## ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Master thesis

Master in Mathematics

# **Multiplicity-free tensor products of irreducible modules over simple algebraic groups in positive characteristic**

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## <span id="page-2-0"></span>**List of notations**



## <span id="page-3-0"></span>**Introduction**

Tensor products are of great interest in representation theory. Determining their structure has been the subject of much research. When the category of representations we are working with is semisimple, the central question is to find the decomposition of a tensor product of simple modules into a direct sum of simple modules. In particular, this is the case for representations of simple Lie algebras over the field of complex numbers, and for representations of simple algebraic groups over the same field. In the case of  $\mathfrak{sl}_2(\mathbb{C})$ , the Clebsch-Gordan formula gives an answer to this question.

Another question that has been the subject of research is to determine whether certain simple modules appear several times in the decomposition of the tensor product, and thus classify tensor products without multiplicity. In the case of simple Lie algebras and simple algebraic groups, this question was resolved in 2003 by Stembridge ([\[Ste03\]](#page-78-0)). In particular, this classification shows that if the tensor product of two simple modules is multiplicity-free, then the highest weight of one of the two modules is a multiple of a fundamental weight. This fact is no longer true in positive characteristic.

When we move on to a field of positive characteristic, the category of representations of a simple algebraic group is no longer semisimple. Other questions, often complicated, may then arise, such as finding indecomposable direct summands and classifying completely reducible tensor products. The classification of multiplicity-free modules is also of interest. Recently, Gruber showed that a multiplicity-free tensor product of simple modules is necessarily completely reducible, which gives another motivation for this classification.

In this project, we will therefore focus on the classification of multiplicity-free tensor products of simple modules over an algebraically closed field of characteristic *p >* 0. Our main question is:

### **Question 1.** *Given a simply connected simple algebraic group G, for which pairs of simple modules*  $L(\lambda)$  *and*  $L(\mu)$  *is the tensor product*  $L(\lambda) \otimes L(\mu)$  *multiplicity-free* ?

We will provide the complete classification in the case of  $SL_2$  and  $SL_3$ , and show a number of important results in the case of  $Sp_4$ . In addition, we will show that, under certain assumptions, being completely reducible implies being multiplicity-free. Using the classification of completely reducible tensor products of simple modules for  $SL_n$  over a field of characteristic 2, established by Gruber ([\[Gru21\]](#page-77-1)), we will answer our question in the case of  $SL_n$  for  $p=2$ .

The first part of this project recalls important notions of representation theory that will be used later. In the second part, we recall some results related to tensor products and show that we can restrict our attention to simple modules with *p*-restricted highest weight in order to answer our question. In the third part, we show some connections between multiplicityfreeness over  $\mathbb C$  and in positive characteristic. In parts 4 to 7, we proceed to the classification of multiplicity-free tensor products in the cases of  $SL_2$ ,  $SL_3$ ,  $Sp_4$  (partial classification only), and  $SL_n$  for  $p = 2$  respectively. This work could be continued on the one hand by completing the classification for Sp<sup>4</sup> , and on the other hand by generalising these results to other simply connected simple algebraic groups.

It should also be noted that we used Magma ([\[BCP97\]](#page-77-2)) in order to compute the composition factors of certain tensor products and to have concrete examples, which enabled us to have a better understanding of the structure of these tensor products.

We assume that the reader is familiar with the representation theory of semisimple Lie algebras, as well as with the basics of algebraic group theory. We will therefore not repeat the relative notions in the preliminaries, but refer the reader to [\[Hum00\]](#page-77-3) for the representation theory of semisimple Lie algebras and to [\[MT11\]](#page-78-1) for the theory of algebraic groups.

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## <span id="page-5-0"></span>**1 Preliminaries**

In this section, we introduce all the notions needed to solve our problem and introduce some arguments which will be used several times in the next parts of this project.

#### <span id="page-5-1"></span>**1.1 Weights and alcoves**

We start by recalling some results about root systems and weights. Then, we will define the notion of an alcove, which has a very important role in the structure of some modules.

Let  $X_{\mathbb{R}}$  be a Euclidean space of dimension *n* with scalar product ( , ):  $X_{\mathbb{R}} \times X_{\mathbb{R}} \to \mathbb{R}$ and let  $\Phi \subseteq X_{\mathbb{R}}$  be an irreducible root system. We fix  $\Pi = {\alpha_1, \dots, \alpha_n}$  a base of  $\Phi$  and denote by  $\Phi^+$  the set of positive roots with respect to Π. Moreover, we fix p a prime.

For  $\alpha \in \Phi$ , we define its *coroot*  $\alpha^{\vee} := \frac{2\alpha}{(\alpha,\alpha)} \in X_{\mathbb{R}}$ . We set

$$
X_{\mathbb{R}}^{+} := \{ x \in X_{\mathbb{R}} | (x, \alpha^{\vee}) \ge 0 \quad \forall \alpha \in \Phi^{+} \},
$$

and define the *weight lattice* of Φ to be the set

$$
X := \{ \lambda \in X_{\mathbb{R}} | \ (\lambda, \alpha^{\vee}) \in \mathbb{Z} \quad \forall \alpha \in \Phi \}.
$$

Moreover we define the set of *dominant weights* to be the set

$$
X^+ := X \cap X^+_{\mathbb{R}} = \{ \lambda \in X | \ (\lambda, \alpha^{\vee}) \ge 0 \quad \forall \alpha \in \Phi^+ \}.
$$

There is a partial order on *X* given by  $\lambda \leq \mu$  if  $\lambda - \mu$  is a N-linear combination of simple roots, i.e.  $\lambda - \mu = \sum$  $\sum_{\alpha \in \Pi} n_{\alpha} \alpha$  with  $n_{\alpha} \in \mathbb{N}$   $\forall \alpha \in \Pi$ .

Since  $\Pi$  is a basis of  $X_{\mathbb{R}}$ , it follows that  $\{\alpha^{\vee} \mid \alpha \in \Pi\}$  is a basis of  $X_{\mathbb{R}}$ . Thus it admits a dual basis with respect to (, ) and there exists a set  $\{\omega_\alpha | \alpha \in \Pi\}$  whose elements satisfy

$$
(\omega_{\alpha}, \alpha^{\vee}) = 1
$$
 and  $(\omega_{\alpha}, \beta^{\vee}) = 0 \quad \forall \beta \in \Pi \setminus {\alpha}.$ 

We observe that the  $\omega_{\alpha} \in X^+$  for all  $\alpha \in \Pi$ . We call the weights  $\omega_{\alpha}$  with  $\alpha \in \Pi$  the *fundamental dominant weights*. One can easily check that every weight is a Z-linear combination of the fundamental dominant weights. To simplify the notation, we set  $\omega_i := \omega_{\alpha_i}$  for  $i \in \{1, \ldots, n\}$ . We will use the numeration of simple roots given in [\[Hum00,](#page-77-3) 11.4], so the numeration of the fundamental dominant weights will correspond to this labelling of Dynkin diagrams.

**Definition 1.1.** *A dominant weight*  $\lambda \in X^+$  *is called p*-restricted *if*  $(\lambda, \alpha^{\vee}) < p$  *for all α* ∈ Π*.*

We denote the highest short root in the root system  $\Phi$  (with respect to  $\Pi$ ) by  $\alpha_h$ and the half sum of all positive roots by  $\rho := \frac{1}{2} \sum_{n=1}^{\infty}$ *α*∈Φ<sup>+</sup> *α*. This element satisfies  $\rho = \sum_{n=1}^{\infty}$  $\sum_{i=1} \omega_i$ ([\[Hum00,](#page-77-3) 13.3]). The *Coxeter number* of the root system  $\Phi$  is  $h := (\rho, \alpha_h^{\vee}) + 1$ .

<span id="page-5-2"></span>**Remark 1.2.** The coroot  $\alpha_h^{\vee}$  is the highest root in the dual root system  $\Phi^{\vee}$  and for any dominant weight  $\lambda \in X^+$ , we have  $(\lambda, \alpha^{\vee}) \leq (\lambda, \alpha^{\vee}_h)$  for all  $\alpha \in \Phi^+$ . More generally, for every  $x \in X_{\mathbb{R}}^+$  $\mathbb{R}^+$  and  $\alpha \in \Phi^+$ , we have  $(x, \alpha^{\vee}) \leq (x, \alpha^{\vee})$ . Recall also that  $(\alpha, \alpha^{\vee}) \geq 0$  for all  $\alpha \in \Phi^+.$ 

**Definition 1.3.** *A set*  $Y \subseteq X$  *is called* saturated *if for all*  $\lambda \in Y$ ,  $\alpha \in \Phi$  *and i* between 0 *and*  $(\lambda, \alpha^{\vee})$ *, we have*  $\lambda - i\alpha \in Y$ *.* 

For  $\alpha \in \Phi$  we define the reflection  $s_{\alpha} \in GL(X_{\mathbb{R}})$  by

$$
s_{\alpha}(v) := v - \frac{2(v, \alpha)}{(\alpha, \alpha)} \alpha \quad \text{for } v \in X_{\mathbb{R}}.
$$

We also define the *Weyl group* of Φ by

$$
W := \langle s_{\alpha} | \alpha \in \Phi \rangle \subseteq GL(X_{\mathbb{R}}).
$$

The group *W* is generated by  $\{s_\alpha | \alpha \in \Pi\}$ . More precisely,  $(W, \{s_\alpha | \alpha \in \Pi\})$  is a Coxeter system ([\[Hum90,](#page-77-4) 1.5]). It contains a unique longest element which we denote by  $w_0 \in W$ ([\[Hum90,](#page-77-4) 1.8]).

For  $\alpha \in \Phi$  and  $m \in \mathbb{Z}$ , we define the affine reflection  $s_{\alpha,mp} \in \text{AGL}(X_{\mathbb{R}})$  by

$$
s_{\alpha,mp}(v) := s_{\alpha}(v) + mp\alpha \quad \text{for } v \in X_{\mathbb{R}}.
$$

The *affine Weyl group* associated to  $\Phi$  and p, denoted by  $W_p$ , is the group

 $W_p := \langle s_{\alpha,mp} | \alpha \in \Phi, m \in \mathbb{Z} \rangle \subseteq \text{AGL}(X_{\mathbb{R}}).$ 

The *dot action* of  $W_p$  on  $X_{\mathbb{R}}$  is the group action given by

$$
w \bullet x := w(x + \rho) - \rho
$$
 for  $w \in W_p$  and  $x \in X_{\mathbb{R}}$ .

For  $\alpha \in \Phi$  and  $m \in \mathbb{Z}$ , we define the *reflection hyperplane* of  $s_{\alpha, mn}$  (for the dot action) to be the set

$$
H_{\alpha,m} := \{ x \in X_{\mathbb{R}} | (x + \rho, \alpha^{\vee}) = mp \}.
$$

Since *W* is a subgroup of  $W_p$ , we can restrict the dot action to *W*. We set

$$
D := \{ \lambda \in X \mid \lambda + \rho \in X^+ \},
$$

which is a fundamental domain for the dot action of *W* on *X*. (This follows from the facts that  $X^+_{\mathbb{R}}$  $\mathbb{R}^+$  is a fundamental domain for the action of *W* on  $X_{\mathbb{R}}$  ([\[Hum90,](#page-77-4) 1.12]) and that *X* is preserved by *W*.)

**Definition 1.4.** *Let*  $n = (n_{\alpha})_{\alpha \in \Phi^+} \in \mathbb{Z}^{|\Phi^+|}$ *. We define* 

 $C_n := \{ x \in X_{\mathbb{R}} \mid (n_\alpha - 1)p < (x + \rho, \alpha^{\vee}) < n_\alpha p \text{ for all } \alpha \in \Phi^+ \}.$ 

We say that  $C_n$  is an alcove if it is a non-empty set. For  $C_n$  an alcove, its upper closure is *the set*

$$
\widehat{C_n} := \{ x \in X_{\mathbb{R}} \mid (n_\alpha - 1)p < (x + \rho, \alpha^\vee) \le n_\alpha p \quad \text{for all } \alpha \in \Phi^+ \},
$$

*and its* closure *is the set*

$$
\overline{C_n} := \{ x \in X_{\mathbb{R}} | (n_\alpha - 1)p \le (x + \rho, \alpha^\vee) \le n_\alpha p \quad \text{for all } \alpha \in \Phi^+ \}.
$$

Alternatively, we can define an alcove to be a connected component of  $X_{\mathbb{R}} \setminus \cup$ *α*∈Φ *m*∈Z *Hα,m*.

**Definition 1.5.** *The* fundamental alcove *is the alcove*

$$
C_1 := \{ x \in X_{\mathbb{R}} \mid 0 < (x + \rho, \alpha^{\vee}) < p \quad \text{for all } \alpha \in \Phi^+ \}.
$$

<span id="page-6-0"></span>**Remark 1.6.** By Remark [1.2,](#page-5-2) we have

$$
C_1 = \{ x \in X_{\mathbb{R}} \mid (x + \rho, \alpha_h^{\vee}) < p \text{ and } 0 < (x + \rho, \alpha^{\vee}) \quad \text{for all } \alpha \in \Pi \}.
$$

<span id="page-7-3"></span>**Lemma 1.7.** Let  $\lambda \in X^+ \cap \widehat{C_1}$  and  $\mu \in X^+$  be such that  $\mu \leq \lambda$ . We have  $\mu \in \widehat{C_1}$ .

*Proof.* Let  $a_1, \ldots, a_n \in \mathbb{N}$  be such that  $\mu = \lambda - \sum_{i=1}^{n}$  $\sum_{i=1}^{n} c_i \alpha_i$ . By assumption,  $\mu \in X^+$ , thus  $(\mu + \rho, \alpha) \geq 0$  for all  $\alpha \in \Pi$ . Thus, using Remark [1.6,](#page-6-0) we only need to show that  $(\mu + \rho, \alpha_h^{\vee}) \leq p$ . Since  $(\lambda + \rho, \alpha_h^{\vee}) \leq p$  and  $(\alpha_i, \alpha_h^{\vee}) \geq 0$  for all  $\alpha_i \in \Pi$ , we have

$$
(\mu + \rho, \alpha_h^{\vee}) = (\lambda - \sum_{i=1}^n c_i \alpha_i + \rho, \alpha_h^{\vee}) = (\lambda + \rho, \alpha_h^{\vee}) - \sum_{i=1}^n c_i (\alpha_i, \alpha_h^{\vee}) \le (\lambda + \rho, \alpha_h^{\vee}) \le p,
$$

 $\Box$ 

so  $\mu \in C_1$ .

**Definition 1.8.** *An alcove C is p*-restricted *if there exists a p-restricted dominant weight*  $\lambda \in X^+$  *such that*  $\lambda \in C$ *.* 

<span id="page-7-4"></span>**Theorem 1.9** ([\[Hum90,](#page-77-4) 4.5 and 4.8])**.** *The affine Weyl group W<sup>p</sup> acts simply transitively on the set of alcoves. Moreover,*  $\overline{C_1}$  *is a fundamental domain for the dot action of*  $W_p$  *on*  $X_{\mathbb{R}}$ *.* 

**Definition 1.10.** Let  $\lambda, \mu \in X$ . The weight  $\lambda$  is linked to  $\mu$  if  $\lambda = \mu$  or if there exist affine  $reflections \, s_{\beta_1,m_1p}, \ldots, s_{\beta_t,m_tp} \in W_p \, such \, that$ 

$$
\lambda \leq s_{\beta_1,m_1p} \bullet \lambda \leq \ldots \leq s_{\beta_t,m_tp} \cdots s_{\beta_1,m_1p} \bullet \lambda = \mu.
$$

*In this case, we write*  $\lambda \uparrow \mu$ *.* 

**Remark 1.11.** The relation ↑ is a partial order on *X*.

#### <span id="page-7-0"></span>**1.2 Chevalley groups and algebraic groups**

In this section, we recall some definitions about linear algebraic groups, following [\[MT11\]](#page-78-1), and we construct Chevalley groups following [\[Ste16\]](#page-78-2).

#### <span id="page-7-1"></span>**1.2.1 Linear algebraic groups**

Let *G* be a linear algebraic group. A *Borel subgroup*  $B \leq G$  is a closed, connected, solvable subgroup of *G* which is maximal with respect to all these properties. The *radical R*(*G*) of *G* is the maximal closed connected solvable normal subgroup of *G*. The group *G* is *semisimple* if  $R(G) = 1$ . A non-trivial semisimple algebraic group G is *simple* if it has no non-trivial proper closed connected normal subgroups. A representation  $\rho : G \to GL(V)$  is *rational* if  $\rho$  is a morphism of algebraic groups.

#### <span id="page-7-2"></span>**1.2.2 Chevalley groups**

We fix a numeration of the roots  $\Phi = {\alpha_1, \ldots, \alpha_m}$  such that  $\text{ht } \alpha_i \leq \text{ht } \alpha_j$  for all  $i \leq j$  (recall that  $\Pi = {\alpha_1, \ldots, \alpha_n}$ ). Let g be the simple Lie algebra associated to  $\Phi$  (over  $\mathbb{C}$ ) with Cartan subalgebra h. We denote by  $\mathcal{U}(\mathfrak{g})$  its universal enveloping algebra and by  $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$  the *Killing form* on  $\mathfrak{g}$ . We fix  $\{e_{\alpha}, h_{\beta} | \alpha \in \Phi, \beta \in \Pi\}$  a Chevalley basis of  $\mathfrak{g}$ which satisfies the following properties:

- (1)  $h_{\alpha}$  is the coroot of  $\alpha$ , i.e.  $h_{\alpha} = \frac{2t_{\alpha}}{\kappa(t_{\alpha}, t_{\alpha})}$  where  $t_{\alpha} \in \mathfrak{h}$  is the unique element such that  $κ(t<sub>α</sub>, h) = α(h)$  for all  $h ∈ θ$ ,
- (2)  $e_{\alpha} \in \mathfrak{g}_{\alpha}$  for all  $\alpha \in \Phi$ ,
- (3)  $[h_{\alpha}h_{\beta}] = 0$  for all  $\alpha, \beta \in \Pi$ ,
- (4)  $[h_{\alpha}e_{\beta}] = (\beta, \alpha^{\vee})e_{\alpha}$
- (5)  $[e_{\alpha}e_{-\alpha}] = h_{\alpha}$  for all  $\alpha \in \Pi$ ,
- (6)  $[e_{\alpha}e_{\beta}] = 0$  if  $\alpha + \beta \notin \Phi$  and  $\beta \neq -\alpha$ ,
- (7) if  $\beta r\alpha, \ldots, \beta + q\alpha$  is the *α*-string through  $\beta$ , then  $[e_{\alpha}e_{\beta}] = N_{\alpha,\beta}e_{\alpha+\beta}$  if  $\alpha + \beta \in \Phi$ with  $N_{\alpha,\beta} = -N_{-\alpha,-\beta} = \pm (r+1)$ .

The existence of this basis is proven in [\[Hum00,](#page-77-3) 25.2], [\[Car89,](#page-77-5) 4.2.1] and [\[Ste16,](#page-78-2) Chapter 1]. For  $\alpha \in \Phi^+$ , we set  $f_{\alpha} := e_{-\alpha}$ . Moreover, for any sequences of non-negative integers  $A = (a_1, \ldots, a_m), B = (b_1, \ldots, b_m) \in \mathbb{N}^m, C = (c_1, \ldots, c_n) \in \mathbb{N}^n$ , we define

$$
E^{A} := \frac{e_{\alpha_1}^{a_1}}{a_1!} \cdots \frac{e_{\alpha_m}^{a_m}}{a_m!} \in \mathcal{U}(\mathfrak{g}),
$$
  

$$
F^{B} := \frac{f_{\alpha_1}^{b_1}}{b_1!} \cdots \frac{f_{\alpha_m}^{b_m}}{b_m!} \in \mathcal{U}(\mathfrak{g}),
$$
  

$$
H^{C} := \begin{pmatrix} h_{\alpha_1} \\ c_1 \end{pmatrix} \cdots \begin{pmatrix} h_{\alpha_n} \\ c_n \end{pmatrix} \in \mathcal{U}(\mathfrak{g}),
$$

where

$$
\binom{h_{\alpha_i}}{c_i} := \frac{h_{\alpha_i}(h_{\alpha_i}-1)\cdots(h_{\alpha_i}-c_i+1)}{c_i!}.
$$

Using the PBW Theorem (see [\[Hum00,](#page-77-3) 17.3] or [\[Ste16,](#page-78-2) Chapter 2]), the set  $\{F^B H^C E^A\}$  is a basis of  $\mathcal{U}(\mathfrak{g})$ . We define  $\mathcal{U}(\mathfrak{g})_{\mathbb{Z}}$  to be the subring of  $\mathcal{U}(\mathfrak{g})$  generated by  $\{\frac{e^a_{\alpha}}{a!} | \alpha \in \Phi, a \in \mathbb{N}\},$  $\mathcal{U}(\mathfrak{g})_{\mathbb{Z}}^{\pm}$  $\frac{1}{\mathbb{Z}}$  to be the subring of  $\mathcal{U}(\mathfrak{g})$  generated by  $\{\frac{e^a}{a!} | a \in \Phi^{\pm}, a \in \mathbb{N}\},\$  and  $\mathcal{U}(\mathfrak{g})_{\mathbb{Z}}^{\circ}$  $\frac{1}{Z}$  the subring of  $\mathcal{U}(\mathfrak{g})$  generated by  $\{ \begin{pmatrix} h_{\alpha} \\ c \end{pmatrix}$  $\begin{bmatrix} a & b \\ c \end{bmatrix}$   $\alpha \in \Phi$ ,  $a \in \mathbb{N}$ . Then  $\{F^B H^C E^A\}$  is a Z-basis of  $\mathcal{U}(\mathfrak{g})_{\mathbb{Z}}$  $([Ste16, Chapter 2]).$  $([Ste16, Chapter 2]).$  $([Ste16, Chapter 2]).$ 

Let *V* be an irreducible finite-dimensional  $\mathcal{U}(\mathfrak{g})$ -module with highest weight  $\lambda$ . There exists  $v^+ \in V$  a maximal vector for  $\mathcal{U}(\mathfrak{g})^{\circ} \mathcal{U}(\mathfrak{g})^+$  of weight  $\lambda$ . We define  $M := \mathcal{U}(\mathfrak{g})_{\mathbb{Z}}^ \bar{z}$   $v^+$ , which is a lattice in *V*. Then *M* is stable under  $\mathcal{U}(\mathfrak{g})_{\mathbb{Z}}$ . For an arbitrary field *k*, we define  $\mathcal{U}(\mathfrak{g})_k := \mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{Z}} k$  and  $V_k := M \otimes_{\mathbb{Z}} k$ , which has thus the structure of a  $\mathcal{U}(\mathfrak{g})_k$ -module.

For  $\alpha \in \Phi$  and  $t \in k$ , we define

$$
x_{\alpha}(t) := \exp(te_{\alpha}) = \sum_{i=0}^{\infty} t^{i} \frac{e_{\alpha}^{i}}{i!}.
$$

Since  $e_{\alpha}$  acts nilpotently on  $V_k$ , the map  $x_{\alpha}(t)$  is well-defined and is an automorphism of  $V_k$ ([\[Ste16,](#page-78-2) Chapter 3]). We call the group

$$
G = G(V, k) = \langle x_{\alpha}(t) | \alpha \in \Phi, t \in k \rangle \subseteq GL(V_{k})
$$

the *Chevalley group* associated to *V* and *k*. The type of *G* is the type of the root system Φ.

We fix  $G = G(V, k)$  a Chevalley group. For  $\alpha \in \Phi$ , we define the *root subgroup* corresponding to  $\alpha$  to be

$$
X_{\alpha} := \{ x_{\alpha}(t) | t \in k \} \le G.
$$

We also define

$$
U:=\langle X_\alpha|\ \alpha\in \Phi^+\rangle\leq G\quad \text{ and }\quad U^-:=\langle X_\alpha|\ \alpha\in \Phi^-\rangle\leq G.
$$

Furthermore, for  $t \in k^*$ , we define

$$
w_{\alpha}(t) := x_{\alpha}(t)x_{-\alpha}(-t^{-1})x_{\alpha}(t) \in G
$$
 and  $h_{\alpha}(t) := w_{\alpha}(t)w_{\alpha}(1)^{-1} \in G$ 

and set

$$
T := \langle h_{\alpha}(t) | \alpha \in \Phi, t \in k^* \rangle \le G.
$$

Finally, we set

$$
B := \langle U, T \rangle \le G.
$$

From now on, we assume that *k* is algebraically closed. Then *G* is a semisimple algebraic group (over  $k$ ) with maximal torus  $T$  and Borel subgroup  $B$  ([\[Ste16,](#page-78-2) Theorem 6]). We call Φ the *root system associated to G*.

Let  $X_G$  be the lattice of all weights appearing in rational representations of  $G$ . Then  $X_G \subseteq X$  ([\[MT11,](#page-78-1) Section 9.2]), and we say that *G* is *simply connected* if  $X_G = X$ . For each type of root system, there exists a unique simply connected Chevalley group of this type (up to isomorphism). For a root system of type  $A_n$ , we have  $G = SL_{n+1}$ , for a root system of type  $B_n$ , we have  $G = \text{Spin}_{2n+1}$ , and for a root system of type  $C_n$ , we have  $G = \text{Sp}_{2n}$  $([Ste16, Chapter 3]).$  $([Ste16, Chapter 3]).$  $([Ste16, Chapter 3]).$ 

If  $G = G(V, k)$  is simply connected and  $G(V', k)$  is another Chevalley group of the same type, there exists a surjective homomorphism  $G \to G(V',k)$  ([\[Ste16,](#page-78-2) Corollary 5]). In particular,  $V'$  has the structure of a *G*-module.

For the rest of this paper, we fix *k* an algebraically closed field of characteristic  $p > 0$ ,  $\Phi$  a root system with base Π, weight lattice *X* and set of dominant weights *X*+, *G* = *G*(*V, k*) the simply connected Chevalley group with root system  $\Phi$ , and  $B, T \leq G$  as in the last section. Moreover, we fix *W* the Weyl group associated to  $\Phi$  and we set  $G_{\mathbb{C}} = G(V, \mathbb{C})$ .

#### <span id="page-9-0"></span>**1.3 Modules**

In this section, we define several notions related to modules for *G*. In particular, we define Weyl modules and tilting modules. All the modules that we consider are finite-dimensional and correspond to rational representations. Moreover, by module, we always mean *G*-module.

#### <span id="page-9-1"></span>**1.3.1 First definitions and irreducible modules**

We start by recalling some basic definitions and the classification of finite-dimensional simple *G*-modules.

Let *M* be a *G*-module. Its *socle*, denoted by soc *M,* is the sum of all its simple submodules and its *radical*, denoted by rad *M*, is the intersection of all its maximal submodules. The socle soc *M* is the largest completely reducible submodule of *M*. The radical rad*M* is the smallest submodule of *M* such that  $M/\text{rad } M$  is completely reducible ([\[Jan03,](#page-78-3) I 2.14]). A vector  $v \in M$  is a *maximal vector* with respect to *B* if  $Bv \subseteq kv$ . For  $\lambda \in X$ , we denote by  $m_M(\lambda) := \dim M_\lambda$  the multiplicity of the weight  $\lambda$  in M, where  $M_\lambda$  is the weight space associated to the weight  $\lambda$  in M. The weight  $\mu$  is called the *highest weight* of M if every  $\nu \in X$  with  $m_M(\nu) > 0$  satisfy  $\nu \leq \mu$ . For  $a \in \mathbb{N}$ , we define the *a*-th symmetric power of M to be

$$
S^a M := M^{\otimes a} / \langle P - \sigma(P) \rangle
$$

where *P* is a pure tensor and  $\sigma(v_1 \otimes \ldots \otimes v_a) = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(a)}$  for  $\sigma \in S_a$ . By multilinear algebra, if  $(v_1, \ldots, v_m)$  is an ordered basis of M, then  $\{v_{i_1} \otimes \ldots \otimes v_{i_a}\}_{1 \leq i_1 \leq \ldots \leq i_a \leq m}$  is a basis of  $S^aV$ .

The irreducible *G*-modules are classified by their highest weight.

**Theorem 1.12** ([\[Hum75,](#page-77-6) 31.3]). Let  $\lambda \in X^+$  be a dominant weight. Up to isomorphism, *there exists a unique irreducible module with highest weight*  $\lambda$  *which we denote by*  $L(\lambda)$ *. This module satisfies*  $m_{L(\lambda)}(\lambda) = 1$ *. Moreover, every irreducible module is of the form*  $L(\nu)$  for *some*  $\nu \in X^+$ *.* 

<span id="page-10-5"></span>**Proposition 1.13** ([\[Pie12,](#page-78-4) 2.4]). Let M be a completely reducible module, and  $N < M$  be a *submodule. Then N and M/N are completely reducible.*

Let *M* be a *G*-module. A *composition series* for *M* is a sequence of submodules

$$
0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M
$$

such that the quotients  $M_i/M_{i-1}$  are simple for all  $i \in \{1, \ldots, n\}$ . For  $\nu \in X^+$ , we write

$$
[M:L(\nu)]:=|\{i\in\{1,\ldots,n\}| \ M_i/M_{i-1}\cong L(\nu)\}|.
$$

Due to Jordan-Hölder Theorem, the value  $[M : L(\nu)]$  does not depend on the choice of the composition series (see for example [\[Erd18,](#page-77-7) Theorem 3.11]). An irreducible module  $L(\nu)$  is called a *composition factor* of *M* if it appears in a composition series for *M*, i.e. if  $[M: L(\nu)] > 0$ . If *M* is a *G*-module with composition series  $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$ , we write this composition series  $[L(\nu_n), \ldots, L(\nu_1)]$  where  $L(\nu_i) \cong M_i/M_{i-1}$ .

We are now ready to define the central notion of this project.

**Definition 1.14.** *A module M is* multiplicity-free *if all composition factors appear with multiplicity* 1*, i.e.*  $[M: L(\nu)] \leq 1$  *for all*  $\nu \in X^+$ *. If a module M is not multiplicity-free, we say that M* has multiplicity*.*

<span id="page-10-6"></span> ${\bf Lemma \ 1.15 \ \ ([{\rm Tes88},\,1.30])}$ .  $Let\ \lambda = \sum^{n}$  $\sum_{i=1}^{n} a_i \omega_i \in X^+$  *with*  $0 \le a_i < p$  *for all*  $i \in \{1, ..., n\}$ *. Then for*  $i = 1, \ldots, n$  *and*  $0 \le r \le a_i$ , *we have* 

$$
m_{L(\lambda)}(\lambda - r\alpha_i) = 1.
$$

#### <span id="page-10-0"></span>**1.3.2 Duality**

Now we define two notions of duality in the category of *G*-modules.

**Definition 1.16.** *Let M be a G-module. Its* dual *M*<sup>∗</sup> *is the usual dual vector space of M with G-action given by*

$$
(gf)(m) = f(g^{-1}m) \qquad for \ f \in M^*, \ g \in G, \ m \in M.
$$

<span id="page-10-1"></span>**Proposition 1.17** ([\[Jan03,](#page-78-3) II 1.16]). *There exists*  $\tau$  *an antiautomorphism of G which satisfies*

$$
\tau^2 = \mathrm{id}_G, \quad \tau|_T = \mathrm{id}_T \quad \text{and} \quad \tau(X_\alpha) = X_{-\alpha} \quad \text{for all } \alpha \in \Phi.
$$

*Moreover, if*  $G = SL_n(k)$  *and*  $T := \{diagonal \ matrices\}$ *, we can take*  $\tau$  *to be the matrix transposition, i.e.*  $\tau(g) = g^t$  *for*  $g \in G$ *.* 

**Definition 1.18.** *Let M be a G-module. Its* contravariant dual *M<sup>τ</sup> is the dual vector space M*<sup>∗</sup> *with action defined by*

$$
(gf)(m) = f(\tau(g)m) \qquad for \ f \in M^*, \ g \in G, \ m \in M,
$$

*where τ is the antiautomorphism from Proposition [1.17.](#page-10-1) The module M is called* contravariantly self-dual *if*  $M^{\tau} \cong M$ .

<span id="page-10-4"></span><span id="page-10-3"></span>**Remark 1.19.** One can easily check that  $M^{\tau} \otimes N^{\tau} \cong (M \otimes N)^{\tau}$  for any *G*-modules *M*, *N*.

<span id="page-10-2"></span>**Remark 1.20** ([\[Jan03,](#page-78-3) II 2.12])**.** Irreducible modules are contravariantly self-dual.

**Lemma 1.21** ([\[Gru22,](#page-77-8) V 4.2])**.** *Let M be a contravariantly self-dual module. If M is multiplicity-free, then M is completely reducible.*

#### <span id="page-11-0"></span>**1.3.3 Weyl modules**

In this section, we will define the so called Weyl modules. In characteristic 0, those modules and the irreducible modules coincide. In positive characteristic, the Weyl modules are no longer irreducible, but are still useful to understand the irreducible modules.

Let  $\lambda \in X^+$ ,  $V'$  the  $\mathcal{U}(\mathfrak{g})$ -module of highest weight  $\lambda$  and  $V'_k$  be as defined in section [1.2.](#page-7-0) The group *G* acts naturally on  $V'_k$ . We call this *G*-module the *Weyl module* of highest weight  $\lambda$ and denote it by  $\Delta(\lambda)$ . The Weyl module  $\Delta(\lambda)$  is generated by a maximal vector for *B* of weight  $\lambda$  and satisfies the following universal property ([\[Jan03,](#page-78-3) II 2.13]):

**Lemma 1.22.** Let V be a *G*-module generated by a maximal vector for *B* of weight  $\lambda \in X^+$ . *There exists a surjective morphism*  $\Delta(\lambda) \rightarrow V$ .

<span id="page-11-2"></span>**Proposition 1.23** ([\[Hum00,](#page-77-3) 21.3]). Let  $\lambda \in X^+$ . The set  $\{\nu \in X | m_{\Delta(\lambda)}(\nu) > 0\}$  is *saturated with highest weight λ.*

To compute weight multiplicities in Weyl modules, we can use Freudenthal's formula, whose proof can be found in [\[Hum00,](#page-77-3) 22.3].

**Theorem 1.24** (Freudenthal's formula). Let  $\lambda \in X^+$  and  $\mu \in X$ . Then

$$
((\lambda + \rho, \lambda + \rho) - (\mu + \rho, \mu + \rho))m_{\Delta(\lambda)}(\mu) = 2 \sum_{\alpha \in \Phi^+} \sum_{i=1}^{\infty} m_{\Delta(\lambda)}(\mu + i\alpha)(\mu + i\alpha, \alpha).
$$

It can be combined with the following proposition.

**Proposition 1.25** ([\[Cav17,](#page-77-9) Proposition A]). *Let*  $\lambda = \sum_{n=1}^n$  $\sum_{i=1}^{n} a_i \omega_i \in X^+$  *and*  $\mu = \lambda - \sum_{i=1}^{n}$  $\sum_{i=1}^{\infty} c_i \alpha_i$ *with*  $c_i \in \mathbb{N}$  *for all i. Suppose the existence of a non-empty subset*  $J \subseteq \{1, \ldots, n\}$  *such that*  $c_j \leq a_j$  *for all*  $j \in J$ *. Let*  $\lambda' = \lambda - \sum$  $\sum_{j \in J} (a_j - c_j) \omega_j$  *and*  $\mu' = \lambda' - \sum_{i=1}^n$  $\sum_{i=1}^{n} c_i \alpha_i$ *.* Then

$$
m_{\Delta(\lambda)}(\mu) = m_{\Delta(\lambda')}(\mu').
$$

For  $\lambda \in X^+$ , we define the *costandard* module of highest weight  $\lambda$  by

$$
\nabla(\lambda) := \Delta(-w_0(\lambda))^*.
$$

The module  $\nabla(\lambda)$  is also called the *induced module* or *dual Weyl module* in the literature, and satisfies  $\nabla(\lambda) \cong \Delta(\lambda)^\tau$  ([\[Jan03,](#page-78-3) II 2.13]).

**Proposition 1.26** ([\[Jan03,](#page-78-3) II 2.4 and 2.14])**.** *Let*  $\lambda \in X^+$ *. We have*  $\Delta(\lambda) / \text{rad } \Delta(\lambda) \cong L(\lambda)$  $and \,\, \text{soc}\,\nabla(\lambda) \cong L(\lambda)$ .

#### <span id="page-11-1"></span>**1.3.4 Filtrations and tilting modules**

Another class of useful modules are the so-called tilting modules, which we define in this subsection. They will be very useful to show that some tensor products have multiplicity. Before that, we define two special kinds of filtrations. We end this section by stating the classification of indecomposable tilting modules.

**Definition 1.27.** *Let M be a G-module. A* Weyl filtration *of M is a sequence*

$$
0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M
$$

*of submodules such that*  $M_i/M_{i-1}$  *is a Weyl module for all*  $i \in \{1, \ldots, n\}$ *, i.e. there exist dominant weights*  $\lambda_1, \ldots, \lambda_n \in X^+$  *such that*  $M_i/M_{i-1} \cong \Delta(\lambda_i)$  *for all*  $i \in \{1, \ldots, n\}$ *.* 

<span id="page-12-3"></span>**Theorem 1.28** ([\[Mat90,](#page-78-6) Theorem 1]). Let  $\lambda$ ,  $\mu \in X^+$  be dominant weights. The tensor *product*  $\Delta(\lambda) \otimes \Delta(\mu)$  *admits a Weyl filtration.* 

**Definition 1.29.** *Let M be a G-module. A* good filtration *of M is a sequence*

$$
0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M
$$

*of submodules such that there exist dominant weights*  $\lambda_1, \ldots, \lambda_n \in X^+$  *with*  $M_i/M_{i-1} \cong \nabla(\lambda_i)$ *for all*  $i \in \{1, ..., n\}$ *.* 

**Remark 1.30.** A *G*-module *M* admits a Weyl filtration if and only if its dual *M*<sup>∗</sup> admits a good filtration if and only if *M<sup>τ</sup>* admits a good filtration.

