SOBOLEV AND HÖLDER ESTIMATES FOR THE $\overline{\partial}$ EQUATION ON PSEUDOCONVEX DOMAINS OF FINITE TYPE IN $\mathbb{C}^2$

ZIMING SHI

Abstract. We prove a homotopy formula which yields almost optimal estimates in all (positive-indexed) Sobolev and Hölder-Zygmund spaces for the $\overline{\partial}$ equation on finite type domains in $\mathbb{C}^2$, extending the earlier results of Fefferman-Kohn (1988), Chang-Nagel-Stein (1992), and Range (1992). The main novelty of our proof is the construction of holomorphic support functions that admit precise estimates when the parameter variable lies in a thin shell outside the domain, which generalizes Range's method.

Contents

1. Introduction

The goal of the present paper is the following:

Theorem 1.1. *Let D be a smoothly bounded pseudoconvex domain in $\mathbb{C}^2$ of finite type m. For each $\eta > 0$, there exist linear operators $\mathcal{H}_i^\eta$, $i = 1, 2$ such that*

(i) $\mathcal{H}_i^\eta : H^{s,p}_{(0,i)}(D) \to H^{s+\frac{1}{m}-\eta,p}_{(0,i-1)}(D)$, *for any* $1 < p < \infty$ *and* $s > \frac{1}{p}$.

(ii) $\mathcal{H}_i^\eta : \Lambda^s_{(0,i)}(\overline{D}) \to \Lambda^{s+\frac{1}{m}-\eta}_{(0,i-1)}(\overline{D})$ *for any* $s > 0$.

(iii) Suppose $\varphi \in H^{s,p}_{(0,1)}(D)$ *and* $\overline{\partial}\varphi \in H^{s,p}_{(0,2)}(D)$ *(resp.* $\varphi \in \Lambda^s_{(0,1)}(\overline{D})$ *and* $\overline{\partial}\varphi \in \Lambda^s_{(0,2)}(\overline{D})$*). Then*

$$\varphi = \overline{\partial}\mathcal{H}_1^\eta\varphi + \mathcal{H}_2^\eta\overline{\partial}\varphi$$

in the sense of distributions. In particular $\mathcal{H}_1^\eta\varphi$ *is a solution to the equation* $\overline{\partial}u = \varphi$ *for any* $\varphi \in H^{s,p}_{(0,1)}(D)$ *(or* $\Lambda^s(\overline{D})$*) with* $\overline{\partial}\varphi = 0$.

Here we use $H^{s,p}_{(0,i)}(D)$ *($i = 1, 2$) to denote the space of $(0, i)$ forms with* $H^{s,p}(D)$ *coefficients, and similarly for* $\Lambda^s_{(0,i)}(\overline{D})$. *Further,* $H^{s,p}(D)$ *is the fractional Sobolev space (see Definition 2.6), and* $\Lambda^s(\overline{D})$ *is the Hölder-Zygmund space (see Definition 2.1).*

2020 *Mathematics Subject Classification.* 32A26, 32T25.
Key words and phrases. $\overline{\partial}$ equation, homotopy formula, finite type domains.

The study of global existence and regularity of the $\overline{\partial}$-equation on pseudoconvex domains is a fundamental problem in several complex variables. Since the early 1960s, two parallel schools of research have developed to solve the $\overline{\partial}$ equation. The first one is by solving the $\overline{\partial}$-Neumann problem. On any pseudoconvex domain of C^∞ boundary, one can define the L^2 canonical solution $\overline{\partial}^*\mathcal{N}\varphi$, where $\mathcal{N}$ is the operator that solves the $\overline{\partial}$-Neumann subelliptic boundary value problem. The solution $\overline{\partial}^*\mathcal{N}\varphi$ is called the canonical solution since it is the unique solution which is orthogonal to Ker$(\overline{\partial})$ under the L^2 inner product. The first global regularity result was obtained by Kohn [Koh64], who showed that on a strongly pseudoconvex domain with C^∞ boundary, $\overline{\partial}^*\mathcal{N}$ is a bounded operator from $H^{s,2}(\Omega)$ to $H^{s+\frac{1}{2},2}(\Omega)$ for $s \geq -\frac{1}{2}$. Later on in their monograph [GS77], Greiner and Stein proves the boundedness $\overline{\partial}^*\mathcal{N} : H^{s,p}_{(0,1)}(\Omega) \to H^{s+\frac{1}{2},p}(\Omega)$ for $s \geq 0$, and $\overline{\partial}^*\mathcal{N} : \Lambda^r_{(0,1)}(\Omega) \to \Lambda^{r+\frac{1}{2}}(\Omega)$ for $r > 0$. Chang [Cha89] extends the (Sobolev space) result of Greiner-Stein to any (p,q) forms.

The second approach for solving the $\overline{\partial}$ equation – the one which we will follow in this paper – is to look for solutions operators in the form of integral formula, in a way that generalize the Cauchy-Green operator that solves the one dimensional $\overline{\partial}$ equation to higher dimensions. The method of integral formula has several advantages. It is more geometric in nature; requires less boundary regularity; and gives L^∞ estimate that is not accessible by the $\overline{\partial}$-Neumann method. The theory was pioneered by Grauert and Lieb, and independently, by Henkin, and has been developed extensively on many classes of pseudoconvex domains. We mention a few notable works in this direction.

On a strictly pseudoconvex domain with C^2 boundary, Henkin and Ramirez [RH71] constructed a solution operator that is bounded from $C^0(\overline{\Omega})$ to $C^{\frac{1}{2}}(\overline{\Omega})$. Assuming the boundary is C^{k+2}, for positive integers k, Siu [Siu74] and Lieb-Range [LR80] constructed solution operators that is bounded from $C^k(\overline{\Omega})$ to $C^{k+\frac{1}{2}}(\overline{\Omega})$. More recently, Gong [Gon19] showed that under the minimal smoothness of C^2 boundary, there exists a solution operator that is bounded from $\Lambda^r(\overline{\Omega})$ to $\Lambda^{r+\frac{1}{2}}(\overline{\Omega})$, for any $r > 1$. In [SY24b], the authors constructed a solution operator which is bounded from $H^{s,p}(\overline{\Omega}) \to H^{s+\frac{1}{2},p}(\overline{\Omega})$ for any $s \in \mathbb{R}$ and $1 < p < \infty$, under the assumption that $b\Omega \in C^\infty$. For convex domains of finite type, sharp estimates have been obtained by [DFFs99], [Ale06] and others. See also the recent work by Yao [Yao24].

Pseudoconvex domains of finite type in $\mathbb{C}^2$ was introduced by Kohn [Koh72], as a natural generalization of strict pseudoconvexity. Kohn showed that if the domain has type m, then the $\overline{\partial}$-Neumann problem is subelliptic of order $1/m$, from which the boundedness of the operator $\overline{\partial}^*\mathcal{N} : H^{s,2} \to H^{s+\frac{1}{m},2}$ follows. Using microlocal analysis, Fefferman and Kohn [FK88] proved the boundedness $\overline{\partial}^*\mathcal{N} : C^s(\overline{\Omega}) \to C^{s+1/m}(\overline{\Omega})$ for any $s > 0$ such that $s + 1/m$ is not an integer, and they also proved the sup norm estimate $\overline{\partial}^*\mathcal{N} : L^\infty(\Omega) \to L^\infty(\Omega)$. Soon after, Chang-Nagel-Stein [CNS92] used method similar to the ones in [GS77] to show that $\overline{\partial}^*\mathcal{N}$ is bounded from $\Lambda^s(\Omega)$ to $\Lambda^{s+\frac{1}{m}}$ for any $s > 0$. They also proved for any smooth complex tangential vector fields L_1 the boundedness of the operator $L_1\overline{\partial}^*\mathcal{N}, \overline{L_1}\overline{\partial}^*\mathcal{N} : W^{k,p}(\Omega) \to W^{k,p}(\Omega)$ for any $1 < p < \infty$ and non-negative integer k.

The method of integral formula has also played an important part in the study of $\overline{\partial}$ equation on pseudoconvex domains of finite type in $\mathbb{C}^2$. As in the case of strictly pseudoconvex, the success of this method depends largely on solving a holomorphic division problem

$$h_1(z,\zeta)(z_1 - \zeta_1) + h_2(z,\zeta)(z_2 - \zeta_2) = \Phi(z,\zeta),$$

where h is holomorphic in $z \in \Omega$, and the function Φ – known as the holomorphic support function – is non-vanishing for $z \in \Omega$ and $\zeta \in b\Omega$. Kohn and Nirenberg [KN73] constructed an example – a finite type pseudoconvex domain on which such function Φ does not exist while vanishing at $z = \zeta \in b\Omega$. Thus in general one cannot expect to obtain h_1, h_2 that are bounded uniformly for $(z,\zeta) \in (\Omega, b\Omega)$. Nevertheless, on a class of domains in $\mathbb{C}^2$ which include the Kohn-Nirenberg example, Fornaess [Fs86] proved the existence of h_1, h_2 with $\Phi \equiv 1$ and h_i, $i = 1, 2$ satisfying a certain weighted integral estimate, which then allows him to prove the sup-norm estimate for the $\overline{\partial}$ equation on such domains. Belanger [Bel93] then refined Fornaess' method and obtained some (far from optimal) Hölder estimates for $\overline{\partial}$ on the same class of domains.

In [Ran90], Range constructed for each $\eta > 0$ an integral solution operator T^η that is bounded from $L^\infty(\overline{\Omega})$ to $C^{\frac{1}{m}-\eta}(\overline{\Omega})$ for any finite type domain Ω in $\mathbb{C}^2$. The estimate is not sharp as in the work of [FK88] and [CNS92], but Range's proof is much simpler and geometric; it is based on the earlier work of Catlin [Cat89] that shows locally one can push out the domain near a boundary point and still remain pseudoconvex. To construct the holomorphic functions h, Range uses Skoda's L^2 division theorem and obtained the following weighted L^2 estimate on the pushed out domain D_p (for fixed $p \in bD$):

$$\int_{D_p} \frac{|h_i^\eta(z,p)|^2}{|z-p|^2} \operatorname{dist}(z, D_p)^{2\eta} \, dV(z) < C_{D,\eta}, \quad \eta > 0.$$

The domain D_p is pseudoconvex and touches bD only at p. In Range's proof, the functions $h_i(z,p)$ is defined only for $p \in bD$. For our proof of Theorem 1.1, we construct the functions $h_i(z,q)$ which are defined for ζ in a thin shell outside the domain, i.e. there exists a small neighborhood $\mathcal{U}$ of $\overline{D}$ such that for each $q \in \mathcal{U} \setminus D$, there is a pseudoconvex domain $D_*(q)$ with $D \Subset D_*(q)$, $q \notin D_*(q)$, and whose boundary is as far away from bD as possible, as measured by certain non-isotropic polydisks with centers in D. The polydisks we use are from [Cat89] with some slight modification. Roughly speaking, the size of these polydisks are determined by the distance of its center and that of q to bD. Using Cauchy integral estimates we can then estimate all z derivatives of the functions h_i for $z \in D$ and $q \in \mathcal{U} \setminus D$.

There is an additional problem of how $h_i(\cdot, q)$ depend on q. This problem arises since we need to integrate in the second variable. Here as in Range, we replace $h(z,q)$ with a smooth function $h(z,\zeta)$, by restricting z in an interior domain $D'_\epsilon \Subset D$ away from the boundary bD, and restricting ζ to a small ball centered at q and whose radius shrinks to 0 as ϵ. In our case we need to control this radius, which can be estimated using the sup norm of $h_i(\cdot, q)$ on $\overline{D}_\epsilon$. Consequently we can show that the radius is small compared to $r(q) + \epsilon$. This allows us to estimate the error arising from the switching of q to ζ, and in the end we obtain a modified $h_i(z,\zeta)$ which is holomorphic in $z \in D'_\epsilon$ and smooth in ζ in $\mathcal{U} \setminus D$.

We can now solve the $\overline{\partial}$ equation on each approximating domain D'_ϵ, using the homotopy operator constructed in [Gon19] and [SY21]. The resulting estimate does not depend on ϵ as $\epsilon \to 0$, so we obtain a solution on the original domain D by taking limits in suitable function spaces. We mention two key auxiliary results which are used in the estimate: the Hardy-Littlewood lemma for Sobolev and Hölder-Zygmund space (Proposition 2.24) which reduces the problem to estimating a weighted integral norm; and the commutator estimate Proposition 2.17 proved in [SY24b] which allows us to harness of the power of the commutator term in the integral operator and essentially dispense with the need for integration by parts.

As in Range [Ran90], the arbitrary small loss of regularity comes from the integrability condition in Skoda's theorem, and it seems unlikely that the current method is inadequate to remove this loss. The following question remains open.

Question 1.2. Let D be a pseudoconvex domain of finite type m in $\mathbb{C}^2$. Does there exist an integral solution operator to the $\overline{\partial}$ equation that is bounded from $H^{s,p}(D)$ to $H^{s+\frac{1}{m},p}(D)$ for any $1 < p < \infty$ and $s > \frac{1}{p}$, and $\Lambda^s(\overline{D})$ to $\Lambda^{s+\frac{1}{m}}(\overline{D})$ for any $s > 0$?

2. Function spaces

In this section, we recall some basic results for the Sobolev space $H^{s,p}(\Omega)$ and the Hölder-Zygmund space $\Lambda^s(\Omega)$.

Definition 2.1 (Hölder-Zygmund space on $\mathbb{R}^N$)**.** The Hölder-Zygmund space on $\mathbb{R}^N$, denoted by $\Lambda^s(\mathbb{R}^N)$ for $s \in \mathbb{R}^+$ is defined as follows

- For $0 < s < 1$, $\Lambda^s(\mathbb{R}^N)$ consists of all $f \in C^0(\mathbb{R}^N)$ such that
$$\|f\|_{\Lambda^s(U)} := \sup_{\mathbb{R}^N} |f| + \sup_{x,y\in\mathbb{R}^N,\, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^s} < \infty.$$
- $\Lambda^1(\mathbb{R}^N)$ consists of all $f \in C^0(\mathbb{R}^N)$ such that
$$\|f\|_{\Lambda^1(\mathbb{R}^N)} := \sup_{\mathbb{R}^N} |f| + \sup_{x,y\in\mathbb{R}^N,\, x\neq y} \frac{|f(x)+f(y)-2f(\frac{x+y}{2})|}{|x-y|} < \infty.$$
- For $s > 1$ recursively, $\Lambda^s(\mathbb{R}^N)$ consists of all $f \in \Lambda^{s-1}(\mathbb{R}^N)$ such that $\nabla f \in \Lambda^{s-1}(\mathbb{R}^N)$. We define $\|f\|_{\Lambda^s(\mathbb{R}^N)} := \|f\|_{\Lambda^{s-1}(\mathbb{R}^N)} + \sum_{j=1}^d \|D_j f\|_{\Lambda^{s-1}(\mathbb{R}^N)}$.
- We define $C^\infty(\mathbb{R}^N) := \bigcap_{s>0} \Lambda^s(\mathbb{R}^N)$ to be the space of bounded smooth functions.

Definition 2.2 (Hölder-Zygmund space on domains)**.** Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. The Hölder-Zygmund space on Ω, denoted by $\Lambda^s(\Omega)$ for $s > 0$, is defined as $\Lambda^s(\Omega) = \{f : \exists\, \widetilde{f} \in \Lambda^s(\mathbb{R}^N)\ s.t.\ \widetilde{f}|_\Omega = f\}$ equipped with the norm:
$$|f|_{\Lambda^s(U)} := \inf_{\widetilde{f}\in\Lambda^s(\mathbb{R}^N),\, \widetilde{f}|_\Omega = f} |\widetilde{f}|_{\Lambda^s(\mathbb{R}^N)}.$$

Remark 2.3. There is an intrinsic equivalent definition – often called the (classical) Besov space – for the space $\Lambda^s(\Omega)$, namely, one which requires only that f is defined in Ω, rather than assuming f is the restriction of a function defined on the whole space. We will not use this definition in this paper. The interested reader can refer to [Tri06, Def. 1.120 and Thm 1.122] or [Gon24, Section 5].

Next we turn to the Sobolev spaces. We denote by $\mathscr{S}(\mathbb{R}^N)$ the space of Schwartz functions, and by $\mathscr{S}'(\mathbb{R}^N)$ the space of tempered distributions. For $g \in \mathscr{S}(\mathbb{R}^N)$, we set the Fourier transform $\widehat{g}(\xi) = \int_{\mathbb{R}^N} g(x) e^{-2\pi i x\cdot\xi} dx$, and the definition extends naturally to tempered distributions.

Definition 2.4. We let $\dot{\mathscr{S}}(\mathbb{R}^N)$ denote the space[1] of all infinite order moment vanishing Schwartz functions. That is, all $f \in \mathscr{S}(\mathbb{R}^N)$ such that $\int x^\alpha f(x) dx = 0$ for all $\alpha \in \mathbb{N}^N$, or equivalently, all $f \in \mathscr{S}(\mathbb{R}^N)$ such that $\widehat{f}(\xi) = O(|\xi|^\infty)$ as $\xi \to 0$.

Definition 2.5 (Sobolev space on $\mathbb{R}^N$)**.** Let $s \in \mathbb{R}$, $1 < p < \infty$. We define $H^{s,p}(\mathbb{R}^N)$ to be the fractional Sobolev space consisting of all (complex-valued) tempered distribution $f \in \mathscr{S}'(\mathbb{R}^N)$ such that $(I-\Delta)^{\frac{s}{2}} f \in L^p(\mathbb{R}^N)$, and equipped with norm
$$\|f\|_{H^{s,p}(\mathbb{R}^N)} := \|(I-\Delta)^{\frac{s}{2}} f\|_{L^p(\mathbb{R}^N)}.$$
Here $(I-\Delta)^{\frac{s}{2}}$ is the Bessel potential operator given by
$$(I-\Delta)^{\frac{s}{2}} f = ((1+4\pi^2|\xi|^2)^{\frac{s}{2}} \widehat{f}(\xi))^\vee. \tag{2.1}$$

As in the case for the Hölder-Zygmund space, we define Sobolev space on domains as restrictions of the Sobolev space on $\mathbb{R}^N$.

Definition 2.6 (Sobolev space on domains)**.** Let $\Omega \subset \mathbb{R}^N$ be an open set.

(i) Define $\mathscr{S}'(\Omega) := \{\tilde{f}|_\Omega : \tilde{f} \in \mathscr{S}'(\mathbb{R}^N)\}$.
(ii) For $s \in \mathbb{R}$ and $1 < p < \infty$, define $H^{s,p}(\Omega) := \{\tilde{f}|_\Omega : \tilde{f} \in H^{s,p}(\mathbb{R}^N)\}$ with norm
$$\|f\|_{H^{s,p}(\Omega)} := \inf_{\widetilde{f}|_\Omega = f} \|\widetilde{f}\|_{H^{s,p}(\mathbb{R}^N)}.$$

[1] In some literature like [Tri83, Section 5.1.2], the notation is $Z(\mathbb{R}^N)$.

(iii) For $s \in \mathbb{R}$ and $1 < p < \infty$, define $H_0^{s,p}(\Omega)$ to be the subspace of $H^{s,p}(\mathbb{R}^N)$ which is the completion of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_{H^{s,p}(\mathbb{R}^N)}$.

Remark 2.7. When $s = k$ is a non-negative integer, the space $H^{s,p}(\mathbb{R}^N)$ is the same as $W^{k,p}(\mathbb{R}^n)$, which consists of complex-valued functions whose derivatives up to order k is in $L^p(\mathbb{R}^n)$.

For computation it is often convenient to use Littlewood-Paley characterizations of the above spaces. This leads to the Triebel-Lizorkin space and the Besov space, which generalize Sobolev space and Hölder-Zygmund space, respectively. decomposition.

Definition 2.8. A *classical dyadic resolution* is a sequence $\lambda = (\lambda_j)_{j=0}^\infty$ of Schwartz functions on $\mathbb{R}^n$, denoted by $\lambda \in \mathfrak{C}$, such that the Fourier transforms $\widehat{\lambda}_j(\xi) = \int_{\mathbb{R}^n} \lambda_j(x) e^{-2\pi i x\cdot\xi} dx$ satisfies

- $\widehat{\lambda}_0 \in C_c^\infty\{|\xi| < 2\}$, $\widehat{\lambda}_0|_{\{|\xi|<1\}} \equiv 1$.
- $\widehat{\lambda}_j(\xi) = \widehat{\lambda}_0(2^{-j}\xi) - \widehat{\lambda}_0(2^{-(j-1)}\xi)$ for $j \geq 1$ and $\xi \in \mathbb{R}^n$.

For any $f \in \mathscr{S}'(\mathbb{R}^n)$, we have the decomposition $f = \sum_{j=0}^\infty (\widehat{\lambda}_j \widehat{f})^\vee = \sum_{j=0}^\infty \lambda_j * f$ and both sums converge as tempered distribution. We can define the Besov and Triebel-Lizorkin spaces using such λ.

A pair (p, q) is said to be *admissible* for the function class $\mathscr{B}_{pq}^s$ if $0 < p, q \leq \infty$, and admissible for the function class $\mathscr{F}_{pq}^s$ if $0 < p < \infty$, $0 < q \leq \infty$ or $p = q = \infty$.

