EXISTENCE AND UNICITY OF PLURIHARMONIC MAPS TO EUCLIDEAN BUILDINGS AND APPLICATIONS

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Abstract. Given a complex smooth quasi-projective variety X, a reductive algebraic group G defined over some non-archimedean local field K and a Zariski dense representation $\varrho : \pi_1(X) \to G(K)$, we construct a ϱ-equivariant pluriharmonic map from the universal cover of X into the Bruhat-Tits building $\Delta(G)$ of G, with appropriate asymptotic behavior. We also establish the uniqueness of such a pluriharmonic map in a suitable sense, and provide a geometric characterization of these equivariant maps. This paper builds upon and extends previous work by the authors jointly with G. Daskalopoulos and D. Brotbek.

Contents

0. Introduction

0.1. **Main results.** We first establish the following existence theorem of equivariant pluriharmonic maps to Bruhat-Tits buildings.

Theorem A. *Let X be a smooth quasi-projective variety and let G be a reductive group defined over a non-archidemean local field K. Let $\Delta(G)$ be the* enlarged *Bruhat-Tits building of G. Denote by $\pi_X : \widetilde{X} \to X$ the universal covering map. If $\varrho : \pi_1(X) \to G(K)$ is a Zariski dense representation, then the following statements hold:*

(i) *There exists a ϱ-equivariant pluriharmonic map $\tilde{u} : \widetilde{X} \to \Delta(G)$ with logarithmic energy growth.*
(ii) *$\tilde{u}$ is harmonic with respect to any Kähler metric on $\widetilde{X}$.*
(iii) *Let $f : Y \to X$ be a morphism from a smooth quasi-projective variety Y. Denote by $\tilde{f} : \widetilde{Y} \to \widetilde{X}$ the lift of f between the universal covers of Y and X. Then the $f^*\varrho$-equivariant map $\tilde{u} \circ \tilde{f} : \widetilde{Y} \to \Delta(G)$ is pluriharmonic and has logarithmic energy growth.*
(iv) *There is a proper Zariski closed subset Ξ of X such that the singular set $\mathcal{S}(\tilde{u})$ of $\tilde{u}$ defined in Definition 1.2 is contained in $\pi_X^{-1}(\Xi)$.*

Note that when X is a compact Kähler manifold, Theorems A.(i) to A.(iii) were established by Gromov-Schoen in [GS92] and Theorem A.(iv) was proved by Eyssidieux in [Eys04]. In the case where G is semisimple, Theorems A.(i) to A.(iii) were proven by the authors with Brotbek in [BDDM22, Theorem A].

In general, the uniqueness of the equivariant pluriharmonic map in Theorem A is not guaranteed, although it can be established under additional assumptions on the representation (cf. [DM23b, BDDM22]). However, we prove the uniqueness in a suitable local setting over a dense open subset of $\widetilde{X}$ that has full Lebesgue measure.

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Theorem B. *Let X be a smooth quasi-projective variety and let G be a reductive group defined over a non-archidemean local field K. For a representation $\varrho : \pi_1(X) \to G(K)$, if $\tilde{u}_0, \tilde{u}_1 : \widetilde{X} \to \Delta(G)$ are two ϱ-equivariant pluriharmonic maps with logarithmic energy growth, then for* almost every *point $x \in \widetilde{X}$, it has an open neighborhood Ω such that*

(i) *there exists an apartment A of $\Delta(G)$ that contains both $\tilde{u}_0(\Omega)$ and $\tilde{u}_1(\Omega)$;*
(ii) *The map $\tilde{u}_0|_\Omega : \Omega \to A$ is a translate of $\tilde{u}_1|_\Omega : \Omega \to A$.*

It is worthwhile mentioning that Theorems A and B were established by Corlette-Simpson [CS08] for the case where $G = \mathrm{PSL}_2$ and the representation ϱ has quasi-unipotent monodromies at infinity. In this setting, the Bruhat-Tits building of G is a tree, and the energy of the ϱ-equivariant pluriharmonic map $\tilde{u}$ is proven to be finite.

Finally, we give a geometric characterization of pluriharmonic maps with logarithmic energy growth in terms of spectral covers.

Theorem C. *Let X, ϱ, G and $\tilde{u}$ be as in Theorem A. Let $\overline{X}$ be a smooth projective compactification of X such that $\Sigma := \overline{X}\backslash X$ is a simple normal crossing divisor. Then we have the following:*

(i) *The ϱ-equivariant pluriharmonic map $\tilde{u}$ induces a multivalued logarithmic 1-form η on the log pair $(\overline{X}, \Sigma)$, satisfying the properties in Lemma 3.8.(ii).*
(ii) *Such η does not depend on the choice of $\tilde{u}$; i.e., if $\tilde{v}$ is another ϱ-equivariant pluriharmonic map with logarithmic energy growth, the multivalued logarithmic 1-form induced by $\tilde{v}$ is η.*
(iii) *There exists a ramified Galois cover $\pi : \overline{X^{\mathrm{sp}}} \to \overline{X}$ such that $\pi^*\eta$ becomes single-valued; i.e., $\pi^*\eta = \{\omega_1, \ldots, \omega_m\} \subset H^0(\overline{X^{\mathrm{sp}}}, \pi^*\Omega_{\overline{X}}(\log \Sigma))$.*
(iv) *Denote by $\Sigma_1 := \overline{X^{\mathrm{sp}}}\backslash X^{\mathrm{sp}}$. Let $\mu : \overline{Y} \to \overline{X^{\mathrm{sp}}}$ be a log resolution of $(\overline{X^{\mathrm{sp}}}, \Sigma_1)$, with $\Sigma_Y := \mu^{-1}(\Sigma_1)$ a simple normal crossing divisor. Then $\{\mu^*\omega_1, \ldots, \mu^*\omega_m\} \in H^0(\overline{Y}, \Omega_{\overline{Y}}(\log \Sigma_Y))$ are* pure imaginary*, i.e., the residue of every $\mu^*\omega_j$ at each irreducible component of Σ_Y is a pure imaginary number.*

We mention that Theorem C.(iv) is analogous to Mochizuki's notion of *pure imaginary harmonic bundles* induced by pluriharmonic maps to symmetric spaces associated with complex semisimple local systems over quasi-projective varieties (cf. [Moc07]). In our case, however, a spectral cover is required to transform the multivalued logarithmic 1-form induced by the pluriharmonic map into single-valued logarithmic 1-forms.

In this paper, we assume that K is a non-archimedean local field endowed with a *discrete* non-archimedean valuation. On the other hand, a recent paper by C. Breiner, B. Dees, and the second author [BDM24] introduces techniques to study the case for a general non-archimedean valuation local field L. In the sequel, we will show that Theorem A, Theorem B, and Theorem C generalize to the case where $\Delta(G)$ is a Bruhat-Tits building associated with a reductive group defined over any non-archimedean field.

0.2. **Notation and Convention.**

(a) Unless otherwise specified, algebraic varieties are assumed to be connected and defined over the field of complex numbers.
(b) Let G be a reductive group defined over a non-archimedean local field K. We denote by $\Delta(G)$ the Bruhat-Tits building of G, which is a non-positively curved (NPC for short) space. Denote by $d(\bullet, \bullet)$ the distance function on $\Delta(G)$. Denote by $\mathscr{D}G$ the derived group of G, which is semisimple.
(c) For a complex space X, denote by X^{norm} the normalization of X.
(d) A *log smooth pair* $(\overline{X}, \Sigma)$ consists of a smooth projective variety $\overline{X}$ and a simple normal crossing divisor Σ. We denote by $X := \overline{X}\backslash\Sigma$, and $\pi_X : \widetilde{X} \to X$ the universal covering map.
(e) Say a function $\tilde{f}$ (resp. a 1-form $\tilde{\eta}$) on $\widetilde{X}$ descends on X if there exists a function f (resp. a 1-form η) on X such that $\tilde{f} = \pi_X^* f$ (resp. $\tilde{\eta} = \pi_X^*\eta$).
(f) Let $\overline{X}$ be a smooth projective variety. A line bundle L on $\overline{X}$ is *sufficiently ample* if there exists a projective embedding $\iota : \overline{X} \hookrightarrow \mathbb{P}^N$ such that $L = \iota^*\mathscr{O}_{\mathbb{P}^N}(d)$ for some $d \geqslant 3$.
(g) A linear representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ with K some field is called *reductive* if the Zariski closure of $\varrho(\pi_1(X))$ is a reductive algebraic group over $\overline{K}$.

If Y is a closed smooth subvariety of X, we denote by $\varrho_Y : \pi_1(Y) \to G(K)$ the composition of the natural homomorphism $\pi_1(Y) \to \pi_1(X)$ and ϱ.

(h) Denote by $\mathbb{D}$ the unit disk in $\mathbb{C}$, and by $\mathbb{D}^*$ the punctured unit disk. We write $\mathbb{D}_r := \{z \in \mathbb{C} \mid |z| < r\}$, $\mathbb{D}_r^* := \{z \in \mathbb{C} \mid 0 < |z| < r\}$, and $\mathbb{D}_{r_1,r_2} := \{z \in \mathbb{C} \mid r_1 < |z| < r_2\}$.

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1. Technical preliminary

For more details of this section, we refer the readers to [BDDM22].

1.1. **Equivariant maps and sections.** Endow X with a Kähler metric g. Let $\varrho : \pi_1(X) \to G(K)$ be a representation where G is a reductive algebraic group over a non-archimedean local field K. The set of all ϱ-equivariant maps into $\Delta(G)$ are in one-to-one correspondence with the set of all sections of the fiber bundle $\Pi : \widetilde{X} \times_\varrho \Delta(G) \to X$. More precisely, for a ϱ-equivariant map $\tilde{f} : \widetilde{X} \to \Delta(G)$, we define a section of Π by setting $f(\pi_X(\tilde{p})) = [(\tilde{p}, \tilde{f}(\tilde{p}))]$, where $\tilde{p}$ is any point in $\widetilde{X}$. We shall use this notation throughout this paper.

One can also define the energy density function $|\nabla \tilde{f}|^2$ of $\tilde{f}$, and we refer the readers to [KS93, BDDM22] for the definition. Since $\tilde{f}$ is equivariant, $|\nabla \tilde{f}|^2$ on $\widetilde{X}$ is a $\pi_1(X)$-invariant function, and thus it descends to a function on X, denoted by $|\nabla f|^2$. We also define the energy of f in any open subset U of X by setting

$$E^f[U] = \int_U |\nabla f|^2 d\mathrm{vol}_g. \tag{1.1}$$

1.2. **Pullback bundles.** Let $f : Y \to X$ be a morphism between smooth quasi-projective varieties. Let C be an NPC space, and let $\varrho : \pi_1(X) \to \mathrm{Isom}(C)$ be a homomorphism. Let $\widehat{Y}$ be a connected component of $\widetilde{X} \times_X Y$. Then we have the following commuting diagram:

$$\begin{tikzcd}
\widetilde{Y} \arrow[d, "\pi_{\widehat{Y}}"] \arrow[dd, bend right=60, "\pi_Y"'] & \\
\widehat{Y} \arrow[r, "\hat{f}"] \arrow[d, "\hat{\pi}_Y"] & \widetilde{X} \arrow[d, "\pi_X"] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

It induces a fiber bundle $\hat{\Pi}_Y : \widehat{Y} \times_{f^*\varrho} C \to Y$, such that one has the following commuting diagram:

$$\begin{tikzcd}
\widehat{Y} \times_{f^*\varrho} C \arrow[r, "F"] \arrow[d, "\hat{\Pi}_Y"] & \widetilde{X} \times_\varrho C \arrow[d, "\Pi_X"] \\
Y \arrow[r, "f"] & X.
\end{tikzcd}$$

By § 1.1, a ϱ-equivariant map $\tilde{u} : \widetilde{X} \to C$ corresponds to a section $u : X \to \widetilde{X} \times_\varrho C$ of Π_X. The composition

$$u \circ f : Y \to \widetilde{X} \times_\varrho C$$

defines a section of the fiber bundle $\widehat{Y} \times_{f^*\varrho} C \simeq f^*(\widetilde{X} \times_\varrho C) \to Y$, which in turn defines a $f^*\varrho$-equivariant map $\hat{u}_f : \hat{Y} \to C$. Define $\widetilde{u_f} := \hat{u}_f \circ \pi_{\widehat{Y}}$, which is an $f^*\varrho$-equivariant map $\widetilde{Y} \to C$. It defines a section

$$u_f : Y \to \widetilde{Y} \times_{f^*\varrho} C.$$

In this paper, we will mainly focus on the special case where Y is a closed smooth subvariety of X and $\iota : Y \to X$ is the inclusion map. In this cases, we will use the notation

$$u_Y : Y \to \widetilde{Y} \times_{\varrho_Y} C. \tag{1.2}$$

in place of u_ι, where $\varrho_Y : \pi_1(Y) \to \mathrm{Isom}(C)$ denotes the composition of $\iota_* : \pi_1(Y) \to \pi_1(X)$ and ϱ. Denote by $\widetilde{u_Y} : \widetilde{Y} \to C$ the corresponding ϱ_Y-equivariant map.

1.3. **Regularity results of Gromov-Schoen.** Let X be a hermitian manifold and let $\tilde{u} : \widetilde{X} \to \Delta(G)$ be a ϱ-equivariant harmonic map. Following § 1.2, let $u : X \to \widetilde{X} \times_\varrho \Delta(G)$ be the section corresponding to $\tilde{u}$. We recall some results in [GS92].

Theorem 1.1 ([GS92], Theorem 2.4)**.** *A harmonic map $\tilde{u} : \widetilde{X} \to \Delta(G)$ is locally Lipschitz continuous.* □

Definition 1.2 (Regular points and singular points)**.** A point $x \in \widetilde{X}$ is said to be a *regular point* of $\tilde{u}$ if there exists a neighborhood $\mathcal{N}$ of x and an apartment $A \subset \Delta(G)$ such that $\tilde{u}(\mathcal{N}) \subset A$. A *singular point* of $\tilde{u}$ is a point in $\widetilde{X}$ that is not a regular point. Note that if $x \in \widetilde{X}$ is a regular point (resp. singular point) of $\tilde{u}$, then every point of $\pi_X^{-1}(\pi_X(x))$ is a regular point (resp. singular point) of $\tilde{u}$. We denote by $\mathcal{R}(\tilde{u})$ (resp. $\mathcal{S}(\tilde{u})$) the set of all regular points (resp. singular points) of $\tilde{u}$ and let $\mathcal{R}(u) = \pi_X(\mathcal{R}(\tilde{u}))$ (resp. $\mathcal{S}(u) = \pi_X(\mathcal{S}(\tilde{u}))$).