**Definition 1.31.** *A module M is a* tilting module *if it admits a Weyl filtration and a good filtration.*

An important result is that the tensor product of two tilting modules is again a tilting module. We will use it several times in the next sections without further reference.

<span id="page-12-5"></span>**Theorem 1.32.** Let  $M, N$  be two tilting modules. Then  $M \otimes N$  is a tilting module.

*Proof.* This is a direct consequence of [\[Mat90,](#page-78-6) Theorem 1].

Like the simple modules, the indecomposable tilting modules are classified by their highest weight.

<span id="page-12-0"></span>**Proposition 1.33** ([\[Jan03,](#page-78-3) II E.6]). Let  $\lambda \in X^+$  be a dominant weight. There exists a unique *indecomposable tilting module*  $T(\lambda)$  *with highest weight*  $\lambda$  *and*  $m_{T(\lambda)}(\lambda) = 1$ *. Moreover, for every tilting module*  $\overline{T}$ *, there exist dominant weights*  $\nu_1, \ldots, \nu_n \in \overline{X^+}$  *such that* 

$$
T \cong \bigoplus_{i=1}^{n} T(\nu_i).
$$

<span id="page-12-1"></span>**Lemma 1.34.** *Every tilting module is contravariantly self-dual.*

*Proof.* If *T* is a tilting module, then  $T^{\tau}$  is a tilting module. By definition,  $(-)^{\tau}$  preserves the weights of the representation. Thus, if *T* is indecomposable with highest weight  $\lambda$ , then so is  $T^{\tau}$ , and we conclude by uniqueness in Proposition [1.33](#page-12-0) that  $T(\lambda)^{\tau} \cong T(\lambda)$ .  $\Box$ 

<span id="page-12-2"></span>**Corollary 1.35.** Let  $T(\lambda)$  be an indecomposable tilting module. Then  $T(\lambda)$  is multiplicity*free if and only if*  $T(\lambda)$  *is irreducible.* 

<span id="page-12-6"></span>*Proof.* This is a direct consequence of Lemmas [1.34](#page-12-1) and [1.21.](#page-10-2)

 $\Box$ 

 $\Box$ 

**Lemma 1.36.** Let *M* be a tilting module. Let  $\eta \in X^+$  be such that  $L(\eta)$  is a composition *factor of M and T*(*η*) *is not irreducible. Then M has multiplicity.*

*Proof.* Using Proposition [1.33,](#page-12-0) there exist  $\nu_1, \ldots, \nu_s \in X^+$  such that  $M \cong \bigoplus_{i=1}^s T(\nu_i)$ . There exists  $\nu_i \geq \eta$  such that  $L(\eta)$  is a composition factor of  $T(\nu_i)$ . If  $\nu_i > \eta$ , then  $T(\nu_i)$  is not irreducible, hence it has multiplicity by Corollary [1.35,](#page-12-2) and so *M* has multiplicity. If  $\nu_i = \eta$ , we conclude using the assumption that  $T(\eta)$  is not irreducible.  $\Box$ 

<span id="page-12-4"></span>**Lemma 1.37.** *Let*  $\lambda \in X^+$ *. If*  $\Delta(\lambda) \cong L(\lambda)$ *, then*  $T(\lambda) \cong \nabla(\lambda) \cong L(\lambda)$ *. Else,*  $T(\lambda)$  *is not irreducible.*

*Proof.* If  $\Delta(\lambda)$  is irreducible, then so is  $\nabla(\lambda)$ . In particular,  $L(\lambda) \cong \Delta(\lambda) \cong \nabla(\lambda)$  admits a Weyl filtration and a good filtration. Thus,  $L(\lambda)$  is a tilting module, and we conclude by uniqueness in Proposition [1.33.](#page-12-0)

Otherwise,  $\Delta(\lambda)$  appear in the Weyl filtration of  $T(\lambda)$ , hence  $T(\lambda)$  is not irreducible.  $\Box$ 

#### <span id="page-13-0"></span>**1.4 Characters**

A lot of information about a *G*-module *M* is given by the dimensions of its weight spaces. These informations are encoded in the character of the module, a notion that we define in this section. Later on, we will use those characters to compute the composition factors of tensor products of simple *G*-modules, and in particular to show that some of them are multiplicity-free.

<span id="page-13-1"></span>**Lemma 1.38** ([\[MT11,](#page-78-1) Lemma 15.3]). Let *M* be an irreducible  $G$ -module,  $\lambda \in X$  and  $w \in W$ . *Then*

$$
m_M(\lambda) = m_M(w\lambda).
$$

**Definition 1.39.** *Let M be a G-module. Its* character *is the formal sum*

$$
\operatorname{ch} M := \sum_{\lambda \in X} m_M(\lambda) e^{\lambda} \in \mathbb{Z}[X],
$$

*where*  $\mathbb{Z}[X]$  *has*  $\mathbb{Z}$ *-basis*  $\{e^{\lambda} | \lambda \in X\}$ *.* 

We denote by  $\mathbb{Z}[X]^W$  the fixed points in  $\mathbb{Z}[X]$  for the natural action of W. By Lemma [1.38,](#page-13-1) we have ch  $M \in \mathbb{Z}[X]^W$  for every *G*-module M.

**Remark 1.40.** Let *M*, *N* be two *G*-modules. We have  $ch(M \oplus N) = ch M + ch N$  and  $ch(M \otimes N) = ch M \cdot ch N$ .

**Notation 1.41.** We denote the character of the Weyl module of highest weight  $\lambda \in X^+$  by

$$
\chi(\lambda) := \operatorname{ch} \Delta(\lambda).
$$

<span id="page-13-2"></span>**Theorem 1.42** (Weyl's character formula). For  $\lambda \in X^+$ , we have

$$
\chi(\lambda) = \frac{\sum_{w \in W} \det(w) e^{w(\lambda + \rho)}}{\sum_{w \in W} \det(w) e^{w(\rho)}}.
$$

<span id="page-13-3"></span>**Corollary 1.43** (Weyl's degree formula). *For*  $\lambda \in X^+$ , *we have* 

$$
\dim \Delta(\lambda) = \frac{\prod_{\alpha \in \Phi^+} (\lambda + \rho, \alpha)}{\prod_{\alpha \in \Phi^+} (\rho, \alpha)}
$$

Proofs of Weyl's character formula and Weyl's degree formula are given in [\[Hum00,](#page-77-3) 24].

*.*

Weyl's character formula allows us to extend our definition of character for non-dominant weights.

**Definition 1.44.** *Let*  $\lambda \in X$ *. The* Weyl character *associated to*  $\lambda$  *is the formal element* 

$$
\chi(\lambda) := \frac{\sum_{w \in W} \det(w) e^{w(\lambda + \rho)}}{\sum_{w \in W} \det(w) e^{w(\rho)}}.
$$

<span id="page-13-4"></span>**Lemma 1.45.** *For*  $\lambda \in X$ *, we have* 

- (1) *<sup>χ</sup>*(*w*·  $\lambda$ ) = det $(w)\chi(\lambda)$   $\forall w \in W$ *,*
- (2)  $\chi(\lambda) = 0 \quad \forall \lambda \in D \setminus X^+$ .

*Proof.* For  $g \in W$ , we have

$$
\chi(g \cdot \lambda) = \frac{\sum_{w \in W} \det(w)e^{w(g \cdot \lambda + \rho)}}{\sum_{w \in W} \det(w)e^{w(\rho)}} = \frac{\sum_{w \in W} \det(w)e^{w(g(\lambda + \rho))}}{\sum_{w \in W} \det(w)e^{w(\rho)}}
$$

$$
= \det(g) \frac{\sum_{w \in W} \det(wg)e^{wg(\lambda + \rho)}}{\sum_{w \in W} \det(w)e^{w(\rho)}} = \det(g) \frac{\sum_{w \in W} \det(w)e^{w(\lambda + \rho)}}{\sum_{w \in W} \det(w)e^{w(\rho)}}
$$

$$
= \det(g)\chi(\lambda).
$$

Now observe that for  $\lambda \in D \setminus X^+$ , there exists  $\alpha \in \Pi$  such that  $s_{\alpha} \bullet \gamma(\lambda) = -\gamma(\lambda)$  so  $\gamma(\lambda) = 0$ *λ* = *λ*. Therefore,  $\chi(\lambda) = -\chi(\lambda)$  so  $\chi(\lambda) = 0$ . □

The following lemma will be useful to compute an explicit decomposition of a product of characters into a sum of irreducible or Weyl characters. It ensures the existence and the uniqueness of such a decomposition. We will use it several times in the next sections without further reference.

<span id="page-14-1"></span>**Lemma 1.46** ([\[Jan03,](#page-78-3) II 5.8])**.** *The set of characters of irreducible modules*  ${\rm \{ch } L(\lambda) \vert \quad \lambda \ \in \ X^+ \}$  is a Z-basis of  $\mathbb{Z}[X]^W$ . Moreover, the set of Weyl characters  $\{\chi(\lambda) | \lambda \in X^+\}$  *is a* Z-basis of  $\mathbb{Z}[X]^W$ .

<span id="page-14-2"></span>**Proposition 1.47** ([\[Ste03,](#page-78-0) Proposition 2.1]). *For*  $\lambda, \mu \in X^+$ *, we have* 

$$
\chi(\lambda)\chi(\mu) = \sum_{\nu \in X} m_{\Delta(\lambda)}(\nu) \cdot \chi(\mu + \nu).
$$

**Corollary 1.48.** *For*  $\lambda, \mu \in X^+$ *, we have* 

$$
\operatorname{ch} L(\lambda) \cdot \chi(\mu) = \sum_{\nu \in X} m_{L(\lambda)}(\nu) \cdot \chi(\mu + \nu).
$$

*Proof.* Since ch  $L(\lambda) \in \mathbb{Z}[X]^W$ , there exist  $\lambda_1, \ldots, \lambda_n \in X^+$  and  $a_1, \ldots, a_n \in \mathbb{Z}$  such that ch  $L(\lambda) = \sum^{n}$  $\sum_{i=1}^{n} a_i \chi(\lambda_i)$  (Lemma [1.46\)](#page-14-1). For  $\nu \in X$ , we have  $m_{L(\lambda)}(\nu) = \sum_{i=1}^{n} a_i$  $\sum_{i=1}^{\infty} a_i m_{\Delta(\lambda_i)}(\nu)$ . Using Proposition [1.47](#page-14-2) in the second equality below, we get

$$
\operatorname{ch} L(\lambda) \cdot \chi(\mu) = \left(\sum_{i=1}^n a_i \chi(\lambda_i)\right) \chi(\mu) = \sum_{i=1}^n a_i \sum_{\nu \in X} m_{\Delta(\lambda_i)}(\nu) \cdot \chi(\mu + \nu)
$$

$$
= \sum_{\nu \in X} \left(\sum_{i=1}^n a_i m_{\Delta(\lambda_i)}(\nu)\right) \cdot \chi(\mu + \nu) = \sum_{\nu \in X} m_{L(\lambda)}(\nu) \cdot \chi(\mu + \nu).
$$

#### <span id="page-14-0"></span>**1.4.1 Jantzen** *p***-sum formula**

As previously claimed, the Weyl modules are not always irreducible in positive characteristic. Thus, it will be useful to compute their composition factors. An important tool for this computation is the so-called Jantzen *p*-sum formula. We will use it to compute the composition factors of Weyl modules with *p*-restricted highest weight.

Let  $m \in \mathbb{N}^*$  be a positive integer. Recall that p is a fixed prime. Let  $a, b \in \mathbb{N}$  be such that  $p \nmid b$  and  $m = p^a b$ . The *p*-adic valuation of *n* is  $\nu_p(m) := a$ .

<span id="page-14-3"></span>**Proposition 1.49** (Jantzen *p*-sum formula, [\[Jan03,](#page-78-3) II 8.19]). Let  $\lambda \in X^+$ . There exists a *filtration*

$$
\Delta(\lambda) \supseteq \Delta(\lambda)^1 \supseteq \Delta(\lambda)^2 \supseteq \dots
$$

*such that*

$$
\sum_{i>0} \text{ch} \,\Delta(\lambda)^i = \sum_{\alpha \in \Phi^+} \sum_{0 < mp < (\lambda + \rho, \alpha^\vee)} \nu_p(mp)\chi(s_{\alpha, mp} \cdot \lambda)
$$

*and*

$$
\Delta(\lambda)/\Delta(\lambda)^1 \cong L(\lambda).
$$

**Notation 1.50.** We set

$$
JSF(\lambda) := \sum_{\alpha \in \Phi^+} \sum_{0 < mp < (\lambda + \rho, \alpha^\vee)} \nu_p(mp)\chi(s_{\alpha, mp} \cdot \lambda).
$$

**Remark 1.51.** Observe that, for  $\lambda \in X^+$  a *p*-restricted weight, we have

$$
(\lambda + \rho, \alpha^{\vee}) \le p \sum_{i=1}^{n} (\omega_i, \alpha^{\vee}) = p(\rho, \alpha^{\vee}) \le p(\rho, \alpha^{\vee}) = p(h - 1).
$$

Therefore, if  $p \geq h - 1$ , all the *p*-adic valuations in JSF( $\lambda$ ) are equal to 1.

**Remark 1.52.** Let  $\lambda \in X^+$ . If there exists  $\mu \in X^+$  such that  $JSF(\lambda) = ch L(\mu)$ , then  $\Delta(\lambda)$ admits two composition factors,  $L(\lambda)$  and  $L(\mu)$ . Since  $\Delta(\lambda)/\text{rad}\,\Delta(\lambda) \cong L(\lambda)$ , it follows that  $\Delta(\lambda)$  admits a unique composition series, given by  $[L(\lambda), L(\mu)]$ .

#### <span id="page-15-0"></span>**1.5 Linkage principle**

Another useful tool to compute the composition factors of a Weyl module is the Strong Linkage Principle. It allows us to show that Weyl modules with highest weight in the fundamental alcove are irreducible.

<span id="page-15-1"></span>**Proposition 1.53** (The Strong Linkage Principle, [\[Jan03,](#page-78-3) II 6.13]). Let  $\lambda, \mu \in X^+$  be *dominant weights. If*

$$
[\Delta(\lambda): L(\mu)] > 0,
$$

<span id="page-15-3"></span>*then*  $\mu \uparrow \lambda$ *.* 

**Proposition 1.54** ([\[Jan03,](#page-78-3) II 6.24]). *Let*  $\lambda \in X^+$  *be a dominant weight. Suppose that*  $\mu \in X$ *is maximal in the set*  $\{v \in X | v \uparrow \lambda, v \neq \lambda\}$  *with respect to the ordering*  $\uparrow$ *. If*  $\mu \in X^+$  *and*  $\mu \notin {\lambda - p\alpha \mid \alpha \in \Phi^+}$ , then

 $[\Delta(\lambda): L(\mu)] = 1.$ 

<span id="page-15-2"></span>**Lemma 1.55.** *For every*  $\lambda \in X^+ \cap \widehat{C}_1$  *we have*  $L(\lambda) \cong \Delta(\lambda)$ *.* 

*Proof.* Let  $\mu \in X^+$  be such that  $L(\mu)$  is a composition factor of  $\Delta(\lambda)$ . By the Strong Linkage Principle (Proposition [1.53\)](#page-15-1), we have  $\mu \uparrow \lambda$ , and in particular,  $\mu \leq \lambda$ , so  $\mu \in \widehat{C}_1$  by Linkage Principie (Proposition 1<br>Lemma [1.7.](#page-7-3) Moreover,  $\mu \in W_p$ . *λ*. Since *C*<sup>1</sup> is a fundamental domain for the dot action of *W<sub>p</sub>* (Lemma [1.9\)](#page-7-4), we have  $\mu \in \overline{C_1} \cap W_p$  • is irreducible  $\lambda = {\lambda}$ . We conclude that  $\mu = \lambda$ , therefore  $\Delta(\lambda)$ is irreducible. □

<span id="page-15-4"></span>**Lemma 1.56.** *Let*  $\lambda \in \widehat{C_1} \cap X^+$  *and*  $\nu \in X^+$  *such that*  $\nu \leq \lambda$ *. Then*  $L(\nu) \cong \Delta(\nu)$ *.* 

*Proof.* This is a direct consequence of Lemmas [1.7](#page-7-3) and [1.55.](#page-15-2)

 $\Box$ 

**Lemma 1.57** ([\[Jan03,](#page-78-3) II 4.16]). *Let*  $\lambda$ ,  $\mu \in X^+$ *. The tensor product*  $\Delta(\lambda) \otimes \Delta(\mu)$  *admits a submodule isomorphic to*  $\Delta(\lambda + \mu)$ *.* 

#### <span id="page-16-0"></span>**1.6 An argument to count multiplicities**

In this subsection, we provide an argument to compute the multiplicities of each composition factor of the tensor product of two simple modules. We will use it several times later to show that some tensor products of two simple modules have multiplicity.

<span id="page-16-1"></span>**Argument 1.** Let  $\lambda, \mu \in X^+$  be dominant weights, and let  $M = L(\lambda) \otimes L(\mu)$ . A vector  $v \otimes w \in M$  is a weight vector of weight  $\nu$  if and only if *v* is a weight vector in  $L(\lambda)$  of weight  $\nu_1$ , *w* is a weight vector in  $L(\mu)$  of weight  $\nu_2$  and  $\nu_1 + \nu_2 = \nu$ . Therefore,

$$
m_M(\nu) = \sum_{\substack{\nu_1, \nu_2 \in X \\ \nu_1 + \nu_2 = \nu}} m_{L(\lambda)}(\nu_1) m_{L(\mu)}(\nu_2).
$$

Suppose that  $\nu_1, \ldots, \nu_s \in X^+$  is the complete list of the dominant weights corresponding to all composition factors of *M* (with multiplicity). For every weight  $\eta \in X$ , we have

$$
m_M(\eta) = \sum_{i=1}^s m_{L(\nu_i)}(\eta).
$$

We compute the  $\nu_i$ 's as follows. We set  $\nu_1 = \lambda + \mu$ . Suppose that we have already  $\nu_1, \ldots, \nu_t$  for  $t < s$ . Let  $\eta \in X^+$  be such that

$$
m_M(\eta) > \sum_{i=1}^t m_{L(\nu_i)}(\eta) \quad \text{ and } \quad m_M(\nu) = \sum_{i=1}^t m_{L(\nu_i)}(\nu) \quad \text{for every dominant weight } \nu > \eta.
$$

It follows that  $m_{L(\nu_i)}(\nu) = 0$  for every  $i > t$  and every  $\nu > \eta$ . In particular,  $\nu_i \ngeq \eta$  for every  $i > t$ . Moreover, there exists  $i > t$  such that  $m_{L(\nu_i)}(\eta) > 0$ , thus  $\nu_i \geq \eta$ . Therefore, we deduce that  $\nu_i = \eta$  for some  $i > t$ , and we can choose  $\nu_{t+1} = \eta$ .

## <span id="page-17-0"></span>**2 Properties of tensor products**

In this section, we establish some properties of tensor products. We state Steinberg's tensor product theorem (see [\[Jan03,](#page-78-3) II 3.16] for a proof), which allows us to restrict our attention to tensor products of irreducible modules with *p*-restricted highest weight in order to answer our question.

#### <span id="page-17-3"></span><span id="page-17-1"></span>**2.1 Steinberg's tensor product theorem**

**Theorem 2.1** ([\[Spr09,](#page-78-7) Theorem 9.4.3]). The Frobenius endomorphism  $k \to k : c \mapsto c^p$ *induces a group endomorphism*  $F: G \to G$  *given by*  $x_\alpha(c) \mapsto x_\alpha(c^p)$  *for all*  $\alpha \in \Phi$ *,*  $c \in k$  *and*  $F(t) = t^p$  *for*  $t \in T$ *.* 

**Notation 2.2.** Let  $\phi: G \to \text{GL}(M)$  be a representation. We denote by  $M^{(p^i)}$  the vector space *M* with *G*-action corresponding to the representation  $\phi \circ F^i : G \to \text{GL}(M)$  where *F* is the group endomorphism described in Theorem [2.1.](#page-17-3)

**Proposition 2.3** ([\[MT11,](#page-78-1) Proposition 16.6]). *For every*  $\lambda \in X^+$ , we have an isomorphism  $of G$ *-modules*  $L(p\lambda) \cong L(\lambda)^{(p)}$ .

<span id="page-17-7"></span>**Theorem 2.4** (Steinberg's tensor product theorem). Let  $\lambda = \lambda_0 + p\lambda_1 + \ldots + p^n\lambda_n \in X^+$  be *a dominant weight such that*  $\lambda_i$  *is p-restricted for all*  $i \in \{0, \ldots, n\}$ *. We have an isomorphism* 

$$
L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_1)^{(p)} \otimes \cdots \otimes L(\lambda_n)^{(p^n)}.
$$

#### <span id="page-17-4"></span><span id="page-17-2"></span>**2.2 Reduction to** *p***-restricted highest weights**

**Lemma 2.5** ([\[Gru21,](#page-77-1) Lemma 4.12]). Let  $\lambda, \mu \in X^+$  be dominant weights. If the *G*-module  $L(\lambda) \otimes L(\mu)$  *is multiplicity-free, then it is completely reducible.* 

*Proof.* Using Remarks [1.19](#page-10-3) and [1.20,](#page-10-4) we have

$$
(L(\lambda) \otimes L(\mu))^{\tau} \cong L(\lambda)^{\tau} \otimes L(\mu)^{\tau} \cong L(\lambda) \otimes L(\mu).
$$

<span id="page-17-5"></span>We conclude by Lemma [1.21.](#page-10-2)

**Theorem 2.6** ([\[Gru21,](#page-77-1) Theorem A]). Let  $\lambda, \mu \in X^+$  be *p*-restricted dominant weights. If the *G*-module  $L(\lambda) \otimes L(\mu)$  *is completely reducible, then all its composition factors are <i>p*-restricted.

<span id="page-17-6"></span>**Corollary 2.7.** *Let*  $\lambda, \mu \in X^+$  *be p*-restricted dominant weights. If the *G*-module  $L(\lambda) \otimes L(\mu)$ *is multiplicity-free, then all its composition factors are p-restricted.*

<span id="page-17-8"></span>*Proof.* This is a direct consequence of Lemma [2.5](#page-17-4) and Theorem [2.6.](#page-17-5)

**Corollary 2.8.** Let  $\lambda, \mu \in X^+$  be *p*-restricted weights. If  $\lambda + \mu$  is not *p*-restricted, then  $L(\lambda) \otimes L(\mu)$  *has multiplicity.* 

*Proof.* We know that  $L(\lambda + \mu)$  is a composition factor of  $L(\lambda) \otimes L(\mu)$ . Thus we conclude by Corollary [2.7.](#page-17-6)  $\Box$ 

<span id="page-17-9"></span>**Proposition 2.9.** Let  $\lambda = \lambda_0 + p\lambda_1 + \ldots + p^n\lambda_n$ ,  $\mu = \mu_0 + p\mu_1 + \ldots + p^n\mu_n \in X^+$  be *dominant weights such that*  $\lambda_i, \mu_i$  *are p-restricted for all*  $i \in \{0, \ldots, n\}$ *. Then*  $L(\lambda) \otimes L(\mu)$ *is multiplicity-free if and only if*  $L(\lambda_i) \otimes L(\mu_i)$  *is multiplicity-free for all*  $i \in \{0, \ldots, n\}$ *.* 

 $\Box$ 

 $\Box$ 

*Proof.* First, using Steinberg's tensor product theorem (Theorem [2.4\)](#page-17-7), we have an isomorphism

$$
L(\lambda) \otimes L(\mu) \cong \bigotimes_{i=0}^n (L(\lambda_i)^{(p^i)} \otimes L(\mu_i)^{(p^i)}) \cong \bigotimes_{i=0}^n (L(\lambda_i) \otimes L(\mu_i))^{(p^i)}.
$$

Clearly, if there exists  $i \in \{0, \ldots, n\}$  such that  $L(\lambda_i) \otimes L(\mu_i)$  has multiplicity, then  $(L(\lambda_i) \otimes L(\mu_i))^{(p^i)}$  has multiplicity, thus  $L(\lambda) \otimes L(\mu)$  has multiplicity.

Now suppose that  $L(\lambda_i) \otimes L(\mu_i)$  is multiplicity-free for all  $i \in \{0, \ldots, n\}$ . By Lemma [2.5](#page-17-4) and Corollary [2.7,](#page-17-6) for each  $i \in \{0, \ldots, n\}$ , there exist  $\nu_i^1, \ldots, \nu_i^{m_i} \in X^+$  distinct and *p*-restricted such that

$$
L(\lambda_i) \otimes L(\mu_i) \cong \bigoplus_{j=1}^{m_i} L(\nu_i^j).
$$

Therefore, we have

$$
L(\lambda) \otimes L(\mu) \cong \bigotimes_{i=0}^{n} \bigoplus_{j=1}^{m_i} L(\nu_i^j)^{(p^i)} \cong \bigoplus_{\vec{j}} \bigotimes_{i=0}^{n} L(\nu_i^{j_i})^{(p^i)},
$$

where  $\vec{j} = (j_0, \ldots, j_n)$  runs over  $\bigtimes_{i=0}^n$  $\{1, \ldots, m_i\}$ . Since all the weights  $\nu_i^j$  $i$ <sup> $j$ </sup> are *p*-restricted, we can use Steinberg's tensor product theorem again to get

$$
L(\lambda) \otimes L(\mu) \cong \bigoplus_{\vec{j}} L(\sum_{i=0}^n p^i \nu_i^{j_i}).
$$

By uniqueness of the *p*-adic expansion of a weight, we conclude that  $L(\lambda) \otimes L(\mu)$  is multiplicityfree.  $\Box$ 

By this proposition, in order to classify multiplicity-free tensor products of simple *G*-modules, we may restrict our attention to the study of the tensor products of simple modules with *p*-restricted highest weights.

## <span id="page-19-0"></span>**3 Connections between characteristic** 0 **and positive characteristic**

In this section, we show some links between multiplicity-free tensor products in characteristic 0 and in positive characteristic. In particular, we show that in positive characteristic, multiplicity-free tensor products and completely reducible tensor products are closely related. This will allow us to classify multiplicity-free tensor products for  $p = 2$  and *G* of type  $A_n$ (see section [7\)](#page-76-0).

**Notation 3.1.** For  $\lambda \in X^+$ , we denote by  $L_{\mathbb{C}}(\lambda)$  the irreducible  $G_{\mathbb{C}}$ -module of highest weight  $\lambda$  (over C). Recall that ch  $L_{\mathbb{C}}(\lambda) = \chi(\lambda)$ .

<span id="page-19-1"></span>**Proposition 3.2.** *Let*  $\lambda, \mu \in X^+$  *be dominant weights such that*  $\Delta(\lambda) \cong L(\lambda)$  *and*  $\Delta(\mu) \cong L(\mu)$ *. Suppose that*  $\Delta(\nu) \cong L(\nu)$  *for all dominant weights*  $\nu \in X^+$  *such that*  $L(\nu)$  *is a composition factor of*  $L(\lambda) \otimes L(\mu)$ *. Then*  $L(\lambda) \otimes L(\mu)$  *is multiplicity-free if and only if*  $L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu)$  *is multiplicity-free.* 

*Proof.* We show that the composition factors in characteristic p and in characteristic 0 are the same. Let  $\nu_1, \ldots, \nu_m \in X^+$  be distinct weights and  $k_1, \ldots, k_m \in \mathbb{Z}_{>0}$  be such that  $ch(L(\lambda) \otimes L(\mu)) = k_1 \, ch \, L(\nu_1) + \ldots + k_m \, ch \, L(\nu_m)$ . By assumption, we have

$$
\chi(\lambda)\chi(\mu) = \text{ch } L(\lambda)\,\text{ch } L(\mu) = \text{ch}(L(\lambda)\otimes L(\mu)) = k_1\,\text{ch } L(\nu_1) + \ldots + k_m\,\text{ch } L(\nu_m)
$$

$$
= k_1\chi(\nu_1) + \ldots + k_m\chi(\nu_m).
$$

By uniqueness of the composition factors and linear independence of the Weyl characters (Lemma [1.46\)](#page-14-1), it follows that  $L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu)$  is multiplicity-free if and only if  $k_i = 1$  for all  $i \in \{1, \ldots, m\}$  if and only if  $L(\lambda) \otimes L(\mu)$  is multiplicity-free.  $\Box$ 

<span id="page-19-2"></span>**Corollary 3.3.** *Let*  $\lambda, \mu \in X^+$  *be dominant weights such that*  $\Delta(\lambda) \cong L(\lambda)$  *and*  $\Delta(\mu) \cong L(\mu)$ *. If*  $\Delta(\nu) \cong L(\nu)$  *for all dominant weights*  $\nu \leq \lambda + \mu$ *, then*  $L(\lambda) \otimes L(\mu)$  *is multiplicity-free if and only if*  $L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu)$  *is multiplicity-free.* 

*Proof.* If  $L(\nu)$  is a composition factor of  $L(\lambda) \otimes L(\nu)$ , then  $\nu$  is dominant and  $\nu \leq \lambda + \mu$ . Thus we can apply Proposition [3.2](#page-19-1)

<span id="page-19-3"></span>**Corollary 3.4.** *Let*  $\lambda, \mu \in X^+$  *be such that*  $\lambda + \mu \in \widehat{C_1}$ *. The module*  $L(\lambda) \otimes L(\mu)$  *is multiplicity-free if and only if*  $L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu)$  *is multiplicity-free.* 

 $\Box$ 

*Proof.* By Lemmas [1.7](#page-7-3) and [1.55,](#page-15-2) Corollary [3.3](#page-19-2) applies in case  $\lambda + \mu \in \widehat{C_1}$ .

The next theorem allows us to find the explicit decomposition of some tensor products as a direct sum of irreducible modules. Then we prove a more general version which allows us to conclude that some tensor products of irreducible modules are multiplicity-free without computing the explicit decomposition.

**Theorem 3.5.** Let  $\lambda, \mu \in X^+$  be *p*-restricted dominant weights such that the following hold:

- $(L) L(\lambda) \cong \Delta(\lambda),$
- $(L) L(\mu) \cong \Delta(\mu),$
- (3)  $L(\lambda) \otimes L(\mu)$  *is completely reducible and*
- (4)  $L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu)$  *is multiplicity-free.*

*Then*  $L(\lambda) \otimes L(\mu)$  *is multiplicity-free. Moreover, if we have the decomposition* 

$$
L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu) \cong \bigoplus_{i=1}^{m} L_{\mathbb{C}}(\nu_m)
$$

*for distinct dominant weights*  $\nu_1, \ldots, \nu_m \in X^+$ , then  $L(\nu_i) \cong \Delta(\nu_i)$  *for all*  $i \in \{1, \ldots, m\}$ *and*

$$
L(\lambda) \otimes L(\mu) \cong \bigoplus_{i=1}^{m} L(\nu_m).
$$

*Proof.* To simplify the notation, we set  $M := L(\lambda) \otimes L(\mu)$ .

Since  $L(\lambda)$  and  $L(\mu)$  are tilting modules, M must be a tilting module. Thus there exist  $\nu_1, \ldots, \nu_m \in X^+$  such that  $M \cong T(\nu_1) \oplus \ldots \oplus T(\nu_m)$  (Proposition [1.33\)](#page-12-0). Since M is completely reducible,  $T(\nu_i)$  must be completely reducible for every  $i \in \{1, \ldots, m\}$ . Therefore,  $T(\nu_i) \cong L(\nu_i) \cong \Delta(\nu_i)$  for every  $i \in \{1, \ldots, m\}$ , so all dominant weights  $\nu$  such that  $L(\nu)$  is a composition factors of *M* satisfy  $\Delta(\nu) \cong L(\nu)$ . Therefore, we can conclude by Proposition [3.2](#page-19-1) (and its proof).  $\Box$ 

**Theorem 3.6.** Let  $\lambda, \mu \in X^+$  be *p*-restricted dominant weights such that the following hold:

- (1)  $L(\lambda) \otimes L(\mu)$  *is completely reducible and*
- (2)  $L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu)$  *is multiplicity-free.*
- *Then*  $L(\lambda) \otimes L(\mu)$  *is multiplicity-free.*

*Proof.* We know that we have a surjection  $\Delta(\lambda) \rightarrow L(\lambda)$  and a surjection  $\Delta(\mu) \rightarrow L(\mu)$ . By right exactness of tensor products, we get a surjection

$$
\phi: \Delta(\lambda) \otimes \Delta(\mu) \to L(\lambda) \otimes \Delta(\mu) \to L(\lambda) \otimes L(\mu).
$$

Using Theorem [1.28,](#page-12-3) we fix a Weyl filtration

$$
0=V_0\subseteq V_1\subseteq\ldots\subseteq V_m=\Delta(\lambda)\otimes\Delta(\mu).
$$

Thus there exist  $\nu_1, \ldots, \nu_m \in X^+$  such that  $V_i/V_{i-1} \cong \Delta(\nu_i)$  for  $i = 1, \ldots, m$ . In particular, we have

$$
\chi(\lambda)\chi(\mu) = \sum_{i=1}^m \chi(\nu_i).
$$

Since  $L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu)$  is multiplicity-free, we deduce that  $\nu_i \neq \nu_j$  for all  $i \neq j$ .

For  $i \in \{1, \ldots, m\}$ , we set  $W_i := \phi(V_i)$  and we denote by  $\phi_i : V_i \to W_i$  the restriction and corestriction of the map  $\phi$ . In particular,  $\phi_i$  is surjective for all  $i \in \{1, \ldots, m\}$ . By construction, we have a filtration

$$
0=W_0\subseteq W_1\subseteq\ldots\subseteq W_m=L(\lambda)\otimes L(\mu)
$$

of  $L(\lambda) \otimes L(\mu)$ .

Now we identify the quotients  $W_i/W_{i-1}$  for  $i = 1, ..., m$ . Let  $\iota_i : V_{i-1} \to V_i$  be the inclusion map. We have the following situation:

$$
\begin{array}{ccc}\n0 & \longrightarrow V_{i-1} & \xrightarrow{\iota_i} & V_i & \longrightarrow \Delta(\nu_i) & \longrightarrow 0 \\
\phi_{i-1} & & \phi_i & & \\
0 & \longrightarrow W_{i-1} & \xrightarrow{\cdots} & W_i & \xrightarrow{\theta_i} & W_i / W_{i-1} & \longrightarrow 0.\n\end{array}
$$

Since  $\theta_i \circ \phi_i \circ \iota_i = 0$ , i.e.  $V_{i-1} \subseteq \ker(\theta_i \circ \phi_i)$ , there exists a unique map  $\psi_i : V_i/V_{i-1} \to W_i/W_{i-1}$ such that the following diagram commutes:

$$
\begin{array}{ccc}\n0 & \longrightarrow V_{i-1} & \xrightarrow{\iota_i} & V_i & \longrightarrow \Delta(\nu_i) & \longrightarrow 0 \\
\phi_{i-1} & & \phi_i & & \psi_i \\
0 & \longrightarrow W_{i-1} & \longrightarrow W_i & \xrightarrow{\theta_i} & W_i/W_{i-1} & \longrightarrow 0.\n\end{array}
$$

In particular,  $\psi_i$  is surjective because  $\theta_i$  and  $\phi_i$  are surjective. By Proposition [1.13,](#page-10-5)  $W_i$  is completely reducible and so  $W_i/W_{i-1}$  is completely reducible. Therefore, rad  $\Delta(\nu_i) \subseteq \text{ker}(\psi_i)$ , so  $\psi_i$  factors as

$$
\Delta(\nu_i) \xrightarrow{\psi_i} W_i/W_{i-1}
$$
\n
$$
\downarrow \qquad \qquad \downarrow
$$
\n
$$
L(\nu_i)
$$

Therefore,  $W_i/W_{i-1}$  is a quotient of  $L(\nu_i)$ , so either  $W_i/W_{i-1} \cong L(\nu_i)$  or  $W_i/W_{i-1} = 0$ . We deduce that  $L(\lambda) \otimes L(\nu)$  is isomorphic to a submodule of  $\bigoplus^m$  $\bigoplus_{i=1}$  *L*(*v*<sub>*i*</sub>), and in particular it is  $\Box$ multiplicity-free since all the  $\nu_i$ 's are distinct.

## <span id="page-22-0"></span>**4** SL<sup>2</sup>

In this section, we establish the classification of multiplicity-free tensor products of simple  $SL_2(k)$ -modules. Recall that  $SL_2(k)$  has root system  $\Phi$  of type  $A_1$ , so  $\Phi = {\alpha, -\alpha}$ . The fundamental weight  $\omega$  satisfies  $\alpha = 2\omega$ . Since all weights are integer multiples of  $\omega$ , we will identify the set of weights with  $\mathbb{Z}$ . Under this identification,  $\omega$  corresponds to 1, the positive root  $\alpha$  to 2, and dominant weights are in bijection with N. Moreover, we have  $\rho = \frac{1}{2}\alpha = \omega$ and it corresponds to 1.

