$$
$$= \chi(1, p - 3) + \chi(-1, p - 3) + \chi(-1, p - 1) + \chi(0, p - 3) + \chi(1, p - 5)$$

Observe that $(1, p - 3) \in F_{2,3}$ and $(0, p - 3), (1, p - 5) \in F_{1,2}$, so $\chi(1, p - 3) = \operatorname{ch} L(1, p - 3), \chi(0, p - 3) = \operatorname{ch} L(0, p - 3)$ and $\chi(1, p - 5) = \operatorname{ch} L(1, p - 5)$ by Lemma 6.4. Moreover, $(-1, p - 3), (-1, p - 1) \in D \setminus X^+$ so $\chi(-1, p - 3) = \chi(-1, p - 1) = 0$ by Lemma 1.45. Thus we get

$$\operatorname{ch} M = \operatorname{ch} L(1, p - 3) + \operatorname{ch} L(0, p - 3) + \operatorname{ch} L(1, p - 5)$$

In particular, M is multiplicity-free.

Proposition 6.27. Let $\lambda = (a, 0), \mu = (0, p - 2) \in X^+$ with $0 < a \leq \frac{p-3}{2}$ (i.e. $\lambda \in \widehat{C_1}$ and $\mu \in F_{2,3}$). Then $L(\lambda) \otimes L(\mu)$ has multiplicity.

Proof. By Lemma 6.4, both $L(\lambda)$ and $L(\mu)$ are tilting modules, thus $L(\lambda) \otimes L(\mu)$ is a tilting module. Moreover, $\lambda + \mu \in C_3$, hence $T(\lambda + \mu)$ is not irreducible. Since $L(\lambda + \mu)$ is a composition factor of $L(\lambda) \otimes L(\mu)$, we conclude by Lemma 1.36.

Proposition 6.28. Let $\lambda = (a, 0), \mu = (0, p - 1) \in X^+$ with $0 < a \leq \frac{p-3}{2}$ (i.e. $\lambda \in \widehat{C_1}$ and $\mu \in F_{3,5}$). Then $L(\lambda) \otimes L(\mu)$ is multiplicity-free if and only if a = 1.

Proof. We set $M := L(\lambda) \otimes L(\mu)$. Observe that $L(\lambda)$ and $L(\mu)$ are tilting modules, thus M is a tilting module.

Suppose that a = 1. Using Corollary 1.48 and the structure of L(1,0), we have

$$ch M = \sum_{\nu \in X} m_{L(1,0)}(\nu) \chi(\mu + \nu)$$
$$= \chi(1, p - 1) + \chi(-1, p - 1) + \chi(-1, p + 1) + \chi(0, p - 1) + \chi(1, p - 3)$$

Observe that $(1, p - 1), (0, p - 1) \in F_{3,5}$ and $(1, p - 3) \in F_{2,3}$, so $\chi(1, p - 1) = \operatorname{ch} L(1, p - 1), \chi(0, p - 1) = \operatorname{ch} L(0, p - 1)$ and $\chi(1, p - 3) = \operatorname{ch} L(1, p - 3)$ by Lemma 6.4. Moreover, $(-1, p - 1), (-1, p + 1) \in D \setminus X^+$ so $\chi(-1, p - 1) = \chi(-1, p + 1) = 0$ by Lemma 1.45. Thus we get

$$\operatorname{ch} M = \operatorname{ch} L(1, p-1) + \operatorname{ch} L(0, p-1) + \operatorname{ch} L(1, p-3).$$

In particular, M is multiplicity-free.

Now suppose a > 1. By Lemma 6.4, we have $L(\lambda) \cong \Delta(\lambda)$, $L(\mu) \cong \Delta(\mu)$ and $L(\lambda + \mu) \cong \Delta(\lambda + \mu)$. We use Argument 1 to show that $L(\lambda + \mu - \alpha_1 - 2\alpha_2)$ is a composition factor of M. By Lemma 6.7, we have