Lemma 1.3 ([GS92], Theorem 6.4)**.** *The set $\mathcal{S}(u)$ is a closed subset of X of Hausdorff codimension at least two.* □

Remark 1.4. B. Dees [Dee22] improved Lemma 1.3 to show that $\mathcal{S}(u)$ is $(n-2)$-countably rectifiable where n is the dimension of the domain.

1.4. **Logarithmic energy growth.** Let X be a smooth quasi-projective variety. Let C be an NPC space. Consider a representation $\varrho : \pi_1(X) \to \mathrm{Isom}(C)$. We define:

Definition 1.5 (Translation length)**.** For an element $\gamma \in \pi_1(X)$, the *translation length* of $\varrho(\gamma)$ is

$$L_{\varrho(\gamma)} := \inf_{P \in C} d(P, \varrho(\gamma)P). \tag{1.3}$$

If there exists $P_0 \in C$ such that

$$\inf_{P \in C} d(P, gP) = d(P_0, gP_0),$$

then $\varrho(\gamma)$ is called a *semisimple isometry*. For notational simplicity, we write L_γ instead of $L_{\varrho(\gamma)}$ if no confusion arises.

The definition of logarithmic energy growth of a harmonic map was introduced in [DM23a, DM24]. A slightly more intrinsic version is provided in [BDDM22], which we recall here.

Definition 1.6 (logarithmic energy growth)**.** Let X be a smooth quasi-projective variety, G be a reductive algebraic group over a non-archimedean local field K, and let $\varrho : \pi_1(X) \to G(K)$ be a Zariski dense representation. A ϱ-equivariant harmonic map $\tilde{u} : \widetilde{X} \to \Delta(G)$ has *logarithmic energy growth* if for any holomorphic map $f : \mathbb{D}^* \to X$ with no essential singularity at the origin (i.e. for some, thus any, smooth projective compactification $\overline{X}$ of X, f extends to a holomorphic map $\bar{f} : \mathbb{D} \to \overline{X}$), there is a positive constant C such that for any $r \in (0, \frac{1}{2})$, one has

$$-\frac{L_\gamma^2}{2\pi} \log r \leq E^{u_f}[\mathbb{D}_{r,\frac{1}{2}}] \leq -\frac{L_\gamma^2}{2\pi} \log r + C, \tag{1.4}$$

where L_γ is the translation length of $\varrho(\gamma)$ with $\gamma \in \pi_1(X)$ corresponding to the loop $\theta \mapsto f(\frac{1}{2}e^{i\theta})$.

1.5. **A Bertini-type theorem.**

Proposition 1.7 ([BDDM22, Proposition 2.11])**.** *Let $(\overline{X}, \Sigma)$ be a log smooth pair with $n := \dim X \geqslant 2$. Fix a projective embedding $\iota : \overline{X} \hookrightarrow \mathbb{P}^N$ and denote by $L := \iota^*\mathscr{O}_{\mathbb{P}^N}(3)$. For any element $s \in H^0(\overline{X}, L)$, we set $\overline{Y_s} := s^{-1}(0)$, $Y_s := \overline{Y_s} \backslash \Sigma$, and denote by $\iota_{Y_s} : Y_s \to X$ the inclusion map. Let*

$$\mathbb{U} = \{s \in H^0(\overline{X}, L) \mid \overline{Y_s} \text{ is smooth and } \overline{Y_s} + \Sigma \text{ is a normal crossing divisor}\}. \tag{1.5}$$

For $q \in X$, consider the subspace

$$V(q) = \{s \in H^0(\overline{X}, L) \mid s(q) = 0\} \text{ and } \mathbb{U}(q) = \mathbb{U} \cap V(q). \tag{1.6}$$

Then

(i) *The set* $\mathbb{U}(q)$ *is non-empty.*
(ii) *For any* $p, q \in X$, *and* $v \in T_pX$, *there exists some* $s \in \mathbb{U}(q)$ *such that* $p \in Y_s$ *and* Y_s *is tangent to* v.
(iii) *For each* $s \in \mathbb{U}$, $\pi_1(Y_s) \to \pi_1(X)$ *is surjective.*

Note that the last assertion follows from the Lefschetz theorem in [Eyr04].

2. Pluriharmonic maps to Euclidean buildings

In this section we prove Theorem A. As a warm-up, we begin by considering the following special case.

Lemma 2.1. *Let* $\varrho : \pi_1(\mathbb{C}^*) \to (\mathbb{R}, +)$ *be a representation. Consider* $\exp : \mathbb{C} \to \mathbb{C}^*$ *as the universal covering map. Then there exists a* ϱ*-equivariant pluriharmonic map* $\tilde{u} : \mathbb{C} \to \mathbb{R}$ *with logarithmic energy growth. Furthermore,*

(i) *the holomorphic 1-form* $\partial\tilde{u} = \exp^*(\zeta d\log z)$ *for some* $\zeta \in \sqrt{-1}\mathbb{R}$.
(ii) *such* $\tilde{u}$ *is unique up to a translation by a constant.*

Proof. Let γ be the equivalent class in $\pi_1(\mathbb{C}^*)$ represented the loop $\theta \mapsto e^{\sqrt{-1}\theta}$ in $\mathbb{C}^*$. Then $\varrho(\gamma)(x) = x + a$ for some $a \in \mathbb{R}$. Define a map

$$\begin{aligned} \tilde{u} &: \mathbb{C} \to \mathbb{R} \\ w &\mapsto \frac{1}{2}\int_0^w (\exp^*(-\sqrt{-1}\frac{a}{2\pi}d\log z + \sqrt{-1}\frac{a}{2\pi}d\log\bar{z})). \end{aligned}$$

Then $\tilde{u}(w) = \frac{a}{2\pi}\mathrm{Im}(w)$. Thus, $\tilde{u}(w + 2\pi\sqrt{-1}) = \tilde{u}(w) + a$, that is a ϱ-equivariant. We have moreover $\partial\tilde{u}(w) = -\sqrt{-1}\frac{a}{4\pi}dw$, which is a holomorphic 1-form on $\mathbb{C}^*$. It follows that $\partial\bar{\partial}\tilde{u} \equiv 0$. Thus, $\tilde{u}$ is pluriharmonic, and

$$\partial\tilde{u} = \exp^*(-\sqrt{-1}\frac{a}{4\pi}d\log z).$$

This proves Item (i).

Endow $\mathbb{D}^*$ with the standard Euclidean metric $\sqrt{-1}\frac{dz\wedge d\bar{z}}{2}$. However, note that the energy is independent of the choice of metric on the Riemann surface. We can easily compute the energy of u in the annulus $\mathbb{D}_{r,1} := \{r < |z| < 1\} \subset \mathbb{C}^*$:

$$\begin{aligned} E^u[\mathbb{D}_{r,1}] &= \int_{\mathbb{D}_{r,1}} |du|^2 \frac{\sqrt{-1}dz \wedge d\bar{z}}{2} \\ &= \int_{\mathbb{D}_{r,1}} |\frac{a}{2\pi}d\theta|^2 r dr \wedge d\theta \\ &= (\frac{a}{2\pi})^2 \int_0^{2\pi} d\theta \int_r^1 d\log r = \frac{a^2}{2\pi}\log\frac{1}{r}. \end{aligned}$$

By Definition 1.5, the translation length $L_\gamma = |a|$. By Definition 1.6, $\tilde{u}$ has logarithmic energy growth. In conclusion, $\tilde{u}$ is a pluriharmonic map with logarithmic energy growth.

Let us prove Item (ii). If $\tilde{v} : \mathbb{C} \to \mathbb{R}$ is another ϱ-equivariant pluriharmonic map with logarithmic energy growth, then $\partial\tilde{v}$ is a holomorphic 1-form, which descends to 1-form η on $\mathbb{C}^*$ such that $\exp^*\eta = \partial\tilde{v}$. By [BDDM22], η is a logarithmic form on $\mathbb{C}^*$. Hence there exists a constant $b = b_1 + \sqrt{-1}b_2$ with $b_i \in \mathbb{R}$ such that $\eta = bd\log z$. Note that $d\tilde{v} = \exp^*(\eta + \bar{\eta})$. It follows that

$$a = \tilde{v}(w + 2\pi\sqrt{-1}) - \tilde{v}(w) = \int_\gamma (\eta + \bar{\eta}) = -4\pi b_2. \tag{2.1}$$

Hence $b_2 = -\frac{a}{4\pi}$.

Let us compute the energy of $\tilde{v}$ on the annulus $\mathbb{D}_{r,1}$. We have

$$E^{v}[\mathbb{D}_{r,1}] = \int_{\mathbb{D}_{r,1}} |dv|^2 \frac{\sqrt{-1}dz \wedge d\bar{z}}{2} \tag{2.2}$$
$$= \int_{\mathbb{D}_{r,1}} |2b_1 d\log r - 2b_2 d\theta|^2 r dr \wedge d\theta$$
$$= \big((\frac{a}{2\pi})^2 + 4b_1^2\big) \int_0^{2\pi} d\theta \int_r^1 d\log r$$
$$= \frac{a^2}{2\pi} \log\frac{1}{r} + 8\pi b_1^2 d\log\frac{1}{r}.$$

By eq. (1.4), $b_1 = 0$. This implies that $\partial\tilde{u} = \partial\tilde{v}$. Hence $d(\tilde{u} - \tilde{v}) = 0$. Therefore, $\tilde{u}$ is unique up to a translation. The lemma is proved. □

Let $(\overline{X}, \Sigma)$ be a log smooth pair. Let us recall the definition of residue of a logarithmic form $\eta \in H^0(\overline{X}, \Omega_{\overline{X}}(\log\Sigma))$ around an irreducible component Σ_i of Σ. We fix an admissible coordinate $(U; z_1 \ldots, z_n)$ centered at some point $x_0 \in \Sigma_i$ away from the crossings of Σ such that $(z_1 = 0) = U \cap \Sigma_i = U \cap \Sigma$. Then we can write $\eta = h_1(z) d\log z_1 + \sum_{i=2}^n h_i(z) dz_i$. We define

$$\mathrm{Res}_{\Sigma_i}\eta := h_1(0). \tag{2.3}$$

Note that such definition does not depend on the choice of local coordinate system.

Definition 2.2 (Pure imaginary logarithmic form)**.** Let $(\overline{X}, \Sigma)$ be a log smooth pair. A logarithmic form η is *pure imaginary* if for each irreducible component Σ_i of Σ, the residue of η at Σ_i is a pure imaginary number.

Note that Definition 2.2 does not depend on the choice of compactification of $X = \overline{X}\backslash\Sigma$.

Proposition 2.3. *Let $(\overline{X}, \Sigma)$ be a log smooth pair. Let $\varrho : \pi_1(X) \to (\mathbb{R}, +)$ be a representation. If there exists a ϱ-equivariant pluriharmonic map $\tilde{u} : \widetilde{X} \to \mathbb{R}$, then $\tilde{u}$ has logarithmic energy growth if and only if $\partial\tilde{u}$ descends to a logarithmic form $\eta \in H^0(\overline{X}, \Omega_{\overline{X}}(\log\Sigma))$, that is pure imaginary.*

Proof. We write $\Sigma = \sum_{i=1}^m \Sigma_i$ into a sum of irreducible components. Fix some $i \in \{1, \ldots, m\}$. Choose a point $x_0 \in \Sigma_i \backslash \cup_{j\neq i} \Sigma_j$. We take a small embedded disk $f : \mathbb{D} \to \overline{X}$ such that $f^{-1}(\Sigma) = f^{-1}(\Sigma_i) = \{0\}$ and f is transverse to Σ_i at x_0. Let $\gamma \in \pi_1(X)$ be the element representing the loop $\theta \mapsto f(\frac{1}{2}e^{i\theta})$. Let $\mathbb{H}$ be the left half plane of $\mathbb{C}$. Then $\exp : \mathbb{H} \to \mathbb{D}^*$ is the universal covering map. Let $\tilde{f} : \mathbb{H} \to \widetilde{X}$ be the lift of f between universal covers. Then $\tilde{u} \circ \tilde{f} : \mathbb{H} \to \mathbb{R}$ is $f^*\varrho$-equivariant pluriharmonic map and let u_f be the section defined in § 1.2.

If $\tilde{u}$ has logarithmic energy growth, then by [BDDM22], $\partial\tilde{u}$ descends to a logarithmic form $\eta \in H^0(\overline{X}, \Omega_{\overline{X}}(\log\Sigma))$. Let us prove that η is pure imaginary. By Definition 1.5, the translation length L_γ is given by

$$L_\gamma = \left|\int_\gamma (f^*\eta + f^*\bar{\eta})\right| = \left|2\pi\sqrt{-1}(\mathrm{Res}_{\Sigma_i}\eta - \overline{\mathrm{Res}_{\Sigma_i}\eta})\right|.$$

Since η has logarithmic poles, there is some $h(z) \in \mathscr{O}(\mathbb{D})$ such that $f^*\eta = h(z) d\log z$. Write $h(z) = h_1(z) + \sqrt{-1}h_2(z)$, where $h_i(z)$ are real harmonic functions on $\mathbb{D}$. Then

$$L_\gamma = |4\pi h_2(0)|. \tag{2.4}$$

The energy

$$
\begin{aligned}
E^{u_f}[\mathbb{D}_{r,1}] &= \int_{\mathbb{D}_{r,1}} |f^*\eta + f^*\bar{\eta}|^2 \frac{\sqrt{-1}dz\wedge d\bar{z}}{2} \\
&= \int_{\mathbb{D}_{r,1}} |h(z)d\log z + \overline{h(z)}d\log\bar{z}|^2 \frac{\sqrt{-1}dz\wedge d\bar{z}}{2} \\
&= \int_{\mathbb{D}_{r,1}} |2h_1(z)d\log t - 2h_2(z)d\theta|^2 tdt\wedge d\theta \\
&= \int_r^1\int_0^{2\pi} |2h_1(te^{\sqrt{-1}\theta})|^2 d\log t\wedge d\theta \\
&+ \int_r^1\int_0^{2\pi} |2h_2(te^{\sqrt{-1}\theta})|^2 d\log t\wedge d\theta
\end{aligned}
\tag{2.5}
$$

Since $|h_i(z)|^2$ are subharmonic functions on $\mathbb{D}$, by the mean value inequality there exists a constant $C>0$ such that

(2.6)

$$8\pi(|h_1(0)|^2+|h_2(0)|^2)\log\frac{1}{r} \leq E^{u_f}[\mathbb{D}_{r,1}] \leq 8\pi(|h_1(0)|^2+|h_2(0)|^2)\log\frac{1}{r}+C, \quad \forall\ r\in(0,1).$$

By Definition 1.6, we have $h_1(0)=0$. Hence η is pure imaginary.