Since there exists a unique positive root in  $\Phi$ , alcoves are in bijection with  $\mathbb{Z}$ , with

$$
C_n = \{ \lambda \in X_{\mathbb{R}} \mid (n-1)p < (\lambda + \rho, \alpha^{\vee}) < np \}.
$$

Using the identification previously described, we identify  $X_{\mathbb{R}}$  with  $\mathbb{R}$ . Thus we get

$$
C_n = \{ \lambda \in \mathbb{R} \mid (n-1)p < \lambda + 1 < np \}.
$$

In particular, there exists a unique  $p$ -restricted alcove, the fundamental alcove  $C_1$ , and by Lemma [1.55,](#page-15-2)  $L(\lambda) \cong \Delta(\lambda)$  for all *p*-restricted dominant weights  $\lambda \in X^+$ .

We start by computing the Weyl characters and the decomposition of the product of two such characters.

<span id="page-22-1"></span>**Lemma 4.1.** *Let*  $\lambda \in X^+$ *. Then* 

$$
\chi(\lambda) = \sum_{i=0}^{\lambda} e^{\lambda - 2i}
$$

*.*

*Proof.* We show this result using Weyl's character formula (Theorem [1.42\)](#page-13-2). We have

$$
(e1 - e-1) \sum_{i=0}^{\lambda} e^{\lambda - 2i} = \sum_{i=0}^{\lambda} e^{\lambda + 1 - 2i} - \sum_{i=0}^{\lambda} e^{\lambda - 1 - 2i}
$$
  
=  $e^{\lambda + 1} + \sum_{i=1}^{\lambda} e^{\lambda + 1 - 2i} - e^{-\lambda - 1} - \sum_{i=0}^{\lambda - 1} e^{\lambda - 1 - 2i}$   
=  $e^{\lambda + 1} - e^{-\lambda - 1}$ .

Therefore

$$
\chi(\lambda) = \frac{e^{\lambda + 1} - e^{-\lambda - 1}}{e^1 - e^{-1}} = \sum_{i=0}^{\lambda} e^{\lambda - 2i}.
$$

<span id="page-22-2"></span>**Proposition 4.2** (Clebsch-Gordan formula). *For*  $\lambda, \mu \in X^+$  *with*  $\lambda \geq \mu$ *, we have* 

$$
\chi(\lambda)\chi(\mu) = \chi(\lambda + \mu) + \chi(\lambda + \mu - 2) + \ldots + \chi(\lambda - \mu + 2) + \chi(\lambda - \mu).
$$

*Proof.* By Proposition [1.47](#page-14-2) and Lemma [4.1,](#page-22-1) we have

$$
\chi(\lambda)\chi(\mu) = \chi(\mu)\chi(\lambda) = \sum_{i=0}^{\mu} \chi(\lambda + \mu - 2i).
$$

We are now ready to state the main result of this section.

<span id="page-22-3"></span>**Proposition 4.3.** *Let*  $\lambda, \mu \in X^+$  *be p*-restricted dominant weights. Then  $L(\lambda) \otimes L(\mu)$  is *multiplicity-free if and only if*  $\lambda + \mu$  *is p-restricted.* 

*Proof.* The "only if" direction is a direct consequence of Corollary [2.8.](#page-17-8) For the "if" direction, observe that  $L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu)$  is multiplicity-free for all  $\lambda, \mu \in X^+$  by Proposition [4.2.](#page-22-2) If  $\lambda + \mu < p$ , then  $\lambda + \mu \in \widehat{C_1}$ , and  $L(\lambda) \otimes L(\mu)$  is multiplicity-free by Corollary [3.4.](#page-19-3)  $\Box$ 

Finally, we state the classification theorem for  $SL_2(k)$ .

**Theorem 4.4.** Let  $\lambda = \lambda_0 + p\lambda_1 + \ldots + p^n\lambda_n$ ,  $\mu = \mu_0 + p\mu_1 + \ldots + p^n\mu_n \in X^+$  be dominant *weights with*  $\lambda_i, \mu_i$  *p*-restricted for all  $i \in \{0, \ldots, n\}$ . Then  $L(\lambda) \otimes L(\mu)$  is multiplicity-free if *and only if*  $\lambda_i + \mu_i$  *is p-restricted for all*  $i \in \{0, \ldots, n\}$ *.* 

*Proof.* By Proposition [2.9,](#page-17-9)  $L(\lambda) \otimes L(\mu)$  is multiplicity-free if and only if  $L(\lambda_i) \otimes L(\mu_i)$  is multiplicity-free for all  $i \in \{0, \ldots, n\}$ . We conclude by Proposition [4.3.](#page-22-3)  $\Box$ 

## <span id="page-24-0"></span>**5** SL<sup>3</sup>

In this section, we establish the classification of multiplicity-free tensor products of simple  $SL_3$ -modules with *p*-restricted highest weight. We fix  $G = SL_3(k)$  with root system  $\Phi$  of type *A*<sub>2</sub> and  $\Pi = {\alpha_1, \alpha_2}$  a base of  $\Phi$ . With respect to this base, we have  $\rho = \alpha_1 + \alpha_2$ . For  $\lambda = x\omega_1 + y\omega_2 \in X_{\mathbb{R}}$ , we write  $\lambda = (x, y)$ . In particular, we have  $\alpha_1 = (2, -1)$ ,  $\alpha_2 = (-1, 2)$ and  $\rho = (1, 1)$ .

We will prove the following theorem:

<span id="page-24-2"></span>**Theorem 5.1.** *Let*  $\lambda = (a, b), \mu = (c, d) \in X^+$  *be non-zero p-restricted dominant weights. Up to the reordering of*  $\lambda$  *and*  $\mu$ *, the module*  $L(\lambda) \otimes L(\mu)$  *is multiplicity-free if and only if one of the following holds:*

- (1)  $b = d = 0$  *and*  $a + c < p$ ,
- (2)  $a = c = 0$  and  $b + d < p$ .
- (3)  $b = c = 0$  *and*  $a + d < p 1$  *or*  $(a, d) \in \{(p 1, 1), (1, p 1)\},\$
- (4)  $b = 0$  *and*  $a + c + d < p 1$ ,
- (5)  $a = 0$  *and*  $b + c + d < p 1$ ,
- (6)  $b = 0, c + d = p 1, a + c < p$  and  $a < c + 2$ .
- (7)  $a = 0, c + d = p 1, b + d < p$  and  $b < d + 2$ .
- (8)  $b = 0, c + d > p 1, a + c < p$  and  $a + d < p$ .
- (9)  $a = 0, c + d > p 1, b + c < p$  and  $b + d < p$  or
- (10)  $a + b < p 1$ ,  $c + d = p 1$ ,  $a + b + c < p$  and  $a + b + d < p$ .

We start by recalling some facts about the structure of those simple modules. Then we consider the relation between characters of simple modules and Weyl characters. Finally, we will establish a sequence of propositions which yield the classification.

#### <span id="page-24-1"></span>**5.1 Alcoves**

In this subsection, we describe the *p*-restricted alcoves of a root system of type  $A_2$ . There are two such alcoves which we define to be the *fundamental alcove*

$$
C_1 := \{ \lambda \in X_{\mathbb{R}} \mid (\lambda + \rho, \alpha_1^{\vee}) > 0, \ (\lambda + \rho, \alpha_2^{\vee}) > 0, \ (\lambda + \rho, (\alpha_1 + \alpha_2)^{\vee}) < p \}
$$

and the *second alcove*

$$
C_2 := \{\lambda \in X_{\mathbb{R}} \mid (\lambda + \rho, \alpha_1^{\vee}) < p, \ (\lambda + \rho, \alpha_2^{\vee}) < p, \ (\lambda + \rho, (\alpha_1 + \alpha_2)^{\vee}) > p\}.
$$

Therefore, all *p*-restricted dominant weights belong to  $\widehat{C_1} \sqcup \widehat{C_2}$ . Using the notation previously defined, we get

$$
\widehat{C}_1 \cap X^+ = \{(a, b) \in \mathbb{N}^2 \mid a + b \le p - 2\}
$$

and

$$
\widehat{C_2} \cap X^+ = \{ (a, b) \in \mathbb{N}^2 \mid a < p, \ b < p, \ a + b > p - 2 \}.
$$

Furthermore, we define the walls

$$
F_{1,2} := \overline{C_1} \cap \overline{C_2} = \{(x, y) \in [-1, p-1]^2 | x+y=p-2\},
$$
  
\n
$$
F_{2,3} := \{\lambda \in \overline{C_2} | (\lambda + \rho, \alpha_1^{\vee}) = p\},
$$
  
\n
$$
F_{2,3'} := \{\lambda \in \overline{C_2} | (\lambda + \rho, \alpha_2^{\vee}) = p\}.
$$

Thus we get

$$
F_{1,2} \cap X^+ = \{(a,b) \in \mathbb{N}^2 \mid a+b=p-2\},\
$$
  

$$
F_{2,3} \cap X^+ = \{(p-1,b) \in \mathbb{N}^2 \mid b \le p-1\},\
$$
  

$$
F_{2,3'} \cap X^+ = \{(a,p-1) \in \mathbb{N}^2 \mid a \le p-1\}.
$$

**Remark 5.2.** Following the labelling of the alcoves from [\[BDM15\]](#page-77-10), we have  $F_{2,i} = \overline{C_2} \cap \overline{C_i}$ for  $i = 3, 3'$ .

Let us illustrate these alcoves with a picture. The blue arrows form the root system. The black arrows are the fundamental weights, generating the weight lattice (in black). The region containing the dominant weights is coloured in green. The red triangles are the walls of the alcoves.



Figure 1: Alcoves for  $A_2$  and  $p = 7$ .

## <span id="page-25-0"></span>**5.2 Structure of Weyl modules and weights in irreducible modules**

In this subsection, we compute the composition factors of Weyl modules with *p*-restricted highest weight and the multiplicity of certain weights in irreducible modules with *p*-restricted highest weight.

<span id="page-26-0"></span>**Lemma 5.3.** *Let*  $\lambda = (a, b) \in X^+$  *be a p-restricted dominant weight.* 

- *If*  $a + b < p 1$  *(i.e.*  $\lambda \in \widehat{C_1}$ *), then*  $\Delta(\lambda) \cong L(\lambda)$ *.*
- *If*  $a = p 1$  *or*  $b = p 1$  *(i.e.*  $\lambda \in F_{2,3}$  *or*  $\lambda \in F_{2,3'}$ *), then*  $\Delta(\lambda) \cong L(\lambda)$ *.*
- *Else* (*i.e. if*  $\lambda \in C_2$ ), then  $\Delta(\lambda)$  *admits exactly two composition factors,*  $L(\lambda)$  *and*  $L(\lambda - (a+b+2-p)\rho)$ .

*In particular,*  $T(\lambda)$  *is irreducible if and only if*  $\lambda \in C_1 \cup F_{2,3} \cup F_{2,3'}$ *.* 

*Proof.* We use the Jantzen *p*-sum formula (Proposition [1.49\)](#page-14-3) to show this result.

- If  $\lambda \in \widehat{C_1}$ , then  $\Delta(\lambda) \cong L(\lambda)$  by Lemma [1.55.](#page-15-2)
- If  $\lambda \in \widehat{C_2}$ , then

$$
JSF(\lambda) = \chi(s_{\alpha_1 + \alpha_2, p} \cdot \lambda) = \chi(\lambda - (a+b+2-p)\rho) = \chi(p-b-2, p-a-2).
$$

If  $a = p - 1$  or  $b = p - 1$ , then  $JSF(\lambda) = 0$  because  $s_{\alpha_1 + \alpha_2, p} \cdot \lambda \in D \setminus X^+$  and  $\Delta(\lambda)$  is irreducible. Else  $\Delta(\lambda)$  admits the unique composition series irreducible. Else,  $\Delta(\lambda)$  admits the unique composition series

$$
[L(\lambda), L(\lambda - (a+b+2-p)\rho)].
$$

The last claim follows directly from Lemma [1.37.](#page-12-4)

We can also prove Lemma [5.3](#page-26-0) using Proposition [1.54,](#page-15-3) see for example [\[Sch19,](#page-78-8) Lemma 2.1.4]. To simplify the notation we define the map

$$
\Theta: \begin{array}{rcl} X & \to & \mathbb{Z} \\ (a, b) & \mapsto & a+b+2-p. \end{array}
$$

<span id="page-26-3"></span>**Remark 5.4.** By the proof of Lemma [5.3,](#page-26-0) we have

$$
ch L(\lambda) = \chi(\lambda) - \chi(\lambda - \Theta(\lambda)\rho)
$$

<span id="page-26-4"></span>for all dominant weights  $\lambda \in \widehat{C_2}$ .

**Lemma 5.5.** Let 
$$
\lambda \in X^+
$$
 be p-restricted. Then  $\lambda \in \widehat{C_2}$  if and only if  $\Theta(\lambda) \geq 1$ .

<span id="page-26-2"></span>*Proof.* This is a direct consequence of the definitions of  $\Theta$  and  $\widehat{C_2}$ .

**Lemma 5.6** ([\[Tes88,](#page-78-5) 1.35]). Let  $\lambda = (a, b) \in X^+$  be p-restricted with  $a \neq 0$  and  $b \neq 0$ . Then

$$
m_{L(\lambda)}(\lambda - \alpha_1 - \alpha_2) = \begin{cases} 1 & \text{if } a + b = p - 1, \\ 2 & \text{otherwise.} \end{cases}
$$

<span id="page-26-1"></span>**Lemma 5.7.** *Let*  $\lambda = (a, 0) \in X^+$  *with*  $a < p$ *. Then* 

$$
m_{L(\lambda)}(\lambda - i\alpha_1 - j\alpha_2) = \begin{cases} 1 & \text{if } 0 \le j \le i \le a, \\ 0 & \text{else.} \end{cases}
$$

*Proof.* First, observe that for  $a < p$ , we have  $\Delta(\lambda) \cong L(\lambda)$  by Lemma [5.3.](#page-26-0) Recall that  $\rho = \omega_1 + \omega_2$ . Using Weyl's degree formula (Corollary [1.43\)](#page-13-3) with  $(\alpha_1, \alpha_1) = 2$ , we have

$$
\dim L(\lambda) = \frac{(\lambda + \rho, \alpha_1)(\lambda + \rho, \alpha_2)(\lambda + \rho, \alpha_1 + \alpha_2)}{(\rho, \alpha_1)(\rho, \alpha_2)(\rho, \alpha_1 + \alpha_2)} = \frac{(a+1) \cdot 1 \cdot (a+2)}{1 \cdot 1 \cdot 2} = \frac{(a+2)(a+1)}{2}.
$$

Observe that  $A = {\lambda - i\alpha_1 - j\alpha_2}_{0 \le i \le i \le a}$  is saturated with highest weight  $\lambda$ . Thus, by Proposition [1.23,](#page-11-2) we have  $m_{L(\lambda)}(\nu) = m_{\Delta(\lambda)}(\nu) \ge 1$  for all  $\nu \in A$ . Moreover,

$$
|A| = \sum_{i=0}^{a} \sum_{j=0}^{i} 1 = \sum_{i=0}^{a} (i+1) = \sum_{i=1}^{a+1} i = \frac{(a+2)(a+1)}{2} = \dim L(\lambda).
$$

Therefore,  $m_{L(\lambda)}(\nu) = 1$  for all  $\nu \in A$ , as claimed.

 $\Box$ 

 $\Box$ 

 $\Box$ 

#### <span id="page-27-0"></span>**5.3 The classification**

In this subsection, we prove several propositions which yield the classification of multiplicity-free tensor products of simple SL3(*k*)-modules with *p*-restricted highest weight (Theorem [5.1\)](#page-24-2). We start by stating a theorem from Stembridge ([\[Ste03,](#page-78-0) Theorem 1.1.A]) which classifies multiplicity-free tensor products of simple  $SL_3(\mathbb{C})$ -modules.

<span id="page-27-3"></span>**Theorem 5.8.** *Let*  $(a, b)$ *,*(*c,d*)  $\in X^+$  *be dominant weights. Then*  $L_{\mathbb{C}}(a, b) \otimes L_{\mathbb{C}}(c, d)$  *is multiplicity-free if and only if*  $a \cdot b \cdot c \cdot d = 0$ .

Before moving on to the classification, recall that  $L(a, b)^* \cong L(b, a)$  for all  $a, b \in \mathbb{N}$  and observe that  $L(\lambda) \otimes L(\mu)$  is multiplicity-free if and only if  $(L(\lambda) \otimes L(\mu))^* \cong L(\lambda)^* \otimes L(\mu)^*$ is multiplicity-free. This allows us to treat several cases simultaneously.

<span id="page-27-1"></span>**5.3.1**  $L(a, 0) ⊗ L(c, 0)$ 

**Proposition 5.9.** *Let*  $\lambda = (a, 0), \mu = (c, 0) \in X^+$  *be p*-restricted. Then  $L(\lambda) \otimes L(\mu)$  *is multiplicity-free if and only if*  $a + c < p$ *.* 

*Proof.* The "only if" direction is a direct consequence of Corollary [2.8.](#page-17-8)

Suppose that  $a + c < p$ . By Lemma [5.3,](#page-26-0) we have  $\Delta(\lambda) \cong L(\lambda)$  and  $\Delta(\mu) \cong L(\mu)$ . If *ν*  $\leq \lambda + \mu$  is a dominant weight, then either *ν* =  $\lambda + \mu$  or *ν* ∈  $\widehat{C_1}$ . We have  $\Delta(\lambda + \mu) \cong L(\lambda + \mu)$ by Lemma [5.3](#page-26-0) and  $\Delta(\nu) \cong L(\nu)$  for all  $\nu \in \widehat{C_1}$  by Lemma [1.56.](#page-15-4) Thus we can apply Corollary [3.3.](#page-19-2) By Theorem [5.8,](#page-27-3)  $L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu)$  is multiplicity-free, thus  $L(\lambda) \otimes L(\mu)$  is multiplicity-free. multiplicity-free.

**Remark 5.10.** By duality,  $L(0, b) \otimes L(0, d)$  is multiplicity-free if and only if  $b + d < p$ .

<span id="page-27-2"></span>**5.3.2**  $L(a, 0) \otimes L(0, d)$ 

**Proposition 5.11.** Let  $\lambda = (a, 0), \mu = (0, d) \in X^+$  be p-restricted with  $a, d \neq 0$ . Then  $L(\lambda) \otimes L(\mu)$  *is multiplicity-free if and only if*  $a + d < p - 1$ *, or*  $(a, d) \in \{(p - 1, 1), (1, p - 1)\}.$ 

*Proof.* To simplify the notation, we set  $M := L(\lambda) \otimes L(\mu)$ .

By Lemma [5.3,](#page-26-0)  $L(\lambda)$  and  $L(\mu)$  are tilting modules. By Theorem [1.32,](#page-12-5) M is a tilting module. We know that  $L(\lambda + \mu)$  is a composition factor of M. If  $\lambda + \mu \in C_2$ , then we conclude that *M* has multiplicity using Lemma [1.36.](#page-12-6)

If  $a + d < p - 1$  (i.e. if  $\lambda + \mu \in \widehat{C_1}$ ), we apply Corollary [3.4](#page-19-3) and Theorem [5.8](#page-27-3) to conclude that *M* is multiplicity-free.

Suppose  $a = p - 1$  and  $d > 1$  (in particular,  $\lambda + \mu \in F_{2,3}$ ). We use Lemmas [1.15](#page-10-6) and [5.7](#page-26-1) and Argument [1](#page-16-1) to compute

$$
m_M(\lambda + \mu) = 1, \qquad m_M(\lambda + \mu - \alpha_1) = 1, \qquad m_M(\lambda + \mu - \alpha_2) = 1,
$$

$$
m_M(\lambda + \mu - \alpha_1 - \alpha_2) = 3.
$$

Since  $m_{L(\lambda+\mu)}(\lambda+\mu-\alpha_1) = m_{L(\lambda+\mu)}(\lambda+\mu-\alpha_2) = 1$  (Lemma [1.15\)](#page-10-6), we conclude that  $L(\lambda + \mu - \alpha_1)$  and  $L(\lambda + \mu - \alpha_2)$  are not composition factors of *M*. Moreover,  $m_{L(\lambda+\mu)}(\lambda+\mu-\alpha_1-\alpha_2) = 2$  by Lemma [5.6,](#page-26-2) thus  $L(\lambda+\mu-\alpha_1-\alpha_2)$  is a composition factor of *M*. But  $\lambda + \mu - \alpha_1 - \alpha_2 = (p - 2, d - 1) \in C_2$ , so  $T(\lambda + \mu - \alpha_1 - \alpha_2)$  is not irreducible by Lemma [5.3.](#page-26-0) By Lemma [1.36,](#page-12-6) we conclude that *M* has multiplicity. The case  $a > 1, d = p - 1$  is symmetric.

Finally consider the case  $a = p - 1$ ,  $d = 1$  (in particular,  $\lambda + \mu \in F_{2,3}$ ). Again, we have

$$
m_M(\lambda + \mu) = 1, \qquad m_M(\lambda + \mu - \alpha_1) = 1, \qquad m_M(\lambda + \mu - \alpha_2) = 1.
$$

Therefore,  $L(\lambda + \mu)$  is a composition factor, and all other composition factors have highest weight *ν* with  $\nu \leq \lambda + \mu - \alpha_1 - \alpha_2$ ,  $\nu \leq \lambda + \mu - 2\alpha_1$  or  $\nu \leq \lambda + \mu - 2\alpha_2$ . In particular,  $\nu \in \widehat{C_1}$ . By Lemma [5.3,](#page-26-0) this implies that for all dominant weights  $\nu$  such that  $L(\nu)$  is a composition factor of *M*, we have  $L(\nu) \cong \Delta(\nu)$ . Using Proposition [3.2](#page-19-1) and Theorem [5.8,](#page-27-3) we conclude that *M* is multiplicity-free. The case  $a = 1$ ,  $d = p - 1$  is symmetric  $\Box$ 

<span id="page-28-0"></span>**5.3.3**  $L(a, 0) ⊗ L(c, d)$ 

**Proposition 5.12.** Let  $\lambda = (a, 0), \mu = (c, d) \in X^+$  be *p*-restricted with  $a, c, d \neq 0$  and  $a + c + d < p - 1$  *(i.e.*  $\lambda + \mu \in \widehat{C_1}$ *). Then*  $L(\lambda) \otimes L(\mu)$  *is multiplicity-free.* 

*Proof.* This is a direct consequence of Corollary [3.4](#page-19-3) and Theorem [5.8.](#page-27-3)

 $\Box$ 

**Proposition 5.13.** Let  $\lambda = (a, 0), \mu = (c, d) \in X^+$  be *p*-restricted with  $a, c, d \neq 0$ ,  $a+c+d \geq p-1$  and  $L(\mu) \cong \Delta(\mu)$  (i.e.  $\lambda + \mu \notin \widehat{C}_1$  and  $\mu \in \widehat{C}_1 \cup F_{2,3} \cup F_{2,3'}$ ). Then  $L(\lambda) \otimes L(\mu)$  *has multiplicity.* 

*Proof.* To simplify the notation, we set  $M := L(\lambda) \otimes L(\mu)$ . By assumption  $\Delta(\lambda) \cong L(\lambda)$  and  $\Delta(\mu) \cong L(\mu)$ , in particular these two modules are tilting modules. By Theorem [1.32,](#page-12-5) M is a tilting module.

If  $a + c \geq p$ , we conclude directly by Corollary [2.8](#page-17-8) that *M* has multiplicity. So we can restrict our attention to the case  $a + c < p$ .

Observe that for  $d = p - 2$ , the condition  $\Delta(\lambda) \cong L(\lambda)$  forces  $c = p - 1$  (Lemma [5.3\)](#page-26-0), so  $a + c \geq p$  and *M* has multiplicity.

Suppose that  $d = p - 1$ . By Lemma [1.15](#page-10-6) and Argument [1,](#page-16-1) we have

 $m_M(\lambda + \mu) = 1,$   $m_M(\lambda + \mu - \alpha_1) = 2.$ 

By Lemma [1.15,](#page-10-6) we have  $m_{L(\lambda+\mu)}(\lambda+\mu-\alpha_1)=1$ , so we deduce that  $L(\lambda+\mu-\alpha_1)$  is a composition factor of *M*. But  $\lambda + \mu - \alpha_1 = (a + c - 2, p)$  is not *p*-restricted, thus *M* has multiplicity by Corollary [2.7.](#page-17-6)

Suppose that  $a + c < p - 1$  and  $d < p - 1$ . By assumption,  $\lambda + \mu \in C_2$ , and  $T(\lambda + \mu)$  is not irreducible by Lemma [5.3.](#page-26-0) Therefore, by Lemma [1.36,](#page-12-6) *M* has multiplicity.

Finally, consider the case  $a + c = p - 1$ ,  $d < p - 2$ . Again, we have

$$
m_M(\lambda + \mu) = 1, \qquad m_M(\lambda + \mu - \alpha_1) = 2,
$$

and  $L(\lambda + \mu - \alpha_1)$  is a composition factor of *M*. Observe that  $\lambda + \mu - \alpha_1 = (p-3, d+1) \in C_2$ , so  $T(\lambda + \mu - \alpha_1)$  is not irreducible. We conclude by Lemma [1.36](#page-12-6) that *M* has multiplicity.  $\Box$ 

**Proposition 5.14.** *Let*  $\lambda = (a, 0), \mu = (c, d) \in X^+$  *be p-restricted with*  $a, c, d \neq 0$ *. If*  $d + \min(a, c) \geq p$ *, then*  $L(\lambda) \otimes L(\mu)$  *has multiplicity.* 

*Proof.* To simplify the notation, we set  $M := L(\lambda) \otimes L(\mu)$ .

If  $a + c \geq p$ , we conclude directly using Corollary [2.8.](#page-17-8) Thus we can assume  $a + c < p$ . We show inductively that  $[M : L(\lambda + \mu - s\alpha_1)] = 1$  for  $0 \le s \le p - d \le \min(a, c)$ . For  $0 \leq s \leq \min(a, c)$ , using Lemma [1.15,](#page-10-6) we have

$$
m_M(\lambda + \mu - s\alpha_1) = \sum_{i=0}^s m_{L(\lambda)}(\lambda - i\alpha_1) \cdot m_{L(\mu)}(\mu - (s - i)\alpha_1) = s + 1.
$$

Moreover, if  $0 \le i \le s \le p - d$ , then  $\lambda + \mu - i\alpha_1 = (a + c - 2i, d + i)$  is *p*-restricted and by Lemma [1.15,](#page-10-6) we have

$$
\sum_{i=0}^{s-1} m_{L(\lambda+\mu-i\alpha_1)}(\lambda+\mu-s\alpha_1) = s.
$$

We know that  $[M: L(\lambda + \mu)] = 1$ , and combining the two previous equations, we conclude inductively that  $[M : L(\lambda + \mu - s\alpha_1)] = 1$  for  $0 \le s \le p - d$ . In particular,  $L(\lambda + \mu - (p - d)\alpha_1)$ is a composition factor of *M*. But  $\lambda + \mu - (p - d)\alpha_1 = (a + c - 2(p - d), p)$  is not *p*-restricted. We conclude by Corollary [2.7](#page-17-6) that *M* has multiplicity.  $\Box$ 

**Proposition 5.15.** *Let*  $\lambda = (a, 0), \mu = (c, d) \in X^+$  *be p-restricted with*  $a, c, d \neq 0$ *,*  $c + d = p - 1$  *and*  $a + c < p$  *(in particular,*  $\mu \in \widehat{C_2}$ *). Then*  $L(\lambda) \otimes L(\mu)$  *is multiplicityfree if and only if*  $a < c + 2$ *.* 

*Proof.* We set  $M := L(\lambda) \otimes L(\mu)$ .

By Remark [5.4,](#page-26-3) we have  $ch L(\mu) = \chi(\mu) - \chi(\mu - \rho)$  and  $ch L(\lambda) = \chi(\lambda)$ . Therefore, by Proposition [1.47,](#page-14-2) we have

ch 
$$
M = \chi(\lambda)(\chi(\mu) - \chi(\mu - \rho)) = \sum_{\nu \in X} m_{\Delta(\lambda)}(\nu)(\chi(\mu + \nu) - \chi(\mu + \nu - \rho)).
$$

Now we use Lemma [5.7](#page-26-1) to rewrite this sum. We get

ch 
$$
M = \sum_{\nu \in X} m_{\Delta(\lambda)}(\nu) (\chi(\mu + \nu) - \chi(\mu + \nu - \rho))
$$
  
\n
$$
= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \chi(\mu + \lambda - i\alpha_1 - j\alpha_2 - \rho)
$$
\n
$$
= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2 - \rho)
$$
\n
$$
= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \sum_{i=1}^{a+1} \sum_{j=1}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2)
$$
\n
$$
= \sum_{i=0}^{a} \chi(\mu + \lambda - i\alpha_1) - \sum_{j=1}^{a+1} \chi(\mu + \lambda - (a+1)\alpha_1 - j\alpha_2)
$$
\n
$$
= \sum_{i=0}^{a} \chi(\mu + \lambda - i\alpha_1) - \sum_{j=0}^{a} \chi(\mu + \lambda - (a+1)\alpha_1 - (a+1 - j)\alpha_2)
$$
\n
$$
= \sum_{i=0}^{a} \chi(\mu + \lambda - i\alpha_1) - \chi(\mu + \lambda - (a+1)\alpha_1 - (a+1 - i)\alpha_2)
$$

$$
= \sum_{i=0}^{a} \chi(\mu + \lambda - i\alpha_1) - \chi(\mu + \lambda - i\alpha_1 - (a+1-i)\rho)
$$
  

$$
= \sum_{i=0}^{a} \chi(\mu + \lambda - i\alpha_1) - \chi(\mu + \lambda - i\alpha_1 - \Theta(\mu + \lambda - i\alpha_1)\rho),
$$
 (1)

where in the last equality, we used that, for  $i \in \{0, \ldots, a\}$ , we have

<span id="page-30-0"></span>
$$
\mu + \lambda - i\alpha_1 = (a + c - 2i, d + i),
$$

so  $\Theta(\mu + \lambda - i\alpha_1) = a + 1 - i$ . Now we dissociate several cases.

First, suppose that  $a \leq c$ . In this case, for  $i \in \{0, \ldots, a\}$  we have  $0 \leq a + c - 2i < p$  and  $0 < d + i \leq d + c < p$ , therefore  $\mu + \lambda - i\alpha_1$  is dominant and *p*-restricted. Moreover,

$$
\Theta(\mu + \lambda - i\alpha_1) = a + 1 - i \ge 1.
$$

By Lemma [5.5,](#page-26-4) we have  $\mu + \lambda - i\alpha_1 \in \widehat{C_2}$  for all  $i \in \{0, \ldots, a\}$ . Using Remark [5.4,](#page-26-3) we get

$$
\operatorname{ch} M = \sum_{i=0}^{a} \chi(\mu + \lambda - i\alpha_1) - \chi(\mu + \lambda - i\alpha_1 - \Theta(\mu + \lambda - i\alpha_1)\rho)
$$

$$
= \sum_{i=0}^{a} \operatorname{ch} L(\mu + \lambda - i\alpha_1). \tag{2}
$$

Thus we conclude that *M* is multiplicity-free.

Suppose that  $a = c + 1$ . Using line [\(1\)](#page-30-0), we have

ch 
$$
M = \sum_{i=0}^{c} \left( \chi(\mu + \lambda - i\alpha_1) - \chi(\mu + \lambda - i\alpha_1 - \Theta(\mu + \lambda - i\alpha_1)\rho) \right)
$$
  
+  $\chi(\mu + \lambda - a\alpha_1) - \chi(\mu + \lambda - a\alpha_1 - \Theta(\mu + \lambda - a\alpha_1)\rho).$ 

In this case, for  $i \in \{0, \ldots, c\}$  we have  $0 \le a + c - 2i < p$  and  $0 < d + i \le d + c < p$  so  $\mu + \lambda - i\alpha_1$  is dominant and *p*-restricted. Moreover,

$$
\Theta(\mu + \lambda - i\alpha_1) = a + 1 - i \ge 2,
$$

so by Lemma [5.5,](#page-26-4)  $\mu + \lambda - i\alpha_1 \in \widehat{C_2}$  for all  $i \in \{0, \ldots, c\}$ . By Remark [5.4,](#page-26-3) we have

$$
\chi(\mu + \lambda - i\alpha_1) - \chi(\mu + \lambda - i\alpha_1 - \Theta(\mu + \lambda - i\alpha_1)\rho) = \text{ch } L(\mu + \lambda - i\alpha_1). \tag{3}
$$

Moreover,

<span id="page-30-2"></span>
$$
\mu + \lambda - a\alpha_1 = (-1, d + a) \in D \setminus X^+,
$$

so by Lemma [1.45](#page-13-4) we have

<span id="page-30-3"></span><span id="page-30-1"></span>
$$
\chi(\mu + \lambda - a\alpha_1) = 0. \tag{4}
$$

Finally,

$$
\mu + \lambda - a\alpha_1 - \Theta(\mu + \lambda - a\alpha_1)\rho = \mu + \lambda - a\alpha_1 - \rho = (-2, d + a - 1) = (-2, p - 1).
$$

Using again Lemma [1.45,](#page-13-4) we get

$$
\chi(-2, p-1) = -\chi(s_{\alpha_1} \bullet (-2, p-1)) = -\chi(0, p-2) = -\operatorname{ch} L(0, p-2)
$$
  
= 
$$
-\operatorname{ch} L(\lambda + \mu - a\alpha_1 - \alpha_2)
$$
 (5)

where in the last equality, we used that  $\lambda + \mu - a\alpha_1 - \alpha_2 = (0, p - 2) \in X^+$  is *p*-restricted and Lemma [5.3.](#page-26-0) Combining lines [\(3\)](#page-30-1), [\(4\)](#page-30-2) and [\(5\)](#page-30-3), we conclude that

$$
\operatorname{ch} M = \sum_{i=0}^{c} \left( \chi(\mu + \lambda - i\alpha_1) - \chi(\mu + \lambda - i\alpha_1 - \Theta(\mu + \lambda - i\alpha_1)\rho) \right)
$$

$$
+ \chi(\mu + \lambda - a\alpha_1) - \chi(\mu + \lambda - a\alpha_1 - \Theta(\mu + \lambda - a\alpha_1)\rho)
$$

$$
= \operatorname{ch} L(\lambda + \mu - a\alpha_1 - \alpha_2) + \sum_{i=0}^{c} \operatorname{ch} L(\mu + \lambda - i\alpha_1), \tag{6}
$$

and *M* is multiplicity-free.

Finally, we consider the case  $a \geq c+2$ . We show that *M* has multiplicity. Using line [\(1\)](#page-30-0), we have

$$
\operatorname{ch} M = \sum_{i=0}^{c} \left( \chi(\mu + \lambda - i\alpha_1) - \chi(\mu + \lambda - i\alpha_1 - \Theta(\mu + \lambda - i\alpha_1)\rho) \right) \tag{7}
$$

<span id="page-31-4"></span><span id="page-31-2"></span><span id="page-31-1"></span><span id="page-31-0"></span>
$$
+\sum_{i=c+1}^{a} \left(\chi(\mu+\lambda-i\alpha_1)-\chi(\mu+\lambda-i\alpha_1-\Theta(\mu+\lambda-i\alpha_1)\rho)\right).
$$
 (8)

By the same argument as in the previous case (see line  $(3)$ ), we write  $(7)$  as

$$
\sum_{i=0}^{c} \left( \chi(\mu + \lambda - i\alpha_1) - \chi(\mu + \lambda - i\alpha_1 - \Theta(\mu + \lambda - i\alpha_1)\rho) \right) = \sum_{i=0}^{c} \text{ch } L(\mu + \lambda - i\alpha_1). \tag{9}
$$

We rewrite the sum of line  $(8)$  by making the expression of the weights explicit. We get

$$
\sum_{i=c+1}^{a} \left( \chi(\mu + \lambda - i\alpha_1) - \chi(\mu + \lambda - i\alpha_1 - \Theta(\mu + \lambda - i\alpha_1)\rho) \right)
$$
  
= 
$$
\sum_{i=c+1}^{a} \left( \chi(a + c - 2i, d + i) - \chi(c - i - 1, d + 2i - a - 1) \right)
$$
  
= 
$$
\sum_{i=1}^{a-c} \left( \chi(a - c - 2i, d + c + i) - \chi(-i - 1, d + 2c + 2i - a - 1) \right)
$$
  
= 
$$
\sum_{i=1}^{a-c} \chi(a - c - 2i, p - 1 + i)
$$
 (10)  
- 
$$
\sum_{i=1}^{a-c} \chi(-i - 1, d + 2c + 2i - a - 1).
$$
 (11)

We show that the sum in line [\(10\)](#page-31-2) is equal to zero. Using Lemma [1.45,](#page-13-4) we have

<span id="page-31-3"></span>
$$
\sum_{i=1}^{a-c} \chi(a-c-2i, p-1+i)
$$
  
= 
$$
-\sum_{i=1}^{a-c} \chi(s_{\alpha_1} \cdot (a-c-2i, p-1+i))
$$
  
= 
$$
-\sum_{i=1}^{a-c} \chi(-a+c+2i-2, p-i+a-c)
$$

$$
= -\sum_{i=1}^{a-c} \chi(-a+c+2(a-c+1-i) - 2, p - (a-c+1-i) + a - c)
$$
  
= 
$$
-\sum_{i=1}^{a-c} \chi(a-c-2i, p-1+i)
$$

Therefore,

<span id="page-32-0"></span>
$$
\sum_{i=1}^{a-c} \chi(a-c-2i, p-1+i) = 0.
$$
 (12)

Now, let us study line [\(11\)](#page-31-3). Using Lemma [1.45,](#page-13-4) we have

$$
\sum_{i=1}^{a-c} \chi(-i-1, d+2c+2i-a-1) = -\sum_{i=1}^{a-c} \chi(s_{\alpha_1} \cdot (-i-1, d+2c+2i-a-1))
$$
  
= 
$$
-\sum_{i=1}^{a-c} \chi(i-1, d+2c+i-a-1) = -\sum_{i=0}^{a-c-1} \chi(i, p-1+c-a+i).
$$

Observe that  $(i, p - 1 + c - a + i) \in X^+$  is *p*-restricted for all  $i \in \{0, ..., a - c - 1\}$ . We dissociate the cases  $a - c$  even and odd.

