

CONVEXITY OF THE MABUCHI FUNCTIONAL IN BIG COHOMOLOGY CLASSES

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ABSTRACT. We study the Mabuchi functional associated to a big cohomology class. We define an invariant associated to transcendental Fujita approximations, whose vanishing is related to the Yau-Tian Donaldson conjecture. Assuming vanishing (finiteness) of this invariant we establish (almost) convexity along weak geodesics. As an application, we give an explicit expression of the distance d_p in the big setting for finite entropy potentials.

1. INTRODUCTION

Let X be a compact Kähler manifold of complex dimension n and fix a Kähler form ω . Let d and d^c be the real differential operators defined as $d := \partial + \bar{\partial}$, $d^c := \frac{i}{2\pi} (\bar{\partial} - \partial)$. By the dd^c -lemma, the space of Kähler forms cohomologous to ω can be identified with the space

$$\mathcal{H} = \{u \in C^\infty(X) : \omega + dd^c u > 0\} / \mathbb{R}.$$

To study canonical Kähler metrics on X , Mabuchi in [52], [53] introduced a natural Riemannian metric g on \mathcal{H} . He defined the Mabuchi functional \mathcal{M} (known as well as K-energy) such that its critical points are constant scalar curvature Kähler (cscK for short) metrics. Furthermore, he demonstrated that the Mabuchi functional is convex along smooth geodesics of (\mathcal{H}, g) . However, (\mathcal{H}, g) is an infinite dimensional Fréchet Riemannian manifold, hence the existence of smooth geodesics is not guaranteed, as shown in [48], [25]. Nevertheless, a natural notion of weak geodesics exists to connect two points in \mathcal{H} . In [3], Berman and Berndtsson proved convexity of the Mabuchi functional along such weak geodesics and, as a consequence they established the uniqueness of the cscK metric in a given cohomology class (whenever it exists).

In the 1990's Tian [57] made an influential conjecture stating that the existence of a cscK metric is equivalent to the properness of the Mabuchi functional.

There were several attempts by many in this direction. The conjecture was first proven in the (Fano) Kähler-Einstein case by Darvas and Rubinstein [26]. The fact that the existence of a cscK metric implies the properness of the K -energy is due to Berman, Darvas and Lu [7], while the reverse implication was proven more recently by Chen and Cheng [14, 15] (see also [51], [49], [55], [54] for some results in the singular setting).

Motivated by the classification problem in birational geometry, in [5], the authors studied the Kähler-Einstein equation as a solution to a similar variational problem, but in the more singular context of a big cohomology class $\{\theta\} \in H^{1,1}(X, \mathbb{R})$. We say that $\{\theta\}$ is big if and only if its volume $\text{Vol}(\theta) > 0$ and we define this notion in Section 2. The notion of big cohomology classes is in fact invariant under bimeromorphic maps, while this is not the case for Kähler classes, and naturally arises in algebraic geometry.

In this paper we thus define and study the (relative) Mabuchi functional in a big cohomology class $\{\theta\}$. We recall that a function $\varphi : X \rightarrow \mathbb{R} \cup \{-\infty\}$ is quasi-plurisubharmonic (qpsh) if it can be locally written as the sum of a plurisubharmonic function and a smooth function. φ is called

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θ -plurisubharmonic (θ -psh) if it is qpsH and $\theta_\varphi := \theta + dd^c\varphi \geq 0$ in the sense of currents. We let $\text{PSH}(X, \theta)$ denote the set of θ -psh functions that are not identically $-\infty$. Thanks to [10], given a θ -psh function φ , the *non-pluripolar Monge-Ampère measure* of φ is well defined and denoted by $\theta_\varphi^n := (\theta + dd^c\varphi)^n$. In $\text{PSH}(X, \theta)$ there exists a “best candidate” which is less singular than any other θ -psh function:

$$V_\theta := \sup\{u \in \text{PSH}(X, \theta), u \leq 0\}.$$

We then say that a θ -psh function φ has minimal singularities if it is relative bounded with respect to V_θ , i.e. $|V_\theta - \varphi| \leq C$, for some $C > 0$.

Also, we have that $\text{Vol}(\theta) = \int_X \theta_{V_\theta}^n$. Thus, the fact that $\{\theta\}$ is big means that the mass of (the Monge-Ampère measure associated to) V_θ is strictly positive. The function V_θ is then the less singular function with “full mass”.

For any other mass $0 < m < \text{Vol}(\theta)$, one can consider *model potentials* ϕ having that mass, $\int_X \theta_\phi^n = m$, which are going to play the role of V_θ in a relative setting. We refer to Section 2.1 for a precise definition.

Taking inspiration from the Kähler setting (and from the big and nef case studied by Di Nezza and Lu in [36]), we define the Mabuchi functional relative to (X, θ_φ) , for any closed and positive $(1, 1)$ -current θ_φ with *well defined Ricci curvature* (see Section 4), as

$$(1.1) \quad \mathcal{M}_{\theta, \varphi}(u) := \bar{S}_\varphi E(\theta; u, \varphi) - nE_{\text{Ric}(\theta_\varphi)}(\theta; u, \varphi) + \text{Ent}(u, \varphi),$$

for any u with the same singularity type of φ , that is $|u - \varphi| \leq C$, for some $C > 0$. We refer to Sections 3 and 4 for the definitions of the energy terms $E, E_{\text{Ric}(\theta_\varphi)}$ and the entropy term Ent appearing in (1.1).

The main result of the paper is the (almost)-convexity of $\mathcal{M}_{\theta, \varphi}$ along weak geodesic. Our strategy is to treat this problem as a limiting case of classes with “prescribed singularities”. The latter notion was introduced in [21], [24], [23] and further developed in [59], [61], [60].

To be more precise, we consider a *monotone transcendental Fujita approximation* of $\{\theta\}$, i.e. a sequence of model potentials $(\phi_k)_k \subset \text{PSH}(X, \theta)$, $\phi_k \nearrow V_\theta$ such that for any k , there exists a modification $\pi_k : Y_k \rightarrow X$, Y_k compact Kähler manifold, and

$$\pi_k^*(\theta + dd^c\phi_k) = (\eta_k + dd^c\tilde{\phi}_k) + [F_k]$$

where F_k is an effective \mathbb{R} -divisor, $\tilde{\phi}_k$ is a potential with minimal singularities and η_k represents a big and nef class. The existence of a sequence $(\phi_k)_k$ is a consequence of [28] (see Lemma 4.14). We refer to Section 4.2 for more details.

There is a key quantity associated to a given monotone transcendental Fujita approximation $(\phi_k)_k$:

$$H(\phi_k) := \liminf_{k \rightarrow +\infty} \{\eta_k^{n-1}\} \cdot K_{Y_k/X}.$$

Note that $H(\phi_k) \geq 0$ and let us stress that it does not depend on the modifications π_k (Lemma 4.17). We then consider

$$H := \inf\{H(\phi_k), (\phi_k)_k \text{ monotone transcendental Fujita approximation}\}.$$

Our main result states as follows:

Theorem 1.1. *Let $u_0, u_1 \in \text{PSH}(X, \theta)$ with minimal singularities and let $(u_t)_{t \in [0, 1]}$ be the weak geodesic connecting u_0 and u_1 . Let $\varphi \in \mathcal{E}(X, \theta)$ be such that $\theta_\varphi^n = \text{Vol}(\theta)\omega^n$, $\sup_X \varphi = 0$. Then u_t has minimal singularities and the function $t \mapsto \mathcal{M}_{\theta, \varphi}(u_t)$ is almost convex in $[0, 1]$, i.e.*

$$(1.2) \quad \mathcal{M}_{\theta, \varphi}(u_t) \leq (1-t)\mathcal{M}_{\theta, \varphi}(u_0) + t\mathcal{M}_{\theta, \varphi}(u_1) + \frac{n\|u_0 - u_1\|_\infty}{2\text{Vol}(\theta)}H.$$

The proof consists in two big steps: first, for any monotone transcendental Fujita approximation $(\phi_k)_k$ and for each k , we prove (almost)-convexity for $\mathcal{M}_{\theta, \varphi_k}$ where φ_k is a suitable θ -psh function such that $|\varphi_k - \phi_k| \leq C$; then we perform a limiting procedure.

Also the inequality in (1.2) is interesting only if the quantity H is finite or equal to zero. As a consequence of our theorem we get the following:

Corollary 1.2. *Assume $H = 0$. Then the function $t \mapsto \mathcal{M}_{\theta, \varphi}(u_t)$ is convex in $[0, 1]$.*

When $\{\theta\}$ is big and nef we find $H = 0$. The convexity of the Mabuchi functional in the big and nef case was proved by Di Nezza and Lu in [36] using the fact that a big and nef class can be approximated by Kähler classes.

In Section 4.4 we examine the condition $H = 0$. Notably, we observe that $H = 0$ when $\{\theta\}$ has a bimeromorphic Zariski decomposition (Theorem 4.23). From this, it follows easily that $\mathcal{M}_{\theta, \varphi}$ is convex in complex dimension 2.

As it was observed in [50], when X is a projective manifold, and α is the cohomology class of a big \mathbb{Q} -divisor, if $H = 0$ then one can solve the Yau-Tian-Donaldson Conjecture (see Remark 4.24). It is conjectured that H is zero [50, Conjecture 4.7]. This raises the natural question of how the convexity of the Mabuchi functional for any big integral class is connected to the Yau-Tian-Donaldson Conjecture.

Once the convexity of the Mabuchi functional is established, we manage to ensure a uniform control of the entropy along the geodesic segment:

Corollary 1.3. *Assume $H < +\infty$. Let $C_1 > 0$ and let $u_0, u_1 \in \text{PSH}(X, \theta)$ be such that $u_0 - u_1$ is bounded. Assume $\text{Ent}(u_0), \text{Ent}(u_1) \leq C_1$. Then there exists a positive constant C_2 such that*

$$\text{Ent}(u_t) \leq C_2$$

for any $t \in [0, 1]$, where C_2 only depends on $C_1, n, X, \{\omega\}, \{\theta\}, \|u_0 - u_1\|_\infty, H$ and on a lower bound of $\text{Vol}(\theta)$.

This observation is the key to show that the distance d_1 admits an explicit expression. Let us recall that, following [19], the distance d_1 (in the big setting) is defined as

$$d_1(u_0, u_1) := E(\theta; u_0, V_\theta) + E(\theta; u_1, V_\theta) - 2E(\theta; P_\theta(u_0, u_1), V_\theta),$$

where E denotes the energy functional as in (1.1) and

$$P_\theta(u_0, u_1) := \sup\{v \in \text{PSH}(X, \theta), v \leq \min(u_0, u_1)\}.$$

Theorem 1.4. *Assume $H < +\infty$. Let $u_0, u_1 \in \text{Ent}(X, \theta)$ and u_t be the weak geodesic connecting u_0 and u_1 . If $u_0 - u_1$ is bounded then*

$$(1.3) \quad d_1(u_0, u_1) = \int_X |\dot{u}_t| \theta_{u_t}^n$$

for any $t \in [0, 1]$.

For the purpose of the introduction, we state the above theorem for d_1 since the latter can be more easily defined in the big setting. But we actually can prove the analogous result for all the Finsler distances d_p , $p \geq 1$, recently defined in the big setting by Gupta [45] (see Section 6 and Theorem 6.1).

Let us recall that (1.3) was first proved by Chen [13] for d_2 in the Kähler case, specifically for smooth Kähler potentials. This equality was instrumental in demonstrating that d_2 is a genuine distance, rather than merely a semi-distance.

The distances d_p on “extended Mabuchi spaces”, the so called Monge-Ampère energies classes \mathcal{E}^p , were initially introduced by Darvas [18] in the Kähler setting. Since then, they have been extensively studied by various authors due to their crucial role in the variational approach to finding Kähler-Einstein or cscK metrics. These distances have been defined in singular contexts through approximation procedures [32], [19], [35], [63], [45].

Having an explicit formulation for d_p is crucial for several applications. For example, it enables us to extend the result of [38] and [36] on Monge-Ampère measures on contact sets (see Proposition 5.2 and Corollary 5.3). Knowing the support of the Monge-Ampère measure of a (singular) θ -psh function is indeed a cornerstone of pluripotential theory.

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2. PRELIMINARIES

We recall results from (relative) pluripotential theory of big cohomology classes. We borrow notation and terminology from [23].

Let (X, ω) be a compact Kähler manifold of dimension n . Let θ be a smooth closed $(1, 1)$ -form on X . A function $\varphi : X \rightarrow \mathbb{R} \cup \{-\infty\}$ is quasi-plurisubharmonic (qpsh) if it can be locally written as the sum of a plurisubharmonic function and a smooth function. φ is called θ -plurisubharmonic (θ -psh) if it is qpsh and $\theta_\varphi := \theta + dd^c\varphi \geq 0$ in the sense of currents. We let $\text{PSH}(X, \theta)$ denote the set of θ -psh functions that are not identically $-\infty$. In the whole paper we will assume that $\{\theta\}$ is *big*, which means that it admits a *Kähler current*, i.e. there exists $\psi \in \text{PSH}(X, \theta)$ such that $\theta + dd^c\psi \geq \varepsilon\omega$ for some small constant $\varepsilon > 0$. Here, d and d^c are real differential operators defined as $d := \partial + \bar{\partial}$, $d^c := \frac{i}{2\pi}(\bar{\partial} - \partial)$.

We say that a θ -psh function φ has *analytic singularities* if there exists a constant $c > 0$ such that locally on X ,

$$(2.1) \quad \varphi = c \log \sum_{j=1}^N |f_j|^2 + g,$$

where g is bounded and f_1, \dots, f_N are local holomorphic functions.

Demailly regularization's theorem ensures that there are plenty of Kähler currents with analytic singularities (see for e.g. [30, Theorem 3.2]).

The *ample locus* $\text{Amp}(\theta)$ of $\{\theta\}$ is the set of points $x \in X$ such that there exists a Kähler current $T \in \{\theta\}$ with analytic singularity smooth in a neighbourhood of x . The ample locus $\text{Amp}(\theta)$ is a Zariski open subset, and it is nonempty [9]. The complement of the ample locus is known as the *non-Kähler locus*, $E_{nK}(\theta)$.

If φ and φ' are two θ -psh functions on X , then φ' is said to be *less singular* than φ , i.e. $\varphi \preceq \varphi'$, if they satisfy $\varphi \leq \varphi' + C$ for some $C \in \mathbb{R}$. We say that φ has the same singularity as φ' , i.e. $\varphi \simeq \varphi'$, if $\varphi \preceq \varphi'$ and $\varphi' \preceq \varphi$. The latter condition is easily seen to yield an equivalence relation, whose equivalence classes are denoted by $[\varphi]$, $\varphi \in \text{PSH}(X, \theta)$.

A θ -psh function φ is said to have *minimal singularity type* if it is less singular than any other θ -psh function. Such θ -psh functions with minimal singularity type always exist, one can consider for example

$$V_\theta := \sup \{ \varphi \text{ } \theta\text{-psh}, \varphi \leq 0 \text{ on } X \}.$$

Trivially, a θ -psh function with minimal singularity type is locally bounded in $\text{Amp}(\theta)$. It follows from [37, Theorem 1.1] that V_θ is $C^{1,1}$ in the ample locus $\text{Amp}(\theta)$.

Given $\theta^1 + dd^c\varphi_1, \dots, \theta^p + dd^c\varphi_p$ positive $(1, 1)$ -currents, where θ^j are closed smooth real $(1, 1)$ -forms, following the construction of Bedford-Taylor [2] in the local setting, it has been shown in [10] that the sequence of currents

$$\mathbf{1}_{\bigcap_j \{\varphi_j > V_{\theta_j} - k\}} (\theta^1 + dd^c \max(\varphi_1, V_{\theta_1} - k)) \wedge \dots \wedge (\theta^p + dd^c \max(\varphi_p, V_{\theta_p} - k))$$

is non-decreasing in k and converges weakly to the so called *non-pluripolar product*

$$(2.2) \quad \langle \theta_{\varphi_1}^1 \wedge \dots \wedge \theta_{\varphi_p}^p \rangle.$$

In the following, with a slight abuse of notation, we will denote the non-pluripolar product simply by $\theta_{\varphi_1}^1 \wedge \dots \wedge \theta_{\varphi_p}^p$. When $p = n$, the resulting positive (n, n) -current is a Borel measure that does not charge pluripolar sets. Pluripolar sets are Borel measurable sets that are contained in some set $\{\psi = -\infty\}$ (as it follows from [5, Corollary 2.11]).