$$\begin{split} m_{L(\lambda)}(\lambda - \alpha_{1}) &= 1, & m_{L(\lambda)}(\lambda - \alpha_{1} - \alpha_{2}) = 1, \\ m_{L(\lambda)}(\lambda - \alpha_{1} - 2\alpha_{2}) &= 1, & m_{L(\mu)}(\mu - \alpha_{2}) = 1, \\ m_{L(\mu)}(\mu - \alpha_{1} - \alpha_{2}) &= 1, & m_{L(\mu)}(\mu - \alpha_{1} - 2\alpha_{2}) = 2, \\ m_{L(\mu)}(\mu - 2\alpha_{2}) &= 1, & m_{L(\lambda + \mu)}(\lambda + \mu - \alpha_{1}) = 1, \\ m_{L(\lambda + \mu)}(\lambda + \mu - \alpha_{2}) &= 1, & m_{L(\lambda + \mu)}(\lambda + \mu - \alpha_{1} - \alpha_{2}) = 2, \\ m_{L(\lambda + \mu)}(\lambda + \mu - 2\alpha_{2}) &= 1, & m_{L(\lambda + \mu)}(\lambda + \mu - \alpha_{1} - 2\alpha_{2}) = 3, \\ m_{L(\lambda + \mu - \alpha_{1} - \alpha_{2})}(\lambda + \mu - \alpha_{1} - 2\alpha_{2}) &= 1. \end{split}$$

Therefore, $m_M(\lambda + \mu - \alpha_1) = m_M(\lambda + \mu - \alpha_2) = m_M(\lambda + \mu - 2\alpha_2) = 1$, $m_M(\lambda + \mu - \alpha_1 - \alpha_2) = 3$ and $m_M(\lambda + \mu - \alpha_1 - 2\alpha_2) = 5$. We deduce that $L(\lambda + \mu - \alpha_1 - 2\alpha_2)$ is a composition factor of M. Observe that $\lambda + \mu - \alpha_1 - 2\alpha_2 = (a, p - 3) \in C_3$, thus $T(\lambda + \mu - \alpha_1 - 2\alpha_2)$ is not irreducible. We conclude by Lemma 1.36 that M has multiplicity. **Proposition 6.29.** Let $\lambda = (a, 0), \mu = (0, b) \in X^+$ be p-restricted with $\frac{p-1}{2} \leq a \leq p-3$ and $a+b \leq p-2$ (i.e. $\lambda \in C_2$ and $\lambda + \mu \in \widehat{C_2}$). Then $L(\lambda) \otimes L(\mu)$ is multiplicity-free.

Proof. In this proof, we use Euclidean coordinates. We set $M := L(\lambda) \otimes L(\mu)$.

ch $M = \chi(b\omega_2)(\chi(a,0) - \chi(p-3-a,0))$

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For $i \in \{0, ..., b\}$, we have $p - 2 - 2a + i \le i$. By Proposition 1.47 and Lemma 6.4, and using Lemma 6.22 in the second equality below, we get

$$\begin{split} &= \sum_{i=0}^{b} \sum_{j=0}^{\min\{i,b-i\}} \chi(a + \frac{b}{2} - i, \frac{b}{2} - j) - \sum_{i=0}^{b} \sum_{j=0}^{\min\{i,b-i\}} \chi(p - 3 - a + \frac{b}{2} - i, \frac{b}{2} - j) \\ &= \sum_{i=0}^{b} \sum_{j=\max\{0,p-2-2a+i\}}^{\min\{i,b-i\}} \chi(a + \frac{b}{2} - i, \frac{b}{2} - j) \\ &+ \sum_{i=0}^{b} \sum_{j=0}^{\min\{b-i,p-3-2a+i\}} \chi(a + \frac{b}{2} - i, \frac{b}{2} - j) \\ &- \sum_{i=0}^{b} \sum_{j=\max\{0,b-i+p-2-2a\}}^{\min\{i,b-i\}} \chi(p - 3 - a + \frac{b}{2} - i, \frac{b}{2} - j) \\ &- \sum_{i=0}^{b} \sum_{j=0}^{\min\{i,b-i+p-3-2a\}} \chi(p - 3 - a + \frac{b}{2} - i, \frac{b}{2} - j) \\ &- \sum_{i=0}^{b} \sum_{j=0}^{\min\{i,b-i+p-3-2a\}} \chi(p - 3 - a + \frac{b}{2} - i, \frac{b}{2} - j) \end{split}$$

$$=\sum_{i=0}^{b}\sum_{j=\max\{0,p-2-2a+i\}}^{\min\{i,b-i\}}\chi(a+\frac{b}{2}-i,\frac{b}{2}-j)$$
(49)