If  $a - c$  is even, we get

$$
\sum_{i=0}^{a-c-1} \chi(i, p-1+c-a+i) = \sum_{i=0}^{\frac{a-c}{2}-1} \chi(a-c-1-i, p-2-i) + \chi(i, p-1+c-a+i)
$$
  
= 
$$
\sum_{i=0}^{\frac{a-c}{2}-1} \chi(a-c-1-i, p-2-i)
$$
  
+ 
$$
\chi((a-c-1-i, p-2-i) - \Theta(a-c-1-i, p-2-i)\rho).
$$

For  $i \in \{0, \ldots, \frac{a-c}{2} - 1\}$ , we have

$$
(a-c-1-i)+(p-2-i) \ge a-c-1+p-(a-c-2)=p+1,
$$

thus  $(a - c - 1 - i, p - 2 - i) \in C_2$ . This implies that

$$
(a-c-1-i, p-2-i) - \Theta(a-c-1-i, p-2-i)\rho \in C_1,
$$

and in particular, by Lemma [5.3](#page-26-0) we have

$$
\chi((a-c-1-i,p-2-i)-\Theta(a-c-1-i,p-2-i)\rho)
$$
  
= ch  $L((a-c-1-i,p-2-i)-\Theta(a-c-1-i,p-2-i)\rho).$ 

Therefore, by Remark [5.4,](#page-26-3) we have

$$
\sum_{i=0}^{\frac{a-c}{2}-1} \chi(a-c-1-i, p-2-i) + \chi((a-c-1-i, p-2-i) - \Theta(a-c-1-i, p-2-i)\rho)
$$
  
= 
$$
\sum_{i=0}^{\frac{a-c}{2}-1} \text{ch } L(a-c-1-i, p-2-i)
$$
  
+ 
$$
2\chi((a-c-1-i, p-2-i) - \Theta(a-c-1-i, p-2-i)\rho)
$$

$$
= \sum_{i=0}^{\frac{a-c}{2}-1} \text{ch } L(a-c-1-i, p-2-i)
$$
  
+ 2ch  $L((a-c-1-i, p-2-i) - \Theta(a-c-1-i, p-2-i)\rho)$   
= 
$$
\sum_{i=0}^{\frac{a-c}{2}-1} \text{ch } L(a-c-1-i, p-2-i) + 2\text{ch } L(i, p-1+c-a+i)
$$
  
= 
$$
\sum_{i=1}^{\frac{a-c}{2}} \text{ch } L(a-c-i, p-1-i) + 2\text{ch } L(i-1, p-2+c-a+i)
$$
  
= 
$$
\sum_{i=1}^{\frac{a-c}{2}} \text{ch } L(\lambda + \mu - c\alpha_1 - i\rho) + 2\text{ch } L(\lambda + \mu - c\alpha_1 - (a+d-i+2-p)\rho).
$$
 (13)

<span id="page-33-0"></span>Similarly, if  $a - c$  is odd, we get

$$
\sum_{i=0}^{a-c-1} \chi(i, p-1+c-a+i)
$$
  
= 
$$
\sum_{i=0}^{\frac{a-c-1}{2}-1} \left( \chi(i, p-1+c-a+i) + \chi(a-c-1-i, p-2-i) \right)
$$
  
+ 
$$
\chi(\frac{a-c-1}{2}, p-1+\frac{c-a-1}{2}).
$$

Since  $(\frac{a-c-1}{2}, p-1+\frac{c-a-1}{2}) \in C_1$ , we get, as in the even case,

<span id="page-33-1"></span>
$$
\sum_{i=0}^{\frac{a-c-1}{2}-1} \left( \chi(i, p-1+c-a+i) + \chi(a-c-1-i, p-2-i) \right)
$$
  
+  $\chi\left(\frac{a-c-1}{2}, p-1+\frac{c-a-1}{2}\right)$   
= 
$$
\sum_{i=0}^{\frac{a-c-1}{2}-1} \text{ch } L(a-c-1-i, p-2-i) + 2 \text{ch } L(i, p-1+c-a+i)
$$
  
+  $\text{ch } L\left(\frac{a-c-1}{2}, p-1+\frac{c-a-1}{2}\right)$   
= 
$$
\sum_{i=1}^{\frac{a-c-1}{2}} \text{ch } L(\lambda + \mu - c\alpha_1 - i\rho) + 2 \text{ch } L(\lambda + \mu - c\alpha_1 - (a+d-i+2-p)\rho)
$$
  
+  $\text{ch } L(\lambda + \mu - c\alpha_1 - \frac{a-c+1}{2}\rho).$  (14)

Combining lines [\(9\)](#page-31-4), [\(12\)](#page-32-0) and [\(13\)](#page-33-0) (respectively [\(14\)](#page-33-1)), we find, if  $a - c$  is even:

ch 
$$
M = \sum_{i=0}^{c} \text{ch } L(\mu + \lambda - i\alpha_{1})
$$
  
+  $\sum_{i=1}^{\frac{a-c}{2}} \text{ch } L(\lambda + \mu - c\alpha_{1} - i\rho) + 2 \text{ch } L(\lambda + \mu - c\alpha_{1} - (a + d - i + 2 - p)\rho),$ 

and if  $a - c$  is odd:

ch 
$$
M =
$$
ch  $L(\lambda + \mu - c\alpha_1 - \frac{a - c + 1}{2}\rho) + \sum_{i=0}^{c} \text{ch } L(\mu + \lambda - i\alpha_1)$   
+  $\sum_{i=1}^{\frac{a - c - 1}{2}} \text{ch } L(\lambda + \mu - c\alpha_1 - i\rho) + 2 \text{ch } L(\lambda + \mu - c\alpha_1 - (a + d - i + 2 - p)\rho).$ 

In both cases, the second sum is non-empty since  $a \geq c + 2$  by assumption, and we deduce  $\Box$ that *M* has multiplicity.

**Proposition 5.16.** *Let*  $\lambda = (a, 0), \mu = (c, d) \in X^+$  *be p-restricted with*  $a, c, d \neq 0$ *,*  $c + d > p - 1$ ,  $a + c < p$  and  $a + d < p$  (in particular,  $\mu \in C_2$ ). Then  $L(\lambda) \otimes L(\mu)$  is *multiplicity-free.*

*Proof.* We set  $M := L(\lambda) \otimes L(\mu)$ , and  $m := \Theta(\mu) = c + d + 2 - p \geq 2$ . Observe that our hypotheses imply *a < c* and *a < d*.

By Remark [5.4](#page-26-3) and Lemma [5.3,](#page-26-0) we have ch  $L(\mu) = \chi(\mu) - \chi(\mu - m\rho)$  and ch  $L(\lambda) = \chi(\lambda)$ . Therefore, by Proposition [1.47,](#page-14-2) we have

ch 
$$
M =
$$
ch  $L(\lambda) \cdot$ ch  $L(\mu) = \chi(\lambda) \cdot (\chi(\mu) - \chi(\mu - m\rho))$   
= 
$$
\sum_{\nu \in X} m_{\Delta(\lambda)}(\nu) (\chi(\mu + \nu) - \chi(\mu + \nu - m\rho)).
$$

We use Lemma [5.7](#page-26-1) to rewrite this sum. We get

ch 
$$
M = \sum_{\nu \in X} m_{\Delta(\lambda)}(\nu) (\chi(\mu + \nu) - \chi(\mu + \nu - m\rho))
$$
  
\n
$$
= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \chi(\mu + \lambda - i\alpha_1 - j\alpha_2 - m\rho)
$$
\n
$$
= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2 - m\rho)
$$
\n
$$
= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \sum_{i=m}^{a+m} \sum_{j=m}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2).
$$
\n(15)

For  $0 \leq j \leq i \leq a$ , we have

<span id="page-34-1"></span><span id="page-34-0"></span>
$$
\mu + \lambda - i\alpha_1 - j\alpha_2 = (a + c + j - 2i, d + i - 2j)
$$
\n(16)

with

$$
0 < c - a \le c + a - 2i \le c + a + j - 2i \le c + a - i \le c + a < p
$$

and

$$
0 < d - a \le d - j \le d + i - 2j \le d + i \le d + a < p.
$$

Therefore,

$$
\mu + \lambda - i\alpha_1 - j\alpha_2 \in X^+ \text{ is } p\text{-restricted for all } 0 \le j \le i \le a. \tag{17}
$$

Moreover, using line [\(16\)](#page-34-0), we have

$$
\Theta(\lambda + \mu - i\alpha_1 - j\alpha_2) = (a + c + j - 2i) + (d + i - 2j) + 2 - p = m + a - i - j, \quad (18)
$$

and if  $0 \le j \le i \le \min\{a, m-1\}$ , then

<span id="page-35-2"></span><span id="page-35-1"></span><span id="page-35-0"></span>
$$
\Theta(\lambda + \mu - i\alpha_1 - j\alpha_2) = m + a - i - j \ge 1.
$$

Using line [\(17\)](#page-35-0) and Lemma [5.5,](#page-26-4) we deduce that

$$
\lambda + \mu - i\alpha_1 - j\alpha_2 \in \widehat{C}_2 \quad \text{ for all } 0 \le j \le i \le \min\{a, m - 1\}. \tag{19}
$$

We dissociate the cases  $a < m$  and  $a \geq m$ . If  $a < m$ , using line [\(15\)](#page-34-1), we have

ch 
$$
M = \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \sum_{i=m}^{a+m} \sum_{j=m}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2)
$$
  
\n
$$
= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \sum_{j=m}^{a+m} \sum_{i=j}^{a+m} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2)
$$
\n
$$
= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \sum_{i=m}^{a+m} \sum_{j=i}^{a+m} \chi(\mu + \lambda - j\alpha_1 - i\alpha_2)
$$
\n
$$
= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \sum_{i=0}^{a} \sum_{j=i}^{a} \chi(\mu + \lambda - j\alpha_1 - i\alpha_2 - m\rho)
$$
\n
$$
= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - (a - j)\alpha_1 - (a - i)\alpha_2 - m\rho)
$$
\n
$$
= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \chi(\mu + \lambda - (a - j)\alpha_1 - (a - i)\alpha_2 - m\rho)
$$
\n
$$
= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \chi(\mu + \lambda - i\alpha_1 - j\alpha_2 - (m + a - i - j)\rho)
$$
\n
$$
= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \chi(\mu + \lambda - i\alpha_1 - j\alpha_2 - \Theta(\mu + \lambda - i\alpha_1 - j\alpha_2)\rho),
$$

where in the last equality, we use line [\(18\)](#page-35-1). Since  $a \leq m - 1$ , we can use line [\(19\)](#page-35-2) and Remark [5.4](#page-26-3) to conclude that

$$
\operatorname{ch} M = \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \chi(\mu + \lambda - i\alpha_1 - j\alpha_2 - \Theta(\mu + \lambda - i\alpha_1 - j\alpha_2)\rho)
$$

$$
= \sum_{i=0}^{a} \sum_{j=0}^{i} \operatorname{ch} L(\mu + \lambda - i\alpha_1 - j\alpha_2). \tag{20}
$$
In particular, *M* is multiplicity-free.

If  $a \geq m$ , using line [\(15\)](#page-34-0), we have

$$
\begin{split}\n\text{ch } M &= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \sum_{i=m}^{a+m} \sum_{j=m}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) \\
&= \sum_{i=0}^{m-1} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) + \sum_{i=m}^{a} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) \\
&- \sum_{i=a+1}^{a+m} \sum_{j=m}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) \\
&= \sum_{i=0}^{m-1} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) + \sum_{i=m}^{a} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) \\
&- \sum_{j=a+1}^{a+m} \sum_{i=j}^{a+m} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \sum_{j=m}^{a} \sum_{i=a+1}^{a+m} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) \\
&= \sum_{i=0}^{m-1} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \sum_{i=0}^{m-1} \sum_{j=i}^{m-1} \chi(\mu + \lambda - j\alpha_{1} - i\alpha_{2} - (a+1)\rho) \\
&+ \sum_{i=m}^{a} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_{1} - j\alpha_{2}) - \sum_{i=m}^{a} \sum_{j=0}^{m-1} \chi(\mu + \lambda - (j+a+1)\alpha_{1} - i\alpha_{2}).\n\end{split} \tag{22}
$$

Let us study line [\(21\)](#page-36-0). We have

<span id="page-36-1"></span><span id="page-36-0"></span>
$$
\sum_{i=0}^{m-1} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \sum_{i=0}^{m-1} \sum_{j=i}^{m-1} \chi(\mu + \lambda - j\alpha_1 - i\alpha_2 - (a+1)\rho)
$$
  
= 
$$
\sum_{i=0}^{m-1} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2)
$$
  

$$
- \sum_{i=0}^{m-1} \sum_{j=0}^{i} \chi(\mu + \lambda - (m-1-j)\alpha_1 - (m-1-i)\alpha_2 - (a+1)\rho)
$$
  
= 
$$
\sum_{i=0}^{m-1} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \chi(\mu + \lambda - i\alpha_1 - j\alpha_2 - (a+m-i-j)\rho)
$$
  
= 
$$
\sum_{i=0}^{m-1} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \chi(\mu + \lambda - i\alpha_1 - j\alpha_2 - \Theta(\mu + \lambda - i\alpha_1 - j\alpha_2)\rho),
$$

where in the last equality, we use line [\(18\)](#page-35-0). Now we can use line [\(19\)](#page-35-1) and Remark [5.4](#page-26-0) to conclude that

<span id="page-36-2"></span>
$$
\sum_{i=0}^{m-1} \sum_{j=0}^{i} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \chi(\mu + \lambda - i\alpha_1 - j\alpha_2 - \Theta(\mu + \lambda - i\alpha_1 - j\alpha_2)\rho)
$$
  
= 
$$
\sum_{i=0}^{m-1} \sum_{j=0}^{i} \text{ch } L(\mu + \lambda - i\alpha_1 - j\alpha_2).
$$
 (23)

Now we study line [\(22\)](#page-36-1). We have

$$
\sum_{i=m}^{a} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \sum_{i=m}^{a} \sum_{j=0}^{m-1} \chi(\mu + \lambda - (j + a + 1)\alpha_1 - i\alpha_2)
$$
\n
$$
= \sum_{i=m}^{a} \sum_{j=0}^{m-1} \chi(\mu + \lambda - i\alpha_1 - j\alpha_2) - \sum_{i=m}^{a} \sum_{j=0}^{m-1} \chi(\mu + \lambda - ((m - 1 - j) + a + 1)\alpha_1 - i\alpha_2)
$$
\n
$$
= \sum_{i=0}^{a-m} \sum_{j=0}^{m-1} \chi(\mu + \lambda - (i + m)\alpha_1 - j\alpha_2) - \sum_{i=0}^{a-m} \sum_{j=0}^{m-1} \chi(\mu + \lambda - (m + a - j)\alpha_1 - (i + m)\alpha_2)
$$
\n
$$
= \sum_{i=0}^{a-m} \sum_{j=0}^{m-1} \chi(\mu + \lambda - (i + m)\alpha_1 - j\alpha_2)
$$
\n
$$
- \sum_{i=0}^{a-m} \sum_{j=0}^{m-1} \chi(\mu + \lambda - (m + a - j)\alpha_1 - ((a - m - i) + m)\alpha_2)
$$
\n
$$
= \sum_{i=0}^{a-m} \sum_{j=0}^{m-1} \chi(\mu + \lambda - (i + m)\alpha_1 - j\alpha_2) - \sum_{i=0}^{a-m} \sum_{j=0}^{m-1} \chi(\mu + \lambda - (m + a - j)\alpha_1 - (a - i)\alpha_2)
$$
\n
$$
= \sum_{i=0}^{a-m} \sum_{j=0}^{m-1} \chi(\mu + \lambda - (i + m)\alpha_1 - j\alpha_2) - \chi(\mu + \lambda - (m + a - j)\alpha_1 - (a - i)\alpha_2)
$$
\n
$$
= \sum_{i=0}^{a-m} \sum_{j=0}^{m-1} \chi(\mu + \lambda - (i + m)\alpha_1 - j\alpha_2) - \chi(\mu + \lambda - (i + m)\alpha_1 - j\alpha_2 - (a - i - j)\rho)
$$
\n
$$
= \sum_{i=0}^{a-m} \sum_{j=0}^{m-1} \chi(\mu + \lambda - (i + m)\alpha_1 - j\alpha_
$$

where in the last equality, we use line [\(18\)](#page-35-0). For  $0 \le i \le a - m$ ,  $0 \le j \le m - 1$ , we have

<span id="page-37-0"></span>
$$
\Theta(\mu + \lambda - (i+m)\alpha_1 - j\alpha_2) = a - i - j \ge 1.
$$

Therefore, using line [\(17\)](#page-35-2) and Lemma [5.5,](#page-26-1) we deduce that  $\mu + \lambda - (i + m)\alpha_1 - j\alpha_2 \in C_2$  for all  $0 \leq i \leq a-m$ ,  $0 \leq j \leq m-1$ . By Remark [5.4,](#page-26-0) we conclude that

$$
\sum_{i=0}^{a-m} \sum_{j=0}^{m-1} \chi(\mu + \lambda - (i+m)\alpha_1 - j\alpha_2)
$$
  
 
$$
- \chi(\mu + \lambda - (i+m)\alpha_1 - j\alpha_2 - \Theta(\mu + \lambda - (i+m)\alpha_1 - j\alpha_2)\rho)
$$
  
 
$$
= \sum_{i=0}^{a-m} \sum_{j=0}^{m-1} \text{ch } L(\mu + \lambda - (i+m)\alpha_1 - j\alpha_2).
$$
 (24)

<span id="page-38-0"></span>We conclude by combining lines [\(23\)](#page-36-2) and [\(24\)](#page-37-0) to obtain

$$
\operatorname{ch} M = \sum_{i=0}^{m-1} \sum_{j=0}^{i} \operatorname{ch} L(\mu + \lambda - i\alpha_1 - j\alpha_2) + \sum_{i=0}^{a-m} \sum_{j=0}^{m-1} \operatorname{ch} L(\mu + \lambda - (i+m)\alpha_1 - j\alpha_2)
$$

$$
= \sum_{i=0}^{m-1} \sum_{j=0}^{i} \operatorname{ch} L(\mu + \lambda - i\alpha_1 - j\alpha_2) + \sum_{i=m}^{a} \sum_{j=0}^{m-1} \operatorname{ch} L(\mu + \lambda - i\alpha_1 - j\alpha_2). \tag{25}
$$

In particular, *M* is multiplicity-free.

**Remark 5.17.** By duality, for  $\lambda = (0, b), \mu = (c, d) \in X^+$  with  $0 < b, c, d < p$ , the tensor product  $L(\lambda) \otimes L(\mu)$  is multiplicity-free if and only if one of the following holds:

- (1)  $b + c + d < p 1$ ,
- (2)  $c + d = p 1$ ,  $b + d < p$  and  $b < d + 2$  or
- (3)  $c + d > p 1$ ,  $b + c < p$  and  $b + d < p$ .

At this step, let us make the following observation, which will be useful in the next proposition.

<span id="page-38-1"></span>**Corollary 5.18.** *Let*  $\lambda = (a, 0), \mu = (c, d) \in X^+$  *be p-restricted with*  $a, c, d \neq 0, c+d \geq p-1$ ,  $a + c < p$  and  $a + d < p$  (in particular  $\mu \in C_2$ ). Let  $m := \Theta(\mu)$ . Then

$$
\operatorname{ch}(L(\lambda) \otimes L(\mu)) = \sum_{i=0}^{a} \sum_{j=0}^{i} \operatorname{ch} L(\lambda + \mu - i\alpha_1 - j\alpha_2) - \sum_{i=m}^{a} \sum_{j=m}^{i} \operatorname{ch} L(\lambda + \mu - i\alpha_1 - j\alpha_2).
$$

*Proof.*

- If  $c+d > p-1$  and  $a < m$ , then the second double sum is empty and we find the same result as in line [\(20\)](#page-35-3) (in the proof of Proposition [5.16\)](#page-34-1).
- If  $c + d > p 1$  and  $a \geq m$ , the result is a consequence of line [\(25\)](#page-38-0) in the proof of Proposition [5.16](#page-34-1) since we have

$$
\operatorname{ch}(L(\lambda) \otimes L(\mu)) = \sum_{i=0}^{m-1} \sum_{j=0}^{i} \operatorname{ch} L(\mu + \lambda - i\alpha_1 - j\alpha_2) + \sum_{i=m}^{a} \sum_{j=0}^{m-1} \operatorname{ch} L(\mu + \lambda - i\alpha_1 - j\alpha_2)
$$

$$
= \sum_{i=0}^{a} \sum_{j=0}^{i} \operatorname{ch} L(\lambda + \mu - i\alpha_1 - j\alpha_2) - \sum_{i=m}^{a} \sum_{j=m}^{i} \operatorname{ch} L(\lambda + \mu - i\alpha_1 - j\alpha_2).
$$

• If  $c + d = p - 1$ , the condition  $a + d < p$  is equivalent to  $a \leq c$ . Moreover, in this case  $m = \Theta(\mu) = 1$ . Therefore, by line [\(2\)](#page-30-0) in the proof of Proposition [5.15,](#page-29-0) we get

$$
ch(L(\lambda) \otimes L(\mu)) = \sum_{i=0}^{a} ch L(\mu + \lambda - i\alpha_1)
$$
  
= 
$$
\sum_{i=0}^{a} \sum_{j=0}^{i} ch L(\lambda + \mu - i\alpha_1 - j\alpha_2) - \sum_{i=1}^{a} \sum_{j=1}^{i} ch L(\lambda + \mu - i\alpha_1 - j\alpha_2)
$$
  
= 
$$
\sum_{i=0}^{a} \sum_{j=0}^{i} ch L(\lambda + \mu - i\alpha_1 - j\alpha_2) - \sum_{i=m}^{a} \sum_{j=m}^{i} ch L(\lambda + \mu - i\alpha_1 - j\alpha_2).
$$

Hence in all cases we are done.

 $\Box$ 

<span id="page-39-0"></span>**5.3.4**  $L(a, b) ⊗ L(c, d)$ 

**Lemma 5.19.** *Let*  $\lambda = (a, b) \in X^+$  *be p*-restricted with  $a, b \neq 0$  and  $a + b < p - 1$  (i.e.  $\lambda \in C_1$ *). Then* 

$$
ch(L(a,0) \otimes L(0,b)) - ch(L(a-1,0) \otimes L(0,b-1)) = ch L(a,b).
$$

*Proof.* First, note that  $(a - 1, 0) + (0, b - 1) = \lambda - \rho$ . By Lemma [5.3,](#page-26-2) we have  $L(a-1,0) \cong \Delta(a-1,0), L(a,0) \cong \Delta(a,0), L(0,b-1) \cong \Delta(0,b-1), L(0,b) \cong \Delta(0,b)$ and  $L(\lambda) \cong \Delta(\lambda)$ . By Proposition [1.47](#page-14-0) and Lemma [5.7,](#page-26-3) and using Lemma [1.45](#page-13-0) in the seventh equality below, we get

$$
\operatorname{ch}(L(a,0) \otimes L(0,b)) - \operatorname{ch}(L(a-1,0) \otimes L(0,b-1)) = \chi(a,0)\chi(0,b) - \chi(a-1,0)\chi(0,b-1)
$$
\n
$$
= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\lambda - i\alpha_1 - j\alpha_2) - \sum_{i=0}^{a-1} \sum_{j=0}^{i} \chi(\lambda - \rho - i\alpha_1 - j\alpha_2)
$$
\n
$$
= \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\lambda - i\alpha_1 - j\alpha_2) - \sum_{i=1}^{a} \sum_{j=1}^{i} \chi(\lambda - i\alpha_1 - j\alpha_2)
$$
\n
$$
= \sum_{i=0}^{a} \chi(\lambda - i\alpha_1)
$$
\n
$$
= \chi(\lambda) + \sum_{i=1}^{a} \chi(\lambda - i\alpha_1)
$$
\n
$$
= \chi(\lambda) + \frac{1}{2} \sum_{i=1}^{a} \chi(\lambda - i\alpha_1) + \chi(\lambda - (a+1-i)\alpha_1)
$$
\n
$$
= \chi(\lambda) + \frac{1}{2} \sum_{i=1}^{a} \chi(\lambda - i\alpha_1) - \chi(s_{\alpha_1} \cdot (\lambda - (a+1-i)\alpha_1))
$$
\n
$$
= \chi(\lambda) + \frac{1}{2} \sum_{i=1}^{a} \chi(\lambda - i\alpha_1) - \chi(\lambda - i\alpha_1)
$$
\n
$$
= \chi(\lambda)
$$
\n
$$
= \operatorname{ch} L(a, b).
$$

<span id="page-39-1"></span>**Proposition 5.20.** *Let*  $\lambda = (a, b)$ ,  $\mu = (c, d) \in X^+$  *be two p-restricted weights with*  $a, b, c, d \neq 0$ . Then  $L(\lambda) \otimes L(\mu)$  *is multiplicity-free if and only if one of the following holds:* 

- (1)  $a + b = p 1$ ,  $c + d < p 1$ ,  $a + c + d < p$  and  $b + c + d < p$  or
- (2)  $a + b < p 1$ ,  $c + d = p 1$ ,  $a + b + c < p$  and  $a + b + d < p$ .

*Proof.* We set  $M := L(\lambda) \otimes L(\mu)$ .

If  $a + c \geq p$  or  $b + d \geq p$ , then *M* has multiplicity by Corollary [2.8.](#page-17-0) Therefore, we can assume  $a + c < p$  and  $b + d < p$ . In particular, it cannot happen that  $a + b \geq p - 1$  and  $c + d > p - 1$ .

Suppose that  $a + b \neq p - 1$  and  $c + d \neq p - 1$ . Using Argument [1,](#page-16-0) we show that  $L(\lambda + \mu - \alpha_1 - \alpha_2)$  has multiplicity at least 2 in *M*. Using Lemmas [1.15](#page-10-0) and [5.6,](#page-26-4) we have

$$
m_M(\lambda + \mu) = 1, \qquad m_M(\lambda + \mu - \alpha_1) = 2, \qquad m_M(\lambda + \mu - \alpha_2) = 2,
$$

 $m_M(\lambda + \mu - \alpha_1 - \alpha_2) = 6.$ 

Using Lemmas [1.15](#page-10-0) and [5.6](#page-26-4) again, we have

$$
m_{L(\lambda+\mu)}(\lambda+\mu-\alpha_1)=1, \qquad m_{L(\lambda+\mu)}(\lambda+\mu-\alpha_2)=1.
$$

Therefore,  $L(\lambda + \mu - \alpha_1)$  and  $L(\lambda + \mu - \alpha_2)$  are composition factors of *M*. Using Lemmas [1.15](#page-10-0) and [5.6](#page-26-4) a third time, we get

$$
m_{L(\lambda+\mu)}(\lambda+\mu-\alpha_1-\alpha_2) \le 2, \qquad m_{L(\lambda+\mu-\alpha_1)}(\lambda+\mu-\alpha_1-\alpha_2) = 1,
$$

 $m_{L(\lambda+\mu-\alpha_2)}(\lambda+\mu-\alpha_1-\alpha_2)=1.$ 

We conclude that  $L(\lambda + \mu - \alpha_1 - \alpha_2)$  is a composition factor with multiplicity

$$
[M:L(\lambda + \mu - \alpha_1 - \alpha_2)] \ge 6 - 1 - 1 - 2 = 2.
$$

Therefore,  $M = L(\lambda) \otimes L(\mu)$  has multiplicity.

Suppose that  $a + b = c + d = p - 1$ . Since we assume  $a + c < p$  and  $b + d < p$ , we only need to consider the case  $\lambda + \mu = (p-1, p-1)$ . Using the same reasoning as in the previous case, we deduce that  $L(\lambda + \mu - \alpha_1)$  is a composition factor of *M*. But  $\lambda + \mu - \alpha_1 = (p-3, p)$ is not *p*-restricted. Therefore, by Corollary [2.7,](#page-17-1) *M* has multiplicity.

Let us consider the case  $a + b < p - 1$ ,  $c + d = p - 1$ ,  $a + b + d \geq p$  (the other remaining cases where we claim that *M* has multiplicity are symmetric). Note that  $b + c + d \geq p$ , therefore  $b + d + \min(a, c) \geq p$ . We show inductively that  $[M : L(\lambda + \mu - s\alpha_1)] = 1$  for  $0 \leq s \leq p - b - d \leq \min(a, c)$ . For  $0 \leq s \leq \min(a, c)$ , using Lemma [1.15,](#page-10-0) we have:

$$
m_M(\lambda + \mu - s\alpha_1) = \sum_{i=0}^s m_{L(\lambda)}(\lambda - i\alpha_1) \cdot m_{L(\mu)}(\mu - (s - i)\alpha_1) = s + 1.
$$

Moreover, if  $i < p - b - d$ , then  $\lambda + \mu - i\alpha_1 = (a + c - 2i, b + d + i)$  is *p*-restricted and we have

$$
\sum_{i=0}^{s-1} m_{L(\lambda+\mu-i\alpha_1)}(\lambda+\mu-s\alpha_1) = s.
$$

We know that  $[M : L(\lambda + \mu)] = 1$ , and combining the two previous equations, we conclude inductively that  $[M : L(\lambda + \mu - s\alpha_1)] = 1$  for  $0 \le s \le p - b - d$ . In particular,  $L(\lambda + \mu - (p - b - d)\alpha_1)$  is a composition factor of M. But  $\lambda + \mu - (p - b - d)\alpha_1 = (a + c - 2(p - b - d), p)$  is not *p*-restricted. We conclude by Corollary [2.7](#page-17-1) that *M* has multiplicity.

Now we consider the cases where we claim that *M* is multiplicity-free. Up to the reordering of the weights and up to symmetry, we can suppose  $a + b < p - 1$ ,  $c + d = p - 1$  and  $c \geq d$ . Thus the conditions in the statement of the proposition are equivalent to a single one:  $a + b + c < p$ . In particular we have  $a < d \leq c$  and  $b < d \leq c$ . We show that M is multiplicity-free by showing the following equality of characters:

ch 
$$
M = \sum_{i=0}^{a} \sum_{j=0}^{b} \text{ch} L(\lambda + \mu - i\alpha_1 - j\alpha_2).
$$

Let  $0 \leq i \leq a, \ 0 \leq j \leq b$ . We claim that  $\lambda + \mu - i\alpha_1 - j\alpha_2 \in \widehat{C_2}$ . We have

$$
\lambda + \mu - i\alpha_1 - j\alpha_2 = (a + c - 2i + j, b + d - 2j + i)
$$

with

$$
a + c - 2i + j \le a + c + b < p,
$$
  
\n
$$
b + d - 2j + i \le b + d + a \le a + b + c < p
$$
 and  
\n
$$
(a + c - 2i + j) + (b + d - 2j + i) = a + b + c + d - i - j \ge c + d = p - 1.
$$

Thus  $\lambda + \mu - i\alpha_1 - j\alpha_2 \in \widehat{C_2}$  for all  $0 \le i \le a, 0 \le j \le b$ . By Lemma [5.19,](#page-39-0) we have

 $ch M = ch L(a, b) ch L(c, d)$  $=$   $\left(\text{ch}(L(a, 0) \otimes L(0, b)) - \text{ch}(L(a - 1, 0) \otimes L(0, b - 1))\right) \text{ch } L(c, d)$  $=$ ch(*L*(*a*, 0) ⊗ *L*(0*, b*) ⊗ *L*(*c, d*)) − ch(*L*(*a* − 1*,* 0) ⊗ *L*(0*, b* − 1) ⊗ *L*(*c, d*))  $=$  ch  $L(a, 0)$  ch( $L(0, b) \otimes L(c, d)$ )  $-$  ch  $L(a - 1, 0)$  ch( $L(0, b - 1) \otimes L(c, d)$ ).

Since  $c + d = p - 1$ ,  $b + d < p$  and  $b \le d$ , we can use line [\(2\)](#page-30-0) in the proof of Lemma [5.15](#page-29-0) to express  $ch(L(0, b) \otimes L(c, d))$  and  $ch(L(0, b - 1) \otimes L(c, d))$ . We set  $\lambda_b := (0, b)$  and  $\lambda_{b-1} := (0, b-1)$ . We get

$$
ch(L(0, b) \otimes L(c, d)) = \sum_{k=0}^{b} ch L(\lambda_b + \mu - k\alpha_2),
$$

and

$$
ch(L(0, b-1) \otimes L(c, d)) = \sum_{k=0}^{b-1} ch L(\lambda_{b-1} + \mu - k\alpha_2).
$$

Observe that  $\lambda_b + \mu - k\alpha_2 = (c + k, b + d - 2k)$ . For  $0 \le k \le b$ , we have

$$
0 < c \le c + k \le c + b < p - 1,
$$
  
\n
$$
0 < d - b \le b + d - 2k \le b + d < p - 1
$$
 and  
\n
$$
(c + k) + (b + d - 2k) \ge c + d = p - 1.
$$
 (26)

Therefore  $\lambda_b + \mu - k\alpha_2 \in C_2$  for all  $k \in \{0, \ldots, b\}$ . Moreover, we have

<span id="page-41-2"></span><span id="page-41-1"></span><span id="page-41-0"></span>
$$
a + c + k \le a + b + c < p \qquad \text{and} \tag{27}
$$

<span id="page-41-5"></span><span id="page-41-4"></span><span id="page-41-3"></span>
$$
a+b+d-2k \le a+b+d \le a+b+c < p. \tag{28}
$$

Similarly, for all  $0 \le k \le b - 1$ , we have  $\lambda_{b-1} + \mu - k\alpha_2 = (c + k, b - 1 + d - 2k) \in C_2$  and

$$
(c+k) + (b-1+d-2k) \ge p-1,
$$
\n(29)

$$
(a-1) + c + k \le a + b + c - 2 < p \qquad \text{and} \tag{30}
$$

$$
(a-1)+b-1+d-2k \le a+b+d-2 \le a+b+c-2 < p. \tag{31}
$$

Let  $m := \Theta(\lambda_b + \mu) = b + c + d + 2 - p = b + 1$ . Observe that  $\Theta(\lambda_b + \mu - k\alpha_2) = m - k$ and  $\Theta(\lambda_{b-1} + \mu - k\alpha_2) = m - k - 1$ . Lines [\(26\)](#page-41-0), [\(27\)](#page-41-1), [\(28\)](#page-41-2), [\(29\)](#page-41-3), [\(30\)](#page-41-4) and [\(31\)](#page-41-5) allow us to use Corollary [5.18](#page-38-1) in the fourth equality below, to get

 $\setminus$ 

 $\setminus$ 

ch 
$$
M =
$$
ch  $L(a, 0)$  ch $(L(0, b) \otimes L(c, d))$  – ch  $L(a - 1, 0)$  ch $(L(0, b - 1) \otimes L(c, d))$   
\n
$$
= ch L(a, 0) \left( \sum_{k=0}^{b} ch L(\lambda_{b} + \mu - k\alpha_{2}) \right) - ch L(a - 1, 0) \left( \sum_{k=0}^{b-1} ch L(\lambda_{b-1} + \mu - k\alpha_{2}) \right)
$$
\n
$$
= \left( \sum_{k=0}^{b} ch L(a, 0) \text{ ch } L(\lambda_{b} + \mu - k\alpha_{2}) \right) - \left( \sum_{k=0}^{b-1} ch L(a - 1, 0) \text{ ch } L(\lambda_{b-1} + \mu - k\alpha_{2}) \right)
$$
\n
$$
= \sum_{k=0}^{b} \sum_{i=0}^{a} \sum_{j=0}^{i} ch L(\lambda + \mu - i\alpha_{1} - (j + k)\alpha_{2})
$$
\n
$$
- \sum_{k=0}^{b} \sum_{i=m-k}^{a} \sum_{j=m-k}^{i} ch L(\lambda + \mu - i\alpha_{1} - (j + k)\alpha_{2})
$$
\n
$$
- \sum_{k=0}^{b-1} \sum_{i=0}^{a-1} \sum_{j=0}^{i} ch L(\lambda + \mu - \rho - i\alpha_{1} - (j + k)\alpha_{2})
$$
\n
$$
+ \sum_{k=0}^{b-1} \sum_{i=0}^{a-1} \sum_{j=m-k-1}^{i} ch L(\lambda + \mu - \rho - i\alpha_{1} - (j + k)\alpha_{2})
$$
\n
$$
= \sum_{k=0}^{b} \sum_{i=0}^{a} \sum_{j=0}^{i} ch L(\lambda + \mu - i\alpha_{1} - (j + k)\alpha_{2})
$$
\n
$$
- \sum_{k=0}^{b} \sum_{i=m-k}^{a} \sum_{j=m-k}^{i} ch L(\lambda + \mu - i\alpha_{1} - (j + k)\alpha_{2})
$$
\n
$$
- \sum_{k=0}^{b-1} \sum_{i=0}^{a} \sum_{j=1}^{i} ch L(\lambda + \mu - i\alpha_{1} - (j + k)\alpha_{2})
$$
\n
$$

$$

To simplify the notation, we set  $Z(i, j) := ch L(\lambda + \mu - i\alpha_1 - j\alpha_2)$ . Thus we get

*k*=0 *i*=*m*−*k j*=*m*−*k*

ch 
$$
M = \sum_{k=0}^{b} \sum_{i=0}^{a} \sum_{j=0}^{i} Z(i, j + k) - \sum_{k=0}^{b} \sum_{i=m-k}^{a} \sum_{j=m-k}^{i} Z(i, j + k)
$$
  
\n
$$
- \sum_{k=0}^{b-1} \sum_{i=1}^{a} \sum_{j=1}^{i} Z(i, j + k) + \sum_{k=0}^{b-1} \sum_{i=m-k}^{a} \sum_{j=m-k}^{i} Z(i, j + k)
$$
\n
$$
= \sum_{k=0}^{b-1} Z(0, k) + \sum_{k=0}^{b-1} \sum_{i=1}^{a} Z(i, k) + \sum_{i=0}^{a} \sum_{j=0}^{i} Z(i, j + b) - \sum_{i=m-b}^{a} \sum_{j=m-b}^{i} Z(i, j + b)
$$

$$
= \sum_{k=0}^{b-1} \sum_{i=0}^{a} Z(i,k) + \sum_{i=0}^{a} Z(i,b) + \sum_{i=1}^{a} \sum_{j=1}^{i} Z(i,j+b) - \sum_{i=m-b}^{a} \sum_{j=m-b}^{i} Z(i,j+b)
$$
  

$$
= \sum_{k=0}^{b} \sum_{i=0}^{a} Z(i,k) + \sum_{i=1}^{a} \sum_{j=1}^{i} Z(i,j+b) - \sum_{i=m-b}^{a} \sum_{j=m-b}^{i} Z(i,j+b).
$$

Finally, observe that  $m - b = 1$ . This allows us to conclude that

$$
\begin{aligned} \n\text{ch}\,M &= \sum_{k=0}^{b} \sum_{i=0}^{a} Z(i,k) + \sum_{i=1}^{a} \sum_{j=1}^{i} Z(i,j+b) - \sum_{i=1}^{a} \sum_{j=1}^{i} Z(i,j+b) \\ \n&= \sum_{k=0}^{b} \sum_{i=0}^{a} Z(i,k) \\ \n&= \sum_{k=0}^{b} \sum_{i=0}^{a} \text{ch}\,L(\lambda + \mu - i\alpha_1 - k\alpha_2). \n\end{aligned} \tag{32}
$$

In particular,  $M = L(\lambda) \otimes L(\mu)$  is multiplicity-free.