We now assume that η is pure imaginary. Let $g:\mathbb{D}\to\overline{X}$ be any holomorphic map such that $g^{-1}(\Sigma)=\{0\}$. Then $g^*\eta = h(z)\log z$ with $h(0)\in\sqrt{-1}\mathbb{R}$. We denote by u_g the section of $\mathbb{D}^*\times_{g^*\varrho}\mathbb{R}\to\mathbb{D}^*$ defined in § 1.2. By the same manner as (2.4) and (2.6), we can show that u_g has logarithmic energy growth. By Definition 1.6, u has logarithmic energy growth. □

We can extend Lemma 2.1 to the case of semi-abelian varieties.

Proposition 2.4. *Let A be a semiabelian variety and let $\varrho:\pi_1(A)\to(\mathbb{R}^N,+)$ be a representation. Then there is a ϱ-equivariant pluriharmonic map $u:\widetilde{A}\to\mathbb{R}^N$ with logarithmic energy growth. Such pluriharmonic map is unique up to translation.*

Proof. Note that there is a short exact sequence

$$0\to(\mathbb{C}^*)^k\xrightarrow{j}A\xrightarrow{\pi}A_0\to 0,$$

where A_0 is an abelian variety. Let $\overline{A}$ be the canonical compactification of A such that $\pi:A\to A_0$ extends to a $(\mathbb{P}^1)^k$-fiber bundle

$$0\to(\mathbb{P}^1)^k\xrightarrow{\bar{j}}\overline{A}\xrightarrow{\bar{\pi}}A_0\to 0.$$

Let $\Sigma:=\overline{A}\setminus A$ which is a smooth divisor. Let $V\subset H^0(\overline{A},\Omega_{\overline{A}}(\log\Sigma))$ be the $\mathbb{R}$-linear subspace consisting of logarithmic forms, whose resides at each irreducible component of Σ are pure imaginary. Let $d:=\dim A_0$.

Claim 2.5. *We have $\dim_{\mathbb{R}}V=2d+k$. The $\mathbb{R}$-linear map*

$$
\begin{aligned}
\Psi: V &\to H^1(A,\mathbb{R}) \\
\eta &\mapsto \{\frac{\eta+\bar{\eta}}{2}\}
\end{aligned}
\tag{2.7}
$$

is an isomorphism of $\mathbb{R}$-vector spaces.

Proof. Note that $\dim_{\mathbb{C}}H^0(\overline{A},\Omega_{\overline{A}}(\log\Sigma))=d+k$ and $\dim_{\mathbb{R}}H^1(A,\mathbb{R})=2d+k$. We choose a $\mathbb{C}$-basis $\eta_1,\ldots,\eta_d;\xi_1,\ldots,\xi_k$ for $H^0(\overline{A},\Omega_{\overline{A}}(\log\Sigma))=d+k$ such that $\{\eta_1,\ldots,\eta_d\}\subset\pi^*H^0(A_0,\Omega_{A_0})$. The residues of η_i at each component of Σ is thus zero. Let $(w_1,\ldots,w_k)$ be the canonical coordinate of $(\mathbb{C}^*)^k$. Then $j^*\xi_m=\sum_{i=1}^k a_{mi}d\log w_i$ with $(a_{m1},\ldots,a_{mk})\in\mathbb{C}^k$. Note that $\bar{j}^*\xi_1,\ldots,\bar{j}^*\xi_m$ is a $\mathbb{C}$-basis of $H^0((\mathbb{P}^1)^k,\Omega_{(\mathbb{P}^1)^k}(\log D))$, where $D:=(\mathbb{P}^1)^k\setminus(\mathbb{C}^*)^k$. Note that $d\log w_1,\ldots,d\log w_k$ is also $\mathbb{C}$-basis of $H^0((\mathbb{P}^1)^k,\Omega_{(\mathbb{P}^1)^k}(\log D))$. We can thus replace $\xi_1,\ldots,\xi_m$ by some $\mathbb{C}$-linear combination such that $\bar{j}^*\xi_i = \sqrt{-1}d\log w_i$ for each $i = 1,\ldots,k$. This implies that each ξ_i has pure imaginary residues at each irreducible component of Σ. Then we have

$$V := \mathrm{Span}_{\mathbb{R}}\{\xi_1,\ldots,\xi_k,\eta_1,\ldots,\eta_d,i\eta_1,\ldots,i\eta_d\}.$$

We can see that Ψ is a $\mathbb{R}$-isomorphism. □

Let the homomorphism $\mathrm{pr}_i : (\mathbb{R}^N,+) \to (\mathbb{R},+)$ be the projection into i-th factor. Then $\mathrm{pr}_i \circ \varrho : \pi_1(A) \to (\mathbb{R},+)$ is a representation which can be identified with an element $\lambda_i \in H^1(A,\mathbb{R})$ as $H^1(A,\mathbb{R}) \simeq \mathrm{Hom}(H_1(A,\mathbb{Z}),\mathbb{R})$. Denote by $\zeta_i := \Psi^{-1}(\lambda_i)$. We define

$$\begin{aligned} \tilde{u}_i : \widetilde{A} &\to \mathbb{R} \\ z &\mapsto \frac{1}{2}\int_0^z \pi_A^*(\zeta_i+\bar{\zeta}_i). \end{aligned}$$

Then we obtain a smooth map $\tilde{u} : \tilde{A} \to \mathbb{R}^N$ defined by $\tilde{u} = (\tilde{u}_1,\ldots,\tilde{u}_N)$. This map is pluriharmonic as $\bar{\partial}\partial\tilde{u} = (\frac{1}{2}\bar{\partial}\pi_A^*\zeta_1,\ldots,\frac{1}{2}\bar{\partial}\pi_A^*\zeta_N) = (0,\ldots,0)$. One can verify that $\tilde{u}$ is ϱ-equivariant. Indeed, for any $x \in \widetilde{A}$ and any $\gamma \in \pi_1(A)$, we have

$$\tilde{u}_i(\gamma.x) - \tilde{u}_i(x) = \int_\gamma \frac{1}{2}(\zeta_i+\bar{\zeta}_i) = \lambda_i(\gamma) = \mathrm{pr}_i \circ \varrho(\gamma)(\tilde{u}_i(x)) - \tilde{u}_i(x). \tag{2.8}$$

Let us prove that $\tilde{u}$ has logarithmic energy growth. Since $\partial\tilde{u}_i = \frac{1}{2}\pi_X^*\zeta_i$, where ζ_i is a pure imaginary logarithmic 1-form, by Proposition 2.3, $\tilde{u}_i : \widetilde{A} \to \mathbb{R}$ is a $\mathrm{pr}_i\circ\varrho$-pluriharmonic map with logarithmic energy growth. Let $f : \mathbb{D} \to \overline{A}$ be any holomorphic map such that $f^{-1}(\Sigma) = \{0\}$. Let γ be the element in $\pi_1(X)$ represented by the loop $\theta \mapsto f(\frac{1}{2}e^{\sqrt{-1}\theta})$. Let L_i be the translation length of $\mathrm{pr}_i \circ \varrho(\gamma)$. It follows that there exists a constant $C > 0$ such that for each $i \in \{1,\ldots,N\}$, we have

$$\frac{L_i^2}{2\pi}\log\frac{1}{r} \leq E^{(u_i)_f}[\mathbb{D}_{r,1}] \leq \frac{L_i^2}{2\pi}\log\frac{1}{r} + C, \quad \forall\ r \in (0,1).$$

Note that

$$E^{u_f}[\mathbb{D}_{r,1}] = \sum_{i=1}^N E^{(u_i)_f}[\mathbb{D}_{r,1}], \quad \text{and} \quad L^2_{\varrho(\gamma)} = \sum_{i=1}^N L_i^2.$$

We thus have

$$\frac{L^2_{\varrho(\gamma)}}{2\pi}\log\frac{1}{r} \leq E^{u_f}[\mathbb{D}_{r,1}] \leq \frac{L^2_{\varrho(\gamma)}}{2\pi}\log\frac{1}{r} + C, \quad \forall\ r \in (0,1).$$

Thus, $\tilde{u}$ is a pluriharmonic map with logarithmic energy growth.

Let us prove the uniqueness assertion. Let $\tilde{v} = (\tilde{v}_1,\ldots,\tilde{v}_N) : \widetilde{A} \to \mathbb{R}^N$ be another ϱ-equivariant pluriharmonic map with logarithmic energy growth. Then for each $i \in \{1,\ldots,N\}$, $\tilde{v}_i : \widetilde{A} \to \mathbb{R}$ is a $\mathrm{pr}_i \circ \varrho$-pluriharmonic map with logarithmic energy growth. By Proposition 2.3, $\partial\tilde{v}_i$ descends to a logarithmic form $\frac{1}{2}\omega_i$ that is pure imaginary. By (2.8), for any $\gamma \in \pi_1(X)$, we have

$$\begin{aligned} \tilde{v}_i(\gamma.x) - \tilde{v}_i(x) &= \int_\gamma \frac{1}{2}(\omega_i + \bar{\omega}_i) = \mathrm{pr}_i \circ \varrho(\gamma)(\tilde{v}_i(x)) - \tilde{v}_i(x) \\ &= \mathrm{pr}_i \circ \varrho(\gamma)(\tilde{u}_i(x)) - \tilde{u}_i(x)\lambda_i(\gamma) = \int_\gamma \frac{1}{2}(\zeta_i+\bar{\zeta}_i). \end{aligned}$$

By Claim 2.5, we have $\zeta_i = \omega_i$. It follows that $d\tilde{u} = d\tilde{v}$. Hence $\tilde{u} - \tilde{v}$ is a constant. The proposition is proved. □

Let us prove Theorem A, except for Theorem A.(iv), whose proof is deferred to § 5.

Proof of Theorem A. Consider the enlarged Bruhat-Tits building $\Delta(G)$. It is indeed the product of the Bruhat-Tits building of $\Delta(\mathscr{D}G)$ where $\mathscr{D}G$ is the derived group of G, with a real Euclidean space $V := \mathbb{R}^N$ such that $G(K)$ acts on V by translation (cf. [KP23]). The fixator of any point in $\Delta(G)$ is an open and bounded subgroup of $G(K)$. Note that there is a natural action of $\mathscr{D}G(K)$ on $\Delta(\mathscr{D}G)$. The action of $G(K)$ on $\Delta(\mathscr{D}G)$ is given by the composition of $G(K) \to \mathscr{D}G(K)$ with the action of of $\mathscr{D}G(K)$ on $\Delta(\mathscr{D}G)$.

We consider the representation $\sigma : \pi_1(X) \to \mathscr{D}G(K)$ induced by ϱ, which is Zariski dense. By [BDDM22], there exists a σ-equivariant pluriharmonic map $\tilde{u}_0 : \widetilde{X} \to \Delta(\mathscr{D}G)$ with logarithmic energy growth.

On the other hand, for the action of $G(K)$ on V, it induces a representation $\tau : \pi_1(X) \to (V, +)$. Let $a : X \to A$ be the quasi-Albanese map, and $\tilde{a} : \widetilde{X} \to \widetilde{A}$ be a lift of a between universal covers. Note that τ factors through a representation $\tau' : \pi_1(A) \to (V, +)$. By Proposition 2.4, there exists a τ'-equivariant pluriharmonic map $\tilde{v} : \widetilde{A} \to V$ which has logarithmic energy growth. Therefore, $\tilde{v} \circ \tilde{a} : \widetilde{X} \to V$ is a τ-equivariant pluriharmonic map. Since $\partial\tilde{v}$ descends to a tuple of logarithmic 1-forms $\{\omega_1, \ldots, \omega_m\}$ on A that are pure imaginary, it implies that $\partial\tilde{v} \circ \tilde{a}$ descends to $\{a^*\omega_1, \ldots, a^*\omega_m\}$, that are also pure imaginary logarithmic 1-forms on $(\overline{X}, \Sigma)$. By Proposition 2.3, $\tilde{v} \circ \tilde{a}$ has logarithmic energy growth. We define

$$\begin{aligned} \tilde{u} : \widetilde{X} &\to \Delta(\mathscr{D}G) \times V \\ x &\mapsto (\tilde{u}_0(x), \tilde{v} \circ \tilde{a}(x)). \end{aligned} \tag{2.9}$$

Since $\varrho = (\sigma, \tau)$, $\tilde{u}$ is ϱ-equivariant pluriharmonic map. Since both $\tilde{u}_0$ and $\tilde{v} \circ \tilde{a}$ have logarithmic energy growth, $\tilde{u}$ also has logarithmic energy growth. The existence assertion in Theorem A.(i) is established.

Let us prove Theorem A.(ii). By [BDDM22, Theorem A], $\tilde{u}_0$ is harmonic with respect to an arbitrary Kähler metric ω on $\widetilde{X}$. The pluriharmonicity of $\tilde{v} \circ \tilde{a}$ yields that $\partial\bar{\partial}\tilde{v} \circ \tilde{a} \equiv 0$. Thus,

$$\Delta\tilde{v} \circ \tilde{a} = -2\sqrt{-1}\Lambda_\omega \partial\bar{\partial}\tilde{v} \circ \tilde{a} \equiv 0,$$

where Λ_ω denotes the contraction with ω. It follows that $\tilde{v} \circ \tilde{a}$ is harmonic with respect to the metric ω. Therefore, $\tilde{u}$ is harmonic with respect to the metric ω.