$$-\sum_{i=0}^{b}\sum_{j=\max\{0,b-i+p-2-2a\}}^{\min\{i,b-i\}}\chi(p-3-a+\frac{b}{2}-i,\frac{b}{2}-j)$$
(50)

$$+\sum_{i=0}^{b}\sum_{j=0}^{\min\{b-i,p-3-2a+i\}}\chi(a+\frac{b}{2}-i,\frac{b}{2}-j)$$
(51)

$$-\sum_{i=0}^{b} \sum_{j=0}^{\min\{i,b-i+p-3-2a\}} \chi(p-3-a+\frac{b}{2}-i,\frac{b}{2}-j).$$
(52)

First, we work on the terms of lines (51) and (52). We claim that they sum to zero. To that end, observe that in line (51), the second sum is empty whenever i < 2a+3-p and in line (52), the second sum is empty whenever i > b + p - 3 - 2a. Thus we get

$$\begin{split} &\sum_{i=0}^{b} \sum_{j=0}^{\min\{b-i,p-3-2a+i\}} \chi(a+\frac{b}{2}-i,\frac{b}{2}-j) - \sum_{i=0}^{b} \sum_{j=0}^{\min\{i,b-i+p-3-2a\}} \chi(p-3-a+\frac{b}{2}-i,\frac{b}{2}-j) \\ &= \sum_{i=2a+3-p}^{b} \sum_{j=0}^{\min\{b-i,p-3-2a+i\}} \chi(a+\frac{b}{2}-i,\frac{b}{2}-j) \\ &\quad - \sum_{i=0}^{b+p-3-2a} \sum_{j=0}^{\min\{i,b-i+p-3-2a\}} \chi(p-3-a+\frac{b}{2}-i,\frac{b}{2}-j) \end{split}$$

$$=\sum_{i=2a+3-p}^{b}\sum_{j=0}^{\min\{b-i,p-3-2a+i\}}\chi(a+\frac{b}{2}-i,\frac{b}{2}-j)$$
$$-\sum_{i=2a+3-p}^{b}\sum_{j=0}^{\min\{i-2a-3+p,b-i\}}\chi(a+\frac{b}{2}-i,\frac{b}{2}-j)$$
$$=0.$$

Therefore, only lines (49) and (50) remain and we get

$$\operatorname{ch} M = \sum_{i=0}^{b} \sum_{j=\max\{0, p-2-2a+i\}}^{\min\{i, b-i\}} \chi(a + \frac{b}{2} - i, \frac{b}{2} - j) - \sum_{i=0}^{b} \sum_{j=\max\{0, b-i+p-2-2a\}}^{\min\{i, b-i\}} \chi(p - 3 - a + \frac{b}{2} - i, \frac{b}{2} - j) = \sum_{i=0}^{b} \sum_{j=\max\{0, p-2-2a+i\}}^{\min\{i, b-i\}} \chi(a + \frac{b}{2} - i, \frac{b}{2} - j) - \sum_{i=0}^{b} \sum_{j=\max\{0, i+p-2-2a\}}^{\min\{i, b-i\}} \chi(p - 3 - a - \frac{b}{2} + i, \frac{b}{2} - j) = \sum_{i=0}^{b} \sum_{j=\max\{0, p-2-2a+i\}}^{\min\{i, b-i\}} \chi(a + \frac{b}{2} - i, \frac{b}{2} - j) - \chi(p - 3 - a - \frac{b}{2} + i, \frac{b}{2} - j).$$
(53)

At this step, observe that if $i > \frac{b+2a+2-p}{2}$, then p-2-2a+i > b-i and the second sum in line (53) is empty. Thus we get

$$\operatorname{ch} M = \sum_{i=0}^{\min\{b, \lfloor \frac{b+2a+2-p}{2} \rfloor\}} \sum_{j=\max\{0, p-2-2a+i\}}^{\min\{i, b-i\}} \chi(a + \frac{b}{2} - i, \frac{b}{2} - j) - \chi(p - 3 - (a + \frac{b}{2} - i), \frac{b}{2} - j).$$