For a θ -psh function φ , the *non-pluripolar complex Monge-Ampère measure* of φ is

$$\theta_{\varphi}^n := (\theta + dd^c\varphi)^n.$$

The volume of a big class $\{\theta\}$ is defined by

$$\text{Vol}(\theta) := \int_{\text{Amp}(\{\theta\})} \theta_{V_{\theta}}^n.$$

For notational convenience in the following we simply write $\text{Vol}(\theta)$, but keeping in mind that the volume is a cohomological constant.

By [10, Theorem 1.16], in the above expression one can replace V_{θ} with any θ -psh function with minimal singularity type. A θ -psh function φ is said to have *full Monge-Ampère mass* if

$$\int_X \theta_{\varphi}^n = \text{Vol}(\theta),$$

and we then write $\varphi \in \mathcal{E}(X, \theta)$.

An important property of the non-pluripolar product is that it is local with respect to the plurifine topology (see [2, Corollary 4.3], [10, Section 1.2]). This topology is the coarsest such that all qpsH functions on X are continuous. For convenience we record the following version of this result for later use.

Lemma 2.1. *Fix closed smooth real big $(1, 1)$ -forms $\theta^1, \dots, \theta^n$. Assume that $\varphi_j, \psi_j, j = 1, \dots, n$ are θ^j -psh functions such that $\varphi_j = \psi_j$ on an open set U in the plurifine topology. Then*

$$\mathbf{1}_U \theta_{\varphi_1}^1 \wedge \dots \wedge \theta_{\varphi_n}^n = \mathbf{1}_U \theta_{\psi_1}^1 \wedge \dots \wedge \theta_{\psi_n}^n.$$

Lemma 2.1 will be referred to as the *plurifine locality property*. We will often work with sets of the form $\{u < v\}$, where u, v are quasi-psh functions. These are always open in the plurifine topology.

The classical Monge-Ampère capacity (see [1], [47], [43]) is defined by

$$\text{Cap}_{\omega}(E) := \sup \left\{ \int_E (\omega + dd^c u)^n : u \in \text{PSH}(X, \omega), -1 \leq u \leq 0 \right\}.$$

A sequence u^k converges in capacity to u if for any $\varepsilon > 0$ we have

$$\lim_{k \rightarrow +\infty} \text{Cap}_{\omega}(\{|u^k - u| \geq \varepsilon\}) = 0.$$

The following extension of [23, Theorem 2.6] will be used several times in the paper.

Theorem 2.2. *For $j \in \{1, \dots, n\}$, let $\{\theta_k^j\}_k$ be a sequence of smooth closed real $(1, 1)$ -forms smoothly converging to smooth forms θ^j representing big cohomology classes. Suppose that for all $j \in \{1, \dots, n\}$ we have $u_j \in \text{PSH}(X, \theta)$, $u_j^k \in \text{PSH}(X, \theta_k^j)$ such that $u_j^k \rightarrow u_j$ in capacity as $k \rightarrow +\infty$.*

Let $\chi_k, \chi \geq 0$ be quasi continuous and uniformly bounded functions such that $\chi_k \rightarrow \chi$ in capacity. Then

$$(2.3) \quad \liminf_{k \rightarrow +\infty} \int_X \chi_k \theta_{k,u_1^k}^1 \wedge \wedge \theta_{k,u_2^k}^2 \wedge \dots \wedge \theta_{k,u_n^k}^n \geq \int_X \chi \theta_{u_1}^1 \wedge \theta_{u_2}^2 \wedge \dots \wedge \theta_{u_n}^n.$$

In addition, if

$$(2.4) \quad \limsup_{k \rightarrow +\infty} \int_X \theta_{k,u_1^k}^1 \wedge \theta_{k,u_2^k}^2 \wedge \dots \wedge \theta_{k,u_n^k}^n \leq \int_X \theta_{u_1}^1 \wedge \theta_{u_2}^2 \wedge \dots \wedge \theta_{u_n}^n$$

then

$$\chi_k \theta_{k,u_1^k}^1 \wedge \theta_{k,u_2^k}^2 \wedge \dots \wedge \theta_{k,u_n^k}^n \rightarrow \chi \theta_{u_1}^1 \wedge \theta_{u_2}^2 \wedge \dots \wedge \theta_{u_n}^n$$

in the weak sense of measure on X .

Proof. Fix j and let $T_j = \theta^j + dd^c \varphi_j$ be a Kähler current in $\{\theta^j\}$ with analytic singularities along the non-Kähler locus of $\{\theta_j\}$ (such a T_j exists thanks to [9, Theorem 3.17]). Let $\varepsilon_j > 0$ such that $T_j \geq 2\varepsilon_j \omega$. For $k \gg 1$ we have $\theta_k^j \geq \theta^j - \varepsilon_j \omega$. In particular, $\varphi_j \in \text{PSH}(X, \theta_k^j)$ and $\theta_k^j + dd^c \varphi_j$ is a Kähler current with analytic singularities. It then follows that for $k \gg 1$ we have $E_{nK}(\theta_k^j) \subseteq E_{nK}(\theta^j)$. Thus (up to take k big enough) we can assume that for any k ,

$$(2.5) \quad \text{Amp}(\theta_k^j) \supseteq \Omega := \bigcap_{j=1}^n \text{Amp}(\theta^j).$$

We then claim that $V_{\theta_k^j}$ converges to V_{θ^j} in capacity. It follows from the smooth convergence $\theta_k^j \rightarrow \theta^j$ that for any $\varepsilon > 0$ there exists $k_0 = k_0(\varepsilon) \gg 1$ such that $\theta^j - \varepsilon \omega \leq \theta_k^j \leq \theta^j - \varepsilon \omega$ for all $k \geq k_0$. It then follows by definition that for all $k \geq k_0$,

$$V_{\theta^j - \varepsilon \omega} \leq V_{\theta_k^j} \leq V_{\theta^j + \varepsilon \omega}.$$

Thus to conclude the claim it is enough to show that $V_{\theta^j - \varepsilon \omega} \nearrow V_{\theta^j}$ and $V_{\theta^j + \varepsilon \omega} \searrow V_{\theta^j}$ as $\varepsilon \searrow 0$. Observe that $\{V_{\theta^j - \varepsilon \omega}\}_\varepsilon$ and $\{V_{\theta^j + \varepsilon \omega}\}_\varepsilon$ are (respectively) increasing/decreasing sequences as $\varepsilon \searrow 0$. We can then consider their point-wise θ^j -psh limits

$$\phi^+ := \lim_{\varepsilon \rightarrow 0} V_{\theta^j + \varepsilon \omega}, \quad \phi^- := \left(\lim_{\varepsilon \rightarrow 0} V_{\theta^j - \varepsilon \omega} \right)^*.$$

We also observe that, by Hartog's lemma, $\sup_X \phi^+ = \sup_X \phi^- = 0$. Now, by construction $\phi^+ \geq V_{\theta^j}$. On the other hand, $V_{\theta^j} \geq \phi^+$ since ϕ^+ is a candidate in the envelope. Hence $\phi^+ = V_{\theta^j}$.

By [38, Corollary 3.4] we know that

$$(2.6) \quad (\theta^j - \varepsilon \omega + dd^c V_{\theta^j - \varepsilon \omega})^n = \mathbf{1}_{\{V_{\theta^j - \varepsilon \omega} = 0\}} (\theta^j - \varepsilon \omega)^n = \mathbf{1}_{\{V_{\theta^j - \varepsilon \omega} = 0\}} ((\theta^j)^n + O(\varepsilon))$$

Since $V_{\theta^j - \varepsilon \omega}$ is increasing to ϕ^- the sets $\{V_{\theta^j - \varepsilon \omega} = 0\}$ increase as ε decreases to 0, outside of a pluripolar set. Let

$$W := \bigcup_{\varepsilon > 0 \text{ small}} \{V_{\theta^j - \varepsilon \omega} = 0\}.$$

Then $\mathbf{1}_{\{V_{\theta^j - \varepsilon \omega} = 0\}}$ is increasing to $\mathbf{1}_W \leq \mathbf{1}_{\{\phi^- = 0\}}$ outside of the same pluripolar set. Using the fact that $(\theta^j - \varepsilon \omega + dd^c V_{\theta^j - \varepsilon \omega})^n$ converges weakly to $(\theta_{\phi^-}^j)^n$, $\phi^- \leq V_{\theta^j} \leq 0$ and that the form $(\theta^j)^n$ is non-negative on the set $\{V_{\theta^j} = 0\}$, from (2.6) we deduce that

$$\left(\theta_{\phi^-}^j \right)^n = \mathbf{1}_W (\theta^j)^n \leq \mathbf{1}_{\{\phi^- = 0\}} (\theta^j)^n \leq \mathbf{1}_{\{V_{\theta^j} = 0\}} (\theta^j)^n = \left(\theta_{V_{\theta^j}}^j \right)^n.$$

On the other hand by continuity of the volume function,

$$\int_X (\theta_{\phi^-}^j)^n = \lim_{\varepsilon \rightarrow 0} \int_X (\theta^j - \varepsilon \omega + dd^c V_{\theta^j - \varepsilon \omega})^n = \lim_{\varepsilon \rightarrow 0} \text{Vol}(\theta^j - \varepsilon \omega) = \text{Vol}(\theta^j) = \int_X \left(\theta_{V_{\theta^j}}^j \right)^n.$$

It then follows that $(\theta_{\phi^-}^j)^n = (\theta_{V_{\theta^j}}^j)^n$. By uniqueness of normalized solutions of Monge-Ampère equations, we infer that $\phi^- = V_{\theta^j}$. The claim is then proved

The proof now proceeds exactly as that in [23, Theorem 2.6] replacing the forms θ^j by θ_k^j . We give the details for the reader's convenience.

Fix an open relatively compact subset U of Ω . By (2.5), we know that the functions $V_{\theta_k^j}$ are bounded on U . Fix $C > 0$, $\varepsilon > 0$ and consider

$$f_j^{k,C,\varepsilon} := \frac{\max(u_j^k - V_{\theta_k^j} + C, 0)}{\max(u_j^k - V_{\theta_k^j} + C, 0) + \varepsilon}, \quad j = 1, \dots, n, \quad k \in \mathbb{N}^*$$

and

$$u_j^{k,C} := \max(u_j^k, V_{\theta_k^j} - C).$$

Observe that for C, j fixed, the functions $u_j^{k,C} \geq V_{\theta_k^j} - C$ are uniformly bounded in U (since $V_{\theta_k^j}$ are uniformly bounded in U) and converge in capacity to $u_j^C := \max(u_j, V_{\theta^j} - C)$ as $k \rightarrow +\infty$ by Lemma 2.3 below.

Moreover $f_j^{k,C,\varepsilon} = 0$ if $u_j^k \leq V_{\theta_k^j} - C$. By locality of the non-pluripolar product we can write

$$f^{k,C,\varepsilon} \chi_k \theta_{k,u_1^k}^1 \wedge \dots \wedge \theta_{k,u_n^k}^n = f^{k,C,\varepsilon} \chi_k \theta_{k,u_1^{k,C}}^1 \wedge \dots \wedge \theta_{k,u_n^{k,C}}^n,$$

where $f^{k,C,\varepsilon} = f_1^{k,C,\varepsilon} \dots f_n^{k,C,\varepsilon}$. For each C, ε fixed the functions $f^{k,C,\varepsilon}$ are quasi-continuous, uniformly bounded (with values in $[0, 1]$) and converge in capacity to $f^{C,\varepsilon} := f_1^{C,\varepsilon} \dots f_n^{C,\varepsilon}$ where $f_j^{C,\varepsilon}$ is defined by

$$f_j^{C,\varepsilon} := \frac{\max(u_j - V_{\theta^j} + C, 0)}{\max(u_j - V_{\theta^j} + C, 0) + \varepsilon}.$$

With the information above we can apply [23, Proposition 2.2] to get that

$$f^{k,C,\varepsilon} \chi_k \theta_{k,u_1^{k,C}}^1 \wedge \dots \wedge \theta_{k,u_n^{k,C}}^n \longrightarrow f^{C,\varepsilon} \chi \theta_{u_1^C}^1 \wedge \dots \wedge \theta_{u_n^C}^n \quad \text{as } k \rightarrow +\infty,$$

in the weak sense of measures on U . In particular since $0 \leq f^{k,C,\varepsilon} \leq 1$ we have that

$$\begin{aligned} \liminf_{k \rightarrow +\infty} \int_X \chi_k \theta_{k,u_1^k}^1 \wedge \dots \wedge \theta_{k,u_n^k}^n &\geq \liminf_{k \rightarrow +\infty} \int_U f^{k,C,\varepsilon} \chi_k \theta_{k,u_1^{k,C}}^1 \wedge \dots \wedge \theta_{k,u_n^{k,C}}^n \\ &\geq \int_U f^{C,\varepsilon} \chi \theta_{u_1^C}^1 \wedge \dots \wedge \theta_{u_n^C}^n. \end{aligned}$$

Now, letting $\varepsilon \rightarrow 0$ and then $C \rightarrow +\infty$, we obtain

$$\liminf_{k \rightarrow +\infty} \int_X \chi_k \theta_{k,u_1^k}^1 \wedge \dots \wedge \theta_{k,u_n^k}^n \geq \int_U \chi \theta_{u_1}^1 \wedge \dots \wedge \theta_{u_n}^n.$$

Finally, letting U increase to Ω and noting that the complement of Ω is pluripoar, we conclude the proof of the first statement of the theorem. Note that in the particular case $\chi_k = \chi \equiv 1$, we have

$$\liminf_{k \rightarrow +\infty} \int_X \theta_{k,u_1^k}^1 \wedge \dots \wedge \theta_{k,u_n^k}^n \geq \int_X \theta_{u_1}^1 \wedge \dots \wedge \theta_{u_n}^n.$$

Thus we actually have equality in (2.4) and the lim sup is a lim.

Now, let $B \in \mathbb{R}$ such that $\chi, \chi_k \leq B$. By (2.3) we get that

$$\liminf_{k \rightarrow +\infty} \int_X (B - \chi_k) \theta_{k,u_1^k}^1 \wedge \dots \wedge \theta_{k,u_n^k}^n \geq \int_X (B - \chi) \theta_{u_1}^1 \wedge \dots \wedge \theta_{u_n}^n.$$

Flipping the signs and using (equality in) (2.4), we conclude the following inequality, finishing the proof:

$$\limsup_{k \rightarrow +\infty} \int_X \chi_k \theta_{k, u_1^k}^1 \wedge \cdots \wedge \theta_{k, u_n^k}^n \leq \int_X \chi \theta_{u_1}^1 \wedge \cdots \wedge \theta_{u_n}^n.$$

□

Lemma 2.3. *Let u_k, v_k be two sequences of quasi-psh functions that converge in capacity respectively to u, v . Then*

$$\max(u_k, v_k) \longrightarrow \max(u, v)$$

in capacity.

Proof. Set $\varphi_k := \max(u_k, v_k)$, $\varphi := \max(u, v)$ and $\psi_k := \sup_{j \geq k} \varphi_j$. Note that $\psi_k \searrow$ and $\psi_k \geq \varphi_k$. We first observe that $\varphi_k \rightarrow \varphi$ in L^1 . Indeed, as for any two real numbers a, b we have $2 \max(a, b) = a + b + |a - b|$, we get

$$\begin{aligned} 2\|\varphi_k - \varphi\|_{L^1} &\leq \|u_k - u\|_{L^1} + \|v_k - v\|_{L^1} + \||u_k - v_k| - |u - v|\|_{L^1} \\ &\leq \|u_k - u\|_{L^1} + \|v_k - v\|_{L^1} + \|u_k - u + v_k - v\|_{L^1} \\ &\leq 2(\|u_k - u\|_{L^1} + \|v_k - v\|_{L^1}). \end{aligned}$$

We deduce that $\varphi_k \rightarrow \varphi$ weakly as $u_k \rightarrow u, v_k \rightarrow v$ weakly.