Finally, we prove Theorem A.(iii). Let $\overline{Y}$ be a smooth projective compactification with $\Sigma_Y := \overline{Y} \backslash Y$ a simple normal crossing divisor such that f extends to morphism $\bar{f} : \overline{Y} \to \overline{X}$. Then by [BDDM22, Theorem A], $\tilde{u}_0 \circ \tilde{f} : \widetilde{Y} \to \Delta(G)$ is a pluriharmonic map with logarithmic energy growth. By the above arguments, $\partial\tilde{v} \circ \tilde{a} \circ \tilde{f} : \widetilde{Y} \to V$ descends to logarithmic forms $\{(a \circ f)^*\omega_1, \ldots, (a \circ f)^*\omega_m\}$ on the log smooth pair $(\overline{Y}, \Sigma_Y)$, that are pure imaginary. By Proposition 2.3, $\tilde{v} \circ \tilde{a} \circ \tilde{f}$ is pluriharmonic with logarithmic energy growth. Thus, $\tilde{u} \circ \tilde{f}$ is pluriharmonic with logarithmic energy growth. The theorem is proved. □

3. Multivalued section and spectral cover

The notion of *multivalued sections* of a holomorphic vector bundle over a complex manifold has appeared in [CDY22, DW24b], and has proven to be important in studying the geometry of complex algebraic varieties that admit a local system over a non-archimedean local field. In this section, we provide a more systematic description of multivalued sections and their properties in a general setting. The construction of multivalued logarithmic 1-forms on log smooth pairs here is equivalent to, though simpler than, that in [CDY22].

3.1. Construction of spectral cover.

We start with the following definition.

Definition 3.1 (Multivalued section)**.** Let X be a complex manifold, and let E be a holomorphic vector bundle on X. A multivalued (holomorphic) section of E on X, denoted by η, is a formal sum $Z_\eta = \sum_{i=1}^m n_i Z_i$ where $n_i \in \mathbb{N}^*$, and each Z_i is an irreducible closed subvariety of E, such that the natural map $Z_i \to X$ is a finite and surjective.

A multivalued section η is *splitting*, if for each point $x \in X$, it has an open neighborhood Ω_x and holomorphic sections $\{\omega_1, \ldots, \omega_m\} \subset \Gamma(\Omega_x, E|_{\Omega_x})$, such that $Z_\eta|_{\Omega_x}$ is the graph of $\{\omega_1, \ldots, \omega_m\}$.

Note that in [CDY22], multivalued sections are splitting ones.

Let X be a complex manifold, and let E be a holomorphic vector bundle on X. Assume that η is a splitting multivalued section of E. Let T be a formal variable. Consider $\prod_{i=1}^m (T - \omega_i) =: T^m + \sigma_1 T^{m-1} + \cdots + \sigma_m$, where $\{\omega_1, \ldots, \omega_m\}$ are local sections of E whose graph is Z_η. Then

σ_i is a local section of $\mathrm{Sym}^i E$. One can see that σ_i is a global section in $H^0(X, \mathrm{Sym}^i E)$. We call $T^m + \sigma_1 T^{m-1} + \cdots + \sigma_m$ the *characteristic polynomial* of η, and denote it by $P_\eta(T)$.

Proposition 3.2. *Let X be a smooth projective variety endowed with a holomorphic vector bundle E. Let $X' \subset X$ be a topological dense open set. Let η be a splitting* multivalued *section of $E|_{X'}$ over X'. Assume that for the characteristic polynomial $P_\eta(T) = T^m + \sigma_1 T^{m-1} + \cdots + \sigma_m$ of η, its coefficient $\sigma_i \in H^0(X', \mathrm{Sym}^i E|_{X'})$ extends to a section in $H^0(X, \mathrm{Sym}^i E)$ for each i. Then η extends to a multivalued section of E.*

Furthermore, there exists a ramified Galois cover $\pi : X^{\mathrm{sp}} \to X$ with Galois group G from a projective normal variety such that $\pi^*\eta$ becomes single-valued, i.e., there exists sections $\{\eta_1, \ldots, \eta_m\} \subset H^0(X^{\mathrm{sp}}, \pi^* E)$ such that $\pi^*\eta = \{\eta_1, \ldots, \eta_m\}$. The group G acts on $\{\eta_1, \ldots, \eta_m\}$ as a permutation.

Definition 3.3. The above Galois cover π is called the *spectral cover* of X with respect to η.

Proof. Denote by $\mu : E \to X$ be projection map. Let $\lambda \in H^0(E, \mu^* E)$ be the Liouville section defined by $\lambda(e) = e$ for any $e \in E$. Consider the section

$$P_\eta(\lambda) := \lambda^m + \mu^*\sigma_1 \lambda^{m-1} + \cdots + \mu^*\sigma_m \in H^0(E, \mu^*\mathrm{Sym}^m E).$$

Let $Z \subset E$ be the zero locus of $P_\eta(\lambda)$ (here we count multiplicities). By assumption, one can see that, $Z|_{X'} = Z_\eta$. Moreover, $\mu|_Z : Z \to X$ is a finite morphism. To show that Z is a multivalued section of E, we need to prove that, for each irreducible component Z_i of Z, $\mu|_{Z_i} : Z_i \to X$ is surjective.

Let Z^{norm} be the normalization of Z which might not be connected. Then the natural morphism $q : Z^{\mathrm{norm}} \to X$ is finite. Consider the locus X° of X such that q is étale. Then X° is a Zariski dense open set of X. One can see that $X^\circ \supset X'$. Set $Z^{\mathrm{norm}}_\circ := q^{-1}(X^\circ)$ and $Z^\circ := (\mu|_Z)^{-1}(X^\circ)$. Note that $Z^{\mathrm{norm}}_\circ$ is the normalization of Z°.

Claim 3.4. *η extends to a splitting multivalued section of E on X°.*

Proof. Note that $q : Z^{\mathrm{norm}}_\circ \to X^\circ$ is étale of degree m. Hence for every $x \in X^\circ$, it has neighborhood Ω_x such that $q^{-1}(\Omega_x)$ is isomorphic to m copy of Ω_x. Thus it gives rise to m natural local sections $s_1, \ldots, s_m : \Omega_x \to Z^{\mathrm{norm}}_\circ$ of $q : Z^{\mathrm{norm}} \to X$ such that $s_1(\Omega_x), \ldots, s_m(\Omega_x)$ correspond to m components of $q^{-1}(\Omega_x)$. Let $\nu_Z : Z^{\mathrm{norm}} \to Z$ be the normalization map. Then $\{\nu_Z \circ s_i : \Omega_x \to Z \subset E\}_{i=1,\ldots,m} \subset H^0(\Omega_x, E|_{\Omega_x})$. Note that the graph of $\{\nu_Z \circ s_1, \ldots, \nu_Z \circ s_m\}$ is $Z|_{\Omega_x}$. The claim is proved. □

We still denote by η the extended multivalued section of $E|_{X^\circ}$.

Claim 3.5. *The étale morphism $q|_{Z^{\mathrm{norm}}_\circ} : Z^{\mathrm{norm}}_\circ \to X^\circ$ gives rise to a representation $\phi : \pi_1(X^\circ) \to \mathfrak{S}_m$ where $\mathfrak{S}_m$ is the symmetric group of m elements. Let $\pi : Y^\circ \to X^\circ$ be the Galois étale cover corresponding to the finite index subgroup* $\ker\phi$ *of $\pi_1(X^\circ)$. Then*

- *the normalization of the base change $Z^\circ \times_{X^\circ} Y^\circ$ is a quasi-projective variety with m connected component such that each component is isomorphic to Y° under the natural map $(Z^\circ \times_{X^\circ} Y^\circ)^{\mathrm{norm}} \to Y^\circ$.*
- *There are sections $\{\eta_1, \ldots, \eta_m\} \subset H^0(Y^\circ, \pi^* E)$ such that $\{\eta_1, \ldots, \eta_m\} = \pi^*\eta$.*
- *G acts on $\{\eta_1, \ldots, \eta_m\}$ as a permutation.*

Proof. We fix a base point $x_0 \in X'$. There exists an open neighborhood Ω_{x_0} of x_0 such that, the multivalued section $\eta|_{\Omega_{x_0}}$ is given by sections $\{s_1, \ldots, s_m\} \subset H^0(\Omega_{x_0}, E|_{\Omega_{x_0}})$. Consider any loop γ of X° based at x_0. Since $q|_{Z^{\mathrm{norm}}_\circ} : Z^{\mathrm{norm}}_\circ \to X^\circ$ is étale, by Definition 3.1, the transport of $\{s_1, \ldots, s_m\}$ along $Z^{\mathrm{norm}}_\circ|_\gamma$ gives a permutation of $\{s_1, \ldots, s_m\}$ hence an element in the symmetric group $\mathfrak{S}_m$ of m elements. We can see that it only depends on the choice of homotopy class of γ and thus it corresponds to a representation $\phi : \pi_1(X^\circ) \to \mathfrak{S}_m$. Let $\pi : Y^\circ \to X^\circ$ be the Galois étale cover with the Galois group $G := \pi_1(X^\circ)/\ker\phi$. Then for any loop $\gamma \in \pi_1(Y^\circ)$, the transport of $\{s_1, \ldots, s_m\}$ along $Z^{\mathrm{norm}}_\circ \times_{X^\circ} Y^\circ|_\gamma$ is a trivial permutation, which thus gives rise to holomorphic sections $\{\eta_1, \ldots, \eta_m\} \subset H^0(Y^\circ, \pi^* E)$. It follows that $\pi^*\eta = \{\eta_1, \ldots, \eta_m\}$. One can see that G acts on $\{\eta_1, \ldots, \eta_m\}$ as a permutation.

Let $W^\circ \subset \pi^*E$ be the graph variety of $\{\eta_1, \ldots, \eta_m\}$. One can see that W° coincides with $Z^\circ \times_{X^\circ} Y^\circ$. Hence the normalization $(Z^\circ \times_{X^\circ} Y^\circ)^{\mathrm{norm}}$ is isomorphic to m copy of Y°. The claim is proved. □

Note that $\pi : Y^\circ \to X^\circ$ extends to a ramified Galois cover $Y \to X$ with the Galois group G, where Y is a projective normal variety. We still denote by $\pi : Y \to X$ the extended cover.

Let $\nu : \pi^*E \to Y$ the natural projection map. We have the following commutative diagram:

$$\begin{tikzcd} \pi^*E \arrow[r, "f"] \arrow[d, "\nu"] & E \arrow[d, "\mu"] \\ Y \arrow[r, "\pi"] & X \end{tikzcd}$$

Let $\lambda' \in H^0(\pi^*E, \nu^*\pi^*E)$ be the Liouville section. Consider the section

$$Q(\lambda') := \lambda'^m + \nu^*\pi^*\sigma_1\lambda'^{m-1} + \cdots + \nu^*\pi^*\sigma_m \in H^0(\pi^*E, \nu^*\pi^*\mathrm{Sym}^m E).$$

Let $W \subset \pi^*E$ be the zero scheme of $Q'(\lambda')$. Note that $W|_{\nu^{-1}(Y^\circ)} = W^\circ$, that is the graph variety of $\{\eta_1, \ldots, \eta_m\}$. Therefore, over $\nu^{-1}(Y^\circ)$, we have

$$Q(\lambda') = \prod_{i=1}^{m} (\lambda' - \eta_i).$$

By continuity, it follows that the above equality holds over the whole π^*E. Since we have $Q(\lambda') = f^*P_\eta(\lambda)$, W is equal to the scheme theoretic inverse image $f^{-1}(Z)$. Note that each irreducible component of W is mapped to Y surjectively. It follows that each irreducible component of Z is mapped to X surjectively. Hence Z is a multivalued section of $E \to X$. We write $\pi : X^{\mathrm{sp}} \to X$ for $\pi : Y \to X$. The proposition is proved. □

3.2. **Invariant 1-forms on Bruhat-Tits buildings.** Let G be a reductive algebraic group over a non-archimedean local field. Then G induces a real Euclidean space V endowed with a Euclidean metric and an affine Weyl group W acting on V isometrically. Such group W is a semidirect product $T \rtimes W^v$, where W^v is the *vectorial Weyl group*, which is a finite group generated by reflections on V, and T is a translation group of V.

For any apartment A in $\Delta(G)$, there exists an isomorphism $i_A : A \to V$, which is called a chart. For two charts $i_{A_1} : A_1 \to V$ and $i_{A_2} : A_2 \to V$, if $A_1 \cap A_2 \neq \varnothing$, it satisfies the following properties:

(a) $Y := i_{A_2}(i_{A_1}^{-1}(V))$ is convex.
(b) There is an element $w \in W$ such that $w \circ i_{A_1}|_{A_1\cap A_2} = i_{A_2}|_{A_1\cap A_2}$.

Let us fix orthonormal coordinates $(x_1, \ldots, x_N)$ for V. Since $W^v \subset \mathrm{GL}(V)$ acts on V isometrically, for any $w \in W^v$, $(w^*x_1, \ldots, w^*x_N)$ are orthonormal coordinates for V. We define a subset of V^* by setting

$$\Phi := \{w^*x_i\}_{i\in\{1,\ldots,N\};w\in W^v}. \tag{3.1}$$

Since W^v is a finite group, then Φ is a finite set. Note that Φ is invariant under the action by W^v. We write $\Phi = \{\beta_1, \ldots, \beta_m\}$.

We define real affine functions

$$\beta_{A,i} := \beta_i \circ i_A(x) \tag{3.2}$$

on A for each i.

Lemma 3.6. *If $A_1 \cap A_2 \neq \varnothing$, then we have*

$$\{d\beta_{A_1,1}, \ldots, d\beta_{A_1,m}\}|_{A_1\cap A_2} = \{d\beta_{A_2,1}, \ldots, d\beta_{A_2,m}\}|_{A_1\cap A_2}.$$

Proof. By Item (b), there exists an element $w \in W$ such that $\beta_k \circ i_{A_2}|_{A_1\cap A_2} = \beta_k \circ w \circ i_{A_1}|_{A_1\cap A_2}$ for any $k = 1, \ldots, m$. Recall that W^v permutes Φ. It follows that there exist $a_1, \ldots, a_m \in \mathbb{R}$ and a permutation σ of m-elements such that

$$\beta_k \circ i_{A_2}|_{A_1\cap A_2} = \beta_k \circ w \circ i_{A_1}|_{A_1\cap A_2} = \beta_{\sigma(k)} \circ i_{A_1}|_{A_1\cap A_2} - a_k \tag{3.3}$$

for any $k = 1, \ldots, m$. This implies the lemma. □

3.3. **Mutivalued 1-forms and spectral 1-forms.** We prove Theorem C, except for Theorem C.(ii), whose proof is defered to § 4.

