

TROPICAL SUBREPRESENTATIONS OF THE BOOLEAN REGULAR REPRESENTATION IN LOW DIMENSION

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1. INTRODUCTION

In [GM20], Giansiracusa–Manaker develop the theory of group representations over a fixed idempotent semifield \mathbb{S} , i.e. *tropical* representation theory. This continues efforts in recent years to pursue ideas in scheme theoretic tropicalization, including [Fre13, GG16, GG18, MR18, Jun18, CGM20, Lor23, BB19, FGGM24] among many others. In this theory a linear representation $G \rightarrow \mathrm{GL}(\mathbb{S}^n)$ of a group G over \mathbb{S} is described by monomial matrices, and a *tropical subrepresentation* $T \subseteq \mathbb{S}^n$ is a G -invariant tropical linear space in \mathbb{S}^n . The constant-coefficient case features tropical representations over the boolean semifield $\mathbb{B} = \{-\infty, 0\}$ consisting of the tropical additive and multiplicative identities. This setting highlights a unique interaction between group theory and matroid theory, since tropical representations $T \subseteq \mathbb{S}^n$ correspond to group homomorphisms $G \rightarrow \mathrm{Aut}(M_T)$ to the automorphism group of the corresponding matroid.

Given a finite group G the *boolean regular representation* $\mathbb{B}[G]$ is the tropicalization of the regular representation $\mathbb{C}[G]$ and can be described per usual by considering the \mathbb{B} linear span

$$\mathbb{B}[G] = \left\{ \sum_{g \in G} c_g \mathbf{e}_g \right\}$$

of basis elements indexed by G under the action determined by left-multiplication in G . As a central example [GM20, Theorem B] the authors begin the study of tropical subrepresentations of $\mathbb{B}[G]$ and their *realizability*, that is, whether they arise as the tropicalization of a classical subrepresentation of $\mathbb{C}[G]$. In particular when $G \cong \mathbb{Z}_p$ is cyclic of prime order they prove that there is only one realizable subrepresentation in each dimension $1 \leq d \leq p$ corresponding to the uniform matroid $U_{d,p}$. In dimension 2 they show that there are no other non-realizable subrepresentations of $\mathbb{B}[\mathbb{Z}_p]$ for a specified infinite collection of primes. They conjecture [GM20, Conjecture 4.1.6] that there is only one two-dimensional subrepresentation of $\mathbb{B}[\mathbb{Z}_n]$ if and only if n is prime.

The goal of this paper is to continue the classification of tropical subrepresentations of $\mathbb{B}[G]$. Our main result in two-dimensions classifies all two-dimensional tropical subrepresentations for any finite group G and provides a direct correspondence with the proper subgroups of G .

Theorem (Theorem 3.5). *Let G be a finite group. Two-dimensional tropical subrepresentations of $\mathbb{B}[G]$ correspond bijectively to proper subgroups $H \subset G$. The set of bases of the corresponding matroids are explicitly presented as a union of G -orbits*

$$\bigcup_{g \in G-H} \{ \{a, ag\} \mid a \in G \}$$

for the induced G -action on subsets of G size 2.

This theorem is proven in Section 3. It completely solves the problem in dimension 2 and confirms conjecture [GM20, Conjecture 4.1.6] as the special case where $G \cong \mathbb{Z}_p$. The union of G -orbits explicitly describes the set of bases for each matroid on the ground set G corresponding to a tropical subrepresentation. Our proof generalizes the methods in [GM20, Section 4] to arbitrary groups, is surprisingly elementary, and demonstrates a direct interplay between the basis exchange axiom and the group structure of G . In higher dimensions, this interplay and the combinatorics involved present a more significant challenge.

To this end, in Section 4 we make progress on the dimension 3 case. Our first result in dimension 3 shows that for any subgroup $H \subseteq G$ of index larger than 2, there exists a tropical subrepresentation of $\mathbb{B}[G]$ with bases of its corresponding matroid identified in a similar way to Theorem 3.5.

Theorem (Theorem 4.5). *Let G be a finite group, and let H be a subgroup of G with $[G : H] > 2$. There exists a three-dimensional tropical subrepresentations of $\mathbb{B}[G]$ for which the set of bases of the corresponding matroid is explicitly presented as a union of G -orbits*

$$\bigcup_{g, h, g^{-1}h \in G-H} \{\{a, ag, ah\} | a \in G\}$$

for the induced G -action on subsets of G size 3.

In contrast to dimension 2, this is not an equivalence – the combinatorics in higher dimension seem to allow for a wider collection of tropical subrepresentations. In Section 4.3 we investigate this further in the cyclic case. Our first result for cyclic groups identifies a common subset of the set of bases for every matroid corresponding to a three-dimensional subrepresentation of $\mathbb{B}[\mathbb{Z}_n]$. Denote by \mathbb{Z}_n^\times the set of units in \mathbb{Z}_n .

Theorem (Theorem 4.8). *Let $G = \mathbb{Z}_n$ be a finite cyclic group. The set of bases \mathcal{B} of any matroid corresponding to a three-dimensional subrepresentation contains the union*

$$\bigcup_{u \in \mathbb{Z}_n^\times} \{\{a, u + a, 2u + a\} | a \in \mathbb{Z}_n\}$$

of orbits for the induced \mathbb{Z}_n -action on subsets of \mathbb{Z}_n of size 3.

Our second result for cyclic groups provides the set of bases for a number of tropical subrepresentations of $\mathbb{B}[\mathbb{Z}_n]$ not corresponding to subgroups. Denote by $\binom{[n]}{3}$ the set of subsets of $[n] = \{1, \dots, n\}$ of size 3. In Theorem 4.9 these sets of bases take the form, for any unit $u \in \mathbb{Z}_n^\times$,

$$\binom{[n]}{3} - \{\{a, u + a, ku + a\} | a \in \mathbb{Z}_n\}$$

for various values of k determined by the combinatorics of the problem. Here we are simply excluding one specific orbit of the induced \mathbb{Z}_n -action. For $n > 5$ these are the bases of non-uniform matroids. This confirms that $\mathbb{B}[\mathbb{Z}_n]$ contains non-uniform three-dimensional tropical subrepresentations for all $n > 5$, even for n a prime, which directly contrasts with the two-dimensional case.

In the boolean setting, tropical representation theory can be recast as a question about group actions on matroids, and this is our approach. Our attention is restricted to the classification question as it provides a fascinating playground in which group theory and

matroid theory seem to interact deeply with interesting combinatorics. Realizability of the subrepresentations constructed here, a continuation of the classification in dimensions 3 and higher, and a comparison with the classical case are all ripe questions that we leave for future work.

Acknowledgements. We are grateful to Andrew Clifford, Noah Giansiracusa, and Thomas Hagedorn for interesting and helpful discussions benefiting this project.

2. BACKGROUND

In this section we briefly summarize the necessary theory from [GM20]. For more detailed background, we refer the reader to this paper, as well as [CGM20, GG18]. We assume a familiarity with matroid theory, see [Oxl11] for further reading.

2.1. The tropical booleans. In [GM20] the finite-dimensional representation theory of groups is developed over an arbitrary idempotent semifield $(\mathbb{S}, +, \cdot, 0, 1)$, meaning additive inverses may not exist and $s + s = s$ for all $s \in \mathbb{S}$. This is an appropriate generalized setting for tropical algebra as the *tropical numbers*

$$\mathbb{T} = (\mathbb{R} \cup \{-\infty\}, \oplus, \odot, -\infty, 0)$$

form an idempotent semifield consisting of the set $\mathbb{R} \cup \{-\infty\}$ under the operations $a \oplus b = \max(a, b)$ and $a \odot b = a + b$. Note that $x \oplus y = \max(x, y) \geq x$ for all $x, y \in \mathbb{T}$ and, aside from $-\infty$, no element has an additive inverse.

The boolean semifield

$$\mathbb{B} = \{0, 1\} \subseteq \mathbb{S}$$

consists of the additive and multiplicative identity elements alone and forms the initial object of the category. We will restrict our attention to the booleans throughout.

2.2. Exterior powers. Let \mathbb{B}^n be a free \mathbb{B} module with basis elements $\mathbf{e}_1, \dots, \mathbf{e}_n$. In [GG18], an exterior algebra formalism for semirings introduces free modules $\bigwedge^d \mathbb{B}^n$ for $d = 1, \dots, n$ given by a basis of the form $\mathbf{e}_{i_1} \wedge \dots \wedge \mathbf{e}_{i_d}$ where each \mathbf{e}_{i_k} are basis elements of \mathbb{B}^n . Recall the two standard properties of wedge product: $\mathbf{v} \wedge \mathbf{v} = 0$ and $\mathbf{v} \wedge \mathbf{w} = -\mathbf{w} \wedge \mathbf{v}$. In this context the second property becomes $\mathbf{v} \wedge \mathbf{w} = \mathbf{w} \wedge \mathbf{v}$.