Thus we infer that $\psi_k \searrow \varphi$, in particular $\psi_k \rightarrow \varphi$ in capacity. Let $\delta > 0$. Clearly

$$(2.7) \quad \{|\varphi_k - \varphi| \geq \delta\} \subset \{\varphi_k \geq \delta + \varphi\} \cup \{\varphi \geq \delta + \varphi_k\}$$

and

$$(2.8) \quad \{\varphi_k \geq \delta + \varphi\} \subseteq \{\psi_k \geq \delta + \varphi\}.$$

Then we set $A := \{u \geq v\}, B := \{v \geq u\}$. On A we have

$$(2.9) \quad \{\varphi \geq \delta + \varphi_k\} = \{u \geq \delta + \varphi_k\} \subseteq \{u \geq \delta + u_k\} = \{|u_k - u| \geq \delta\}$$

as $\varphi_k \geq u_k$. Similarly, on B we get

$$(2.10) \quad \{\varphi \geq \delta + \varphi_k\} \subseteq \{|v_k - v| \geq \delta\}.$$

Combining (2.9) and (2.10) we get

$$(2.11) \quad \{\varphi \geq \delta + \varphi_k\} = (\{\varphi \geq \delta + \varphi_k\} \cap A) \cup (\{\varphi \geq \delta + \varphi_k\} \cap B) \subset \{|u_k - u| \geq \delta\} \cup \{|v_k - v| \geq \delta\}.$$

Thus (2.7), (2.8) and (2.11) leads to

$$\{|\varphi_k - \varphi| \geq \delta\} \subseteq \{\psi_k - \varphi \geq \delta\} \cup \{|u_k - u| \geq \delta\} \cup \{|v_k - v| \geq \delta\}.$$

As $u_k \rightarrow u, v_k \rightarrow v, \psi_k \rightarrow \varphi$ in capacity, we conclude that $\varphi_k \rightarrow \varphi$ in capacity thanks to the subadditivity property of the Monge-Ampère capacity. □

2.1. Envelopes and model potentials. If f is a function on X , we define the envelope of f in the class $\text{PSH}(X, \theta)$ by

$$P_\theta(f) := (\sup\{u \in \text{PSH}(X, \theta) : u \leq f\})^*,$$

with the convention that $\sup \emptyset = -\infty$. Observe that $P_\theta(f) \in \text{PSH}(X, \theta)$ if and only if there exists some $u \in \text{PSH}(X, \theta)$ lying below f . Note also that $V_\theta = P_\theta(0)$, and that $P_\theta(f + C) = P_\theta(f) + C$ for any constant C .

In the particular case $f = \min(\psi, \phi)$, we denote the envelope as $P_\theta(\psi, \phi) := P_\theta(\min(\psi, \phi))$. We observe that $P_\theta(\psi, \phi) = P_\theta(P_\theta(\psi), P_\theta(\phi))$, so w.l.o.g. we can assume ψ, ϕ are two θ -psh functions.

Starting from the ‘‘rooftop envelope’’ $P_\theta(\psi, \phi)$ we introduce

$$P_\theta[\psi](\phi) := \left(\lim_{C \rightarrow \infty} P_\theta(\psi + C, \phi) \right)^*.$$

It is easy to see that $P_\theta[\psi](\phi)$ only depends on the singularity type of ψ . When $\phi = V_\theta$, we will simply write $P_\theta[\psi] := P_\theta[\psi](V_\theta)$, and we refer to this potential as the *envelope of the singularity type* $[\psi]$.

Since $\psi - \sup_X \psi \leq P_\theta[\psi]$, we have that $[\psi] \leq [P_\theta[\psi]]$ and typically equality does not happen. When $[\psi] = [P_\theta[\psi]]$, we say that ψ has *model singularity type*. In the (more particular) case $\psi = P_\theta[\psi]$ we say that ψ is a *model potential*.

It is worth to mention that given any θ -psh function ψ with positive mass, the associated envelope $P_\theta[\psi]$ is in fact a model potential [23, Theorem 3.14].

From now on, (otherwise stated), ϕ will denote a model potential with strictly positive mass, i.e. $\int_X \theta_\phi^n > 0$. We say that a θ -psh function φ has ϕ -relative minimal singularities if $\varphi \simeq \phi$.

Remark 2.4. Let $\psi \in \text{PSH}(X, \theta)$ with $\sup_X \psi = 0$, for $N \in \mathbf{N}$ we set $P_N := P_\theta(\psi + N, V_\theta)$. On one hand we have $P_N \sim \psi$, on the other hand P_N is increasing to $P[\psi]$, then by Remark 2.6 $\theta_{P_N}^n$ converges weakly to $\theta_{P[\psi]}^n$, hence $\int_X \theta_N^n = \int_X \theta_{P[\psi]}^n$.

Definition 2.5. Given a model potential ϕ , the relative full mass class $\mathcal{E}(X, \theta, \phi)$ is the set of all θ -psh functions u such that u is more singular than ϕ and $\int_X \theta_u^n = \int_X \theta_\phi^n$. We will denote simply by $\mathcal{E}(X, \theta)$ the space $\mathcal{E}(X, \theta, V_\theta)$.

As pointed out in [44], for potential theoretic reasons, it is natural to consider weighted subspaces of $\mathcal{E}(X, \theta, \phi)$.

A weight is a continuous strictly increasing function $\chi : [0, +\infty) \rightarrow [0, +\infty)$ such that $\chi(0) = 0$ and $\chi(+\infty) = +\infty$. Denote by χ^{-1} its inverse function, i.e. such that $\chi(\chi^{-1}(t)) = t$ for all $t \geq 0$.

We fix ϕ a model potential and we let $\mathcal{E}_\chi(X, \theta, \phi)$ denote the set of all $u \in \mathcal{E}(X, \theta, \phi)$ such that

$$E_\chi(u, \phi) := \int_X \chi(|u - \phi|) \theta_u^n < \infty.$$

When $\phi = V_\theta$, we denote $\mathcal{E}(X, \theta) = \mathcal{E}(X, \theta, V_\theta)$, $\mathcal{E}_\chi(X, \theta) = \mathcal{E}_\chi(X, \theta, V_\theta)$ and $E_\chi(u) = E_\chi(u, V_\theta)$. Compared to [44], we have changed the sign of the weight, but the weighted classes are the same. Also, in the special case $\chi(t) = t^p$, $p > 0$, we simply denote the relative energy class with $\mathcal{E}^p(X, \theta, \phi)$ and the corresponding relative energy $E_p(u, \phi)$.

Remark 2.6. Under the assumptions of Theorem 2.2 we further assume that for all $j \in \{1, \dots, n\}$ and for k large enough, u_j^k is more singular than u_j , then

$$\theta_{u_1^k}^1 \wedge \theta_{u_2^k}^2 \wedge \dots \wedge \theta_{u_n^k}^n \rightarrow \theta_{u_1}^1 \wedge \theta_{u_2}^2 \wedge \dots \wedge \theta_{u_n}^n$$

in the weak sense of measures. Indeed, by [62, Theorem 1.2], [23, Theorem 3.3], if u_j^k is more singular than u_j we have

$$\int_X \theta_{u_1^k}^1 \wedge \theta_{u_2^k}^2 \wedge \dots \wedge \theta_{u_n^k}^n \leq \int_X \theta_{u_1}^1 \wedge \theta_{u_2}^2 \wedge \dots \wedge \theta_{u_n}^n.$$

This means that the second statement of Theorem 2.2 above holds with $\chi_k = \chi \equiv 1$.

The same conclusion holds if $u_j^k, u_j \in \mathcal{E}(X, \theta, \phi_j)$ (where ϕ_j are model potentials) since by [19, Proposition 3.1] we know that

$$\int_X \theta_{u_1^k}^1 \wedge \theta_{u_2^k}^2 \wedge \dots \wedge \theta_{u_n^k}^n = \int_X \theta_{u_1}^1 \wedge \theta_{u_2}^2 \wedge \dots \wedge \theta_{u_n}^n.$$

2.2. Plurisubharmonic geodesics. We next recall the definition/construction in [21] of plurisubharmonic geodesics.

For a curve $(0, 1) \ni t \mapsto u_t \in \text{PSH}(X, \theta)$ we define

$$(2.12) \quad X \times A \ni (x, z) \mapsto U(x, z) := u_{\log|z|}(x),$$

where $A = \{z \in \mathbb{C}, 1 < |z| < e\}$ and $\pi : X \times A \rightarrow X$ is the projection on the first factor.

Definition 2.7. We say that $t \mapsto u_t$ is a subgeodesic if $(x, z) \mapsto U(x, z)$ is a $\pi^*\theta$ -psh function on $X \times A$.

Definition 2.8. For $\varphi_0, \varphi_1 \in \text{PSH}(X, \theta)$, we let $\mathcal{S}_{[0,1]}(\varphi_0, \varphi_1)$ denote the set of all subgeodesics $(0, 1) \ni t \mapsto u_t$ such that $\limsup_{t \rightarrow 0^+} u_t \leq \varphi_0$ and $\limsup_{t \rightarrow 1^-} u_t \leq \varphi_1$.

Let $\varphi_0, \varphi_1 \in \text{PSH}(X, \theta)$. For $(x, z) \in X \times A$ we define

$$\Phi(x, z) := \sup\{U(x, z) : U \in \mathcal{S}_{[0,1]}(\varphi_0, \varphi_1)\}.$$

The curve $t \mapsto \varphi_t$ constructed from Φ via (2.12) is called the plurisubharmonic (psh) geodesic segment connecting φ_0 and φ_1 .

Geodesic segments connecting two general θ -psh functions may not exist. If $\varphi_0, \varphi_1 \in \mathcal{E}^p(X, \theta)$, it was shown in [21, Theorem 2.13] that $P(\varphi_0, \varphi_1) \in \mathcal{E}^p(X, \theta)$. Since $P(\varphi_0, \varphi_1) \leq \varphi_t$, we obtain that $t \rightarrow \varphi_t$ is a curve in $\mathcal{E}^p(X, \theta)$. Also, each subgeodesic segment is convex in t :

$$\varphi_t \leq (1-t)\varphi_0 + t\varphi_1, \quad \forall t \in [0, 1].$$

Consequently the upper semicontinuous regularization (with respect to both variables x, z) of Φ is again in $\mathcal{S}_{[0,1]}(\varphi_0, \varphi_1)$, hence so is Φ . If φ_0, φ_1 have the same singularities then the geodesic φ_t is Lipschitz on $[0, 1]$ (see [21, Lemma 3.1]):

$$(2.13) \quad |\varphi_t - \varphi_s| \leq |t - s| \sup_X |\varphi_0 - \varphi_1|, \quad \forall t, s \in [0, 1].$$

3. ENTROPY

We recall that given two positive probability measures μ, ν , the relative entropy $\text{Ent}(\mu, \nu)$ is defined as

$$\text{Ent}(\mu, \nu) := \int_X \log \left(\frac{d\mu}{d\nu} \right) d\mu,$$

if μ is absolutely continuous with respect to ν , and $+\infty$ otherwise.

Remark 3.1. Let μ, ν positive probability measure with $\mu := f\nu$ absolutely continuous with respect to ν . Then $\text{Ent}(\mu, \nu) < +\infty$ if and only if $f \log f \in L^1(X, \nu)$, in fact if $f < 1$, $f \log f$ is bounded, and if $f \geq 1$, $f \log f \geq 0$.

Once for all, we normalize the Kähler form ω such that $\int_X \omega^n = 1$. We consider $u \in \text{PSH}(X, \theta)$ such that $\theta_u^n = f\omega^n$, $0 \leq f$ and $m_u := \int_X \theta_u^n > 0$. Then $u \in \mathcal{E}(X, \theta, \phi)$, where ϕ is the model potential with $\int_X \theta_\phi^n = \int_X \theta_u^n$, and $m_u^{-1}\theta_u^n$ is a probability measure. We then define the θ -entropy of u as

$$(3.1) \quad \text{Ent}_\theta(u) := \text{Ent}(m_u^{-1}\theta_u^n, \omega^n) = \int_X \log \left(\frac{m_u^{-1}\theta_u^n}{\omega^n} \right) m_u^{-1}\theta_u^n = m_u^{-1} \int_X f \log f \omega^n - \log m_u.$$

By Jensen inequality we have $\text{Ent}_\theta(u) \geq 0$. Also, observe that the definition of the θ -entropy does depend on the chosen volume form ω^n but its finiteness does not.

Also, the expression in (3.1) coincides with the definition of entropy in [34] when $P[u] = V_\theta$, i.e. when $u \in \mathcal{E}(X, \theta)$. The definition in (3.1) is indeed a generalisation that allows to consider any θ -psh function not necessarily of full mass.

More generally, given two θ -psh functions u, v with $m_u, m_v > 0$ we define

$$\text{Ent}_\theta(u, v) := \text{Ent}(m_u^{-1}\theta_u^n, m_v^{-1}\theta_v^n).$$

Also, if no confusion can arise, we simply write $\text{Ent}(u)$ and $\text{Ent}(u, v)$.

Definition 3.2. We say that $u \in \text{PSH}(X, \theta)$ with $m_u > 0$ has *finite θ -entropy* if $\text{Ent}_\theta(u) < +\infty$. We denote by $\text{Ent}(X, \theta)$ the set of all θ -psh functions having finite θ -entropy.

By (3.1), $\text{Ent}_\theta(u) < +\infty$ if and only if $\int_X f \log f \omega^n < +\infty$ or equivalently $\int_X (f+1) \log(f+1) \omega^n < +\infty$.

We recall the following result from [39] which will be useful in the following:

Lemma 3.3. *Let $\pi : Y \rightarrow X$ a bimeromorphic and holomorphic map and assume that $\tilde{\omega}$ is Kähler form on Y normalized with volume equal to 1. Then*

- (i) $\text{Ent}(\mu, \nu) = \text{Ent}(\pi^*\mu, \pi^*\nu)$, for any two non-pluripolar probability measures μ, ν .
- (ii) If $\text{Ent}_\theta(\varphi) < +\infty$, then $\text{Ent}(m_\varphi^{-1}\pi^*\theta_\varphi^n, \tilde{\omega}^n) < +\infty$. In particular $\text{Ent}_{\pi^*\theta}(\pi^*\varphi) < +\infty$.

By $\pi^*\mu$ we mean the pushforward by π^{-1} of $\mathbf{1}_{X \setminus Z} \mu$ where Z is the indeterminacy locus of π^{-1} (see also [4, lines after Definition 1.3]). We refer to [39, Lemma 2.11] for a proof.

4. CONVEXITY OF THE MABUCHI FUNCTIONAL IN BIG CLASSES

Assume θ to be a smooth form such that $\{\theta\}$ is big. W.l.o.g. we normalize the Kähler form ω such that $\int_X \omega^n = 1$. Following [6, Section 2], given a θ -psh function φ , we say that θ_φ has well-defined Ricci curvature if its Monge-Ampère measure θ_φ^n corresponds to a singular metric on K_X , i.e. if locally

$$\theta_\varphi^n = e^{-f} i^{n^2} \Omega \wedge \bar{\Omega}$$

with $f \in L^1_{loc}$ and where Ω is any nowhere zero local holomorphic section of K_X . The Ricci curvature is then locally given by

$$\text{Ric}(\theta_\varphi) := \text{Ric}(\theta_\varphi^n) = dd^c f.$$

The local currents $dd^c f$ glue together and define a (global) closed real $(1, 1)$ -current $\text{Ric}(\theta_\varphi)$, which recovers the usual definition of the Ricci curvature when θ_φ is Kähler. With this choice of normalization, if θ and θ_φ have well-defined Ricci curvature we have

$$\text{Ric}(\theta_\varphi) = \text{Ric}(\theta) - dd^c \log \left(\frac{\theta_\varphi^n}{\theta^n} \right).$$

Moreover for every current θ_φ with well defined Ricci curvature the cohomology class of $\text{Ric}(\theta_\varphi)$ is always the first Chern class. We say that θ_φ has *good Ricci curvature* if it has well-defined Ricci curvature and there exists real closed positive $(1, 1)$ -currents T_1, T_2 such that

$$\text{Ric}(\theta_\varphi) = T_1 - T_2.$$

Note that if θ_φ has good Ricci curvature then $\int_X \theta_\varphi^n > 0$ and $\theta_\varphi^n = e^g \omega^n$ where g is difference of quasi-plurisubharmonic functions. In particular, [42, Theorem 1.1] implies that θ_φ^n has L^p density for $p > 1$. Hence [20, Theorem 1.4] gives that φ has $\phi := P_\theta[\varphi]$ -relative minimal singularities, where ϕ is the model type envelope such that $\varphi \in \mathcal{E}(X, \theta, \phi)$.