Definition 2.9. Let $(\lambda_j)_{j=0}^\infty \in \mathfrak{C}$, $s \in \mathbb{R}$ and let (p, q) be admissible. We define the following norms:

$$\|f\|_{\mathscr{F}_{pq}^s(\lambda)} := \|(2^{js}\lambda_j * f)_{j=0}^\infty\|_{L^p(\ell^q)} = \left\| \|(2^{js}\lambda_j * f)_{j=0}^\infty\|_{\ell^q(\mathbb{N})} \right\|_{L^p(\mathbb{R}^n)}.$$

The *Triebel-Lizorkin space* $\mathscr{F}_{pq}^s(\mathbb{R}^n)$ is the set of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that $\|f\|_{\mathscr{F}_{pq}^s(\lambda)} < \infty$. One can show that the spaces do not depend on the choice of $\lambda \in \mathfrak{C}$. Therefore we will henceforth use the norm $\|\cdot\|_{\mathscr{F}_{pq}^s(\mathbb{R}^n)} = \|\cdot\|_{\mathscr{F}_{pq}^s(\lambda)}$ for an (implicitly chosen) $\lambda \in \mathfrak{C}$.

Let $\Omega \subseteq \mathbb{R}^n$ be an arbitrary open subset. We define $\mathscr{F}_{pq}^s(\Omega) := \{\tilde{f}|_\Omega : \tilde{f} \in \mathscr{F}_{pq}^s(\mathbb{R}^n)\}$ as subspaces of distributions in Ω, with norms

$$\|f\|_{\mathscr{F}_{pq}^s(\Omega)} := \inf\{\|\tilde{f}\|_{\mathscr{F}_{pq}^s(\mathbb{R}^n)} : \tilde{f}|_\Omega = f\}. \tag{2.2}$$

Notice that the above definition is not intrinsic to the domain since it requires the distribution to have an extension on the whole space. To define an intrinsic norm we need to impose certain conditions on the domain so it admits extension operators bounded on the function classes. We will use the Lipschitz boundary condition and Rychkov (universal) extension operator.

We now define the notion of special and bounded Lipschitz domains.

Definition 2.10. A *special Lipschitz domain* is an open set $\omega \subset \mathbb{R}^N$ of the form $\omega = \{(x', x_N) : x_N > \rho(x')\}$ with $\|\nabla\rho\|_{L^\infty} < L$. A *bounded Lipschitz* domain is a bounded open set Ω whose boundary is locally the graph of some Lipschitz function. In other words, $b\Omega = \bigcup_{\nu=1}^M U_\nu$, where for each $1 \leq \nu \leq M$, there exists an invertible linear transformation $\Phi_\nu : \mathbb{R}^N \to \mathbb{R}^N$ and a special Lipschitz domain ω_ν such that

$$U_\nu \cap \Omega = U_\nu \cap \Phi_\nu(\omega_\nu).$$

Fix any such covering $\{U_\nu\}$, we define the *Lipschitz norm of* Ω *with respect to* U_ν*, denoted as* $\mathrm{Lip}_{U_\nu}(\Omega)$, to be $\sup_\nu \|D\Phi_\nu\|_{C^0}$.

Definition 2.11. Let $\omega \subset \mathbb{R}^N$ be a *special Lipschitz domain* of the form $\omega = \{(x', x_N) : x_N > g(x')\}$ for some $g : \mathbb{R}^{N-1} \to \mathbb{R}$ such that $\|\nabla\rho\|_{L^\infty} < L$.

The *Rychkov's universal extension operator* $E = E_\omega$ for ω is given as follows:

$$\mathcal{E}_\omega f := \sum_{j=0}^\infty \psi_j * (\mathbf{1}_\omega \cdot (\phi_j * f)), \qquad f \in \mathscr{S}'(\omega). \tag{2.3}$$

Here $(\psi_j)_{j=0}^\infty$ and $(\phi_j)_{j=0}^\infty$ are families of Schwartz functions that satisfy the following properties:

(2.4) *Scaling condition*: $\phi_j(x) = 2^{(j-1)N}\phi_1(2^{j-1}x)$ and $\psi_j(x) = 2^{(j-1)N}\psi_1(2^{j-1}x)$ for $j \geq 2$.
(2.5) *Moment condition*: $\int \phi_0 = \int \psi_0 = 1$, $\int x^\alpha \phi_0(x)dx = \int x^\alpha \psi_0(x)dx = 0$ for all multi-indices $|\alpha| > 0$, and $\int x^\alpha \phi_1(x)dx = \int x^\alpha \psi_1(x)dx = 0$ for all $|\alpha| \geq 0$.
(2.6) *Approximate identity*: $\sum_{j=0}^\infty \phi_j = \sum_{j=0}^\infty \psi_j * \phi_j = \delta_0$ is the Direc delta measure.
(2.7) *Support condition*: ϕ_j, ψ_j are all supported in the negative cone $-\mathbb{K}^L := \{(x', x_N) : x_N < -L|x'|\}$.

We call (ϕ, ψ) a $\mathbb{K}^L$*-Littlewood-Paley pair.*

Note that condition ((2.7)) implies that there exists some $c_0 > 0$ such that $\operatorname{supp}\phi_0 \subset \{x_N < -c_0'\}$. By the Scaling condition ((2.4)), we have

$$\operatorname{supp}\phi_j \subset \{x_N < -c_0 2^{-j}\}, \quad c_0 = 2c_0'. \tag{2.8}$$

Since a bounded Lipschitz domain Ω is locally a special Lipschitz domain, we can apply a partition of unity and patch together the extension operators for the special Lipschitz domains to obtain an Rychkov extension operator $\mathcal{E}_\Omega$ for Ω. We omit the exact expression for $\mathcal{E}_\Omega$ (see for example [SY21, (3.3)]) and we simply note that $\mathcal{E}_\Omega$ have the same properties as $\mathcal{E}_\omega$.

Given a family $\phi = (\phi_j)_{j=0}^\infty$ satisfying properties (2.4) - (2.7), we can define the following intrinsic norm:

$$\|f\|_{\mathscr{F}_{pq}^{s,in}(\phi)} := \|(2^{js}\lambda_j * f)_{j=0}^\infty\|_{L^p(\ell^q)} = \left\| \|(2^{js}\lambda_j * f)_{j=0}^\infty\|_{\ell^q(\mathbb{N})} \right\|_{L^p(\mathbb{R}^n)}.$$

For a fixed ϕ, we define $\mathscr{F}_{p,q}^{s,in}(\phi)$ as the space of distributions f on ω with finite norm $\|f\|_{\mathscr{F}_{pq}^{s,in}(\phi)}$. By using partition of unity, we can similarly define the space $\mathscr{F}_{p,q}^{s,in}(\Omega)$ for any bounded Lipschitz domain Ω.

Rychkov in the paper [Ryc99] proves the following remarkable extension theorem.

Proposition 2.12 (Rychkov)**.** *Let ω be a special Lipschitz domain. For any $s \in \mathbb{R}$, the operator $\mathcal{E}_\omega$ is $\mathscr{F}_{pq}^{s,in}(\omega) \to \mathscr{F}_{pq}^s(\mathbb{R}^N)$ bounded, provided that either $p < \infty$ or $p = q = \infty$. Using partition of unity, same properties hold for $\mathcal{E}_\Omega$ for any bounded Lipschitz domain Ω.*

Corollary 2.13. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let (p, q) be a pair such that either $p < \infty$ or $p = q = \infty$. Then the intrinsic norm $\|\cdot\|_{\mathscr{F}_{pq}^{s,in}(\Omega)}$ is equivalent to the extrinsic norm $\|\cdot\|_{\mathscr{F}_{pq}^s(\Omega)}$ (see Definition (2.2)), and therefore $\mathscr{F}_{pq}^{s,in}(\Omega) = \mathscr{F}_{pq}^s(\Omega)$.*

Both the Sobolev space and the Hölder-Zygmund space are special cases of the Triebel-Lizorkin spaces, as shown by the following result.

Proposition 2.14. *Let Ω be either $\mathbb{R}^N$ or a bounded Lipschitz domain. Let $\phi = (\phi_j)_{j=0}^\infty$ be any Littlewood-Paley family satisfying properties (2.4) - (2.7).*

(i) For all $1 < p < \infty$ and $s \in \mathbb{R}$, $H^{s,p}(\Omega) = \mathscr{F}_{p2}^s(\Omega) = \mathscr{F}_{p2}^{s,in}(\phi)$.
(ii) For all $s > 0$, $\Lambda^s(\Omega) = \mathscr{F}_{\infty\infty}^s(\Omega) = \mathscr{F}_{\infty\infty}^{s,in}(\phi)$.

Proof. For the statement $H^{s,p}(\mathbb{R}^N) = \mathscr{F}_{p,2}^s(\mathbb{R}^N)$ for $s \in \mathbb{R}$ and $1 < p < \infty$, the proof is in [Tri83, p. 88]. For $\Lambda^s(\mathbb{R}^N) = \mathscr{F}_{\infty\infty}^s(\mathbb{R}^N)$, $s > 0$, the proof can be found in [Tri83, p. 90]. For bounded Lipschitz domain Ω, the spaces $H^{s,p}(\Omega)$, $\Lambda^s(\overline{\Omega})$ $\mathscr{F}_{pq}^s(\Omega)$ are defined as the restrictions of the corresponding spaces on $\mathbb{R}^n$, so the statements follow from that of $\mathbb{R}^n$. The assertions $\mathscr{F}_{p2}^s(\Omega) = \mathscr{F}_{p2}^{s,in}$ and $\mathscr{F}_{\infty\infty}^s(\Omega) = \mathscr{F}_{\infty\infty}^{s,in}(\phi)$ follow immediately from Corollary 2.13. □

As a consequence to Proposition 2.12 and Proposition 2.14, we have the following extension result for the Sobolev and Hölder-Zygmund space.

Corollary 2.15. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Then the Rychkov extension operator $\mathcal{E}_\Omega$ defined on Ω is bounded between the following spaces:*

(i) $\mathcal{E}_\Omega : H^{s,p}(\Omega) \to H^{s,p}(\mathbb{R}^N)$, *for any* $s \in \mathbb{R}$ *and* $1 < p < \infty$.
(ii) $\mathcal{E}_\Omega : \Lambda^s(\overline{\Omega}) \to \Lambda^s(\mathbb{R}^N)$, *for any* $s > 0$.

Proposition 2.16 (Equivalence of norms)**.** *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Let $0 < p < \infty$, $0 < q \leq \infty$ or $p = q = \infty$. Then for any non-negative integer k, the following norms are equivalent.*

$$\|f\|_{\mathscr{F}^s_{pq}(\Omega)} \approx \sum_{|\alpha| \leq k} \|D^\alpha f\|_{\mathscr{F}^{s-k}_{pq}(\Omega)}.$$

Proof. This is [SY24a, Thm 1.1]. □

The following commutator estimate plays a crucial role in our proof.

Proposition 2.17. [SY21, Proposition 5.8] *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded Lipschitz domain, and let $\mathcal{E} = \mathcal{E}_\Omega$ be the Rychkov extension operator on Ω. Then for $1 < p < \infty$ and $s > 0$, the following estimates hold.*

$$\|\delta^{1-s}[D, \mathcal{E}]f\|_{L^p(\overline{\Omega}^c)} \leq C_{s,p}\|f\|_{H^{s,p}(\Omega)}, \quad \forall f \in H^{s,p}(\Omega);$$
$$\|\delta^{1-s}[D, \mathcal{E}]f\|_{L^\infty(\overline{\Omega}^c)} \leq C_s\|f\|_{\Lambda^s(\Omega)}, \quad \forall f \in \Lambda^s(\Omega).$$

We collect a few more useful facts about the spaces:

Definition 2.18. Let X_0, X_1 be two Banach spaces that belong to a larger ambient space. For $0 < \theta < 1$. The *complex interpolation space* $\left[X_0, X_1\right]_\theta$ is defined to be the space consisting of all $f(\theta) \in X_0 + X_1$, where $f : \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\} \to X_0 + X_1$ is a continuous map that is analytic in the interior, such that $f(it) \in X_0$ and $f(1 + it) \in X_1$ for all $t \in \mathbb{R}$. The norm is given by

$$\|u\|_{[X_0,X_1]_\theta} = \inf_f \{\sup_{t\in\mathbb{R}}(\|f(it)\|_{X_0} + \|f(1 + it)\|_{X_1}) : u = f(\theta)\}.$$

Proposition 2.19. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $1 < p < \infty$ and $s_0, s_1 \in \mathbb{R}$. Set $s_\theta = (1 - \theta)s_0 + \theta s_1$, for $0 < \theta < 1$. Then*

$$\left[H^{s_0,p}(\Omega), H^{s_1,p}(\Omega)\right]_\theta = H^{s_\theta,p}(\Omega);$$
$$\left[\Lambda^{s_0}(\Omega), \Lambda^{s_1}(\Omega)\right]_\theta = \Lambda^{s_\theta}(\Omega).$$

The reader can find the proof for the first statement in [Tri06, p. 70, Corollary 1.111], and the proof of the second statement in [BL76, p. 152, Theorem 6.4.5.]. Notice that in the reference B^s_{pq} denotes the Besov space and we use the identification $\Lambda^s(\Omega) = \mathscr{F}^s_{\infty,\infty}(\Omega) = B^s_{\infty,\infty}(\Omega)$.

Proposition 2.20 (Complex interpolation theorem)**.** *Let X_0, X_1, Y_0, Y_1 be Banach spaces that belong to some larger ambient spaces. Suppose $T : X_0 + X_1 \to Y_0 + Y_1$ is a linear operator such that for each $i = 0, 1$, $\|Tu\|_{Y_i} \leq C_0\|u\|_{X_i}$ for all $u \in X_i$. Then $T : [X_0, X_1]_\theta \to [Y_0, Y_1]_\theta$ is bounded linear with $\|Tu\|_{[Y_0,Y_1]_\theta} \leq C_0^{1-\theta}C_1^\theta\|u\|_{[X_0,X_1]_\theta}$ for all $u \in [X_0, X_1]_\theta$.*

Proof. See [Tri95, Theorem 1.9.3(a)] □

Lemma 2.21. *Let $R_0 \in Z_+$ and let $\omega = \{x_N < g(x')\}$ be a special Lipschitz domain with $\|\nabla g\|_{L^\infty} \leq L - 2^{-R_0}$. Then for every $j \in \mathbb{N}$ and $a \in \mathbb{R}$,*

$$\left[-\mathbb{K}^L \cap \{x_N < -2^{-j}\}\right] + \{x_N - g(x') < a\} \subset \{x \in \mathbb{R}^N : x_N - g(x') < a - bL^{-1}2^{-R_0-j}\}.$$

Proof. Let $u \in \{x_N - g(x') < a\}$ and $v \in -\mathbb{K}^L \cap \{x_N < -b2^{-j}\}$, for $a \in \mathbb{R}$ and $b > 0$. Then $u_N - g(u') < a$ and $v_N < -\max(L|v'|, b2^{-j})$. Using $\sup |\nabla \rho| \le L - 2^{-R_0}$, we get

$$\begin{aligned} u_N + v_N - g(u' + v') &\le u_N - g(u') + v_N + |g(u') - g(u' + v')| \\ &\le u_N - g(u') + v_N + \big(L - 2^{-R_0}\big)|v'| \\ &\le a + bL^{-1}2^{-R_0} v_N \le a - bL^{-1}2^{-R_0-j}. \end{aligned}$$

□

Lemma 2.22. *Let $1 \le p \le \infty$ and $s > 0$. Then for any $f \in \mathscr{F}^t_{p,\infty}(\mathbb{R}^N)$ with $f \equiv 0$ on $\overline{\Omega}^c$,*

$$\|f\|_{L^p(\Omega,\delta^{-t})} \le C_{\Omega,t}\|f\|_{\mathscr{F}^t_{p,\infty}(\mathbb{R}^N)}.$$

The constant $C_{\Omega,t}$ depends only on t and the Lipschitz norm of Ω.

Proof. The reader can find a proof in [Yao24, Prop. 5.3]. We include a slightly modified version here for the reader's convenience. We will prove for the case Ω is a special Lipschitz domain and the general case follows by standard partition of unity argument. Let $\omega = \{x_N < g(x')\}$, where $x' = (x_1, \dots, x_{N-1})$. Assume that $|g|_{L^\infty(\mathbb{R}^{N-1})} \le L$. Then

$$\omega + \big\{x_N > L|x'|\big\} \subset \omega.$$

We partition Ω into dyadic strips.

$$\omega = \bigcup_{k \in \mathbb{Z}} S_k := \bigcup_{k \in \mathbb{Z}} \Big\{(x', x_N) : -2^{\frac{1}{2}-k} < x_N - g(x') < -2^{-\frac{1}{2}-k}\Big\}$$

Since $\mathbb{K}_L \subset \omega$, we can show that

$$\frac{1}{\sqrt{1+L^2}}|x_N - g(x')| < \operatorname{dist}(x, b\omega) < |x_N - g(x')|.$$

Denote $c_L := (1 + L^2)^{-\frac{1}{2}}$. Then

$$c_L 2^{-\frac{1}{2}-k} < \operatorname{dist}(x, b\omega) < 2^{\frac{1}{2}-k}, \quad \text{for } x \in S_k. \tag{2.9}$$

Let f satisfy the hypothesis of lemma, and in particular $\operatorname{supp} f \in \omega$. By (2.7) and (2.8), $\operatorname{supp} \phi_j \subset -\mathbb{K}^L \cap \{x_N < -c_0 2^{-j}\}$. In view of Lemma 2.21, there exists $R_0 > 0$ such that

$$\operatorname{supp} \phi_j + \omega \subset \big[-\mathbb{K}^L \cap \{x_N < -c_0 2^{-j}\}\big] + \{x_N - g(x') < 0\} \subset \{x \in \mathbb{R}^N : x_N < -c_0 L^{-1} 2^{-R_0-j}\}.$$

By (2.6) we have $f = \sum_{j=0}^\infty \phi_j * f$. Observe that

$$\operatorname{supp}(\phi_j * f) = \operatorname{supp} \phi_j + \operatorname{supp} f \subset 2^{-j} + \omega \subset \{x \in \mathbb{R}^N : x_N < -c_0 L^{-1} 2^{-R_0-j}\},$$

which is disjoint from S_k if $-c_0 L^{-1} 2^{-R_0-j} < -2^{\frac{1}{2}-k}$, or $j < k - R_0 - \frac{1}{2} + \log_2(c_0 L^{-1})$. Let $R_1 = \lceil R_0 - \frac{1}{2} + \log_2(c_0 L^{-1})\rceil$. Then $S_k \cap \operatorname{supp}(\phi_j * f) = \emptyset$ if $j < k - R_1$. Consequently we have $f(x) = \sum_{j=k-R_1}^\infty \phi_j * f$ for $x \in S_k$. Using (2.9), we have

$$\begin{aligned} \|f\|^p_{L^p(\omega,\delta^{-t})} &= \sum_{k=0}^\infty \|f\|^p_{L^p(\omega,\delta^{-t})} \le \sum_{k=0}^\infty c_L^{-tp} 2^{(k+\frac{1}{2})tp} \|f\|^p_{L^p(S_k)} \\ &\le c_L^{-tp} \sum_{k=0}^\infty 2^{(k+\frac{1}{2})tp} \Big\| \sum_{j=k-R_1}^\infty |\phi_j * f| \Big\|^p_{L^p(S_k)} \\ &= c_L^{-tp} \sum_{k=0}^\infty \Big\| \sum_{j=k-R_1}^\infty 2^{(k+\frac{1}{2}-j)t} 2^{jt} |\phi_j * f| \Big\|^p_{L^p(S_k)} \end{aligned}$$

Let $l = j - k$, and the above expression is bounded by

$$c_L^{-tp} \sum_{k=0}^{\infty} \Big\| \sum_{l=-R_1}^{\infty} 2^{(1-l)t} 2^{jt} |\phi_j * f| \Big\|_{L^p(S_k)}^p \leq c_L^{-tp} \sum_{k=0}^{\infty} \sum_{l=-R_1}^{\infty} 2^{(1-l)t} \Big\| \sup_{j \geq k-R_1} 2^{jt} |\phi_j * f| \Big\|_{L^p(S_k)}^p$$

$$\leq c_L^{-tp} C_{R_1} \sum_{k=0}^{\infty} \Big\| \sup_{j\in\mathbb{N}} 2^{jt} |\phi_j * f| \Big\|_{L^p(\omega)}^p = C_{\omega,t} \|f\|_{\mathscr{F}^t_{p\infty}(\phi)}.$$

Here we use the convention $\phi_j * f = 0$ for $j \leq -1$. Notice that the constant $C_{\omega,t}$ blows up as $L, t \to \infty$. □

Let $\mathring{\mathscr{F}}_{pq}(\mathbb{R}^N)$ be the norm closure of $C_c^\infty(\mathbb{R}^N)$ in $\mathscr{F}^s_{pq}(\mathbb{R}^N)$ and let $\mathring{\mathscr{F}}_{pq}(\overline{\Omega})$ be the subspace of $\mathring{\mathscr{F}}_{pq}(\mathbb{R}^N)$ defined by $\mathring{\mathscr{F}}_{pq}(\overline{\Omega}) = \big\{ f \in \mathring{\mathscr{F}}_{pq}(\mathbb{R}^N) : f|_{\overline{\Omega}^c} = 0 \big\}$.