For $0 \le i \le \min\{b, \lfloor \frac{b+2a+2-p}{2} \rfloor\}$ and $\max\{0, p-2-2a+i\} \le j \le \min\{i, b-i\}$, we have

$$a + \frac{b}{2} - i \ge a + \frac{b}{2} - (\frac{b + 2a + 2 - p}{2}) = \frac{p - 2}{2} > \frac{p - 3}{2},$$

$$\frac{b}{2} - j \ge 0 \qquad \text{and} \qquad (a + \frac{b}{2} - i) + (\frac{b}{2} - j) = a + b - i - j \le a + b \le p - 2.$$

Therefore, $(a + \frac{b}{2} - i, \frac{b}{2} - j) \in \widehat{C_2}$ for all $0 \leq i \leq \min\{b, \lfloor \frac{b+2a+2-p}{2} \rfloor\}$ and $\max\{0, p-2-2a+i\} \leq j \leq \min\{i, b-i\}$ and by Remark 6.5, we get

$$\operatorname{ch} M = \sum_{i=0}^{\min\{b, \lfloor \frac{b+2a+2-p}{2} \rfloor\}} \sum_{j=\max\{0, p-2-2a+i\}}^{\min\{i, b-i\}} \operatorname{ch} L(a + \frac{b}{2} - i, \frac{b}{2} - j).$$

In particular, M is multiplicity-free.
Proposition 6.30. Let $\lambda = (a, 0), \mu = (0, b) \in X^+$ be *p*-restricted with $\frac{p-1}{2} \le a \le p-3$ and a+b > p-2 (i.e. $\lambda \in C_2$ and $\lambda + \mu \in \widehat{C_3} \cup \widehat{C_4}$). Then $L(\lambda) \otimes L(\mu)$ has multiplicity.

Proof. In this proof, we use Euclidean coordinates. We set $M := L(\lambda) \otimes L(\mu)$. By Proposition 1.47, Lemma 6.22 and Remark 6.5, we have

ch $M = \chi(\lambda)\chi(\mu) - \chi(s_{\alpha_1+\alpha_2,p} \bullet \lambda)\chi(\mu)$

$$=\sum_{i=0}^{b}\sum_{j=0}^{\min\{i,b-i\}}\chi(a+\frac{b}{2}-i,\frac{b}{2}-j)-\sum_{i=0}^{b}\sum_{j=0}^{\min\{i,b-i\}}\chi(p-3-a+\frac{b}{2}-i,\frac{b}{2}-j).$$
(54)

We set

$$A := \{ (a + \frac{b}{2} - i, \frac{b}{2} - j) | 0 \le i \le b, \ 0 \le j \le \min\{i, b - i\} \}$$

and

$$B := \{ (p-3-a+\frac{b}{2}-i,\frac{b}{2}-j) | \ 0 \le i \le b, \ 0 \le j \le \min\{i,b-i\} \}.$$

By Lemma 6.22, we have $A, B \subseteq D \cup W \bullet \overline{C_1}$.

If a + b is even, we set $t := \frac{a+b-p+1}{2}$, $\nu_3 := (a + \frac{b}{2} - t, \frac{b}{2} - t) \in C_3$ and $\nu_2 := s_{\alpha_1+2\alpha_2,p} \bullet \nu_3 = (a + \frac{b}{2} - (t+1), \frac{b}{2} - (t+1)) \in C_2.$

If a + b is odd, we set $t := \frac{a+b-p}{2}$, $\nu_3 := (a + \frac{b}{2} - (t+1), \frac{b}{2} - t) \in C_3$ and $\nu_2 := s_{\alpha_1+2\alpha_2,p} \bullet \nu_3 = (a + \frac{b}{2} - (t+2), \frac{b}{2} - (t+1)) \in C_2.$

In both cases, we show that ν_2 has multiplicity 2 in M. Observe that $\nu_2, \nu_3 \in A$ (this follows from $t+1 \leq \frac{b}{2}$). Moreover, we claim that $\nu_2, \nu_3 \notin B$. We check ν_3 in case a+b even, the other cases are similar. Suppose that $(a + \frac{b}{2} - t, \frac{b}{2} - t) = (p - 3 - a + \frac{b}{2} - k, \frac{b}{2} - r)$ for some k, r. Then r = t and $k = \frac{p+b-5-3a}{2} < r$, thus $\nu_3 \notin B$.

