COMPACTIFICATIONS OF $\mathbb{C}^n$ AND THE PROJECTIVE SPACE

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## 1. Introduction

In his famous problem list [Hi54], Hirzebruch asked to classifiy all (smooth) compactifications $X$ of $\mathbb{C}^n$ with $b_2(X) = 1$. The condition on $b_2(X)$ is equivalent to saying that the divisor at $\infty$ is irreducible. For an overview on this problem, see [PS89]. In any case, without additional assumptions, such a classification seems to be possible only in low dimensions.

A particularly interesting case is when $X$ is Kähler and the divisor at infinity is smooth. A folklore conjecture states that then $X$ must be projective space. In this paper, we confirm this conjecture. Somehow more generally, we show

**1.1. Theorem.** *Let $X$ be a compact Kähler manifold of dimension $n$ and $Y \subset X$ a smooth connected hypersurface such that*

$$H_p(Y,\mathbb{Z}) \to H_p(X,\mathbb{Z})$$

*is bijective for all $0 \leq p \leq 2n-2$. Then $X \simeq \mathbb{P}_n$ and $Y$ is a hyperplane.*

**1.2. Corollary.** *Let $X$ be a compact Kähler manifold of dimension $n$ and $Y \subset X$ a smooth connected hypersurface such that $X \setminus Y$ is biholomorphic to $\mathbb{C}^n$. Then $X \simeq \mathbb{P}_n$ and $Y$ is a hyperplane.*

These results had been proved for $n \leq 5$ by van de Ven [vdV62] and for $n \leq 6$ by Fujita [Fu80].

## 2. Proof of Theorem 1.1

We fix $X$ and $Y$ as in Theorem 1.1 and first collect some basic properties of $X$ and $Y$ and refer to Fujita [Fu80] and Sommese [So76].

**2.1. Proposition.**

*a) The cohomology ring $H^*(X,\mathbb{C})$ is isomorphic as graded ring to $H^*(\mathbb{P}_n,\mathbb{C})$.*
*b) The cohomology ring $H^*(Y,\mathbb{C})$ is isomorphic as graded ring to $H^*(\mathbb{P}_{n-1},\mathbb{C})$.*
*c) The restrictions $H^q(X,\mathbb{C}) \to H^q(Y,\mathbb{C})$ are bijective for $0 \leq q \leq 2n-2$.*
*d) $X$ is a Fano manifold. Further $\mathrm{Pic}(X) \simeq \mathbb{Z}$, with ample generator $\mathcal{O}_X(Y)$.*

Thus we may regard all Chern numbers of $X$ and $Y$ as numbers and intersection is just multiplication. Further let $r$ denote the index of $X$, so that $-K_X = \mathcal{O}_X(rY)$. In other words,

$$c_1(X) = r.$$

Note that it is not necessary to assume $Y$ to be ample, since $\mathrm{Pic}(X) \simeq \mathbb{Z}$ holds actually for any smooth compactification with $b_2 = 1$ as well as (1) and (2) in the proposition.

By Corollary 2.5 of Libgober-Wood [LW90], we have

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**2.2. Proposition.**

$$r \cdot c_{n-1}(X) = c_1(X) \cdot c_{n-1}(X) = \frac{1}{2}\ n(n+1)^2 \tag{1}$$

*and*

$$(r-1) \cdot c_{n-2}(Y) = c_1(Y) \cdot c_{n-1}(Y) = \frac{1}{2}\ (n-1)n^2. \tag{2}$$

We now start the proof of Theorem 1.1 and observe first that $r \neq 1$ by Equation (2). The tangent bundle sequence

$$0 \to T_Y \to T_X|Y \to N_{Y/X} \ \to 0$$

yields (in terms of numbers!)

$$c_{n-1}(X) = c_{n-1}(T_X|Y) = c_{n-1}(Y) + c_{n-2}(Y) \cdot c_1(N_{Y/X}) = c_{n-1}(Y) + c_{n-2}(Y).$$

Since

$$c_{n-1}(Y) = \chi_{\text{top}}\ (Y) = \chi_{\text{top}}(\mathbb{P}_{n-1}) = n,$$

it follows

$$c_{n-1}(X) = n + c_{n-2}(Y). \tag{3}$$

Replacing $c_{n-1}(X)$ by $n + c_{n-2}(Y)$ in Equation (1) and putting in

$$c_{n-2}(Y) = \frac{1}{2(r-1)}(n-1)n^2$$

by virtue of Equation (2), we obtain

$$r\big(1 + \frac{1}{2(r-1)}(n-1)n\big) = \frac{1}{2}(n+1)^2. \tag{4}$$

Fixing $r$, we obtain a quadratic equation for $n$ with solutions

$$n = r - 1$$

and

$$n = 2r - 1.$$

In the first case $X \simeq \mathbb{P}_n$ and we are done.

Thus the second case has to be ruled out. So assume $n = 2r - 1$, in particular $n$ is odd.

Assume first that $r = \frac{n+1}{2}$ is even and write $r = 2m$ so that $n = 4m - 1$. By Equation (1), $n-1$ divides $n(n+1)^2$. In other words, $4m-2$ divides $(4m-1)(4m)^2$. Thus $4m - 2$ divides 4, so that $m = 1$ and $n = 3$. This case is already settled by [vdV62].

If $r$ is odd, write $r = 2m + 1$ so that $n = 4m + 3$. By Equation (2), $n - 3$ divides $(n-1)n^2$, hence $4m$ divides $(4m+2)(4m+3)^2$. Hence $4m$ divides 18 which is impossible.

**2.3. Remark.** Theorem 1.1 actually holds for compact manifolds which are bimeromorphic to a Kähler manifold. In fact, the result of Libgober-Wood rests on Hodge decomposition which is valid in this class. The only thing to observe that in Equation (3) we might have a change in sign, so that either Equation (3) holds or

$$c_{n-1}(X) = -n + c_{n-2}(Y).$$

The latter case leads to a contradiction by analogous computations as above.

Of course it is expected that Theorem 1.1 remains true without any Kähler assumption. This is true in dimension three, see [PS89], but seems out of reach in general. E.g., the divisor $Y$ could be homologous to 0.

## References


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