Proposition 2.23. *Let Ω be a bounded Lipschitz domain or $\mathbb{R}^N$. Then for $s \in \mathbb{R}$ and $1 \leq p \leq \infty$, $\mathscr{F}^{-s}_{p1}(\Omega) = \big[\mathring{\mathscr{F}}^s_{p'\infty}(\Omega)\big]'$.*

Proof. By [Tri20, Remark 1.5], we have $\mathscr{F}^{-s}_{p1}(\mathbb{R}^N) = \mathring{\mathscr{F}}^s_{p'\infty}(\mathbb{R}^N)'$ for all $s \in \mathbb{R}$, $1 \leq p \leq \infty$, and $p' = \frac{p}{p-1}$. Using the same argument as in [Tri95, Theorem 4.3.2/1], we have $\mathring{\mathscr{F}}_{pq}(\overline{\Omega})$ is the norm closure of $C_c^\infty(\Omega)$ in $\mathscr{F}_{pq}(\mathbb{R}^N)$. It follows that

$$\begin{aligned}
\big[\mathring{\mathscr{F}}^s_{p'\infty}(\overline{\Omega})\big]' &= \mathscr{F}^{-s}_{p1}(\mathbb{R}^N)/\big\{ f : \langle f, \phi \rangle = 0,\ \forall\, \phi \in \mathring{\mathscr{F}}^s_{p'\infty}(\mathbb{R}^N), \phi|_{\overline{\Omega}^c} = 0 \big\} \\
&= \mathscr{F}^{-s}_{p1}(\mathbb{R}^N)/\big\{ f : \langle f, \phi \rangle = 0,\ \forall\, \phi \in C_c^\infty(\Omega) \big\} \\
&= \mathscr{F}^{-s}_{p1}(\mathbb{R}^N)/\{ f : f|_\Omega = 0 \} = \mathscr{F}^{-s}_{p1}(\Omega).
\end{aligned}$$
□

Proposition 2.24. *Let $1 < p < \infty$, $k \in \mathbb{N}$ and $s \in \mathbb{R}$ with $s \leq k$. Let Ω be a bounded Lipschitz domain and denote by $\delta(x)$ the distance from x to the boundary $b\Omega$.*

(i) There exists a constant C such that for all $u \in W^{k,p}_{\text{loc}}(\Omega)$ with $\|\delta^{k-s} D^k u\|_{L^p(\Omega)} < \infty$, the following inequality holds

$$\|u\|_{H^{s,p}(\Omega)} \leq C \sum_{|\beta| \leq k} \|\delta^{k-s} D^\beta u\|_{L^p(\Omega)}. \tag{2.10}$$

(ii) There exists a constant C such that for all $u \in W^{k,p}_{\text{loc}}(\Omega)$ with $\|\delta^{k-s} D^k u\|_{L^p(\Omega)} < \infty$, the following inequality holds

$$\|u\|_{\Lambda^s(\Omega)} \leq C \sum_{|\beta| \leq k} \|\delta^{k-s} D^\beta u\|_{L^\infty(\Omega)}. \tag{2.11}$$

Proof. (i) See [SY24b, Prop. A.2].

(ii) By Proposition 2.16 we have the following equivalence of norms:

$$|u|_{\Omega,s} \approx \sum_{|\alpha| \leq k} |D^\alpha u|_{\Omega,s-k}, \quad \forall\, s \in \mathbb{R},\ k \in \mathbb{N}.$$

It is clear that (2.11) is an immediate consequence of the following statement: for all $u \in L^\infty_{\text{loc}}(\Omega)$,

$$|u|_{\Omega,-t} \leq C \|\delta^t u\|_{L^\infty(\Omega)}, \quad t > 0.$$

By Proposition 2.23, we have $\Lambda^{-t}(\Omega) = \mathscr{F}^{-t}_{\infty,\infty}(\Omega) = \big[\mathring{\mathscr{F}}^t_{1,1}(\Omega)\big]'$, where $\mathring{\mathscr{F}}^t_{1,1}(\Omega)$ denotes the norm closure of $C_c^\infty(\Omega)$ in $\mathscr{F}^t_{1,1}(\mathbb{R}^N)$. By definition we see that

$$\|f\|_{\mathscr{F}^t_{1,\infty}(\Omega)} = \Big\| \sup_{j\in\mathbb{N}} 2^{jt} |\lambda_j * f| \Big\|_{L^1(\Omega)} \leq \int_\Omega \sum_{j\in\mathbb{N}} 2^{jt} |\lambda_j * f(x)|\, dx = \sum_{j\in\mathbb{N}} 2^{jt} \|\lambda_j * f\|_{L^1(\Omega)} = \|f\|_{\mathscr{F}^t_{1,1}(\Omega)}.$$

Thus $\mathscr{F}^t_{1,1}(\Omega) \subset \mathscr{F}^t_{1,\infty}(\Omega)$ and $\mathring{\mathscr{F}}^t_{1,1}(\Omega) \subset \mathring{\mathscr{F}}^t_{1,\infty}(\Omega)$. It is clear that $\mathring{\mathscr{F}}^t_{1,\infty}(\Omega) \subset \{f \in \mathscr{F}^t_{1,\infty}(\mathbb{R}^N) : f \equiv 0 \text{ on } \overline{\Omega}^c\}$. Now, by Lemma 2.22, we have

$$\{f \in \mathscr{F}^t_{1,\infty}(\mathbb{R}^N) : f \equiv 0 \text{ on } \overline{\Omega}^c\} \subset L^1\big(\Omega, \delta^{-t}\big), \quad t > 0.$$

In summary we have $\mathring{\mathscr{F}}^t_{1,1}(\Omega) \subset L^1(\Omega, \delta^{-t})$, and $|f|_{L^1(\Omega,\delta^{-t})} \le C|f|_{\mathscr{F}^t_{1,1}(\Omega)}$ for any $f \in \mathring{\mathscr{F}}^t_{1,1}(\Omega)$. It follows that

$$L^\infty(\Omega, \delta^t) = \big[L^1(\Omega, \delta^{-t})\big]' \subset \big[\mathring{\mathscr{F}}^t_{1,1}(\Omega)\big]' = \Lambda^{-t}(\Omega).$$

and $|u|_{\Omega,-t} \le C|u|_{L^\infty(\Omega,\delta^t)}$ for all u. The proof is complete. □

3. Construction of the holomorphic support function

We modify Range's construction. Let D be a bounded pseudoconvex domain with C^∞ boundary in $\mathbb{C}^2$ of finite type m. Let r be the defining function of D such that $D = \{z \in U : r(z) < 0\}$. Fix $p_0 \in bD$ and a small neighborhood U_0 of p_0. For each $p \in U_0$, we introduce a special holomorphic coordinate system $\phi_p : \mathbb{C}^2 \to \mathbb{C}^2$ as in Catlin. The map ϕ_p is constructed as follows. Assume $\frac{\partial r}{\partial z_2}(p) \neq 0$. We define $\Phi^1 : \zeta^{(1)} \mapsto z$ by

$$\zeta_1 = z_1 - p_1$$
$$\zeta_2 = 2\frac{\partial r}{\partial z_1}(p)(z_1 - p_1) + 2\frac{\partial r}{\partial z_2}(p)(z_2 - p_2)$$

Denote $\rho^{(1)}(\zeta) = r \circ \Phi^{(1)}(\zeta)$. Then $\rho^{(1)}(0) = r(p)$, and

$$\rho^{(1)}(\zeta) = r(p) + \operatorname{Re}\zeta_2 + O(|\zeta|^2).$$

In general, suppose that for $2 \le l < m$, we have $\rho^{(l-1)} = r \circ \Phi^{(1)} \circ \cdots \Phi^{(l-1)}$, where

$$\rho^{(l-1)}(\zeta) = r(p) + \operatorname{Re} w_2 + \sum_{\substack{j+k\le l-1\\ j,k\ge 1}} a_{j,k}(p) w_1^j \overline{w}_1^k + \operatorname{Re}(2b_l(p)w_1^l) + O(|w_1|^l + |w_2||w|).$$

We then define $\Phi^l : \zeta \mapsto w$ by

$$\zeta_1 = w_1, \quad \zeta_2 = w_2 + 2b_l(p)w_1^l$$

So then $\rho^{(l)}(\zeta) = r \circ \Phi^{(1)} \cdots \circ \Phi^{(l)}(\zeta)$ takes the form

$$\rho^{(l+1)}(\zeta) = r(p) + \operatorname{Re}\zeta_2 + \sum_{\substack{j+k\le l\\ j,k\ge 1}} a_{j,k}(p)\zeta_1^j\overline{\zeta}_1^k + O(|\zeta_1|^{l+1} + |\zeta_2||\zeta|).$$

Let $\phi_p = \Phi^{(1)} \cdots \circ \Phi^{(m)}$. Then $\rho_p := r \circ \phi_p - r(p)$ takes the form

$$\rho_p(\zeta) = \operatorname{Re}\zeta_2 + \sum_{\substack{j,k\ge 1\\ j+k\le m}} a_{j,k}(p)\zeta_1^j\overline{\zeta}_1^k + O(|\zeta_1|^{m+1} + |\zeta_2||\zeta|). \tag{3.1}$$

ρ_p is the defining function of the domain $\Omega_p = \phi_p^{-1}(D)$. The coefficients $a_{j,k}(p)$ are linear combinations of products of derivatives of r at p up to order m and depend smoothly on $p \in U_0$.

For $l = 2, \dots, m$ and $\delta > 0$, set

$$A_l(q) = \max\{|a_{j,k}(q)| : j + k = l\},$$

and

$$\tau(q, \delta) = \min\left\{\left(\frac{\delta}{A_l(q)}\right)^{\frac{1}{l}} : 2 \le l \le m\right\}.$$

By the assumption that $p_0 \in bD$ is of finite type m, we know that at least one of the functions $a_{j,k}(p_0)$ is non-zero (otherwise the curve $t \mapsto (\phi_{p_0}(t), 0)$ has order of tangency with bD greater than m at p_0.) Hence by continuity and shrinking U_0, we can assume that the same function $a_{j,k}(q) \neq 0$ for all $q \in U_0$, and in particular $A_l(q) > 0$ for some $2 \le l \le m$. It follows that

$$\delta^{\frac{1}{2}} \lesssim \tau(q,\delta) \lesssim \delta^{\frac{1}{m}}, \quad q \in U_0. \tag{3.2}$$

The definition of $\tau(q,\delta)$ also implies that if $\delta' < \delta''$, then

$$(\delta/\delta'')^{\frac{1}{2}} \tau(q,\delta'') \le \tau(q,\delta') \le (\delta/\delta'')^{\frac{1}{m}} \tau(q',\delta''). \tag{3.3}$$

For $\delta \ge 0$ we define

$$J_\delta(p,\zeta) = \left[\delta^2 + |\zeta_2|^2 + \sum_{k=2}^{m} A_k(p)^2 |\zeta_1|^{2k} \right]^{\frac{1}{2}}. \tag{3.4}$$

For $a > 0$ and $\gamma \ge 0$, we define the generalized "nonisotropic" polydisk $P^{(a)}_{\delta,\gamma}(\zeta')$ centered at ζ' by

$$P^{(a)}_{\delta,\gamma}(\zeta') = \left\{ \zeta \in \mathbb{C}^2 : |\zeta_2 - \zeta_2'| < a(J_\delta(p,\zeta') + \gamma),\ |\zeta_1 - \zeta_1'| < \tau\big(p, aJ_\delta(p,\zeta')\big) + a\gamma \right\}.$$

Lemma 3.1. *Let $\tilde\zeta \in \{|\zeta| < a\} \setminus \Omega_p$ be such that $\operatorname{dist}(\tilde\zeta, \Omega_p) \approx |\tilde\zeta|$. Suppose that there exists $\tilde\delta > 0$ such that $\tilde\zeta \in S_{\tilde\delta} = \{\rho^{\varepsilon}_{\tilde\delta} = 0\}$, where $\varepsilon < C(p,m)$. Then $\tilde\delta \approx |\tilde\zeta|$.*

Proof. Since $\operatorname{dist}(\tilde\zeta, \Omega_p) = |\tilde\zeta|$, we can assume that for some $\lambda_0, \lambda_1 > 0$, $\rho(\tilde\zeta) \in (\lambda_0|\tilde\zeta|, \lambda_1|\tilde\zeta|)$. Since $\tilde\zeta \in S_{\tilde\delta}$, we have $\rho(\tilde\zeta) + \varepsilon H_{p,\tilde\delta}(\tilde\zeta) = 0$. By [Cat89, Prop. 4.1.] there exist constants $c_0, C_0 > 0$ such that $c_0 J_{\tilde\delta}(p,\tilde\zeta) < -H_{p,\tilde\delta}(\tilde\zeta) < C_0 J_{\tilde\delta}(p,\tilde\zeta)$. Hence $\rho(\tilde\zeta) \in (c_0\varepsilon J_{\tilde\delta}(p,\tilde\zeta), C_0\varepsilon J_{\tilde\delta}(p,\tilde\zeta))$. This implies that

$$c_0\varepsilon J_{\tilde\delta}(p,\tilde\zeta) < \lambda_1|\tilde\zeta|, \quad \lambda_0|\tilde\zeta| < C_0\varepsilon J_{\tilde\delta}(p,\tilde\zeta).$$

Since $\tilde\delta \le J_{\tilde\delta}(p,\tilde\zeta)$, the first inequality above shows that $\tilde\delta \le (c_0\varepsilon)^{-1}\lambda_1|\tilde\zeta|$. On the other hand, by the definition of J_δ, there exists $C_{\rho,p}$ such that $J_{\tilde\delta}(p,\tilde\zeta) \le \tilde\delta + C_{\rho,p}|\tilde\zeta|$, so the second inequality shows that $\lambda_0|\tilde\zeta| \le C_0\varepsilon(\tilde\delta + C_{\rho,p}|\tilde\zeta|)$. If $\varepsilon \le \frac{\lambda_0}{2C_0C_{\rho,p}}$, then $|\tilde\zeta| \le C^{-1}_{\rho,p}\tilde\delta$. ☐

Lemma 3.2. *There exist a sequence of points $\{\zeta_j\} \to 0$, and constant c_0, C_0 such that $\operatorname{dist}(\zeta_j, b\Omega_p) \approx |\zeta_j|$, $c_0 2^{-j} \le |\rho(\zeta_j)| \le C_0 2^{-(j-1)}$, and $\rho(\zeta_j) + \varepsilon H_{p,\delta_j}(\zeta_j) = 0$, with $\delta_j = 2^{-j}$. The constant c_0, C_0 depend only on ρ, p and ε.*

Proof. Denote $\rho^{\varepsilon}_\delta(\zeta) = \rho(\zeta) + \varepsilon H_{p,\delta}(\zeta)$. Take ζ to be of the form $(x_1, y_1, x_2, y_2) = (0, 0, x_2, 0)$. Since $\nabla\rho(0) = \frac{\partial\rho}{\partial x_2}(0) = 1$, we have $\operatorname{dist}(\zeta, \Omega_p) \approx x_2$ if x_2 sufficiently small. There exist some $\lambda_0, \lambda_1 > 0$ such that $\rho(\zeta) \in (\lambda_0 x_2, \lambda_1 x_2)$. By [Cat89, Prop. 4.1.], there exist $0 < c < C$ such that $-CJ_\delta(p,\zeta) < H_{p,\delta}(\zeta) < -cJ_\delta(p,\zeta)$. We seek ζ' in the form $\zeta' = (0,0,x_2',0)$ such that $\rho(\zeta') + \varepsilon H_{p,\delta}(\zeta') > 0$. Since

$$H_{p,\delta}(\zeta') > -CJ_\delta(p,\zeta') = -C(\delta^2 + (x_2')^2)^{\frac{1}{2}} \ge -C(\delta + x_2'),$$

it suffices to find ζ' with $\rho(\zeta') - \varepsilon C(\delta + x_2') = x_2' - \varepsilon C(\delta + x_2') > 0$, or equivalently,

$$x_2' > \frac{\varepsilon C}{1 - \varepsilon C}\delta.$$

On the other hand, we seek ζ'' in the form $(0,0,x_2'',0)$ such that $\rho(\zeta'') + \varepsilon H_{p,\delta}(\zeta'') < 0$. Since

$$H_{p,\delta}(\zeta'') < -cJ_\delta(p,\zeta'') = -c(\delta^2 + (x_2'')^2)^{\frac{1}{2}} < -c'(\delta + x_2''), \quad c' = c/\sqrt{2},$$

it suffices to find ζ'' with $\rho(\zeta'') - \varepsilon c'(\delta + x_2'') = x_2'' - \varepsilon c'(\delta + x_2'') < 0$, or equivalently,

$$x_2'' < \frac{\varepsilon c'}{1 - \varepsilon c'}\delta.$$

To summarize, we choose $\zeta' = (0,0,x_2',0)$ and $\zeta'' = (0,0,x_2'',0)$ such that

$$|\zeta''| = x_2'' < \frac{\varepsilon c'}{1-\varepsilon c'}\delta < \frac{\varepsilon C}{1-\varepsilon C}\delta < x_2' = |\zeta'|.$$

For example, we can choose $x_2'' = \frac{1}{2}\frac{\varepsilon c}{1-\varepsilon c}\delta$ and $x_2' = 2\frac{\varepsilon C}{1-\varepsilon C}\delta$. Since $\rho^\varepsilon_\delta(\zeta)$ is a continuous function in ζ, there exists $a_* \in \left(\frac{1}{2}\frac{\varepsilon c'}{1-\varepsilon c'}\delta, 2\frac{\varepsilon C}{1-\varepsilon C}\delta\right)$ such that $\rho^\varepsilon_\delta\big((0,0,a^*,0)\big) = 0$. We now set $\delta = \delta_j = 2^{-j}$, and let $\zeta_j = (0,0,a_j^*,0)$ be such that $\rho^\varepsilon_{\delta_j}\big((0,0,a_j^*,0)\big) = 0$. In other words $\rho(\zeta_j) + \varepsilon H_{p,\delta_j}(\zeta_j) = 0$. Setting $c_0 = \frac{1}{2}\frac{\varepsilon c'}{1-\varepsilon c'}$ and $C_0 = \frac{\varepsilon C}{1-\varepsilon C}$, we then have $|\zeta_j| = a_j^* \in (c_0 2^{-j}, C_0 2^{-(j-1)})$. Since $\rho(\zeta_j) \approx \operatorname{dist}(\zeta_j, \Omega_p) = |\zeta_j|$, by adjusting the constants c_0, C_0 we get $\rho(\zeta_j) \in (c_0 2^{-j}, C_0 2^{-(j-1)})$. □

Lemma 3.3. *There exists a constant $a > 0$ (independent of p, ζ', δ), such for $\zeta' \in \Omega_p \cap \{|\zeta| < a\}$ and $\zeta \in P^{(a)}_{\delta,\gamma}(\zeta')$, the following estimates hold*

$$J_\delta(p,\zeta) \le C_{m,\rho}\big(J_\delta(p,\zeta') + a\gamma\big);$$
$$J_\delta(p,\zeta') \le C_{m,\rho}\big(J_\delta(p,\zeta) + a\gamma\big).$$

Proof. For simplicity we write $J_\delta(\zeta') = J_\delta(p,\zeta')$ and $\tau' = \tau(p, aJ_\delta(p,\zeta'))$. The condition $\zeta \in P^{(a)}_{\delta,\gamma}(\zeta')$ implies that

$$\big||\zeta_2| - |\zeta_2'|\big| \le a\big(J_\delta(\zeta') + \gamma\big), \quad \big||\zeta_1| - |\zeta_1'|\big| \le \tau' + a\gamma \tag{3.5}$$

Thus for $\zeta \in P^{(a)}_{\delta,\gamma}(\zeta')$, we have

$$\begin{aligned} J^2_\delta(\zeta) &= \delta^2 + |\zeta_2|^2 + \sum_{k=2}^m \big[A_k(p)\big]^2 |\zeta_1|^{2k} \\ &\le C_m\Big[\delta^2 + |\zeta_2'|^2 + \big(aJ_\delta(\zeta')\big)^2 + (a\gamma)^2 + \sum_{k=2}^m A_k(p)^2\left(|\zeta_1'|^{2k} + (\tau')^{2k} + (a\gamma)^{2k}\right)\Big] \end{aligned} \tag{3.6}$$

By definition

$$\tau' = \tau(p, aJ_\delta(\zeta')) \le \left(\frac{aJ_\delta(\zeta')}{A_k(p)}\right)^{\frac{1}{k}} \quad \text{for } 2 \le k \le m. \tag{3.7}$$

Plugging the above into (3.6) and using the fact that $(a\gamma)^{2k} < (a\gamma)^2$ $(a,\gamma{<}1)$, we get

$$\begin{aligned} J^2_\delta(\zeta) &\le C_{m,\rho}\Big[\delta^2 + |\zeta_2'|^2 + \big(aJ_\delta(\zeta')\big)^2 + (a\gamma)^2 + \sum_{k=2}^m \big[A_k(p)\big]^2 |\zeta_1'|^{2k}\Big] \\ &\le C'_{m,\rho}\big(\big[J_\delta(\zeta')\big]^2 + (a\gamma)^2\big). \end{aligned}$$