For every weight $\nu \in X$, we fix $w_{\nu} \in W$ such that $w_{\nu} \cdot \nu \in D$ (recall that D is a fundamental domain for the dot action of W on X). Moreover, we take the convention that $\Delta(\nu) = 0$ for every $\nu \in D \setminus X^+$. Using line (54) and Lemma 1.45, we have

$$[M:L(\nu_2)] = \sum_{\eta \in A} \det(w_\eta) [\Delta(w_\eta \bullet \eta) : L(\nu_2)] - \sum_{\eta \in B} \det(w_\eta) [\Delta(w_\eta \bullet \eta) : L(\nu_2)].$$

By the Strong Linkage Principle (Proposition 1.53), $[\Delta(\eta) : L(\nu_2)] = 0$ unless $\nu_2 \uparrow \eta$. By Lemma 6.4, we have $[\Delta(\eta) : L(\nu_2)] = 0$ for all $\eta \in C_4$. Thus, if η is *p*-restricted and such that $[\Delta(\eta) : L(\nu_2)] \neq 0$, we have either $\eta = \nu_2$ or $\eta = \nu_3$. At this step, recall that every $\eta \in (A \cup B) \cap X^+$ is *p*-restricted. By Lemma 6.22, if $\eta \in A \cup B$ and $w_\eta \neq id$, then $w_\eta \bullet \eta \in \overline{C_1}$ and $[\Delta(w_\eta \bullet \eta) : \nu_2] = 0$. Therefore, we have

$$[M:L(\nu_2)] = \sum_{\eta \in A \cap \{\nu_2, \nu_3\}} [\Delta(\eta):L(\nu_2)] - \sum_{\eta \in B \cap \{\nu_2, \nu_3\}} [\Delta(\eta):L(\nu_2)].$$

By the previous observations, $B \cap \{\nu_2, \nu_3\} = \emptyset$ and $A \cap \{\nu_2, \nu_3\} = \{\nu_2, \nu_3\}$. Therefore,

$$[M: L(\nu_2)] = [\Delta(\nu_2): L(\nu_2)] + [\Delta(\nu_3): L(\nu_2)] = 2$$

and M has multiplicity.

6.3.4 $L(a,b) \otimes L(0,d)$

Proposition 6.31. Let $\lambda = (a, b), \mu = (0, d) \in X^+$ with $1 \le a < p$ and $2 \le b, d < p$. If $2a + b + 2 \not\equiv 0 \mod p$ and $a + b \neq p - 1$, then $L(\lambda) \otimes L(\mu)$ has multiplicity.

Proof. We set $M := L(\lambda) \otimes L(\mu)$. Using Argument 1, we show that either $[M : L(\lambda + \mu - \alpha_1 - \alpha_2)] \ge 2$ or $[M : L(\lambda + \mu - \alpha_1 - 2\alpha_2)] \ge 2$. Using Lemmas 1.15, 6.6, 6.7 and 6.8, we have

 $m_M(\lambda + \mu) = 1,$ $m_M(\lambda + \mu - \alpha_1) = 1,$ $m_M(\lambda + \mu - \alpha_2) = 2,$ $m_M(\lambda + \mu - \alpha_1 - \alpha_2) = 4,$ $m_M(\lambda + \mu - 2\alpha_2) = 3,$ $m_M(\lambda + \mu - \alpha_1 - 2\alpha_2) = 9.$

Therefore

$$[M: L(\lambda + \mu)] = [M: L(\lambda + \mu - \alpha_2)] = [M: L(\lambda + \mu - 2\alpha_2)] = 1$$

and

$$[M: L(\lambda + \mu - \alpha_1)] = 0.$$

If $m_{L(\lambda+\mu)}(\lambda+\mu-\alpha_1-\alpha_2) = 1$, then $[M: L(\lambda+\mu-\alpha_1-\alpha_2)] = 2$ and M has multiplicity. If $m_{L(\lambda+\mu)}(\lambda+\mu-\alpha_1-\alpha_2) = 2$, then $[M: L(\lambda+\mu-\alpha_1-\alpha_2)] = 1$. In this case, we have

$$m_{L(\lambda+\mu)}(\lambda+\mu-\alpha_1-2\alpha_2) \leq 3$$
 and $m_{L(\lambda+\mu-\alpha_2)}(\lambda+\mu-\alpha_1-2\alpha_2) \leq 2.$

Therefore, $[M: L(\lambda + \mu - \alpha_1 - 2\alpha_2)] \ge 9 - 3 - 2 - 1 - 1 = 2$ and M has multiplicity. \Box

Proposition 6.32. Let $\lambda = (a, b), \mu = (0, 1) \in X^+$ with $1 \le a, b < p$. Then $L(\lambda) \otimes L(\mu)$ is multiplicity-free if and only if $\lambda \in C_1 \cup C_2 \cup C_3 \cup C_4$.