Theorem 3.7. *Let $(\overline{X}, \Sigma)$ be a smooth log pair. Let ϱ, G and $\tilde{u}$ be as in Theorem A. Then*

(i) *the pluriharmonic map $\tilde{u}$ induces a multivalued logarithmic 1-form η on the log pair $(\overline{X}, \Sigma)$.*
(ii) *There exists a ramified Galois cover $\pi : \overline{X^{\rm sp}} \to \overline{X}$ such that $\pi^*\eta$ becomes single-valued; i.e., $\pi^*\eta := \{\omega_1, \ldots, \omega_m\} \subset H^0(\overline{X^{\rm sp}}, \pi^*\Omega_{\overline{X}}(\log \Sigma))$.*
(iii) *Denote by $X^{\rm sp} = \pi^{-1}(X)$, and let $\Sigma_1 := \overline{X^{\rm sp}} \backslash X^{\rm sp}$. Let $\mu : \overline{Y} \to \overline{X^{\rm sp}}$ be a log resolution of $(\overline{X^{\rm sp}}, \Sigma_1)$, with $\Sigma_Y := \mu^{-1}(\Sigma_1)$ a simple normal crossing divisor. Then logarithmic forms $\{\mu^*\omega_1, \ldots, \mu^*\omega_m\}$ are* pure imaginary.

Proof. **Step 1.** We assume that G is semi-simple. Let u be the corresponding section of $\widetilde{X} \times_\varrho \Delta(G) \to X$ of $\tilde{u}$ defined in § 1.2. Let $\mathcal{R}(u) \subset X$ be the regular locus of u defined in Definition 1.2. Then $X \backslash \mathcal{R}(u)$ is an open subset of X of Hausdorff codimension at least two by Lemma 1.3.

For any regular point $x \in \mathcal{R}(u)$ of u (cf. Definition 1.2), one can choose a simply-connected open neighborhood U of x such that

(1) the inverse image $\pi_X^{-1}(U) = \coprod_{\alpha \in I} U_\alpha$ is a union of disjoint open sets in $\widetilde{X}$, each of which is mapped isomorphically onto U by $\pi_X : \widetilde{X} \to X$.
(2) For some $\alpha \in I$, there is an apartment A_α of $\Delta(G)$ such that $u(U_\beta) \subset A_\beta$.

Let $\Phi = \{\beta_1, \ldots, \beta_m\}$ be the subset of V^* defined in (3.1). For each apartment A of $\Delta(G)$, $\{\beta_{A,1}, \ldots, \beta_{A,m}\}$ are the affine functions on A defined in (3.2). For each $j \in \{1, \ldots, m\}$, we define a real function

$$u_{\alpha,j} = \beta_{A_\alpha,j} \circ \tilde{u} \circ (\pi_X|_{U_\alpha})^{-1} : U \to \mathbb{R}. \tag{3.4}$$

By the pluriharmonicity of $\tilde{u}$, we have $\partial\bar{\partial} u_{\alpha,j} = 0$ for each j. Hence $\partial u_{\alpha,j}$ is a holomorphic 1-form on U. By [BDDM22, §4.2], the set of holomorphic 1-forms $\{\partial u_{\alpha,1}, \ldots, \partial u_{\alpha,m}\}$ on U will glue together into a splitting multivalued 1-forms η over $\mathcal{R}(u)$. Moreover, for the characteristic polynomial $P_\eta(T) := T^m + \sigma_1 T^{m-1} + \cdots + \sigma_m$ of η defined in § 3.1, each σ_i extends to a logarithmic 1-form in $H^0(\overline{X}, \Omega_{\overline{X}}(\log \Sigma))$. Hence conditions in Proposition 3.2 are fulfilled. It implies that, η extends to a multivalued logarithmic 1-form over $(\overline{X}, \Sigma)$, and there exists a spectral cover $\pi : \overline{X^{\rm sp}} \to \overline{X}$ with respect to η. The first two assertions of the theorem are proved.

We denote by $\bar{f} : (\overline{Y}, \Sigma_Y) \to (\overline{X}, \Sigma)$ be the morphism between log smooth pairs, that is the composition of μ and π. Let $f : Y \to X$ be the restriction of $\bar{f}$ over Y. Then by Theorem A, $\tilde{u} \circ \tilde{f} : \widetilde{Y} \to \Delta(G)$ is an $f^*\varrho$-equivariant pluriharmonic map with logarithmic energy growth. Here we denote by $\tilde{f} : \widetilde{Y} \to \widetilde{X}$ the lift of f between the universal covers. Write $\tilde{v} := \tilde{u} \circ \tilde{f}$ and let v be the corresponding section.

We fix an irreducible component Σ_o of Σ_Y. Since $\mathcal{S}(u) := X \backslash \mathcal{R}(u)$ has Hausdorff codimension at least two, we can choose an embedded transverse disk $g : \mathbb{D} \to \overline{Y}$, such that $g^{-1}(\Sigma_o) = g^{-1}(\Sigma_Y) = \{0\}$, and $g(\mathbb{D})$ intersects with Σ_o transversely. Furthermore, $(f \circ g)^{-1}(\mathcal{S}(u))$ has Hausdorff dimension 0.

We fix the Euclidean metric $\frac{\sqrt{-1}}{2} dz \wedge d\bar{z}$ over $\mathbb{D}^*$. By the above construction, the multivalued 1-forms associated with the equivariant pluriharmonic maps we defined are functorial. In other words,

$$f^*\eta = \{\mu^*\omega_1, \ldots, \mu^*\omega_m\} \subset H^0(\overline{Y}, \Omega_{\overline{Y}}(\log \Sigma_Y))$$

corresponds to the multivalued 1-form induced by $\tilde{v}$. Thus, applying (3.8.(i)) below, after rescaling of η by multiplying it by $\frac{1}{\sqrt{|W^v|}}$, we obtain the following over $f^{-1}(\mathcal{R}(u))$:

$$|\nabla v|^2 = \sum_{i=1}^m |\mu^*\omega_i + \mu^*\bar{\omega}_i|^2,$$

where $|\nabla v|^2$ is the energy density function of v. Since $|\nabla v|^2 \in L^1_{\rm loc}$, we conclude that the above equality holds over the whole Y.

Let v_g be the section of $\mathbb{D}^* \times_{(f\circ g)^*\varrho} \Delta(G) \to \mathbb{D}^*$ defined in § 1.2. On the other hand, since $(f\circ g)^{-1}(\mathcal{S}(u))$ has Hausdorff dimension 0, by the same argument as above, we can conclude that

$$|\nabla v_g|^2 = \sum_{i=1}^m |g^*\mu^*\omega_i + g^*\mu^*\bar{\omega}_i|^2.$$

Write $g^*\mu^*\omega_i = (a_i(z) + \sqrt{-1}b_i(z))d\log z$, where $a_i(z)$ and $b_i(z)$ are real harmonic functions on $\mathbb{D}$. Then by the same computation as in (2.5) and (2.6), there exists a constant $C > 0$ such that

(3.5)

$$8\pi(\sum_{i=1}^m |a_i(0)|^2 + |b_i(0)|^2)\log\frac{1}{r} \leq E^{v_g}[\mathbb{D}_{r,1}] \leq 8\pi(\sum_{i=1}^m |a_i(0)|^2 + |b_i(0)|^2)\log\frac{1}{r} + C, \quad \forall\ r \in (0,1).$$

Let $\gamma \in \pi_1(Y)$ be the element representing the loop $\theta \mapsto g(\frac{1}{2}e^{\sqrt{-1}\theta})$. Since v has logarithmic energy growth, by Definition 1.6, we have

$$L^2_{f^*\varrho(\gamma)} = 16\pi^2(\sum_{i=1}^m |a_i(0)|^2 + |b_i(0)|^2). \tag{3.6}$$

Since $(f\circ g)^{-1}(\mathcal{S}(u))$ has Hausdorff dimension 0, by [Shi68, Corollary 1] there exists a subset $I \subset (0,1)$ of Lebesgue measure 1, such that for each $r \in I$, the loop ℓ_r in $\mathbb{D}^*$ defined by $\theta \mapsto re^{\sqrt{-1}\theta}$, does not intersect with $(f\circ g)^{-1}(\mathcal{S}(u))$. Let $\gamma \in \pi_1(Y)$ be the element representing the loop $\theta \mapsto g(re^{\sqrt{-1}\theta})$. In this case, the translation length $L_{f^*\varrho(\gamma)}$ satisfies that, for any $r \in I$, we have

$$\begin{aligned} L_{f^*\varrho(\gamma)} &\leq \oint_{\ell_r} \sqrt{\sum_{i=1}^m |(g^*\mu^*\omega_i + g^*\mu^*\bar{\omega}_i)(\frac{\partial}{\partial\theta})|^2}d\theta \\ &= \int_0^{2\pi} \sqrt{\sum_{i=1}^m |2b_i(re^{\sqrt{-1}\theta})|^2}d\theta. \end{aligned}$$

If letting $r \in I$ tends to 0, we have

$$L^2_{f^*\varrho(\gamma)} \leq 16\pi^2 \sum_{i=1}^m b_i^2(0).$$

It follows from (3.6) that $a_i(0) = 0$ for each i. Thus, for each $i \in \{1,\ldots,m\}$, we have $\mathrm{Res}_{\Sigma_o}\mu^*\omega_i = \sqrt{-1}b_i(0)$, which is pure imaginary. Since Σ_o is an arbitrary irreducible component of Σ_Y, it follows that $\mu^*\omega_1,\ldots,\mu^*\omega_m$ are pure imaginary logarithmic forms. The theorem is thus proved when G is semisimple.

Step 2. We assume that G is reductive. We shall use the notation introduced in the proof of Theorem A without recalling them explicitly. Recall that $\Delta(G) = \Delta(\mathscr{D}G) \times V$, where V is isometric to $\mathbb{R}^N$, and $G(K)$ acts on V by translation. Note that $\tilde{u}$ is the product of a σ-equivariant pluriharmonic map $\tilde{u}_0 : \widetilde{X} \to \Delta(\mathscr{D}G)$ with logarithmic energy growth, and a τ-equivariant pluriharmonic map $\tilde{v}\circ\tilde{a} : \widetilde{X} \to V$, also with logarithmic energy growth. Thus, the multivalued 1-form η induced by $\tilde{u}$ is merely the union of the multivalued 1-form η_0 induced by $\tilde{u}_0$, and the logarithmic 1-forms $\{\zeta_1,\ldots,\zeta_k\} \subset H^0(\overline{X}, \Omega_{\overline{X}}(\log\Sigma))$ induced by $\partial(\tilde{v}\circ\tilde{a})$. Hence, the spectral cover $\pi : \overline{X^{\mathrm{sp}}} \to \overline{X}$ with respect to η coincides with the spectral cover with respect to η_0, whose existence is ensured by Step 1. The first two items are proved.

By Step 1, $f^*\eta_0$ is a finite set of pure imaginary logarithmic 1-forms. Recall that in Theorem A, we prove that $\{\zeta_1,\ldots,\zeta_k\}$ are pure imaginary. Thus, $f^*\zeta_i$ is also pure imaginary for each i. We conclude that $f^*\eta = f^*\eta_0 \cup \{f^*\zeta_1,\ldots,f^*\zeta_k\}$ is a set of pure imaginary logarithmic 1-forms. This completes the proof of the theorem. □

By the proof of Theorem 3.7.(i), if G is semi-simple, at each point x_0 of $\mathcal{R}(u)$, there exists an open neighborhood U of x_0 such that, η is given by holomorphic 1-forms $\{\partial u_{\alpha,1},\ldots,\partial u_{\alpha,m}\}$, where $u_{\alpha,j} : U \to \mathbb{R}$ is defined in (3.4). Let U_α be a connected component of $\pi_X^{-1}(U)$ introduced in item (1). Note that

$$|\nabla u|^2 = \sum_{i=1}^{N} |dx_i \circ i_{A_\alpha} \circ \tilde{u} \circ (\pi_X|_{U_\alpha})^{-1}|^2,$$

where $\{x_1, \ldots, x_N\}$ is some orthogonal coordinates for V defined in § 3.2. For any $w \in W$, note that $\{w^* dx_1, \ldots, w^* dx_N\}$ is a orthogonal basis for TV^*. Hence, by the definition of Φ defined in (3.1), we have

$$\begin{aligned}
\sum_{j=1}^{m} |\partial u_{\alpha,j}| &= \sum_{w\in W^v} \sum_{i=1}^{N} |w^*\partial x_i \circ i_{A_\alpha} \circ \tilde{u} \circ (\pi_X|_{U_\alpha})^{-1}|^2 \\
&= |W^v| \cdot \sum_{i=1}^{N} |\partial x_i \circ i_{A_\alpha} \circ \tilde{u} \circ (\pi_X|_{U_\alpha})^{-1}|^2 \\
&= \frac{|W^v|}{2} |\nabla u|^2. \qquad (3.7)
\end{aligned}$$

The following result will be used in § 5.

Lemma 3.8. *Let X, G, ϱ and $\tilde{u}$ be as in Theorem A. Then there exists a multivalued logarithmic 1-form η on $(\overline{X}, \Sigma)$, that is splitting over $\mathcal{R}(u)$, such that for any point $x \in \mathcal{R}(u)$, it has a simply connected open neighborhood U satisfying:*

(i) *over U, η is represented by some holomorphic 1-forms $\{\omega_1, \ldots, \omega_{N\ell}\}$ on Ω, and*

$$|\nabla u|^2 = 2 \sum_{j=1}^{N\ell} |\omega_j|^2, \qquad (3.8)$$

where N is the K-rank of G, and ℓ is the cardinality of the vectorial Weyl group W^v of $\mathscr{D}G$.

(ii) *There exists a partition of $\sqcup_{i=1}^{\ell} \{\omega_{i,1}, \ldots, \omega_{i,N}\} = \{\omega_1, \ldots, \omega_{\ell N}\}$ satisfying*
   - *for each $i = 2, \ldots, \ell$, there exists a constant matrix $M_i \in \mathrm{O}(N, \mathbb{R})$ such that*

$$\begin{bmatrix}\omega_{i,1}, & \cdots, & \omega_{i,N}\end{bmatrix} = \begin{bmatrix}\omega_{1,1}, & \cdots, & \omega_{1,N}\end{bmatrix} \cdot M_i. \qquad (3.9)$$

   - *If there exists some apartment A of $\Delta(G)$, such that $\tilde{u}(U_\alpha) \subset A$, where U_α is some connected component of $\pi_X^{-1}(U)$, then for any isometry $i : A \to \mathbb{R}^N$, denoting $(u_1, \ldots, u_N) = i \circ \tilde{u} \circ (\pi_X|_{U_\alpha})^{-1} : U \to \mathbb{R}^N$, we have*

$$\begin{bmatrix}\partial u_1, & \cdots, & \partial u_N\end{bmatrix} = \begin{bmatrix}\omega_{1,1}, & \cdots, & \omega_{1,N}\end{bmatrix} \cdot M \cdot \frac{1}{\sqrt{\ell}} \qquad (3.10)$$

   for some constant matrix $M \in \mathrm{O}(N, \mathbb{R})$.