For the other direction, we have

$$\begin{aligned} \big[J_\delta(\zeta')\big]^2 &= \delta^2 + |\zeta_2'|^2 + \sum_{k=2}^m A_k(p)^2 |\zeta_1'|^{2k} \\ &\le C_m\Big[\delta^2 + |\zeta_2|^2 + \big(aJ_\delta(\zeta')\big)^2 + (a\gamma)^2 + \sum_{k=2}^m A_k(p)^2\left(|\zeta_1|^{2k} + (\tau')^{2k} + (a\gamma)^{2k}\right)\Big] \\ &= C_{m,\rho}\left(\big[J_\delta(\zeta)\big]^2 + \big(aJ_\delta(\zeta')\big)^2 + (a\gamma)^2\right) \end{aligned}$$

Taking $a < \frac{1}{2}$, we get $\big[J_\delta(\zeta')\big]^2 \le C'_{m,\rho}\Big(\big[J_\delta(\zeta)\big]^2 + (a\gamma)^2\Big)$. Taking square root we are done. □

Proposition 3.4. *There are positive constants a_0 and c, and for each $p \in U_0 \cap bD$, there is a family of pseudoconvex domains $\{\Omega_p^\delta\}_{0<\delta<\delta_0}$ with the following properties:*

(i) $0 \in b\Omega_p^*$*, where* $\Omega_p^* := int\big[\cap_{0<\delta<\delta_0}\Omega_p^\delta\big]$
(ii) $\{\zeta \in \overline{\Omega}_p : 0 < |\zeta| < c\} \subset \Omega_p^* \subset \Omega_p^\delta$*;*
(iii) For $\zeta' \in \Omega_p$ *with* $|\zeta'| < c$ *one has*

$$P_{\delta,\delta}^{(a_0)}(\zeta') \subset \Omega_p^\delta.$$

The constant a_0 *depends only on* ρ *not is independent of* δ *and* ζ'*.*

Proof. For simplicity we will write $J_\delta(p,\zeta)$ as $J_\delta(\zeta)$. Part (i) and (ii) are the same as in [Ran90, Prop. 2.4, (i),(ii)], and we shall include the details here for the reader's convenience. For small s and $\delta > 0$, define

$$W_{s,\delta}(p) = \{\zeta \in \mathbb{C}^2 : |\zeta| < c \text{ and } |\rho(\zeta)| < sJ_\delta(\zeta)\}$$

Catlin (see the proof of [Cat89, Prop. 4.2]) shows that c, s, ε_0 and δ_0 can be chosen so that for all $0 < \varepsilon < \varepsilon_0$, $0 < \delta \le \delta_0$, the set

$$S_\delta = \{\zeta \in W_{s,\delta}(p) : \rho_\delta^\varepsilon(\zeta) = 0\}, \quad \rho_\delta^\varepsilon(\zeta) := \rho(\zeta) + \varepsilon H_{p,\delta}(\zeta)$$

is a smooth pseudoconvex hypersurface (from the side $\rho_\delta^\varepsilon < 0$), and that the constants can be chosen independently of $p \in U_0 \cap bD$. We now fix $\varepsilon_0 = \varepsilon_0(\rho)$ which is independent of δ. It follows that

$$\Omega_p^\delta = \{|\zeta| < c : \rho(\zeta) < 0\} \cup \{\zeta \in W_{s,\delta}(p) : \rho_\delta^{\varepsilon_0}(\zeta) < 0\}$$

is a pseudoconvex domain which satisfies

$$\overline{\Omega}_p \cap \{|\zeta| < c\} \subset \Omega_p^\delta \subset \big(\Omega_p \cap \{|\zeta| < c\}\big) \cup W_{s,\delta}(p).$$

Define

$$\Omega_p^* = int \bigcap_{0<\delta<\delta_0} \Omega_p^\delta.$$

Then Ω_p^* is pseudoconvex. Since $\rho_\delta^{\varepsilon_0}(0) = \rho(0) + \varepsilon_0 H_{p,\delta}(0) \approx -J_\delta(p,0) = -\delta$, it follows that $\operatorname{dist}(0, S_\delta) \lesssim \delta$ and thus $0 \in b\Omega_p^*$, which proves (i).

Next we prove (ii). Suppose $\zeta \in \overline{\Omega_p}$ and $0 < |\zeta| < c$. Then $\rho(\zeta) \le 0$.

$$\rho_\delta^{\varepsilon_0}(\zeta) = \rho(\zeta) + \varepsilon_0 H_{p,\delta}(\zeta) \le -\varepsilon_0 c J_\delta(\zeta) \le -c\varepsilon_0\Big(|\zeta_1|^2 + \sum_{k=2}^m [A_k(p)]^2|\zeta_2|^{2k}\Big)^{\frac{1}{2}}.$$

For fixed $\zeta \ne 0$, the expression on the right-hand side is a negative constant independent of δ. Also it is clear that $\zeta \in W_{s,\delta}(p)$ since $\rho(\zeta) \le 0 < c_\zeta \le sJ_\delta(\zeta)$ uniformly for all δ. Hence $\zeta \in \Omega_p^*$. This proves (ii).

We now prove (iii). First, we show that for $\zeta' \in \Omega_p \cap \{|\zeta| < c\}$, we have

$$|\rho(\zeta) - \rho(\zeta')| < C_\rho a_0^{1/m} J_\delta(\zeta'), \quad \forall\, \zeta \in P_{\delta,\delta}^{(a_0)}(\zeta'), \tag{3.8}$$

where $a_0 > 0$ is some number to be determined. Denote

$$R_b(p) = \big\{\zeta \in \mathbb{C}^2 : |\zeta_1| < \tau(p,b),\ |\zeta_2| < b\big\},$$

We observe that $\zeta \in R_{J_\delta(\zeta)}(p)$. Indeed, we have $|\zeta_2| < J_\delta(\zeta)$ and for some $2 \le k_* \le m$,

$$\tau(p, J_\delta(\zeta)) = \left(\frac{J_\delta(\zeta)}{A_{k_*}(p)}\right)^{\frac{1}{k_*}} \ge \left(\frac{A_{k_*}(p)|\zeta_1|^{k_*}}{A_{k_*}(p)}\right)^{\frac{1}{k_*}} = |\zeta_1|.$$

By [Cat89, Prop. 1.2], we have

$$|D_1^l\rho(\eta)| \lesssim b\big[\tau(\zeta', b)\big]^{-l}, \quad \eta \in R_b(\zeta').$$

Apply the above estimate with $l = 1$ and $b = J_\delta(\eta)$ to get

$$|D_1\rho(\eta)| \lesssim J_\delta(\eta)\big[\tau(p, J_\delta(\eta))\big]^{-1}, \quad \eta \in \phi_p^{-1}(U_0). \tag{3.9}$$

In particular the above holds for $\eta \in P^{(a_0)}_{\delta,\delta}(\zeta')$. We now compare $\overline{C}J_\delta(\eta)$ and $\overline{C}J_\delta(\zeta')$ for $\eta \in P^{(a_0)}_{\delta,\delta}(\zeta')$. By Lemma 3.3 we have

$$J_\delta(\eta) \le C_{m,\rho}\big(J_\delta(\zeta') + a_0\delta\big) \le C'_{m,\rho}J_\delta(\zeta');$$
$$J_\delta(\eta) \ge \big(C^{-1}_{m,\rho}J_\delta(\zeta') - a_0\delta\big) \ge c_{m,\rho}J_\delta(\zeta'),$$

where we chose $a_0 < (2C_{m,\rho})^{-1}$. Hence

$$|D_1\rho(\eta)| \le C_{m,\rho}J_\delta(\eta)\big[\tau(p, J_\delta(\eta))\big]^{-1} \le C_{m,\rho}J_\delta(\zeta')\big[\tau(p, J_\delta(\zeta'))\big]^{-1}, \quad \eta \in P^{(a_0)}_{\delta,\delta}(\zeta').$$

By the mean value theorem and (3.9), we have for $\zeta \in P^{(a_0)}_{\delta,\delta}(\zeta')$,

$$\begin{aligned} |\rho(\zeta) - \rho(\zeta')| &\le C_\rho J_\delta(\zeta')\tau(p, J_\delta(\zeta'))^{-1}|\zeta_1 - \zeta'_1| + C_\rho|\zeta_2 - \zeta'_2| \\ &\le C_\rho J_\delta(\zeta')\tau(p, J_\delta(\zeta'))^{-1}\big[\tau(p, a_0J_\delta(\zeta')) + a_0\delta\big] + C_\rho a_0\big[J_\delta(\zeta') + \delta\big]. \end{aligned} \tag{3.10}$$

By (3.3), we have $\tau(p, a_0J_\delta(\zeta')) \le a_0^{1/m}\tau(p, J_\delta(\zeta'))$. Also $J_\delta(\zeta')\tau(p, J_\delta(\zeta'))^{-1} \le 1$ by (3.2) (assume that $J_\delta(\zeta') < 1$). Thus (3.10) implies $|\rho(\zeta) - \rho(\zeta')| \le C_\rho\big[a_0^{1/m}J_\delta(\zeta') + a_0\delta\big] \le C''_\rho a_0^{1/m}J_\delta(\zeta')$ for all $\zeta \in P^{(a_0)}_{\delta,\delta}(\zeta')$. This proves (3.8).

To finish the proof we consider two different cases.

Case 1. $\zeta' \in \overline{\Omega}_p \cap \{|\zeta| < c\} \setminus W_{s,\delta}(p)$.

By definition of $W_{s,\delta}(p)$ we have $\rho(\zeta') < -sJ_\delta(\zeta')$. Together with (3.8) this implies that

$$\rho(\zeta) \le \rho(\zeta') + |\rho(\zeta) - \rho(\zeta')| \le -sJ_\delta(\zeta') + C_\rho a_0^{\frac{1}{m}}J_\delta(\zeta') < 0, \quad \zeta \in P^{(a_0)}_{\delta,\delta}(\zeta'),$$

where we choose $a_0 < [s/C_\rho]^m$. Hence $\zeta \in \Omega_p \subset \Omega^\delta_p$.

Case 2. $\zeta' \in \overline{\Omega}_p \cap \{|\zeta| < c\} \cap W_{s,\delta}(p)$. It suffices to consider $\zeta \in P^{(a_0)}_{\delta,\delta}(\zeta') \setminus \overline{\Omega}_p$, in which case $\rho(\zeta) \ge 0$. By (3.8), we have

$$0 \le \rho(\zeta) \le \rho(\zeta') + |\rho(\zeta) - \rho(\zeta')| \le C_\rho a_0^{1/m}J_\delta(\zeta').$$

By our choice of $a_0 < [s/C_\rho]^m$, the last quantity is bounded by $sJ_\delta(\zeta')$, which means $\zeta \in W_{s,\delta}(p)$. On the other hand, Lemma 3.3 shows that $\rho(\zeta) \le C_\rho a_0^{1/m}J_\delta(\zeta') < C'_\rho a_0^{1/m}J_\delta(\zeta)$. Hence

$$\rho^{\varepsilon_0}_\delta(\zeta) := \rho(\zeta) + \varepsilon_0 H_{p,\delta}(\zeta) \le C_\rho a_0^{\frac{1}{m}}J_\delta(\zeta) - \varepsilon_0 cJ_\delta(\zeta), \quad \zeta \in P^{(a_0)}_{\delta,\tilde{\delta}}(\zeta').$$

The last expression is negative if we choose $a_0 < \big[\varepsilon_0 c(C_\rho)^{-1}\big]^m$. This shows that $P^{(a_0)}_{\delta,\delta}(\zeta') \subset \Omega^\delta_p$. $\square$

Proposition 3.5. *Let $p \in U_0 \cap bD$ and let Ω^*_p be the domain from Proposition 3.4. There are positive constants c' and γ which are independent of p such that*

$$r(z) \ge c'|z - p|^m \quad \text{for} \quad |z - p| \le \gamma \quad \text{and} \quad z \notin \phi_p(\Omega^*_p).$$

Proof. It suffices to show that if $|\zeta_0| < \gamma'$ for some $\gamma' > 0$ and $\zeta_0 \notin \Omega^*_p$, then $\rho(\zeta_0) \ge c|\zeta|^m$. By definition of Ω^*_p, there exists $\delta_0 > 0$ such that $\zeta_0 \notin \Omega^{\delta_0}_p$, where

$$\Omega^{\delta_0}_p = \{|\zeta| < c : \rho(\zeta) < 0\} \cup \{\zeta \in W_{s,\delta_0}(p) : \rho^{\varepsilon_0}_{\delta_0}(\zeta) < 0\}.$$

Hence $0 < \rho^{\varepsilon_0}_{\delta_0}(\zeta_0) := \rho(\zeta_0) + \varepsilon_0 H_{p,\delta_0}(\zeta_0)$, or $\rho(\zeta_0) > -\varepsilon_0 H_{p,\delta_0}(\zeta_0) \approx \varepsilon_0 J_{\delta_0}(p, \zeta_0)$. By (3.4), we have $J_{\delta_0}(p, \zeta_0) \gtrsim |\zeta_2| + |\zeta_1|^m \gtrsim |\zeta|^m$, and the proof is done. $\square$

Since a pseudoconvex domain $D \subset \mathbb{C}^2$ of finite type is regular in the sense of Diederich and Fornaess [DF77], it follows that there is a pseudoconvex domain $\widehat{D}$ with

$$\overline{D} \subset \widehat{D} \Subset \{z : r(z) < c\gamma^m\}.$$

Let $\mu = \sup\{r(z) : z \in b\widehat{D}\}$; then $0 < \mu < c\gamma^m$, and we can choose $\gamma' > 0$, such that $0 < \mu < c(\gamma')^m < c\gamma^m$. For each $0 < \delta < \delta_0$, we define the domain D_δ by

$$D_\delta := \big[\widehat{D} \cap \{z : |z-p| < \gamma\} \cap \phi_p\big(\Omega_p^\delta\big)\big] \cup \{\widehat{D} \cap \{z : |z-p| > \gamma'\}. \tag{3.11}$$

We also define the domain D_* by

$$D_* := \big[\widehat{D} \cap \{z : |z-p| < \gamma\} \cap \phi_p\big(\Omega_p^*\big)\big] \cup \{\widehat{D} \cap \{z : |z-p| > \gamma'\}. \tag{3.12}$$

Proposition 3.6. *For each $0 < \delta < \delta_0$, the domain D_δ is pseudoconvex. The same holds for D_*.*

Proof. The proof is a slight modification of [Ran90, Lem. 3.15]. Denote

$$D_\delta^1 = \widehat{D} \cap \{z : |z-p| < \gamma\} \cap \phi_p\big(\Omega_p^\delta\big), \quad D_\delta^2 = \{\widehat{D} \cap \{z : |z-p| > \gamma'\}.$$

We show that D_δ is pseudoconvex at any boundary point $w \in bD_\delta$. If $|w-p| < \gamma'$ (resp. $|w-p| > \gamma$), then $w \in bD_\delta^2$ (resp. bD_δ^1). Assume now that $\gamma' \le |w-p| \le \gamma$. Suppose that $w \notin \phi_p(\Omega_p^\delta)$, then in particular $w \notin \phi_p(\Omega_p^*)$ and Proposition 3.5 shows that $r(w) \ge c|w-p|^m \ge c(\gamma')^m$. On the other hand, we know that $w \in \overline{\widehat{D}}$, so that $r(w) \le \mu$, which is a contradiction as we had chosen $\mu < c(\gamma')^m$. Hence we must have $w \in \phi_p(\Omega_p^\delta)$. This forces $w \in b\widehat{D}$, and thus D_δ is pseudoconvex at w. The proof for D_* is similar and we leave the details to the reader. □

We denote by $P^{b_1,b_2}(\zeta')$ the polydisk

$$P^{b_1,b_2}(\zeta') = \big\{\zeta \in \mathbb{C}^2 : |\zeta_1 - \zeta_1'| < b_1, |\zeta_2 - \zeta_2'| < b_2\big\}.$$

Proposition 3.7. *Let $h \in \mathcal{O}(\Omega)$ and $\zeta' \in \Omega$ be such that $P^{(b_1,b_2)}(\zeta') \subset \Omega$. Then the following estimate holds*

$$|D^\alpha h(\zeta')| \le \frac{C_\alpha}{b_1^{\alpha_1+1} b_2^{\alpha_2+1}} \|h\|_{L^2(P^{(b_1,b_2)}(\zeta'))}. \tag{3.13}$$

Proof. Let $0 < r_1 < b_1$ and $0 < r_2 < b_2$. Denote $P^{(b_1,b_2)} := \{\zeta \in \mathbb{C}^2 : |\zeta_1| < r_1, |\zeta_2| < r_2\}$. By the Cauchy integral formula,

$$h(z) = \frac{1}{(2\pi i)^2} \int_{bP^{(b_1,b_2)}(z)} \frac{h(\zeta)}{(\zeta_1 - z_1)(\zeta_2 - z_2)}\, d\zeta_1\, d\zeta_2.$$

Taking derivatives and writing in polar coordinates $\zeta_j = z_j + r_j e^{i\theta_j}$ with $\theta_j \in [0, 2\pi]$, we get

$$D^\alpha h(z) = \frac{\alpha! i^2}{(2\pi i)^2} \int_{[0,2\pi]^2} \frac{h(\zeta)(r_1 e^{i\theta_1})(r_2 e^{i\theta_2})}{(\zeta_1 - z_1)^{\alpha_1+1}(\zeta_2 - z_2)^{\alpha_2+2}}\, d\theta_1\, d\theta_2.$$

Multiply $r_1^{\alpha_1+1} r_2^{\alpha_2+1}$ on both sides:

$$|D^\alpha h(z)| r_1^{\alpha_1+1} r_2^{\alpha_2+1} \le \frac{\alpha!}{(2\pi)^2} \int_{[0,2\pi]^2} |h(\zeta)| r_1 r_2\, d\theta_1\, d\theta_2$$

Now integrate r_j from 0 to b_j and apply Hölder's inequality to get

$$\begin{aligned}
\frac{b_1^{\alpha_1+2}}{\alpha_1+2} \frac{b_2^{\alpha_1+2}}{\alpha_2+2} |D^\alpha h(z)| &\le \frac{\alpha!}{(2\pi)^2} \int_{r_1=0}^{b_1} \int_{r_2=0}^{b_2} \int_{\theta_1=0}^{2\pi} \int_{\theta_1=0}^{2\pi} |h(\zeta)| r_1 r_2\, dr_1\, dr_2\, d\theta_1\, d\theta_2 \\
&\le \frac{\alpha!}{(2\pi)^2} \Big(\int_{\substack{0 \le r_j \le b_j \\ 0 \le \theta_j \le 2\pi}} |h(\zeta)|^2 (r_1 r_2)^2\, dr_1\, dr_2\, d\theta_1\, d\theta_2 \Big)^{\frac{1}{2}} (b_1 b_2)^{\frac{1}{2}} (2\pi) \\
&\le \frac{\alpha!}{2\pi} b_1 b_2 \left(\int_{P^{(b_1,b_2)}(\zeta')} |h(\zeta)|^2\, dV(\zeta) \right)^{\frac{1}{2}},
\end{aligned}$$

which proves (3.13). □

Let $\tilde{\zeta} \notin \Omega_p^\delta$ and $|\tilde{\zeta}| < C\delta$. Denote $d_\delta(z) := \operatorname{dist}(z, b\Omega_p^\delta)$. Suppose $h \in \mathcal{O}(\Omega_p^\delta)$ satisfies

$$\left[M^\eta_{\tilde{\zeta},\delta}(h)\right]^2 := \int_{\Omega_p^\delta} \frac{|h(\zeta)|^2 (d_\delta)^{2\eta}(\zeta)}{|\zeta - \tilde{\zeta}|^2}\, dV(\zeta) < \infty.$$

Denote

$$\begin{aligned} \beta_2(\zeta') &= (a_0/2)\big(J_\delta(p,\zeta') + \delta\big) \approx a_0 J_\delta(p,\zeta'), \\ \beta_1(\zeta') &= \tau\big(p, (a_0/2) J_\delta(p,\zeta')\big) + (a_0/2)\delta \approx \tau\big(p, a_0 J_\delta(p,\zeta')\big). \end{aligned} \tag{3.14}$$

We now apply Proposition 3.7 to $P^{(\beta_1,\beta_2)}(\zeta') = P^{(a_0/2)}_{\delta,\delta}(\zeta') \subset \Omega_p^\delta$ for $\zeta' \in \Omega_p \cap \{|\zeta| < c\}$, and we obtain the following:

$$|D^\alpha h(\zeta')| \le C_\alpha \frac{1}{\beta_1^{\alpha_1+1}} \frac{1}{\beta_2^{\alpha_2+1}} \left[\int_{P^{(\beta_1,\beta_2)}(\zeta')} |h(\zeta)|^2\, dV(\zeta)\right]^{1/2}. \tag{3.15}$$

We estimate

$$\begin{aligned} \int_{P^{(\beta_1,\beta_2)}(\zeta')} |h(\zeta)|^2\, dV(\zeta) &= \int_{P^{(\beta_1,\beta_2)}(\zeta')} \frac{|h(\zeta)|^2 (d_\delta)^{2\eta}(\zeta)}{|\zeta - \tilde{\zeta}|^2} \frac{|\zeta - \tilde{\zeta}|^2}{(d_\delta)^{2\eta}(\zeta)}\, dV(\zeta) \\ &< \left[M^\eta_{\tilde{\zeta},\delta}(h)\right]^2 \sup_{\zeta \in P^{(\beta_1,\beta_2)}(\zeta')} \frac{|\zeta - \tilde{\zeta}|^2}{(d_\delta)^{2\eta}(\zeta)}. \end{aligned} \tag{3.16}$$

To proceed, we need to estimate $|\zeta - \tilde{\zeta}|$ from above and $d_\delta(\zeta)$ from below, for $\zeta \in P^{(\beta_1,\beta_2)}(\zeta')$. We have

$$\begin{aligned} |\zeta - \tilde{\zeta}| &\le |\zeta - \zeta'| + |\zeta'| + |\tilde{\zeta}| \\ &\le \beta_1(\zeta') + \beta_2(\zeta') + |\zeta'| + C\delta \\ &\le C_\rho \tau\big(p, J_\delta(p,\zeta')\big) + |\zeta'|, \end{aligned} \tag{3.17}$$

where we used the assumption $|\tilde{\zeta}| < C\delta$. Note that that the constant C_ρ depends on a_0. By Proposition 3.4, we have

$$P^{(a_0)}_{\delta,\delta}(\zeta') = \left\{\zeta \in \mathbb{C}^2 : |\zeta_2 - \zeta_2'| < a_0(J_\delta(p,\zeta') + \delta),\ |\zeta_1 - \zeta_1'| < \tau\big(p, a_0 J_\delta(p,\zeta')\big) + a_0\delta\right\} \subset \Omega_p^\delta.$$

Thus for $w \in b\Omega_p^\delta$, we must either have
(i) $|w_2 - \zeta_2'| \ge a_0\big(J_\delta(p,\zeta') + \delta\big) = 2\beta_2$;
or
(ii) $|w_1 - \zeta_1'| \ge \tau\big(p, a_0 J_\delta(p,\zeta')\big) + a_0\delta$.