Let now φ be a θ -psh function with good Ricci curvature and let $\phi = P_\theta[\varphi]$ be the model potential associated to φ .

For $u, v \in \text{PSH}(X, \theta)$ with $|u - v|$ bounded, we then consider

$$(4.1) \quad E(\theta; u, v) := \frac{1}{(n+1)m_\phi} \sum_{k=0}^n \int_X (u-v) \theta_u^k \wedge \theta_v^{n-k}$$

and

$$(4.2) \quad E_{\text{Ric}(\theta_\varphi)}(\theta; u, v) := \frac{1}{n m_\phi} \sum_{k=0}^{n-1} \int_X (u-v) \theta_u^k \wedge \theta_v^{n-k-1} \wedge \text{Ric}(\theta_\varphi),$$

where each integral in the left-hand side is defined as

$$(4.3) \quad \int_X (u-v) \theta_u^k \wedge \theta_v^{n-k-1} \wedge T_1 - \int_X (u-v) \theta_u^k \wedge \theta_v^{n-k-1} \wedge T_2.$$

Similarly, given $\alpha = T_1 - T_2$, with T_i closed positive $(1, 1)$ -currents, we set

$$E_\alpha(\theta; u, v) := \frac{1}{n m_\phi} \sum_{k=0}^{n-1} \left(\int_X (u-v) \theta_u^k \wedge \theta_v^{n-k-1} \wedge T_1 - \int_X (u-v) \theta_u^k \wedge \theta_v^{n-k-1} \wedge T_2 \right).$$

To avoid any ambiguity, we recall that the wedge products between positive currents has to be understood as the non-pluripolar product. We observe that the above definition is independent of the choice of the two positive currents T_1, T_2 . Note also that thanks to [23, Lemma 5.6] if ϕ is a model potential and $u \in \mathcal{E}(X, \theta, \phi)$ then $E(\theta, u, \phi)$ is finite if and only if $u \in \mathcal{E}^1(X, \theta, \phi)$.

Taking inspiration from the Kähler setting (see also [36]), we define the Mabuchi functional relative to a (X, θ_φ) as

$$(4.4) \quad \mathcal{M}_{\theta, \varphi}(u) := \bar{S}_\varphi E(\theta; u, \varphi) - n E_{\text{Ric}(\theta_\varphi)}(\theta; u, \varphi) + \text{Ent}(u, \varphi), \quad u \simeq \varphi$$

for any $u \in \mathcal{E}(X, \theta, \phi)$ with ϕ -relative minimal singularities where $\text{Ent}(u, \varphi) := \text{Ent}(\theta_u^n m_\phi^{-1}, \theta_\varphi^n m_\phi^{-1})$ and

$$\bar{S}_\varphi := \frac{n}{m_\phi} \int_X \text{Ric}(\theta_\varphi) \wedge \theta_\varphi^{n-1}.$$

We start with a Proposition which generalizes [21, Theorem 3.12]:

Proposition 4.1. *Let $u_0, u_1 \in \mathcal{E}^1(X, \theta)$ and let u_t be the psh geodesic joining u_0, u_1 . Then $t \rightarrow E(\theta; u_t, V_\theta)$ is linear.*

Proof. For $i = 0, 1$ and $C > 0$, we set $u_i^C := \max(u_i, V_\theta - C)$. Let u_t^C be the psh geodesic joining u_0^C and u_1^C . Observe that u_t^C has minimal singularities.

We claim that u_t^C decreases to u_t . Indeed, since $u_i^C \geq u_i$, by comparison principle, u_t^C is a decreasing sequence such that $u_t^C \geq u_t$. Hence $u_t^C \searrow w_t$ for some θ -psh function $w_t \geq u_t$ joining u_0 and u_1 . By maximality of the geodesic we infer that $w_t = u_t$. By [21, Theorem 3.12], $t \rightarrow E(\theta; u_t^C, V_\theta)$ is linear. Moreover by [23, Lemma 5.7] we know that $E(\theta; u_t^C, V_\theta)$ converges to $E(\theta; u_t, V_\theta)$. It then follows that $t \rightarrow E(\theta; u_t, V_\theta)$, as limit of a linear function, is linear as well. \square

The next statement slightly generalizes the cocycle property in [23, Theorem 5.3]:

Proposition 4.2. *Let ϕ be a model potential. Then for any $u, v \in \mathcal{E}^1(X, \theta, \phi)$, we have*

$$(4.5) \quad E(\theta; u, \phi) - E(\theta; v, \phi) = E(\theta; u, v).$$

For the proof we adapt the arguments in [21, Proposition 2.5] We will refer to (4.5) as the *cocycle property*.

Proof. By [23, Corollary 3.16] for $k \in \{0, \dots, n\}$,

$$\int_X \theta_u^k \wedge \theta_\phi^{n-k} = \int_X \theta_v^k \wedge \theta_\phi^{n-k}$$

so we can assume that $\max(u, v) \leq \phi \leq 0$. For $C > 0$, we set $u^C := \max(u, \phi - C)$, $v^C := \max(v, \phi - C)$. By locality we have $\{0, \dots, n\}$

$$\mathbf{1}_{\{u > \phi - C\}} \theta_{u^C}^n = \mathbf{1}_{\{u > \phi - C\}} \theta_u^n,$$

and more generally, for any $k \in \{0, \dots, n\}$

$$(4.6) \quad \mathbf{1}_{\{\min(u,v) > \phi - C\}} \theta_{u^C}^k \wedge \theta_{v^C}^{n-k} = \mathbf{1}_{\{\min(u,v) > \phi - C\}} \theta_u^k \wedge \theta_v^{n-k}.$$

Since $\int_X \theta_u^n = \int_X \theta_{u^C}^n$ we can write

$$(4.7) \quad \begin{aligned} \lim_{C \rightarrow +\infty} C \int_{\{u \leq \phi - C\}} \theta_{u^C}^n &= \lim_{C \rightarrow +\infty} C \int_{\{u \leq \phi - C\}} \theta_u^n \\ &\leq \lim_{C \rightarrow +\infty} \int_{\{u \leq \phi - C\}} (\phi - u) \theta_u^n = \int_{\{u = -\infty\}} (\phi - u) \theta_u^n = 0. \end{aligned}$$

By [23, Theorem 5.3] we have that

$$E(\theta; u^C, \phi) - E(\theta; v^C, \phi) = E(\theta; u^C, v^C).$$

By [20, Lemma 4.12] we already know that $E(\theta; u^C, \phi)$ and $E(\theta; v^C, \phi)$ decrease to $E(\theta; u, \phi)$ and $E(\theta; v, \phi)$, respectively. We want to prove that $E(\theta; u^C, v^C)$ decreases to $E(\theta; u, v)$, i.e. that for any $k \in \{0, \dots, n\}$,

$$(4.8) \quad \lim_{C \rightarrow +\infty} \int_X (u^C - v^C) \theta_{u^C}^k \wedge \theta_{v^C}^{n-k} = \int_X (u - v) \theta_u^k \wedge \theta_v^{n-k}.$$

Clearly, it suffices to check that

$$(4.9) \quad \lim_{C \rightarrow +\infty} \int_X (u^C - \phi) \theta_{u^C}^k \wedge \theta_{v^C}^{n-k} = \int_X (u - \phi) \theta_u^k \wedge \theta_v^{n-k}$$

and that

$$(4.10) \quad \lim_{C \rightarrow +\infty} \int_X (\phi - v^C) \theta_{u^C}^k \wedge \theta_{v^C}^{n-k} = \int_X (\phi - v) \theta_u^k \wedge \theta_v^{n-k}.$$

In the following we prove (4.9). The same arguments will give (4.10).

We decompose the integral into two parts $\int_{\{\min(u,v) > \phi - C\}}$ and $\int_{\{\min(u,v) \leq \phi - C\}}$, by the locality property (4.6) we have

$$\begin{aligned} &\int_{\{\min(u,v) > \phi - C\}} (u^C - \phi) \theta_{u^C}^k \wedge \theta_{v^C}^{n-k} \\ &= \int_{\{\min(u,v) > \phi - C\}} (u - \phi) \theta_u^k \wedge \theta_v^{n-k} \rightarrow \\ &\quad \int_X (u - \phi) \theta_u^k \wedge \theta_v^{n-k} \end{aligned}$$

as $C \rightarrow +\infty$. Noting that $\{\min(u, v) \leq \phi - C\} \subseteq \{u \leq \phi - C\} \cup \{v \leq \phi - C\}$, we see that proving (4.9) boils down to showing that

$$(4.11) \quad \lim_{C \rightarrow +\infty} C \int_{\{u \leq \phi - C\}} \theta_{u^C}^k \wedge \theta_{v^C}^{n-k} = 0, \text{ and } \lim_{C \rightarrow +\infty} C \int_{\{v \leq \phi - C\}} \theta_{u^C}^k \wedge \theta_{v^C}^{n-k} = 0, \forall k.$$

We will prove the first equality and the same arguments apply to prove the second one. Observing that $\phi - C \leq v^C \leq \phi$ we have the inclusion

$$\{u \leq \phi - C\} \subset \left\{ u^C \leq \frac{v^C + \phi - C}{2} \right\} \subset \{u \leq \phi - C/2\}.$$

Using the partial comparison principle [23, Proposition 3.22] and that

$$2^{k-n} \theta_{v^C}^{n-k} \leq \left(\frac{\theta}{2} + dd^c \frac{v^C}{2} \right)^{n-k} \leq \theta_{\frac{v^C + \phi - C}{2}}^{n-k}$$

we get

$$\begin{aligned}
C \int_{\{u \leq \phi - C\}} \theta_{u^C}^k \wedge \theta_{v^C}^{n-k} &\leq 2^{n-k} C \int_{\{u^C \leq \frac{v^C + \phi - C}{2}\}} \theta_{u^C}^k \wedge \theta_{\frac{v^C + \phi - C}{2}}^{n-k} \\
&\leq 2^{n-k} C \int_{\{u^C \leq \frac{v^C + v_\theta - C}{2}\}} \theta_{u^C}^n \\
&\leq 2^{n-k} C \int_{\{u \leq \phi - C/2\}} \theta_{u^C}^n
\end{aligned}$$

From the above and (4.7) we obtain (4.11), completing the proof. \square

4.1. From model to divisorial singularities. We introduce a set of model potentials with which we will work through the paper.

Definition 4.3. Let $\mathcal{N}_\theta \subset \text{PSH}(X, \theta)$ be the set of all model potential ϕ such that there exists a modification (i.e. bimeromorphic holomorphic map) $\pi : Y \rightarrow X$ from Y a compact Kähler manifold of dimension n such that

$$\pi^* \theta_\phi = [F] + S$$

for an effective \mathbb{R} -divisor F and a closed, positive current S with minimal singularities, representing a big and nef class.

Remark 4.4. In the definition above we restrict our attention to model potentials since if a θ -psh function u admits a modification $\pi : Y \rightarrow X$ such that $\pi^* \theta_u = [F] + S$, for an effective \mathbb{R} -divisor F and a closed, positive current S with minimal singularities, representing a big and nef class, then u is of model type.

Indeed, since $\pi^* \theta_u = [F] + S$ and $\{S\}$ is big, we have $\int_X \theta_u^n = \int_Y S^n > 0$. By [20, Theorem 1.3] $u \in \mathcal{E}(X, \theta, P[u])$. Moreover, by [23, Lemma 5.1] we infer that u and $P[u]$ have the same multiplier ideal sheaf and in particular $u \circ \pi$ and $P[u] \circ \pi$ have the same Lelong numbers [11, Theorem A]. Thus $\pi^* \theta_{P[u]} - [F]$ is a positive and closed (1,1)-current in the cohomology class $\{S\}$. Thus $\pi^* \theta_{P[u]} = [F] + \tilde{S}$, where $\{\tilde{S}\} = \{S\}$. Since $P[u]$ is less singular than u , \tilde{S} is less singular than S , hence it has minimal singularities.

The following lemma lists some properties of \mathcal{N}_θ .

Lemma 4.5. *The followings hold:*

i) *For $\phi \in \mathcal{N}_\theta$ we have*

$$m_\phi = \text{Vol}(S)$$

where $\text{Vol}(S)$ is the volume of the nef and big class $\{S\}$. In particular $m_\phi > 0$, and letting η be a smooth and closed form representing $\{S\}$ we have $m_\phi = \int_Y \eta^n$.

ii) *if $\psi \in \text{PSH}(X, \theta)$ is a function with analytic singularities such that $\int_X \theta_\psi^n > 0$, then $\phi := P_\theta[\psi] \in \mathcal{N}_\theta$ and any associated big and nef class admits bounded potentials.*

Proof. We start proving (i). As the non-pluripolar product does not charge pluripolar sets such as divisors we have

$$m_\phi = \int_X \theta_\phi^n = \int_Y S^n = \text{Vol}(S)$$

where the last equality follows from the fact that S has minimal singularities. As the class $\{S\}$ is big we deduce that $m_\phi > 0$ and the equality $\text{Vol}(S) = \int_Y \eta^n$ for a smooth and closed form η

representing $\{S\}$ follows from the fact that the class is nef.

We now prove (ii). The local holomorphic functions f_1, \dots, f_N such that

$$\psi = c \log \sum_{j=1}^N |f_j|^2 + g$$

as in (2.1) generates a coherent ideal sheaf \mathcal{I} . Taking a log resolution of (X, \mathcal{I}) yields a modification $\pi : Y \rightarrow X$ such that

$$\pi^* \theta_\psi = T + c[F]$$

where F is an effective divisor such that $\mathcal{O}_Y(-F) = \pi^{-1}\mathcal{I}$ and $T = \eta + dd^c \tilde{\psi}$ for a smooth and closed form η and for $\tilde{\psi} \in \text{PSH}(Y, \eta) \cap L^\infty(Y)$.

Indeed,

$$\psi \circ \pi = c \log \sum_{j=1}^N |f_j \circ \pi|^2 + g \circ \pi = c \log \left(\prod_{l=1}^M |u_l|^2 \sum_{j=1}^N |v_j|^2 \right) + g \circ \pi = c \sum_{l=1}^M \log |u_l|^2 + c \log \sum_{j=1}^N |v_j|^2 + g \circ \pi$$

where $u_l, l = 1, \dots, M$ are the holomorphic functions which divides all the $f_j \circ \pi$ and which define the divisor F . Note that $v_j, j = 1, \dots, N$, do not any common zeros by construction, and so $\log \sum_j |v_j|^2$ is bounded.

Next, by [23, Proposition 5.24], we can infer that ψ has model type singularities, i.e. $|P_\theta[\psi] - \psi| \leq C, C > 0$. Set $\phi := P_\theta[\psi]$. By [23, Lemma 5.1] we infer that ψ and ϕ have the same multiplier ideal sheaf and in particular $\psi \circ \pi$ and $\phi \circ \pi$ have the same Lelong numbers [11, Theorem A]. Thus $\pi^* \theta_\phi - c[F]$ is a positive and closed $(1, 1)$ -current in the cohomology class $\{T\}$. Since $\psi \circ \pi$ and $\phi \circ \pi$ have the same singularities, we have

$$\pi^* \theta_\phi = S + c[F]$$

for a closed and positive current S with the same singularities of T , i.e. $S = \eta + dd^c \tilde{\phi}$ for $\tilde{\phi} \in \text{PSH}(Y, \eta) \cap L^\infty(Y)$. In particular S is a current with minimal singularities, hence by combining [9, Propositions 3.2 and 3.6] we find that the class η is nef. Finally we have

$$0 < \int_X \theta_\psi^n = \int_X \theta_\phi^n = \int_Y S^n = \text{Vol}(S)$$

by the same calculation performed in the proof of (i). Hence $\{S\}$ is big, which concludes the proof. \square

Now, suppose $\phi \in \mathcal{N}_\theta$, and consider $\pi : Y \rightarrow X$ a modification such that

$$(4.12) \quad \pi^* \theta_\phi = \eta_{\tilde{\phi}} + [F]$$

for an effective \mathbb{R} -divisor F , for η closed smooth $(1, 1)$ -form such that $\{\eta\}$ is big and nef, and for $\tilde{\phi} \in \text{PSH}(Y, \eta)$ normalized such that $\sup_Y \tilde{\phi} = 0$. Let also $\tilde{\omega}$ be a fixed Kähler form on Y normalized such that $\int_Y \tilde{\omega} = 1$.