For any positive integer n , set $[n] = \{1, \dots, n\}$. For any finite set A denote by $\binom{A}{d}$ the set of subsets of A of cardinality d . The basis of $\bigwedge^d \mathbb{B}^n$ corresponds to the set $\binom{[n]}{d}$ by taking indices, and vectors in $\bigwedge^d \mathbb{B}^n$ simply indicate a subset of the basis vectors of $\bigwedge^d \mathbb{B}^n$. This gives a correspondence between vectors $v \in \bigwedge^d \mathbb{B}^n$ and subsets of $\binom{[n]}{d}$. More generally given any finite group G as an index set, the same correspondence can be formed between vectors $v \in \bigwedge^d \mathbb{B}^{|G|}$ and subsets of $\binom{G}{d}$. Moreover, the effect of projectivization on $\bigwedge^d \mathbb{B}^n$ disappears since $\mathbb{B}^* = \mathbb{B} - 0 = \{1\}$, so $\mathbb{P}(\bigwedge^d \mathbb{B}^n) \cong \bigwedge^d \mathbb{B}^n$.

2.3. Tropical Linear Spaces and Matroids. A nonzero vector

$$\mathbf{v} = \sum_{I \in \binom{[n]}{d}} v_I \mathbf{e}_I \in \bigwedge^d \mathbb{B}^n$$

is a *tropical Plücker vector over* \mathbb{B} of rank d if it satisfies the *Tropical Plücker Relations* over \mathbb{B} , that is, for all $A \in \binom{[n]}{d+1}$, $B \in \binom{[n]}{d-1}$, $j \in A - B$,

$$\sum_{i \in A-B} v_{A-\{i\}} v_{B \cup \{i\}} = \sum_{i \in A-B-j} v_{A-\{i\}} v_{B \cup \{i\}}.$$

Formally, a vector will satisfy the tropical Plücker relations if whenever $v_{A-i} v_{B+i} = 1$ for some i , then for some $j \neq i$ we must have $v_{A-j} v_{B+j} = 1$. Indeed, tropical Plücker vectors in $\bigwedge^d \mathbb{B}^n$ are precisely the vectors which correspond to choices of subsets in $\binom{[n]}{d}$ that satisfy the strong basis exchange axiom, which we recall here: for all $X, Y \in \mathcal{B}$, $i \in X - Y$, there is some $j \in Y - X$ with $X - i + j$, $Y - j + i \in \mathcal{B}$. Thus \mathbf{v} is a tropical Plücker vector if and only if it is the basis indicator vector of a matroid over the ground set $[n]$, i.e. $\mathcal{B} = \{I \mid V_I = 1\}$ is the basis set of a matroid.

The set of all tropical Plücker vectors are in a bijective correspondence with submodules $L \subseteq \mathbb{B}^n$. These submodules are the *tropical linear spaces* in \mathbb{B}^n . In the boolean setting this correspondence takes a satisfying form. A tropical Plücker vector $\mathbf{v} \in \bigwedge^d \mathbb{B}^n$ corresponds to a matroid $M_{\mathbf{v}} = ([n], \mathcal{B}_{\mathbf{v}})$ with bases $\mathcal{B}_{\mathbf{v}}$ indicated by the vector \mathbf{v} , and this data corresponds to a tropical linear subspace $L_{\mathbf{v}} \subseteq \mathbb{B}^n$ given by the \mathbb{B} -linear span of the vectors $\sum_i w_i \mathbf{e}_i$ indicating the *cocircuits* of $M_{\mathbf{v}}$. Thus

$$\mathbf{v} \in \bigwedge^d \mathbb{B}^n \iff M_{\mathbf{v}} = ([n], \mathcal{B}_{\mathbf{v}}) \iff L_{\mathbf{v}} \subseteq \mathbb{B}^n.$$

See [GG18] for a more thorough discussion.

2.4. Linear representations and G -fixed Plücker vectors. Let G be an arbitrary group and let $V \cong \mathbb{B}^n$ be a free module of rank n and let $[n]$ be the index set for the chosen basis of V . A (linear) *tropical representation of G over \mathbb{B}* is a group homomorphism $\rho : G \rightarrow \mathrm{GL}(V)$. Note that $\mathrm{GL}(V)$ consists solely of permutation matrices, so over the booleans the theory boils down to permutation representations. A *tropical subrepresentation* of ρ is given by restricting the G -action to a G -invariant tropical linear subspace $L \subseteq V$. The dimension of a tropical subrepresentation is the rank of the tropical linear subspace L . This also equals the rank of the corresponding matroid under the correspondence described in Section 2.3.

For each $1 \leq d \leq n$, a given tropical representation $\rho : G \rightarrow \mathrm{GL}(V)$ induces a representation $G \rightarrow \mathrm{GL}(\bigwedge^d V)$ which restricts to a G -action on the *Dressian* $\mathrm{Dr}(d, n) \subseteq \bigwedge^d V$, that is, the subset consisting of tropical Plücker vectors of rank d . We also have a compatibility of G -actions: whenever $L_{\mathbf{v}} \subseteq V$ is a tropical linear space in V associated to the tropical Plücker vector $\mathbf{v} \in \bigwedge^d V$ then for all $g \in G$ we have $L_{g \cdot \mathbf{v}} = g \cdot L_{\mathbf{v}}$. Thus tropical subrepresentations of dimension d correspond to G fixed tropical Plücker vectors $\mathbf{v} \in \bigwedge^d V$. This crucial observation from [CGM20] is shown for all idempotent semifields and is central to our approach. We highlight it here in the Boolean case.

Theorem 2.1 ([GM20] Theorem A(1)). *For any $1 \leq d \leq n$, the induced linear representation on $\bigwedge^d \mathbb{B}^n$ restricts to a linear action on the Dressian $\mathrm{Dr}(d, n) \subseteq \bigwedge^d \mathbb{B}^n$, and d -dimensional tropical subrepresentations in \mathbb{B}^n are equivalent to G -fixed points in $\mathrm{Dr}(d, n)$.*

Thus the problem of classifying d -dimensional tropical subrepresentation of ρ is equivalent to identifying all rank d tropical Plücker vectors fixed by the induced G -action. Moreover, this reduces further to a problem in matroid theory. We recall that an automorphism of a matroid is a permutation of the ground set sending independent sets to independent sets, and

thus bases to bases and cocircuits to cocircuits. Tropical linear subspaces $L_{\mathbf{v}} \subseteq V$ correspond to matroids $M_{\mathbf{v}}$ over the ground set $[n]$. A tropical representation $\rho : G \rightarrow \text{GL}(V)$ induces a G -action on $[n]$, and ρ factors through the automorphism group of $M_{\mathbf{v}}$ if and only if $L_{\mathbf{v}}$ is G -invariant. Since a tropical Plücker vector is the basis indicator vector for $M_{\mathbf{v}}$, a G -fixed tropical Plücker vector \mathbf{v} corresponds to a set of bases $\mathcal{B}_{\mathbf{v}}$ invariant under the G -action on $[n]$. Thus to classify d -dimensional tropical subrepresentation of ρ , it is enough to classify matroids rank d matroids $([n], \mathcal{B})$ for which $\mathcal{B} \subseteq \binom{[n]}{d}$ is invariant under the induced G -action.

2.5. The regular representation. Let G be a finite group. For each $g \in G$ we write \mathbf{e}_g for a basis element represented by the group element g and denote by

$$\mathbb{B}[G] = \left\{ \sum_{g \in G} c_g \mathbf{e}_g \right\}$$

the \mathbb{B} -linear span of these basis elements. The regular representation is given by taking the G action by left multiplication on the basis elements of $\mathbb{B}[G]$:

$$g \cdot \mathbf{e}_a = \mathbf{e}_{ga}, \quad g, a \in G.$$

Fix $1 \leq d \leq n$. We can extend this action to the basis of $\bigwedge^d \mathbb{B}[G]$ in the following manner:

$$g \cdot (\mathbf{e}_{a_1} \wedge \cdots \wedge \mathbf{e}_{a_d}) = \mathbf{e}_{ga_1} \wedge \cdots \wedge \mathbf{e}_{ga_d}, \quad g \in G, \{a_1, \dots, a_d\} \in \binom{G}{d}.$$

This extends linearly to an action on $\bigwedge^d \mathbb{B}[G]$.

In parallel, consider the G action on itself by left multiplication:

$$g \cdot a = ga, \quad g, a \in G.$$

This extends to $\binom{G}{d}$ in the following manner:

$$g \cdot \{a_1, \dots, a_d\} = \{ga_1, \dots, ga_d\}, \quad g \in G, \{a_1, \dots, a_d\} \in \binom{G}{d}.$$

There is an obvious bijection from the basis of $\bigwedge^d \mathbb{B}[G]$ to the set of subsets of G of size d along which these two actions are equivalent. The bijection extends to one between $\bigwedge^d \mathbb{B}[G]$ and the power set of $\binom{G}{d}$, by sending the d -vector

$$\sum_{I \in \binom{G}{d}} v_I \mathbf{e}_I \in \bigwedge^d \mathbb{B}[G]$$

to the set of subsets $\{I | v_I = 1\}$ it indicates. Hence terms in the tropical Plücker vector $\mathbf{v} = \sum_{I \in \binom{G}{d}} v_I \mathbf{e}_I$ associated with the matroid $M_{\mathbf{v}} = (G, \mathcal{B}_{\mathbf{v}})$ will have

$$v_I = 1 \iff I \in \mathcal{B}$$

and thus can be written as $\mathbf{v} = \sum_{I \in \mathcal{B}_{\mathbf{v}}} \mathbf{e}_I$.