Proof. In this proof, we take the convention that $\operatorname{ch} L(\nu) = 0$ for all $\nu \notin X^+$. Moreover, for $\nu = (c, d) \in X$, we set

$$\delta_2(\nu) := \begin{cases} 0 & \text{if } 2c + d + 2 = p, \\ 1 & \text{else.} \end{cases} \qquad \delta_3(\nu) := \begin{cases} 0 & \text{if } c + d = p - 1, \\ 1 & \text{else.} \end{cases}$$

$$\delta_4(\nu) := \begin{cases} 0 & \text{if } 2c + d + 2 = 2p, \\ 1 & \text{else.} \end{cases}$$

We set $M := L(\lambda) \otimes L(\mu)$. By Lemma 6.4, $L(\mu)$ is a tilting module.

- If $\lambda \in F_{1,2} \cup F_{2,3} \cup F_{3,4}$ then $L(\lambda)$ is a tilting module so M is a tilting module. In this case $\lambda + \mu \in C_2 \cup C_3 \cup C_4 \cup (F_{4,7} \setminus F_{4,6})$, hence $T(\lambda + \mu)$ is not irreducible by Lemma 6.4. We conclude by Lemma 1.36 that M has multiplicity.
- If $\lambda \in F_{3,5} \cup F_{4,7}$, then b = p-1 so $\lambda + \mu$ is not *p*-restricted. We conclude by Corollary 2.8 that *M* has multiplicity.
- If $\lambda \in F_{4,6}$ (i.e. a = p 1), observe that $L(\lambda + \mu \alpha_2)$ is a composition factor of M (by Argument 1). But $\lambda + \mu \alpha_2$ is not p-restricted, hence we conclude by Corollary 2.7 that M has multiplicity.
- If $\lambda \in C_1$, then $\lambda + \mu \in \widehat{C_1}$ and we apply Corollary 3.4 and Theorem 6.16 to conclude that M is multiplicity-free.

• If $\lambda \in C_2$, let $\lambda_1 := s_{\alpha_1 + \alpha_2, p} \bullet \lambda \in C_1$ and $\delta_2 := \delta_2(\lambda)$. By Proposition 1.47 and Lemma 6.4 we have

$$\begin{split} \operatorname{ch} M &= \chi(\mu)(\chi(\lambda) - \chi(\lambda_1)) \\ &= \chi(\lambda + \mu) + \chi(\lambda + \mu - \alpha_2) + \chi(\lambda + \mu - \alpha_1 - \alpha_2) + \chi(\lambda + \mu - \alpha_1 - 2\alpha_2) \\ &- \chi(\lambda_1 + \mu) - \chi(\lambda_1 + \mu - \alpha_2) - \chi(\lambda_1 + \mu - \alpha_1 - \alpha_2) \\ &- \chi(\lambda_1 + \mu - \alpha_1 - 2\alpha_2) \\ &= (\chi(\lambda + \mu) - \chi(\lambda_1 + \mu - \alpha_1 - \alpha_2)) + (\chi(\lambda + \mu - \alpha_2) - \chi(\lambda_1 + \mu - \alpha_1 - 2\alpha_2)) \\ &+ (\chi(\lambda + \mu - \alpha_1 - \alpha_2) - \chi(\lambda_1 + \mu)) \\ &+ (\chi(\lambda + \mu - \alpha_1 - 2\alpha_2) - \chi(\lambda_1 + \mu - \alpha_2)) \\ &= \operatorname{ch} L(\lambda + \mu) + \operatorname{ch} L(\lambda + \mu - \alpha_2) + \delta_2 \cdot \operatorname{ch} L(\lambda + \mu - \alpha_1 - \alpha_2) \\ &+ \delta_2 \cdot \operatorname{ch} L(\lambda + \mu - \alpha_1 - 2\alpha_2). \end{split}$$

Thus M is multiplicity-free.