Let E_1, \dots, E_m be the exceptional divisors of $\pi : Y \rightarrow X, a_j > 0$ such that $K_{Y/X} = \sum_{j=1}^m a_j E_j$. Recall that at the level of cohomology classes we have

$$(4.13) \quad K_Y = \pi^* K_X + K_{Y/X}.$$

Let h_j be smooth metrics on $\mathcal{O}_Y(E_j)$ such that the curvature form $\Theta := \sum_{j=1}^m a_j \Theta(h_j)$ satisfies

$$\Theta = \pi^* \text{Ric}(\omega) - \text{Ric}(\tilde{\omega}).$$

Then there exist holomorphic sections s_j of $\mathcal{O}_Y(E_j)$ such that

$$(4.14) \quad \text{Ric}(\pi^* \omega) = \pi^* \text{Ric}(\omega) - \Theta - \sum_{j=1}^m a_j dd^c \log |s_j|_{h_j}^2 = \text{Ric}(\tilde{\omega}) - \sum_{j=1}^m a_j dd^c \log |s_j|_{h_j}^2.$$

Set $f := \sum_{j=1}^m a_j \log |s_j|_{h_j}^2$. Then $\Theta + dd^c f = \sum_{j=1}^m a_j [E_j]$ and $\pi^* \omega^n = e^f \tilde{\omega}^n$. Also, observe that $\pi|_{Y \setminus \cup_j E_j}$ is a biholomorphism.

The goal of this section is to prove the following result:

Theorem 4.6. *Let $\varphi \in \mathcal{E}(X, \theta, \phi)$ such that $\theta_\varphi^n = m_\phi \omega^n$, let $u_0, u_1 \in \text{PSH}(X, \theta)$ with ϕ -relative minimal singularities and let $(u_t)_{t \in [0,1]}$ be the psh geodesic connecting u_0 and u_1 . Then*

$$(4.15) \quad \mathcal{M}_{\theta, \varphi}(u_t) \leq t \mathcal{M}_{\theta, \varphi}(u_1) + (1-t) \mathcal{M}_{\theta, \varphi}(u_0) + \frac{n \|u_0 - u_1\|_\infty}{2m_\phi} \{\eta^{n-1}\} \cdot K_{Y/X}$$

for any $t \in [0, 1]$.

In this first Lemma we show how to go from (X, θ, ϕ) to (Y, η) and back.

Lemma 4.7. *There exists a unique map*

$$\mathbf{L} : \text{PSH}(X, \theta, \phi) \rightarrow \text{PSH}(Y, \eta)$$

such that for $u \in \text{PSH}(X, \theta)$ we have

$$u \circ \pi + \tilde{\phi} = \mathbf{L}(u) + \phi \circ \pi$$

where the function $\tilde{\phi}$ is defined in (4.12), (Sometimes for convenience we will simply write $\mathbf{L}(u) := (u - \phi) \circ \pi + \tilde{\phi}$; note however that this equality makes sense only at points where $\phi \circ \pi \neq -\infty$). Moreover:

- (i) \mathbf{L} is a bijection;
- (ii) if $t \rightarrow u_t$ is a psh geodesic joining $u_0, u_1 \in \mathcal{E}^1(X, \theta, \phi)$ then $t \rightarrow v_t := \mathbf{L}(u_t)$ is a psh geodesic in $\mathcal{E}^1(Y, \eta)$ joining $v_0 := \mathbf{L}(u_0), v_1 := \mathbf{L}(u_1)$;
- (iii) The map \mathbf{L} produces a bijection between $\mathcal{E}(X, \theta, \phi)$ and $\mathcal{E}(Y, \eta)$ (resp. $\mathcal{E}^p(X, \theta, \phi)$ and $\mathcal{E}^p(Y, \eta)$ for any $p \geq 1$);
- (iv) $E(\theta; u, w) = E(\eta; \mathbf{L}(u), \mathbf{L}(w))$ for any $u, w \in \mathcal{E}^1(X, \theta, \phi)$;
- (v) $\text{Ent}_\theta(u, w) = \text{Ent}_\eta(\mathbf{L}(u), \mathbf{L}(w))$ for any $u, w \in \mathcal{E}(X, \theta, \phi)$.
- (vi) Let $u_X \in \text{PSH}(X, \theta, \phi)$ and $v_Y := \mathbf{L}(u_X)$. If the current η_{v_Y} has good Ricci curvature then θ_{u_X} does too. Furthermore

$$(4.16) \quad \text{Ric}(\eta_{v_Y}) = \pi^*(\text{Ric}(\theta_{u_X})) - [K_{Y/X}],$$

and

$$(4.17) \quad E_{\text{Ric}(\theta_{u_X})}(\theta; u, w) = E_{\text{Ric}(\eta_{v_Y})}(\eta; \mathbf{L}(u), \mathbf{L}(w))$$

for any u, w with ϕ -relative minimal singularities.

In the above statement the energy functionals in (iv) and in (4.17) are defined in (4.1) and (4.2).

Proof. The proof of the first three points proceeds as in the Kähler case ([60, Lemma 4.6] and [58, Proposition 3.10]). We give nevertheless some details for the reader's convenience.

We start proving that \mathbf{L} is well defined. In fact, set $\tilde{v} := (u - \phi) \circ \pi + \tilde{\phi}$, then by (4.12)

$$\eta + dd^c((u - \phi) \circ \pi + \tilde{\phi}) = \eta_{\tilde{v}} + \pi^* \theta_u - \pi^* \theta_\phi = \pi^* \theta_u - [F].$$

The above means that the restriction of $\eta_{\tilde{v}}$ to $Y \setminus F$ is positive, moreover, since u is more singular than ϕ , we infer that \tilde{v} is bounded from above on $Y \setminus F$. Hence there exists a unique η -psh function v on Y which equals \tilde{v} almost everywhere with respect to the Lebesgue measure. Therefore the quasi-psh functions $u \circ \pi + \tilde{\phi}$ and $v + \phi \circ \pi$ coincide almost everywhere on Y , hence they coincide everywhere.

By construction \mathbf{L} is clearly injective. We now show the surjectivity. For any $v \in \text{PSH}(Y, \eta)$ we claim that $v + \phi \circ \pi - \tilde{\phi}$ coincides almost everywhere with a $\pi^* \theta$ -psh. Indeed by (4.12), $\pi^* \theta + dd^c(v + \phi \circ$

$\pi - \tilde{\phi}) = \eta_v + [F]$. Thus, since the fibers of π are connected and compact, there exists $u \in \text{PSH}(X, \theta)$ such that $u \circ \pi := v + \phi \circ \pi - \tilde{\phi}$ almost everywhere on X (see for instance [8, Proposition 1.2.7.(ii)]). Using the same arguments as above we see that we must have $\mathbf{L}(u) = v$. This concludes the proof of (i).

As the non-pluripolar product does not put mass on pluripolar sets, we have

$$(4.18) \quad \pi^*(\theta_u^j \wedge \theta_w^{n-j}) = (\pi^*\theta_u)^j \wedge (\pi^*\theta_w)^{n-j} = \eta_{\mathbf{L}(u)}^j \wedge \eta_{\mathbf{L}(w)}^{n-j}$$

for any $j = 0, \dots, n$ and any $u \in \text{PSH}(X, \theta, \phi)$. Thus

$$\int_X (u-w)\theta_u^j \wedge \theta_w^{n-j} = \int_{Y \setminus F} ((u-\phi) \circ \pi + \tilde{\phi} - (w-\phi) \circ \pi - \tilde{\phi}) \eta_{\mathbf{L}(u)}^j \wedge \eta_{\mathbf{L}(w)}^{n-j} = \int_Y (\mathbf{L}(u) - \mathbf{L}(w)) \eta_{\mathbf{L}(u)}^j \wedge \eta_{\mathbf{L}(w)}^{n-j}.$$

Similarly, for any $p \geq 1$ we have

$$\int_X |u - \phi|^p \theta_u^n = \int_Y |\mathbf{L}(u) - \mathbf{L}(\phi)|^p \eta_{\mathbf{L}(u)}^n.$$

Then (iii) and (iv) follow.

Let us prove (ii). Let $p_X : X \times A \rightarrow X$ and $p_Y : Y \times A \rightarrow Y$ be the projections on the first factors, and consider

$$U(x, z) := u_{\log|z|}(x) \in \text{PSH}(X \times A, p_X^*\theta), \quad V(y, z) := v_{\log|z|}(y) = (u_{\log|z|}(x) - \phi(x)) \circ \pi + \tilde{\phi}(y) \in \text{PSH}(Y \times A, p_Y^*\eta).$$

Then we find:

$$(4.19) \quad (\pi \times \text{Id})^*(p_X^*\theta + dd_{(x,z)}^c U) = p_Y^*\eta + dd_{(y,z)}^c V + p_Y^*[F].$$

On the other hand, by assumption there exists $M > 0$ such that $\max(u_0, u_1) \leq \phi + M$; then by convexity $u_{\log|z|} \leq \phi + M$ for all $z \in A$. It follows that the function V is bounded from above and it is $p_Y^*\eta$ -psh on $Y \times A \setminus p_Y^{-1}(F)$. Then V extends to an $p_Y^*\eta$ -psh function on $Y \times A$, i.e. $t \rightarrow v_t$ (with $t = \log|z|$) is a psh subgeodesic. Note that the argument above says that \mathbf{L} produces a injective map from psh subgeodesics joining u_0, u_1 and psh subgeodesics joining v_0, v_1 . Such map is actually a bijection as the surjectivity follows reading (4.19) backwards, using the fact that F is effective and taking the pushforward by $\pi \times \text{Id}$. Moreover, this correspondence between psh subgeodesics respects the partial order \leq . Namely, for any couple of psh subgeodesics U_1, U_2 joining u_0, u_1 such that $U_1 \leq U_2$, the corresponding psh subgeodesics $V_1(y, z) = (\pi \times \text{Id})^*(U_1(x, t) - \phi(x)) + \tilde{\phi}(y)$, $V_2(y, z) = (\pi \times \text{Id})^*(U_2(x, t) - \phi(x)) + \tilde{\phi}(y)$ clearly satisfy $V_1 \leq V_2$, and vice-versa. The proof of (ii) follows from the maximality of psh geodesics.

As seen above, for any θ -psh function $u \in \text{PSH}(X, \theta, \phi)$ we have $\pi^*\theta_u^n = \eta_{\mathbf{L}(u)}^n$. Also, we already observed that $m_\phi = \text{Vol}(\eta)$. Then for any $u, w \in \mathcal{E}(X, \theta, \phi)$ we have $m_u = m_w = m_\phi = \text{Vol}(\eta)$. The entropy formula in (v) then follows from the first item of Lemma 3.3.

We now prove (vi). Let $p \in Y$, and Ω a nowhere zero holomorphic section of K_Y near p , and Ω' a nowhere zero local holomorphic section of K_X near $\pi(p)$. Note that $\pi^*\Omega' = h\Omega$ for some holomorphic function vanishing on each E_j . If $\eta_{v_Y}^n = g i^{n^2} \Omega \wedge \bar{\Omega}$ for some function $g \geq 0$, then on $X \setminus \pi(\cup_j E_j)$

$$(4.20) \quad \pi_* \eta_{v_Y}^n = \theta_{u_X}^n = (g \circ \pi^{-1}) i^{n^2} \pi_*(\Omega \wedge \bar{\Omega}) = (g \circ \pi^{-1}) |h \circ \pi^{-1}|^{-2} i^{n^2} \Omega' \wedge \bar{\Omega}'.$$

We then observe that $\log(g \circ \pi^{-1}) \in L^1(\omega^n)$ since

$$\int_X |\log(g \circ \pi^{-1})| \omega^n = \int_Y |\log g| e^f \tilde{\omega}^n$$

and the latter integral is finite since $\log g \in L^1(\tilde{\omega}^n)$ and e^f is bounded.

Therefore θ_{u_X} has well defined Ricci if so does η_{v_Y} . Moreover, from the identity in (4.20) we find that on $X \setminus \pi(\cup_j E_j)$

$$\text{Ric}(\theta_{u_X}) = \pi_* \text{Ric}(\eta_{v_Y}) + \pi_* dd^c \log |h|^2.$$

Note that, since $dd^c \log |h|^2 = 0$ on $Y \setminus \cup_j E_j$ and π is a biholomorphism there, we obtain that $\pi^* \text{Ric}(\theta_{u_X}) - \text{Ric}(\eta_{v_Y}) = 0$ on $Y \setminus \cup_j E_j$, or equivalently $\pi^* \text{Ric}(\theta_{u_X}) - \text{Ric}(\eta_{v_Y})$ is a $(1, 1)$ -current supported on the singular locus. Since $K_Y = \pi^* K_X + \sum_j a_j E_j$, $\pi^* \{\text{Ric}(\theta_{u_X})\} = -\pi^* c_1(K_X)$ and $\{\text{Ric}(\eta_{v_Y})\} = -c_1(K_Y)$ we obtain (4.16). Observe that the last claim holds since the information on the cohomology transfers at the level of forms since E_j are numerically independent. Indeed by Hironaka [46] any modification $\pi : Y \rightarrow X$ can be dominated by a map $p : Z \rightarrow X$ given by a sequence of blow-ups along smooth centers, i.e. there exists $\tau : Z \rightarrow Y$ such that $p = \pi \circ \tau$. In particular any linear combination of (classes of) π -exceptional divisors becomes a linear combination of p -exceptional divisors after pulling-back through τ , thus the numerical independence follows from [41, Page 605].

Also, since the non-pluripolar product does not put mass on the pluripolar sets, we deduce that for any $k = 0, \dots, n-1$

$$\int_X (u - w) \theta_u^k \wedge \theta_w^{n-k-1} \wedge \text{Ric}(\theta_{u_X}) = \int_Y (\mathbf{L}(u) - \mathbf{L}(w)) \eta_{\mathbf{L}(u)}^k \wedge \eta_{\mathbf{L}(w)}^{n-k-1} \wedge \text{Ric}(\eta_{v_Y})$$

for any $u, w \in \mathcal{E}(X, \theta, \phi)$ with ϕ -relative minimal singularities. This yields (4.17) and concludes the proof. \square

Remark 4.8. It is worth to mention that given two positive real closed $(1, 1)$ -currents T (on X) and S (on Y) whose cohomology classes are big and such that $\pi^* T^n = S^n$, the same arguments in the above lemma ensure that T has good Ricci curvature if so does S .

Proposition 4.9. *Let $\varphi \in \mathcal{E}(X, \theta, \phi)$ such that $\theta_\varphi^n = m_\phi \omega^n$, and let η_w be such that $\eta_w^n = m_\phi \tilde{\omega}^n$. Using the same notations of Lemma 4.7, we set $\hat{\varphi} := \mathbf{L}(\varphi)$ and $v := \mathbf{L}(u)$ for $u \in \mathcal{E}(X, \theta, \phi)$ with ϕ -relative minimal singularities. If $\text{Ent}_\theta(u) < +\infty$, then*

$$(4.21) \quad \mathcal{M}_{\theta, \varphi}(u) = \mathcal{M}_{\eta, w}(v) - \mathcal{M}_{\eta, w}(\hat{\varphi}) + \frac{n}{\text{Vol}(\eta)} \sum_{E_j \notin E_{nK}(\eta)} a_j \text{Vol}(\eta|_{E_j}) \left\{ E(\eta; v, \hat{\varphi}) - E_{E_j}(\eta|_{E_j}; v|_{E_j}, \hat{\varphi}|_{E_j}) \right\}$$

where

$$E_{E_j}(\eta|_{E_j}; v|_{E_j}, \hat{\varphi}) := \frac{1}{n \text{Vol}(\eta|_{E_j})} \sum_{k=0}^{n-1} \int_{E_j} (v - \hat{\varphi}) \eta_v^k \wedge \eta_{\hat{\varphi}}^{n-k-1}$$

is the energy relative to the smooth submanifold E_j .

Here $\mathcal{M}_{\theta, \varphi}$ and $\mathcal{M}_{\eta, w}$ denote the Mabuchi functionals relative to $\varphi, \hat{\varphi}$ defined in (4.4).