3. 2-DIMENSIONAL SUBREPRESENTATIONS OF $\mathbb{B}[G]$

In this section we classify all 2-dimensional tropical subrepresentations of $\mathbb{B}[G]$ for an arbitrary finite group G and provide some examples. By [GM20, Theorem A(1)] these correspond to rank 2 tropical Plücker vectors fixed by the induced action of G on $\bigwedge^2 \mathbb{B}[G]$. They are studied by taking unions of orbits of the equivalent action on $\binom{G}{2}$. This amounts to searching for unions of orbits of the G action on $\binom{G}{2}$ satisfying the strong basis exchange axiom.

3.1. Orbits and their properties. To identify the orbits of the G action on $\binom{G}{2}$ note that for any subset $\{a, b\} \in \binom{G}{2}$, the product $a^{-1}b$ is preserved under the G action on $\binom{G}{2}$ since

$$(ga)^{-1}gb = a^{-1}g^{-1}gb = a^{-1}b.$$

We think of $a^{-1}b$ as a generalized “difference” (see the example of \mathbb{Z}_n in Section 3.2 below). Moreover, we have that every pair with the same generalized difference is in the same orbit. To see this, let $\{a_1, b_1\}, \{a_2, b_2\} \in \binom{G}{2}$ with

$$a_1^{-1}b_1 = a_2^{-1}b_2,$$

giving

$$a_2a_1^{-1}b_1 = b_2.$$

Then $a_2a_1^{-1}$ acting on $\{a_1, b_1\}$ gives

$$a_2a_1^{-1} \cdot \{a_1, b_1\} = \{a_2a_1^{-1}a_1, a_2a_1^{-1}b_1\} = \{a_2, b_2\}.$$

Thus the orbits are given by the sets

$$f_g = \{\{a, ag\} | a \in G\} = \{\{a, b\} | a, b \in G, a^{-1}b = g\}$$

consisting of pairs $\{a, b\}$ with a fixed difference $g = a^{-1}b$ between the two elements. Note that $g = e$ gives $f_e = \emptyset$ and each orbit f_g determines a G fixed vector $\mathbf{f}_g \in \bigwedge^2 \mathbb{B}[G]$. Denote by \mathcal{O}_G the set of orbits of the G action on $\binom{G}{2}$. Each orbit determines a G fixed vector in $\bigwedge^2 \mathbb{B}[G]$, denoted

$$\mathbf{f}_g = \sum_{a \in G} \mathbf{e}_a \wedge \mathbf{e}_{ag}.$$

We now highlight salient properties of the orbits f_g .

Proposition 3.1. *For all $g \in G$, $f_g = f_{g^{-1}}$.*

Proof. To show $f_g \subseteq f_{g^{-1}}$, let $\{a, b\}$ be an arbitrary set in f_g . Then $g = a^{-1}b$ by definition and $\{a, b\} = \{b, a\} \in f_{b^{-1}a} = f_{(a^{-1}b)^{-1}} = f_{g^{-1}}$. The containment in the other direction is similar. \square

For reference, the matroid basis axioms are:

- (1) $\mathcal{B} \neq \emptyset$
- (2) if $A, B \in \mathcal{B}$ then $|A| = |B|$
- (3) if $A, B \in \mathcal{B}$ and $x \in A - B$, then there is some $y \in B - A$ with $A - x + y \in \mathcal{B}$ (strong basis exchange).

Let $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$ be bases. There are three cases:

Case 1: $A = B$. In this case there is no x in $A - B$ and basis exchange succeeds.

Case 2: A, B are unequal but not disjoint, so they share exactly one element since they both have size 2. Without loss of generality, let $a_1 = b_1$. Then $x = a_2, y = b_2$, and this forces $A - a_2 + b_2$ to be in \mathcal{B} . However, this set is $\{a_1, b_2\}$, but since $a_1 = b_1$ the set is just $\{b_1, b_2\} = B$.

Case 3: A, B are disjoint. Without loss of generality, pick $x = a_2$; since the sets are disjoint, y could be either b_1 or b_2 . Then we have $\{a_1, b_1\}$ or $\{a_1, b_2\}$ is in \mathcal{B} .

Since only the third case results in including sets not already in \mathcal{B} , we will only consider basis exchange between disjoint subsets of size 2 when dealing with dimension 2. The following proposition shows how basis exchange can be applied to the orbits of the G action on $\binom{G}{2}$.

Proposition 3.2. *Let \mathcal{B} be the set of bases of a matroid $M_{\mathbf{v}} = (G, \mathcal{B})$ corresponding to a G -fixed tropical Plücker vector $\mathbf{v} \in \wedge^2 \mathbb{B}[G]$. Then*

$$f_{gh} \subseteq \mathcal{B} \implies f_g \subseteq \mathcal{B} \text{ or } f_h \subseteq \mathcal{B}.$$

Proof. We note that $\{e, gh\}, \{g^{-1}, h\} \in f_{gh}$. These sets are disjoint as long as $g^{-1}, h \neq e, gh \neq g^{-1}$ and $gh \neq h$. In any of these cases, $f_{gh} = f_g$ or f_h and the proposition is true. Otherwise, the sets are disjoint and we can use basis exchange to deduce

$$\{e, g^{-1}\} \in \mathcal{B} \text{ or } \{e, h\} \in \mathcal{B}.$$

But $\{e, g^{-1}\} \in f_{g^{-1}} = f_g$ by Proposition 3.1 and $\{e, h\} \in f_h$. Therefore we have $f_{gh} \subseteq \mathcal{B}$ implies $f_g \subseteq \mathcal{B}$ or $f_h \subseteq \mathcal{B}$ as desired since \mathcal{B} must be a union of orbits. \square

For any subset $S \subseteq G$ of indices we introduce the notation

$$f_S = \bigcup_{g \in S} f_g$$

for a union of orbits. In the proofs below we may sometimes only consider cases where $S \subseteq G - \{e\}$, but since $f_e = \emptyset$ this will never be an issue when quantifying over all possible nonempty unions. Each subset S determines in this way a candidate set of bases of a matroid corresponding to a 2-dimensional tropical subrepresentation. A nonempty union f_S will satisfy the matroid basis exchange axioms if and only if the corresponding 2-vector

$$\mathbf{f}_S = \sum_{g \in S} \mathbf{f}_g$$

satisfies the tropical Plücker relations.

3.2. Example: \mathbb{Z}_n . The case $G = \mathbb{Z}_n$ is studied in part in [GM20, Section 4]. Here the action on subsets of size 2 becomes

$$g \cdot \{a, b\} = \{g + a, g + b\}, \quad g \in \mathbb{Z}_n, \quad \{a, b\} \in \binom{\mathbb{Z}_n}{2}.$$

Note that the difference $b - a$ is preserved under the \mathbb{Z}_n -action, and the distinct orbits are precisely the sets

$$f_i = \{\{a \bmod n, a + i \bmod n\} \mid a \in \mathbb{Z}_n\}$$

consisting of subsets of \mathbb{Z}_n of size 2 with a fixed difference i modulo n between the two elements. It is clear from the definition that $f_i = f_j$ whenever i and j are congruent modulo n and $f_i = f_{-i}$ for all i . With n odd, the distinct \mathbb{Z}_n -orbits are

$$\mathcal{O}_{\mathbb{Z}_n} = \left\{ f_1, \dots, f_{\frac{n-1}{2}} \right\}$$

since $n - \frac{n+1}{2} = \frac{n-1}{2}$. With n even, the distinct orbits are

$$\mathcal{O}_{\mathbb{Z}_n} = \{f_1, \dots, f_{\frac{n}{2}}\}.$$

Denote by $\mathbb{Z}_n^\times \subseteq \mathbb{Z}_n$ the multiplicative group of units, and let p be a prime. In [GM20, Theorem 4.1.3], Giansiracusa & Manaker prove that the only subrepresentation of $\mathbb{B}[\mathbb{Z}_p]$ in dimension two is uniform when 2 is a primitive root mod p or $p \cong 7 \pmod{8}$ and 2 has order $\frac{p-1}{2} \pmod{p}$. Their argument appeals to studying when 2 generates the multiplicative quotient group $\mathbb{Z}_n^\times / \langle p-1 \rangle$, and applying a weaker version of Proposition 3.2 to build a chain of inclusions for the set of bases of a matroid. They conjecture [GM20, Conjecture 4.1.6] that p is prime if and only if the only subrepresentation of $\mathbb{B}[\mathbb{Z}_p]$ in dimension two corresponds to the uniform rank 2 matroid on $[n]$. We prove this conjecture as a special case of Theorem 3.5 in Section 3.3, but as a brief example we can obtain one direction by showing that every two-dimensional tropical subrepresentation of $\mathbb{B}[\mathbb{Z}_n]$ contains $f_{\mathbb{Z}_n^\times}$ in its set of bases.