Now Theorem [5.1](#page-24-0) follows from the previous sequence of propositions.

### **5.4 Decomposition of multiplicity-free tensor products**

Since multiplicity-free tensor products of simple modules are completely reducible (Lemma [3.5\)](#page-19-0), we can specify the structure of those modules. For some of the cases, we still need to compute the decomposition in characteristic zero, which we do in the next lemma.

<span id="page-43-4"></span>**Lemma 5.21.** *Let*  $\lambda = (a, 0), \mu = (c, d) \in X^+$ *. Then* 

<span id="page-43-5"></span>
$$
\chi(\lambda)\chi(\mu) = \sum_{i=0}^{a} \sum_{j=\max\{0,i-c\}}^{\min\{i,d\}} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2).
$$

*Moreover, all weights appearing in the sum are dominant.*

*Proof.* By Proposition [1.47](#page-14-0) and Lemma [5.7,](#page-26-3) we get

$$
\chi(\lambda)\chi(\mu) = \sum_{i=0}^{a} \sum_{j=0}^{i} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2)
$$
  
= 
$$
\sum_{i=0}^{a} \sum_{j=\max\{0, i-c\}}^{\min\{i, d\}} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2)
$$
 (33)

<span id="page-43-0"></span>
$$
+\sum_{i=0}^{a} \sum_{j=0}^{\min\{i-c-1,d\}} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2)
$$
\n(34)

$$
+\sum_{i=0}^{a} \sum_{j=d+1}^{i} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2). \tag{35}
$$

Observe that

<span id="page-43-3"></span><span id="page-43-2"></span><span id="page-43-1"></span> $\lambda + \mu - i\alpha_1 - j\alpha_2 = (a + c - 2i + j, d + i - 2j).$  (36)

In particular, for  $i \le a$  and  $j \ge i - c$ , we have

$$
a + c - 2i + j \ge a - i \ge 0,
$$

and for  $i \geq 0$  and  $j \leq \min\{i, d\}$ , we have

$$
d + i - 2j \ge (d - j) + (i - j) \ge 0.
$$

Therefore, all weights appearing in line [33](#page-43-0) are dominant.

We show that lines [\(34\)](#page-43-1) and [\(35\)](#page-43-2) are equal to zero. We start by line [\(35\)](#page-43-2). Using Lemma [1.45,](#page-13-0) and line [\(36\)](#page-43-3) in the second equality below, we have

$$
\sum_{j=d+1}^{i} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2) = -\sum_{j=d+1}^{i} \chi(s_{\alpha_2} \cdot (\lambda + \mu - i\alpha_1 - j\alpha_2))
$$
  
= 
$$
-\sum_{j=d+1}^{i} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2 - (d + i - 2j + 1)\alpha_2)
$$
  
= 
$$
-\sum_{j=d+1}^{i} \chi(\lambda + \mu - i\alpha_1 - (d + i - j + 1)\alpha_2)
$$
  
= 
$$
-\sum_{j=d+1}^{i} \chi(\lambda + \mu - i\alpha_1 - (d + i - (d + 1 + i - j) + 1)\alpha_2)
$$
  
= 
$$
-\sum_{j=d+1}^{i} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2).
$$

Therefore,

$$
\sum_{j=d+1}^{i} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2) = 0
$$

and

<span id="page-44-0"></span>
$$
\sum_{i=0}^{a} \sum_{j=d+1}^{i} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2) = 0.
$$

Now we work on line [\(34\)](#page-43-1). If  $i < c + 1$ , then  $i - c - 1 < 0$ . Thus we have

$$
\sum_{i=0}^{a} \sum_{j=0}^{\min\{i-c-1,d\}} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2) = \sum_{i=c+1}^{a} \sum_{j=0}^{\min\{i-c-1,d\}} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2)
$$

$$
= \sum_{j=0}^{\min\{a-c-1,d\}} \sum_{i=c+1+j}^{a} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2). \tag{37}
$$

We fix *j* and work on the second sum in line [\(37\)](#page-44-0). Using Lemma [1.45,](#page-13-0) and line [\(36\)](#page-43-3) in the second equality below, we get

$$
\sum_{i=c+1+j}^{a} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2)
$$
  
= 
$$
-\sum_{i=c+1+j}^{a} \chi(s_{\alpha_1} \cdot (\lambda + \mu - i\alpha_1 - j\alpha_2))
$$

$$
= - \sum_{i=c+1+j}^{a} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2 - (a + c - 2i + j + 1)\alpha_1)
$$
  
= 
$$
- \sum_{i=c+1+j}^{a} \chi(\lambda + \mu - (a + c - i + j + 1)\alpha_1 - j\alpha_2)
$$
  
= 
$$
- \sum_{i=c+1+j}^{a} \chi(\lambda + \mu - (a + c - (a + c + 1 + j - i) + j + 1)\alpha_1 - j\alpha_2)
$$
  
= 
$$
- \sum_{i=c+1+j}^{a} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2).
$$

Therefore,

$$
\sum_{i=c+1+j}^{a} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2) = 0
$$

so

$$
\sum_{j=0}^{\min\{a-c-1,d\}} \sum_{i=c+1+j}^{a} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2) = 0
$$

and

$$
\sum_{i=0}^{a} \sum_{j=0}^{\min\{i-c-1,d\}} \chi(\lambda + \mu - i\alpha_1 - j\alpha_2) = 0.
$$

**Remark 5.22.** Let  $(a, b)$ ,  $(c, d) \in X^+$  be dominant weights. Lemma [5.21](#page-43-4) shows that  $L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu)$  is multiplicity-free if  $a \cdot b \cdot c \cdot d = 0$ . Moreover, using Argument [1](#page-16-0) like in the proof of Proposition [5.20,](#page-39-1) one can prove that  $[L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu) : L_{\mathbb{C}}(\lambda + \mu - \alpha_1 - \alpha_2)] = 2$  if  $a \cdot b \cdot c \cdot d \neq 0$ . These two facts provide a proof of Theorem [5.8.](#page-27-0)

**Remark 5.23.** The computations in Lemma [5.21](#page-43-4) can be done using the Littlewood-Richardson rule, see [\[LR34,](#page-78-0) Theorem III] or [\[FH04,](#page-77-0) Proposition 15.25] for this specific case.

**Corollary 5.24.** Let  $\lambda = (a, b), \mu = (c, d) \in X^+$  be non-zero and p-restricted such that  $L(\lambda) \otimes L(\mu)$  *is multiplicity-free. We have the following decompositions:* 

(1) *If*  $b = d = 0$ *,*  $a + c < p$  *and*  $a \leq c$ *, then* 

$$
L(\lambda) \otimes L(\mu) \cong \bigoplus_{i=0}^{a} L(\mu + \lambda - i\alpha_1).
$$

(2) *If*  $b = c = 0$ *,*  $a \leq d$  *and*  $a + d < p - 1$  *or*  $(a, d) = (1, p - 1)$ *, then* 

$$
L(\lambda) \otimes L(\mu) \cong \bigoplus_{i=0}^{a} L(\mu + \lambda - i\rho).
$$

(3) *If*  $b = 0$  *and*  $a + c + d < p - 1$ *, then* 

$$
L(\lambda) \otimes L(\mu) \cong \bigoplus_{i=0}^{a} \bigoplus_{j=\max\{0,i-c\}}^{\min\{i,d\}} L(\mu+\lambda-i\alpha_1-j\alpha_2).
$$

(4) *If*  $b = 0$ ,  $c + d = p - 1$ ,  $a + c < p$  and  $a \leq c$ , then

$$
L(\lambda) \otimes L(\mu) \cong \bigoplus_{i=0}^{a} L(\mu + \lambda - i\alpha_1).
$$

(5) If  $b = 0$ ,  $c + d = p - 1$ ,  $a + c < p$  and  $a = c + 1$ , then

$$
L(\lambda) \otimes L(\mu) \cong L(\lambda + \mu - a\alpha_1 - \alpha_2) \oplus \bigoplus_{i=0}^{c} L(\mu + \lambda - i\alpha_1).
$$

(6) If  $b = 0$ ,  $c + d > p - 1$ ,  $a + c < p$  and  $a + d < p$ , then

$$
L(\lambda) \otimes L(\mu) \cong \bigoplus_{i=0}^{a} \bigoplus_{j=0}^{\min\{i, \Theta(\mu)-1\}} L(\mu+\lambda-i\alpha_1-j\alpha_2).
$$

(7) If  $a \cdot b \cdot c \cdot d \neq 0$ ,  $a + b < p - 1$ ,  $c + d = p - 1$ ,  $a + b + c < p$  and  $a + b + d < p$ , then

$$
L(\lambda) \otimes L(\mu) \cong \bigoplus_{k=0}^{b} \bigoplus_{i=0}^{a} L(\lambda + \mu - i\alpha_1 - k\alpha_2).
$$

 $\Box$ 

*Proof.*

- (1) This follows from Lemma [5.21](#page-43-4) and Proposition [5.9.](#page-27-1)
- (2) This follows from Lemma [5.21](#page-43-4) and Proposition [5.11.](#page-27-2)
- (3) This follows from Lemma [5.21](#page-43-4) and Proposition [5.12.](#page-28-0)
- (4) This follows from line [\(2\)](#page-30-0) in the proof of Proposition [5.15.](#page-29-0)
- (5) This follows from line [\(6\)](#page-31-0) in the proof of Proposition [5.15.](#page-29-0)
- (6) This follows from lines [\(20\)](#page-35-3) and [\(25\)](#page-38-0) in the proof of Proposition [5.16.](#page-34-1)
- (7) This follows from line [\(32\)](#page-43-5) in the proof of Proposition [5.20.](#page-39-1)

# **6** Sp<sup>4</sup>

In this section, we establish a number of results for the classification of multiplicity-free tensor products of simple  $Sp_4$ -modules with *p*-restricted highest weight. We fix  $G = Sp_4(k)$ with root system  $\Phi$  of type  $B_2 = C_2$  and  $\Pi = {\alpha_1, \alpha_2}$  a base of  $\Phi$  with  $\frac{(\alpha_1, \alpha_1)}{(\alpha_2, \alpha_2)} = 2$ . With respect to this base, we have  $\rho = \frac{3}{2}\alpha_1 + 2\alpha_2$ . We assume  $p \ge 5$ , so that there exist weights inside the alcoves. Since the Coxeter number of a root system  $\Phi$  of type  $B_2$  is  $h = 4$ , we have in particular  $p \geq h$ . For  $\lambda = x\omega_1 + y\omega_2 \in X$ , we write  $\lambda = (x, y)$ . In particular, we have  $\alpha_1 = (2, -2), \ \alpha_2 = (-1, 2) \text{ and } \rho = (1, 1).$ 

In this section, we will also use Euclidean coordinates several times. We fix an orthogonal basis  $(\epsilon_1, \epsilon_2)$  of  $\mathbb{R}^2$  with  $\epsilon_1 = \omega_1 = \alpha_1 + \alpha_2$  and  $\epsilon_2 = \alpha_2 = 2\omega_2 - \omega_1$ . With respect to this basis, we have  $\alpha_1 = (1, -1)$ ,  $\alpha_2 = (0, 1)$ ,  $\alpha_1 + \alpha_2 = (1, 0)$ ,  $\alpha_1 + 2\alpha_2 = (1, 1)$  and  $\rho = (\frac{3}{2}, \frac{1}{2})$  $(\frac{1}{2})$ . Since the notation ( *,* ) might be confusing, we will always explicitly state when we use Euclidean coordinates. If it is not mentioned, then it means that we use coordinates with respect to the fundamental weights basis. In particular, coordinates with respect to the fundamental weights are used in all statements of propositions in the classification (section [6.3\)](#page-55-0).

#### **6.1 Alcoves**

In this subsection, we describe the four *p*-restricted alcoves of a root system of type *B*2. We start by defining a numeration of those alcoves.

#### **Definition 6.1.** *We set*

$$
C_1 := \{ \lambda \in X_{\mathbb{R}} \mid (\lambda + \rho, \alpha_1^{\vee}) > 0, (\lambda + \rho, \alpha_2^{\vee}) > 0, (\lambda + \rho, (\alpha_1 + \alpha_2)^{\vee}) < p \},
$$
  
\n
$$
C_2 := \{ \lambda \in X_{\mathbb{R}} \mid (\lambda + \rho, \alpha_2^{\vee}) > 0, (\lambda + \rho, (\alpha_1 + \alpha_2)^{\vee}) > p, (\lambda + \rho, (\alpha_1 + 2\alpha_2)^{\vee}) < p \},
$$
  
\n
$$
C_3 := \{ \lambda \in X_{\mathbb{R}} \mid (\lambda + \rho, \alpha_2^{\vee}) < p, (\lambda + \rho, (\alpha_1 + \alpha_2)^{\vee}) < 2p, (\lambda + \rho, (\alpha_1 + 2\alpha_2)^{\vee}) > p \},
$$
  
\n
$$
C_4 := \{ \lambda \in X_{\mathbb{R}} \mid (\lambda + \rho, \alpha_1^{\vee}) < p, (\lambda + \rho, \alpha_2^{\vee}) < p, (\lambda + \rho, (\alpha_1 + \alpha_2)^{\vee}) > 2p \}.
$$

*We called*  $C_1$  *the* fundamental alcove,  $C_2$  *the* second alcove,  $C_3$  *the* third alcove *and*  $C_4$  *the* fourth alcove*.*

*We also set*

$$
F_{i,i+1} := \overline{C_i} \cap \overline{C_{i+1}}
$$

*for*  $i = 1, 2, 3$ *, i.e.*  $F_{i,i+1}$  *is the wall between the alcove*  $C_i$  *and*  $C_{i+1}$ *. Finally, we set* 

$$
F_{3,5} := \{ \lambda \in \overline{C_3} | \ (\lambda + \rho, \alpha_2^{\vee}) = p \},
$$
  

$$
F_{4,6} := \{ \lambda \in \overline{C_4} | \ (\lambda + \rho, \alpha_1^{\vee}) = p \} \ and
$$
  

$$
F_{4,7} := \{ \lambda \in \overline{C_4} | \ (\lambda + \rho, \alpha_2^{\vee}) = p \}.
$$

We set  $C_5 := s_{\alpha_2, p} \cdot C_3$ ,  $C_6 := s_{\alpha_1, p} \cdot C_4$  and  $C_7 := s_{\alpha_2, p} \cdot C_4$  $C_4$ . Then  $F_{i,j} = C_i \cap C_j$  for  $(i, j) \in \{(3, 5), (4, 6), (4, 7)\}.$ 

Using coordinates with respect to the fundamental weights, we have  $\widehat{C_1} \cap X^+ = \{ (a, b) \in \mathbb{N}^2 \mid 2a + b \leq p - 3 \},\$  $\widehat{C_2} \cap X^+ = \{ (a, b) \in \mathbb{N}^2 \mid 2a + b > p - 3, \ a + b \leq p - 2 \},\$  $\widehat{C_3} \cap X^+ = \{ (a, b) \in \mathbb{N}^2 \mid a+b > p-2, \ 2a+b \le 2p-3, \ b \le p-1 \},\$  $\widehat{C_4} \cap X^+ = \{ (a, b) \in \mathbb{N}^2 \mid 2a + b > 2p - 3, \ a \leq p - 1, \ b \leq p - 1 \}.$ 

Using Euclidean coordinates, we have

$$
X = \{(a, b) \in (\frac{1}{2}\mathbb{Z})^2 | a + b \in \mathbb{Z}\},
$$
  
\n
$$
\widehat{C}_1 \cap X^+ = \{(a, b) \in (\frac{1}{2}\mathbb{N})^2 | a + b \in \mathbb{N}, a \le \frac{p-3}{2}, b \le a\},
$$
  
\n
$$
\widehat{C}_2 \cap X^+ = \{(a, b) \in (\frac{1}{2}\mathbb{N})^2 | a + b \in \mathbb{N}, a > \frac{p-3}{2}, a + b \le p-2\},
$$
  
\n
$$
\widehat{C}_3 \cap X^+ = \{(a, b) \in (\frac{1}{2}\mathbb{N})^2 | a + b \in \mathbb{N}, a + b > p-2, a \le p-\frac{3}{2}, b \le \frac{p-1}{2}\},
$$
  
\n
$$
\widehat{C}_4 \cap X^+ = \{(a, b) \in (\frac{1}{2}\mathbb{N})^2 | a + b \in \mathbb{N}, a > p-\frac{3}{2}, b \le \frac{p-1}{2}, a-b \le p-1\}.
$$
  
\nMoreover, a might (a, b)  $\in X$  is dominant if and only if 0  $\le b \le a$  and (a, b)  $\in D$ 

Moreover, a weight  $(a, b) \in X$  is dominant if and only if  $0 \leq b \leq a$ , and  $(a, b) \in D$  if and only if  $-\frac{1}{2}$  ≤ *b* ≤ *a* + 1.

Let us illustrate these alcoves with a picture. The blue arrows form the root system. The black arrows are the fundamental weights, generating the weight lattice (in black). The region containing the dominant weights is coloured in green. The red triangles are the walls of the alcoves.



<span id="page-49-1"></span>**Remark 6.2.** In Euclidean coordinates, we have

- $(1)$   $s_{\alpha_1}$   $(a, b) = (b - 1, a + 1),$
- $(2)$   $s_{\alpha_2}$   $(a, b) = (a, -b - 1),$
- $(3)$   $s_{\alpha_1+\alpha_2,p}$  $(a, b) = (p - 3 - a, b),$
- $(4)$   $s_{\alpha_1+2\alpha_2,p}$  $(a, b) = (a - m, b - m)$  with  $m = (a + b + 2 - p)$ .

**Lemma 6.3.** *Let*  $\lambda = (a, b), \mu = (c, d) \in X$  *with*  $\mu \leq \lambda$ *. Then*  $c + d \leq a + b$ *.* 

*Proof.* Let  $s, t \in \mathbb{N}$  be such that  $\mu = \lambda - s\alpha_1 - t\alpha_2$ . We have  $\mu = (a - 2s + t, b - 2t + 2s) =: (c, d)$  $\Box$ and  $c + d = a + b - t \leq a + b$ .

#### **6.2 Structure of Weyl modules and weights in irreducible modules**

In this subsection, we compute the composition factors of Weyl modules with *p*-restricted highest weight and the multiplicity of certain weights in irreducible modules with *p*-restricted highest weight.

<span id="page-49-0"></span>**Lemma 6.4.** *Let*  $\lambda = (a, b) \in X^+$  *be a p-restricted dominant weight.* 

- *If*  $\lambda \in C_1$ *, then*  $\Delta(\lambda) \cong L(\lambda)$ *.*
- *If*  $\lambda \in C_i$  for  $i \in \{2,3,4\}$ , then  $\Delta(\lambda)$  admits the unique composition series  $[L(\lambda), L(\mu)]$ *where*  $\mu \in C_{i-1}$  *is the unique weight linked to*  $\lambda$ *.*
- *If*  $\lambda \in F_{i,j}$  *with*  $(i, j) \neq (4, 7)$  *then*  $\Delta(\lambda) \cong L(\lambda)$ *.*
- *If*  $\lambda \in F_{4,7} \setminus F_{4,6}$ *, then*  $\Delta(\lambda)$  *admits the unique composition series*  $[L(\lambda), L(\mu)]$  *where*  $\mu \in F_{3.5}$  *is the unique weight linked to*  $\lambda$ *.*

*In particular,*  $T(\lambda)$  *is irreducible if and only if*  $\lambda \in \widehat{C_1} \cup F_{2,3} \cup F_{3,4} \cup F_{3,5} \cup F_{4,6}$ *.* 

*Proof.* We prove this lemma using Jantzen *p*-sum formula (Proposition [1.49\)](#page-14-1). For a weight  $\lambda \in C_i$ we prove this lemma using Januzen p-sum it<br>, we write  $\lambda_j$  for the unique weight in  $(W_p \bullet)$ <br>, 1.45 several times without further reference  $\lambda$ ) ∩  $C_j$ . We will use Remark [1.52](#page-15-0) and Lemma [1.45](#page-13-0) several times without further reference.

- If  $\lambda \in \widehat{C_1}$  then  $\Delta(\lambda) \cong L(\lambda)$  by Lemma [1.55.](#page-15-1)
- If  $\lambda \in C_2$ , then

$$
JSF(\lambda) = \chi(s_{\alpha_1 + \alpha_2, p} \cdot \lambda) = \chi(\lambda_1) = \text{ch } L(\lambda_1).
$$

Therefore,  $\chi(\lambda_2) = \text{ch } L(\lambda_2) + \text{ch } L(\lambda_1)$  and  $\text{ch } L(\lambda_2) = \chi(\lambda_2) - \chi(\lambda_1)$ .

• If  $\lambda \in C_3$ , then

$$
JSF(\lambda) = \chi(s_{\alpha_1 + \alpha_2, p} \cdot \lambda) + \chi(s_{\alpha_1 + 2\alpha_2, p} \cdot \lambda) = -\chi(\lambda_1) + \chi(\lambda_2) = \text{ch } L(\lambda_2).
$$

Therefore,  $\chi(\lambda_3) = \text{ch } L(\lambda_3) + \text{ch } L(\lambda_2)$  and  $\text{ch } L(\lambda_3) = \chi(\lambda_3) - \chi(\lambda_2) + \chi(\lambda_1)$ .

• If  $\lambda \in C_4$ , then

$$
JSF(\lambda) = \chi(s_{\alpha_1 + \alpha_2, p} \cdot \lambda) + \chi(s_{\alpha_1 + \alpha_2, 2p} \cdot \lambda) + \chi(s_{\alpha_1 + 2\alpha_2, p} \cdot \lambda) = \chi(\lambda_1) + \chi(\lambda_3) - \chi(\lambda_2)
$$
  
= ch  $L(\lambda_3)$ .

• If  $\lambda \in F_{2,3}$ , then

$$
JSF(\lambda) = \chi(s_{\alpha_1 + \alpha_2, p} \cdot \lambda) = 0
$$

because  $s_{\alpha_1+\alpha_2,p} \bullet \lambda \in D \setminus X^+$ . Thus  $\Delta(\lambda)$  is irreducible.

• If  $\lambda \in F_{3,4}$ , then

$$
JSF(\lambda) = \chi(s_{\alpha_1 + \alpha_2, p} \cdot \lambda) + \chi(s_{\alpha_1 + 2\alpha_2, p} \cdot \lambda) = 0
$$

because  $s_{\alpha_1+2\alpha_2,p} \bullet \lambda \in D \setminus X^+$  and  $s_{\alpha_1}s_{\alpha_1+\alpha_2,p} \bullet \lambda \in D \setminus X^+$ . Thus  $\Delta(\lambda)$  is irreducible.

• If  $\lambda \in F_{3,5}$ , then

$$
JSF(\lambda) = \chi(s_{\alpha_1 + \alpha_2, p} \cdot \lambda) + \chi(s_{\alpha_1 + 2\alpha_2, p} \cdot \lambda) = -\chi(s_{\alpha_1} s_{\alpha_1 + \alpha_2, p} \cdot \lambda) + \chi(s_{\alpha_1 + 2\alpha_2, p} \cdot \lambda) = 0
$$

because *sα*<sup>1</sup>  $s_{\alpha_1+\alpha_2,p} \bullet \lambda = s_{\alpha_1+2\alpha_2,p} \bullet$ *λ*. Thus ∆(*λ*) is irreducible.

• If  $\lambda \in F_{4,6}$ , then

$$
JSF(\lambda) = \chi(s_{\alpha_1 + \alpha_2, p} \cdot \lambda) + \chi(s_{\alpha_1 + \alpha_2, 2p} \cdot \lambda) + \chi(s_{\alpha_1 + 2\alpha_2, p} \cdot \lambda)
$$

$$
= \chi(s_{\alpha_1 + \alpha_2, 2p} \cdot \lambda) - \chi(s_{\alpha_2} s_{\alpha_1 + 2\alpha_2, p} \cdot \lambda) = 0
$$

where we used that  $s_{\alpha_1+\alpha_2}$  $s_{\alpha_1+\alpha_2,p} \bullet \lambda \in D \setminus X^+$  in the second equality and in the last equality Thus  $\Lambda(\lambda)$  is irreducible  $s_{\alpha_1+\alpha_2,2p}$  $\lambda = s_{\alpha_2}$ *s*<sub> $\alpha_1+\alpha$ </sub> $s_{\alpha_1+2\alpha_2,p}$  $\lambda$  in the last equality. Thus  $\Delta(\lambda)$  is irreducible.

• If  $\lambda \in F_{4,7} \setminus F_{4,6}$ , then

$$
JSF(\lambda) = \chi(s_{\alpha_1 + \alpha_2, p} \cdot \lambda) + \chi(s_{\alpha_1 + \alpha_2, 2p} \cdot \lambda) + \chi(s_{\alpha_1 + 2\alpha_2, p} \cdot \lambda)
$$
  
=  $\chi(s_{\alpha_1} s_{\alpha_1 + \alpha_2} s_{\alpha_1 + \alpha_2, p} \cdot \lambda) + \chi(s_{\alpha_1 + \alpha_2, 2p} \cdot \lambda) - \chi(s_{\alpha_2} s_{\alpha_1 + 2\alpha_2, p} \cdot \lambda)$   
=  $\chi(s_{\alpha_1 + \alpha_2, 2p} \cdot \lambda)$ 

where we used that  $s_{\alpha_1} s_{\alpha_1 + \alpha_2}$  $s_{\alpha_1+\alpha_2,p}$ <sup>•</sup><br>*r*<sub>80</sub>  $\sqrt{s}$  $\lambda = s_{\alpha_2}$  $s_{\alpha_1+2\alpha_2,p}$ <br>•  $\lambda$ ) – ch L *λ* in the last equality. Now where we used that  $s_{\alpha_1}s_{\alpha_1+\alpha_2}s_{\alpha_1+\alpha_2,p} \bullet \lambda = s_{\alpha_2}s_{\alpha_1+2\alpha_2,p} \bullet \lambda$  in the last observe that  $s_{\alpha_1+\alpha_2,2p} \bullet \lambda \in F_{3,5}$ , so  $\chi(s_{\alpha_1+\alpha_2,2p} \bullet \lambda) = \text{ch } L(s_{\alpha_1+\alpha_2,2p} \bullet \lambda)$ *λ*) by one of the previous cases and we are done.

The last claim follows directly from Lemma [1.37.](#page-12-0)

 $\Box$ 

Combining Lemma [6.4](#page-49-0) and Remark [6.2,](#page-49-1) we get the following remark.

<span id="page-50-0"></span>**Remark 6.5.** In this remark, we use Euclidean coordinates. Let  $\lambda = (a, b) \in X^+$ . We set  $m := a + b + 2 - p.$ 

- If  $\lambda \in \widehat{C_2}$ , then ch  $L(\lambda) = \chi(a, b) \chi(p 3 a, b)$ .
- If  $\lambda \in C_3$ , then  $\chi(a, b) = \text{ch } L(a, b) + \text{ch } L(a m, b m)$ .

<span id="page-50-1"></span>**Lemma 6.6** ([\[Tes88,](#page-78-1) 1.35].). Let  $\lambda = (a, b) \in X^+$  be p-restricted with  $a \neq 0$  and  $b \neq 0$ . *Then*

$$
m_{L(\lambda)}(\lambda - \alpha_1 - \alpha_2) = \begin{cases} 1 & \text{if } 2a + b + 2 \equiv 0 \bmod p, \\ 2 & \text{otherwise.} \end{cases}
$$

$\lambda=(a,b)$		$a \geq 1, b \geq 2 \mid a \geq 1, b = 1 \mid a = 0, b \geq 2 \mid a \geq 1, b = 0 \mid a = 0, b = 1$	
$m_{\Delta(\lambda)}(\lambda)$			
$m_{\Delta(\lambda)}(\lambda-\alpha_1)$			
$m_{\Delta(\lambda)}(\lambda-\alpha_2)$			
$m_{\Delta(\lambda)}(\lambda-\alpha_1-\alpha_2)$			
$m_{\Delta(\lambda)}(\lambda-2\alpha_2)$			
$m_{\Delta(\lambda)}(\lambda-\alpha_1-2\alpha_2)$			

<span id="page-51-0"></span>**Lemma 6.7.** *Let*  $\lambda = (a, b) \in X^+$ *. We have the following weight multiplicities:* 

*Moreover, for*  $\lambda = (a, 0) \in X^+$  *with*  $a \geq 2$ *, we have* 

$$
m_{\Delta(a,0)}(\lambda - \alpha_1 - \alpha_2) = 1, \qquad m_{\Delta(a,0)}(\lambda - \alpha_1 - 2\alpha_2) = 1, \qquad m_{\Delta(a,0)}(\lambda - 2\alpha_1 - \alpha_2) = 1,
$$
  

$$
m_{\Delta(a,0)}(\lambda - 2\alpha_1 - 2\alpha_2) = 2.
$$

<span id="page-51-1"></span>*Proof.* This follows from Proposition [1.25](#page-11-0) and tables of dominant weights ([\[Bre85\]](#page-77-1)).

**Lemma 6.8.** *Let*  $\lambda = (a, b) \in X^+$  *with*  $1 \leq a < p$  *and*  $2 \leq b < p$ *. Then* 

$$
m_{L(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) = \begin{cases} 2 & \text{if } 2a + b + 2 \equiv 0 \bmod p, \\ 2 & \text{if } a + b = p - 1, \\ 3 & \text{otherwise.} \end{cases}
$$

*Proof.*

• If  $2a + b + 2 = p$ , then  $\lambda \in C_2$  and by Lemma [6.4,](#page-49-0)  $\Delta(\lambda)$  admits the unique composition series  $[L(\lambda), L(\lambda - \alpha_1 - \alpha_2)]$ . By Lemma [6.7,](#page-51-0) we get

$$
m_{L(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) = m_{\Delta(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) - m_{\Delta(\lambda - \alpha_1 - \alpha_2)}(\lambda - \alpha_1 - 2\alpha_2) = 3 - 1 = 2.
$$

• If  $2a + b + 2 = 2p$ , then  $\lambda \in C_4 \cup (F_{4,7} \setminus F_{4,6})$  and by Lemma [6.4,](#page-49-0)  $\Delta(\lambda)$  admits the unique composition series  $[L(\lambda), L(\lambda - \alpha_1 - \alpha_2)]$ . By Lemmas [6.7](#page-51-0) and [1.15,](#page-10-0) we get

$$
m_{L(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) = m_{\Delta(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) - m_{L(\lambda - \alpha_1 - \alpha_2)}(\lambda - \alpha_1 - 2\alpha_2) = 3 - 1 = 2.
$$

• If  $a + b = p - 1$ , then  $\lambda \in C_3$  and by Lemma [6.4,](#page-49-0)  $\Delta(\lambda)$  admits the unique composition series  $[L(\lambda), L(\lambda - \alpha_1 - 2\alpha_2)]$ . By Lemma [6.7,](#page-51-0) we get

$$
m_{L(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) = m_{\Delta(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) - m_{\Delta(\lambda - \alpha_1 - 2\alpha_2)}(\lambda - \alpha_1 - 2\alpha_2) = 3 - 1 = 2.
$$

• In all other cases, either  $\Delta(\lambda) \cong L(\lambda)$  or all composition factors of  $\Delta(\lambda)$  except  $L(\lambda)$ have highest weight  $\nu \ngeq \lambda - \alpha_1 - 2\alpha_2$ , thus

$$
m_{L(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) = m_{\Delta(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) = 3
$$

by Lemma [6.7.](#page-51-0)

**Lemma 6.9.** *Let*  $\lambda = (a, 1) \in X^+$  *with*  $1 \leq a \leq p$ *. Then* 

$$
m_{L(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) = \begin{cases} 1 & \text{if } 2a + 3 = p, \\ 2 & \text{otherwise.} \end{cases}
$$

 $\Box$ 

*Proof.*

• If  $2a + 3 = p$ , then  $\lambda \in C_2$  and by Lemma [6.4,](#page-49-0)  $\Delta(\lambda)$  admits the unique composition series  $[L(\lambda), L(\lambda - \alpha_1 - \alpha_2)]$ . By Lemma [6.7,](#page-51-0) we get

$$
m_{L(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) = m_{\Delta(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) - m_{\Delta(\lambda - \alpha_1 - \alpha_2)}(\lambda - \alpha_1 - 2\alpha_2) = 2 - 1 = 1.
$$

• In all other cases, either  $\Delta(\lambda) \cong L(\lambda)$  or all composition factors of  $\Delta(\lambda)$  except  $L(\lambda)$ have highest weight  $\nu \not\geq \lambda - \alpha_1 - 2\alpha_2$ , thus

$$
m_{L(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) = m_{\Delta(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) = 2
$$

by Lemma [6.7.](#page-51-0)

<span id="page-52-0"></span>**Lemma 6.10.** *Let*  $2 \le a < p$  *and set*  $W := \Delta(1,0)$ *. Then*  $\chi(a,0) = \text{ch } S^a(W) - \text{ch } S^{a-2}(W)$  $and \Delta(0, a) \cong S^a(\Delta(0, 1))$ *. In particular*  $\chi(0, a) = \text{ch } S^a(\Delta(0, 1))$ *.* 

*Proof.* For  $\chi(a,0)$ , see [\[FH04,](#page-77-0) §19.5]. For  $\Delta(0,a)$ , observe that  $S^a(\Delta(0,1))$  admits a maximal vector of weight  $(0, a)$ . By Lemma [1.22,](#page-11-1) there exists a non-zero morphism  $\theta$  :  $\Delta(0, a) \to S^a(\Delta(0, 1))$ . By Lemma [6.4,](#page-49-0)  $\Delta(0, a)$  is irreducible, thus  $\theta$  is injective. To show that  $\theta$  is surjective and thus an isomorphism, it is enough to show that  $\dim \Delta(0, a) =$ dim  $S^a(\Delta(0,1))$ . Using usual multilinear algebra, we have dim  $S^a(\Delta(0,1)) = \binom{a+3}{3}$  $\binom{+3}{3}$ . On the other hand, using Weyl's degree formula (Corollary [1.43\)](#page-13-1), with the choice  $(\alpha_2, \alpha_2) = 2$  (and hence  $(\alpha_1, \alpha_1) = 4$ ), we have

$$
\dim \Delta(0, a) = \frac{(\lambda + \rho, \alpha_1)(\lambda + \rho, \alpha_2)(\lambda + \rho, \alpha_1 + \alpha_2)(\lambda + \rho, \alpha_1 + 2\alpha_2)}{(\rho, \alpha_1)(\rho, \alpha_2)(\rho, \alpha_1 + \alpha_2)(\rho, \alpha_1 + 2\alpha_2)}
$$

$$
= \frac{2(a + 1)(a + 3)(2a + 4)}{2 \cdot 1 \cdot 3 \cdot 4} = \binom{a + 3}{3},
$$

hence we are done.