If $|w_2 - \zeta_2'| \ge 2\beta_2$, then

$$|w_2 - \zeta_2| \ge |w_2 - \zeta_2'| - |\zeta_2' - \zeta_2| \ge 2\beta_2 - \beta_2 = \beta_2 \approx a_0 J_\delta(p,\zeta'). \tag{3.18}$$

If $|w_1 - \zeta_1'| \ge \tau\big(p, a_0 J_\delta(p,\zeta')\big) + a_0\delta$, then by using (3.3),

$$\begin{aligned} |w_1 - \zeta_1| &\ge |w_1 - \zeta_1'| - |\zeta_1' - \zeta_1| \\ &\ge \tau\big(p, a_0 J_\delta(p,\zeta')\big) + a_0\delta - \Big[\tau\big(p, \frac{a_0}{2} J_\delta(p,\zeta')\big) + \frac{a_0}{2}\delta\Big] \\ &\ge (2^{1/m} - 1)\tau\Big(p, \frac{a_0}{2} J_\delta(p,\zeta')\Big) + \frac{a_0}{2}\delta \\ &\ge C_\rho\Big[\frac{a_0}{2} J_\delta(p,\zeta')\Big]^{1/2} \ge C_\rho' J_\delta(p,\zeta'). \end{aligned} \tag{3.19}$$

Together, (3.18) and (3.19) imply that $|w - \zeta| \ge C_\rho J_\delta(p,\zeta')$, which means

$$d_\delta(\zeta) \ge C_\rho J_\delta(p,\zeta') \quad \text{for} \quad \zeta \in P^{(a_0)}_{\delta,\delta}(\zeta'). \tag{3.20}$$

Using (3.17) and (3.20) in (3.16), we get

$$\int_{P_{\delta,\delta}^{(a_0/2)}(\zeta')} |h(\zeta)|^2\, dV(\zeta) \le C_\rho \big[M^\eta_{\tilde\zeta,\delta}(h)\big]^2 \frac{\big[\tau(p, J_\delta(p,\zeta') + |\zeta'|\big]^2}{\big[J_\delta(p,\zeta')\big]^{2\eta}}.$$

Using the above estimate and (3.14) in (3.15), and denoting $\tau = \tau(p, J_\delta(p,\zeta'))$, we get

$$\begin{aligned} |D^\alpha h(\zeta')| &\le C_{\rho,\alpha}\frac{1}{\tau^{\alpha_1+1}}\frac{1}{\big[J_\delta(p,\zeta')\big]^{\alpha_2+1}}\frac{\tau+|\zeta'|}{\big[J_\delta(p,\zeta')\big]^{\eta}} M^\eta_{\tilde\zeta,\delta}(h) \\ &\le C_{\rho,\alpha}\left(\frac{1}{\tau^{\alpha_1}[J_\delta(p,\zeta')]^{\alpha_2+1+\eta}} + \frac{|\zeta'|}{\tau^{\alpha_1+1}[J_\delta(p,\zeta')]^{\alpha_2+1+\eta}}\right) M^\eta_{\tilde\zeta,\delta}(h). \end{aligned} \tag{3.21}$$

By the Taylor expansion of ρ ((3.1)), we have

$$J_\delta(p,\zeta') \ge c_\rho\big(\delta + |\rho(\zeta')| + |(\zeta')_2| + |\zeta'|^m\big), \tag{3.22}$$

which then implies

$$\tau = \tau\big(p, J_\delta(p,\zeta')\big) \ge c_\rho\big[J_\delta(p,\zeta')\big]^{1/2} \ge c_\rho J_\delta(p,\zeta') \ge c'_\rho\big(\delta + |\rho(\zeta')| + |(\zeta')_2|\big).$$

On the other hand, by definition we have for some $2 \le l_0 \le m$,

$$\tau\big(p, J_\delta(p,\zeta')\big) = \left(\frac{J_\delta(p,\zeta')}{A_{l_0}(p)}\right)^{1/l_0} \ge \left(\frac{A_{l_0}(p)|(\zeta')_1|^{l_0}}{A_{l_0}(p)}\right)^{1/l_0} = |(\zeta')_1|.$$

Hence

$$\tau\big(p, a_0 J_\delta(p,\zeta')\big) \ge c_\rho\big[\delta + |\rho(\zeta')| + (\zeta')_1 + (\zeta')_2\big] \ge c_\rho\big(\delta + |\rho(\zeta')| + |\zeta'|\big). \tag{3.23}$$

Combining (3.21) with (3.22), we obtain the following

Lemma 3.8. *Let $\tilde\zeta \notin \Omega_p^\delta$ and $|\tilde\zeta| < C\delta$. Suppose $h \in \mathcal{O}(\Omega_p^\delta)$ and $M^\eta_{\tilde\zeta,\delta}(h) < \infty$ for some $\eta > 0$. Then for $\zeta' \in \Omega_p$ with $|\zeta'| < c$ the following estimate holds:*

$$|D^\alpha h(\zeta')| \le C_{\rho,\alpha}\frac{M^\eta_{\tilde\zeta,\delta}(h)}{\big(\delta + |\rho(\zeta')| + |\zeta'|\big)^{\alpha_1}\big(\delta + |\rho(\zeta')| + |\zeta'_2| + |\zeta'_1|^m\big)^{\alpha_2+1+\eta}}. \tag{3.24}$$

We now pull back the estimates onto the original domain via the biholomorphic map ϕ_p. For $q \notin D_\delta$, we denote

$$I^\eta_{q,\delta}[h] := \left[\int_{D_\delta}\frac{|h(\zeta)|^2}{|\zeta - q|^2}\,\mathrm{dist}^{2\eta}_{bD_q}(\zeta) dV(\zeta)\right]^{\frac12}, \quad L_i = d\phi_p\left(\frac{\partial}{\partial\zeta_i}\right), \quad i = 1, 2.$$

The following result is an immediate consequence of the lemma above. Recall that D_δ given by (3.11).

Lemma 3.9. *Let $q \notin D_\delta$ and suppose that $|q - p| < C\delta$. Let $h \in \mathcal{O}(D_\delta)$ with $I^\eta_{q,\delta}[h] < \infty$ for some $\eta > 0$. Then for $z \in D$ with $|z - p| < c$ the following estimate holds:*

$$|L_1^{\alpha_1}L_2^{\alpha_2}h(z)| \le C_{r,\alpha}\frac{I^\eta_{q,\delta}(h)}{\big[\delta + |r(z)| + |z-p|\big]^{\alpha_1}\big[\delta + |r(z)| + |g(p,z)| + |z-p|^m\big]^{\alpha_2+1+\eta}}.$$

The following result is known as the Skoda's division theorem, which we shall use to construct holomoprhic support functions.

Proposition 3.10. [Sko72, Thm 1] *Let Ω be an pseudoconvex domain in $\mathbb{C}^n$, and ψ be a plurisubharmonic function in Ω. Let $g_1, g_2, \dots, g_m$ be a system of m functions holomorphic in Ω. Let $\alpha > 1$ and $\mu = \min\{n, m-1\}$. Let f be a holomorphic function in Ω such that*

$$\int_\Omega |f|^2 |g|^{-2\alpha\mu-2} e^{-\psi}\, dV(z) < \infty.$$

Then there exist $h_1, h_2, \dots, h_m$ holomorphic in Ω such that $f = \sum_{i=1}^m g_i h_i$ and

$$\int_\Omega |h|^2 |g|^{-2\alpha\mu} e^{-\psi}\, dV(z) \le \frac{\alpha}{\alpha-1} \int_\Omega |f|^2 |g|^{-2\alpha\mu-2} e^{-\psi}\, dV(z).$$

For $q \in U_0 \setminus D$ with $|q-p| < Cr(q)$, we want to apply Skoda's theorem to obtain holomorphic functions on a domain $D_*(q)$ associated with q, and satisfying

$$\int_{D_*(q)} \frac{|h_{\eta,i}(q,z)|^2}{|z-q|^2} \big[\mathrm{dist}(z, D_*(q))\big]^{2\eta}\, dV(z) < C,$$

for some constant C independent of q.

We now describe how to construct such domain $D_*(q)$. Fix $p \in bD$, and let $\{\zeta_j\} \notin \Omega_p$, $\{\delta_j\} > 0$ be the sequence given by Lemma 3.2, such that $\zeta_j \in b\Omega_p^{\delta_j}$ and $|\rho(\zeta_j)| \approx \delta_j = 2^{-j}$. Denote $q_j := \phi_p(\zeta_j)$, so then $|r(q_j)| \approx \delta_j$, for all j. For each $q \in U_0 \setminus D$ with $|q-p| < Cr(q)$, let $j_* = j_*(q) = \max\{j : r(q_j) \le r(q) < r(q_{j-1})\}$, so that $r(q) \ge r(q_{j_*})$. Since $r(q_j) \in \big(c_0 2^{-j}, C_0 2^{-(j-1)}\big)$ for all j, we have

$$r(q) \le r(q_{j_*-1}) \le C_0 2^{-(j_*-2)} = 4C_0 c_0^{-1} c_0 2^{-j_*} \le C_1 r(q_{j_*}), \quad C_1 = 4C_0 c_0^{-1}.$$

Hence $r(q) \approx r(q_{j_*})$, which implies that $|q-p| < Cr(q) \approx Cr(q_{j_*}) \approx C\delta_{j_*}$.

Let $D_*(q) := D_{\delta_{j_*}}$ be given by (3.11). By Proposition 3.6, $D_*(q)$ is pseudoconvex, so we can apply Proposition 3.10 to $D_*(q)$.

Proposition 3.11. *For each $\eta > 0$ there is a constant $C_{D,\eta}$ such that for $q \in U_0 \setminus \overline{D}$ there are functions $h_{\eta,i}(q,\cdot) \in \mathcal{O}(D_*(q))$, $i = 1,2$ such that*

$$h_{\eta,1}(q,z)(z_1-q_1) + h_{\eta,2}(q,z)(z_2-q_2) = 1, \quad z \in D_*(q). \tag{3.25}$$

Furthermore, we have

$$\int_{D_*(q)} \frac{|h_{\eta,i}(q,z)|^2}{|z-q|^2} \big[\mathrm{dist}(z, D_*(q))\big]^{2\eta}\, dV(z) < C_{D,\eta}. \tag{3.26}$$

Similarly, there are holomorphic functions $h_{\eta,i}(p,\cdot) \in \mathcal{O}(D_*)$, $i = 1,2$ which satisfy (3.25) and (3.26). The domain D_* is given by (3.12)

Proof. Since $D_*(q)$ is pseudoconvex, the function $-\log \mathrm{dist}(z, D_*(q))$ is a plurisubharmonic function in $D_*(q)$. We show that $q \notin D_*(q) = D_{\delta_{j_*}}$, or $r(q) + \varepsilon \tilde{H}_{p,\delta_{j_*}}(q) \ge 0$. Denote $\zeta_0 = \phi_p^{-1}(q)$ and $\zeta_{j_*} = \phi_p^{-1}(q_{j_*})$. Since $\zeta_{j_*} \in b\Omega_p^{\delta_{j_*}}$, we have $\rho_{\delta_{j_*},\varepsilon}(\zeta_{j_*}) = 0$. Note that $J_{\delta_j}(p,\zeta_0) \gtrsim J_{\delta_j}(p,\zeta_j)$, since $\zeta_j = (0,0,x_2^j,0)$, $\zeta_0 = (x_1,y_1,x_2,y_2)$, and $x_2 \approx r(q) \approx r(q_j) \approx x_2^j > 0$. Using $-C_0 J(p,\zeta_j) \le H_{p,\delta_j}(\zeta_j) < -c_0 J_{\delta_j}(p,\zeta_j)$, we obtain

$$H_{p,\delta_{j_*}}(\zeta_0) \ge -C_0 J_{\delta_{j_*}}(p,\zeta_0) \gtrsim -C_0 J_{\delta_{j_*}}(p,\zeta_j) = (C_0 c_0^{-1})\big[-c_0 J_{\delta_{j_*}}(p,\zeta_j)\big] \ge c_1 H_{p,\delta_{j_*}}(\zeta_j).$$

It follows that

$$\rho(\zeta_0) + \varepsilon H_{p,\delta_{j_*}}(\zeta_0) \ge c_1 \rho(\zeta_{j_*}) + \varepsilon c_1 H_{p,\delta_{j_*}}(\zeta_{j_*}) = c_1 \rho_{\delta_{j_*},\varepsilon}(\zeta_{j_*}) = 0,$$

which shows that $q \notin D_*(q)$. We wish to apply Proposition 3.10 to $D_*(q)$ with $\alpha = 1 + \eta/2$, $\mu = 1$, $f \equiv 1$, $g = z - q$ and $\psi = -2\eta \log \mathrm{dist}(\cdot, D_*(q))$. Notice that since $q \notin D_*(q)$, $\mathrm{dist}(z, D_*(q)) \le |z-q|$ for all $z \in D_*(q)$. It follows that

$$\int_{D_*(q)} |f|^2 |g|^{-2\alpha\mu-2} e^{-\psi}\, d\lambda = \int_{D_*(q)} |z-q|^{-4-\eta} \big[\mathrm{dist}(z, D_*(q))\big]^{2\eta}(z) \lesssim \int_{D_*(q)} |z-q|^{-4+\eta} < C_{D,\eta},$$

where the constant $C_{D,\eta}$ is independent of q and q_{j_*}. Hence there exist functions $h_{\eta,i}(q,\cdot)$ that are holomorphic in $D_*(q)$, $\sum_{i=1}^2 h_{\eta,i}(q,z)(z_i - q_i) = 1$, and $h_{\eta,i}(q,\cdot)$ satisfies the estimate:

$$\int_{D_*(q)} \frac{|h_{\eta,i}(q,z)|^2}{|z-q|^2} \big[\operatorname{dist}(z, D_*(q))\big]^{2\eta}(z)\, dV(z) \le C_{D,\eta}. \qquad \square$$

The proof for the case $p \in bD$ is similar we will leave the details to the reader.

Combining Proposition 3.11 with Lemma 3.9, we get

Proposition 3.12. *Let $q \in U_0 \setminus \overline{D}$ and suppose that $|q-p| < Cr(q)$. There exist functions $h_{\eta,i}(q,\cdot) \in \mathcal{O}(D_*(q))$, $i = 1,2$ such that*

$$h_{\eta,1}(q,z)(z_1 - q_1) + h_{\eta,2}(q,z)(z_2 - q_2) = 1, \quad z \in D_*(q).$$

Furthermore, for $z \in D$ with $|z-p| < c$ the following estimate holds:

(3.27)

$$|L_1^{\alpha_1} L_2^{\alpha_2} h_{\eta,i}(q,z)| \le C_{D,\alpha} \frac{I_q^\eta(h)}{\big[r(q) + |r(z)| + |z-p|\big]^{\alpha_1} \big[r(q) + |r(z)| + |g(p,z)| + |z-p|^m\big]^{\alpha_2+1+\eta}}.$$

In particular,

$$|h(q,z)| \le C_D \frac{I_q^\eta(h)}{\big[r(q) + |r(z)| + |g(p,z)| + |z-p|^m\big]^{1+\eta}}. \tag{3.28}$$

Proof. We take $h_{\eta,i}$ to be the ones constructed in Proposition 3.11. In view of (3.26), and since $q \notin D_*(q)$, $|q-p| < Cr(q) < C'\delta_{j_*}$, we can apply Lemma 3.9 with D_δ replaced by $D_*(q) = D_{\delta_{j_*}}$, and δ replaced by δ_{j_*}. The estimate then follows by the fact that $\delta_{j_*} \approx r(q_{j_*}) \approx r(q)$. $\square$

Note that the functions $h_{\eta,i}$ come from Skoda's theorem applied to the domain $D_*(q)$, and it is not known how the functions $h_{\eta,i}(q,\cdot)$ depend on q. To address this issue we apply the trick of Range to show that if z is restricted to a compact subset of D, then $h_{\eta,i}$ can be replaced by functions which are smooth in $q \in U_0 \setminus D$, while essentially preserving all the relevant properties. In our case, since q does not need to be the boundary, we need to impose the additional condition that $|q-p| < Cr(q)$, for some fixed $C > 1$.