Proposition 3.3. *Let \mathcal{B} be the set of bases of a matroid $M_{\mathbf{v}} = ([n], \mathcal{B})$ corresponding to a \mathbb{Z}_n -fixed tropical Plücker vector $\mathbf{v} \in \bigwedge^2 \mathbb{B}[\mathbb{Z}_n]$. Let m be a positive integer.*

$$(1) f_{mi} \subseteq \mathcal{B} \implies f_i \subseteq \mathcal{B}.$$

$$(2) \text{ Let } d \mid n \text{ and } k \in \mathbb{Z}_n \text{ with } k \neq 0. \text{ Then } f_{kd} \subseteq \mathcal{B} \implies f_{d\mathbb{Z}_n^\times} \subseteq \mathcal{B}.$$

Proof. We prove (1) by induction. It is a tautology in the case $m = 1$. Assume $f_{(m-1)i} \subseteq \mathcal{B} \implies f_i \subseteq \mathcal{B}$. Then by Proposition 3.2,

$$f_{mi} = f_{(m-1)i+i} \subseteq \mathcal{B} \implies f_{(m-1)i} \subseteq \mathcal{B} \text{ or } f_i \subseteq \mathcal{B}.$$

The latter case is immediate. In the former case, we know $f_i \subseteq \mathcal{B}$ by the inductive hypothesis.

For (2), we have by (1) that $f_{kd} \subseteq \mathcal{B} \implies f_d \subseteq \mathcal{B}$. Then for any $u \in \mathbb{Z}_n^\times$ we must have

$$f_d = f_{u^{-1}ud} \subseteq \mathcal{B} \implies f_{ud} \subseteq \mathcal{B}.$$

Thus $f_{d\mathbb{Z}_n^\times} \subseteq \mathcal{B}$ □

In the setting of the above proposition, since \mathcal{B} must be a non-empty union of orbits, we know that for some $k \in \mathbb{Z}_n$, $f_k \subseteq \mathcal{B}$. Taking $d = 1$ in (2) gives $f_{\mathbb{Z}_n^\times} \subseteq \mathcal{B}$. We get the following Corollary.

Corollary 3.4. *Every two-dimensional tropical subrepresentation of $\mathbb{B}[\mathbb{Z}_n]$ contains $f_{\mathbb{Z}_n^\times}$ in its corresponding set of bases. When n is prime then $f_{\mathbb{Z}_n^\times} = f_{[n]} = \binom{[n]}{2}$ and the only 2-dimensional tropical subrepresentation corresponds to the uniform rank 2 matroid $U_{2,n}$ on the ground set $[n]$.*

3.3. Classification in dimension 2. Our first theorem classifies all two-dimensional tropical subrepresentations of $\mathbb{B}[G]$.

Theorem 3.5. *The two-dimensional tropical subrepresentations of $\mathbb{B}[G]$ correspond to the Plücker vectors \mathbf{f}_{G-H} for H a proper subgroup of G .*

Proof. We begin by showing that, for any proper subgroup H of G , the union of orbits f_{G-H} forms the bases of a matroid. Since H is proper, we know f_{G-H} is a non-empty subset of $\binom{G}{2}$. Let $\{a_1, a_2\}$ and $\{b_1, b_2\}$ in f_{G-H} . Without loss of generality, it is enough to show $\{a_1, b_1\}$ or $\{a_1, b_2\}$ is in f_{G-H} . Assume on the contrary $\{a_1, b_1\}$ and $\{a_1, b_2\} \notin f_{G-H}$. Notice that since H is a subgroup, both H and $G-H$ are closed under taking inverses so $f_{G-H} \cap f_H = \emptyset$

and $f_{G-H} \cup f_H = \binom{G}{2}$, giving a disjoint union. Thus $\{a_1, b_1\} \in f_H$ and $\{a_1, b_2\} \in f_H$. Then $a_1^{-1}b_1, a_1^{-1}b_2 \in H$, so

$$(a_1^{-1}b_1)^{-1}a_1^{-1}b_2 = b_1^{-1}a_1a_1^{-1}b_2 = b_1^{-1}b_2 \in H.$$

Thus $\{b_1, b_2\} \in f_H$ contradicting our assumption that $\{b_1, b_2\} \in f_{G-H}$.

For the converse, we will now show that orbit unions of the form f_{G-H} with H a proper subgroup of G are the only G -fixed matroidal subsets of $\binom{G}{2}$. Let $S \subseteq G - \{e\}$ with S nonempty and assume the union of orbits $f_S = \bigcup_{i \in S} f_i$ is the set of bases indicated by a G fixed Plücker vector, and thus matroidal. Our goal is to show the complement $S^C = G - S$ is a proper subgroup of G .

In the case $S = G - \{e\}$ we have $S^C = \{e\}$, which is a proper subgroup as desired. In this case, $f_S = \binom{G}{2}$ and thus \mathbf{f}_S corresponds to the uniform matroid.

Assume now S is a proper nonempty subset of $G - \{e\}$. Let $a, b \in S^C$. Then $f_a, f_b \subseteq f_{S^C}$. We will use the subgroup test to show that S^C is a subgroup of G . Assume on the contrary that $ab^{-1} \notin S^C$. Then $ab^{-1} \in S$ giving $f_{ab^{-1}} \subseteq f_S$. Thus f_a or $f_{b^{-1}} = f_b \subseteq f_S$ by Proposition 3.2. So a or $b \in S$ which contradicts our assumption $a, b \in S^C$. Thus S^C is a subgroup of G . \square

Remark 3.6. The uniform matroid corresponds to choosing $H = \{e\}$.

In the cyclic case $G = \mathbb{Z}_n$, setting n to be a prime gives $f_{\mathbb{Z}_n^\times} = \binom{[n]}{2}$ and confirms Giansiracusa & Manaker's Conjecture.

Corollary 3.7. [GM20, Conjecture 4.1.6] *For $\mathbb{B}[\mathbb{Z}_n]$, n is prime if and only if the only two-dimensional subrepresentation corresponds to the uniform matroid.*

3.4. Examples.

Example 3.8 ($\mathbb{B}[Q_8]$). We use the usual presentation

$$Q_8 = \langle -1, i, j, k \mid (-1)^2 = 1, i^2 = j^2 = k^2 = ijk = -1 \rangle$$

of the quaternions. There are 4 orbits of the action of Q_8 on subsets of Q_8 of size 2 by left multiplication:

$$\begin{aligned} f_{-1} &= \{\{1, -1\}, \{i, -i\}, \{j, -j\}, \{k, -k\}\}, \\ f_i &= \{\{1, i\}, \{-1, -i\}, \{i, -1\}, \{-i, 1\}, \{j, -k\}, \{-j, k\}, \{k, j\}, \{-k, -j\}\}, \\ f_j &= \{\{1, j\}, \{-1, -j\}, \{i, k\}, \{-i, -k\}, \{j, -1\}, \{-j, 1\}, \{k, -i\}, \{-k, i\}\}, \\ f_k &= \{\{1, k\}, \{-1, -k\}, \{i, -j\}, \{-i, j\}, \{j, i\}, \{-j, -i\}, \{k, -1\}, \{-k, 1\}\}. \end{aligned}$$

By Proposition 3.2, $f_{-1} \subseteq \mathcal{B} \implies f_i \subseteq \mathcal{B}$ since $i^2 = -1$. Combining this with similar statements for j and k , we obtain $f_{-1} \subseteq \mathcal{B} \implies f_i \cup f_j \cup f_k \subseteq \mathcal{B}$.

We can write $i = jk$, so by Proposition 3.2 we know that $f_i \subseteq \mathcal{B} \implies f_j$ or $f_k \subseteq \mathcal{B}$. Similarly for f_j and f_k , we obtain

$$f_j \subseteq \mathcal{B} \implies f_i \text{ or } f_k \subseteq \mathcal{B},$$

and

$$f_k \subseteq \mathcal{B} \implies f_i \text{ or } f_j \subseteq \mathcal{B}.$$

Combining all of these statements together, it can be checked that the only matroidal sums of orbits are the following:

$$\begin{aligned} f_{Q_8-\langle k \rangle} &= f_i \cup f_j \\ f_{Q_8-\langle j \rangle} &= f_i \cup f_k \\ f_{Q_8-\langle i \rangle} &= f_j \cup f_k \\ f_{Q_8-\langle -1 \rangle} &= f_i \cup f_j \cup f_k \\ f_{Q_8-\langle 1 \rangle} &= f_{-1} \cup f_i \cup f_j \cup f_k \end{aligned}$$

These correspond to the subgroups of Q_8 .

Example 3.9 ($\mathbb{B}[D_n]$). We use the presentation $D_n = \langle \rho, \sigma \mid \rho^n = \sigma^2 = e, \sigma\rho\sigma = \rho^{-1} \rangle$. The orbits of the left action of D_n on $\binom{D_n}{2}$ are as follows:

- (1) $f_{\rho^i} = \{ \{e, \rho^i\}, \{\rho, \rho^{i+1}\}, \dots, \{\rho^{n-1}, \rho^{i-1}\}, \{\sigma, \sigma\rho^i\}, \{\sigma\rho, \sigma\rho^{i+1}\}, \dots, \{\sigma\rho^{n-1}, \sigma\rho^{i-1}\} \}$ for $0 < i < \frac{n}{2}$,
- (2) $f_{\rho^{\frac{n}{2}}} = \{ \{e, \rho^{\frac{n}{2}}\}, \{\rho, \rho^{\frac{n}{2}+1}\}, \dots, \{\rho^{\frac{n}{2}-1}, \rho^{n-1}\}, \{\sigma, \sigma\rho^{\frac{n}{2}}\}, \{\sigma\rho, \sigma\rho^{\frac{n}{2}+1}\}, \dots, \{\sigma\rho^{\frac{n}{2}-1}, \sigma\rho^{n-1}\} \}$ if n is even,
- (3) $f_{\sigma\rho^i} = \{ \{e, \sigma\rho^i\}, \{\rho, \sigma\rho^{i-1}\}, \dots, \{\rho^{n-1}, \sigma\rho^{i+1}\} \}$, for $0 \leq i < n$.