(iii) *For each $p \in \{1, \ldots, n\}$, η induces a multivalued section η^p on $\Omega^p_{\overline{X}}(\log \Sigma)$.*

Proof. We shall use the notations in Step 2 of the proof of Theorem 3.7. For each point x_0 of $\mathcal{R}(u_0)$, there exists a simply connected neighborhood U of x_0 such that, for some connected component U_α of $\pi_X^{-1}(U)$, $\tilde{u}_0(U_\alpha)$ is contained in some apartment A of $\Delta(\mathscr{D}G)$. Let (W^v, V) be data of $\mathscr{D}G$ defined in § 3.2. Let N' be the dimension of $\Delta(\mathscr{D}G)$ and ℓ be the cardinality of W^v. Fix orthonormal coordinates $(x_1, \ldots, x_{N'})$ for V.

We use the notations in § 3.2. Define a set of holomorphic 1-forms on U with a partition as follows:

(3.11)

$$\sqcup_{w\in W^v} \{\frac{1}{\sqrt{\ell}} w^*\partial x_1 \circ i_A \circ \tilde{u}_0 \circ (\pi_X|_{U_\alpha})^{-1}, \ldots, \frac{1}{\sqrt{\ell}} w^*\partial x_{N'} \circ i_A \circ \tilde{u}_0 \circ (\pi_X|_{U_\alpha})^{-1}, \frac{1}{\sqrt{\ell}}\xi_1, \ldots, \frac{1}{\sqrt{\ell}}\xi_k\},$$

where $\xi_1, \ldots, \xi_k$ are logarithmic 1-forms on $(\overline{X}, \Sigma)$ induced by the pluriharmonic map $\tilde{v} \circ \tilde{a}$. Thus we have $N = N' + k$, as N is also the dimension of $\Delta(G)$. By Step 1 of the proof of Theorem 3.7.(i), (3.11) gives rise to a multivalued logarithmic 1-form on $(\overline{X}, \Sigma)$, denoted by η. By (3.7), we have

$$|\nabla u_0|^2 = 2\sum_{i=1}^{N'} \frac{1}{|W^v|} \left|\partial x_1 \circ i_A \circ \tilde{u}_0 \circ (\pi_X|_{U_\alpha})\right|^2 + \cdots + \left|\partial x_{N'} \circ i_A \circ \tilde{u}_0 \circ (\pi_X|_{U_\alpha})\right|^2$$

Note that $\partial\tilde{v} \circ \tilde{a} = (\pi_X^*\xi_1, \ldots, \pi_X^*\xi_k)$. Since $|\nabla u|^2 = |\nabla u_0|^2 + |\nabla \tilde{v} \circ \tilde{a}|^2$, it yields (3.8).

Note that w is an isometry of V. Lemma 3.8.(ii) follows directly from the construction of η in (3.11).

Let us prove Lemma 3.8.(iii). For each $I = \{i_1, \ldots, i_p\}$ with $1 \le i_1 < \cdots < i_p \le n$, we define a set of holomorphic p-forms with a partition given by

$$\sqcup_{j=1}^{\ell}\{\pm\omega_{j,i_1} \wedge \cdots \wedge \omega_{j,i_p}\}_{1\le i_1<\cdots<i_p\le N}.$$

By (3.11), this is a well-defined splitting multivalued p-form on $\mathcal{R}(u)$, denoted by η^p.

We fix a smooth hermitian metric h for the vector bundle $\Omega_{\overline{X}}(\log\Sigma)$. It induces a hermitian metric h_p on $\Omega^p_{\overline{X}}(\log\Sigma)$. Since the support $|Z_\eta|$ is compact, there exists a uniform constant $C > 0$ such that

$$\left|\omega_{j,i_1} \wedge \cdots \wedge \omega_{j,i_p}(x)\right|_{h^p} \le C, \quad \forall\, x \in U \cap \mathcal{R}(u)$$

for each I. Let $P_{\eta^p}(T) = T^M + \sigma_1 T^{M-1} + \cdots + \sigma_M$ be the characteristic polynomial of η^p defined in § 3.1, with $\sigma_i \in H^0(\mathcal{R}(u), \mathrm{Sym}^i\Omega^p_{\overline{X}}(\log\Sigma)|_{\mathcal{R}(u)})$. Then the norm of σ_i with respect to the metric h^p is uniformly bounded. By the Hartogs theorem in [Shi68], each σ_i extends to a section of $\mathrm{Sym}^i\Omega^p_{\overline{X}}(\log\Sigma)$ on $\overline{X}$. The conditions in Proposition 3.2 are fulfilled. We conclude that ϕ^p extends to a multivalued section of $\Omega^p_{\overline{X}}(\log\Sigma)$ on $\overline{X}$.The last assertion is proved. □

Remark 3.9. When X is a compact Kähler manifold, spectral covers associated with equivariant harmonic maps to Euclidean buildings were systematically studied by Eyssidieux in [Eys04]. The construction of spectral covers presented here follows the approach of Klingler [Kli13], while the definition of multivalued 1-forms builds on the ideas of [Eys04], which differ slightly from those in [BDDM22, §4].

4. Unicity of pluriharmonic maps

4.1. **Uniqueness of energy density function.** Throughout this subsection, G is a reductive algebraic group defined over a non-archimedean local field K. We begin with the following definition.

Definition 4.1 (Directional energy)**.** Let $u : \Omega \to \Delta(G)$ be a locally finite energy map from a Riemannian domain Ω. For $V \in \Gamma(\Omega, T_\Omega)$, the *directional energy* defined in [KS93, Theorem 1.9.6] is denoted by $|u_*(V)|^2$. By [KS93, Lemma 1.9.3 and Theorem 2.3.2],

$$|u_*(V)|^2(p) = \lim_{t\to 0} \frac{d^2(u(p), u(\exp_p(tV)))}{t^2} \quad \text{for a.e. } p \in \Omega.$$

Remark 4.2. Let M be a Riemannian manifold, $\varrho : \pi_1(M) \to G(K)$ be a representation and $u : \widetilde{M} \to \Delta(G)$ be a ϱ-equivariant map. Given a vector field V defined on M, lift it to $\tilde{M}$ and denote it again by V. Then the energy density function $|u_*(V)|^2$ is a $\pi_1(M)$-invariant function on $\tilde{M}$ and thus descends to a well-defined L^1_{loc}-function on M.

Proposition 4.3. *Let X be a smooth quasi-projective variety of dimension n and $\varrho : \pi_1(X) \to G(K)$ be a representation. If $\tilde{u}, \tilde{v} : \widetilde{X} \to \Delta(G)$ are two ϱ-equivariant pluriharmonic maps of logarithmic energy growth, then we have*

(i) *$d(\tilde{u}, \tilde{v}) = c$ for some constant $c \ge 0$;*
(ii) *$|\tilde{u}_*(V)|^2 = |\tilde{v}_*(V)|^2$ for any holomorphic vector field $V \in \Gamma(\Omega, T_\Omega)$, where $\Omega \subset \widetilde{X}$ is an open set.*

Proof. If $\dim_{\mathbb{C}} X = 1$, then the proposition follows from [DM23a, Lemma 5.8]. Assume by induction that the assertions are both true if $\dim X = n - 1$. We take a smooth projective compactification $\overline{X}$ for X such that $\Sigma := \overline{X}\backslash X$ is a simple normal crossing divisor.

We fix an projective embedding $\iota : \overline{X} \hookrightarrow \mathbb{P}^N$ and denote by $L := \iota^*\mathscr{O}_{\mathbb{P}^N}(3)$. Let $\mathbb{U}(q) \subset H^0(\overline{X}, L)$ be defined in Proposition 1.7. For any element $s \in H^0(\overline{X}, L)$, let $\iota_{Y_s} : Y_s \to X$ be the inclusion map defined in Proposition 1.7.

Choose any $q \in X$, and any $\tilde{q} \in \widetilde{X}$ such that $\pi_X(\tilde{q}) = q$. By Proposition 1.7.(iii), for any section $s \in \mathbb{U}(q)$, letting $\widetilde{\iota_{Y_s}} : \widetilde{Y_s} \to \widetilde{X}$ be the lift of ι_{Y_s} between universal covers, we have $\pi_X^{-1}(q) \subset \widetilde{\iota_{Y_s}}(\widetilde{Y_s})$. Hence there exists $\widetilde{q_s} \in \widetilde{Y_s}$ such that $\widetilde{\iota_{Y_s}}(\widetilde{q_s}) = \tilde{q}$. By Theorem A, the ϱ_{Y_s}-equivariant maps $\widetilde{u_{Y_s}}$ and $\widetilde{v_{Y_s}}$ defined in § 1.2 are pluriharmonic maps of logarithmic energy. The inductive hypothesis implies that there exists a constant $c_{Y_s} \geq 0$ such that $d\left(\widetilde{u_{Y_s}}(y), \widetilde{v_{Y_s}}(y)\right) = c_{Y_s}$ for each $y \in \widetilde{Y_s}$. Since $\widetilde{u_{Y_s}} = \tilde{u} \circ \widetilde{\iota_{Y_s}}$ and $\widetilde{v_{Y_s}} = \tilde{v} \circ \widetilde{\iota_{Y_s}}$, it follows that for any other $s' \in \mathbb{U}(q)$, we have

$$c_{Y_s} = d\left(\widetilde{u_{Y_s}}(\tilde{q}_s), \widetilde{v_{Y_s}}(\tilde{q}_s)\right) = d\left(\tilde{u}(\tilde{q}), \tilde{v}(\tilde{q})\right) = d\left(\widetilde{u_{Y_{s'}}}(\tilde{q}_{s'}), \widetilde{v_{Y_{s'}}}(\tilde{q}_{s'})\right) = c_{Y_{s'}}.$$

Hence c_{Y_s} does not depend on the choice of $s \in \mathbb{U}(q)$, which we shall denote by c.

Let p be any other point in X. Then by Proposition 1.7.(ii), there exists $s \in \mathbb{U}(q)$ such that $p \in Y_s$. By Proposition 1.7.(iii), for any $\tilde{p} \in \pi_X^{-1}(p)$, there exists $\widetilde{p_s} \in \widetilde{Y_s}$ such that $\widetilde{\iota_{Y_s}}(\widetilde{p_s}) = \tilde{p}$. It follows that

$$d\left(\tilde{u}(\tilde{p}), \tilde{v}(\tilde{p})\right) = d\left(\widetilde{u_{Y_s}}(\tilde{p}_s), \widetilde{v_{Y_s}}(\tilde{p}_s)\right) = c.$$

Thus, we conclude that $d(\tilde{u}(x), \tilde{v}(x)) \equiv c$ for each $x \in \widetilde{X}$.

Let us prove the second assertion. For any local smooth vector field V on $\widetilde{X}$, we know that $|\tilde{u}_*(V)|^2, |\tilde{v}_*(V)|^2 \in L^1_{\mathrm{loc}}$, and thus it suffices to prove Proposition 4.3.(ii) over the dense open subset $\mathcal{R}(\tilde{u}) \cap \mathcal{R}(\tilde{v})$. Since $\tilde{u}$ and $\tilde{v}$ are both smooth over $\mathcal{R}(\tilde{u}) \cap \mathcal{R}(\tilde{v})$, it suffices to prove that for any point $\tilde{q} \in \mathcal{R}(\tilde{u}) \cap \mathcal{R}(\tilde{v})$, and any $V \in T_{\tilde{q}}\widetilde{X}$, we have

$$|\tilde{u}_*(V)|^2 = |\tilde{v}_*(V)|^2.$$

Set $q = \pi_X(\tilde{q})$. By Proposition 1.7.(ii), there exists $s \in \mathbb{U}(q)$ such that $(\pi_X)_*V \in T_qY_s$. Hence $V \in T_{\tilde{q}}(\widetilde{Y_s})$.

By the inductive hypothesis, we have

$$|\tilde{u}_*(V)|^2 = |(\widetilde{u_{Y_s}})_*(V)|^2 = |(\widetilde{v_{Y_s}})_*(V)|^2 = |\tilde{v}_*(V)|^2.$$

This yields the second assertion. The proposition is proved. □

4.2. **Proof of unicity theorem.** Recall the following definition from [GS92].

Definition 4.4 ([GS92], Section 6)**.** We say that a nonpositively curved N-dimensional complex $\mathcal{F}$ is *F-connected* if any two adjacent simplices are contained in a totally geodesic subcomplex A which is isometric to a subset of the Euclidean space $\mathbb{R}^N$.

The regular set and the singular set of a harmonic map into a F-connected complex is defined analogously as in Definition 1.2.

A neighborhood of a point $P_0 \in \Delta(G)$ is isometric to a neighborhood of the origin in the tangent cone $T_{P_0}\Delta(G)$. Two simplices (which are actually simplicial cones) in $T_{P_0}\Delta(G)$ are contained in a totally geodesic subcomplex $T_{P_0}A$ where A is an apartment of $\Delta(G)$. In other words, $T_{P_0}\Delta(G)$ is an N-dimensional F-connected complex. Thus, when we study the local behavior of harmonic maps $u : \Omega \to \Delta(G)$ at a point $x_0 \in \Omega$, we can assume that u maps into the N-dimensional, F-connected complex $T_{P_0}\Delta(G)$ where $P_0 = u(x_0)$.