Observe that $\mathcal{M}_{\eta, w}(\hat{\varphi}) < +\infty$ since $\eta_{\hat{\varphi}}^n = \pi^* \omega^n = e^f \tilde{\omega}^n$ has L^p density for $p > 1$ with respect to $\eta_w^n = \text{Vol}(\eta) \tilde{\omega}^n$.

It is crucial to stress that $\int_{E_j} \eta^{n-1} > 0$ if and only if $E_j \notin E_{nK}(\eta)$ by the main result in [16]. In particular for such j , $\{\eta|_{E_j}\}$ is a big and nef class on E_j and $\text{Vol}(\eta|_{E_j}) = \int_{E_j} \eta^{n-1}$. Moreover, in this case any $v \in \text{PSH}(Y, \eta)$ with minimal singularities restricts to a function $v|_{E_j} \in \text{PSH}(E_j, \eta|_{E_j})$ with full Monge-Ampère mass as a consequence of the following Lemma.

Lemma 4.10. *Let η be a smooth and closed $(1, 1)$ -form representing a big and nef class and let $v \in \text{PSH}(Y, \eta)$ with minimal singularities. Assume also that $Z \subset Y$ is a positive dimensional connected submanifold (we also allow $Z = Y$) such that $Z \notin E_{nK}(\eta)$ and let Γ be a semipositive smooth and closed (p, p) -form, $0 \leq p \leq \dim Z$. Then $v|_Z \in \text{PSH}(Z, \eta|_Z)$ satisfies*

$$\int_Z \langle \Gamma|_Z \wedge (\eta|_Z + dd^c v|_Z)^{\dim Z - p} \rangle = \int_Z \Gamma|_Z \wedge \eta|_Z^{\dim Z - p},$$

where at the LHS we have the non-pluripolar product on Z , while at the RHS we consider the usual wedge product between smooth forms on Z .

Proof. To lighten notations, we merely write $\langle \Gamma \wedge (\eta + dd^c v)^{\dim Z-p} \rangle$ instead of $\langle \Gamma|_Z \wedge (\eta|_Z + dd^c v|_Z)^{\dim Z-p} \rangle$. More generally, we drop the notation for the restriction over Z .

By [9, Theorem 3.17] there exists a function $\psi \in \text{PSH}(Y, \eta)$ with analytic singularities along the non-Kähler locus $E_{nK}(\eta)$ such that $T := \eta + dd^c \psi \geq \varepsilon \tilde{\omega}$ where $\tilde{\omega}$ is a fixed Kähler form on Y . By nefness of $\{\eta\}$, for any $\delta > 0$ there exists also a Kähler form $\eta + \delta \tilde{\omega} + dd^c \varphi_\delta$. Then we set $u_\delta := \frac{\varepsilon}{\delta + \varepsilon} \varphi_\delta + \frac{\delta}{\delta + \varepsilon} \psi$. Such a function is η -psh as

$$\eta + dd^c u_\delta = \frac{\varepsilon}{\delta + \varepsilon} (\eta + \delta \tilde{\omega} + dd^c \varphi_\delta) + \frac{\delta}{\delta + \varepsilon} (\eta - \varepsilon \tilde{\omega} + dd^c \psi) \geq 0.$$

Since by assumption $Z \not\subset E_{nK}(\eta)$, the function $u_\delta|_Z$ is a well-defined $\eta|_Z$ -psh function. Thus as $v \in \text{PSH}(Y, \eta)$ has minimal singularities we have $v|_Z \geq u_\delta|_Z - C$ and from [20, Theorem 1.1] it follows that

$$\begin{aligned} \int_Z \langle \Gamma \wedge (\eta + dd^c v)^{\dim Z-p} \rangle &\geq \int_Z \langle \Gamma \wedge (\eta + dd^c u_\delta)^{\dim Z-p} \rangle \\ &\geq \left(\frac{\varepsilon}{\delta + \varepsilon} \right)^{\dim Z-p} \int_Z \langle \Gamma \wedge (\eta + \delta \tilde{\omega} + dd^c \varphi_\delta)^{\dim Z-p} \rangle \\ &\geq \left(\frac{\varepsilon}{\delta + \varepsilon} \right)^{\dim Z-p} \int_Z \langle \Gamma \wedge \eta^{\dim Z-p} \rangle \end{aligned}$$

where the last equality follows from Stokes' theorem and the positivity of $\tilde{\omega}$. Letting $\delta \rightarrow 0$, we find that

$$\int_Z \langle \Gamma \wedge (\eta + dd^c v)^{\dim Z-p} \rangle \geq \int_Z \langle \Gamma \wedge \eta^{\dim Z-p} \rangle.$$

For the reverse inequality we observe that for any $\varepsilon > 0$, φ_δ is with minimal singularities in $\eta + \varepsilon \tilde{\omega}$ so, again from [20, Theorem 1.1] and Stokes' Theorem, it follows that

$$\begin{aligned} \int_Z \langle \Gamma \wedge (\eta + dd^c v)^{\dim Z-p} \rangle &\leq \int_Z \langle \Gamma \wedge (\eta + \varepsilon \tilde{\omega} + dd^c v)^{\dim Z-p} \rangle \\ &\leq \int_Z \langle \Gamma \wedge (\eta + \varepsilon \tilde{\omega} + dd^c \varphi_\delta)^{\dim Z-p} \rangle \\ &= \int_Z \langle \Gamma \wedge (\eta + \varepsilon \tilde{\omega})^{\dim Z-p} \rangle \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ then concludes the proof. □

We are now ready to prove Proposition 4.9.

Proof of Proposition 4.9. We first assume that θ_u^n has bounded density, i.e. $\theta_u^n = g\omega^n$ with g bounded.

Step 1: A first formula connecting $\mathcal{M}_{\theta, \varphi}(u)$ and $\mathcal{M}_{\eta, w}(v)$.

Lemma 4.7 gives

$$E(\theta; u, \varphi) = E(\eta; v, \hat{\varphi}).$$

Moreover, as $\pi^* \text{Ric}(\omega) = \text{Ric}(\tilde{\omega}) + \Theta$ by (4.14), $\text{Ric}(\omega) = \text{Ric}(\theta_\varphi)$, $\text{Ric}(\tilde{\omega}) = \text{Ric}(\eta_w)$ we similarly have

$$\begin{aligned} E_{\text{Ric}(\theta_\varphi)}(\theta; u, \varphi) &= E_{\pi^* \text{Ric}(\omega)}(\eta; v, \hat{\varphi}) \\ &= E_{\text{Ric}(\tilde{\omega})}(\eta; v, \hat{\varphi}) + E_\Theta(\eta; v, \hat{\varphi}) \\ &= E_{\text{Ric}(\eta_w)}(\eta; v, \hat{\varphi}) + E_\Theta(\eta; v, \hat{\varphi}). \end{aligned}$$

Furthermore, since $\eta_{\hat{\varphi}}^n = m_{\phi} e^f \tilde{\omega}^n$ for $f = \sum_{j=1}^n a_j \log |s_j|_{h_j}^2$, by Lemma 4.7 and the definition of the entropy we have

$$\text{Ent}_{\theta}(u, \varphi) = \text{Ent}_{\eta}(v, \hat{\varphi}) = \text{Ent}_{\eta}(v) - \frac{1}{\text{Vol}(\eta)} \int_Y f \eta_v^n.$$

Combining all the above and using the cocycle property for the energies (e.g. $E(\eta; v, \hat{\varphi}) = E(\eta; v, w) - E(\eta; \hat{\varphi}, w)$), we obtain

$$(4.22) \quad \begin{aligned} \mathcal{M}_{\theta, \varphi}(u) &= \bar{S}_{\varphi} E(\eta; v, \hat{\varphi}) - n E_{\text{Ric}(\eta_w)}(\eta; v, \hat{\varphi}) - n E_{\Theta}(\eta; v, \hat{\varphi}) + \text{Ent}_{\eta}(v) - \frac{1}{\text{Vol}(\eta)} \int_Y f \eta_v^n \\ &= \mathcal{M}_{\eta, w}(v) - \mathcal{M}_{\eta, w}(\hat{\varphi}) + (\bar{S}_{\varphi} - \bar{S}_w) E(\eta; v, \hat{\varphi}) - n E_{\Theta}(\eta; v, \hat{\varphi}) + \text{Ent}_{\eta}(\hat{\varphi}) - \frac{1}{\text{Vol}(\eta)} \int_Y f \eta_v^n. \end{aligned}$$

Step 2: The twisted energy $E_{\Theta}(\eta; v, \hat{\varphi})$.

Using the same ideas in the proof of [36, Theorem 4.2], we approximate $v, \hat{\varphi}$ by decreasing sequences $v_j, \hat{\varphi}_j$ of bounded $(\eta + \varepsilon_j \tilde{\omega})$ -psh functions such that the respective entropy converges and $v_j - \hat{\varphi}_j$ is uniformly bounded.

Observe that $\eta_{\hat{\varphi}}^n = m_{\phi} e^f \tilde{\omega}^n$ and $\eta_v^n = \pi^* \theta_u^n = (g \circ \pi) \pi^* \omega^n = g \circ \pi e^f \tilde{\omega}^n$. In particular, the measures $\eta_{\hat{\varphi}}^n, \eta_v^n$ both have bounded density.

We set $\eta_j := \eta + \frac{1}{j} \tilde{\omega}$, and let $\alpha > 0$ be small enough so that $\sup_{w \in \text{PSH}(Y, \eta)} \int_Y e^{-2\alpha(w - \sup_Y w)} \tilde{\omega}^n < \infty$. We then define $v_j \in \mathcal{E}(Y, \eta_j), \hat{\varphi}_j \in \mathcal{E}(Y, \eta_j)$ as the unique bounded solutions of

$$(4.23) \quad \begin{aligned} (\eta_j + dd^c v_j)^n &= e^{\alpha(v_j - v)} (\eta + dd^c v)^n = e^{\alpha(v_j - v)} g \circ \pi e^f \tilde{\omega}^n \\ (\eta_j + dd^c \hat{\varphi}_j)^n &= e^{\alpha(\hat{\varphi}_j - \hat{\varphi})} (\eta + dd^c \hat{\varphi})^n = e^{\alpha(\hat{\varphi}_j - \hat{\varphi})} m_{\phi} e^f \tilde{\omega}^n. \end{aligned}$$

Note that, since $\{\eta_j\}$ is a Kähler class, the existence of bounded $v_j, \hat{\varphi}_j$ follows from [10] since $e^{-\alpha(v - \sup_X v)} g \circ \pi e^f$ and $e^{-\alpha(\hat{\varphi} - \sup_X \hat{\varphi})} e^f$ are in L^2 .

From the comparison principle (see for instance [21, Lemma 2.5]) we obtain that $v_j, \hat{\varphi}_j$ are decreasing sequences converging respectively to v and $\hat{\varphi}$. Moreover [33, Theorem 1.9] implies that v_j and $\hat{\varphi}_j$ are uniformly bounded, from which we deduce that $v_j - \hat{\varphi}_j$ is uniformly bounded.

As the functions $v_j, \hat{\varphi}_j$ are bounded, and $v, \hat{\varphi}$ have minimal singularities, it follows from Lemma 4.10 that for any smooth closed semipositive $(1, 1)$ -form Γ and for any $k = 0, \dots, n-1$ we have

$$(4.24) \quad \int_Y \langle \Gamma \wedge \eta_{j, v_j}^k \wedge \eta_{j, \hat{\varphi}_j}^{n-k-1} \rangle = \int_Y \Gamma \wedge \eta_j^{n-1} \longrightarrow \int_Y \Gamma \wedge \eta^{n-1} = \int_Y \langle \Gamma \wedge \eta_v^k \wedge \eta_{\hat{\varphi}}^{n-k-1} \rangle,$$

as $j \rightarrow +\infty$. In the above, we emphasized by $\langle \dots \rangle$ the use of the non-pluripolar product vs the usual wedge product between smooth forms.

Thus, writing Θ as difference of two semipositive closed smooth $(1, 1)$ -forms it follows from Theorem 2.2 that for any $k = 0, \dots, n-1$,

$$(4.25) \quad \int_Y (v_j - \hat{\varphi}_j) \Theta \wedge \eta_{j, v_j}^k \wedge \eta_{j, \hat{\varphi}_j}^{n-k-1} \longrightarrow \int_Y (v - \hat{\varphi}) \Theta \wedge \eta_v^k \wedge \eta_{\hat{\varphi}}^{n-k-1}$$

as $j \rightarrow +\infty$.

It also follows from Lemma 4.10 that if $E_l \not\subset E_{nK}(\eta)$, for any $k = 0, \dots, n-1$ we have

$$\int_{E_l} \langle \eta_{j, v_j}^k \wedge \eta_{j, \hat{\varphi}_j}^{n-k-1} \rangle = \int_{E_l} \eta_j^{n-1} \longrightarrow \int_{E_l} \eta^{n-1} = \int_{E_l} \langle \eta_v^k \wedge \eta_{\hat{\varphi}}^{n-k-1} \rangle.$$

Therefore, Theorem 2.2 ensures that for any $E_l \not\subset E_{nK}(\eta)$ and any $k = 0, \dots, n-1$

$$(4.26) \quad \int_{E_l} (v_j - \hat{\varphi}_j) \eta_{j, v_j}^k \wedge \eta_{j, \hat{\varphi}_j}^{n-k-1} \longrightarrow \int_{E_l} (v - \hat{\varphi}) \eta_v^k \wedge \eta_{\hat{\varphi}}^{n-k-1}$$

as $j \rightarrow +\infty$. If instead $E_l \subset E_{nK}(\eta)$ then

$$(4.27) \quad \left| \int_{E_l} (v_j - \hat{\varphi}_j) \eta_{j,v_j}^k \wedge \eta_{j,\hat{\varphi}_j}^{n-k-1} \right| \leq \|v_j - \hat{\varphi}_j\|_\infty \text{Vol}(\eta_j|_{E_l}) \rightarrow 0$$

as $j \rightarrow +\infty$ since $v_j - \hat{\varphi}_j$ is uniformly bounded while $\text{Vol}(\eta_j|_{E_l}) \rightarrow 0$. The last claim is a consequence of the fact that $E_{nK}(\eta) = \bigcup \{V \subset X : \int_V \eta^{\dim V} = 0\}$ by [16].