If n is odd, there are $\frac{n-1}{2}$ of the first type of orbit, each of which has order $2n$, while there are n of the third type of orbit, each of which has order n . This gives a total of $2n\binom{\frac{n-1}{2}}{2} + n(n) = n(2n-1) = \binom{2n}{2}$ elements. If n is even, there are $\frac{n}{2} - 1$ of the first type of orbit, each of which has order $2n$, one of the second type of orbit of order n , and n of the third type of orbit, which has order n . This gives us a total of $2n\binom{\frac{n}{2}-1}{2} + 1(n) + n(n) = n(2n-1) = \binom{2n}{2}$ elements.

By Theorem 3.5, matroidal unions of orbits correspond to subgroups of D_n . Subgroups of D_n are classified as follows (see for instance [Con19, Theorem 3.1] or [Cav75]):

- (i) $\langle \rho^d \rangle$, $d \mid n$,
- (ii) $\langle \rho^d, \sigma\rho^i \rangle$, $d \mid n$ $0 \leq i < d$.

Since the subgroups depend on the factorization of n , we restrict our example to the case D_p where p is prime for simplicity. The corresponding unions of orbits are:

$$\begin{aligned} d = 1: \langle \rho \rangle &\text{ corresponds to } f_{D_p-\langle \rho \rangle} = f_\sigma \cup f_{\sigma\rho} \cup \dots \cup f_{\sigma\rho^{p-1}}; \\ d = p: \langle e \rangle &\text{ corresponds to the uniform matroid, } f_{D_p-\langle e \rangle} = f_{D_p}, \text{ and} \\ &\langle \sigma\rho^i \rangle, 0 \leq i < p, \text{ corresponds to } f_{D_p-\langle \sigma\rho^i \rangle} = f_{D_p} - f_{\sigma\rho^i}. \end{aligned}$$

For a specific example of the composite case, we turn to D_4 :

$$\begin{aligned} d = 1: \langle \rho \rangle &\text{ corresponds to } f_{D_4-\langle \rho \rangle} = f_\sigma \cup f_{\sigma\rho} \cup f_{\sigma\rho^2} \cup f_{\sigma\rho^3}; \\ d = 2: \langle \rho^2 \rangle &\text{ corresponds to } f_{D_4-\langle \rho^2 \rangle} = f_\rho \cup f_\sigma \cup f_{\sigma\rho} \cup f_{\sigma\rho^2} \cup f_{\sigma\rho^3}; \\ &\langle \rho^2, \sigma \rangle \text{ corresponds to } f_{D_4-\langle \rho^2, \sigma \rangle} = f_\rho \cup f_{\sigma\rho} \cup f_{\sigma\rho^3}; \\ &\langle \rho^2, \sigma\rho \rangle \text{ corresponds to } f_{D_4-\langle \rho^2, \sigma\rho \rangle} = f_\rho \cup f_\sigma \cup f_{\sigma\rho^2}; \\ d = 4: \langle e \rangle &\text{ corresponds to } f_{D_4-\langle e \rangle} = f_\rho \cup f_{\rho^2} \cup f_\sigma \cup f_{\sigma\rho} \cup f_{\sigma\rho^2} \cup f_{\sigma\rho^3} \text{ (uniform);} \\ &\langle \sigma \rangle \text{ corresponds to } f_{D_4-\langle \sigma \rangle} = f_\rho \cup f_{\rho^2} \cup f_{\sigma\rho} \cup f_{\sigma\rho^2} \cup f_{\sigma\rho^3}; \\ &\langle \sigma\rho \rangle \text{ corresponds to } f_{D_4-\langle \sigma\rho \rangle} = f_\rho \cup f_{\rho^2} \cup f_\sigma \cup f_{\sigma\rho^2} \cup f_{\sigma\rho^3}; \\ &\langle \sigma\rho^2 \rangle \text{ corresponds to } f_{D_4-\langle \sigma\rho^2 \rangle} = f_\rho \cup f_{\rho^2} \cup f_\sigma \cup f_{\sigma\rho} \cup f_{\sigma\rho^3}; \\ &\langle \sigma\rho^3 \rangle \text{ corresponds to } f_{D_4-\langle \sigma\rho^3 \rangle} = f_\rho \cup f_{\rho^2} \cup f_\sigma \cup f_{\sigma\rho} \cup f_{\sigma\rho^2}. \end{aligned}$$

4. 3-DIMENSIONAL SUBREPRESENTATIONS

The problem in dimension 3 offers more combinatorial complexity. We present in this section some properties of orbits and computations of three dimensional subrepresentations of $\mathbb{B}[G]$. We identify tropical subrepresentations of $\mathbb{B}[G]$ corresponding to subgroups of index larger than 2. In the cyclic case we find additional tropical subrepresentations of $\mathbb{B}[\mathbb{Z}_n]$ that do not fit into this correspondence. We also identify a specific collection of orbits, indexed by \mathbb{Z}_n^\times , that are contained in the set of bases of any matroid corresponding to a subrepresentation of $\mathbb{B}[\mathbb{Z}_n]$.

4.1. Orbits and their properties. As in dimension 2, the G action on $\mathbb{B}[G]$ extends to the basis of $\bigwedge^3 \mathbb{B}[G]$ as

$$g \cdot \mathbf{e}_a \wedge \mathbf{e}_b \wedge \mathbf{e}_c = \mathbf{e}_{ga} \wedge \mathbf{e}_{gb} \wedge \mathbf{e}_{gc}, \quad g \in G, \{a, b, c\} \in \binom{G}{3}$$

and extends linearly to an action on $\bigwedge^3 \mathbb{B}[G]$. The equivalent G action on $\binom{G}{3}$ is similar. Following the approach in dimension 2, G -orbits in this setting are given by sets of the form

$$f_{g,h} = \{\{a, ag, ah\} | a \in G\} = \{\{a, b, c\} | a^{-1}b = g \text{ and } a^{-1}c = h\}$$

for any $g, h \neq e$ with $g \neq h$. It is obvious from the definition that $f_{g,h} = f_{h,g}$. We write $\mathbf{f}_{g,h} \in \bigwedge^3 \mathbb{B}[G]$ for the corresponding Plücker vector. We will investigate similar properties of the orbits that can be deduced in the 3-dimensional setting. The first is analogous to Proposition 3.1.

Proposition 4.1. *For all $g, h \in G$, $f_{g,h} = f_{g^{-1},g^{-1}h} = f_{h^{-1},h^{-1}g}$.*

Proof. To show $f_{g,h} \subseteq f_{g^{-1},g^{-1}h}$, let $\{a, b, c\} \in f_{g,h}$ be given. Then $g = a^{-1}b$ and $h = a^{-1}c$. Notice $g^{-1} = b^{-1}a$ and

$$\begin{aligned} g^{-1}h &= (b^{-1}a)(a^{-1}c) \\ &= b^{-1}c \end{aligned}$$

thus $\{a, b, c\} = \{b, a, c\} \in f_{g^{-1},g^{-1}h}$. The other containment is similar. Finally, a similar proof gives the third equality. \square

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ in \mathcal{B} be bases of a matroid (G, \mathcal{B}) . There are now four cases when analyzing basis exchange to replace a_3 :

Case 1: $A = B$. In this case $A - B$ is empty and basis exchange succeeds.

Case 2: A, B share two elements, none of which are a_3 . Without loss of generality, let $a_1 = b_1$ and $a_2 = b_2$. Then $x = a_3$, $y = b_3$, and this forces $A - a_3 + b_3$ to be in \mathcal{B} . However, this set is $\{a_1, a_2, b_3\}$, but since $a_1 = b_1$ and $a_2 = b_2$ the set is just $\{b_1, b_2, b_3\} = B$.

Case 3: A, B share a single element that isn't a_3 . Without loss of generality, let $a_1 = b_1$. Then replacing a_3 gives $\{a_1, a_2, b_2\}$ or $\{a_1, a_2, b_3\} \in \mathcal{B}$.

Case 4: A, B are disjoint. Then replacing a_3 gives $\{a_1, a_2, b_1\}$, $\{a_1, a_2, b_2\}$ or $\{a_1, a_2, b_3\} \in \mathcal{B}$.

Note that only cases 3 and 4 result in the inclusion of new bases. The following proposition shows how basis exchange in this setting interacts with the orbits of the G -action.