Lemma 4.5 ([GS92], proof of Proposition 2.2)**.** *Let $u : \Omega \to \mathcal{F}$ be a harmonic map from an n-dimensional Riemannian domain to a F-connected complex and $x_0 \in \Omega$. Then there exists a constant $c > 0$ and $\sigma_0 > 0$ such that*

$$\sigma \to \frac{e^{c\sigma^2}\sigma \displaystyle\int_{B_\sigma(x_0)} |\nabla u|^2 d\mu}{\displaystyle\min_{Q \in \Delta(G)} \int_{\partial B_\sigma(x_0)} d^2(u, Q) d\Sigma}$$

is a non-decreasing functions in the interval $(0, \sigma_0)$. □

Definition 4.6. For u and x_0 as in Lemma 4.5, we set

$$\mathrm{Ord}^u(x_0) = \lim_{\sigma\to 0} \frac{e^{c\sigma^2}\sigma \displaystyle\int_{B_\sigma(x_0)} |\nabla u|^2 d\mu}{\displaystyle\min_{Q\in\Delta(G)} \int_{\partial B_\sigma(x_0)} d^2(u,Q) d\Sigma}.$$

As a limit of non-decreasing sequence of functions, $x \mapsto \mathrm{Ord}^u(x)$ is a upper semicontinuous function. Thus, we have the following:

(a) By [GS92, Lemma 1.3], $\mathrm{Ord}^u(x) \geq 1$ for all $x \in \Omega$.
(b) By [GS92, Theorem 6.3.(i)], if $x_i \to x$ and $\mathrm{Ord}^u(x_i) > 1$, then $\mathrm{Ord}^u(x) > 1$.

Lemma 4.7 ([GS92], proof of Theorem 6.4)**.** *Let u be as in Lemma 4.5 and $\tilde{\mathcal{S}}_0(u)$ to be the set of points $x \in \Omega$ such that* $\mathrm{Ord}^u(x) > 1$. *Then $\tilde{\mathcal{S}}_0(u)$ is a closed set such that* $\dim_{\mathcal{H}}(\tilde{\mathcal{S}}_0(u)) \leq n-2$. □

Lemma 4.8 ([GS92], proof of Proposition 2.2, Theorem 2.3)**.** *Let u and x_0 be as in Lemma 4.5 and let $\alpha := \mathrm{Ord}^u(x_0)$. There exists a constant $c > 0$ and $\sigma_0 > 0$ such that*

$$\sigma \to \frac{e^{c\sigma^2}}{\sigma^{n-1+2\alpha}} \int_{\partial B_\sigma(x_0)} d^2(u, u(x_0)) d\Sigma$$

and

$$\sigma \to \frac{e^{c\sigma^2}}{\sigma^{n-2+2\alpha}} \int_{B_\sigma(x_0)} |\nabla u|^2 d\mu$$

are non-decreasing functions in the interval $(0, \sigma_0)$. □

Remark 4.9. For a finite energy map $u : \Omega \to \mathcal{F}$ into a F-connected complex, $|\nabla u|^2 \in L^1_{\mathrm{loc}}$ is not necessarily defined at all points of Ω. On the other hand, it follows from Lemma 4.8 that for a harmonic map u, we can define $|\nabla u|^2$ at every point of $x_0 \in \Omega$ by setting

$$|\nabla u|^2(x_0) = \lim_{\sigma\to 0} \frac{1}{c_n \sigma^n} \int_{B_\sigma(x_0)} |\nabla u|^2 \, d\mu$$

where $c_n\sigma^n$ is the volume of a ball or radius σ in Euclidean space.

Let $u : \Omega \to \Delta(G)$ be a harmonic map and $x_0 \in \Omega$. Use normal coordinates centered at x_0 to identify $x_0 = 0$ and let $\mathbb{B}_r(0) = \{x = (x^1, \dots, x^n) \in \mathbb{R}^n : |x| < r\}$. As mentioned above, we can identify a neighborhood of $u(0)$ with a neighborhood of the origin O of the tangent cone $T_{u(0)}\Delta(G)$. For $\mu > 0$ and $P \in T_{u(0)}\Delta(G)$, denote by μP to be the point in $T_{u(0)}\Delta(G)$ on the geodesic ray emanating from O and going through P at a distance $\mu d(O, P)$ from O. Let

$$\mu(\sigma) = \left(\sigma^{1-n} \int_{\partial B_\sigma(0)} d^2(u, u(0)) d\Sigma\right)^{-1}.$$

Definition 4.10. The *blow up map* is defined by

$$u_\sigma : \mathbb{B}_1(0) \to T_{u(0)}\Delta(G), \quad u_\sigma(x) = \mu(\sigma) u(\sigma x).$$

By [GS92, Proposition 3.3] and the paragraph proceeding it, there exists a sequence $\sigma_i \to 0$ such that u_{σ_i} converges locally uniformly to a non-constant homogeneous harmonic map u_* of degree $\alpha := \mathrm{Ord}^u(x_0)$.

If $\mathrm{Ord}^u(x_0) = 1$, then have the following:

(a) By [GS92, Proposition 3.1], there exists $m \in \{1, \dots, \min\{n, N\}\}$ such that

$$u_* = J \circ v\big|_{B_1(0)}$$

for an isometric and totally geodesic embedding $J : \mathbb{R}^m \to T_{u(0)}\Delta(G)$ and a linear map $v : \mathbb{R}^n \to \mathbb{R}^m$ of full rank.

(b) By [GS92, Lemma 6.2], the union of all N-flats of $T_{u(0)}\Delta(G)$ containing $J(\mathbb{R}^m)$ is isometric to $\mathbb{R}^m \times \mathcal{F}$ where $\mathcal{F}$ is a $(N-m)$-dimensional, F-connected complex.

(c) By [GS92, Theorem 6.3], there exists $\sigma_0 > 0$ such that $u(B_{\sigma_0}(x_0)) \subset \mathbb{R}^m \times \mathcal{F}$. If we write

$$u = (u^1, u^2) : B_{\sigma_0}(x_0) \to \mathbb{R}^m \times \mathcal{F}, \tag{4.1}$$

then $u^1 : B_{\sigma_0}(x_0) \to \mathbb{R}^m$ is a smooth harmonic map of rank m and $u^2 : B_{\sigma_0}(x_0) \to \mathcal{F}$ is a harmonic map with $\alpha_2 := \mathrm{Ord}^{u^2}(x_0) \geq 1 + \epsilon$ for $\epsilon > 0$.

Definition 4.11. Let $u : \Omega \to \Delta(G)$ be a harmonic map. For $x_0 \in \tilde{\mathcal{S}}_0(u)$, define $m_{x_0} = 0$. For $x_0 \in \Omega \backslash \tilde{\mathcal{S}}(u)$, let m_{x_0} be the integer m in eq. (4.1). Let $M := \sup_{x_0 \in \Omega} m_{x_0}$. We say the point $x_0 \in \Omega$ is a *critical point* if $m_{x_0} < M$. We denote the set of critical points by $\tilde{\mathcal{S}}(u)$. Define $\tilde{\mathcal{R}}(u) = \Omega \backslash \tilde{\mathcal{S}}(u)$.

Lemma 4.12. *If $u : \Omega \to \Delta(G)$ is a non-constant harmonic map, then $\tilde{\mathcal{R}}(u) \subset \mathcal{R}(u)$.*

Proof. Let $x_0 \in \tilde{\mathcal{R}}(u)$; i.e. $m_{x_0} = M$ where M is as in Definition 4.11 and there exists $\sigma > 0$ such that we can write

$$u = (u^1, u^2) : B_\sigma(x_0) \to \mathbb{R}^M \times \mathcal{F}. \tag{4.2}$$

By choosing $\sigma > 0$ smaller if necessary, we can assume that u^1 is of rank M at all points $x \in B_{\sigma_0}(x_0)$. Therefore, the restriction of (4.2) to $B_r(x)$ is an expression of u as $u = (u_1, u_2)$ as in eq. (4.1) in $B_r(x) \subset B_{\sigma_0}(x_0)$. By eq. (4.3), $|\nabla u^2|^2(x) = 0$ for all $x \in B_{\sigma_0}(x_0)$. Thus, we conclude that $u^2 \equiv P_0$ for some $P_0 \in \mathcal{F}$. Hence $u(B_\sigma(x_0)) \subset \mathbb{R}^M \times \{P\}$ which implies $B_\sigma(x_0) \subset \mathcal{R}(u)$. □

Lemma 4.13. *Let u and x_0 be as in Lemma 4.5. Then*

$$\mathrm{Ord}^u(x_0) > 1 \quad \Leftrightarrow \quad |\nabla u|^2(x_0) = 0.$$

Proof. First, assume $\alpha := \mathrm{Ord}^u(x_0) > 1$. Lemma 4.8 implies that there exists a constant $C > 0$ and $\sigma_0 > 0$ such that for $\sigma \in (0, \sigma_0)$

$$\int_{\partial B_\sigma(x_0)} d^2(u, u(x_0)) d\Sigma \leq C\sigma^{n-1+2\alpha}.$$

By Remark 4.9, the above inequality and $\alpha > 1$ imply (with c_n equal to the volume of the unit ball in $\mathbb{R}^n$)

$$|\nabla u|^2(x_0) = \lim_{\sigma \to 0} \frac{1}{c_n \sigma^n} \int_{B_\sigma(x_0)} |\nabla u|^2 \, d\mu = \lim_{\sigma \to 0} \frac{1}{c_n \sigma^{n+1}} \int_{\partial B_\sigma(x_0)} d^2(u, u(x_0)) d\Sigma = 0 \tag{4.3}$$

Next, assume $\mathrm{Ord}^u(x_0) = 1$. Use normal coordinates centered at x_0 and write $u = (u^1, u^2)$ as in eq. (4.1). Define ${}_\theta u = ({}_\theta u^1, {}_\theta u^2)$ by setting ${}_\theta u(x) = \theta^{-1} u(\theta x)$. From [GS92, (5.14)], ${}_\theta u \to L$ uniformly on compact subsets to a non-constant homogeneous degree 1 map L. Furthermore, since $\alpha_2 := \mathrm{Ord}^{u^2}(x_0) > 1$ (cf. (c)), arguing analogously as (4.3), we get

$$\lim_{\theta \to 0} \int_{\partial B_1(0)} d^2({}_\theta u^2, {}_\theta u^2(0)) d\Sigma = \lim_{\theta \to 0} \frac{1}{\theta^{n+1}} \int_{\partial B_\theta(0)} d^2(u^2, u^2(0)) d\Sigma = \lim_{\theta \to 0} C\theta^{2\alpha_2 - 2} = 0.$$

By the maximum principle, this implies that ${}_\theta u^2 \to {}_\theta u^2(0) = u^2(0)$ uniformly on compact subsets of $B_1(0)$. This in turn implies that ${}_\theta u^1 \to L$ uniformly on compact subsets of $B_1(0)$. Since ${}_\theta u^1$ is a smooth harmonic map, ${}_\theta u^1 \to L$ in C^k for any k in any compact subset of $B_1(0)$. Since $|\nabla L|^2(0) > 0$, we also have $|\nabla u^1|^2(0) = |\nabla {}_\theta u^1|^2(0) > 0$. Therefore, $|\nabla u|^2 > 0$. □

Lemma 4.14. *The set of critical points $\tilde{\mathcal{S}}(u)$ is a closed set of Hausdorff dimension at most $n-2$.*

Proof. By Lemma 4.12, $\tilde{\mathcal{S}}(u) = \mathcal{S}(u) \cup (\tilde{\mathcal{S}}(u) \cap \mathcal{R}(u))$. By [GS92, Theorem 6.4], $\mathcal{S}(u)$ is a closed set of Hausdorff dimension at most $n-2$. Thus, the assertion follows from the fact that the Hausdorff dimension of the set of critical points of a harmonic map into Euclidean space is at most $n-2$. □

Lemma 4.15. *For a non-constant harmonic map $u : \Omega \to \Delta(G)$, let $\Omega^*(u)$ be the set of points $x \in \Omega$ such that there exists $r > 0$ and a chamber C such that $u(B_r(x)) \subset \overline{C}$. Then $\Omega^*(u)$ is an open set of full measure in Ω.*

Proof. The openness of $\Omega^*(u)$ follows from its definition. Denote the complement of $\Omega^*(u)$ by $\Omega^*(u)^c$. We want to show that $\Omega^*(u)^c$ has zero measure. Since $\tilde{\mathcal{S}}(u)$ has zero measure, it is sufficient to show that $\Omega^*(u)^c \cap \tilde{\mathcal{R}}(u)$ has zero measure.

On the contrary, assume that $\Omega^*(u)^c \cap \tilde{\mathcal{R}}(u)$ has positive measure. Then there exists a point $x_0 \in \Omega^*(u)^c \cap \tilde{\mathcal{R}}(u)$ such that

$$\lim_{\varrho\to 0} \frac{\mu(B_\varrho(x_0)\cap\Omega^*(u)^c)}{\mu(B_\varrho(x_0))} = \lim_{\varrho\to 0} \frac{\mu(B_\varrho(x_0)\cap\Omega^*(u)^c\cap\tilde{\mathcal{R}}(u))}{\mu(B_\varrho(x_0))} = 1. \tag{4.4}$$

Since $x_0 \in \tilde{\mathcal{R}}(u)$, Lemma 4.12 implies $x_0 \in \mathcal{R}(u)$. Thus, there exists a neighborhood $\mathcal{N}$ of x_0 and a totally geodesic subcomplex A_{x_0} isometric to $\mathbb{R}^N$ such that $u(\mathcal{N}) \subset A$. By choosing $\mathcal{N}$ smaller if necessary, we can assume that $u|_{\mathcal{N}}$ has no critical points since $x_0 \in \tilde{\mathcal{R}}(u)$.

For $k = 0, \ldots, N$, denote the k-skeleton of $\Delta(G)$ by $\Delta(G)^{(k)}$. Let k be the smallest integer such that $u(\mathcal{N}) \subset \Delta(G)^{(k)}$. Since u is not locally constant (cf. [GS92, Proposition 4.3]), $k \geq 1$.

First, assume $u(\mathcal{N}) \cap \Delta(G)^{(k-1)} = \emptyset$. Let C be a chamber such that $u(x_0) \in \bar{C}$. Since $u(\mathcal{N}) \subset \Delta(G)^{(k)}$ and $u(\mathcal{N}) \cap \Delta(G)^{(k-1)} = \emptyset$, we conclude that $u(\mathcal{N})$ is a k-dimensional face of $\bar{C}$. Thus, $u(\mathcal{N}) \subset \bar{C}$. This shows that, for $\varrho > 0$ sufficiently small $B_\varrho(x_0) \subset \mathcal{N} \subset \Omega^*(u)$, contradicting eq. (4.4).