**Notation 6.11.** Let  $\lambda = x\epsilon_1 + y\epsilon_2 \in X$ . We define its 1-norm

$$
\|\lambda\| := |x| + |y|
$$

and its ∞-norm

$$
|\lambda|_{\infty} := \max\{|x|, |y|\}.
$$

<span id="page-52-1"></span>**Lemma 6.12.** *Let*  $a \in \mathbb{N}$  *and*  $\lambda \in X$ *. We have* 

$$
m_{\Delta(a,0)}(\lambda) = \begin{cases} \lfloor \frac{a - ||\lambda||}{2} \rfloor + 1 & \text{if } ||\lambda|| \in \mathbb{N} \text{ and } ||\lambda|| \le a, \\ 0 & \text{else.} \end{cases}
$$

*Proof.* In this proof, we use Euclidean coordinates. We set  $W := \Delta(1,0)$ . Let  $k \in \mathbb{N}$ . We compute  $m_{S^kW}(\lambda)$  and then use Lemma [6.10.](#page-52-0)

Let  $i := \lfloor \frac{k - ||\lambda||}{2} \rfloor$  $\frac{d|\mathcal{A}|}{2}$ . We claim that

$$
m_{S^k W}(\lambda) = \begin{cases} \binom{2+i}{2} & \text{if } \|\lambda\| \in \mathbb{N} \text{ and } \|\lambda\| \le k, \\ 0 & \text{else.} \end{cases}
$$

We know that dim  $W = 5$  and W admits the five weights  $0, \pm \epsilon_1, \pm \epsilon_2$ , all of them with multiplicity 1. We fix  $(v_{-2}, v_{-1}, v_0, v_1, v_2)$  an ordered basis of *W* with  $v_0$  a weight vector associated to 0 and  $v_{\pm i}$  a weight vector associated to  $\pm \epsilon_i$  for  $i = 1, 2$ . By multilinear algebra,  $S^kW$  admits the basis  $\{v_{i_1} \otimes \cdots \otimes v_{i_k}\}_{i_1 \leq i_2 \leq \ldots \leq i_k}$  and there exists a natural bijection between

 $\Box$ 

this basis and the set  $\{(x, y, z, s, t) \in \mathbb{N}^5 \mid x + y + z + s + t = k\}$  where *x* is the number of  $v_{i_j}$ with  $i_j = -2$ , *y* is the number of  $v_{i_j}$  with  $i_j = -1$ , and so on. Moreover, under this bijection, a basis vector associated to  $(x, y, z, s, t)$  is a weight vector with weight  $(t - x)\epsilon_2 + (s - y)\epsilon_1$ . Thus, to compute  $m_{S^kW}(\lambda)$ , we will count the number of 5-tuples  $(x, y, z, s, t)$  with associated weight  $\lambda$ . Let us write  $\lambda = f\epsilon_1 + g\epsilon_2$ . Then  $(x, y, z, s, t)$  is associated to  $\lambda$  if  $t - x = g$  and *s* − *y* = *f*. In particular, we need  $\|\lambda\|$  ∈ N and  $\|\lambda\|$  ≤ *k*. In this case, we get

$$
m_{S^k W}(\lambda) = |\{(x, y, z, s, t) \in \mathbb{N}^5 \mid x + y + z + s + t = k, t - x = g, s - y = f\}|.
$$

Up to symmetry we can assume that both  $f$  and  $g$  are non-negative. We get

$$
m_{S^k W}(\lambda) = |\{(x, y, z, s, t) \in \mathbb{N}^5 \mid x + y + z + s + t = k, t - x = g, s - y = f\}|
$$
  
=  $|\{(x, y, z) \in \mathbb{N}^3 \mid 2x + g + 2y + f + z = k\}|$   
=  $|\{(x, y, z) \in \mathbb{N}^3 \mid 2x + 2y + z = k - ||\lambda||\}|.$ 

If  $k - ||\lambda||$  is odd, then so is *z*, and we have

$$
|\{(x, y, z) \in \mathbb{N}^3 \mid 2x + 2y + z = k - ||\lambda||\}| = |\{(x, y, z) \in \mathbb{N}^3 \mid 2x + 2y + (z - 1) = k - ||\lambda|| - 1\}|.
$$
  
In this case, let  $z' := (z - 1)/2$ . We get  

$$
m_{S^k W}(\lambda) = |\{(x, y, z') \in \mathbb{N}^3 \mid 2x + 2y + 2z' = k - ||\lambda|| - 1\}|
$$

$$
= |\{(x, y, z') \in \mathbb{N}^3 \mid x + y + z' = (k - ||\lambda|| - 1)/2\}|
$$

$$
= \binom{2 + (k - ||\lambda|| - 1)/2}{2}
$$

$$
= \binom{2 + \lfloor (k - ||\lambda||)/2 \rfloor}{2}
$$

where the third equality is a well-known combinatorial result, see for example [\[MN08,](#page-78-2) 3.3]. Now suppose that  $k - ||\lambda||$  and *z* are even, and set  $z' := z/2$ . Using the same reasoning we get

$$
m_{S^k W}(\lambda) = |\{(x, y, z') \in \mathbb{N}^3 | 2x + 2y + 2z' = k - ||\lambda||\}|
$$
  
=  $|\{(x, y, z') \in \mathbb{N}^3 | x + y + z' = (k - ||\lambda||)/2\}|$   
=  $\binom{2 + (k - ||\lambda||)/2}{2}$   
=  $\binom{2 + \lfloor (k - ||\lambda||)/2 \rfloor}{2}$ .

Hence in both cases we are done with our claim.

Now let  $i := \lfloor \frac{a - ||\lambda||}{2} \rfloor$  $\frac{\|\lambda\|}{2}$  and observe that  $\frac{(a-2)-\|\lambda\|}{2}$  $\frac{1-\|\lambda\|}{2}$  = *i* - 1. For  $\|\lambda\| \in \mathbb{N}$  and  $\|\lambda\| \leq a-2$ , by Lemma [6.10,](#page-52-0) we have

$$
m_{\Delta(a,0)}(\lambda) = m_{S^aW}(\lambda) - m_{S^{a-2}W}(\lambda) = {\binom{2+i}{2}} - {\binom{1+i}{2}} = \frac{(2+i)!}{2!i!} - \frac{(1+i)!}{2!(i-1)!}
$$

$$
= \frac{(2+i)! - i(1+i)!}{2!i!} = \frac{2(i+1)!}{2!i!} = i+1 = \lfloor \frac{a-||\lambda||}{2} \rfloor + 1.
$$

If  $\|\lambda\| \in \{a, a-1\}$ , then  $i = 0$  and

$$
m_{\Delta(a,0)}(\lambda) = m_{S^aW}(\lambda) = \binom{2}{2} = 1.
$$

<span id="page-54-0"></span>If  $\|\lambda\| \notin \mathbb{N}$  or if  $\|\lambda\| > a$ , we conclude that  $m_{\Delta(a,0)}(\lambda) = 0$ .

**Corollary 6.13.** *Let*  $a \in \mathbb{N}$ *. We set* 

$$
Y(i,j) := \lfloor \frac{a - |i| - |j|}{2} \rfloor + 1.
$$

*We have*

$$
\chi(a,0) = \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} Y(i,j) e^{j\epsilon_1 + i\epsilon_2} + \sum_{i=1}^{a} \sum_{j=i-a}^{a-i} Y(i,j) e^{j\epsilon_1 - i\epsilon_2}.
$$

<span id="page-54-1"></span>*Proof.* This is a direct consequence of Lemma [6.12.](#page-52-1)

**Corollary 6.14.** *Let*  $a = \frac{p-1}{2}$  $\frac{-1}{2}$  and  $\lambda \in X$ *. Then* 

$$
m_{L(a,0)}(\lambda) = \begin{cases} 1 & \text{if } ||\lambda|| \in \mathbb{N} \text{ and } ||\lambda|| \leq a, \\ 0 & \text{else.} \end{cases}
$$

*In particular, we have*

ch 
$$
L(a, 0) = \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} e^{j\epsilon_1 + i\epsilon_2} + \sum_{i=1}^{a} \sum_{j=i-a}^{a-i} e^{j\epsilon_1 - i\epsilon_2}.
$$

*Proof.* Observe that  $(a, 0) \in C_2$ . Moreover,  $s_{\alpha_1+\alpha_2,p}$ .<br>the only weight in the first alcove linked to  $(a, 0)$ . By  $(a, 0) = (a - 2, 0),$  so  $(a - 2, 0) \in C_1$  is the only weight in the first alcove linked to  $(a, 0)$ . By Lemmas [6.4](#page-49-0) and [6.13,](#page-54-0) we have

$$
m_{L(a,0)}(\lambda) = m_{\Delta(a,0)}(\lambda) - m_{\Delta(a-2,0)}(\lambda)
$$
  
= 
$$
\begin{cases} \lfloor \frac{a-\|\lambda\|}{2} \rfloor + 1 - (\lfloor \frac{a-2-\|\lambda\|}{2} \rfloor + 1) & \text{if } \|\lambda\| \in \mathbb{N} \text{ and } \|\lambda\| \le a-2, \\ \lfloor \frac{a-\|\lambda\|}{2} \rfloor + 1 & \text{if } \|\lambda\| \in \{a, a-1\}, \\ 0 & \text{else.} \end{cases}
$$

Observe that  $\frac{a-\|\lambda\|}{2}$  $\left\lfloor \frac{\|\mathcal{A}\|}{2} \right\rfloor + 1 = 1$  for  $\|\mathcal{A}\| \in \{a, a-1\}$  and

$$
\lfloor \frac{a - \|\lambda\|}{2} \rfloor + 1 - \left( \lfloor \frac{a - 2 - \|\lambda\|}{2} \rfloor + 1 \right) = \lfloor 1 + \frac{a - 2 - \|\lambda\|}{2} \rfloor - \lfloor \frac{a - 2 - \|\lambda\|}{2} \rfloor = 1,
$$

<span id="page-54-2"></span>which allows us to conclude.

**Lemma 6.15.** *Let*  $0 \leq b < p$  *and*  $\lambda \in X$ *. Then* 

$$
m_{L(0,b)}(\lambda) = m_{\Delta(0,b)}(\lambda) = \begin{cases} \left(\frac{b}{2} - |\lambda|_{\infty}\right) + 1 & \text{if } \frac{b}{2} - |\lambda|_{\infty} \in \mathbb{N}, \\ 0 & \text{else.} \end{cases}
$$

*Proof.* Let  $V := \Delta(0,1)$ . First, observe that  $L(0,b) = \Delta(0,b) = S^b V$  by Lemmas [6.4](#page-49-0) and [6.10.](#page-52-0) We know that  $\dim V = 4$  and *V* admits the four weights  $\frac{1}{2}(\pm \epsilon_1 \pm \epsilon_2)$ , all of them with multiplicity 1. We fix a basis of weight vectors  $(v_{-2}, v_{-1}, v_1, v_2)$  with  $v_{\pm 2}$  associated to  $\pm \frac{1}{2}(\epsilon_1 + \epsilon_2)$  and  $v_{\pm 1}$  to  $\pm \frac{1}{2}(\epsilon_1 - \epsilon_2)$ . By multilinear algebra,  $S^bV$  admits the basis  $\{v_{i_1} \otimes \cdots \otimes v_{i_k}\}_{i_1 \leq i_2 \leq \ldots \leq i_k}$  and there exists a natural bijection between this basis and the set  $\{(x, y, s, t) \in \mathbb{N}^4 \mid x + y + s + t = b\}$  where *x* is the number of  $v_{i_j}$  with  $i_j = -2$ , *y* is

 $\Box$ 

 $\Box$ 

the number of  $v_{i_j}$  with  $i_j = -1$ , and so on. Moreover, under this bijection, a basis vector associated to  $(x, y, s, t)$  is a weight vector with weight

$$
(t-x)\frac{1}{2}(\epsilon_1+\epsilon_2) + (s-y)\frac{1}{2}(\epsilon_1-\epsilon_2) = \frac{1}{2}(t+s-x-y)\epsilon_1 + (t+y-x-s)\epsilon_2.
$$

Thus, to compute  $m_{S^bV}(\lambda)$ , we will count the number of 4-tuples  $(x, y, s, t)$  with associated weight  $\lambda$ . Let us write  $\lambda = f(\frac{1}{2})$  $\frac{1}{2}(\epsilon_1 + \epsilon_2) + g\frac{1}{2}$  $\frac{1}{2}(\epsilon_1 - \epsilon_2)$ . Then  $(x, y, s, t)$  is associated to  $\lambda$  if  $t - x = f$  and  $s - y = g$ . Therefore we get

$$
m_{S^{b}V}(\lambda) = |\{(x, y, s, t) \in \mathbb{N}^4 \mid x + y + s + t = b, t - x = f, s - y = g\}|.
$$

Up to symmetry we can assume that both  $f$  and  $g$  are non-negative. In particular, this implies that  $|\lambda|_{\infty} = \frac{1}{2}$  $\frac{1}{2}(f+g)$ , so we get

$$
m_{S^{b}V}(\lambda) = |\{(x, y, s, t) \in \mathbb{N}^{4} | x + y + s + t = b, t - x = f, s - y = g\}|
$$
  
=  $|\{(x, y) \in \mathbb{N}^{2} | 2x + f + 2y + g = b\}|$   
=  $|\{(x, y) \in \mathbb{N}^{2} | 2x + 2y = b - 2|\lambda|_{\infty}\}|$   
=  $|\{(x, y) \in \mathbb{N}^{2} | x + y = \frac{1}{2}b - |\lambda|_{\infty}\}|.$ 

Clearly, the equality  $x + y = \frac{1}{2}$  $\frac{1}{2}b - |\lambda|_{\infty}$  cannot be satisfied for  $x, y \in \mathbb{N}$  if  $\frac{1}{2}b - |\lambda|_{\infty} \notin \mathbb{N}$ . Thus we can restrict our attention to the case  $\frac{1}{2}b - |\lambda|_{\infty} \in \mathbb{N}$ , and using the combinatorial result ([\[MN08,](#page-78-2) 3.3]) again, we get

$$
m_{S^{b}V}(\lambda) = |\{(x, y) \in \mathbb{N}^{2} \mid x + y = \frac{1}{2}b - |\lambda|_{\infty}\} = \binom{\frac{1}{2}b - |\lambda|_{\infty} + 1}{1} = \frac{1}{2}b - |\lambda|_{\infty} + 1,
$$

<span id="page-55-0"></span>thus we are done.

## **6.3 Classification results**

In all statements of this section, unless stated otherwise, we use coordinates with respect to the fundamental weights.

We start by stating a theorem from Stembridge ([\[Ste03,](#page-78-3) Theorem 1.1.B]) which classifies multiplicity-free tensor products of simple  $Sp_4(\mathbb{C})$ -modules.

<span id="page-55-1"></span>**Theorem 6.16.** *Let*  $\lambda = (a, b), \mu = (c, d) \in X^+$  *be dominant weights. Up to the reordering of*  $\lambda$  *and*  $\mu$ ,  $L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu)$  *is multiplicity-free if and only if one of the following holds:* 

- $(1)$   $a = 0$  and  $b = 1$ ,
- $(2)$   $a = 1$  *and*  $b = 0$ ,
- $(3)$   $a = d = 0$ ,
- $(4)$   $a = c = 0$ ,
- (5)  $b = 0$  *and*  $d = 1$ , *or*
- (6)  $b = d = 0$ .

**Proposition 6.17.** *Let*  $\lambda \in C_2$  *and*  $\mu \in \widehat{C_1}$ *. If*  $\lambda + \mu \in C_3$ *, then*  $L(\lambda) \otimes L(\mu)$  *has multiplicity.* 



*Proof.* Let  $\lambda' := s_{\alpha_1 + \alpha_2, p}$ .<br>  $[L(\lambda) | L(\lambda')]$  and let  $n \sim s$  $\lambda \in C_1$ , so that  $\Delta(\lambda)$  admits the unique composition series  $[ L(\lambda), L(\lambda')]$  and let  $\eta := s_{\alpha_1+\alpha_2,p} \bullet \lambda \in C_1,$ <br>  $[ L(\lambda), L(\lambda')]$  and let  $\eta := s_{\alpha_1+\alpha_2,p} \bullet \lambda$ <br>
composition series  $[ L(\lambda + \mu) L(\eta) ]$  (Let  $(\lambda + \mu) \in C_2$ , so that  $\Delta(\lambda + \mu)$  admits the unique composition series  $[L(\lambda + \mu), L(\eta)]$  (Lemma [6.4\)](#page-49-0). We have a short exact sequence

$$
0 \longrightarrow L(\lambda') \longrightarrow \Delta(\lambda) \longrightarrow L(\lambda) \longrightarrow 0.
$$

By Lemma [6.4,](#page-49-0) we have  $L(\mu) \cong \Delta(\mu)$  and  $L(\lambda') \cong \Delta(\lambda')$ . Taking the tensor product with  $L(\mu)$ , we get the short exact sequence

$$
0 \longrightarrow \Delta(\lambda') \otimes \Delta(\mu) \stackrel{\phi}{\longrightarrow} \Delta(\lambda) \otimes \Delta(\mu) \stackrel{\psi}{\longrightarrow} L(\lambda) \otimes L(\mu) \longrightarrow 0.
$$

By Lemma [1.57,](#page-15-2)  $\Delta(\lambda) \otimes \Delta(\mu)$  admits a submodule isomorphic to  $\Delta(\lambda + \mu)$ . We will abuse the notation and denote it by  $\Delta(\lambda + \mu)$ . Thus, we can restrict our exact sequence to

$$
0 \longrightarrow \phi^{-1}(\Delta(\lambda + \mu)) \longrightarrow \phi^{\phi}(\lambda + \mu) \longrightarrow \psi(\Delta(\lambda + \mu)) \longrightarrow 0.
$$

Suppose for contradiction that  $L(\lambda) \otimes L(\mu)$  is multiplicity-free. Then in particular it is completely reducible, and  $\psi(\Delta(\lambda + \mu))$  is completely reducible (Proposition [1.13\)](#page-10-1). Therefore,  $\text{rad }\Delta(\lambda + \mu) \subseteq \text{ker}(\psi)$  and  $\psi(\Delta(\lambda + \mu)) \cong L(\lambda + \mu)$  or  $\psi(\Delta(\lambda + \mu)) = 0$ . We claim that the second case is impossible. By exactness, it would imply  $\phi^{-1}(\Delta(\lambda + \mu)) \cong \Delta(\lambda + \mu)$ , but  $\lambda + \mu \nleq \lambda' + \mu$ , so it cannot appear as a submodule of  $\Delta(\lambda') \otimes \Delta(\mu)$ . Thus,  $\phi^{-1}(\Delta(\lambda + \mu)) \cong L(\eta)$  and it is a submodule of  $\Delta(\lambda') \otimes \Delta(\mu)$ .

Using Theorem [1.28,](#page-12-1) we fix

$$
0 = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_m = \Delta(\lambda') \otimes \Delta(\mu)
$$

a Weyl filtration. Thus there exist  $\nu_1, \ldots, \nu_m \in X^+$  such that  $V_i/V_{i-1} \cong \Delta(\nu_i)$  for  $i \in \{1, \ldots, m\}$ . We set  $W_i := V_i \cap \phi^{-1}(\Delta(\lambda + \mu))$ . Since  $\phi^{-1}(\Delta(\lambda + \mu))$  is irreducible, then  $W_i = 0$  or  $W_i \cong L(\eta)$ . Let *j* be minimal such that  $W_j \cong L(\eta)$  (in particular  $W_{j-1} = 0$ ). We have the following situation

$$
0 \longrightarrow V_{j-1} \longrightarrow V_j \xrightarrow{\pi_j} \Delta(\nu_j) \longrightarrow 0
$$
  

$$
\uparrow
$$
  

$$
L(\eta)
$$

Since ker  $\pi_j = V_{j-1}$  and  $W_j \cap V_{j-1} = W_{j-1} = 0$ , we have an injective map  $L(\eta) \to \Delta(\nu_j)$ , so *L*(*η*) is a submodule of  $\Delta(\nu_j)$ . In particular, *L*(*η*) is a composition factor of  $\Delta(\nu_j)$ , so  $\eta \uparrow \nu_j$ by the Strong Linkage Principle (Proposition [1.53\)](#page-15-3). Recall at this step that  $\eta \uparrow \lambda + \mu$ . Now, observe that  $\nu_j \leq \lambda' + \mu < \lambda + \mu$ , so  $\nu_j \neq \lambda + \mu$ . By the geometry of alcoves, it follows that  $\nu_j \in C_2$ , so  $\eta = \nu_j$ . But  $L(\eta)$  is not a submodule of  $\Delta(\eta)$  (Lemma [6.4\)](#page-49-0), so we get a contradiction. Therefore,  $L(\lambda) \otimes L(\mu)$  has multiplicity.  $\Box$ 

**6.3.1**  $L(0,b) ⊗ L(0,d)$ 

**Proposition 6.18.** *Let*  $\lambda = (0, b), \mu = (0, d) \in X^+$  *be p-restricted dominant weights with*  $0 < b, d < p$ . Then  $L(\lambda) \otimes L(\mu)$  is multiplicity-free if and only if one of the following holds:

- (1)  $b + d \leq p 3$  (*i.e.*  $\lambda + \mu \in \widehat{C_1}$ ) or
- (2)  $(b, d) \in \{(1, p 2), (p 2, 1)\}.$

*Proof.* We set  $M := L(\lambda) \otimes L(\mu)$ . By Lemma [6.4,](#page-49-0)  $L(\lambda)$  and  $L(\mu)$  are tilting modules, so M is a tilting module.

If  $b + d \geq p$ , we conclude directly from Corollary [2.8](#page-17-0) that *M* has multiplicity.

If  $b + d \leq p - 3$  (i.e.  $\lambda + \mu \in \widehat{C_1}$ ), we apply Corollary [3.4](#page-19-1) and Theorem [6.16](#page-55-1) to conclude that *M* is multiplicity-free.

Suppose that  $b + d = p - 2$ . By Lemma [1.15,](#page-10-0) we have

 $m_{L(\lambda)}(\lambda - \alpha_2) = 1$ ,  $(m_L(\lambda - \alpha_2)) = 1,$   $m_{L(\mu)}(\mu - \alpha_2) = 1,$   $m_{L(\lambda + \mu)}(\lambda + \mu - \alpha_2) = 1.$ 

Using Argument [1,](#page-16-0) we have

$$
m_M(\lambda + \mu - \alpha_2) = 1 + 1 = 2,
$$

and we deduce that  $L(\lambda + \mu - \alpha_2)$  is a composition factor of M. Observe that  $\lambda + \mu - \alpha_2 = (1, p - 4) \in C_2$ , so  $T(\lambda + \mu - \alpha_2)$  is not irreducible by Lemma [6.4.](#page-49-0) We can thus conclude by Lemma [1.36](#page-12-2) that *M* has multiplicity.

If  $b + d = p - 1$  and  $b \neq 1, d \neq 1$ , we use Argument [1.](#page-16-0) By Lemma [1.15,](#page-10-0) we have

 $m_{L(\lambda)}(\lambda - \alpha_2) = 1,$   $m_{L(\lambda)}$  $m_{L(\lambda)}(\lambda - 2\alpha_2) = 1$  $m_{L(\mu)}(\mu - \alpha_2) = 1,$   $m_{L(\mu)}$  $m_{L(\mu)}(\mu - 2\alpha_2) = 1,$  $m_{L(\lambda+\mu)}(\lambda+\mu-\alpha_2)=1,$   $m_{L(\lambda+\mu)}$  $m_{L(\lambda+\mu)}(\lambda+\mu-2\alpha_2)=1,$  $m_{L(\lambda + \mu - \alpha_2)}(\lambda + \mu - \alpha_2) = 1,$   $m_{L(\lambda + \mu - \alpha_2)}$  $m_{L(\lambda+\mu-\alpha_2)}(\lambda+\mu-2\alpha_2)=1.$ 

Therefore, we get

$$
m_M(\lambda + \mu - \alpha_2) = 2, \qquad m_M(\lambda + \mu - 2\alpha_2) = 3.
$$

We deduce that  $L(\lambda + \mu - 2\alpha_2)$  is a composition factor of *M*. Observe that  $\lambda + \mu - 2\alpha_2 = (2, p - 5) \in C_2$ , so  $T(\lambda + \mu - \alpha_2)$  is not irreducible by Lemma [6.4.](#page-49-0) As in the previous case, we conclude by Lemma [1.36](#page-12-2) that *M* has multiplicity.

Finally, consider the case  $b = 1, d = p - 2$  (the case  $b = p - 2, d = 1$  is symmetric). By Proposition [1.47](#page-14-0) and Lemmas [6.4](#page-49-0) and [1.45,](#page-13-0) we have

ch 
$$
M = \chi(\lambda)\chi(\mu) = \chi(0, p - 1) + \chi(1, p - 3) + \chi(0, p - 3) + \chi(-1, p - 1)
$$
  
= ch  $L(0, p - 1)$  + ch  $L(1, p - 3)$  + ch  $L(0, p - 3)$ .

Therefore, *M* is multiplicity-free.

<span id="page-57-0"></span>**6.3.2**  $L(a, 0) ⊗ L(c, 0)$ 

**Lemma 6.19.** *Let*  $a, c \in \mathbb{N}$ *. We use Euclidean coordinates. For*  $i, j \in \mathbb{Z}$ *, we set* 

$$
\delta(a,i,j) := \begin{cases} 1 & \text{if } a-i-j \text{ is even,} \\ 0 & \text{if } a-i-j \text{ is odd.} \end{cases}
$$

*Then*

$$
\chi(a\omega_1)\chi(c\omega_1) = \sum_{i=0}^a \sum_{j=i-a}^{a-i} \delta(a,i,j)\chi(c+j,i).
$$

*Moreover, if*  $a \leq c$ *, then all the weights on the right hand side of the equality are dominant.* 

*Proof.* In this proof we use Euclidean coordinates. Like in Corollary [6.13,](#page-54-0) we set

$$
Y(i,j) := \lfloor \frac{a - |i| - |j|}{2} \rfloor + 1.
$$

Using Proposition [1.47](#page-14-0) and Corollary [6.13](#page-54-0) in the first equality below, Lemma [1.45](#page-13-0) in the second one and Remark [6.2](#page-49-1) in the third one, we get

$$
\chi(a\omega_1)\chi(c\omega_1) = \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} Y(i,j)\chi(c+j,i) + \sum_{i=1}^{a} \sum_{j=i-a}^{a-i} Y(i,j)\chi(c+j,-i)
$$
  
\n
$$
= \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} Y(i,j)\chi(c+j,i) - \sum_{i=1}^{a} \sum_{j=i-a}^{a-i} Y(i,j)\chi(s_{\alpha_2} \cdot (c+j,-i))
$$
  
\n
$$
= \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} Y(i,j)\chi(c+j,i) - \sum_{i=1}^{a} \sum_{j=i-a}^{a-i} Y(i,j)\chi(c+j,i-1)
$$
  
\n
$$
= \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} Y(i,j)\chi(c+j,i) - \sum_{i=0}^{a-1} \sum_{j=i+1-a}^{a-i-1} Y(i+1,j)\chi(c+j,i)
$$
  
\n
$$
= \sum_{i=0}^{a-1} \sum_{j=i+1-a}^{a-i-1} (Y(i,j) - Y(i+1,j))\chi(c+j,i)
$$
  
\n
$$
+ \sum_{i=0}^{a-1} Y(i,a-i)\chi(c+a-i,i) + Y(i,i-a)\chi(c+i-a,i)
$$
  
\n
$$
+ Y(a,0)\chi(c,a).
$$

For  $i \in \{0, \ldots, a-1\}$ , we have  $Y(i, a - i) = Y(i, i - a) = 1 = \delta(a, i, \pm (a - i))$ , and  $Y(a, 0) = 1 = \delta(a, a, 0)$ . Moreover

$$
Y(i,j) - Y(i+1,j) = \left( \lfloor \frac{a - |i| - |j|}{2} \rfloor + 1 \right) - \left( \lfloor \frac{a - |i+1| - |j|}{2} \rfloor + 1 \right) = \delta(a,i,j).
$$

Therefore, we get

$$
\chi(a\omega_1)\chi(c\omega_1) = \sum_{i=0}^{a-1} \sum_{j=i+1-a}^{a-i-1} \delta(a,i,j)\chi(c+j,i) \n+ \sum_{i=0}^{a-1} \delta(a,i,a-i)\chi(c+a-i,i) + \delta(a,i,i-a)\chi(c+i-a,i) \n+ \delta(a,a,0)\chi(c,a) \n= \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \delta(a,i,j)\chi(c+j,i).
$$
\n(38)

<span id="page-58-0"></span>Finally, if  $0 \le i$ ,  $a \le c$  and  $i - a \le j$ , then  $0 \le i \le i + (c - a) \le c + j$ , hence all the weights appearing in line [\(38\)](#page-58-0) are dominant.  $\Box$ 

<span id="page-58-1"></span>**Proposition 6.20.** *Let*  $\lambda = (a, 0), \mu = (c, 0) \in X^+$  *be two p-restricted dominant weights with*  $0 < a < c, c \geq \frac{p-1}{2}$  $\frac{-1}{2}$  and  $a + c < p - 1$  (i.e.  $\lambda \in C_1, \mu \in C_2$  and  $\lambda + \mu \in C_2$ ). Then  $L(\lambda) \otimes L(\mu)$  *is multiplicity-free.* 

*Proof.* In this proof we use Euclidean coordinates. We set  $M := L(\lambda) \otimes L(\mu)$  and  $m := 2c - p + 2$ . Like in Lemma [6.19,](#page-57-0) for  $i, j \in \mathbb{Z}$ , we set

$$
\delta(a, i, j) := \begin{cases} 1 & \text{if } a - i - j \text{ is even,} \\ 0 & \text{if } a - i - j \text{ is odd.} \end{cases}
$$

Note that  $\delta(a, i, j) = \delta(a, i, -j)$ .

By Remark [6.5](#page-50-0) and Proposition [1.47,](#page-14-0) and using Lemma [6.19](#page-57-0) in the second equality below, we have

ch 
$$
M = \chi(a, 0)(\chi(c, 0) - \chi(p - 3 - c, 0))
$$
  
\n
$$
= \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \delta(a, i, j)\chi(c + j, i) - \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \delta(a, i, j)\chi(p - 3 - c + j, i)
$$
\n
$$
= \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \delta(a, i, j)\chi(c + j, i) - \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \delta(a, i, -j)\chi(p - 3 - c - j, i)
$$
\n
$$
= \sum_{i=0}^{a} \sum_{j=\max\{i-a, a-i-m\}}^{a-i} \delta(a, i, j)\chi(c + j, i) + \sum_{i=0}^{a} \sum_{j=i-a}^{a-i-m-1} \delta(a, i, j)\chi(c + j, i)
$$
\n
$$
- \sum_{i=0}^{a} \sum_{j=\max\{i-a, a-i-m\}}^{a-i-m} \delta(a, i, -j)\chi(p - 3 - c - j, i)
$$
\n
$$
- \sum_{i=0}^{a} \sum_{j=i-a}^{a-i-m-1} \delta(a, i, -j)\chi(p - 3 - c - j, i)
$$
\n
$$
= \sum_{i=0}^{a} \sum_{j=\max\{i-a, a-i-m\}}^{a-i} \delta(a, i, j)(\chi(c + j, i) - \chi(p - 3 - c - j, i))
$$
\n(39)

<span id="page-59-1"></span><span id="page-59-0"></span>
$$
+\sum_{i=0}^{a} \left( \sum_{j=i-a}^{a-i-m-1} \delta(a,i,j) \chi(c+j,i) - \sum_{j=i-a}^{a-i-m-1} \delta(a,i,j) \chi(p-3-c-j,i) \right) \tag{40}
$$

We show that line [\(40\)](#page-59-0) is equal to zero. We have

$$
\sum_{j=i-a}^{a-i-m-1} \delta(a,i,j)\chi(c+j,i) - \sum_{j=i-a}^{a-i-m-1} \delta(a,i,j)\chi(p-3-c-j,i)
$$
  
= 
$$
\sum_{j=i-a}^{a-i-m-1} \delta(a,i,j)\chi(c+j,i)
$$
  

$$
-\sum_{j=i-a}^{a-i-m-1} \delta(a,i,-m-1-j)\chi(p-3-c-(-m-1-j),i)
$$
  
= 
$$
\sum_{j=i-a}^{a-i-m-1} \delta(a,i,j)\chi(c+j,i) - \sum_{j=i-a}^{a-i-m-1} \delta(a,i,-m-1-j)\chi(c+j,i)
$$
  
= 
$$
\sum_{j=i-a}^{a-i-m-1} (\delta(a,i,j)-\delta(a,i,-m-1-j))\chi(c+j,i) = 0,
$$

where in the last equality, we use that *m* is odd, thus  $-m-1$  is even and  $\delta(a, i, -m-1-j) = \delta(a, i, j)$ . Therefore

$$
\sum_{i=0}^{a} \left( \sum_{j=i-a}^{a-i-m-1} \delta(a,i,j) \chi(c+j,i) - \sum_{j=i-a}^{a-i-m-1} \delta(a,i,j) \chi(p-3-c-j,i) \right) = \sum_{i=0}^{a} 0 = 0.
$$

Now we work on line [\(39\)](#page-59-1). For  $0 \leq i \leq a$  and  $\max\{i-a, a-i-m\} \leq j \leq a-i$ , we claim that  $(c + j, i) \in \widehat{C_2}$ . Indeed, we have  $i \geq 0$  and

$$
(c+j) + i \le c + (a-i) + i \le p-2
$$

by assumption. Thus it remains to show that  $c + j > \frac{p-3}{2}$ . If  $i - a \ge \frac{p-1}{2} - c$ , then

$$
c + j \ge c + i - a \ge \frac{p-1}{2} > \frac{p-3}{2},
$$

hence we are done. If  $i - a < \frac{p-1}{2} - c$ , then  $a - i > c - \frac{p-1}{2}$  $\frac{-1}{2}$  and

$$
c+j \ge c+a-i-m > 2c - \frac{p-1}{2} - m = 2c - \frac{p-1}{2} - (2c - p + 2) = \frac{p-3}{2}.
$$

so we are done. Using Remark [6.5,](#page-50-0) we get

ch 
$$
M = \sum_{i=0}^{a} \sum_{j=\max\{i-a,a-i-m\}}^{a-i} \delta(a,i,j)(\chi(c+j,i) - \chi(p-3-c-j,i))
$$
  

$$
= \sum_{i=0}^{a} \sum_{j=\max\{i-a,a-i-m\}}^{a-i} \delta(a,i,j) \operatorname{ch} L(c+j,i).
$$

We conclude that *M* is multiplicity-free.

**Proposition 6.21.** *Let*  $\lambda = (a, 0), \mu = (c, 0) \in X^+$  *be p-restricted dominant weights with*  $0 < a \leq c$ . Then  $L(\lambda) \otimes L(\mu)$  *is multiplicity-free if and only if one of the following holds:* 

 $(1)$   $a + c \leq \frac{p-3}{2}$  $\frac{-3}{2}$  (*i.e.*  $\lambda + \mu \in C_1$ ),

(2) 
$$
c \geq \frac{p-1}{2}
$$
 and  $a + c < p - 1$  or

$$
(3) \ \ a = c = \frac{p-1}{2}.
$$

*Proof.* We set  $M := L(\lambda) \otimes L(\mu)$ .

If  $a + c > p$ , we use Corollary [2.8](#page-17-0) to conclude that *M* has multiplicity.

If  $a + c \leq \frac{p-3}{2}$  $\frac{-3}{2}$  (i.e.  $\lambda + \mu \in C_1$ ), we apply Corollary [3.4](#page-19-1) and Theorem [6.16](#page-55-1) to conclude that  $M$  is multiplicity-free.

Suppose that  $c \leq \frac{p-3}{2}$  $\frac{-3}{2}$  and  $a + c > \frac{p-3}{2}$  (i.e.  $\mu \in \widehat{C_1}$  and  $\lambda + \mu \in C_2$ ). In this case,  $L(\lambda)$ and  $L(\mu)$  are tilting modules by Lemma [6.4,](#page-49-0) hence *M* is a tilting module. Since  $T(\lambda + \mu)$  is not irreducible by Lemma [6.4,](#page-49-0) we conclude that *M* has multiplicity by Lemma [1.36.](#page-12-2)

Consider the case  $a = 1, c = p - 2$  (in particular  $\mu \in F_{2,3}, \lambda \in \widehat{C_1}$  and  $\lambda + \mu \in F_{4,6}$ ). By Lemma [6.4,](#page-49-0) we have  $L(\lambda)$  and  $L(\mu)$  are tilting modules, therefore M is a tilting module. We use Argument [1.](#page-16-0) By Lemma [1.15,](#page-10-0) we have

$$
m_{L(\lambda)}(\lambda - \alpha_1) = 1,
$$
  $m_{L(\mu)}(\mu - \alpha_1) = 1,$   $m_{L(\lambda + \mu)}(\lambda + \mu - \alpha_1) = 1.$ 

Since  $m_M(\lambda + \mu - \alpha_1) = 2$ , we deduce that  $L(\lambda + \mu - \alpha_1)$  is a composition factor of M. Observe that  $\lambda + \mu - \alpha_1 = (p-3, 2) \in C_3$ , so  $T(\lambda + \mu - \alpha_1)$  is not irreducible by Lemma [6.4.](#page-49-0) We conclude by Lemma [1.36](#page-12-2) that *M* has multiplicity.