• If $\lambda \in C_3$, let $\lambda_2 := s_{\alpha_1+2\alpha_2,p} \bullet \lambda \in C_2$, $\delta_2 := \delta_2(\lambda_2)$ and $\delta_3 := \delta_3(\lambda)$. By Proposition 1.47, Lemma 6.4 and the previous case, we have

$$\begin{split} \operatorname{ch} M &= \chi(\mu)(\chi(\lambda) - \operatorname{ch} L(\lambda_2)) \\ &= \chi(\lambda + \mu) + \chi(\lambda + \mu - \alpha_2) + \chi(\lambda + \mu - \alpha_1 - \alpha_2) + \chi(\lambda + \mu - \alpha_1 - 2\alpha_2) \\ &- \operatorname{ch} L(\lambda_2 + \mu) - \operatorname{ch} L(\lambda_2 + \mu - \alpha_2) - \delta_2 \cdot \operatorname{ch} L(\lambda_2 + \mu - \alpha_1 - \alpha_2) \\ &- \delta_2 \cdot \operatorname{ch} L(\lambda_2 + \mu - \alpha_1 - 2\alpha_2) \\ &= (\chi(\lambda + \mu) - \delta_2 \cdot \operatorname{ch} L(\lambda_2 + \mu - \alpha_1 - 2\alpha_2)) \\ &+ (\chi(\lambda + \mu - \alpha_2) - \operatorname{ch} L(\lambda_2 + \mu - \alpha_2)) \\ &+ (\chi(\lambda + \mu - \alpha_1 - \alpha_2) - \delta_2 \cdot \operatorname{ch} L(\lambda_2 + \mu - \alpha_1 - \alpha_2)) \\ &+ (\chi(\lambda + \mu - \alpha_1 - 2\alpha_2) - \operatorname{ch} L(\lambda_2 + \mu)) \\ &= \operatorname{ch} L(\lambda + \mu) + \operatorname{ch} L(\lambda + \mu - \alpha_2) + \operatorname{ch} L(\lambda + \mu - \alpha_1 - \alpha_2) \\ &+ \delta_3 \cdot \operatorname{ch} L(\lambda + \mu - \alpha_1 - 2\alpha_2). \end{split}$$

Thus M is multiplicity-free.

• If $\lambda \in C_4$, let $\lambda_3 := s_{\alpha_1 + \alpha_2, 2p} \bullet \lambda \in C_3$, $\delta_3 := \delta_3(\lambda_3)$ and $\delta_4 := \delta_4(\lambda)$. By Proposition 1.47, Lemma 6.4 and the previous case, we have

$$ch M = \chi(\mu)(\chi(\lambda) - ch L(\lambda_3))$$

$$= \chi(\lambda + \mu) + \chi(\lambda + \mu - \alpha_2) + \chi(\lambda + \mu - \alpha_1 - \alpha_2) + \chi(\lambda + \mu - \alpha_1 - 2\alpha_2)$$

$$- ch L(\lambda_3 + \mu) - ch L(\lambda_3 + \mu - \alpha_2) - ch L(\lambda_3 + \mu - \alpha_1 - \alpha_2)$$

$$- \delta_3 \cdot ch L(\lambda_3 + \mu - \alpha_1 - 2\alpha_2)$$

$$= (\chi(\lambda + \mu) - \operatorname{ch} L(\lambda_3 + \mu - \alpha_1 - \alpha_2)) + (\chi(\lambda + \mu - \alpha_2) - \delta_3 \cdot \operatorname{ch} L(\lambda_3 + \mu - \alpha_1 - 2\alpha_2)) + (\chi(\lambda + \mu - \alpha_1 - \alpha_2) - \operatorname{ch} L(\lambda_3 + \mu)) + (\chi(\lambda + \mu - \alpha_1 - 2\alpha_2) - \operatorname{ch} L(\lambda_3 + \mu - \alpha_2)) = \operatorname{ch} L(\lambda + \mu) + \operatorname{ch} L(\lambda + \mu - \alpha_2) + \delta_4 \cdot \operatorname{ch} L(\lambda + \mu - \alpha_1 - \alpha_2) + \delta_4 \cdot \operatorname{ch} L(\lambda + \mu - \alpha_1 - 2\alpha_2).$$

Thus M is multiplicity-free.