For $0 < \epsilon < \epsilon_0$, let

$$D'_\epsilon = \{\delta_{bD'_\epsilon}(z) < 0\}, \quad \delta_{bD'_\epsilon}(z) := r(z) + \epsilon.$$

To simplify notation we will fix $\eta > 0$ and drop the subscript η in $h_{\eta,i}$. For $q \in U_0 \setminus D$, define

$$\Phi_q(\zeta, z) := \sum_{i=1}^2 h_i(q,z)(z_i - \zeta_i).$$

Then $\Phi_q \in C^\infty\big(\mathbb{C}^2 \times D_*(q)\big) \subset C^\infty\big(\mathbb{C}^2 \times \overline{D}\big)$, $h_i(q,\cdot) \in \mathcal{O}(D_*(q)) \subset \mathcal{O}(D)$, and $\Phi_q(q,z) \equiv 1$ on $D_*(q)$ and in particular on D. Since $h_i(q,\cdot)$ is continuous on $D_*(q)$ and $D'_\epsilon \Subset D_*(q)$, we have $|h_i(q,z)| \le C_{q,\epsilon}(h)$ for all $z \in \overline{D'_\epsilon}$, where $C_{q,\epsilon}(h)$ is an upper bound of $|h_i(q,\cdot)|$ on $\overline{D'_\epsilon}$. This implies that

$$|\Phi_q(\zeta,z) - \Phi_q(q,z)| = \left| \sum_{i=1}^2 h_i(q,z)(q_i - \zeta_i) \right| \le 2C_{q,\epsilon}(h)|\zeta - q|.$$

Hence if we take $|\zeta - q| < \varepsilon_{q,\varepsilon} = \big[4C_{q,\epsilon}(h)\big]^{-1}$, then $|\Phi_q(\zeta,z) - \Phi_q(q,z)| < 1/2$. In other words,

$$|\Phi_q(\zeta,z)| \ge \frac{1}{2} \quad \text{for} \quad (\zeta,z) \in B(q,\varepsilon_{q,\epsilon}) \times \overline{D'_\epsilon}.$$

We now set

$$h_{q,i}(\zeta,z)=\frac{h_i(q,z)}{\Phi_q(\zeta,z)},\quad \Phi_q(\zeta,z):=\sum_{i=1}^{2}h_i(q,z)(z_i-\zeta_i). \tag{3.29}$$

Then $h_{q,i}\in C^\infty\big(B(q,\varepsilon_{q,\epsilon})\times\overline{D'_\epsilon}\big)$, holomorphic in $z\in\overline{D'_\epsilon}$, and

$$\sum_{i=1}^{2}h_{q,i}(\zeta,z)(\zeta_i-z_i)=1\quad\text{on}\quad B(q,\varepsilon_{q,\epsilon})\times\overline{D'_\epsilon}.$$

By Proposition 3.12 and (3.29), we have for $\zeta\in B(q,\varepsilon_{q,\epsilon})$ and $z\in\overline{D'_\epsilon}$ with $|z-p|<c$,

$$|h_{q,i}(\zeta,z)|\lesssim|h_i(q,z)|\le\frac{C_{D,\eta}}{\big[r(q)+\epsilon+|r(z)|-\epsilon+|g(p,z)|+|z-p|^m\big]^{1+\eta}}. \tag{3.30}$$

We need to change q to ζ on the above right-hand side. For this we need to estimate $\varepsilon_{q,\epsilon}:=c_*[C_{q,\epsilon}(h)]^{-1}$, where $C_{q,\epsilon}(h)$ is any upper bound of $h_i(q,\cdot)$ on $\overline{D'_\epsilon}$ (as defined right below), and c_* is some small constant to be determined. By (3.28), we have

$$|h_i(q,z)|\le\frac{C_{D,\eta}}{\big[r(q)+|r(z)|\big]^{1+\eta}}\le\frac{C_{D,\eta}}{\big[r(q)+\epsilon\big]^{1+\eta}}:=C_{q,\epsilon}(h),\quad z\in\overline{D'_\epsilon}. \tag{3.31}$$

Hence if we set $c_*<\big(8\|r\|_1\big)^{-1}$, then

$$\varepsilon_{q,\epsilon}=c_*[C_{q,\epsilon}(h)]^{-1}=c_*C_{D,\eta}^{-1}\big[r(q)+\epsilon\big]^{1+\eta}<\frac{1}{8\|r\|_1}\big(r(q)+\epsilon\big),$$

where we assume that $C_{D,\eta}>1$. This implies that $|r(\zeta)-r(q)|\le\|r\|_1|\zeta-q|<\frac{1}{8}(r(q)+\epsilon)$ for $\zeta\in B(q,\varepsilon_{q,\epsilon})$, and consequently

$$r(q)+\epsilon\ge r(\zeta)-|r(\zeta)-r(q)|+\epsilon\ge r(\zeta)-\frac{1}{8}r(q)+\frac{7}{8}\epsilon.$$

or $(9/8)r(q)+\epsilon\ge r(\zeta)+(7/8)\epsilon$. This further implies that $(9/8)(r(q)+\epsilon)\ge(7/8)(r(\zeta)+\epsilon)$ and

$$\frac{1}{2}(r(q)+\epsilon)\ge\frac{1}{3}(r(\zeta)+\epsilon) \tag{3.32}$$

By similar reasoning, if we set $\varepsilon_{q,\epsilon}:=c_*[C_{q,\epsilon}(h)]^{-1}$ with $c_*<\big[8\|g\|_1\big]^{-1}$, $\|g\|_1:=\sup_{|z-p|<a_1}\|g(\cdot,z)\|_1$ we get

$$|g(q,z)|\ge|g(\zeta,z)|-\frac{1}{8}\big(r(q)+\epsilon\big),\quad\zeta\in B(q,\varepsilon_{q,\epsilon}), \tag{3.33}$$

We would like to bound $g(p,z)$ below by $g(q,z)-\frac{1}{8}\big(r(q)+\epsilon\big)$. However this would require that $|p-q|$ to be less than $\big(8\|g\|_1\big)^{-1}\big(r(q)+\epsilon\big)$ which is clearly impossible since $|p-q|\approx r(q)$ $(r(p)=0)$. To fix this issue, we replace $|g(p,z)|$ by the smaller quantity $|\operatorname{Im}g(p,z)|$, which in view of Lemma 3.14 below can be used as local coordinate on bD. As a result, if we choose $|p^T-q^T|<\big(8\|g\|_1\big)^{-1}(r(q)+\epsilon)$, where p^T denotes the projection of p on bD, then

$$|\operatorname{Im}g(p,z)-\operatorname{Im}g(q,z)|\le\|g\|_1|p^T-q^T|\le\frac{1}{8}(r(q)+\epsilon),$$

which implies $|\operatorname{Im}g(p,z)|\ge|\operatorname{Im}g(q,z)|-\frac{1}{8}\big(r(q)+\epsilon\big)$. Now (3.33) clearly holds with g replaced by $\operatorname{Im}g$. Thus

$$|\operatorname{Im}g(p,z)|\ge|\operatorname{Im}g(\zeta,z)|-\frac{1}{4}\big(r(q)+\epsilon\big),\quad\zeta\in B(q,\varepsilon_{q,\epsilon}). \tag{3.34}$$

Now, $|z-p| < a_1$, $|p-q| < Cr(q) < c_1$ and $|q-\zeta| < \varepsilon_{q,\epsilon}$, so there exists $C_1 > 0$ such that $\||\cdot - z|^m\|_1 \le C_m$ on some fixed ball centered at p and contains z, q, ζ in the given range. Hence if we set $\varepsilon_{q,\epsilon} = c_*[C_{q,\epsilon}(h)]^{-1}$ with $c_* < (8C_m)^{-1}$, and argue the same way as for (3.33), we get

$$|z^T - q^T|^m \ge |z^T - \zeta^T|^m - \frac{1}{8}\big(r(q)+\epsilon\big), \quad \zeta \in B(q, \varepsilon_{q,\epsilon}). \tag{3.35}$$

Likewise, by choosing $|q^T - p^T| < (8C_m)^{-1}\big(r(q)+\epsilon\big)$, we get

$$\big||z^T-p^T|^m - |z^T - q^T|^m\big| \le C_m (8C_m)^{-1}\big(r(q)+\epsilon\big) \le \frac{1}{8}\big(r(q)+\epsilon\big).$$

Thus we have

$$|z^T - p^T|^m \ge |z^T - q^T|^m - \frac{1}{8}\big(r(q)+\epsilon\big), \quad |q-p| < Cr(q). \tag{3.36}$$

Together, (3.35) and (3.36) imply that

$$|z^T - p^T|^m \ge |z^T - \zeta^T|^m - \frac{1}{4}\big(r(q)+\epsilon\big), \quad \zeta \in B(q,\varepsilon_{q,\epsilon}), \quad |q^T - p^T| < (8C_m)^{-1}\big(r(q)+\epsilon\big). \tag{3.37}$$

For each $p \in U_0 \cap bD$, we define the set

$$U'_{p,\epsilon} = \left\{ q \in U_0 \setminus D : |p-q| < Cr(q) \quad \text{and} \quad |p^T - q^T| < \frac{1}{8C'_m}\big(r(q)+\epsilon\big) \right\},$$

where $C'_m = \max\{C_m, \|g\|_1\}$. Here the important fact is that for all $\epsilon \to 0$, the set $U'_{p,\epsilon}$ contains the "cone":

$$\left\{ q \in U_p : |p^T - q^T| < c\, r(q) \right\}.$$

Without loss of generality we assume that $C'_m > 1$ so that $|p^T - q^T| < \frac{1}{8}\big(r(q)+\epsilon\big)$ for $q \in U'_{p,\epsilon}$.

Using estimates (3.32), (3.37), (3.34) and (3.38) in (3.30), we conclude that if $\varepsilon_{q,\epsilon} = c_*[C_{q,\epsilon}(h)]^{-1} < c_*\big(r(q)+\epsilon\big)$, where c_* is a sufficiently small constant depending only on r and p, then for each $q \in U'_{p,\epsilon}$,

$$|h_{q,i}(\zeta, z)| \le \frac{C_{r,\eta}}{\big[r(\zeta)+\epsilon + |r(z)| - \epsilon + |g(\zeta,z)| + |z-\zeta|^m\big]^{1+\eta}}.$$

If ϵ is sufficiently small, $\delta_{bD'_\epsilon}(z) := r(z)+\epsilon$ is a defining function for the domain D'_ϵ, and

$$|r(z)| - \epsilon = -r(z) - \epsilon = -\delta_{bD'_\epsilon}(z), \quad z \in D'_\epsilon. \tag{3.38}$$

Thus we have

$$|h_{q,i}(\zeta,z)| \le \frac{C_{r,\eta}}{\big[\Gamma_\epsilon(\zeta,z)\big]^{1+\eta}}, \quad \Gamma_\epsilon(\zeta,z) := \delta_{bD'_\epsilon}(\zeta) - \delta_{bD'_\epsilon}(z) + |\operatorname{Im} g(\zeta,z)| + |z-\zeta|^m.$$

Next we estimate the z-derivatives of $h_{q,i}$. By (3.29) and (3.27) we have

$$\begin{aligned} |L_1^{\alpha_1} L_2^{\alpha_2} h_{q,i}(\zeta,z)| \lesssim |L_1^{\alpha_1} L_2^{\alpha_2} h_i(q,z)| &\lesssim \frac{C_{\alpha,\eta}}{\big[r(q)+|r(z)|+|z-p|\big]^{\alpha_1}\big[\Gamma_\epsilon(\zeta,z)\big]^{\alpha_2+1+\eta}} \\ &= \frac{C_{\alpha,\eta}}{\big[r_\epsilon(q) - \delta_{bD'_\epsilon}(z) + |z-p|\big]^{\alpha_1}\big[\Gamma_\epsilon(\zeta,z)\big]^{\alpha_2+1+\eta}}, \end{aligned} \tag{3.39}$$

where we used that $r(q) + |r(z)| = r(q) + \epsilon + |r(z)| - \epsilon = r_\epsilon(q) - \delta_{bD'_\epsilon}(z)$. By (3.32) we get $\frac{1}{2}(r(q)+\epsilon) \ge \frac{1}{3}\big(r(\zeta)+\epsilon\big)$, and similar estimate as in (3.35) shows that $|z^T - q^T| \ge |z^T - \zeta^T| - \frac{1}{8}\big(r(q)+\epsilon\big)$ for $\zeta \in B(q, \varepsilon_{q,\epsilon})$. Since

$$\big||z^T - p^T| - |z^T - q^T|\big| \le |p^T - q^T| < \frac{1}{8}\big(r(q)+\epsilon\big), \quad q \in U'_{p,\epsilon},$$

we get

$$|z^T - p^T| \ge |z^T - q^T| - \frac{1}{8}\big(r(q)+\epsilon\big), \quad q \in U'_{p,\epsilon}.$$

Consequently

$$|z^T - p^T| \geq |z^T - \zeta^T| - \frac{1}{4}\big(r(q) + \epsilon\big), \quad q \in U'_{p,\epsilon} \quad \text{and} \quad \zeta \in B(q, \varepsilon_{q,\epsilon}).$$

It follows from (3.39) that for z, q, ζ in the chosen ranges,

$$|D_z^\alpha h_{q,i}(\zeta, z)| \lesssim \frac{C_{\alpha,\eta}}{\big[\delta_{bD'_\epsilon}(\zeta) - \delta_{bD'_\epsilon}(z) + |z^T - \zeta^T|\big]^{\alpha_1}\big[\Gamma_\epsilon(\zeta, z)\big]^{\alpha_2+1+\eta}} \lesssim \frac{C_{\alpha,\eta}}{|z-\zeta|^{\alpha_1}\big[\Gamma_\epsilon(\zeta, z)\big]^{\alpha_2+1+\eta}}.$$

By compactness, finitely many sets $B(q_1, \varepsilon_{q_1,\epsilon}), \dots, B(q_N, \varepsilon_{q_N,\epsilon})$ will cover $\overline{U'_{p,\epsilon}}$. By our choice of c_*, the value $C_{q,\epsilon}$ given by (3.31) and the fact that all the q_i-s are in $U_0 \setminus D$, the constants $\varepsilon_{q_i,\epsilon} := c_*[C_{q_i,\epsilon}(h)]^{-1}$ are bounded uniformly below by a constant depending only on ρ and ϵ.

Choose a partition of unity $\chi_\nu \in C_c^\infty(B(q_\nu, \varepsilon_{q_\nu,\epsilon}))$ with $0 \leq \chi_\nu \leq 1$, for $\nu = 1, \dots, N$ and $\sum_{\nu=1}^N \chi_\nu \equiv 1$ on $\overline{U}_0 \setminus D$. Set

$$h_i^\epsilon(\zeta, z) = \sum_{\nu=1}^m \chi_\nu(\zeta) h_{q_\nu,i}(\zeta, z), \quad \text{for } \zeta \in U'_{p,\epsilon} \text{ and } z \in D'_\epsilon\ .$$

We now summarize the properties of h_i^ϵ.

Proposition 3.13. *For each sufficiently small $\epsilon > 0$ there are functions $h_i^\epsilon \in C^\infty\big((U'_{p,\epsilon}) \times D'_\epsilon\big)$, $i = 1, 2$ with the following properties:*

(i) $$h_1^\epsilon(\zeta, z)(z_1 - \zeta_1) + h_2^\epsilon(\zeta, z)(z_2 - \zeta_2) = 1;$$

(ii) $$h_i^\epsilon(\zeta, \cdot) \in \mathcal{O}(D'_\epsilon), \quad \textit{for } \zeta \in U'_{p,\epsilon};$$

(iii) $$\Big|D_z^\alpha h_i^{(\epsilon)}(\zeta, z)\Big| \lesssim \frac{C_{\alpha,\eta}}{|z-\zeta|^{\alpha_1}\big[\Gamma_\epsilon(\zeta, z)\big]^{\alpha_2+1+\eta}}, \quad |z - \zeta| < c.$$

$$\Gamma_\epsilon(\zeta, z) = \delta_{bD'_\epsilon}(\zeta) - \delta_{bD'_\epsilon}(z) + |\operatorname{Im} g(\zeta, z)| + |z - \zeta|^m.$$

For fixed ϵ, and $z \in D'_\epsilon$, the functions $h_i^{(\epsilon)}(\cdot, z)$ are defined in a neighborhood $U'_{p,\epsilon}$ of an arbitrary point $p \in bD$. By compactness of bD, we can use a partition of unity in ζ to patch together the functions $h_i^{(\epsilon)}$ to obtain smooth functions $w_i^{(\epsilon)}$ on $(\mathcal{U} \setminus \overline{D}) \times D'_\epsilon$, where $\mathcal{U}$ is some fixed neighborhood of $\overline{D}$ that does not shrink with ϵ. Furthermore, $w_i^{(\epsilon)}$ satisfies the same properties (i)-(iii) as $h_i^{(\epsilon)}$ in Proposition 3.13.

Lemma 3.14. *Let $p_0 \in bD$, one can find a neighborhood $\mathcal{V}$ of p_0 such that for all $z \in \mathcal{V}$, there exists a coordinate map $\phi_z : \mathcal{V} \to \mathbb{C}^n$ given by $\phi_z : \zeta \in \mathcal{V} \to (s, t) = (s_1, s_2, t_1, t_2)$, where $s_1 = \delta_{bD'_\epsilon}(\zeta)$. Moreover, for $z \in \mathcal{V} \cap D'_\epsilon$ and $\zeta \in \mathcal{V} \setminus \overline{D}'_\epsilon$, the function $\Gamma_\epsilon(\zeta, z)$ satisfies*

$$|\Gamma_\epsilon(z, \zeta)| \geq c\left(|\delta_{bD'_\epsilon}(z)| + s_1 + |s_2| + |t|^m\right), \quad \delta_{bD'_\epsilon}(z) := \operatorname{dist}(z, bD'_\epsilon), \tag{3.40}$$

$$|\zeta - z| \geq c(s_1 + |s_2| + |t|), \tag{3.41}$$

for some constant c depending on the domain.

Proof. It suffices to show that for fixed z sufficiently close to a boundary point $p_0 \in bD$, the function $\zeta \to \operatorname{Im} g(\zeta, z)$ can be introduced as a (real) local coordinate on bD. Let U_0 be the neighborhood of p_0 such that corresponding to each $p \in U_0$, there exists a biholomorphic map ϕ_p given in the beginning of Section 3. We are done if we can show that

$$dr(p_0) \wedge d_\zeta \operatorname{Im} g(p_0, p_0) \neq 0.$$

Recall that $g(p,z) = \left(\phi_p^{-1}\right)^{(2)}(z)$. Since $\phi_p^{-1}(p) = 0$, we have $g(p,p) = 0$ for all $p \in U_0$. Thus $\nabla_\zeta g(p,p) = -\nabla_z g(p,p)$. Hence it suffices to show that

$$dr(p_0) \wedge d_z \operatorname{Im} g(p_0,p_0) \neq 0. \tag{3.42}$$

Without loss of generality, we can assume that $dr(p_0) = dx_2$. Denote $z' = \phi_{p_0}^{-1}(z)$. Then $r(z) = \rho(z')$, and

$$\rho(z') = \operatorname{Re} z'_2 + O(|z'|^2), \quad z'_2 = (\phi_{p_0}^{-1})^{(2)}(z) = g(p_0,z).$$

we have

$$\frac{1}{2} = \frac{\partial r}{\partial z_2}(p_0) = \frac{\partial \rho}{\partial z'_2}(0)\frac{\partial z'_2}{\partial z_2}(p_0) = \frac{1}{2}\frac{\partial z'_2}{\partial z_2}(p_0), \quad \text{and} \quad 0 = \frac{\partial r}{\partial z_1}(p_0) = \frac{\partial \rho}{\partial z'_2}(0)\frac{\partial z'_2}{\partial z_1}(p_0) = \frac{1}{2}\frac{\partial z'_2}{\partial z_1}(p_0).$$

Thus $\frac{\partial z'_2}{\partial z_2}(p_0) = 1$ and $\frac{\partial z'_2}{\partial z_1}(p_0) = 0$. This implies that

$$d_z \operatorname{Im} g(p_0,p_0) = d_z \operatorname{Im} z'_2(p_0) = \frac{1}{2i}\left[d_z z'_2(p_0) - d_z \overline{z}'_2(p_0)\right] = \frac{1}{2i}(dz_2 + d\overline{z}_2) = dy_2.$$

Hence $dr(p_0) \wedge d_z \operatorname{Im} g(p_0,p_0) = dx_2 \wedge dy_2 \neq 0$. This shows that if U_0 is a sufficiently small neighborhood of p_0, then for each $z \in U_0$, $\zeta \to \operatorname{Im} g(\zeta,z)$ can be used as a coordinate on $U_0 \cap bD$. In fact, by proving (3.42) we have also shown that for fixed $\zeta \in U_0$, the function $z \to \operatorname{Im} g(\zeta,z)$ can be introduced as a (real) local coordinate on $U_0 \cap bD$. □

4. Estimates for the $\overline{\partial}$ homotopy operator

The Leray maps w_i^ϵ defined on $(\zeta,z) \in \mathcal{U} \times D'_\epsilon$ allow us to construct a $\overline{\partial}$ homotopy formula on the domain D'_ϵ. Let $W^{(\epsilon)}(z,\zeta) = w_1^{(\epsilon)}(\zeta,z)d\zeta_1 + w_2^{(\epsilon)}(\zeta,z)d\zeta_2$.

Let Ω be a pseudoconvex domain of finite type m in $\mathbb{C}^2$. Suppose φ is a $(0,1)$-form such that φ and $\overline{\partial}\varphi$ are in $C^\infty(\overline{\Omega})$. We will show that the following $\overline{\partial}$ homotopy formula

$$\varphi = \overline{\partial}\mathcal{H}_q\varphi + \mathcal{H}_{q+1}\overline{\partial}\varphi$$

holds in the sense of distributions. Our homotopy operator $\mathcal{H}$ has the form

$$\mathcal{H}\varphi = \mathcal{H}^0\varphi + \mathcal{H}^1\varphi, \tag{4.1}$$

where

$$\mathcal{H}^0\varphi(z) := \int_{\mathcal{U}} K^0(z,\cdot) \wedge \mathcal{E}\varphi, \quad \mathcal{H}^1\varphi(z) := \int_{\mathcal{U}\setminus\overline{\Omega}} K^{01}(z,\cdot) \wedge [\overline{\partial},\mathcal{E}]\varphi. \tag{4.2}$$

Here $\mathcal{E}$ is an extension operator satisfying the boundedness properties in Corollary 2.15, and $\operatorname{supp}\mathcal{E}\varphi \subset \mathcal{U}$ for all φ. To achieve this we simply multiply the Rychkov extension operator with a smooth cut-off function χ and denote this new operator by $\mathcal{E}$.

$$K^0(z,\zeta) = \frac{1}{(2\pi i)^n}\frac{\langle \overline{\zeta} - \overline{z}\,,\, d\zeta\rangle}{|\zeta - z|^2} \wedge \left(\overline{\partial}_{\zeta,z}\frac{\langle \overline{\zeta} - \overline{z}\,,\, d\zeta\rangle}{|\zeta - z|^2}\right), \quad \overline{\partial}_{\zeta,z} = \overline{\partial}_\zeta + \overline{\partial}_z; \tag{4.3}$$

$$K^{01}(z,\zeta) = \frac{1}{(2\pi i)^2}\frac{\langle \overline{\zeta} - \overline{z}\,,\, d\zeta\rangle}{|\zeta - z|^2} \wedge \frac{\langle W, d\zeta\rangle}{\langle W,\, \zeta - z\rangle} = \frac{1}{(2\pi i)^2}\frac{\langle \overline{\zeta} - \overline{z}\,,\, d\zeta\rangle}{|\zeta - z|^2} \wedge \langle W, d\zeta\rangle \tag{4.4}$$

We set $K^1_{0,-1} = 0$ and $K^{0,1}_{0,-1} = 0$.