If  $a + c = p - 1$  and  $a \neq 1, \frac{p-1}{2}$  $\frac{-1}{2}$ , then  $\frac{p-1}{2} < c < p-2$  (i.e. λ ∈  $\widehat{C_1}$  and *µ* ∈  $C_2$ ) and by Lemma [6.4,](#page-49-0) we have  $L(\lambda) \cong \Delta(\lambda)$ . Moreover,  $\Delta(\mu)$  admits the unique composition series  $[L(\mu), L(\eta)]$  with  $\eta = s_{\alpha_1 + \alpha_2, p}$ .<br>  $n < \mu - 2\alpha_1 - 2\alpha_2$ , thus  $m \chi_{\alpha_1}(\mu) - m \chi_{\alpha_2}(\mu)$ .  $\mu \in C_1$ . Since  $c \geq \frac{p+1}{2}$  $\frac{+1}{2}$ , we have in particular  $\eta < \mu - 2\alpha_1 - 2\alpha_2$ , thus  $m_{L(\mu)}(\nu) = m_{\Delta(\mu)}(\nu)$  for all weights  $\nu \geq \lambda - 2\alpha_1 - 2\alpha_2$ . We use Argument [1](#page-16-0) to show that  $[M : L(\lambda + \mu - 2\alpha_1 - 2\alpha_2)] = 2$ . Using Lemmas [6.7](#page-51-0) and [1.15](#page-10-0) (note that  $a \geq 2$ , we have

$$
m_M(\lambda + \mu) = 1, \t m_M(\lambda + \mu - \alpha_1) = 2,
$$
  
\n
$$
m_M(\lambda + \mu - 2\alpha_1) = 3, \t m_M(\lambda + \mu - \alpha_2) = m_M(\lambda + \mu - 2\alpha_2) = 0,
$$
  
\n
$$
m_M(\lambda + \mu - \alpha_1 - \alpha_2) = 2, \t m_M(\lambda + \mu - \alpha_1 - 2\alpha_2) = 2,
$$
  
\n
$$
m_M(\lambda + \mu - 2\alpha_1 - \alpha_2) = 4, \t m_M(\lambda + \mu - 2\alpha_1 - 2\alpha_2) = 7.
$$

We deduce that

 $[M: L(\lambda + \mu)] = [M: L(\lambda + \mu - \alpha_1)] = [M: L(\lambda + \mu - 2\alpha_1)] = 1,$ 

$$
[M:L(\lambda + \mu - \alpha_2)] = [M:L(\lambda + \mu - 2\alpha_2)] = 0.
$$

By Lemma [6.4,](#page-49-0)  $L(\lambda + \mu) \cong \Delta(\lambda + \mu)$ . Moreover,  $\lambda + \mu - \alpha_1 = (p-3, 2)$ . Thus, using Lemmas [1.15](#page-10-0) and [6.7,](#page-51-0) we get

$$
[M:L(\lambda+\mu-\alpha_1-\alpha_2)]=0, \qquad \text{and} \qquad [M:L(\lambda+\mu-\alpha_1-2\alpha_2)]=0.
$$

Since  $2(p-3) + 2 + 2 \neq 0 \text{ mod } p$ , we have  $m_{L(\lambda + \mu - \alpha_1)}(\lambda + \mu - 2\alpha_1 - \alpha_2) = 2$  by Lemma [6.6.](#page-50-1) By Lemma [1.15,](#page-10-0) we have  $m_{L(\lambda+\mu-2\alpha_1)}(\lambda+\mu-2\alpha_1-\alpha_2)=1$ . Moreover, by Lemma [6.7,](#page-51-0) we have  $m_{L(\lambda+\mu)}(\lambda+\mu-2\alpha_1-\alpha_2)=1$ . Therefore,

$$
[M:L(\lambda+\mu-2\alpha_1-\alpha_2)]=0.
$$

Finally observe that  $(p-3)+2=p-1$ . Thus, using Lemmas [6.7,](#page-51-0) [6.8](#page-51-1) and [1.15,](#page-10-0) we have

$$
m_{L(\lambda+\mu)}(\lambda+\mu-2\alpha_1-2\alpha_2) = 2, \qquad m_{L(\lambda+\mu-\alpha_1)}(\lambda+\mu-2\alpha_1-2\alpha_2) = 2,
$$
  

$$
m_{L(\lambda+\mu-2\alpha_1)}(\lambda+\mu-2\alpha_1-2\alpha_2) = 1.
$$

We conclude that

$$
[M:L(\lambda+\mu-2\alpha_1-2\alpha_2)]=7-2-2-1=2.
$$

In particular, *M* has multiplicity.

If  $c \geq \frac{p-1}{2}$  $\frac{-1}{2}$  and  $a + c < p - 1$  then *M* is multiplicity-free by Proposition [6.20.](#page-58-1)

Finally, suppose that  $a = c = \frac{p-1}{2}$  $\frac{-1}{2}$  (i.e.  $\lambda, \mu \in C_2$  and  $\lambda + \mu \in F_{4,6}$ ). We show that *M* is multiplicity-free. We have

$$
s_{\alpha_1+\alpha_2,p} \bullet \mu = \mu - 2(\alpha_1+\alpha_2).
$$

For the rest of this proof we use Euclidean coordinates. Using Corollary [1.48](#page-14-2) and Lemma [6.4](#page-49-0) in the first equality below, Corollary [6.14](#page-54-1) in the second one and Lemma [1.45](#page-13-0) in the third one, we get

ch 
$$
M = \sum_{\nu \in X} m_{L(\lambda)}(\nu) (\chi(\mu + \nu) - \chi(\mu - 2(\alpha_1 + \alpha_2) + \nu))
$$
  
\n
$$
= \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \chi(a+j,i) - \chi(a-2+j,i) + \sum_{i=1}^{a} \sum_{j=i-a}^{a-i} \chi(a+j,-i) - \chi(a-2+j,-i)
$$
\n
$$
= \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \chi(a+j,i) - \sum_{i=1}^{a} \sum_{j=i-a}^{a-i} \chi(s_{\alpha_2} \cdot (a+j,-i))
$$
\n
$$
- \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \chi(a-2+j,i) + \sum_{i=1}^{a} \sum_{j=i-a}^{a-i} \chi(s_{\alpha_2} \cdot (a-2+j,-i))
$$
\n
$$
= \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \chi(a+j,i) - \sum_{i=1}^{a} \sum_{j=i-a}^{a-i} \chi(a+j,i-1)
$$
\n
$$
- \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \chi(a-2+j,i) + \sum_{i=1}^{a} \sum_{j=i-a}^{a-i} \chi(a-2+j,i-1)
$$
\n
$$
= \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \chi(a+2+j,i) + \sum_{i=0}^{a-1} \sum_{j=i+1-a}^{a-i-1} \chi(a+2+j,i)
$$
\n
$$
- \sum_{i=0}^{a} \sum_{j=i-a}^{a-i} \chi(a-2+j,i) + \sum_{i=0}^{a-1} \sum_{j=i+1-a}^{a-i-1} \chi(a-2+j,i)
$$
\n
$$
= \left(\sum_{i=0}^{a-1} \chi(i,i) + \chi(2a-i,i)\right) + \chi(a,a)
$$
\n
$$
- \left(\sum_{i=0}^{a-1} \chi(i-2,i) + \chi(2a-2-i,i)\right) - \chi(a-2,a)
$$
\n
$$
= \sum_{i=0}^{a} \chi(i,i) + \chi(2a,0) + \sum_{i=1}^{a-1} \chi(2a-i,i)
$$
\n
$$
- \sum_{i=0}^{a-2} \chi(
$$

At this step, recall that we use Euclidean coordinates and observe that  $(i, i) \in C_1 \cup F_{2,3} \cup F_{3,5}$ for  $0 \leq i \leq a = \frac{p-1}{2}$  $\frac{-1}{2}$ . Thus, by Lemma [6.4,](#page-49-0) we have  $\chi(i, i) = \text{ch } L(i, i)$ . Similarly,  $(2a, 0) = (p - 1, 0) \in F_{4,6}$ , hence  $\chi(2a, 0) = \text{ch } L(2a, 0)$  by Lemma [6.4.](#page-49-0) Using those facts and Lemma [1.45,](#page-13-0) we get

ch 
$$
M = \sum_{i=0}^{a} \text{ch } L(i, i) + \text{ch } L(2a, 0) + \sum_{i=1}^{a-1} \chi(2a - i, i) - \sum_{i=1}^{a-1} \chi(2a - 1 - i, i - 1)
$$
  
+ 
$$
\sum_{i=1}^{a-1} \chi(s_{\alpha_1} \cdot (i - 2, i)) + \chi(s_{\alpha_1 + \alpha_2} \cdot (-2, 0)) - \chi(a - 1, a - 1) + \chi(s_{\alpha_1} \cdot (a - 2, a))
$$

$$
= \sum_{i=0}^{a} \text{ch } L(i, i) + \text{ch } L(2a, 0) + \sum_{i=1}^{a-1} \chi(2a - i, i) - \sum_{i=1}^{a-1} \chi(2a - 1 - i, i - 1)
$$
  
+ 
$$
\sum_{i=1}^{a-1} \chi(i - 1, i - 1) + \chi(-1, 0) - \chi(a - 1, a - 1) + \chi(a - 1, a - 1)
$$
  
= 
$$
\sum_{i=0}^{a} \text{ch } L(i, i) + \text{ch } L(2a, 0) + \sum_{i=1}^{a-1} \chi(2a - i, i)
$$
  
- 
$$
\left(\sum_{i=1}^{a-1} \chi(2a - 1 - i, i - 1) - \chi(i - 1, i - 1)\right),
$$

where in the last equality we use that  $(-1,0) \in D \setminus X^+$  so  $\chi(-1,0) = 0$  by Lemma [1.45.](#page-13-0) If  $i \in \{1, ..., a-1\}$ , then  $(2a - 1 - i, i - 1) \in C_2$ . Thus, by Remark [6.2,](#page-49-1) we have

$$
s_{\alpha_1+\alpha_2,p} \bullet (2a-1-i, i-1) = (p-3-(2a-1-i), i-1) = (i-1, i-1),
$$

and by Remark [6.5](#page-50-0) we get

$$
\chi(2a-1-i, i-1) - \chi(i-1, i-1) = \text{ch } L(2a-1-i, i-1).
$$

Moreover,  $(2a - i, i) \in C_3$  and  $2a - i + i - (p - 2) = 1$ , so by Remark [6.5,](#page-50-0) we have

$$
\chi(2a - i, i) = \text{ch } L(2a - i, i) + \text{ch } L(2a - i - 1, i - 1).
$$

We can thus conclude that

$$
\begin{aligned} \operatorname{ch} M &= \sum_{i=0}^{a} \operatorname{ch} L(i,i) + \operatorname{ch} L(2a,0) + \sum_{i=1}^{a-1} \chi(2a-i,i) - \sum_{i=1}^{a-1} \operatorname{ch} L(2a-1-i,i-1) \\ &= \sum_{i=0}^{a} \operatorname{ch} L(i,i) + \operatorname{ch} L(2a,0) + \sum_{i=1}^{a-1} \operatorname{ch} L(2a-i,i) \\ &= \sum_{i=0}^{a} \operatorname{ch} L(i,i) + \sum_{i=0}^{a-1} \operatorname{ch} L(2a-i,i). \end{aligned}
$$

In particular, *M* is multiplicity-free.

<span id="page-63-0"></span>**6.3.3** 
$$
L(a, 0) \otimes L(0, d)
$$

**Lemma 6.22.** *Let*  $a, b \in \mathbb{N}$ *. In Euclidean coordinates, we have* 

$$
\chi(a\omega_1)\chi(b\omega_2) = \sum_{i=0}^b \sum_{j=0}^{\min\{i,b-i\}} \chi(a+\frac{b}{2}-i,\frac{b}{2}-j).
$$

*Moreover, if*  $b < p$  *and*  $a < p$ *, then all weights appearing on the right hand side of the equality belong to*  $D \cup W$  • *C*<sup>1</sup> *and the dominant ones are p-restricted.*

*Proof.* In this proof, we use Euclidean coordinates. Let  $\lambda = a\omega_1$  and  $\mu = b\omega_2$ . We set  $X(i, j) := m_{\Delta(\mu)}(\frac{b}{2} - i, \frac{b}{2} - j) = \frac{b}{2} - \max\{|\frac{b}{2} - i|, |\frac{b}{2} - j|\} + 1$  (Lemma [6.15\)](#page-54-2). We get

$$
\chi(\lambda)\chi(\mu) = \sum_{\nu \in X} m_{\Delta(\mu)}(\nu)\chi(\lambda + \nu) = \sum_{i=0}^{b} \sum_{j=0}^{b} X(i,j)\chi(a + \frac{b}{2} - i, \frac{b}{2} - j).
$$

We dissociate the cases *b* even and *b* odd. First suppose that *b* is even. Using Lemma [1.45](#page-13-0) in the second equality below and Remark [6.2](#page-49-1) in the third one, we get

$$
\chi(\lambda)\chi(\mu) = \sum_{i=0}^{b} \sum_{j=0}^{\frac{b}{2}} X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j) + \sum_{i=0}^{b} \sum_{j=\frac{b}{2}+1}^{b} X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j)
$$
  
\n
$$
= \sum_{i=0}^{b} \sum_{j=0}^{\frac{b}{2}} X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j) - \sum_{i=0}^{b} \sum_{j=\frac{b}{2}+1}^{b} X(i,j)\chi(s_{\alpha_{2}} \cdot (a+\frac{b}{2}-i,\frac{b}{2}-j))
$$
  
\n
$$
= \sum_{i=0}^{b} \sum_{j=0}^{\frac{b}{2}} X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j) - \sum_{i=0}^{b} \sum_{j=\frac{b}{2}+1}^{b} X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}+j-1)
$$
  
\n
$$
= \sum_{i=0}^{b} \sum_{j=0}^{\frac{b}{2}} X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j)
$$
  
\n
$$
- \sum_{i=0}^{b} \sum_{j=1}^{\frac{b}{2}} X(i,b-j+1)\chi(a+\frac{b}{2}-i,\frac{b}{2}+(b-j+1)-1)
$$
  
\n
$$
= \sum_{i=0}^{b} \sum_{j=0}^{\frac{b}{2}} X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j) - \sum_{i=0}^{b} \sum_{j=1}^{\frac{b}{2}} X(i,b-j+1)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j)
$$
  
\n
$$
= \sum_{i=0}^{b} X(i,0)\chi(a+\frac{b}{2}-i,\frac{b}{2})
$$
  
\n
$$
+ \sum_{i=0}^{b} \sum_{j=1}^{\frac{b}{2}} (X(i,j) - X(i,b-j+1))\chi(a+\frac{b}{2}-i,\frac{b}{2}-j).
$$
 (41)

If *b* is odd, then for any *i*, we have

<span id="page-64-0"></span>
$$
s_{\alpha_2} \bullet (a + \frac{b}{2} - i, \frac{b}{2} - \frac{b+1}{2}) = s_{\alpha_2} \bullet (a + \frac{b}{2} - i, -\frac{1}{2}) = (a + \frac{b}{2} - i, -\frac{1}{2}),
$$

so  $\chi(a + \frac{b}{2} - i, \frac{b}{2} - \frac{b+1}{2})$  $\frac{+1}{2}$  = 0 by Lemma [1.45.](#page-13-0) Similarly to the case where *b* is even, we get

$$
\chi(\lambda)\chi(\mu) = \sum_{i=0}^{b} \sum_{j=0}^{\frac{b-1}{2}} X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j) + \sum_{i=0}^{b} \sum_{j=\frac{b+3}{2}}^{b} X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j)
$$
  

$$
= \sum_{i=0}^{b} \sum_{j=0}^{\frac{b-1}{2}} X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j) - \sum_{i=0}^{b} \sum_{j=\frac{b+3}{2}}^{b} X(i,j)\chi(s_{\alpha_2} \bullet (a+\frac{b}{2}-i,\frac{b}{2}-j))
$$
  

$$
= \sum_{i=0}^{b} \sum_{j=0}^{\frac{b-1}{2}} X(i,j)\chi(a+\frac{b}{2}-i,\frac{b}{2}-j) - \sum_{i=0}^{b} \sum_{j=\frac{b+3}{2}}^{b} X(i,j)\chi(a+\frac{b}{2}-i,-\frac{b}{2}+j-1)
$$

$$
= \sum_{i=0}^{b} \sum_{j=0}^{\frac{b-1}{2}} X(i,j) \chi(a + \frac{b}{2} - i, \frac{b}{2} - j)
$$
  
\n
$$
- \sum_{i=0}^{b} \sum_{j=1}^{\frac{b-1}{2}} X(i,b - j + 1) \chi(a + \frac{b}{2} - i, -\frac{b}{2} + (b - j + 1) - 1)
$$
  
\n
$$
= \sum_{i=0}^{b} \sum_{j=0}^{\frac{b-1}{2}} X(i,j) \chi(a + \frac{b}{2} - i, \frac{b}{2} - j) - \sum_{i=0}^{b} \sum_{j=1}^{\frac{b-1}{2}} X(i,b - j + 1) \chi(a + \frac{b}{2} - i, \frac{b}{2} - j)
$$
  
\n
$$
= \sum_{i=0}^{b} X(i,0) \chi(a + \frac{b}{2} - i, \frac{b}{2})
$$
  
\n
$$
+ \sum_{i=0}^{b} \sum_{j=1}^{\frac{b-1}{2}} (X(i,j) - X(i,b - j + 1)) \chi(a + \frac{b}{2} - i, \frac{b}{2} - j).
$$
 (42)

At this step, observe that  $X(i, 0) = 1$ , and set  $Y(i, j) := X(i, j) - X(i, b - j + 1)$ . Using lines [\(41\)](#page-64-0) and [\(42\)](#page-65-0), for all *b*, we have

$$
\chi(\lambda)\chi(\mu) = \sum_{i=0}^{b} \chi(a + \frac{b}{2} - i, \frac{b}{2}) + \sum_{i=0}^{b} \sum_{j=1}^{\lfloor \frac{b}{2} \rfloor} Y(i, j)\chi(a + \frac{b}{2} - i, \frac{b}{2} - j)
$$
  

$$
= \sum_{i=0}^{b} \chi(a + \frac{b}{2} - i, \frac{b}{2}) + \sum_{i=0}^{\lfloor \frac{b}{2} \rfloor} \sum_{j=1}^{\lfloor \frac{b}{2} \rfloor} Y(i, j)\chi(a + \frac{b}{2} - i, \frac{b}{2} - j)
$$
(43)

<span id="page-65-2"></span><span id="page-65-1"></span><span id="page-65-0"></span>
$$
+\sum_{i=\lfloor \frac{b}{2} \rfloor+1}^{b} \sum_{j=1}^{\lfloor \frac{b}{2} \rfloor} Y(i,j) \chi(a+\frac{b}{2}-i,\frac{b}{2}-j). \tag{44}
$$

We compute the value of  $Y(i, j)$  in lines  $(43)$  and  $(44)$ . Recall that  $Y(i, j) = X(i, j) - X(i, b - j + 1)$  and  $X(i, j) = \frac{b}{2} - \max\{\left|\frac{b}{2} - i\right|, \left|\frac{b}{2} - j\right|\} + 1$ . Therefore, for  $j \leq \frac{b}{2}$  $\frac{b}{2}$ , we have

$$
Y(i,j) = \left(\frac{b}{2} - \max\{\left|\frac{b}{2} - i\right|, \left|\frac{b}{2} - j\right|\} + 1\right) - \left(\frac{b}{2} - \max\{\left|\frac{b}{2} - i\right|, \left|\frac{b}{2} - (b - j + 1)\right|\} + 1\right)
$$
  
\n
$$
= \max\{\left|\frac{b}{2} - i\right|, \left|j - \frac{b}{2} - 1\right|\} - \max\{\left|\frac{b}{2} - i\right|, \left|\frac{b}{2} - j\right|\}
$$
  
\n
$$
= \max\{\left|\frac{b}{2} - i\right|, \left|\frac{b}{2} - j + 1\right|\} - \max\{\left|\frac{b}{2} - i\right|, \left|\frac{b}{2} - j\right|\}
$$
  
\n
$$
= \max\{\left|\frac{b}{2} - i\right|, \frac{b}{2} - j + 1\} - \max\{\left|\frac{b}{2} - i\right|, \frac{b}{2} - j\}.
$$

If  $|\frac{b}{2} - i| \ge \frac{b}{2} - j + 1$ , then

$$
\max\{|\frac{b}{2} - i|, \frac{b}{2} - j + 1\} = \max\{|\frac{b}{2} - i|, \frac{b}{2} - j\} = |\frac{b}{2} - i|
$$

so  $Y(i, j) = 0$ . If  $|\frac{b}{2} - i| < \frac{b}{2} - j + 1$ , then  $|\frac{b}{2} - i| \leq \frac{b}{2} - j$  and we have

$$
\max\{|\frac{b}{2} - i|, \frac{b}{2} - j + 1\} = \frac{b}{2} - j + 1 \quad \text{and} \quad \max\{|\frac{b}{2} - i|, \frac{b}{2} - j\} = \frac{b}{2} - j,
$$

so  $Y(i, j) = 1$ . Therefore, we have

<span id="page-66-1"></span><span id="page-66-0"></span>
$$
Y(i,j) = \begin{cases} 0 & \text{if } \left| \frac{b}{2} - i \right| \ge \frac{b}{2} - j + 1 \\ 1 & \text{if } \left| \frac{b}{2} - i \right| < \frac{b}{2} - j + 1. \end{cases}
$$
(45)

If  $i \leq \frac{b}{2}$  $\frac{b}{2}$ , as in line [\(43\)](#page-65-1), then

<span id="page-66-2"></span>
$$
|\frac{b}{2} - i| = \frac{b}{2} - i \ge \frac{b}{2} - j + 1 \iff j \ge i + 1.
$$
 (46)

If  $i > \frac{b}{2}$ , as in line [\(44\)](#page-65-2), then

$$
|\frac{b}{2} - i| = i - \frac{b}{2} \ge \frac{b}{2} - j + 1 \iff j \ge b - i + 1.
$$
 (47)

Combining lines  $(45)$ ,  $(46)$  and  $(47)$  with lines  $(43)$  and  $(44)$ , we get *b*

$$
\chi(\lambda)\chi(\mu) = \sum_{i=0}^{b} \chi(a + \frac{b}{2} - i, \frac{b}{2}) + \sum_{i=0}^{\lfloor \frac{b}{2} \rfloor} \sum_{j=1}^{i} \chi(a + \frac{b}{2} - i, \frac{b}{2} - j)
$$
  
+ 
$$
\sum_{i=\lfloor \frac{b}{2} \rfloor+1}^{b} \sum_{j=1}^{b-i} \chi(a + \frac{b}{2} - i, \frac{b}{2} - j)
$$
  
= 
$$
\sum_{i=0}^{b} \chi(a + \frac{b}{2} - i, \frac{b}{2}) + \sum_{i=0}^{b} \sum_{j=1}^{\min\{i, b-i\}} \chi(a + \frac{b}{2} - i, \frac{b}{2} - j)
$$
  
= 
$$
\sum_{i=0}^{b} \sum_{j=0}^{\min\{i, b-i\}} \chi(a + \frac{b}{2} - i, \frac{b}{2} - j),
$$
 (48)

establishing the claim of the first statement.

Finally, assume that  $a < p$  and  $b < p$ . We do a change of basis to express our weights in coordinates with respect to the fundamental weights. We have

<span id="page-66-3"></span>
$$
(a + \frac{b}{2} - i)\epsilon_1 + (\frac{b}{2} - j)\epsilon_2 = (a - i + j)\omega_1 + (b - 2j)\omega_2.
$$

For the rest of the proof, we use coordinates with respect to the fundamental weights. Let 0 ≤ *i* ≤ *b* < *p* and  $0 \le j \le \min\{i, b - i\}$ . We set  $\nu := (a - i + j, b - 2j)$ . We have

$$
a - i + j \le a < p \qquad \text{and} \qquad b - 2j \le b < p,
$$

so dominant weights appearing in line [\(48\)](#page-66-3) are *p*-restricted. Moreover,  $j \leq \frac{b}{2}$ , thus  $b-2j \geq 0$ . If  $(a - i + j) \ge -1$ , then  $\nu \in D$  and we are done. Assume  $(a - i + j) < -1$ . We have

$$
s_{\alpha_1} \bullet (a - i + j, b - 2j) = (i - a - j - 2, b + 2a - 2i + 2) =: \nu'.
$$

By assumption, we have

$$
i - a - j - 2 = -(a - i + j) - 2 > 1 - 2 = -1.
$$

If  $b + 2a - 2i + 2 \ge -1$ , then  $\nu' \in D$  and

$$
2(i - a - j - 2) + (b + 2a - 2i + 2) = b - 2j - 2 \le b - 2 \le p - 3,
$$

so  $\nu' \in \overline{C_1}$  and we are done.

Finally, suppose  $b + 2a - 2i + 2 < -1$ . Then we have

$$
s_{\alpha_2} \bullet (i - a - j - 2, b + 2a - 2i + 2) = (b + a - i - j + 1, 2i - b - 2a - 4) =: \eta.
$$

By assumption, we have

$$
2i - b - 2a - 4 = -(b + 2a - 2i + 2) - 2 > 1 - 2 = -1
$$

and

$$
b + a - i - j + 1 = (b - i) - j + a + 1 \ge a + 1 > 0,
$$

hence  $\eta \in D$ . Moreover, we have

$$
2(b+a-i-j+1) + (2i - b - 2a - 4) = b - 2j - 2 \le p - 3
$$

so  $\eta \in \overline{C_1}$ . Therefore in all cases we are done.

**Proposition 6.23.** *Let*  $\lambda = (p-2, 0), \mu = (0, d) \in X^+$  *be p-restricted with*  $d > 0$ *. Then*  $L(\lambda) \otimes L(\mu)$  *is multiplicity-free if and only if*  $d = 1$ *.* 

*Proof.* We set  $M := L(\lambda) \otimes L(\mu)$ . Observe that  $\lambda \in F_{2,3}$ , so  $\Delta(\lambda) \cong L(\lambda)$  by Lemma [6.4.](#page-49-0)

Suppose that  $d = 1$ . Using Corollary [1.48](#page-14-2) and the structure of  $L(0,1)$ , we have

ch 
$$
M = \sum_{\nu \in X} m_{L(\mu)}(\nu) \chi(\lambda + \nu)
$$
  
=  $\chi(p-2, 1) + \chi(p-3, 1) + \chi(p-2, -1) + \chi(p-1, -1)$ .

Observe that  $(p-2, 1) \in F_{3,4}$  and  $(p-3, 1) \in F_{2,3}$  so  $\chi(p-2, 1) = \text{ch } L(p-2, 1)$  and  $\chi(p-3,1) = \text{ch } L(p-3,1)$  by Lemma [6.4.](#page-49-0) Moreover,  $(p-2,-1), (p-1,-1) \in D \setminus X^+$  so  $\chi(p-2,-1) = \chi(p-1,-1) = 0$  by Lemma [1.45.](#page-13-0) Thus we get

$$
ch M = ch L(p-2, 1) + ch L(p-3, 1).
$$

In particular, *M* is multiplicity-free.

Now suppose that  $d > 1$ . In particular,  $\lambda + \mu \in C_4 \cup (F_{4,7} \setminus F_{4,6})$ . By Lemma [6.4,](#page-49-0) both  $L(\lambda)$  and  $L(\mu)$  are tilting modules, so M is a tilting module. Since  $L(\lambda + \mu)$  is a composition factor of *M* but  $T(\lambda + \mu)$  is not irreducible, we conclude by Lemma [1.45](#page-13-0) that *M* has multiplicity.  $\Box$ 

<span id="page-67-0"></span>**Proposition 6.24.** *Let*  $\lambda = (p-1,0), \mu = (0,d) \in X^+$  *be p-restricted with*  $d > 0$ *. Then*  $L(\lambda) \otimes L(\mu)$  *is multiplicity-free if and only if*  $d = 1$ *.* 

*Proof.* We set  $M := L(\lambda) \otimes L(\mu)$ . Observe that  $\lambda \in F_{4,6}$ , so  $\Delta(\lambda) \cong L(\lambda)$  by Lemma [6.4.](#page-49-0) Moreover,  $\lambda + \mu \in F_{4,6}$ .

Suppose that  $d = 1$ . Using Corollary [1.48](#page-14-2) and the structure of  $L(0,1)$ , we get

ch 
$$
M = \sum_{\nu \in X} m_{L(\mu)}(\nu) \chi(\lambda + \nu)
$$
  
=  $\chi(p-1, 1) + \chi(p-2, 1) + \chi(p-1, -1) + \chi(p, -1)$ .

Observe that  $(p-1,1) \in F_{4,6}$  and  $(p-2,1) \in F_{3,4}$  so  $\chi(p-1,1) = \text{ch } L(p-1,1)$  and  $\chi(p-2,1) = \text{ch } L(p-2,1)$  by Lemma [6.4.](#page-49-0) Moreover,  $(p-1,-1), (p,-1) \in D \setminus X^+$  so  $\chi(p-1,-1) = \chi(p,-1) = 0$  by Lemma [1.45.](#page-13-0) Thus we get

$$
ch M = ch L(p-1, 1) + ch L(p-2, 1).
$$

In particular, *M* is multiplicity-free.

Now suppose that  $d > 1$ . By Lemma [6.4,](#page-49-0) both  $L(\lambda)$  and  $L(\mu)$  are tilting modules, so M is a tilting module. We use Argument [1](#page-16-0) to show that  $L(\lambda + \mu - \alpha_1 - \alpha_2)$  is a composition factor of *M*. Using Lemmas [6.12,](#page-52-1) [6.15,](#page-54-2) [1.15](#page-10-0) and [6.6,](#page-50-1) we have

$$
m_{L(\lambda)}(\lambda - \alpha_1) = 1, \t m_{L(\lambda)}(\lambda - \alpha_1 - \alpha_2) = 1,
$$
  
\n
$$
m_{L(\mu)}(\mu - \alpha_2) = 1, \t m_{L(\lambda + \mu)}(\lambda + \mu - \alpha_1) = 1, \t m_{L(\lambda + \mu)}(\lambda + \mu - \alpha_1 - \alpha_2) = 2.
$$
  
\n
$$
m_{L(\lambda + \mu)}(\lambda + \mu - \alpha_1 - \alpha_2) = 2.
$$
  
\n
$$
m_{L(\lambda + \mu)}(\lambda + \mu - \alpha_1 - \alpha_2) = 2.
$$

Therefore  $m_M(\lambda + \mu - \alpha_1) = m_M(\lambda + \mu - \alpha_2) = 1$  and  $L(\lambda + \mu - \alpha_1)$ ,  $L(\lambda + \mu - \alpha_2)$  are not composition factors of *M*. Moreover,  $m_M(\lambda + \mu - \alpha_1 - \alpha_2) = 3$ , thus  $L(\lambda + \mu - \alpha_1 - \alpha_2)$  is a composition factor of *M*. Observe that  $\lambda + \mu - \alpha_1 - \alpha_2 = (p-2, d) \in C_4$ . By Lemma [6.4,](#page-49-0)  $T(\lambda + \mu - \alpha_1 - \alpha_2)$  is not irreducible and by Lemma [1.36,](#page-12-2) *M* has multiplicity.  $\Box$ 

**Proposition 6.25.** *Let*  $\lambda = (a, 0), \mu = (0, d) \in X^+$  *with*  $2a + d \leq p - 3$  *(i.e.*  $\lambda + \mu \in \widehat{C_1}$ *). Then*  $L(\lambda) \otimes L(\mu)$  *is multiplicity-free.* 

*Proof.* We apply Corollary [3.4](#page-19-1) and Theorem [6.16](#page-55-1) to conclude that  $L(\lambda) \otimes L(\mu)$  is multiplicityfree. 口

**Proposition 6.26.** *Let*  $\lambda = (a, 0), \mu = (0, d) \in X^+$  *with*  $0 < a \leq \frac{p-3}{2}$  $\frac{-3}{2}$ , 0 <  $d \leq p-3$  *and*  $2a + d > p - 3$  *(i.e.*  $\lambda, \mu \in \widehat{C_1}$  *and*  $\lambda + \mu \notin \widehat{C_1}$ *). Then*  $L(\lambda) \otimes L(\mu)$  *is multiplicity-free if and only if*  $(a, d) = (1, p - 3)$ *.* 

*Proof.* We set  $M := L(\lambda) \otimes L(\mu)$ . Observe that  $\lambda + \mu \in C_2 \cup C_3 \cup F_{2,3}$ . Moreover,  $L(\lambda)$  and  $L(\mu)$  are tilting modules, so *M* is a tilting module.

Suppose that  $a + d \neq p - 2$  (i.e.  $\lambda + \mu \notin F_{2,3}$ , so  $\lambda + \mu \in C_2 \cup C_3$ ). The tilting module  $T(\lambda + \mu)$  is thus not irreducible, and since  $L(\lambda + \mu)$  is a composition factor of *M*, we conclude by Lemma [1.36](#page-12-2) that *M* has multiplicity.

Suppose that  $a + d = p - 2$  and  $a \neq 1$ . Using the same argument as in the proof of Proposition [6.24,](#page-67-0) we get that  $L(\lambda + \mu - \alpha_1 - \alpha_2)$  is a composition factor of *M*. Observe that  $\lambda + \mu - \alpha_1 - \alpha_2 = (a - 1, d) \in C_2$  so  $T(\lambda + \mu - \alpha_1 - \alpha_2)$  is not irreducible. We conclude by Lemma [1.36](#page-12-2) that *M* has multiplicity.

Finally, suppose that  $(a, d) = (1, p - 3)$ . Using Proposition [1.47](#page-14-0) and the structure of  $L(1,0)$ , we get

ch 
$$
M = \sum_{\nu \in X} m_{L(1,0)}(\nu) \chi(\mu + \nu)
$$
  
=  $\chi(1, p - 3) + \chi(-1, p - 3) + \chi(-1, p - 1) + \chi(0, p - 3) + \chi(1, p - 5)$ 

Observe that  $(1, p − 3) ∈ F_{2,3}$  and  $(0, p − 3)$ ,  $(1, p − 5) ∈ F_{1,2}$ , so  $\chi(1, p − 3) = \text{ch } L(1, p − 3)$ ,  $\chi(0, p-3) = \text{ch } L(0, p-3)$  and  $\chi(1, p-5) = \text{ch } L(1, p-5)$  by Lemma [6.4.](#page-49-0) Moreover,  $(-1, p-3)$ ,  $(-1, p-1) \in D \setminus X^+$  so  $\chi(-1, p-3) = \chi(-1, p-1) = 0$  by Lemma [1.45.](#page-13-0) Thus we get

$$
ch M = ch L(1, p - 3) + ch L(0, p - 3) + ch L(1, p - 5)
$$

In particular, *M* is multiplicity-free.

**Proposition 6.27.** *Let*  $\lambda = (a, 0), \mu = (0, p - 2) \in X^+$  *with*  $0 < a \leq \frac{p-3}{2}$  $\frac{-3}{2}$  (*i.e.*  $\lambda \in C_1$  *and*  $\mu \in F_{2,3}$ ). Then  $L(\lambda) \otimes L(\mu)$  has multiplicity.

*Proof.* By Lemma [6.4,](#page-49-0) both  $L(\lambda)$  and  $L(\mu)$  are tilting modules, thus  $L(\lambda) \otimes L(\mu)$  is a tilting module. Moreover,  $\lambda + \mu \in C_3$ , hence  $T(\lambda + \mu)$  is not irreducible. Since  $L(\lambda + \mu)$  is a composition factor of  $L(\lambda) \otimes L(\mu)$ , we conclude by Lemma [1.36.](#page-12-2) П

**Proposition 6.28.** *Let*  $\lambda = (a, 0), \mu = (0, p - 1) \in X^+$  *with*  $0 < a \leq \frac{p-3}{2}$  $\frac{-3}{2}$  (*i.e.*  $\lambda \in C_1$  *and*  $\mu \in F_{3,5}$ ). Then  $L(\lambda) \otimes L(\mu)$  is multiplicity-free if and only if  $a = 1$ .

*Proof.* We set  $M := L(\lambda) \otimes L(\mu)$ . Observe that  $L(\lambda)$  and  $L(\mu)$  are tilting modules, thus M is a tilting module.

Suppose that  $a = 1$ . Using Corollary [1.48](#page-14-2) and the structure of  $L(1,0)$ , we have

ch 
$$
M = \sum_{\nu \in X} m_{L(1,0)}(\nu) \chi(\mu + \nu)
$$
  
=  $\chi(1, p - 1) + \chi(-1, p - 1) + \chi(-1, p + 1) + \chi(0, p - 1) + \chi(1, p - 3).$ 

Observe that  $(1, p-1)$ ,  $(0, p-1) \in F_{3,5}$  and  $(1, p-3) \in F_{2,3}$ , so  $\chi(1, p-1) = \text{ch } L(1, p-1)$ ,  $\chi(0, p - 1) = \text{ch } L(0, p - 1)$  and  $\chi(1, p - 3) = \text{ch } L(1, p - 3)$  by Lemma [6.4.](#page-49-0) Moreover, (−1*, p* − 1)*,*(−1*, p* + 1) ∈ *D* \ *X*<sup>+</sup> so *χ*(−1*, p* − 1) = *χ*(−1*, p* + 1) = 0 by Lemma [1.45.](#page-13-0) Thus we get

$$
ch M = ch L(1, p - 1) + ch L(0, p - 1) + ch L(1, p - 3).
$$

In particular, *M* is multiplicity-free.