Proposition 4.2. *Let \mathcal{B} be the set of bases of a matroid $M_{\mathbf{v}} = (G, \mathcal{B})$ corresponding to a G -fixed tropical Plücker vector $\mathbf{v} \in \bigwedge^3 \mathbb{B}[G]$. Then $f_{g,h} \cup f_{g',h'} \subseteq \mathcal{B} \implies f_{g,g'}$ or $f_{g,h'} \subseteq \mathcal{B}$.*

Proof. First consider the case where g, h, g' , and h' are distinct. We know $\{0, g, h\} \in f_{g,h}$ and $\{0, g', h'\} \in f_{g',h'}$. Basis exchange gives $\{0, g, g'\}$ or $\{0, g, h'\} \in \mathcal{B}$ and $f_{g,g'}$ or $f_{g,h'} \subseteq \mathcal{B}$. In the case where g, h, g', h' are not distinct, there are 6 cases:

- the cases $g = h$ and $g' = h'$ aren't possible based on how the orbits are defined,
- $g = g' \implies f_{g',h'} = f_{g,h'}$,
- $g = h' \implies f_{g',h'} = f_{g,g'}$,
- $h = g' \implies f_{g,h} = f_{g,g'}$,
- $h = h' \implies f_{g,h} = f_{g,h'}$.

□

As an immediate corollary, we have a property of orbits that is analogous to the product property of Proposition 3.2.

Corollary 4.3. *Let \mathcal{B} be the set of bases of a matroid $M_{\mathbf{v}} = (G, \mathcal{B})$ corresponding to a G -fixed tropical Plücker vector $\mathbf{v} \in \bigwedge^3 \mathbb{B}[G]$. Then $f_{g,gh} \subseteq \mathcal{B} \implies f_{g,g^{-1}}$ or $f_{g,h} \subseteq \mathcal{B}$.*

Proof. By Proposition 4.1, we know $f_{g,gh} = f_{g^{-1},h}$. The result follows directly from applying Proposition 4.2. □

A straightforward application this basis exchange property provides an important reduction of exponents in the indices of the orbits.

Proposition 4.4. *Let \mathcal{B} be the set of bases of a matroid $M_{\mathbf{v}} = (G, \mathcal{B})$ corresponding to a G -fixed tropical Plücker vector $\mathbf{v} \in \bigwedge^3 \mathbb{B}[G]$. Then $f_{g,g^k} \subseteq \mathcal{B} \implies f_{g,g^2} \subseteq \mathcal{B}$ for $k \geq 2$.*

Proof. We proceed by induction. The case $k = 2$ is a tautology. Assume for induction hypothesis that $f_{g,g^{k-1}} \subseteq \mathcal{B} \implies f_{g,g^2} \subseteq \mathcal{B}$. Then

$$f_{g,g^k} \subseteq \mathcal{B} \implies f_{g,g^{-1}} \text{ or } f_{g,g^{k-1}} \subseteq \mathcal{B}$$

by Corollary 4.3. In the first case, we know $f_{g,g^{-1}} = f_{g^{-1},g}$ and

$$f_{g^{-1},g} = f_{g,g^2}$$

using Proposition 4.1. The second case follows immediately from the induction hypothesis. □

4.2. Subrepresentations in dimension 3. Recall that we defined a notation $f_S = \bigcup_{g \in S} f_g$ for the union of orbits in dimension 2 indexed by a subset $S \subseteq G$. We adopt a similar notation in dimension 3:

$$f_S = \bigcup_{g,h,g^{-1}h \in S} f_{g,h}.$$

Our main theorem in dimension 3 gives an attempt to extend our dimension 2 classification to three dimensional tropical subrepresentations of $\mathbb{B}[G]$ coming from proper subgroups of G .

Theorem 4.5. *Let G be a finite group, and let H be a subgroup of G with $[G : H] > 2$. Then there is a matroid $M_{\mathbf{v}} = (G, f_{G-H})$ corresponding to a G -fixed tropical Plücker vector $\mathbf{v} \in \bigwedge^3 \mathbb{B}[G]$.*

Proof. We prove this by contradiction. Assume on the contrary for some $X = \{x_1, x_2, x_3\}, Y = \{y_1, y_2, y_3\} \in \mathcal{B}$ we can apply basis exchange to force the inclusion of an orbit containing a difference in H . Note that by definition, $x_1^{-1}x_2, x_1^{-1}x_3, x_2^{-1}x_3, y_1^{-1}y_2, y_1^{-1}y_3, y_2^{-1}y_3$ must be in $G - H$. Their inverses must also be in $G - H$, since it's closed under taking inverses.

Note that neither X nor Y contains a difference in H by our assumption. The case where $[G : H] = 2$ is impossible here. If H has index 2, then there are two cosets: H and aH . Assume on the contrary that $x_1^{-1}x_2, x_1^{-1}x_3, x_2^{-1}x_3$ in $G - H = aH$. Then $x_1^{-1}x_2 = ah_1$ and $x_1^{-1}x_3 = ah_2$ with $h_1, h_2 \in H$. Then, $x_2^{-1}x_3 = x_2^{-1}x_1x_1^{-1}x_3 = (x_1^{-1}x_2)^{-1}x_1^{-1}x_3 = (ah_1)^{-1}ah_2 = h_1^{-1}h_2 \in H$.

Now, we assume $[G : H] > 2$.

- (1) X and Y share one element, $x_1 = y_1$. $\{x_1, x_2, x_3\}, \{y_1, y_2, y_3\} \in \mathcal{B} \implies \{x_1, x_2, y_2\}$ or $\{x_1, x_2, y_3\} \in \mathcal{B}$. Since we've assumed we can force the inclusion of an orbit containing a difference in H , there must be a difference in H in both sets. So the first set must include a difference in H . There are three possibilities:

- (a) $x_1^{-1}x_2 \in H$ is a contradiction;
- (b) $x_1^{-1}y_2 \in H$ is also a contradiction since $x_1^{-1}y_2 = y_1^{-1}y_2 \in H$
- (c) $x_2^{-1}y_2 \in H$ is the only remaining case.

Similarly we can show that $x_2^{-1}y_3 \in H$. However, then $y_2^{-1}y_3 = y_2^{-1}x_2x_2^{-1}y_3 = (x_2^{-1}y_2)^{-1}x_2^{-1}y_3 \in H$ giving a contradiction.

- (2) X and Y are disjoint. $\{x_1, x_2, x_3\}, \{y_1, y_2, y_3\} \in \mathcal{B} \implies \{x_1, x_2, y_1\}$ or $\{x_1, x_2, y_2\}$ or $\{x_1, x_2, y_3\} \in \mathcal{B}$. Since d must divide a difference in each set, it must divide a difference in the first set; there are again three possibilities:

- (a) $x_1^{-1}x_2 \in H$ is a contradiction;
- (b) if $x_1^{-1}y_1 \in H$, note that $x_1^{-1}y_2$ or $x_1^{-1}y_3 \in H$ would give a contradiction as in case (c) above, so it must be the case that $x_2^{-1}y_2$ and $x_2^{-1}y_3 \in H$, which gives us the contradiction we were trying to avoid; and finally
- (c) the case where $x_2^{-1}y_1 \in H$ gives a similar contradiction.

□

Unlike in Theorem 3.5, this is not an equivalence. Indeed there exist matroids in dimension three that do not come from complements of subgroups, suggesting the theory is more involved. In Section 4.3 we will study the cyclic case to identify some of these subrepresentations.

4.3. Results in the cyclic case. Our final theorems in dimension 3 are restricted to the cyclic case $G = \mathbb{Z}_n$. We classify a number of tropical subrepresentations that appear and give a collection of orbits indexed by the generators of the group that will be always be subsets of the bases of a matroid corresponding to a tropical subrepresentations. While not a complete classification, these results make apparent the combinatorics that appear in higher dimension and hopefully provide a path for further results in higher dimensions.

As in the dimension 2 example given in Section 3.2, we will work using additive notation. The orbits take the form

$$f_{i,j} = \{\{a, a+i, a+j\} | a \in \mathbb{Z}_n\} = \{\{a, b, c\} | b-a = i \text{ and } c-a = j\}$$

for $0 < i < j < n$. We begin with two lemmas to aid in the proof of Theorem 4.8. This first lemma provides a useful bound on the indices of $f_{i,j}$.

Lemma 4.6. *Let $G = \mathbb{Z}_n$ be a cyclic group and $u \in \mathbb{Z}_n^\times$ a unit. Let $f_{i,j}$ be an orbit of the G action on $\binom{G}{3}$, with $0 < i < j < n$. Let $I = \{k \mid f_{ku,lu} = f_{i,j}, 0 < k < l < n\}$. Then I has an element s with $3s \leq n$.*

Proof. Recall $f_{i,j} = f_{-i,j-i} = f_{-j,i-j}$ by Proposition 4.1. Note also that $f_{-i,j-i} = f_{j-i,-i}$ since we can switch the order of indices.

Write $i = au$, $j - i = bu$, $-j = cu$. We know $0 < a, b < n$ since $0 < i < j < n$. We also know $-n < c < 0$ since $0 < j < n$, so we let $c' = c + n$ to ensure $0 < c' < j$. Then, $a, b, c' \in I$.

We also know that $i + (j - i) - i = 0$, so $a + b + c = 0$. Then $a + b + c' = a + b + c + n = n$.

Let s be the smallest element of $\{a, b, c'\} \subseteq I$. Note that $0 < s$. Then, $0 < 3s \leq a + b + c' = n$. \square

In the proof of Theorem 4.8 we will need to split into even and odd cases for n . The next lemma will help us deal with the even case.