Next, assume $u(\mathcal{N}) \cap \Delta(G)^{(k-1)} \neq \emptyset$. Since $\mathcal{N}$ has no critical points of u, $(u|_{\mathcal{N}})^{-1}(\Delta(G)^{(k-1)})$ is a union of smooth $(n-1)$-dimensional submanifolds. For any point $x \in \mathcal{N}\backslash(u|_{\mathcal{N}})^{-1}(\Delta(G)^{(k-1)})$, there exists $r > 0$ and a chamber C such that $u(B_r(x)) \subset \overline{C}$. Thus, $\mathcal{N}\backslash(u|_{\mathcal{N}})^{-1}(\Delta(G)^{(k-1)}) \subset \Omega^*(u)$, again contradicting eq. (4.4). □

Remark 4.16. Note that Lemma 4.15 is of independent interest. For instance, it played a crucial role in [DW24a] in the study of Kollár's conjecture on the positivity of the holomorphic Euler characteristic for varieties with large fundamental groups.

Proposition 4.17. *Let $u_0, u_1 : \Omega \to \Delta(G)$ be harmonic maps from a bounded Riemannian domain. If $d(u_0, u_1) = c$ for some constant $c \geq 0$ and $|\nabla u_0|^2 = |\nabla u_1|^2$, then for almost all points $x \in \widetilde{X}$, there exists $r > 0$ satisfying the following:*

(i) *There is a N-flat A containing both $u_0(B_r(x))$ and $u_1(B_r(x))$;*
(ii) *If we fix an isometry $\nu : A \to \mathbb{R}^N$, then $\nu \circ u_0 : B_r(x) \to \mathbb{R}^N$ is a translation of $\nu \circ u_1 : B_r(x) \to \mathbb{R}^N$.*

Proof. For $i = 0, 1$, let $\Omega^*(u_i)$ be the open set of full measure as in Lemma 4.15. Thus, $\Omega^*(u_0) \cap \Omega^*(u_1)$ is of full measure. Lemma 4.15 implies that, for any $x_0 \in \Omega^*(u_0) \cap \Omega^*(u_1)$, there exists $r > 0$ and a chamber C_i such that $u_i(B_r(x_0)) \subset C_i$ for $i = 0, 1$. Let A be N-flat containing chambers C_0 and C_1 and $\nu : A \to \mathbb{R}^N$ be an isometry. Thus, $\nu \circ u_0$ and $\nu \circ u_1$ are harmonic maps into $\mathbb{R}^N$. The assumption that $d(u_0, u_1) = c$ implies that $|\nu \circ u_0(x) - \nu \circ u_1(x)| = c$. Thus, $0 = \Delta|\nu \circ u_0 - \nu \circ u_1|^2 = 2\,|\nabla(\nu \circ u_0 - \nu \circ u_1)|^2$ which implies $\nu \circ u_0$ is a translation of $\nu \circ u_1$. □

We are able to prove Theorem B.

Proof of Theorem B. The assertion follows immediately from Proposition 4.3 and Proposition 4.17 below. □

Proof of Theorem C.(ii). By Theorem B, there exists a dense open subset $\widetilde{X}^\circ \subset \widetilde{X}$ of full Lebesgue measure such that, for any $x \in \widetilde{X}^\circ$,

(a) there exists an open neighborhood Ω of x and an apartment A of $\Delta(G)$ such that $\tilde{u}_i(\Omega) \subset A$ for $i = 0, 1$;
(b) the map $\tilde{u}_0|_\Omega : \Omega \to A$ is a translate of $\tilde{u}_1|_\Omega : \Omega \to A$

By the construction in [BDDM22], the multivalued 1-forms η_i induced $\tilde{u}_i$ for $i = 0, 1$ are equal over $\widetilde{X}^\circ$, and splitting over $\widetilde{X}^\circ$. By Definition 3.1, we conclude that $\eta_1 = \eta_2$ over the entire X. The claim is proved. □

5. On the singular set of harmonic maps into Euclidean buildings

In this section, we apply Lemma 3.8 and the results from § 4.2 to prove Theorem A.(iv), following the idea by Eyssidieux in [Eys04, Proposition 1.3.3].

Theorem 5.1 (=Theorem A.(iv)). *Let X, ϱ, G and $\tilde{u}$ be as in Theorem A. Then the singular set $\mathcal{S}(u)$ defined in Definition 1.2 is contained in a proper Zariski closed subset of X.*

Proof. We assume that $\tilde{u}$ is non-constant. We shall use the notions in § 4.2 with Ω being $\widetilde{X}$. Let M be the positive integer defined in Definition 4.11. Let η be the logarithmic multivalued 1-form induced by $\tilde{u}$ defined in Lemma 3.8. Let $Z_\eta = \sum_{i=1}^{k} n_i Z_i$ be the formal sum corresponding to η defined in Definition 3.1, where each Z_i is an irreducible closed subvariety of E such that the natural map $Z_i \to X$ is surjective and finite. Let $|Z_\eta| = \cup_{i=1}^{k} Z_i \subset \Omega_{\overline{X}}(\log \Sigma)$ be the support of Z_η.

Let M be the positive integer defined in Definition 4.11. Consider the holomorphic bundle $E := \Omega_{\overline{X}}^{M}(\log \Sigma)$ on $\overline{X}$. By Lemma 3.8.(iii), η induces a multivalued section η^M of E. Let $|Z_{\eta^M}| \subset E$ be the support of the formal sum Z_{η^M} induced by η^M defined in Definition 3.1. Let $\overline{X}^\circ$ be the set of points x in $\overline{X}$ such that $|Z_{\eta^M}|_x \not\subset \{0\}$. We shall prove that the Zariski open subset $\overline{X}^\circ$ is dense in $\overline{X}$.

By our definition of M in Definition 4.11 and Lemma 4.12, for any point $x_0 \in \tilde{\mathcal{R}}(u)$, there exists $r > 0$ such that:

(1) for some connected component Ω of $\pi_X^{-1}(B_r(x_0))$, $\pi_X|_\Omega : \Omega \to B_r(x_0)$ is an isomorphism, where $B_r(x_0)$ is the geodesic ball centered at x_0 of radius r.
(2) We have the decomposition

$$\tilde{u} \circ (\pi_X^{-1}|_\Omega)|_{B_r(x_0)} = (u^1, u^2) : B_r(x_0) \to \mathbb{R}^M \times \{P_0\},$$

where u^1 is a harmonic map with rank M at each point of $B_r(x_0)$.

By Lemma 3.8.(ii), η is represented by ∂u^1 up to some orthogonal transformation and rescaling. It follows that $|Z_{\eta^M}|_x$ is not $\{0\}$ for every $x \in B_r(x_0)$. Hence we have

$$\tilde{\mathcal{R}}(u) \subset \overline{X}^\circ, \tag{5.1}$$

which implies that $\overline{X}^\circ$ is non-empty. Since $\overline{X}^\circ$ is Zariski open in $\overline{X}$, it follows that $X^\circ := X \cap \overline{X}^\circ$ is a dense and Zariski open subset of X. The theorem follows from Lemma 4.12 together with Lemma 5.2 below. □

Lemma 5.2. *We have $\tilde{\mathcal{S}}(u) = X \backslash X^\circ$.*

Proof. Let $x_0 \in X$. If $\mathrm{Ord}^u(x_0) > 1$, then $|\nabla u|^2(x_0) = 0$ by Lemma 4.13. If $\mathrm{Ord}^u(x_0) = 1$, then we apply Item (c) above (4.2). Thus, in either case, there exists $r > 0$ and an F-connected complex $\mathcal{F}$ such that

(a) for some connected component Ω of $\pi_X^{-1}(B_r(x_0))$, $\pi_X|_\Omega : \Omega \to B_r(x_0)$ is an isomorphism.
(b) We have

$$\tilde{u} \circ (\pi_X^{-1}|_\Omega)|_{B_r(x_0)} = (u^1, u^2) : B_r(x_0) \to \mathbb{R}^k \times \mathcal{F}, \tag{5.2}$$

such that $u^1 : B_r(x_0) \to \mathbb{R}^k$ is a smooth pluriharmonic map with rank at each point of $B_\sigma(x_0)$ equal to k (see the proof of Lemma 4.12) and $u^2 : B_r(x_0) \to \mathcal{F}$ is a pluriharmonic map with $\mathrm{Ord}^{u^2}(x_0) \geq 1 + \varepsilon$ for some $\varepsilon > 0$ and $|\nabla u^2|(x_0) = 0$ by Lemma 4.13. Here, we are using the following convention: If $k = M$, then u^2 is a constant map, and if $k = 0$, then $(u^1, u^2) = u^2$.

Note that $\mathcal{F}$ has an Euclidean building structure. By the proof of Theorem 3.7.(i) and Lemma 3.8, the pluriharmonic map u^2 in (5.2) induces a multivalued 1-form ψ_0 on $B_r(x_0)$ satisfying the properties in Lemma 3.8. Then for each $x_1 \in B_r(x_0) \cap \mathcal{R}(u)$, it has a neighborhood Ω_{x_1} over which the multivalued 1-form ψ are given by holomorphic 1-forms $\sqcup_{i=1}^{\ell} \{\psi_{i,1}, \ldots, \psi_{i,N-k}\}$, that is the partition of ψ_0 in Lemma 3.8.(ii). By (3.8), one has

$$|\nabla u^2|^2 = 2 \sum_{i=1}^{\ell} \sum_{j=1}^{N-k} |\psi_{i,j}|^2. \tag{5.3}$$

We define $\psi_{i,N-k+j} := \frac{1}{\sqrt{\ell}} \partial u_j^1$ for each $i \in \{1, \ldots, \ell\}$ and $j \in \{1, \ldots, k\}$. Therefore, $\sqcup_{i=1}^{\ell} \{\psi_{i,1}, \ldots, \psi_{i,N}\}$ is a multivalued 1-form associated with (u^1, u^2) defined in Lemma 3.8.

We can shrink Ω_{x_1} such that η is given by holomorphic 1-forms $\sqcup_{i=1}^{\ell'}\{\omega_{i,1},\ldots,\omega_{i,N}\}$ on Ω_{x_1}, that is the partition of η in Lemma 3.8.(ii). Hence, by (3.9) and (3.10), for each i and j, there exists a constant matrix $M_{i,j}\in \mathrm{O}(N,\mathbb{R})$ such that

$$\begin{bmatrix}\omega_{i,1}, & \ldots, & \omega_{i,N}\end{bmatrix} = \begin{bmatrix}\psi_{j,1}, & \cdots, & \psi_{j,N}\end{bmatrix}\cdot M_{i,j}\cdot\frac{\sqrt{\ell'}}{\sqrt{\ell}}. \tag{5.4}$$

By (3.9) and the definition of η^M in Lemma 3.8.(iii), over Ω_{x_1}, the multivalued section η^M of E is given by

$$\sqcup_{j=1}^{\ell'}\{\pm\omega_{j,i_1}\wedge\cdots\wedge\omega_{j,i_M}\}_{1\le i_1<\cdots<i_M\le N}.$$

On the other hand, by Lemma 3.8.(iii), (u^1,u^2) induces another multivalued section of $E|_{B_r(x_0)}$, which is locally represented by

$$\sqcup_{j=1}^{\ell}\{\pm\psi_{j,i_1}\wedge\cdots\wedge\psi_{j,i_M}\}_{1\le i_1<\cdots<i_M\le N}.$$

For notational simplicity, for each $I=(i_1,\ldots,i_M)\subset\{1,\ldots,N\}$ with $1\le i_1<\cdots<i_M\le N$, we write

$$\omega_{j,I}:=\omega_{j,i_1}\wedge\cdots\wedge\omega_{j,i_M},\quad \forall j\in\{1,\ldots,\ell'\},$$

and

$$\psi_{j,I}:=\psi_{j,i_1}\wedge\cdots\wedge\psi_{j,i_M},\quad \forall j\in\{1,\ldots,\ell\}.$$

Therefore, by (5.4) there exists a constant matrix of $\tilde{M}_{i,j}\in \mathrm{O}(\binom{N}{M},\mathbb{R})$ such that

$$[\omega_{j,I}]_{1\le i_1<\cdots<i_M\le N}=[\psi_{i,I}]_{1\le i_1<\cdots<i_M\le N}\cdot\tilde{M}_{i,j}(\frac{\sqrt{\ell'}}{\sqrt{\ell}})^M.$$

Thus, we have the following equality, which holds over the entire $\mathcal{R}(u)\cap B_r(x_0)$:

$$\sum_{j=1}^{\ell'}\sum_{1\le i_1<\cdots<i_M\le N}|\omega_{j,I}|^2_{h_E}=\frac{(\ell')^{M+1}}{\ell^M}\sum_{1\le i_1<\cdots<i_M\le N}|\psi_{1,I}|^2_{h_E}, \tag{5.5}$$

Note that there exists a constant $C>1$ such that

$$|\psi_{i,N-k+j}(x)|=|\frac{1}{\sqrt{\ell}}\partial u^1_j|\le C,\quad \forall\, x\in B_r(x_0) \tag{5.6}$$

for each $i\in\{1,\ldots,\ell\}$ and $j\in\{1,\ldots,k\}$.

We now consider the cases of $x_0\in\tilde{\mathcal{S}}(u)$ and $x_0\in\tilde{\mathcal{R}}(u)$ separately:

(a) If $x_0\in\tilde{\mathcal{S}}(u)$, then $M>k$. By (5.3) and (5.6), for any $x\in\Omega_{x_1}$, we have

$$|\psi_{1,I}|^2\le C^{2k}|\nabla u^2|^{2\lambda(I)}, \tag{5.7}$$

where $\lambda(I)$ denotes the cardinality of $I\cap\{1,\ldots,N-k\}$, that is a positive integer. Recall that $|\nabla u^2(x_0)|^2=0$. Since C is a constant independent of $x_1\in B_r(x_0)\cap\mathcal{R}(u)$, it then follows from (5.5) and (5.7) that

$$\lim_{x\in\mathcal{R}(u),x\to x_0}\sum_{j=1}^{\ell'}\sum_{1\le i_1<\cdots<i_M\le N}|\omega_{j,I}|^2(x)=0.$$

Since the multivalued section η^M on E is locally represented by $\sqcup_{j=1}^{\ell'}\{\pm\omega_{j,I}\}_{1\le i_1<\cdots<i_M\le m}$, it follows that that $|Z_{\eta^M}|_{x_0}\subset\{0\}$. In other words, $x_0\notin X^\circ$.

(b) If $x_0\in\tilde{\mathcal{R}}(u)$, by (5.1), we have $x_0\in X^\circ$.

In conclusion, we have $\tilde{\mathcal{R}}(u)=X^\circ$. The lemma is proved. □

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