The classification of multiplicity-free tensor products of simple modules with *p*-restricted highest weight for an algebraic group of type B_2 is not completed in this thesis. In the previous sequence of propositions, we fully treated the following cases:

- $\lambda = (a, b), \ \mu = (c, d)$ with $a \cdot b = 0$ and $c \cdot d = 0$,
- $\lambda = (a, b), \ \mu = (0, 1).$

It remains to consider the following cases:

- $\lambda = (a, b), \ \mu = (0, d) \text{ with } a \neq 0, \ b \neq 0 \text{ and } d \geq 2,$
- $\lambda = (a, b), \ \mu = (c, 0)$ with $a \neq 0, \ b \neq 0$ and $c \neq 0$,
- $\lambda = (a, b), \ \mu = (c, d) \text{ with } a \cdot b \cdot c \cdot d \neq 0.$

7 SL_n for p = 2

In this section, we classify multiplicity-free tensor products of simple SL_n -modules with p-restricted highest weight when p = 2. To that end, we will use the classification of completely reducible tensor products of simple SL_n -modules with p-restricted highest weight for p = 2 ([Gru21, Theorem 7.12]).

Theorem 7.1. Let G be of type A_n and p = 2. Let $\lambda, \mu \in X^+$ be nonzero 2-restricted dominant weights. Up to the reordering of λ and μ , $L(\lambda) \otimes L(\mu)$ is completely reducible if and only if one of the following holds:

- (1) $\lambda = \omega_1$ and $\mu = \omega_{i_1} + \ldots + \omega_{i_r}$ for even numbers $1 < i_1 < \ldots < i_r \leq n$,
- (2) $\lambda = \omega_n$ and $\mu = \omega_{i_1} + \ldots + \omega_{i_r}$ for certain $i_1 < \ldots < i_r < n$ such that $n + 1 i_j$ is even for all $j \in \{1, \ldots, r\}$,
- (3) $\lambda = \omega_2$ and $\mu = \omega_j$ for some $2 < j \le n$ with $j 2 \equiv 3 \mod 4$,
- (4) $\lambda = \omega_{n-1}$ and $\mu = \omega_j$ for some $1 \le j < n-1$ with $n-1-j \equiv 3 \mod 4$.

Theorem 7.2 ([Ste03, Theorem 1.1.A]). Let G be of type A_n , $\lambda \in X^+$ and $i \in \{1, \ldots, n\}$. Then $L_{\mathbb{C}}(\omega_i) \otimes L_{\mathbb{C}}(\lambda)$ is multiplicity-free.

Theorem 7.3. Let G be of type A_n and p = 2. Let $\lambda, \mu \in X^+$ be nonzero 2-restricted dominant weights. Up to the reordering of λ and μ and up to duality, $L(\lambda) \otimes L(\mu)$ is multiplicityfree if and only if one of the following holds:

- (1) $\lambda = \omega_1$ and $\mu = \omega_{i_1} + \ldots + \omega_{i_r}$ for even numbers $1 < i_1 < \ldots < i_r \leq n$,
- (2) $\lambda = \omega_n$ and $\mu = \omega_{i_1} + \ldots + \omega_{i_r}$ for certain $i_1 < \ldots < i_r < n$ such that $n + 1 i_j$ is even for all $j \in \{1, \ldots, r\}$,
- (3) $\lambda = \omega_2$ and $\mu = \omega_j$ for some $2 < j \le n$ with $j 2 \equiv 3 \mod 4$,
- (4) $\lambda = \omega_{n-1}$ and $\mu = \omega_j$ for some $1 \le j < n-1$ with $n-1-j \equiv 3 \mod 4$.

Proof. Suppose that λ and μ do not satisfy the conditions. Then $L(\lambda) \otimes L(\mu)$ is not completely reducible by Theorem 7.1, and hence it has multiplicity by Lemma 2.5.

Now suppose that λ , μ verify the condition of the theorem. By Theorem 7.1, $L(\lambda) \otimes L(\mu)$ is completely reducible. Moreover, $L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu)$ is multiplicity-free by Theorem 7.2. Therefore, $L(\lambda) \otimes L(\mu)$ is multiplicity-free by Theorem 3.6.

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