Next, we recall that for any j , as $v_j, \hat{\varphi}_j$ are bounded, the non-pluripolar product coincides with the Bedford-Taylor wedge product. So, since $\Theta + dd^c f = [K_{Y/X}] = \sum_{l=1}^m a_l [E_l]$, we obtain

$$(4.28) \quad \langle \Theta \wedge \eta_{j,v_j}^k \wedge \eta_{j,\hat{\varphi}_j}^{n-k-1} \rangle = \Theta \wedge \eta_{j,v_j}^k \wedge \eta_{j,\hat{\varphi}_j}^{n-k-1} = \sum_{l=1}^m a_l [E_l] \wedge \eta_{j,v_j}^k \wedge \eta_{j,\hat{\varphi}_j}^{n-k-1} - dd^c f \wedge \eta_{j,v_j}^k \wedge \eta_{j,\hat{\varphi}_j}^{n-k-1},$$

where the products $[E_l] \wedge \eta_{j,v_j}^k \wedge \eta_{j,\hat{\varphi}_j}^{n-k-1}$, $dd^c f \wedge \eta_{j,v_j}^k \wedge \eta_{j,\hat{\varphi}_j}^{n-k-1}$ make sense thanks to [27, Section 2]. Observe also that by definition we have

$$dd^c f \wedge \eta_{j,v_j}^k \wedge \eta_{j,\hat{\varphi}_j}^{n-k-1} := dd^c \left(f \eta_{j,v_j}^k \wedge \eta_{j,\hat{\varphi}_j}^{n-k-1} \right).$$

The above is a (n, n) -current which acts on smooth functions. However [27, Theorem 2.2] implies that such action can be extended to bounded quasi-psh functions. It then follows (basically) by definition that

$$(4.29) \quad \begin{aligned} \int_X (v_j - \hat{\varphi}_j) dd^c f \wedge \eta_{j,v_j}^k \wedge \eta_{j,\hat{\varphi}_j}^{n-k-1} &= \int_X f dd^c (v_j - \hat{\varphi}_j) \wedge \eta_{j,v_j}^k \wedge \eta_{j,\hat{\varphi}_j}^{n-k-1} \\ &= \int_X f \eta_{j,v_j}^{k+1} \wedge \eta_{j,\hat{\varphi}_j}^{n-k-1} - \int_X f \eta_{j,v_j}^k \wedge \eta_{j,\hat{\varphi}_j}^{n-k}. \end{aligned}$$

Finally, by (4.23) and Monotone Convergence Theorem

$$(4.30) \quad \int_Y f \eta_{j,v_j}^n = \int_Y f e^{\alpha(v_j - v)} \eta_v^n \rightarrow \int_Y f \eta_v^n$$

and

$$\int_Y f \eta_{j,\hat{\varphi}_j}^n = \int_Y f e^{\alpha(\hat{\varphi}_j - \hat{\varphi})} \eta_{\hat{\varphi}}^n \rightarrow \int_Y f \eta_{\hat{\varphi}}^n.$$

Note also that the integrals at the RHS are finite since $\eta_v^n, \eta_{\hat{\varphi}}^n$ have bounded density and $f \in L^1(\tilde{\omega}^n)$. Combining all the above we have

$$\begin{aligned} n \text{Vol}(\eta) E_\Theta(\eta; v, \hat{\varphi}) &\stackrel{(4.25)}{=} \lim_{j \rightarrow +\infty} \sum_{k=0}^{n-1} \int_X (v_j - \hat{\varphi}_j) \langle \Theta \wedge \eta_{j,v_j}^k \wedge \eta_{j,\hat{\varphi}_j}^{n-k-1} \rangle \\ &\stackrel{(4.28)}{=} \lim_{j \rightarrow +\infty} \left(\sum_{l=1}^m a_l \sum_{k=0}^{n-1} \int_{E_l} (v_j - \hat{\varphi}_j) \eta_{j,v_j}^k \wedge \eta_{j,\hat{\varphi}_j}^{n-k-1} - \sum_{k=0}^{n-1} \int_Y (v_j - \hat{\varphi}_j) dd^c f \wedge \eta_{j,v_j}^k \wedge \eta_{j,\hat{\varphi}_j}^{n-k-1} \right) \\ &\stackrel{(4.29)}{=} \lim_{j \rightarrow +\infty} \left(\sum_{l=1}^m a_l \sum_{k=0}^{n-1} \int_{E_l} (v_j - \hat{\varphi}_j) \eta_{j,v_j}^k \wedge \eta_{j,\hat{\varphi}_j}^{n-k-1} - \int_Y f \eta_{j,v_j}^n + \int_Y f \eta_{j,\hat{\varphi}_j}^n \right) \\ &= \sum_{E_j \not\subset E_{nK}(\eta)} a_l \sum_{k=0}^{n-1} \int_{E_l} (v - \hat{\varphi}) \eta_v^k \wedge \eta_{\hat{\varphi}}^{n-k-1} - \int_Y f \eta_v^n + \int_Y f \eta_{\hat{\varphi}}^n \\ &= \sum_{E_j \not\subset E_{nK}(\eta)} n a_l \text{Vol}(\eta|_{E_l}) E|_{E_l}(\eta|_{E_l}; v|_{E_l}, \hat{\varphi}|_{E_l}) - \int_Y f \eta_v^n + \text{Vol}(\eta) \text{Ent}_\eta(\hat{\varphi}). \end{aligned}$$

where in the fourth equality we used (4.26), (4.27) and (4.30); in the last equality we also used that $\text{Ent}_\eta(\hat{\varphi}) = \frac{1}{\text{Vol}(\eta)} \int_Y f \eta_{\hat{\varphi}}^n$.

Thus, (4.22) writes as

$$(4.31) \quad \mathcal{M}_{\theta, \varphi}(u) = \mathcal{M}_{\eta, w}(v) - \mathcal{M}_{\eta, w}(\hat{\varphi}) + (\bar{S}_\varphi - \bar{S}_w)E(\eta; v, \hat{\varphi}) - \frac{n}{\text{Vol}(\eta)} \sum_{j=1}^m a_j \text{Vol}(\eta|_{E_j}) E|_{E_j}(\eta|_{E_j}; v|_{E_j}, \hat{\varphi}|_{E_j}).$$

Step 3: Computing $\bar{S}_\varphi - \bar{S}_w$.

Since $\pi^* \text{Ric}(\omega) = \text{Ric}(\tilde{\omega}) + \Theta$, by linearity, the proof of Lemma 4.7 and again Lemma 4.10 we have

$$\begin{aligned} \bar{S}_\varphi &= \frac{n}{m_\phi} \int_X \langle \text{Ric}(\omega) \wedge \theta_\varphi^{n-1} \rangle \\ &= \frac{n}{\text{Vol}(\eta)} \int_Y \langle (\text{Ric}(\tilde{\omega}) + \Theta) \wedge \eta_\phi^{n-1} \rangle \\ &= \frac{n}{\text{Vol}(\eta)} \int_Y \langle \text{Ric}(\tilde{\omega}) \wedge \eta_w^{n-1} \rangle + \frac{n}{\text{Vol}(\eta)} \int_Y \Theta \wedge \eta^{n-1} \\ &= \bar{S}_w + \frac{n}{\text{Vol}(\eta)} \{\Theta\} \cdot \{\eta^{n-1}\} \\ &= \bar{S}_w + \frac{n}{\text{Vol}(\eta)} \sum_{E_l \notin E_{nK}(\eta)} a_l \text{Vol}(\eta|_{E_l}) \end{aligned}$$

where in the above we used several times that $\hat{\varphi}$ and w have minimal singularities and $\{\Theta\} = \sum_{l=1}^m a_l \{E_l\}$.

Plugging this into (4.31) concludes the proof for $u \in \mathcal{E}(X, \theta, \phi)$ with ϕ -relative minimal singularities such that θ_u^n has bounded density with respect to ω^n .

Step 4: General case.

Let $u \in \mathcal{E}(X, \theta, \phi)$ with ϕ -relative minimal singularities such that $\text{Ent}_\theta(u) < +\infty$ and let $g \geq 0$ such that $\theta_u^n = g\omega^n$. We fix $\alpha > 0$ small enough so that $e^{-\alpha u} \in L^2(\omega^n)$ and we define $u_k \in \mathcal{E}(X, \theta, \phi)$ to be the solution of

$$\theta_{u_k}^n = e^{\alpha(u_k - u)} \min(g, k) \omega^n$$

(see [20, Theorem 1.4] for the existence of such potentials). Observe that u_k has ϕ -relative minimal potential by [22, Theorem A].

Set $v_k := \mathbf{L}(u_k)$. Our goal is to prove that for any $E_j \notin E_{nK}(\eta)$ we have

$$\begin{aligned} \mathcal{M}_{\theta, \varphi}(u_k) &\rightarrow \mathcal{M}_{\theta, \varphi}(u), \\ \mathcal{M}_{\eta, w}(v_k) &\rightarrow \mathcal{M}_{\eta, w}(v), \\ E(\eta; v_k, \hat{\varphi}) &\rightarrow E(\eta; v, \hat{\varphi}), \\ E_{E_j}(\eta|_{E_j}; v_k|_{E_j}, \hat{\varphi}|_{E_j}) &\rightarrow E_{E_j}(\eta|_{E_j}; v|_{E_j}, \hat{\varphi}|_{E_j}) \end{aligned}$$

as $k \rightarrow +\infty$. By Lemma 4.7 we have

$$(4.32) \quad \eta_{v_k}^n = e^{\alpha(v_k - v)} \min(g \circ \pi, k) e^f \tilde{\omega}^n$$

and $v_k \in \mathcal{E}(Y, \eta)$. Then it follows from the comparison principle in [21, Lemma 2.5] that $v_k \searrow \tilde{v}$ and $\tilde{v} \geq v$. We claim that $\tilde{v} = v$. This is indeed the case since from (4.32) we find that

$$\eta_{\tilde{v}}^n = e^{\alpha(\tilde{v} - v)} g \circ \pi e^f \tilde{\omega}^n = e^{\alpha(\tilde{v} - v)} \eta_v^n.$$

The comparison principle once again gives $\tilde{v} = v$. By construction, the previous fact is equivalent to $u_k \searrow u$. As u has ϕ -relative minimal singularities, we deduce that any difference $u_k - \tilde{u}$ for $\tilde{u} \in \mathcal{E}(X, \theta, \phi)$ with ϕ -relative minimal singularities is uniformly bounded in k , and the analogous holds for $v_k - \tilde{v}$, $\tilde{v} = \mathbf{L}(u)$. In particular, since φ has ϕ -relative minimal singularities and since

$\hat{\varphi}, w$ have minimal singularities, combining Lemma 4.10 with Theorem 2.2 we infer the following convergences of energies:

$$\begin{aligned} E(\theta; u_k, \varphi) &\longrightarrow E(\theta; u, \varphi), & E(\eta; v_k, \hat{\varphi}) &\longrightarrow E(\eta; v, \hat{\varphi}), & E(\eta; v_k, w) &\longrightarrow E(\eta; v, w), \\ E_{\text{Ric}(\theta_\varphi)}(\theta; u_k, \varphi) &\longrightarrow E_{\text{Ric}(\theta_\varphi)}(\theta; u, \varphi), & E_{\text{Ric}(\eta_w)}(\eta; v_k, w) &\longrightarrow E_{\text{Ric}(\eta_w)}(\eta; v, w) \\ E_{E_j}(\eta|_{E_j}; v_k|_{E_j}, \hat{\varphi}|_{E_j}) &\longrightarrow E_{E_j}(\eta|_{E_j}; v|_{E_j}, \hat{\varphi}|_{E_j}) \end{aligned}$$

for any $E_j \not\subset E_{nK}(\eta)$. Note that we also used that $\text{Ric}(\theta_\varphi) = \text{Ric}(\omega)$, $\text{Ric}(\eta_w) = \text{Ric}(\tilde{\omega})$ are smooth. It remains to prove that

$$\text{Ent}_\theta(u_k) \rightarrow \text{Ent}_\theta(u), \quad \text{Ent}_\eta(v_k) \rightarrow \text{Ent}_\eta(v).$$

In order to do so, we observe that if $0 \leq h \leq g$, an elementary calculation gives

$$\begin{aligned} |h \log h| &\leq \mathbf{1}_{\{h < 1\}}(-h \log h) + \mathbf{1}_{\{h \geq 1\}}h \log h \\ &\leq e^{-1} + \mathbf{1}_{\{g \geq 1\}}g \log g \\ &\leq e^{-1} + g \log g + \mathbf{1}_{\{g < 1\}}(-g \log g) \\ &\leq 2e^{-1} + g \log g \end{aligned}$$

as the function $\mathbb{R}_{\geq 0} \ni x \rightarrow x \log x$ is non-positive on $[0, 1]$ with a minimum at $x = e^{-1}$ while it is positive on $\{x > 1\}$. Set $h_k := m_\phi^{-1} e^{\alpha(u_k - u)} \min(g, k)$ and note that $h_k \leq m_\phi^{-1} e^C g$ for a uniform constant $C > 0$. By the above we have

$$\begin{aligned} \text{Ent}_\theta(u_k) &= \int_X h_k \log(h_k) \omega^n \\ &\leq 2e^{-1} + \int_X \log\left(\frac{e^C g}{m_\phi}\right) \frac{e^C g}{m_\phi} \omega^n \\ &= 2e^{-1} + e^C \left(\int_X \log(g m_\phi^{-1}) \frac{g \omega^n}{m_\phi} + C \int_X \frac{g \omega^n}{m_\phi} \right) \\ &= 2e^{-1} + e^C (\text{Ent}_\theta(u) + C_1) < +\infty. \end{aligned}$$

Thanks to Lebesgue Dominated Convergence Theorem we can infer that

$$\text{Ent}_\theta(u_k) = \int_X h_k \log(h_k) \omega^n \longrightarrow \text{Ent}_\theta(u) = \int_X g \log(g) \omega^n.$$

Very similar arguments show the convergence $\text{Ent}_\eta(v_k) \rightarrow \text{Ent}_\eta(v)$. This concludes the proof \square

We can now prove Theorem 4.6.

Proof of Theorem 4.6. Let $(u_t)_{t \in [0, 1]}$ be the psh geodesic connecting $u_0, u_1 \in \text{PSH}(X, \theta)$, functions with ϕ -relative minimal singularities. As $u_t \geq P_\theta(u_0, u_1)$, u_t is with ϕ -minimal singularities for all $t \in [0, 1]$. Set $v_t = \mathbf{L}(u_t)$, $t \in [0, 1]$. By construction v_t is an η -psh function with minimal singularities on Y . Moreover, Lemma 4.7(ii) ensures that v_t is a psh geodesic in Y joining v_0 and v_1 . By [36, Theorem 4.2] we know that $t \rightarrow \mathcal{M}_{\eta, w}(v_t)$ is convex in t , while [21, Theorem 3.12] ensures that $t \rightarrow E(\eta; v_t, \hat{\varphi}) = E(\eta; v_t, 0) - E(\eta; \hat{\varphi}, 0)$ is linear. Thus from Proposition 4.9 it follows that

$$(4.33) \quad \mathcal{M}_{\theta, \varphi}(u_t) - t\mathcal{M}_{\theta, \varphi}(u_1) - (1-t)\mathcal{M}_{\theta, \varphi}(u_0) \leq \frac{n}{\text{Vol}(\eta)} \sum_{E_j \not\subset E_{nK}(\eta)} a_j \text{Vol}(\eta|_{E_j}) \left(tE_{E_j}(\eta|_{E_j}; v_1|_{E_j}, \hat{\varphi}|_{E_j}) + (1-t)E_{E_j}(\eta|_{E_j}; v_0|_{E_j}, \hat{\varphi}|_{E_j}) - E_{E_j}(\eta|_{E_j}; v_t|_{E_j}, \hat{\varphi}|_{E_j}) \right).$$

Set $\mathcal{E}_t := \left(tE_{E_j}(\eta|_{E_j}; v_1|_{E_j}, \hat{\varphi}|_{E_j}) + (1-t)E_{E_j}(\eta|_{E_j}; v_0|_{E_j}, \hat{\varphi}|_{E_j}) - E_{E_j}(\eta|_{E_j}; v_t|_{E_j}, \hat{\varphi}|_{E_j}) \right)$. By the cocycle property of the Monge-Ampère energy (see e.g. Proposition 4.5), we then have for any

$j = 1, \dots, m$,

$$E_{E_j}(\eta|_{E_j}; v_1|_{E_j}, \hat{\varphi}|_{E_j}) - E_{E_j}(\eta|_{E_j}; v_t|_{E_j}, \hat{\varphi}|_{E_j}) = E_{E_j}(\eta|_{E_j}; v_1|_{E_j}, v_t|_{E_j}) \leq \|v_1 - v_t\|_\infty$$

and similarly replacing v_1 by v_0 . Therefore, as $\|v_s - v_t\|_\infty = \|u_s - u_t\|_\infty \leq |s - t| \|u_0 - u_1\|_\infty$, we get

$$(4.34) \quad \mathcal{E}_t \leq 2t(1-t) \|u_0 - u_1\|_\infty.$$

Since $t(1-t) \leq 1/4$, combining (4.33) with (4.34) yields

$$\mathcal{M}_{\theta, \varphi}(u_t) \leq t\mathcal{M}_{\theta, \varphi}(u_1) + (1-t)\mathcal{M}_{\theta, \varphi}(u_0) + \frac{n\|u_0 - u_1\|_\infty}{2\text{Vol}(\eta)} \sum_{E_j \notin E_{nK}(\eta)} a_j \text{Vol}(\eta|_{E_j}).$$

The proof is finished since $\sum_{j=1}^m a_j [E_j] = [K_{Y/X}]$. \square

We conclude this subsection with the following important consequence of Theorem 4.6.

Corollary 4.11. *Let $C_1 > 0$ and let $u_0, u_1 \in \text{PSH}(X, \theta)$ with ϕ -relative minimal singularities such that $\text{Ent}_\theta(u_0), \text{Ent}_\theta(u_1) \leq C_1$. Then there exist positive constants C_2, C_3 such that*

$$\text{Ent}_\theta(u_t) \leq C_1 + C_2 + C_3 \{\eta^{n-1}\} \cdot K_{Y/X}$$

for any $t \in [0, 1]$. Moreover C_2, C_3 only depend on $n, X, \{\omega\}, \{\theta\}, \|u_0 - u_1\|_\infty$ and on a lower bound of m_ϕ .