Lemma 4.7. *Let \mathcal{B} be the set of bases of a matroid $M_{\mathbf{v}} = ([n], \mathcal{B})$ corresponding to a \mathbb{Z}_n -fixed tropical Plücker vector $\mathbf{v} \in \bigwedge^3 \mathbb{B}[\mathbb{Z}_n]$. If $n = 2k$, $k > 2$, and $u \in \mathbb{Z}_n^\times$, then*

$$f_{u,ku} \subseteq \mathcal{B} \iff f_{u,(k+1)u} \subseteq \mathcal{B}.$$

Proof. Assuming $f_{u,ku} \subseteq \mathcal{B}$, we know by Proposition 4.1 that

$$f_{u,ku} = f_{-u,(k-1)u} = f_{ku,(k+1)u}$$

and

$$f_{u,(k+1)u} = f_{-u,ku} = f_{(k-1)u,ku}$$

(using the assumption that $n = 2k$ and working modulo n in the indices). When $k > 2$, $u \neq ku \neq -u \neq (k-1)u$, and we can use basis exchange in the following way:

$$f_{u,ku} \cup f_{-u,(k-1)u} \subseteq \mathcal{B} \implies f_{-u,ku} \text{ or } f_{(k-1)u,ku} \subseteq \mathcal{B}$$

Notice the orbits on the left are $f_{u,ku}$ and the ones on the right are $f_{u,(k+1)u}$, so we know

$$f_{u,ku} \subseteq \mathcal{B} \implies f_{u,(k+1)u} \subseteq \mathcal{B}$$

The other direction is similar. \square

Theorem 4.8. *Let $G = \mathbb{Z}_n$ be a cyclic group, and let \mathcal{B} be the set of bases of a matroid $M_{\mathbf{v}} = ([n], \mathcal{B})$ corresponding to a \mathbb{Z}_n -fixed tropical Plücker vector $\mathbf{v} \in \bigwedge^3 \mathbb{B}[\mathbb{Z}_n]$. Then, for any $u \in \mathbb{Z}_n^\times$, $f_{u,2u} \subseteq \mathcal{B}$.*

Proof. Let $I = \{k \mid f_{ku,lu} \subseteq \mathcal{B} \text{ for } 0 < k < l < n\}$. Since I must be non-empty as the set of bases is nonempty and contains at least one orbit, we know that I must have a least element s . We will show that $s = 1$. Assume on the contrary $s > 1$.

We know $3s \leq n$ by Lemma 4.6. Since $1 < s$ we have $2s + 1 < 3s \leq n$. Since $s \in I$ there is an orbit $f_{su,tu} \subseteq \mathcal{B}$ for some $t > s > 1$, so we can simply long divide $t = sq + r$ for some $1 \leq q$, $0 \leq r < s$.

In the case $r = 0$, we have $t = sq$ and thus $f_{su,(sq)u} \subseteq \mathcal{B}$ for some $2 \leq q$. Proposition 4.4 gives $f_{su,(2s)u} \subseteq \mathcal{B}$, so we must have basis elements

$$\{0, su, (2s)u\}, \{u, (s+1)u, (2s+1)u\} \in f_{su,(2s)u} \subseteq \mathcal{B}.$$

Since $2s + 1 < n$ these are disjoint. Basis exchange gives

$$\{0, u, (s+1)u\} \in \mathcal{B} \text{ or } \{u, su, (s+1)u\} \in \mathcal{B} \text{ or } \{u, (s+1)u, (2s)u\} \in \mathcal{B}.$$

If $\{0, u, (s+1)u\} \in \mathcal{B}$ then $f_{u,(s+1)u} \subseteq \mathcal{B}$ and $1 \in I$. If $\{u, su, (s+1)u\} \in \mathcal{B}$ then $f_{(s-1)u,su} \subseteq \mathcal{B}$. By Proposition 4.1, $f_{(s-1)u,su} = f_{(1-s)u,u} = f_{u,(1-s)u} \subseteq \mathcal{B}$ and again $1 \in I$. Finally, if $\{u, (s+1)u, (2s)u\} \subseteq \mathcal{B}$, then $f_{su,(2s-1)u} \subseteq \mathcal{B}$. By Proposition 4.1,

$$f_{su,(2s-1)u} = f_{-su,(s-1)u} = f_{(s-1)u,-su} \subseteq \mathcal{B}$$

and $s-1 \in I$. In all three cases of basis exchange, we obtain a contradiction since s is not the least element of I .

In the case $r \neq 0$, we know $f_{su,(sq+r)u} \subseteq \mathcal{B}$. Thus by Corollary 4.3 we must have $f_{su,(-sq)u}$ or $f_{su,ru} \subseteq \mathcal{B}$. If $f_{su,(-sq)u}$, we can again use Proposition 4.4 to show $f_{su,(2s)u} \subseteq \mathcal{B}$ and the logic in the previous case to arrive at a contradiction. Otherwise, we know $r < s \in I$, again giving a contradiction.

So it must be the case that $s = 1$, and $f_{u,tu} \subseteq \mathcal{B}$ for some $t > 1$. Thus $f_{u,2u} \subseteq \mathcal{B}$ by Proposition 4.4. □

In particular, When $n = p$ is a prime this theorem tells us that all orbits of the form f_{a,a^2} for $a = 1, \dots, p-1$ are contained in the set of bases for every matroid corresponding to a fixed tropical Plücker vector $\mathbf{v} \in \bigwedge^3 \mathbb{B}[\mathbb{Z}_n]$. However, not all orbits are of this form. Our next two result provides the full set of bases in several cases for tropical subrepresentations of $\mathbb{B}[\mathbb{Z}_n]$.

Theorem 4.9. *For a tropical Plücker vector $\mathbf{v} \in \bigwedge^3 \mathbb{B}[\mathbb{Z}_n]$, write $\mathcal{B}_{\mathbf{v}}$ for the set of bases of its corresponding matroid.*

- (1) *For any $u \in \mathbb{Z}_n^\times$, if n is odd, there exists a tropical Plücker vector $\mathbf{v} \in \bigwedge^3 \mathbb{B}[\mathbb{Z}_n]$ with $\mathcal{B}_{\mathbf{v}} = \binom{[n]}{3} - f_{u,ku}$ if and only if $k \neq -1, 2, \frac{n+1}{2}$;*
- (2) *For any $u \in \mathbb{Z}_n^\times$, if n is even, there exists a tropical Plücker vector $\mathbf{v} \in \bigwedge^3 \mathbb{B}[\mathbb{Z}_n]$ with $\mathcal{B}_{\mathbf{v}} = \binom{[n]}{3} - f_{u,ku}$ if and only if $k \neq -1, 2, \frac{n}{2}, \frac{n}{2} + 1$.*

Proof. To begin, assume on the contrary $\mathcal{B} = \binom{[n]}{3} - f_{u,ku}$ form the bases of a matroid for at least one of the cases $k = -1, k = 2$, or $k = \frac{n}{2}, k = \frac{n}{2} + 1$ for n even, or $k = \frac{n+1}{2}, |G|$ for n odd. First, $k = -1, k = 2$ give an immediate contradiction since $f_{u,-u} = f_{u,2u} \subseteq \mathcal{B}$ by Proposition 4.4. When n is even, either of $k = \frac{n}{2}, k = \frac{n}{2} + 1$ give a contradiction since $f_{u,(\frac{n}{2})u} \subseteq \mathcal{B} \iff f_{u,(\frac{n}{2}+1)u} \subseteq \mathcal{B}$ by Lemma 4.7. Finally when n is odd, if $k = \frac{n+1}{2}$, then $2(\frac{n+1}{2})u$ is congruent to u modulo n . Since $\frac{n+1}{2} \in \mathbb{Z}_n^\times$, we know by Proposition 4.4 that $f_{u,(\frac{n+1}{2})u} = f_{(\frac{n+1}{2})u,u} = f_{(\frac{n+1}{2})u,2(\frac{n+1}{2})u} \subseteq \mathcal{B}$.

In the opposite direction, we need to show $\binom{[n]}{3} - f_{u,ku}$ is a matroid provided $k \neq -1, k \neq 2; k \neq \frac{n}{2}$ and $k \neq \frac{n}{2} + 1$ for n even; and $k \neq \frac{n+1}{2}$ for n odd. We assume on the contrary that $\binom{[n]}{3} - f_{u,ku}$ is not a matroid, and show that k has to be one of the above values. Since $\binom{[n]}{3} - f_{u,ku}$ is nonempty, by the strong basis exchange axiom $\binom{[n]}{3} - f_{u,ku}$ is not a matroid if and only if for some $X = \{x_1, x_2, x_3\}$, some $Y = \{y_1, y_2, y_3\} \in \binom{[n]}{3} - f_{u,ku}$, and some $i \in X - Y$, we have that for all $j \in Y - X$, $X - i + j \in f_{u,ku}$.

If X and Y share more than one element, no new bases must be included (see the discussion in Section 4.1). Thus two cases remain:

- (1) X and Y share one element; without loss of generality set $x_1 = y_1$. In this case, basis exchange is not satisfied if and only if $\{x_1, x_2, y_2\}, \{x_1, x_2, y_3\} \in f_{u,ku}$.

- (2) X and Y are disjoint. In this case, basis exchange is not satisfied if and only if $\{x_1, x_2, y_1\}, \{x_1, x_2, y_2\}, \{x_1, x_2, y_3\} \in f_{u,ku}$.