Proposition 4.1. *For any $s > 0$ and $1 < p < \infty$, the operator $\mathcal{H}_0$ is $H^{s,p}(D'_\epsilon) \to H^{s+1,p}(D'_\epsilon)$ bounded and $\Lambda^s(D'_\epsilon) \to \Lambda^{s+1}(D'_\epsilon)$ bounded. More precisely, there exists a constant C depending only on s, the dimension n, and $\operatorname{dist}(D,\mathcal{U})$ such that*

$$\|\mathcal{H}^0\varphi\|_{H^{s+1,p}(D)} \le C\|\varphi\|_{H^{s,p}(D)}, \quad \|\mathcal{H}^0\varphi\|_{\Lambda^{s+1}(\overline{D})} \le C\|\varphi|_{\Lambda^s(\overline{D})}. \tag{4.5}$$

Proof. Let f be a coefficient function of $\mathcal{E}\varphi$. In view of (4.3), $\mathcal{H}^0\varphi$ can be written as a finite linear combination of

$$\int_{\mathcal{U}} \frac{\overline{\zeta^i - z^i}}{|\zeta - z|^4} f(\zeta)\, dV(\zeta) = \frac{1}{n-1}\int_{\mathcal{U}} \partial_{z_i}\left(|\zeta - z|^{-2}\right) f(\zeta)\, dV(\zeta) = c_0 \partial_{z_i}(N * f)(z),$$

where N denotes the Newtonian potential. The estimates are now standard. We first show that for any non-negative integer k and $0 < \alpha < 1$,

$$\|N * f\|_{W^{k+2+\alpha,p}(\mathcal{U})} \le C_{n,k,p}\|f\|_{W^{k+\alpha,p}(\mathcal{U})}, \quad \|N * f\|_{\Lambda^{k+2+\alpha}(\mathcal{U})} \le C_n\|f\|_{\Lambda^{k+\alpha,p}(\mathcal{U})}. \tag{4.6}$$

For $k = 0$, the reader can refer to [GT01, Thm 4.6] for the case of Hölder space, and [GT01, Thm 9.9, p. 230] for the case of Sobolev space. For $k \ge 1$, one can apply integration by parts and use the fact that f is compactly supported in $\mathcal{U}$ to move derivatives from the kernel to f (see for example [Shi21, Prop. 3.2]), we leave the details to the reader. Once (4.6) is established, the general case follows from interpolation (Proposition 2.20). □

Lemma 4.2. *Let $\beta \ge 0$, $\alpha > -1$, and let $0 < \delta < \frac{1}{2}$. If $\alpha < \beta - \frac{1}{m}$, then*

$$\int_0^1\int_0^1\int_0^1 \frac{s_1^\alpha t\, ds_1\, ds_2\, dt}{(\delta + s_1 + s_2 + t^m)^{2+\beta}(\delta + s_1 + s_2 + t)} \le C_{\alpha,\beta}\delta^{\alpha-\beta+\frac{1}{m}}.$$

Proof. Partition the domain of integration into seven regions:

$R_1 : t > t^m > \delta, s_1, s_2$. We have

$$I \le \int_{\delta^{\frac{1}{m}}}^1 \frac{t}{t^{2m+m\beta}t}\left(\int_0^{t^m} s_1^\alpha\, ds_1\right)\left(\int_0^{t^m} ds_2\right) dt \le C\int_{\delta^{\frac{1}{m}}}^1 t^{m(\alpha-\beta)}\, dt \le C\delta^{\alpha-\beta+\frac{1}{m}}.$$

$R_2 : t > \delta > t^m, s_1, s_2$. We have

$$I \le \delta^{-2-\beta}\left(\int_\delta^{\delta^{\frac{1}{m}}} \frac{t}{t}\, dt\right)\left(\int_0^\delta s_1^\alpha\, ds_1\right)\left(\int_0^\delta ds_2\right) \le C\delta^{\alpha-\beta+\frac{1}{m}}.$$

$R_3 : t > s_1 > \delta, t^m, s_2$. We have

$$I \le \int_\delta^1 \frac{s_1^\alpha}{s_1^{2+\beta}}\left(\int_0^{s_1^{1/m}} \frac{t}{t}\, dt\right)\left(\int_0^{s_1} ds_2\right) ds_1 \le C\int_\delta^1 s_1^{\alpha-\beta+\frac{1}{m}-1}\, ds_1 \le C\delta^{\alpha-\beta+\frac{1}{m}}.$$

$R_4 : t > s_2 > \delta, t^m, s_1$. We have

$$I \le \int_\delta^1 \frac{1}{s_2^{2+\beta}}\left(\int_0^{s_2^{1/m}} \frac{t}{t}\, dt\right)\left(\int_0^{s_2} s_1^\alpha\, ds_1\right) ds_2 \le C\int_\delta^1 s_2^{\alpha-\beta+\frac{1}{m}-1}\, ds_2 \le C\delta^{\alpha-\beta+\frac{1}{m}}.$$

$R_5 : \delta > t, t^m, s_1, s_2$. We have

$$I \le \delta^{-2-\beta}\delta^{-1}\left(\int_0^\delta t\, dt\right)\left(\int_0^\delta s_1^\alpha\, ds_1\right)\left(\int_0^\delta ds_2\right) \le C\delta^{\alpha-\beta+1}.$$

$R_6 : s_1 > \delta, t, t^m, s_2$. We have

$$I \le \int_\delta^1 \frac{s_1^\alpha}{s_1^{2+\beta}s_1}\left(\int_0^{s_1} t\, dt\right)\left(\int_0^{s_1} ds_2\right) ds_1 \le C\int_\delta^1 s_1^{\alpha-\beta}\, ds_1.$$

$R_7 : s_2 > \delta, t, t^m, s_1$. We have

$$I \le \int_\delta^1 \frac{1}{s_2^{2+\beta}s_2}\left(\int_0^{s_2} t\, dt\right)\left(\int_0^{s_2} s_1^\alpha\, ds_1\right) ds_2 \le C\int_\delta^1 s_2^{\alpha-\beta}\, ds_2.$$

Here the constants depend only on α and β. For R_6 and R_7, we have

$$\int_\delta^1 r^{\alpha-\beta} dr \le \begin{cases} C, & \alpha-\beta > -1, \\ C(1+|\log\delta|), & \alpha-\beta=-1, \\ C\delta^{\alpha-\beta+1}, & \alpha-\beta<-1, \end{cases}$$

which is bounded by $C\delta^{\alpha-\beta+\frac{1}{m}}$ in all cases. □

Lemma 4.3. *Let $\beta \ge 0$, $\alpha > -1$, $\alpha < \beta - \frac{1}{m}$, and let $0 < \delta < \frac{1}{2}$. Denote $\delta_{bD'_\epsilon}(z) := \operatorname{dist}(z, bD'_\epsilon)$. Then for any $z, \zeta \in \mathcal{U}$:*

$$\int_{\mathcal{U}\setminus D'_\epsilon} \frac{\left[\delta_{bD'_\epsilon}(\zeta)\right]^\alpha dV(\zeta)}{|\Gamma_\epsilon(z,\zeta)|^{2+\beta}|\zeta-z|} \le C\left[\delta_{bD'_\epsilon}(z)\right]^{\alpha-\beta+\frac{1}{m}}; \quad \int_{D'_\epsilon} \frac{\left[\delta_{bD'_\epsilon}(z)\right]^\alpha dV(z)}{|\Gamma_\epsilon(z,\zeta)|^{2+\beta}|\zeta-z|} \le C\left[\delta_{bD'_\epsilon}(\zeta)\right]^{\alpha-\beta+\frac{1}{m}},$$

where the constants depends only on α, β, D and $\mathcal{U}$, and is independent of ϵ.

Proof. By Lemma 3.14, near each $\zeta_0 \in b\Omega$ there exists a neighborhood $\mathcal{V}_{\zeta_0}$ of ζ_0 and a coordinate system $\zeta \mapsto (s = (s_1, s_2), t) \in \mathbb{R}^2 \times \mathbb{R}^2$ such that

$$|\Gamma_\epsilon(z,\zeta)| \gtrsim \operatorname{dist}(z, bD'_\epsilon) + |s_1| + |s_2| + |t|^m, \quad |\zeta - z| \gtrsim |(s_1, s_2, t)| \tag{4.7}$$

for $z \in \mathcal{V}_{\zeta_0} \cap \Omega$ and $\zeta \in \mathcal{V}_{\zeta_0} \setminus \Omega$, with $|s_1(\zeta)| \approx \delta_{bD'_\epsilon}(\zeta)$. By switching the roles of z and ζ, we can also find a coordinate system $z \mapsto (\tilde{s} = (\tilde{s}_1, \tilde{s}_2), \tilde{t}) \in \mathbb{R}^2 \times \mathbb{R}^2$ such that

$$|\Gamma_\epsilon(z,\zeta)| \gtrsim \operatorname{dist}(\zeta, bD'_\epsilon) + |\tilde{s}_1| + |\tilde{s}_2| + |\tilde{t}|^2, \quad |\zeta - z| \gtrsim |(\tilde{s}_1, \tilde{s}_2, t)| \tag{4.8}$$

for $z \in \mathcal{V}_{\zeta_0} \cap D'_\epsilon$ and $\zeta \in \mathcal{V}_{\zeta_0} \setminus D'_\epsilon$, with $|\tilde{s}_1(z)| \approx \delta_{bD'_\epsilon}(z)$. By partition of unity in both z and ζ variables and (4.7), we have

$$\int_{\mathcal{U}\setminus D'_\epsilon} \frac{\left[\delta_{bD'_\epsilon}(\zeta)\right]^\alpha dV(\zeta)}{|\Gamma_\epsilon(z,\zeta)|^{2+\beta}|\zeta-z|} \le C_D \int_0^1\int_0^1\int_0^1 \frac{s_1^\alpha t\, ds_1\, ds_2\, dt}{(\delta_{bD'_\epsilon}(z)+s_1+s_2+t^2)^{2+\beta}(\delta_{bD'_\epsilon}(z)+s_1+s_2+t)}.$$

By Lemma 4.2, the integral is bounded by $C_{n,\alpha,\beta}\left[\delta_{bD'_\epsilon}(z)\right]^{\alpha-\beta+\frac{1}{m}}$, which proves the first inequality. The second inequality follows by the same way. □

Proposition 4.4. *Let $D'_\epsilon \subset \mathbb{C}^2$ be the sequence of domains given by $D'_\epsilon = \{z \in \mathcal{U} : r(z) < -\epsilon\} = \{z \in \mathcal{U} : r_\epsilon(z) < 0\}$. Let $\mathcal{H}^1\varphi$ be given by (4.2). Then the following statements are true.*

(i) For any $1 < p < \infty$, $s > \frac{1}{p}$ and non-negative integer $k > s + \frac{1}{m} - \eta$, there exists a constant $C = C(D, k, s, p)$ such that for all $\varphi \in H^{s,p}_{(0,1)}(D'_\epsilon)$,

$$\|\delta_{bD'_\epsilon}^{k-(s+\frac{1}{m}-\eta)} D^k\mathcal{H}^1\varphi\|_{L^p(D'_\epsilon)} \le C\|\varphi\|_{H^{s,p}(D'_\epsilon)}, \tag{4.9}$$

where $\delta_{bD'_\epsilon}(z) := \operatorname{dist}(z, bD'_\epsilon)$.

(ii) For any $s > 0$ and any non-negative integer $k > s + 1 + \frac{1}{m} - \eta$, there exists a constant $C = C(D, k, s)$ such that for all $\varphi \in \Lambda^s_{(0,1)}(D'_\epsilon)$,

$$\|\delta_{bD'_\epsilon}^{k-(s+\frac{1}{m}-\eta)} D^k\mathcal{H}^1\varphi\|_{L^\infty(D'_\epsilon)} \le C\|\varphi\|_{\Lambda^s(D'_\epsilon)}. \tag{4.10}$$

In particular, all the constants C are independent of ϵ.

Proof. (i) We estimate the integral

$$\int_{D'_\epsilon} \left[\delta_{bD'_\epsilon}(z)\right]^{k-(s+\frac{1}{m}-\eta)p} \left| \int_{\mathcal{U}\setminus\overline{D'_\epsilon}} D_z^k K(z,\cdot) \wedge [\overline{\partial}, \mathcal{E}]\varphi \right|^p dV(z). \tag{4.11}$$

Let f be a coefficient function of $[\overline{\partial}, \mathcal{E}]\varphi$ so that $f \in H^{s-1,p}(\mathcal{U} \setminus \overline{\Omega})$.

We now estimate the inner integral in (4.11) which we shall denote by $\mathcal{K}f$. In view of (4.4), we can write $\mathcal{K}f(z)$ as a linear combination of

$$\int_{\mathcal{U}\setminus\overline{D'_\epsilon}} f(\zeta)\big[D_z^\gamma W^{(\epsilon)}(\zeta,z)\big] D_z^{k-\gamma}\left(\frac{\overline{\zeta_i}-\overline{z}_i}{|\zeta-z|^2}\right) dV(\zeta), \quad 0\le\mu\le k.$$

In view of (3.13) (iii), and $\Gamma_\epsilon(\zeta,z)<|z-\zeta|$ for $(\zeta,z)\in\mathcal{U}\times D'_\epsilon$, the worst term is when $\gamma=\gamma_2=k$. Thus

$$|\mathcal{K}f(z)|\lesssim\left|\int_{\mathcal{U}\setminus\overline{D'_\epsilon}} f(\zeta)\frac{D_{z_2}^k W^{(\epsilon)}(z,\zeta)}{|\zeta-z|}\,dV(\zeta)\right|. \tag{4.12}$$

To simplify notation we will denote the kernel of the above integral by $B(z,\zeta)$. By (3.13) (iii), we have

$$|B(z,\zeta)|\le\frac{C_{D,\eta}}{\big[\Gamma_\epsilon(\zeta,z)\big]^{k+1+\eta}|\zeta-z|}. \tag{4.13}$$

Let μ be some number to be chosen. We have

$$\begin{aligned}|\mathcal{K}f(z)|&\lesssim\int_{\mathcal{U}\setminus\overline{D'_\epsilon}}|B(z,\zeta)|^{\frac1p}|B(z,\zeta)|^{\frac1{p'}}|f(\zeta)|\,dV(\zeta)\\&=\int_{\mathcal{U}\setminus\overline{D'_\epsilon}}\big[\delta_{bD'_\epsilon}(\zeta)\big]^{-\mu}B(z,\zeta)|^{\frac1p}\big[\delta_{bD'_\epsilon}(\zeta)\big]^{\mu}B(z,\zeta)|^{\frac1{p'}}|f(\zeta)|\,dV(\zeta).\end{aligned}$$

By Hölder's inequality we get

$$|\mathcal{K}f(z)|^p\le\left[\int_{\mathcal{U}\setminus\overline{D'_\epsilon}}\big[\delta_{bD'_\epsilon}(\zeta)\big]^{-\mu p+\mu}|B(z,\zeta)||f(\zeta)|^p\,dV(\zeta)\right]\left[\int_{\mathcal{U}\setminus\overline{D'_\epsilon}}\big[\delta_{bD'_\epsilon}(\zeta)\big]^{\mu}|B(z,\zeta)|\,dV(\zeta)\right]^{\frac{p}{p'}}. \tag{4.14}$$

By (4.13),

$$\int_{\mathcal{U}\setminus\overline{D'_\epsilon}}\big[\delta_{bD'_\epsilon}(\zeta)\big]^{\mu}|B(z,\zeta)|\,dV(\zeta)\lesssim C_{D,\eta}\int_{\mathcal{U}\setminus\overline{D'_\epsilon}}\frac{\big[\delta_{bD'_\epsilon}(\zeta)\big]^{\mu}}{\big[\Gamma_\epsilon(\zeta,z)\big]^{k+1+\eta}|\zeta-z|}\,dV(\zeta).$$

To estimate the integral we apply Lemma 4.3 with $\alpha=\mu$ and $\beta=k-1+\eta$.

$$\int_{\mathcal{U}\setminus\overline{D'_\epsilon}}\big[\delta_{bD'_\epsilon}(\zeta)\big]^{\mu}|B(z,\zeta)|\,dV(\zeta)\lesssim C_{D,\eta}\big[\delta_{bD'_\epsilon}(z)\big]^{\mu-(k-1+\eta)+\frac1m}. \tag{4.15}$$

The hypothesis of Lemma 4.3 requires that we choose

$$-1<\mu<k+\eta-1-\frac1m. \tag{4.16}$$

Using (4.15) in (4.14) and applying Fubini's theorem, we get

$$\begin{aligned}\int_{D'_\epsilon}\big[\delta_{bD'_\epsilon}(z)\big]^{(k-s-\frac1m+\eta)p}|\mathcal{K}f(z)|^p\,dV(z)&\lesssim\int_{D'_\epsilon}\big[\delta_{bD'_\epsilon}(z)\big]^{\sigma}\left(\int_{\mathcal{U}\setminus\overline{D'_\epsilon}}\big[\delta_{bD'_\epsilon}(\zeta)\big]^{-\mu p+\mu}|B(z,\zeta)||f(\zeta)|^p\,dV(\zeta)\right)dV(z)\\&=\int_{\mathcal{U}\setminus\overline{D'_\epsilon}}\big[\delta_{bD'_\epsilon}(\zeta)\big]^{-(p-1)\mu}\left(\int_{D'_\epsilon}\big[\delta_{bD'_\epsilon}(z)\big]^{\sigma}|B(z,\zeta)|\,dV(z)\right)|f(\zeta)|^p\,dV(\zeta),\end{aligned} \tag{4.17}$$

where we set

$$\begin{aligned}\sigma&=\Big(k-s-\frac1m+\eta\Big)p+\frac{p}{p'}\Big(\mu-\eta-k+1+\frac1m\Big)\\&=(p-1)\mu+(1-s)p+\Big(k+\eta-1-\frac1m\Big).\end{aligned} \tag{4.18}$$

By (4.13),

$$\int_{D'_\epsilon} \big[\delta_{bD'_\epsilon}(z)\big]^\sigma |B(z,\zeta)|\, dV(z) \lesssim C_{D,\eta} \int_{D'_\epsilon} \frac{\big[\delta_{bD'_\epsilon}(z)\big]^\sigma}{\big[\Gamma_\epsilon(\zeta,z)\big]^{k+1+\eta}|\zeta - z|}\, dV(z). \tag{4.19}$$

We would like to apply $\alpha = \sigma$ and $\beta = k-1+\eta$. For this we need to choose $-1 < \sigma < k+\eta-1-\frac{1}{m}$. In view of (4.18), μ needs to satisfy

$$\frac{1}{p-1}\Big[p\big(s-1\big) - \eta - k + \frac{1}{m}\Big] < \mu < \frac{p}{p-1}(s-1). \tag{4.20}$$

We need to check that the intersection of (4.16) and (4.20) is non-empty. There are two inequalities to check:

$$\frac{p}{p-1}(s-1) > -1;$$
$$\frac{1}{p-1}\Big[p\big(s-1\big) - \eta - k + \frac{1}{m}\Big] < k + \eta - 1 - \frac{1}{m}.$$

An easy computation shows that the first inequality gives $s > \frac{1}{p}$, while the second inequality reduces to $s < k + \eta + \frac{1}{p} - \frac{1}{m}$. By letting k be arbitrarily large integers, the admissible range of s is $\big(\frac{1}{p}, \infty\big)$. Assuming these conditions hold, we can apply Lemma 4.3 to (4.19) and get

$$\int_{D'_\epsilon} \big[\delta_{bD'_\epsilon}(z)\big]^\sigma |B(z,\zeta)|\, dV(\zeta) \lesssim C_{D,\eta}\big[\delta_{bD'_\epsilon}(\zeta)\big]^{\sigma-(k-1+\eta)+\frac{1}{m}} = C_{D,\eta}\big[\delta_{bD'_\epsilon}(\zeta)\big]^{\sigma-(k-1+\eta)+\frac{1}{m}}.$$

Substituting the above estimate into (4.17) we obtain

$$\int_{D'_\epsilon} \big[\delta_{bD'_\epsilon}(z)\big]^{(k-s-\frac{1}{m}+\eta)p} |\mathcal{K}f(z)|^p\, dV(z) \lesssim \int_{\mathcal{U}\setminus\overline{D'_\epsilon}} \big[\delta_{bD'_\epsilon}(\zeta)\big]^{(1-s)p} |f(\zeta)|^p\, dV(z).$$

By Proposition 2.17, the last expression is bounded by

$$\int_{\mathcal{U}\setminus\overline{D'_\epsilon}} \big[\delta_{bD'_\epsilon}(\zeta)\big]^{(1-s)p} \big|[\overline{\partial}, \mathcal{E}]\varphi(\zeta)\big|^p\, dV(z) \le C_s |\varphi|_{H^{s,p}(D'_\epsilon)}.$$

(ii) We follow the same notation as in (i). By (4.12) and (4.13), we get

$$|\mathcal{K}f(z)| \lesssim C_{D,\eta} \int_{\mathcal{U}\setminus\overline{D'_\epsilon}} \frac{|f(\zeta)|}{\big[\Gamma_\epsilon(\zeta,z)\big]^{k+1+\eta}|\zeta - z|}\, dV(\zeta).$$

where f is a cofficient function of $[\overline{\partial}, \mathcal{E}]\varphi$. Now by Proposition 2.17 we know that $\|\delta_{bD'_\epsilon}(\zeta)^{1-s} f\|_{L^\infty(\mathcal{U}\setminus\overline{D'_\epsilon})} \le C|\varphi|_{\Lambda^s(D'_\epsilon)}$. Hence the above integral is bounded by

$$|\mathcal{K}f(z)| \lesssim C_{D,\eta}|\varphi|_{\Lambda^s(D'_\epsilon)} \int_{\mathcal{U}\setminus\overline{D'_\epsilon}} \frac{\big[\delta_{bD'_\epsilon}(\zeta)\big]^{s-1}}{\big[\Gamma_\epsilon(\zeta,z)\big]^{k+1+\eta}|\zeta - z|}\, dV(\zeta).$$

Let $\alpha = s-1$ and $\beta = k-1+\eta$. The condition $-1 < \alpha < \beta - \frac{1}{m}$ becomes

$$0 < s < k + \eta - 1 - \frac{1}{m}. \tag{4.21}$$

Assuming (4.21) holds, we can apply Lemma 4.3 to the above integral to get

$$|\mathcal{K}f(z)| \lesssim C_{D,\eta}|\varphi|_{\Lambda^s(D'_\epsilon)}\big[\delta_{bD'_\epsilon}(z)\big]^{s-k-\eta+\frac{1}{m}}, \quad z \in D'_\epsilon.$$

In other words, $\sup_{z\in\overline{D'_\epsilon}}\big[\delta_{bD'_\epsilon}(z)\big]^{k-\left(s+\frac{1}{m}-\eta\right)} \big|\mathcal{H}^1\varphi(z)\big| \lesssim C_{D,\eta}|\varphi|_{\Lambda^s(\overline{D'_\epsilon})}$. ∎

We now combine Proposition 4.1 and Proposition 4.4 and Lemma 2.24 to obtain the following estimate for $\mathcal{H}^{(\epsilon)}\varphi$ on D'_ϵ.