Now suppose  $a > 1$ . By Lemma [6.4,](#page-49-0) we have  $L(\lambda) \cong \Delta(\lambda)$ ,  $L(\mu) \cong \Delta(\mu)$  and  $L(\lambda + \mu) \cong \Delta(\lambda + \mu)$ . We use Argument [1](#page-16-0) to show that  $L(\lambda + \mu - \alpha_1 - 2\alpha_2)$  is a composition factor of *M*. By Lemma [6.7,](#page-51-0) we have

 $m_{L(\lambda)}(\lambda - \alpha_1) = 1,$   $m_{L(\lambda)}$  $m_{L(\lambda)}(\lambda - \alpha_1 - \alpha_2) = 1$  $m_{L(\lambda)}(\lambda - \alpha_1 - 2\alpha_2) = 1,$   $m_{L(\mu)}$  $m_{L(\mu)}(\mu - \alpha_2) = 1,$  $m_{L(\mu)}(\mu - \alpha_1 - \alpha_2) = 1,$   $m_{L(\mu)}$  $m_{L(\mu)}(\mu - \alpha_1 - 2\alpha_2) = 2,$  $m_{L(\mu)}(\mu - 2\alpha_2) = 1,$   $m_{L(\lambda + \mu)}$  $m_{L(\lambda+\mu)}(\lambda+\mu-\alpha_1)=1,$  $m_{L(\lambda+\mu)}(\lambda+\mu-\alpha_2)=1,$   $m_{L(\lambda+\mu)}$  $m_{L(\lambda+\mu)}(\lambda+\mu-\alpha_1-\alpha_2)=2,$  $m_{L(\lambda+\mu)}(\lambda+\mu-2\alpha_2)=1,$   $m_{L(\lambda+\mu)}$  $m_{L(\lambda+\mu)}(\lambda+\mu-\alpha_1-2\alpha_2)=3,$  $m_{L(\lambda + \mu - \alpha_1 - \alpha_2)}(\lambda + \mu - \alpha_1 - 2\alpha_2) = 1.$ 

Therefore,  $m_M(\lambda + \mu - \alpha_1) = m_M(\lambda + \mu - \alpha_2) = m_M(\lambda + \mu - 2\alpha_2) = 1$ ,  $m_M(\lambda + \mu - \alpha_1 - \alpha_2) = 3$ and  $m_M(\lambda + \mu - \alpha_1 - 2\alpha_2) = 5$ . We deduce that  $L(\lambda + \mu - \alpha_1 - 2\alpha_2)$  is a composition factor of *M*. Observe that  $\lambda + \mu - \alpha_1 - 2\alpha_2 = (a, p - 3) \in C_3$ , thus  $T(\lambda + \mu - \alpha_1 - 2\alpha_2)$  is not irreducible. We conclude by Lemma [1.36](#page-12-2) that *M* has multiplicity.  $\Box$ 

$$
\Box
$$

**Proposition 6.29.** *Let*  $\lambda = (a, 0), \mu = (0, b) \in X^+$  *be p*-restricted with  $\frac{p-1}{2} \le a \le p-3$  *and*  $a + b \leq p - 2$  *(i.e.*  $\lambda \in C_2$  *and*  $\lambda + \mu \in \widehat{C_2}$ *). Then*  $L(\lambda) \otimes L(\mu)$  *is multiplicity-free.* 

*Proof.* In this proof, we use Euclidean coordinates. We set  $M := L(\lambda) \otimes L(\mu)$ .

ch  $M = \chi(b\omega_2)(\chi(a, 0) - \chi(p - 3 - a, 0))$ 

−

For  $i \in \{0, \ldots, b\}$ , we have  $p - 2 - 2a + i \leq i$ . By Proposition [1.47](#page-14-0) and Lemma [6.4,](#page-49-0) and using Lemma [6.22](#page-63-0) in the second equality below, we get

$$
\begin{split}\n&= \sum_{i=0}^{b} \sum_{j=0}^{\min\{i,b-i\}} \chi(a+\frac{b}{2}-i,\frac{b}{2}-j) - \sum_{i=0}^{b} \sum_{j=0}^{\min\{i,b-i\}} \chi(p-3-a+\frac{b}{2}-i,\frac{b}{2}-j) \\
&= \sum_{i=0}^{b} \sum_{j=\max\{0,p-2-2a+i\}}^{\min\{i,b-i\}} \chi(a+\frac{b}{2}-i,\frac{b}{2}-j) \\
&+ \sum_{i=0}^{b} \sum_{j=0}^{\min\{b-i,p-3-2a+i\}} \chi(a+\frac{b}{2}-i,\frac{b}{2}-j) \\
&- \sum_{i=0}^{b} \sum_{j=\max\{0,b-i+p-2-2a\}}^{\min\{i,b-i\}} \chi(p-3-a+\frac{b}{2}-i,\frac{b}{2}-j) \\
&- \sum_{i=0}^{b} \sum_{j=0}^{\min\{i,b-i+p-3-2a\}} \chi(p-3-a+\frac{b}{2}-i,\frac{b}{2}-j) \\
&= \sum_{i=0}^{b} \sum_{j=0}^{\min\{i,b-i\}} \chi(p-3-a+\frac{b}{2}-i,\frac{b}{2}-j)\n\end{split}
$$

$$
=\sum_{i=0}^{b} \sum_{j=\max\{0,p-2-2a+i\}}^{\min\{i,b-1\}} \chi(a+\frac{b}{2}-i,\frac{b}{2}-j) \tag{49}
$$

<span id="page-70-3"></span><span id="page-70-2"></span>
$$
-\sum_{i=0}^{b} \sum_{j=\max\{0,b-i+p-2-2a\}}^{\min\{i,b-i\}} \chi(p-3-a+\frac{b}{2}-i,\frac{b}{2}-j) \tag{50}
$$

$$
+\sum_{i=0}^{b} \sum_{j=0}^{\min\{b-i, p-3-2a+i\}} \chi(a+\frac{b}{2}-i, \frac{b}{2}-j) \tag{51}
$$

<span id="page-70-1"></span><span id="page-70-0"></span>
$$
-\sum_{i=0}^{b} \sum_{j=0}^{\min\{i, b-i+p-3-2a\}} \chi(p-3-a+\frac{b}{2}-i, \frac{b}{2}-j). \tag{52}
$$

First, we work on the terms of lines [\(51\)](#page-70-0) and [\(52\)](#page-70-1). We claim that they sum to zero. To that end, observe that in line [\(51\)](#page-70-0), the second sum is empty whenever  $i < 2a+3-p$  and in line [\(52\)](#page-70-1), the second sum is empty whenever  $i > b + p - 3 - 2a$ . Thus we get

$$
\sum_{i=0}^{b} \sum_{j=0}^{\min\{b-i, p-3-2a+i\}} \chi(a+\frac{b}{2}-i, \frac{b}{2}-j) - \sum_{i=0}^{b} \sum_{j=0}^{\min\{i, b-i+p-3-2a\}} \chi(p-3-a+\frac{b}{2}-i, \frac{b}{2}-j)
$$
  

$$
= \sum_{i=2a+3-p}^{b} \sum_{j=0}^{\min\{b-i, p-3-2a+i\}} \chi(a+\frac{b}{2}-i, \frac{b}{2}-j)
$$
  

$$
- \sum_{i=0}^{b+p-3-2a} \sum_{j=0}^{\min\{i, b-i+p-3-2a\}} \chi(p-3-a+\frac{b}{2}-i, \frac{b}{2}-j)
$$

$$
= \sum_{i=2a+3-p}^{b} \sum_{j=0}^{\min\{b-i,p-3-2a+i\}} \chi(a+\frac{b}{2}-i,\frac{b}{2}-j)
$$

$$
-\sum_{i=2a+3-p}^{b} \sum_{j=0}^{\min\{i-2a-3+p,b-i\}} \chi(a+\frac{b}{2}-i,\frac{b}{2}-j)
$$

$$
= 0.
$$

Therefore, only lines [\(49\)](#page-70-2) and [\(50\)](#page-70-3) remain and we get

ch 
$$
M = \sum_{i=0}^{b} \sum_{j=\max\{0,p-2-2a+i\}}^{\min\{i,b-i\}} \chi(a+\frac{b}{2}-i,\frac{b}{2}-j)
$$
  
\n
$$
-\sum_{i=0}^{b} \sum_{j=\max\{0,b-i+p-2-2a\}}^{\min\{i,b-i\}} \chi(p-3-a+\frac{b}{2}-i,\frac{b}{2}-j)
$$
\n
$$
=\sum_{i=0}^{b} \sum_{j=\max\{0,p-2-2a+i\}}^{\min\{i,b-i\}} \chi(a+\frac{b}{2}-i,\frac{b}{2}-j)
$$
\n
$$
-\sum_{i=0}^{b} \sum_{j=\max\{0,i+p-2-2a\}}^{\min\{i,b-i\}} \chi(p-3-a-\frac{b}{2}+i,\frac{b}{2}-j)
$$
\n
$$
=\sum_{i=0}^{b} \sum_{j=\max\{0,i+p-2-2a+i\}}^{\min\{i,b-i\}} \chi(a+\frac{b}{2}-i,\frac{b}{2}-j) - \chi(p-3-a-\frac{b}{2}+i,\frac{b}{2}-j).
$$
\n(53)

At this step, observe that if  $i > \frac{b+2a+2-p}{2}$ , then  $p-2-2a+i > b-i$  and the second sum in line [\(53\)](#page-71-0) is empty. Thus we get

$$
\mathrm{ch}\, M=\sum_{i=0}^{\min\{b,\lfloor\frac{b+2a+2-p}{2}\rfloor\}}\sum_{j=\max\{0,p-2-2a+i\}}^{\min\{i,b-i\}}\chi(a+\frac{b}{2}-i,\frac{b}{2}-j)-\chi(p-3-(a+\frac{b}{2}-i),\frac{b}{2}-j).
$$

For  $0 \le i \le \min\{b, \lfloor \frac{b+2a+2-p}{2} \rfloor\}$  $\left\{\frac{+2-p}{2}\right\}$  and  $\max\{0, p-2-2a+i\} \leq j \leq \min\{i, b-i\}$ , we have

<span id="page-71-0"></span>
$$
a + \frac{b}{2} - i \ge a + \frac{b}{2} - (\frac{b + 2a + 2 - p}{2}) = \frac{p - 2}{2} > \frac{p - 3}{2},
$$
  

$$
\frac{b}{2} - j \ge 0 \qquad \text{and}
$$
  

$$
(a + \frac{b}{2} - i) + (\frac{b}{2} - j) = a + b - i - j \le a + b \le p - 2.
$$

Therefore,  $(a + \frac{b}{2} - i, \frac{b}{2} - j) \in \widehat{C}_2$  for all  $0 \le i \le \min\{b, \lfloor\frac{b+2a+2-p}{2}\rfloor\}$  $\binom{+2-p}{2}$  and  $\max\{0, p - 2 - 2a + i\} \leq j \leq \min\{i, b - i\}$  and by Remark [6.5,](#page-50-0) we get

$$
\operatorname{ch} M = \sum_{i=0}^{\min\{b, \lfloor \frac{b+2a+2-p}{2} \rfloor\}} \sum_{j=\max\{0, p-2-2a+i\}}^{\min\{i, b-i\}} \operatorname{ch} L(a+\frac{b}{2}-i, \frac{b}{2}-j).
$$

In particular, *M* is multiplicity-free.
**Proposition 6.30.** *Let*  $\lambda = (a, 0), \mu = (0, b) \in X^+$  *be p*-restricted with  $\frac{p-1}{2} \le a \le p-3$  *and*  $a + b > p - 2$  *(i.e.*  $\lambda \in C_2$  *and*  $\lambda + \mu \in \widehat{C_3} \cup \widehat{C_4}$ *). Then*  $L(\lambda) \otimes L(\mu)$  *has multiplicity.* 

*Proof.* In this proof, we use Euclidean coordinates. We set  $M := L(\lambda) \otimes L(\mu)$ . By Proposition [1.47,](#page-14-0) Lemma [6.22](#page-63-0) and Remark [6.5,](#page-50-0) we have

ch  $M = \chi(\lambda)\chi(\mu) - \chi(s_{\alpha_1+\alpha_2,p})$ *λ*)*χ*(*µ*)

$$
= \sum_{i=0}^{b} \sum_{j=0}^{\min\{i, b-i\}} \chi(a + \frac{b}{2} - i, \frac{b}{2} - j) - \sum_{i=0}^{b} \sum_{j=0}^{\min\{i, b-i\}} \chi(p - 3 - a + \frac{b}{2} - i, \frac{b}{2} - j). \tag{54}
$$

We set

<span id="page-72-0"></span>
$$
A := \{ (a + \frac{b}{2} - i, \frac{b}{2} - j) | 0 \le i \le b, 0 \le j \le \min\{i, b - i\} \}
$$

and

$$
B := \{ (p-3-a+\frac{b}{2}-i, \frac{b}{2}-j) | 0 \le i \le b, 0 \le j \le \min\{i, b-i\} \}.
$$

By Lemma [6.22,](#page-63-0) we have  $A, B \subseteq D \cup W$  • *C*1.

If  $a + b$  is even, we set  $t := \frac{a+b-p+1}{2}$ If  $a + b$  is even, we set  $t := \frac{a+b-p+1}{2}$ ,  $\nu_3 := (a + \frac{b}{2} - t, \frac{b}{2} - t) \in C_3$  and  $\nu_2 := s_{\alpha_1+2\alpha_2,p} \bullet \nu_3 = (a + \frac{b}{2} - (t+1), \frac{b}{2} - (t+1)) \in C_2$ .  $\nu_3 = (a + \frac{b}{2} - (t+1), \frac{b}{2} - (t+1)) \in C_2.$ 

If  $a + b$  is odd, we set  $t := \frac{a+b-p}{2}$ If  $a + b$  is odd, we set  $t := \frac{a+b-p}{2}$ ,  $\nu_3 := (a + \frac{b}{2} - (t + 1), \frac{b}{2} - t) \in C_3$  and  $\nu_2 := s_{\alpha_1+2\alpha_2,p} \cdot \nu_3 = (a + \frac{b}{2} - (t + 2), \frac{b}{2} - (t + 1)) \in C_2$ .  $\nu_3 = (a + \frac{b}{2} - (t + 2), \frac{b}{2} - (t + 1)) \in C_2.$ 

In both cases, we show that  $\nu_2$  has multiplicity 2 in *M*. Observe that  $\nu_2, \nu_3 \in A$  (this follows from  $t+1 \leq \frac{b}{2}$  $\frac{b}{2}$ ). Moreover, we claim that  $\nu_2, \nu_3 \notin B$ . We check  $\nu_3$  in case  $a + b$  even, the other cases are similar. Suppose that  $(a + \frac{b}{2} - t, \frac{b}{2} - t) = (p - 3 - a + \frac{b}{2} - k, \frac{b}{2} - r)$  for some  $k, r$ . Then  $r = t$  and  $k = \frac{p+b-5-3a}{2} < r$ , thus  $\nu_3 \notin B$ .

For every weight  $\nu \in X$ , we fix  $w_{\nu} \in W$  such that  $w_{\nu} \in \mathbb{R}$  amental domain for the dot action of  $W$  on  $X$ ). Moreover *ν* ∈ *D* (recall that *D* is a fundamental domain for the dot action of *W* on *X*). Moreover, we take the convention that  $\Delta(\nu) = 0$  for every  $\nu \in D \setminus X^+$ . Using line [\(54\)](#page-72-0) and Lemma [1.45,](#page-13-0) we have

$$
[M:L(\nu_2)] = \sum_{\eta \in A} \det(w_{\eta})[\Delta(w_{\eta} \bullet \eta) : L(\nu_2)] - \sum_{\eta \in B} \det(w_{\eta})[\Delta(w_{\eta} \bullet \eta) : L(\nu_2)].
$$

By the Strong Linkage Principle (Proposition [1.53\)](#page-15-0),  $[\Delta(\eta): L(\nu_2)] = 0$  unless  $\nu_2 \uparrow \eta$ . By Lemma [6.4,](#page-49-0) we have  $[\Delta(\eta): L(\nu_2)]=0$  for all  $\eta \in C_4$ . Thus, if  $\eta$  is *p*-restricted and such that  $[\Delta(\eta): L(\nu_2)] \neq 0$ , we have either  $\eta = \nu_2$  or  $\eta = \nu_3$ . At this step, recall that every *n* ∈ (*A*∪*B*)∩*X*<sup>+</sup> is *p*-restricted. By Lemma [6.22,](#page-63-0) if  $\eta \in A \cup B$  and  $w_{\eta} \neq id$ , then  $w_{\eta} \bullet$  and  $[A(w_{\eta}, \bullet n) : u_0] = 0$ . Therefore, we have *η* ∈ *C*<sup>1</sup>  $\eta \in (A \cup B)^{\dagger}$ <br>and  $[\Delta(w_{\eta} \bullet$  $\eta$ ) :  $\nu_2$ ] = 0. Therefore, we have

$$
[M:L(\nu_2)] = \sum_{\eta \in A \cap \{\nu_2, \nu_3\}} [\Delta(\eta) : L(\nu_2)] - \sum_{\eta \in B \cap \{\nu_2, \nu_3\}} [\Delta(\eta) : L(\nu_2)].
$$

By the previous observations,  $B \cap {\nu_2, \nu_3} = \emptyset$  and  $A \cap {\nu_2, \nu_3} = {\nu_2, \nu_3}$ . Therefore,

$$
[M:L(\nu_2)] = [\Delta(\nu_2): L(\nu_2)] + [\Delta(\nu_3): L(\nu_2)] = 2
$$

and *M* has multiplicity.

 $\Box$ 

**6.3.4**  $L(a, b) \otimes L(0, d)$ 

**Proposition 6.31.** *Let*  $\lambda = (a, b), \mu = (0, d) \in X^+$  *with*  $1 \le a < p$  *and*  $2 \le b, d < p$ *. If*  $2a + b + 2 \not\equiv 0 \mod p$  and  $a + b \not\equiv p - 1$ , then  $L(\lambda) \otimes L(\mu)$  has multiplicity.

*Proof.* We set  $M := L(\lambda) \otimes L(\mu)$ . Using Argument [1,](#page-16-0) we show that either  $[M: L(\lambda + \mu - \alpha_1 - \alpha_2)] \geq 2$  or  $[M: L(\lambda + \mu - \alpha_1 - 2\alpha_2)] \geq 2$ . Using Lemmas [1.15,](#page-10-0) [6.6,](#page-50-1) [6.7](#page-51-0) and [6.8,](#page-51-1) we have

 $m_M(\lambda + \mu) = 1,$   $m_M(\lambda + \mu - \alpha_1) = 1,$   $m_M(\lambda + \mu - \alpha_2) = 2,$  $m_M(\lambda + \mu - \alpha_1 - \alpha_2) = 4, \qquad m_M(\lambda + \mu - 2\alpha_2) = 3, \qquad m_M(\lambda + \mu - \alpha_1 - 2\alpha_2) = 9.$ 

Therefore

$$
[M:L(\lambda + \mu)] = [M:L(\lambda + \mu - \alpha_2)] = [M:L(\lambda + \mu - 2\alpha_2)] = 1
$$

and

$$
[M:L(\lambda+\mu-\alpha_1)]=0.
$$

If  $m_{L(\lambda+\mu)}(\lambda+\mu-\alpha_1-\alpha_2)=1$ , then  $[M:L(\lambda+\mu-\alpha_1-\alpha_2)]=2$  and M has multiplicity. If  $m_{L(\lambda+\mu)}(\lambda+\mu-\alpha_1-\alpha_2)=2$ , then  $[M:L(\lambda+\mu-\alpha_1-\alpha_2)]=1$ . In this case, we have

$$
m_{L(\lambda+\mu)}(\lambda+\mu-\alpha_1-2\alpha_2) \leq 3
$$
 and  $m_{L(\lambda+\mu-\alpha_2)}(\lambda+\mu-\alpha_1-2\alpha_2) \leq 2$ .

Therefore,  $[M: L(\lambda + \mu - \alpha_1 - 2\alpha_2)] \geq 9 - 3 - 2 - 1 - 1 = 2$  and M has multiplicity.  $\Box$ 

**Proposition 6.32.** *Let*  $\lambda = (a, b), \mu = (0, 1) \in X^+$  *with*  $1 \leq a, b < p$ *. Then*  $L(\lambda) \otimes L(\mu)$  *is multiplicity-free if and only if*  $\lambda \in C_1 \cup C_2 \cup C_3 \cup C_4$ .

*Proof.* In this proof, we take the convention that ch  $L(\nu) = 0$  for all  $\nu \notin X^+$ . Moreover, for  $\nu = (c, d) \in X$ , we set

$$
\delta_2(\nu) := \begin{cases}\n0 & \text{if } 2c + d + 2 = p, \\
1 & \text{else.}\n\end{cases}
$$
\n $\delta_3(\nu) := \begin{cases}\n0 & \text{if } c + d = p - 1, \\
1 & \text{else.}\n\end{cases}$ 

$$
\delta_4(\nu) := \begin{cases} 0 & \text{if } 2c + d + 2 = 2p, \\ 1 & \text{else.} \end{cases}
$$

We set  $M := L(\lambda) \otimes L(\mu)$ . By Lemma [6.4,](#page-49-0)  $L(\mu)$  is a tilting module.

- If  $\lambda \in F_{1,2} \cup F_{2,3} \cup F_{3,4}$  then  $L(\lambda)$  is a tilting module so M is a tilting module. In this case  $\lambda + \mu \in C_2 \cup C_3 \cup C_4 \cup (F_{4,7} \backslash F_{4,6})$ , hence  $T(\lambda + \mu)$  is not irreducible by Lemma [6.4.](#page-49-0) We conclude by Lemma [1.36](#page-12-0) that *M* has multiplicity.
- If  $\lambda \in F_{3,5} \cup F_{4,7}$ , then  $b = p-1$  so  $\lambda + \mu$  is not *p*-restricted. We conclude by Corollary [2.8](#page-17-0) that *M* has multiplicity.
- If  $\lambda \in F_{4,6}$  (i.e.  $a = p-1$ ), observe that  $L(\lambda + \mu \alpha_2)$  is a composition factor of M (by Argument [1\)](#page-16-0). But  $\lambda + \mu - \alpha_2$  is not *p*-restricted, hence we conclude by Corollary [2.7](#page-17-1) that *M* has multiplicity.
- If  $\lambda \in C_1$ , then  $\lambda + \mu \in \widehat{C_1}$  and we apply Corollary [3.4](#page-19-0) and Theorem [6.16](#page-55-0) to conclude that *M* is multiplicity-free.

• If  $\lambda \in C_2$ , let  $\lambda_1 := s_{\alpha_1 + \alpha_2, p}$  • <br>Lemma 6.4 we have  $\lambda \in C_1$  and  $\delta_2 := \delta_2(\lambda)$ . By Proposition [1.47](#page-14-0) and Lemma [6.4](#page-49-0) we have

ch 
$$
M = \chi(\mu)(\chi(\lambda) - \chi(\lambda_1))
$$
  
\n
$$
= \chi(\lambda + \mu) + \chi(\lambda + \mu - \alpha_2) + \chi(\lambda + \mu - \alpha_1 - \alpha_2) + \chi(\lambda + \mu - \alpha_1 - 2\alpha_2)
$$
\n
$$
- \chi(\lambda_1 + \mu) - \chi(\lambda_1 + \mu - \alpha_2) - \chi(\lambda_1 + \mu - \alpha_1 - \alpha_2)
$$
\n
$$
- \chi(\lambda_1 + \mu - \alpha_1 - 2\alpha_2)
$$
\n
$$
= (\chi(\lambda + \mu) - \chi(\lambda_1 + \mu - \alpha_1 - \alpha_2)) + (\chi(\lambda + \mu - \alpha_2) - \chi(\lambda_1 + \mu - \alpha_1 - 2\alpha_2))
$$
\n
$$
+ (\chi(\lambda + \mu - \alpha_1 - \alpha_2) - \chi(\lambda_1 + \mu))
$$
\n
$$
+ (\chi(\lambda + \mu - \alpha_1 - 2\alpha_2) - \chi(\lambda_1 + \mu - \alpha_2))
$$
\n
$$
= ch L(\lambda + \mu) + ch L(\lambda + \mu - \alpha_2) + \delta_2 \cdot ch L(\lambda + \mu - \alpha_1 - \alpha_2)
$$
\n
$$
+ \delta_2 \cdot ch L(\lambda + \mu - \alpha_1 - 2\alpha_2).
$$

Thus *M* is multiplicity-free.

• If  $\lambda \in C_3$ , let  $\lambda_2 := s_{\alpha_1 + 2\alpha_2, p}$ <br>1.47 Lemma 6.4 and the previo  $\lambda \in C_2$ ,  $\delta_2 := \delta_2(\lambda_2)$  and  $\delta_3 := \delta_3(\lambda)$ . By Proposition [1.47,](#page-14-0) Lemma [6.4](#page-49-0) and the previous case, we have

ch 
$$
M = \chi(\mu)(\chi(\lambda) - \text{ch } L(\lambda_2))
$$
  
\n
$$
= \chi(\lambda + \mu) + \chi(\lambda + \mu - \alpha_2) + \chi(\lambda + \mu - \alpha_1 - \alpha_2) + \chi(\lambda + \mu - \alpha_1 - 2\alpha_2)
$$
\n
$$
- \text{ch } L(\lambda_2 + \mu) - \text{ch } L(\lambda_2 + \mu - \alpha_2) - \delta_2 \cdot \text{ch } L(\lambda_2 + \mu - \alpha_1 - \alpha_2)
$$
\n
$$
- \delta_2 \cdot \text{ch } L(\lambda_2 + \mu - \alpha_1 - 2\alpha_2)
$$
\n
$$
= (\chi(\lambda + \mu) - \delta_2 \cdot \text{ch } L(\lambda_2 + \mu - \alpha_1 - 2\alpha_2))
$$
\n
$$
+ (\chi(\lambda + \mu - \alpha_2) - \text{ch } L(\lambda_2 + \mu - \alpha_2))
$$
\n
$$
+ (\chi(\lambda + \mu - \alpha_1 - \alpha_2) - \delta_2 \cdot \text{ch } L(\lambda_2 + \mu - \alpha_1 - \alpha_2))
$$
\n
$$
+ (\chi(\lambda + \mu - \alpha_1 - 2\alpha_2) - \text{ch } L(\lambda_2 + \mu))
$$
\n
$$
= \text{ch } L(\lambda + \mu) + \text{ch } L(\lambda + \mu - \alpha_2) + \text{ch } L(\lambda + \mu - \alpha_1 - \alpha_2)
$$
\n
$$
+ \delta_3 \cdot \text{ch } L(\lambda + \mu - \alpha_1 - 2\alpha_2).
$$

Thus *M* is multiplicity-free.

• If  $\lambda \in C_4$ , let  $\lambda_3 := s_{\alpha_1 + \alpha_2, 2p}$  • 1.47 Lemma 6.4 and the previo  $\lambda \in C_3$ ,  $\delta_3 := \delta_3(\lambda_3)$  and  $\delta_4 := \delta_4(\lambda)$ . By Proposition [1.47,](#page-14-0) Lemma [6.4](#page-49-0) and the previous case, we have

ch 
$$
M = \chi(\mu)(\chi(\lambda) - \text{ch } L(\lambda_3))
$$
  
\n
$$
= \chi(\lambda + \mu) + \chi(\lambda + \mu - \alpha_2) + \chi(\lambda + \mu - \alpha_1 - \alpha_2) + \chi(\lambda + \mu - \alpha_1 - 2\alpha_2)
$$
\n
$$
- \text{ch } L(\lambda_3 + \mu) - \text{ch } L(\lambda_3 + \mu - \alpha_2) - \text{ch } L(\lambda_3 + \mu - \alpha_1 - \alpha_2)
$$
\n
$$
- \delta_3 \cdot \text{ch } L(\lambda_3 + \mu - \alpha_1 - 2\alpha_2)
$$

$$
= (\chi(\lambda + \mu) - \text{ch } L(\lambda_3 + \mu - \alpha_1 - \alpha_2))
$$
  
+ (\chi(\lambda + \mu - \alpha\_2) - \delta\_3 \cdot \text{ch } L(\lambda\_3 + \mu - \alpha\_1 - 2\alpha\_2))  
+ (\chi(\lambda + \mu - \alpha\_1 - \alpha\_2) - \text{ch } L(\lambda\_3 + \mu))  
+ (\chi(\lambda + \mu - \alpha\_1 - 2\alpha\_2) - \text{ch } L(\lambda\_3 + \mu - \alpha\_2))  
= \text{ch } L(\lambda + \mu) + \text{ch } L(\lambda + \mu - \alpha\_2) + \delta\_4 \cdot \text{ch } L(\lambda + \mu - \alpha\_1 - \alpha\_2)   
+ \delta\_4 \cdot \text{ch } L(\lambda + \mu - \alpha\_1 - 2\alpha\_2).

Thus *M* is multiplicity-free.

The classification of multiplicity-free tensor products of simple modules with *p*-restricted highest weight for an algebraic group of type *B*<sup>2</sup> is not completed in this thesis. In the previous sequence of propositions, we fully treated the following cases:

- $\lambda = (a, b), \mu = (c, d) \text{ with } a \cdot b = 0 \text{ and } c \cdot d = 0,$
- $\lambda = (a, b), \mu = (0, 1).$

It remains to consider the following cases:

- $\lambda = (a, b), \mu = (0, d)$  with  $a \neq 0, b \neq 0$  and  $d \geq 2$ ,
- $\lambda = (a, b), \mu = (c, 0)$  with  $a \neq 0, b \neq 0$  and  $c \neq 0$ ,
- $\lambda = (a, b), \mu = (c, d) \text{ with } a \cdot b \cdot c \cdot d \neq 0.$

 $\Box$ 

## **7**  $SL_n$  for  $p = 2$

In this section, we classify multiplicity-free tensor products of simple SL*n*-modules with *p*-restricted highest weight when  $p = 2$ . To that end, we will use the classification of completely reducible tensor products of simple SL*n*-modules with *p*-restricted highest weight for  $p = 2$  ([\[Gru21,](#page-77-0) Theorem 7.12]).

<span id="page-76-0"></span>**Theorem 7.1.** Let G be of type  $A_n$  and  $p = 2$ . Let  $\lambda, \mu \in X^+$  be nonzero 2-restricted *dominant weights. Up to the reordering of*  $\lambda$  *and*  $\mu$ ,  $L(\lambda) \otimes L(\mu)$  *is completely reducible if and only if one of the following holds:*

- (1)  $\lambda = \omega_1$  and  $\mu = \omega_{i_1} + \ldots + \omega_{i_r}$  for even numbers  $1 < i_1 < \ldots < i_r \leq n$ ,
- (2)  $\lambda = \omega_n$  and  $\mu = \omega_{i_1} + \ldots + \omega_{i_r}$  for certain  $i_1 < \ldots < i_r < n$  such that  $n + 1 i_j$  is *even for all*  $j \in \{1, \ldots, r\}$ ,
- (3)  $\lambda = \omega_2$  *and*  $\mu = \omega_j$  *for some*  $2 < j \le n$  *with*  $j 2 \equiv 3 \mod 4$ ,
- <span id="page-76-1"></span>(4)  $\lambda = \omega_{n-1}$  *and*  $\mu = \omega_j$  *for some*  $1 \leq j \leq n-1$  *with*  $n-1-j \equiv 3 \mod 4$ .

**Theorem 7.2** ([\[Ste03,](#page-78-0) Theorem 1.1.A]). Let *G* be of type  $A_n$ ,  $\lambda \in X^+$  and  $i \in \{1, \ldots, n\}$ . *Then*  $L_{\mathbb{C}}(\omega_i) \otimes L_{\mathbb{C}}(\lambda)$  *is multiplicity-free.* 

**Theorem 7.3.** Let G be of type  $A_n$  and  $p = 2$ . Let  $\lambda, \mu \in X^+$  be nonzero 2-restricted domi*nant weights. Up to the reordering of*  $\lambda$  *and*  $\mu$  *and up to duality,*  $L(\lambda) \otimes L(\mu)$  *is multiplicityfree if and only if one of the following holds:*

- (1)  $\lambda = \omega_1$  and  $\mu = \omega_{i_1} + \ldots + \omega_{i_r}$  for even numbers  $1 < i_1 < \ldots < i_r \leq n$ ,
- (2)  $\lambda = \omega_n$  and  $\mu = \omega_{i_1} + \ldots + \omega_{i_r}$  for certain  $i_1 < \ldots < i_r < n$  such that  $n + 1 i_j$  is *even for all*  $j \in \{1, \ldots, r\}$ ,
- (3)  $\lambda = \omega_2$  *and*  $\mu = \omega_j$  *for some*  $2 < j \leq n$  *with*  $j 2 \equiv 3 \mod 4$ ,
- (4)  $\lambda = \omega_{n-1}$  and  $\mu = \omega_j$  for some  $1 \leq j < n-1$  with  $n-1-j \equiv 3 \mod 4$ .

*Proof.* Suppose that  $\lambda$  and  $\mu$  do not satisfy the conditions. Then  $L(\lambda) \otimes L(\mu)$  is not completely reducible by Theorem [7.1,](#page-76-0) and hence it has multiplicity by Lemma [2.5.](#page-17-2)

Now suppose that  $\lambda$ ,  $\mu$  verify the condition of the theorem. By Theorem [7.1,](#page-76-0)  $L(\lambda) \otimes L(\mu)$  is completely reducible. Moreover,  $L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu)$  is multiplicity-free by Theorem [7.2.](#page-76-1) Therefore,  $L(\lambda) \otimes L(\mu)$  is multiplicity-free by Theorem [3.6.](#page-20-0)  $\Box$ 

## **References**

- [BCP97] Wieb Bosma, John Cannon, and Catherine Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997.
- [BDM10] C. Bowman, S. R. Doty, and S. Martin. Decomposition of tensor products of modular irreducible representations for  $SL_3$  (with an appendix by C. M. Ringel). *arXiv.org*, 2010.
- [BDM15] C. Bowman, S. R. Doty, and S. Martin. Decomposition of tensor products of modular irreducible representations for  $SL_3$ : The  $p \geq 5$  case. *International electronic journal of algebra*, 17(17):105–105, 2015.
- [Bre85] M.R. Bremmer. *Tables of dominant weight multiplicities for representations of simple Lie algebras*. Pure and applied mathematics. Monographs and textbooks in pure and applied mathematics 90. M. Dekker, New York, 1985.
- [Car89] Roger W. Carter. *Simple groups of Lie type*. Wiley classics library. Wiley, London, 1989.
- [Cav17] Mikaël Cavallin. An algorithm for computing weight multiplicities in irreducible modules for complex semisimple Lie algebras. *Journal of algebra*, 471:492–510, 2017.
- [DH05] Stephen Doty and Anne Henke. Decomposition of tensor products of modular irreducibles for SL2. *Quarterly journal of mathematics*, 56(2):189–207, 2005.
- [Dot85] Stephen R. Doty. The submodule structure of certain Weyl modules for groups of type *An*. *Journal of algebra*, 95(2):373–383, 1985.
- [Erd18] Karin. Erdmann. *Algebras and Representation Theory*. Springer Undergraduate Mathematics Series. Springer International Publishing, Cham, 1st ed. 2018. edition, 2018.
- [FH04] William Fulton and Joe Harris. *Representation Theory: A First Course*, volume 129 of *Readings in Mathematics*. Springer New York, New York, NY, 2004.
- <span id="page-77-0"></span>[Gru21] Jonathan Gruber. On complete reducibility of tensor products of simple modules over simple algebraic groups. *Transactions of the American Mathematical Society. Series B*, 8(8):249–276, 2021.
- [Gru22] Jonathan Gruber. Generic direct summands of tensor products for simple algebraic groups and quantum groups at roots of unity, 2022.
- [Hum75] James E. Humphreys. *Linear algebraic groups*. Graduate texts in mathematics 21. Springer, New York, 1975.
- [Hum90] James E. Humphreys. *Reflection groups and Coxeter groups*. Cambridge studies in advanced mathematics 29. Cambridge University Press, Cambridge, 1990.
- [Hum00] James E. Humphreys. *Introduction to Lie algebras and representation theory*. Graduate texts in mathematics 9, [Ed. 2000]. Springer, New York, [repr.] edition, 2000.
- [Jan77] Jens Carsten Jantzen. Darstellungen halbeinfacher gruppen und kontravariante formen. *Journal f¨ur die reine und angewandte Mathematik*, 1977(290):117–141, 1977.
- [Jan03] Jens Carsten Jantzen. *Representations of algebraic groups*. Mathematical surveys and monographs vol. 107. American Mathematical Society, Providence R.I, 2nd ed. edition, 2003.
- [LR34] D.E. Littlewood and Archibald Read Richardson. Group characters and algebra. *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character*, 233:99–141, 1934.
- [Lü01] Frank Lübeck. Small degree representations of finite Chevalley groups in defining characteristic. *LMS J. Comput. Math.*, 4:135–169, 2001.
- [Mat90] Olivier Mathieu. Filtrations of *G*-modules. *Annales scientifiques de l'Ecole normale ´ sup´erieure*, 23(4):625–644, 1990.
- [MN08] Jiří Matousek and Jaroslav Nesetřil. *Invitation to Discrete Mathematics*. Oxford University Press, Incorporated, Oxford, 2008.
- [MT11] Gunter Malle and Donna Testerman. *Linear algebraic groups and finite groups of Lie type*. Cambridge studies in advanced mathematics 133. Cambridge University Press, Cambridge, 2011.
- [Pie12] R.S. Pierce. *Associative Algebras*. Graduate Texts in Mathematics. Springer, 2012.
- [Sch19] Nathan Scheinmann. Maximal subgroups acting with two composition factors on irreducible representations of exceptional algebraic groups, 2019.
- [Spr09] Tonny Albert Springer. *Linear algebraic groups*. Modern Birkhäuser Classics. Birkhäuser, Boston, second edition, 2009.
- [Ste01] John R. Stembridge. Multiplicity-free products of Schur functions. *Annals of combinatorics*, 5(2):113–121, 2001.
- <span id="page-78-0"></span>[Ste03] John R. Stembridge. Multiplicity-free products and restrictions of Weyl characters. *Representation theory*, 7(18):404–439, 2003.
- [Ste16] Robert Steinberg. *Lectures on Chevalley groups*. University lectures series vol. 66. American Mathematical Society, Providence R.I, 2016.
- [Tes88] Donna M. Testerman. Irreducible subgroups of exceptional algebraic groups. *Memoirs of the American Mathematical Society*, 75(390), 1988.