In the first case, we know

$$\{x_1, x_2, y_2\}, \{x_1, x_2, y_3\} \in f_{u,ku} = f_{-u,(k-1)u} = f_{-ku,(1-k)u}$$

by Proposition 4.1. Notice that the difference $x_2 - x_1$ is in at least two of the following 3 sets: $\{u, ku\}, \{-u, (k-1)u\}, \{-ku, (1-k)u\}$. Since $u \in \mathbb{Z}_n^\times$, we can divide out by u and write the 12 possibilities in the following way:

- $1 = -k$ and $k = -1$ give $k = -1$.
- $1 = -k + 1$ and $-1 = k - 1$ give $k = 0$, which isn't possible.
- $1 = k$ and $-1 = -k$ give $k = 1$, which isn't possible.
- $1 = k - 1$ and $-1 = -k + 1$ give $k = 2$.
- $k = -k$ gives $2k = 0$; since $k \neq 0$ this case only occurs when n is even and $k = \frac{n}{2}$.
- $k = -k + 1$ and $k - 1 = -k$ give $2k = 1$, which only occurs when n is odd and $k = \frac{n+1}{2}$.
- $k - 1 = -k + 1$ gives $2k = 2$; since $k \neq 1$ this case only occurs when n is even and $k = \frac{n}{2} + 1$.

Thus these are the only possible values of k :

- $k = -1$
- $k = 2$
- $k = \frac{n}{2}, |G|$ even
- $k = \frac{n+1}{2}, |G|$ odd
- $k = \frac{n}{2} + 1, |G|$ even

In the second case case, we know

$$\{x_1, x_2, y_1\}, \{x_1, x_2, y_2\}, \{x_1, x_2, y_3\} \in f_{u,ku} = f_{-u,(k-1)u} = f_{-ku,(1-k)u}$$

by Proposition 4.1. Notice that the difference $x_2 - x_1 \in \{u, ku\}, \{-u, (k-1)u\}$, and $\{(1-k)u, -ku\}$. This is a special case of the first case, so no new values of k arise. \square

Remark 4.10. This guarantees non-uniform matroids but does not preclude there from being other types of non-uniform matroids. Based on SAGE computations, the first example of a prime with matroids excluding more than one orbit is $\mathbb{B}[\mathbb{Z}_{13}]$.

4.4. Examples. We will examine some groups of small order using our results to find all dimension 3 subrepresentations.

Example 4.11 ($\mathbb{B}[\mathbb{Z}_6]$). This example was included in [GM20]. We arrive at the same result using our methods. There are four orbits:

$$\begin{aligned} f_{1,2} = f_{1,5} = f_{4,5} &= \{\{0, 1, 2\}, \{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}, \{0, 4, 5\}, \{0, 1, 5\}\} \\ f_{1,3} = f_{2,5} = f_{3,4} &= \{\{0, 1, 3\}, \{1, 2, 4\}, \{2, 3, 5\}, \{0, 3, 4\}, \{1, 4, 5\}, \{0, 2, 5\}\} \\ f_{1,4} = f_{2,3} = f_{3,5} &= \{\{0, 1, 4\}, \{1, 2, 5\}, \{0, 2, 3\}, \{1, 3, 4\}, \{2, 4, 5\}, \{0, 3, 5\}\} \\ f_{2,4} &= \{\{0, 2, 4\}, \{1, 3, 5\}\}. \end{aligned}$$

We know by Theorem 4.8 that $f_{1,2} = f_{4,5} \subseteq \mathcal{B}$ for every matroid. By Proposition 4.2, $f_{1,2} \subseteq \mathcal{B} \implies f_{1,4}$ or $f_{2,4} \subseteq \mathcal{B}$. Furthermore, by Lemma 4.7, $f_{1,3} \subseteq \mathcal{B} \iff f_{1,4} \subseteq \mathcal{B}$. It can

be checked that the following are the only matroidal unions of orbits.

$$\begin{aligned} f_{1,2} \cup f_{2,4} \\ f_{1,2} \cup f_{1,3} \cup f_{1,4} \\ f_{1,2} \cup f_{1,3} \cup f_{1,4} \cup f_{2,4} \end{aligned}$$

Note that the first matroid corresponds to the subgroup with index 3 in \mathbb{Z}_6 as in Theorem 4.5.

Example 4.12 ($\mathbb{B}[\mathbb{Z}_7]$). The orbits in this example are:

$$\begin{aligned} f_{1,2} = f_{1,6} = f_{5,6} &= \{\{0, 1, 2\}, \{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}, \{4, 5, 6\}, \{0, 5, 6\}, \{0, 1, 6\}\} \\ f_{1,3} = f_{2,6} = f_{4,5} &= \{\{0, 1, 3\}, \{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{0, 4, 5\}, \{1, 5, 6\}, \{0, 2, 6\}\} \\ f_{1,4} = f_{3,4} = f_{3,6} &= \{\{0, 1, 4\}, \{1, 2, 5\}, \{2, 3, 6\}, \{0, 3, 4\}, \{1, 4, 5\}, \{2, 5, 6\}, \{0, 3, 6\}\} \\ f_{1,5} = f_{2,3} = f_{4,6} &= \{\{0, 1, 5\}, \{1, 2, 6\}, \{0, 2, 3\}, \{1, 3, 4\}, \{2, 4, 5\}, \{3, 5, 6\}, \{0, 4, 6\}\} \\ f_{2,4} = f_{2,5} = f_{3,5} &= \{\{0, 2, 4\}, \{1, 3, 5\}, \{2, 4, 6\}, \{0, 3, 5\}, \{1, 4, 6\}, \{0, 2, 5\}, \{1, 3, 6\}\}. \end{aligned}$$

By Theorem 4.8, $f_{1,2}$, $f_{2,4} = f_{3,5}$, and $f_{3,6} = f_{1,4} \subseteq \mathcal{B}$. Again, basis exchange can be used to show that $f_{1,2}$ and $f_{2,4} \subseteq \mathcal{B} \implies f_{1,3}$ or $f_{1,5} \subseteq \mathcal{B}$. It can be checked that the following are the only matroidal unions of orbits.

$$\begin{aligned} f_{1,2} \cup f_{1,3} \cup f_{1,4} \cup f_{2,4} \\ f_{1,2} \cup f_{1,4} \cup f_{1,5} \cup f_{2,4} \\ f_{1,2} \cup f_{1,3} \cup f_{1,4} \cup f_{1,5} \cup f_{2,4} \end{aligned}$$

Example 4.13 ($\mathbb{B}[S_3]$). We use the presentation $S_3 = \langle \rho, \sigma \mid \rho^3 = \sigma^2 = e, \sigma\rho\sigma = \rho^{-1} \rangle$. The orbits of the left action of S_3 on $\binom{S_3}{3}$ are as follows.

$$\begin{aligned} f_{\rho, \rho^2} &= \{\{e, \rho, \rho^2\}, \{\sigma, \sigma\rho, \sigma\rho^2\}\} \\ f_{\rho, \sigma} = f_{\rho^2, \sigma\rho} = f_{\sigma, \sigma\rho} &= \{\{e, \rho, \sigma\}, \{\rho, \rho^2, \sigma\rho^2\}, \{e, \rho^2, \sigma\rho\}, \{e, \sigma, \sigma\rho\}, \{\rho^2, \sigma\rho, \sigma\rho^2\}, \{\rho, \sigma, \sigma\rho^2\}\} \\ f_{\rho, \sigma\rho} = f_{\rho^2, \sigma\rho^2} = f_{\sigma\rho, \sigma\rho^2} &= \{\{e, \rho, \sigma\rho\}, \{\rho, \rho^2, \sigma\}, \{e, \rho^2, \sigma\rho^2\}, \{\rho, \sigma, \sigma\rho\}, \{e, \sigma\rho, \sigma\rho^2\}, \{\rho^2, \sigma, \sigma\rho^2\}\} \\ f_{\rho, \sigma\rho^2} = f_{\rho^2, \sigma} = f_{\sigma, \sigma\rho^2} &= \{\{e, \rho, \sigma\rho^2\}, \{\rho, \rho^2, \sigma\rho\}, \{e, \rho^2, \sigma\}, \{\rho^2, \sigma, \sigma\rho\}, \{\rho, \sigma\rho, \sigma\rho^2\}, \{e, \sigma, \sigma\rho^2\}\} \end{aligned}$$

Based on SAGE computations, there are 5 matroidal sums of orbits.

$$\begin{aligned} f_{\rho, \rho^2} \cup f_{\rho, \sigma} &= f_{S_3 - \langle \sigma\rho^2 \rangle} \\ f_{\rho, \rho^2} \cup f_{\rho, \sigma\rho} &= f_{S_3 - \langle \sigma \rangle} \\ f_{\rho, \rho^2} \cup f_{\rho, \sigma\rho^2} &= f_{S_3 - \langle \sigma\rho \rangle} \\ f_{\rho, \sigma} \cup f_{\rho, \sigma\rho} \cup f_{\rho, \sigma\rho^2} & \\ f_{\rho, \rho^2} \cup f_{\rho, \sigma} \cup f_{\rho, \sigma\rho} \cup f_{\rho, \sigma\rho^2} & \text{ (uniform)} \end{aligned}$$

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