Theorem 4.5. *Let $D \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain in $\mathbb{C}^2$ of finite type m, with a defining function r. Let $\mathcal{H}^{(\epsilon)}$ defined by (4.1)-(4.2) with Ω replaced by D'_ϵ. Then $\mathcal{H}^{(\epsilon)}$ is bounded on the following spaces:*

(i) $\mathcal{H}^{(\epsilon)} : H^{s,p}_{(0,1)}(D'_\epsilon) \to H^{s+\frac{1}{m}-\eta,p}(D'_\epsilon)$, for any $1 < p < \infty$ and $s > \frac{1}{p}$.

(ii) $\mathcal{H}^{(\epsilon)} : \Lambda^s_{(0,1)}(\overline{D'_\epsilon}) \to \Lambda^{s+\frac{1}{m}-\eta}(\overline{D'_\epsilon})$ for any $s > 0$.

More specifically, there exists a constant $C = C(D, \eta, s)$ which is independent of ϵ such that

$$|\mathcal{H}^{(\epsilon)}\varphi|_{H^{s+\frac{1}{m}-\eta,p}(D'_\epsilon)} \le C|\varphi|_{H^{s,p}(D'_\epsilon)}, \quad |\mathcal{H}^{(\epsilon)}\varphi|_{\Lambda^{s+\frac{1}{m}-\eta}(\overline{D'_\epsilon})} \le C|\varphi|_{\Lambda^s(\overline{D'_\epsilon})}. \tag{4.22}$$

Proof. The first statement in (4.22) follows from (2.10), (4.9) and (4.5); the second statement follows from (2.11), (4.10) and (4.5). □

Corollary 4.6 (Homotopy formula)**.** *Under the assumptions of Theorem 4.5, suppose either*

1) $\varphi \in H^{s,p}_{(0,1)}(D)$ and $\overline{\partial}\varphi \in H^{s,p}_{(0,2)}(D)$, for $s > \frac{1}{p}$, or

2) $\varphi \in \Lambda^s_{(0,1)}(D)$ and $\overline{\partial}\varphi \in \Lambda^s_{(0,2)}(D)$, for $s > 0$.

Then the following homotopy formula holds in the sense of distributions:

$$\varphi = \overline{\partial}\mathcal{H}_1^{(\epsilon)}\varphi + \mathcal{H}_2^{(\epsilon)}\overline{\partial}\varphi, \quad \text{on } D'_\epsilon. \tag{4.23}$$

In particular if $\varphi \in H^{s,p}_{(0,1)}(D)$ (resp. $\varphi \in \Lambda^s_{(0,1)}(\overline{D'_\epsilon})$) is $\overline{\partial}$-closed, then we have $\overline{\partial}\mathcal{H}^{(\epsilon)}\varphi = \varphi$ on D'_ϵ and $\mathcal{H}^{(\epsilon)}\varphi \in H^{s+\frac{1}{m}-\eta,p}(D'_\epsilon)$ (resp. $\mathcal{H}^{(\epsilon)}\varphi \in \Lambda^{s+\frac{1}{m}-\eta}(\overline{D'_\epsilon})$), with $\mathcal{H}^{(\epsilon)}$ satisfying estimates (4.22).

Proof. By [Gon19, Prop. 2.1], Formula (4.23) is valid for $\varphi \in C^\infty_{(0,1)}(\overline{\Omega})$. For general $\varphi \in H^{s,p}_{(0,1)}(\Omega)$ such that $\overline{\partial}\varphi \in H^{s,p}_{(0,2)}(\Omega)$, we use approximation. By [SY24b, Prop. A.3], there exists a sequence $\varphi_\varepsilon \in C^\infty(\overline{\Omega})$ such that

$$\varphi_\varepsilon \xrightarrow{\varepsilon\to 0} \varphi \quad \text{in } H^{s,p}(\Omega)\ ,$$
$$\overline{\partial}\varphi_\varepsilon \xrightarrow{\varepsilon\to 0} \overline{\partial}\varphi \quad \text{in } H^{s,p}(\Omega).$$

By Theorem 4.5 we have for any $s > \frac{1}{p}$,

$$\begin{aligned}\|\overline{\partial}\mathcal{H}^{(\epsilon)}(\varphi_\varepsilon - \varphi)\|_{H^{s+\frac{1}{m}-\eta-1,p}(\Omega)} &\le \|\mathcal{H}^{(\epsilon)}(\varphi_\varepsilon - \varphi)\|_{H^{s+\frac{1}{m}-\eta,p}(\Omega)} \\ &\le \|\varphi_\varepsilon - \varphi\|_{H^{s,p}},\end{aligned}$$

and also

$$\|\mathcal{H}^{(\epsilon)}\overline{\partial}(\varphi_\varepsilon - \varphi)\|_{H^{s+\frac{1}{m}-\eta,p}(\Omega)} \le \|\overline{\partial}(\varphi_\varepsilon - \varphi)\|_{H^{s,p}(\Omega)}.$$

Letting $\varepsilon \to 0$ we get (4.23). □

We can now use Corollary 4.6 to prove Theorem 1.1.

Proof of Theorem 1.1. Let $\{\epsilon_j\}$ be a sequence of small positive numbers tending to 0. Consider the sequence of functions $\{\mathcal{E}_j\mathcal{H}_1^{(\epsilon_j)}\varphi\}$, and $\{\mathcal{E}_j\mathcal{H}_2^{(\epsilon_j)}\overline{\partial}\varphi\}$, $\mathcal{E}_j$ being the Rychkov extension operator on D'_{ϵ_j}. Then by Corollary 2.15 and (4.22), we have

$$|\mathcal{E}_j\mathcal{H}_1^{(\epsilon_j)}\varphi|_{H^{s+\frac{1}{m}-\eta,p}(\mathbb{C}^n)} \le C_{s,\eta}|\mathcal{H}_1^{(\epsilon_j)}\varphi|_{H^{s+\frac{1}{m}-\eta,p}(D'_{\epsilon_j})} \le C_{s,\eta}|\varphi|_{H^{s,p}(D)};$$

$$|\mathcal{E}_j\mathcal{H}_2^{(\epsilon_j)}\overline{\partial}\varphi|_{H^{s+\frac{1}{m}-\eta,p}(\mathbb{C}^n)} \le C_{s,\eta}|\mathcal{H}_2^{(\epsilon_j)}\overline{\partial}\varphi|_{H^{s+\frac{1}{m}-\eta,p}(D'_{\epsilon_j})} \le C_{s,\eta}|\overline{\partial}\varphi|_{H^{s,p}(D)}.$$

By the Banach-Alaouglu theorem, there exists a subsequence $\{\mathcal{E}_{j_\alpha}\mathcal{H}_1^{(\epsilon_{j_\alpha})}\varphi\}$ which converges weakly to some limit function $\mathcal{H}_1\varphi$ in $H^{s+\frac{1}{m}-\eta,p}(\mathbb{C}^n)$. Similarly, by extracting another subsequence from $\{\mathcal{E}_{j_\alpha}\mathcal{H}_2^{(\epsilon_{j_\alpha})}\overline{\partial}\varphi\}$, and denoting this subsequence still by $\{\mathcal{E}_j\mathcal{H}_2^{(\epsilon_j)}\overline{\partial}\varphi\}$, we get $\mathcal{E}_j\mathcal{H}_2^{(\epsilon_j)}\overline{\partial}\varphi$ converges weakly to a limit function $\mathcal{H}_2\overline{\partial}\varphi$.

Now, by Corollary 4.6, we have $\varphi = \overline{\partial}\mathcal{H}_1^{(\epsilon_j)}\varphi + \mathcal{H}_2^{(\epsilon_j)}\overline{\partial}\varphi$ on D'_{ϵ_j}. Let $\phi = \phi_1\widehat{d\overline{z}_1} + \phi_2\widehat{d\overline{z}_2}$ be a $(2,1)$ form with coefficients in $C_c^\infty(D)$. Then there exists a large J such that $\operatorname{supp}\phi \subset D'_{\epsilon_j}$, for all $j > J$. It follows that

$$\begin{aligned}\langle\varphi,\phi\rangle_D = \langle\varphi,\phi\rangle_{D'_{\epsilon_j}} &= \left\langle\overline{\partial}\mathcal{E}_j\mathcal{H}_1^{(\epsilon_j)}\varphi + \mathcal{E}_j\mathcal{H}_2^{\epsilon_j}\overline{\partial}\varphi,\phi\right\rangle_{D'_{\epsilon_j}} = \left\langle\overline{\partial}\mathcal{E}_j\mathcal{H}_1^{(\epsilon_j)}\varphi,\phi\right\rangle_{D'_{\epsilon_j}} + \left\langle\mathcal{E}_j\mathcal{H}_2^{\epsilon_j}\overline{\partial}\varphi,\phi\right\rangle_{D'_{\epsilon_j}} \\ &= -\left\langle\mathcal{E}_j\mathcal{H}_1^{(\epsilon_j)}\varphi,\overline{\partial}\phi\right\rangle_{D'_{\epsilon_j}} + \left\langle\mathcal{E}_j\mathcal{H}_2^{\epsilon_j}\overline{\partial}\varphi,\phi\right\rangle_{D'_{\epsilon_j}} = -\left\langle\mathcal{E}_j\mathcal{H}_1^{(\epsilon_j)}\varphi,\overline{\partial}\phi\right\rangle_D + \left\langle\mathcal{E}_j\mathcal{H}_2^{\epsilon_j}\overline{\partial}\varphi,\phi\right\rangle_D.\end{aligned}$$

Taking limit as $j\to\infty$, and using that

$$\lim_{j\to\infty} -\left\langle\mathcal{E}_j\mathcal{H}_1^{(\epsilon_j)}\varphi,\overline{\partial}\phi\right\rangle_D = -\left\langle\mathcal{H}_1\varphi,\overline{\partial}\phi\right\rangle_D, \quad \lim_{j\to\infty}\left\langle\mathcal{E}_j\mathcal{H}_2^{\epsilon_j}\overline{\partial}\varphi,\phi\right\rangle_D = \left\langle\mathcal{H}_2\overline{\partial}\varphi,\phi\right\rangle_D,$$

We get from above

$$\langle\varphi,\phi\rangle_D = -\left\langle\mathcal{H}_1\varphi,\overline{\partial}\phi\right\rangle_D + \left\langle\mathcal{H}_2\overline{\partial}\varphi,\phi\right\rangle_D = \left\langle\overline{\partial}\mathcal{H}_1\varphi + \mathcal{H}_2\overline{\partial}\varphi,\phi\right\rangle_D.$$

Next we prove the statement for the Hölder-Zygmund space. Let ϵ_j and $\mathcal{E}_j$ be as above. The sequence $\mathcal{E}_j\mathcal{H}^{(\epsilon_j)}\varphi$ satisfies the estimate

$$|\mathcal{E}_j\mathcal{H}_1^{(\epsilon_j)}\varphi|_{\Lambda^{s+\frac{1}{m}-\eta}(\mathbb{C}^n)} \le C_s|\mathcal{H}_1^{(\epsilon_j)}\varphi|_{\Lambda^{s+\frac{1}{m}-\eta}(\overline{D'_{\epsilon_j}})} \le C_{s,\eta}|\varphi|_{\Lambda^s(\overline{D})};$$

$$|\mathcal{E}_j\mathcal{H}_2^{(\epsilon_j)}\overline{\partial}\varphi|_{\Lambda^{s+\frac{1}{m}-\eta}(\mathbb{C}^n)} \le C_s|\mathcal{H}_2^{(\epsilon_j)}\overline{\partial}\varphi|_{\Lambda^{s+\frac{1}{m}-\eta}(\overline{D'_{\epsilon_j}})} \le C_{s,\eta}|\overline{\partial}\varphi|_{\Lambda^s(\overline{D})}.$$

In particular, $\{\mathcal{E}_j\mathcal{H}_1^{(\epsilon_j)}\varphi\},\{\mathcal{E}_j\mathcal{H}_2^{(\epsilon_j)}\overline{\partial}\varphi\}$ are families of equicontinuous functions on $\overline{D}$. By the Ascoli-Arzela theorem and by taking two subsequence successively, we may assume that $\mathcal{E}_j\mathcal{H}_1^{(\epsilon_j)}\varphi$ and $\mathcal{E}_j\mathcal{H}_2^{(\epsilon_j)}\overline{\partial}\varphi$ converge uniformly to some limit functions $\mathcal{H}_1\varphi,\mathcal{H}_2\overline{\partial}\varphi$ respectively, which are in $\Lambda^{s+\frac{1}{m}-\eta}(\overline{D})$. Furthermore, we have $\varphi = \overline{\partial}\mathcal{H}_1\varphi + \mathcal{H}_2\overline{\partial}\varphi$ in the sense of distributions on D. The proof is now complete.

References

[Ale06] William Alexandre, *C^k-estimates for the $\overline{\partial}$-equation on convex domains of finite type*, Math. Z. **252** (2006), no. 3, 473–496. MR 2207755

[Bel93] Jay Belanger, *Hölder estimates for $\overline{\partial}$ in $\mathbf{C}^2$*, Several complex variables (Stockholm, 1987/1988), Math. Notes, vol. 38, Princeton Univ. Press, Princeton, NJ, 1993, pp. 98–125. MR 1207856

[BL76] Jöran Bergh and Jörgen Löfström, *Interpolation spaces. An introduction*, Grundlehren der Mathematischen Wissenschaften, No. 223, Springer-Verlag, Berlin-New York, 1976. MR 0482275

[Cat89] David W. Catlin, *Estimates of invariant metrics on pseudoconvex domains of dimension two*, Math. Z. **200** (1989), no. 3, 429–466. MR 978601

[Cha89] Der-Chen E. Chang, *Optimal L^p and Hölder estimates for the Kohn solution of the $\overline{\partial}$-equation on strongly pseudoconvex domains*, Trans. Amer. Math. Soc. **315** (1989), no. 1, 273–304. MR 937241

[CNS92] D.-C. Chang, A. Nagel, and E. M. Stein, *Estimates for the $\overline{\partial}$-Neumann problem in pseudoconvex domains of finite type in $\mathbf{C}^2$*, Acta Math. **169** (1992), no. 3-4, 153–228. MR 1194003

[DF77] Klas Diederich and John Erik Fornaess, *Pseudoconvex domains: existence of Stein neighborhoods*, Duke Math. J. **44** (1977), no. 3, 641–662. MR 447634

[DFFs99] Klas Diederich, Bert Fischer, and John Erik Fornæ ss, *Hölder estimates on convex domains of finite type*, Math. Z. **232** (1999), no. 1, 43–61. MR 1714279

[FK88] Charles L. Fefferman and Joseph J. Kohn, *Hölder estimates on domains of complex dimension two and on three-dimensional CR manifolds*, Adv. in Math. **69** (1988), no. 2, 223–303. MR 946264

[Fs86] John Erik Fornæ ss, *Sup-norm estimates for $\bar\partial$ in* $\mathbf{C}^2$, Ann. of Math. (2) **123** (1986), no. 2, 335–345. MR 835766

[Gon19] Xianghong Gong, *Hölder estimates for homotopy operators on strictly pseudoconvex domains with C^2 boundary*, Math. Ann. **374** (2019), no. 1-2, 841–880. MR 3961327

[Gon24] ———, *On regularity of $\overline{\partial}$-solutions on a_q domains with C^2 boundary in complex manifolds*, ArXiv preprint (2024).

[GS77] P. C. Greiner and Elias M. Stein, *Estimates for the $\overline{\partial}$-Neumann problem*, Princeton University Press, Princeton, N.J., 1977, Mathematical Notes, No. 19. MR 0499319

[GT01] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition. MR 1814364

[KN73] J. J. Kohn and L. Nirenberg, *A pseudo-convex domain not admitting a holomorphic support function*, Math. Ann. **201** (1973), 265–268. MR 330513

[Koh64] J. J. Kohn, *Harmonic integrals on strongly pseudo-convex manifolds. II*, Ann. of Math. (2) **79** (1964), 450–472. MR 208200

[Koh72] ———, *Boundary behavior of δ on weakly pseudo-convex manifolds of dimension two*, J. Differential Geometry **6** (1972), 523–542. MR 322365

[LR80] Ingo Lieb and R. Michael Range, *Lösungsoperatoren für den Cauchy-Riemann-Komplex mit C^k-Abschätzungen*, Math. Ann. **253** (1980), no. 2, 145–164. MR 597825

[Ran90] R. Michael Range, *Integral kernels and Hölder estimates for $\overline{\partial}$ on pseudoconvex domains of finite type in* $\mathbf{C}^2$, Math. Ann. **288** (1990), no. 1, 63–74. MR 1070924

[RH71] A. V. Romanov and G. M. Henkin, *Exact Hölder estimates of the solutions of the $\bar\delta$-equation*, Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 1171–1183. MR 0293121

[Ryc99] Vyacheslav S. Rychkov, *On restrictions and extensions of the Besov and Triebel-Lizorkin spaces with respect to Lipschitz domains*, J. London Math. Soc. (2) **60** (1999), no. 1, 237–257. MR 1721827

[Shi21] Ziming Shi, *Weighted Sobolev L^p estimates for homotopy operators on strictly pseudoconvex domains with C^2 boundary*, J. Geom. Anal. **31** (2021), no. 5, 4398–4446. MR 4244873

[Siu74] Yum Tong Siu, *The $\bar\partial$ problem with uniform bounds on derivatives*, Math. Ann. **207** (1974), 163–176. MR 330515

[Sko72] Henri Skoda, *Application des techniques L^2 à la théorie des idéaux d'une algèbre de fonctions holomorphes avec poids*, Ann. Sci. École Norm. Sup. (4) **5** (1972), 545–579. MR 333246

[SY21] Ziming Shi and Liding Yao, *Sobolev $\frac{1}{2}$ estimates for $\overline{\partial}$ equations on strictly pseudoconvex domains with C^2 boundary*, to appear in *Amer. J. Math.* (2021).

[SY24a] ———, *New estimates of Rychkov's universal extension operator for Lipschitz domains and some applications*, Math. Nachr. **297** (2024), no. 4, 1407–1443. MR 4734977

[SY24b] ———, *A solution operator for the $\overline{\partial}$ equation in Sobolev spaces of negative index*, Trans. Amer. Math. Soc. **377** (2024), no. 2, 1111–1139. MR 4688544

[Tri83] Hans Triebel, *Theory of function spaces*, Monographs in Mathematics, vol. 78, Birkhäuser Verlag, Basel, 1983. MR 781540

[Tri95] ———, *Interpolation theory, function spaces, differential operators*, second ed., Johann Ambrosius Barth, Heidelberg, 1995. MR 1328645

[Tri06] ———, *Theory of function spaces. III*, Monographs in Mathematics, vol. 100, Birkhäuser Verlag, Basel, 2006. MR 2250142

[Tri20] ———, *Theory of function spaces IV*, Monographs in Mathematics, vol. 107, Birkhäuser/Springer, Cham, [2020] ©2020. MR 4298338

[Yao24] Liding Yao, *Sobolev and Hölder estimates for homotopy operators of the $\overline{\partial}$-equation on convex domains of finite multitype*, J. Math. Anal. Appl. **538** (2024), no. 2, Paper No. 128238, 41. MR 4739361

Department of Mathematics, University of California-Irvine, Irvine, CA, 92697
Email address: zimings3@uci.edu