Proof. Using (4.15) we obtain

$$\begin{aligned} \text{Ent}_\theta(u_t) &\leq (1-t)\text{Ent}_\theta(u_0) + t\text{Ent}_\theta(u_1) + \frac{n\|u_0 - u_1\|_\infty}{2m_\phi} \{\eta^{n-1}\} \cdot K_{Y/X} + \\ &\quad + \bar{S}_\varphi(1-t) (E(\theta; u_t, \varphi) - E(\theta; u_0, \varphi)) + \bar{S}_\varphi t (E(\theta; u_t, \varphi) - E(\theta; u_1, \varphi)) \\ &\quad + n(1-t) (E_{\text{Ric}(\omega)}(\theta; u_t, \varphi) - E_{\text{Ric}(\omega)}(\theta; u_0, \varphi)) + nt (E_{\text{Ric}(\omega)}(\theta; u_t, \varphi) - E_{\text{Ric}(\omega)}(\theta; u_1, \varphi)). \end{aligned}$$

By the cocycle property [23, Theorem 5.3]

$$\begin{aligned} E(\theta; u_t, \varphi) - E(\theta; u_i, \varphi) &= \frac{1}{(n+1)m_\phi} \sum_{k=0}^n \int_X (u_t - u_i) \theta_{u_t}^k \wedge \theta_{u_i}^{n-k} \\ &\leq \frac{\|u_t - u_i\|_\infty}{m_\phi} \int_X \theta_\varphi^n \leq \|u_t - u_i\|_\infty. \end{aligned}$$

Again by the cocycle property and the fact that $\text{Ric}(\omega) \leq C\omega$ we get

$$\begin{aligned} &E_{\text{Ric}(\omega)}(\theta; u_t, \varphi) - E_{\text{Ric}(\omega)}(\theta; u_i, \varphi) \\ &= \frac{1}{n m_\phi} \sum_{k=0}^{n-1} \int_X (u_t - u_i) \text{Ric}(\omega) \wedge \theta_{u_t}^k \wedge \theta_{u_i}^{n-k-1} \\ &\leq C \frac{\|u_t - u_i\|_\infty}{m_\phi} \int_X \omega \wedge \theta_\varphi^{n-1} \leq C' \|u_t - u_i\|_\infty. \end{aligned}$$

As $\|u_t - u_s\|_\infty \leq |t - s| \|u_1 - u_0\|_\infty$ for any $s, t \in [0, 1]$, the conclusion follows. \square

4.2. Transcendental Fujita Approximation. We give the following transcendental definition of the well-known Fujita approximation of big line bundle on projective varieties [40].

Definition 4.12. We say that a sequence of model type envelopes $(\phi_k)_k \subset \text{PSH}(X, \theta)$ is a *transcendental Fujita approximation* of $\{\theta\}$ if

- i) $\phi_k \in \mathcal{N}_\theta$ for any $k \in \mathbb{N}$;
- ii) $\int_X \theta_{\phi_k}^n \rightarrow \text{Vol}(\theta)$ as $k \rightarrow +\infty$.

We also say that a transcendental Fujita approximation $(\phi_k)_k$ is *monotone* if $\phi_k \nearrow V_\theta$.

We note that as an immediate consequence of Theorem 2.2, any $(\phi_k)_k$ such that $\phi_k \nearrow V_\theta$ satisfies the condition in (ii).

The following result gives another interpretation of such transcendental Fujita approximation:

Lemma 4.13. *There exists a transcendental Fujita approximation of $\{\theta\}$ if and only if there exists a sequence of data (π_k, β_k, F_k) , where $\pi_k : Y_k \rightarrow X$ is a modification from Y_k compact Kähler manifold of complex dimension n , β_k is a big and nef class, F_k is an effective \mathbb{R} -divisor, such that*

- i) $\pi_k^*\{\theta\} = \beta_k + \{F_k\}$ for any $k \in \mathbb{N}$;
- ii) $\text{Vol}(\beta_k) \rightarrow \text{Vol}(\theta)$ as $k \rightarrow +\infty$.

Proof. By definition of \mathcal{N}_θ if $(\phi_k)_k$ is a transcendental Fujita approximation then there exist modifications $\pi_k : Y_k \rightarrow X$, currents with minimal singularities S_k representing big and nef classes β_k and effective \mathbb{R} -divisors F_k such that $\pi_k^*\theta_{\phi_k} = S_k + [F_k]$ for any $k \in \mathbb{N}$. Thus one implication follows simply observing that $\int_X \theta_{\phi_k}^n = \int_{Y_k} S_k^n = \text{Vol}(\beta_k)$.

Vice-versa, assume to have a sequence of data $(\pi_k, \beta_k, F_k)_k$ as in the statement. Since F_k is effective, for any S_k current with minimal singularities in β_k there exists a unique current $T_k = \theta + dd^c u_k$ such that

$$\pi_k^*(\theta + dd^c u_k) = S_k + [F_k]$$

(see [8, Proposition 1.2.7.(ii)]). Set $\phi_k := P_\theta[u_k]$. By [23, Lemma 5.1] we know that ϕ_k and u_k have the same multiplier ideal sheaf and in particular they have the same Lelong numbers on any modification of X . Thus, since S_k has minimal singularities and ϕ_k is less singular than u_k , we infer that

$$\pi_k^*(\theta + dd^c \phi_k) = \tilde{S}_k + [F_k]$$

for a positive and closed current with minimal singularities \tilde{S}_k in β_k , i.e. $\phi_k \in \mathcal{N}_\theta$. Moreover, as noticed above, $\int_X \theta_{\phi_k}^n = \int_{Y_k} S_k^n = \text{Vol}(\beta_k)$. \square

The existence of a monotone transcendental Fujita approximation is basically a consequence of [28]:

Lemma 4.14. *There exists a monotone transcendental Fujita approximation of $\{\theta\}$.*

Proof. By Lemma 4.5(ii) and the lines below Definition 4.12 it is enough to produce a sequence of θ -psh functions with analytic singularities $\hat{\psi}_k$ such that $\int_X \theta_{\hat{\psi}_k}^n > 0$ and such that $\phi_k := P_\theta[\hat{\psi}_k]$ increases to V_θ .

An immediate consequence of the proof of Demailly's approximation theorem [28, Proposition 3.7] is that for any θ_ψ Kähler current there exists $\psi' \geq \psi$ such that $\theta_{\psi'}$ is a Kähler current with analytic singularities (each element of the approximating sequence satisfies this property). Moreover observe that if $\theta_{\psi_1}, \theta_{\psi_2}$ are Kähler currents, then $\theta_{\max(\psi_1, \psi_2)}$ is a Kähler current as well since by [23, Lemma 2.9]

$$\theta_{\max(\psi_1, \psi_2)} \geq \mathbf{1}_{\{\psi_2 \leq \psi_1\}} \theta_{\psi_1} + \mathbf{1}_{\{\psi_1 < \psi_2\}} \theta_{\psi_2}.$$

Now let θ_ψ be a Kähler current and let $\psi_k := \frac{1}{k}\psi + (1 - \frac{1}{k})V_\theta$. Then

$$\theta_{\psi_k} = \frac{1}{k}\theta_\psi + \left(1 - \frac{1}{k}\right)\theta_{V_\theta}$$

is a Kähler current and $\int_X \theta_{\psi_k}^n \rightarrow \int_X \theta_{V_\theta}^n$ as k goes to $+\infty$.

Let $\hat{\psi}_1$ be a θ -psh function such that $\theta_{\hat{\psi}_1}$ is a Kähler current with analytic singularities and $\hat{\psi}_1 \geq \psi_1$. Inductively let $\hat{\psi}_{k+1}$ be a θ -psh function such that $\theta_{\hat{\psi}_{k+1}}$ is a Kähler current with analytic singularities and $\hat{\psi}_{k+1} \geq \max(\hat{\psi}_k, \psi_{k+1})$. Then by construction we have that $\hat{\psi}_k$ is an increasing sequence, $\hat{\psi}_k \geq \psi_k$ and $\theta_{\hat{\psi}_k}$ is a Kähler current with analytic singularities.

By [62, Theorem 1.2] we know that $\int_X \theta_{\psi_k}^n \leq \int_X \theta_{\hat{\psi}_k}^n \leq \text{Vol}(\theta)$. Thus $\int_X \theta_{\psi_k}^n \rightarrow \int_X \theta_{V_\theta}^n$ as k goes to $+\infty$. We then consider $\phi_k := P_\theta[\hat{\psi}_k]$. This sequence is increasing, it has analytic singularity type and $|\phi_k - \hat{\psi}_k| \leq C_k$, for some $C_k > 0$. Moreover, by construction, $\sup_X \phi_k = 0$ and $\phi_k \leq V_\theta$. Let $\phi = (\sup_k \phi_k)^* \in \text{PSH}(X, \theta)$, where $*$ is the upper semicontinuous regularization. Then $\sup_X \phi = 0$ and by [23, Remark 2.4] we have

$$\int_X \theta_{\phi_k}^n = \int_X \theta_{\psi_k}^n \rightarrow \int_X \theta_{V_\theta}^n$$

as $k \rightarrow +\infty$. Hence,

$$\int_X \theta_{V_\theta}^n = \int_X \theta_\phi^n.$$

On the other hand ϕ is an increasing limit of model type envelopes, hence by [24, Corollary 4.7] it is a model type envelope, therefore $\phi = V_\theta$, concluding the proof. \square

4.3. (Almost) Convexity of the Mabuchi functional. Let $\{\phi_k\}_{k \in \mathbb{N}}$ be a monotone transcendental Fujita approximation of $\{\theta\}$ (see Definition 4.12) and set $V_k := \int_X \theta_{\phi_k}^n$, $V := \text{Vol}(\theta)$. We note that by definition we have $V_k > 0$ for any k . Let $u \in \mathcal{E}(X, \theta)$ and let us consider the function

$$u_k := P_\theta[\phi_k](u).$$

We wish to collect some properties of the correspondence $u \rightarrow u_k$.

Lemma 4.15. *Let $u \in \mathcal{E}(X, \theta)$, then the correspondence $u \rightarrow u_k$ has the following properties:*

- (i) $u_k \in \mathcal{E}(X, \theta, \phi_k)$.
- (ii) *The sequence u_k increases to u as k goes to $+\infty$ outside of a pluripolar set.*
- (iii) *The set*

$$\begin{aligned} S &:= \{x \in X : u_k(x) = u(x) \text{ for some } k \geq 1\} \\ &= \{x \in X : \text{for some } k \geq 1, \text{ and for all } l \geq k \text{ we have } u_l(x) = u(x)\} \end{aligned}$$

has full mass with respect to θ_u^n .

- (iv) *If $u, v \in \text{PSH}(X, \theta)$ satisfy $|u - v| \leq C$ then $|u_k - v_k| \leq C$ for all $k \geq 1$.*
- (v) *We have $\theta_{u_k}^n = g_k \theta_u^n$ with $0 \leq g_k \leq 1$ and g_k increasing almost everywhere with respect to θ_u^n to the constant function 1 as $k \rightarrow +\infty$. In particular if $\text{Ent}_\theta(u)$ is finite, then $\text{Ent}_\theta(u_k)$ is uniformly bounded independent of $k \geq 1$.*
- (vi) *Assume $u_0, u_1 \in \mathcal{E}^1(X, \theta)$ have the same singularity type. Let $t \rightarrow u_t$ be the psh geodesic defined in the interval $[0, 1]$ joining u_0, u_1 , and let $t \rightarrow u_{t,k}$ be the psh geodesic joining $u_{0,k}, u_{1,k}$. Then for all $t \in [0, 1]$ the sequence $u_{t,k}$ is increasing to u_t outside of a pluripolar set as k goes to $+\infty$.*

Proof. Observe that $u + C \in \mathcal{E}(X, \theta)$ for all $C \in \mathbb{R}$. Since $\int_X \theta_{\phi_k}^n > 0$, $\phi_k = P_\theta[\phi_k]$ and $P_\theta(V_\theta) = V_\theta$, if we apply [24, Proposition 5.3] with $\Phi = \phi_k$, and $\Psi = V_\theta$ we obtain that $P_\theta(u + C, \phi_k) \in \mathcal{E}(X, \theta, \phi_k)$. By definition $P_\theta[\phi_k](u) \leq \phi_k$ and it is the increasing limit of $P_\theta(u + C, \phi_k)$, hence

$$\int_X \theta_{\phi_k}^n = \int_X \theta_{P_\theta(u+C, \phi_k)}^n \leq \int_X \theta_{P_\theta[\phi_k](u)}^n \leq \int_X \theta_{\phi_k}^n,$$

this proves (i).

Let \hat{u} be the upper semi-continuous regularization of the limit of the increasing sequence u_k , then $\hat{u} \leq u$ and the set $E = \{x \in X : \sup_k u_k < \hat{u}(x) \text{ or } \hat{u}(x) = -\infty\}$ is pluripolar, hence it has θ_u^n measure 0. Now $\theta_{\hat{u}}^n$ is the weak limit of $\theta_{u_k}^n$ by Remark 2.6. By (i), $\int_X \theta_{u_k}^n = \int_X \theta_{\phi_k}^n \rightarrow \text{Vol}(\theta)$ as k goes to $+\infty$. Since $\hat{u} \leq u$, it follows that $\int_X \theta_{\hat{u}}^n = \int_X \theta_u^n = \text{Vol}(\theta)$. By [20, Theorem 3.8] we have

$$\theta_{u_k}^n \leq \mathbf{1}_{\{u_k = u\}} \theta_u^n,$$

therefore

$$\theta_{\hat{u}}^n \leq \mathbf{1}_S \theta_u^n.$$

Since u and \hat{u} have the same mass,

$$\theta_{\hat{u}}^n = \theta_u^n$$

and S has full mass with respect to θ_u^n . Therefore \hat{u} and u differ by a constant (see for example [23, Theorem 3.13]). Now if $x_0 \in S \setminus E$, we have $u(x_0) = \hat{u}(x_0) > -\infty$, hence $u = \hat{u}$. This proves (ii) and (iii). The map $u \rightarrow P_\theta[\phi_k](u)$ is monotone and $P_\theta[\phi_k](u + C) = P_\theta[\phi_k](u) + C$ for all $C \in \mathbb{R}$. Then (iv) follows. Now, the inequality $\theta_{u_k}^n \leq \mathbf{1}_{\{u_k=u\}} \theta_u^n$ also says that

$$\theta_{u_k}^n = g_k \theta_u^n,$$

with $0 \leq g_k \leq 1$. Moreover for $j \leq k$ we have $\phi_j \leq \phi_k$ and

$$P[\phi_j](u_k) = P[\phi_j](P[\phi_k](u)) = P[\phi_j](u) = u_j.$$

[20, Theorem 3.8] implies that g_k is increasing in k since

$$g_j \theta_u^n = \theta_{u_j}^n \leq \mathbf{1}_{\{u_j=u_k\}} \theta_{u_k}^n \leq \theta_{u_k}^n = g_k \theta_u^n.$$

Now by the above $\theta_{u_k}^n$ converges weakly to θ_u^n , hence g_k converges to the constant function 1 almost everywhere with respect to θ_u^n . Moreover since $\log(m_{\phi_k})$ is uniformly bounded we derive (v).

In order to prove (vi) we note that $u_{i,k-1} \leq u_{i,k}$, $i = 0, 1$, hence $u_{t,k-1}$ is a subgeodesic with respect to the end points $u_{0,k}, u_{1,k}$. This means that the sequence $u_{t,k}$ is increasing in k . Moreover by the t -convexity of the geodesic and (iv) we have $u_{t,k} \leq \phi_k + C \leq V_\theta + C$ for some positive constant C (independent of k), hence $u_{t,k}$ increases to a psh subgeodesic segment $t \rightarrow v_t$ that by (ii) is joining u_0 and u_1 . Now by maximality $u_t \geq P_\theta(u_0, u_1)$, then each u_t has the same singularity type of u_0 and u_1 . We claim that $v_t = u_t$.

Combining Lemma 4.7(iv), Propositions 4.1 and 4.2 we find that

$$t \rightarrow E(\theta; u_{t,k}, \phi_k)$$

is linear. Again by the cocycle property (Proposition 4.2) we have $E(\theta; u_{t,k}, \phi_k) = E(\theta; u_{t,k}, u_{0,k}) + E(\theta; u_{0,k}, \phi_k)$. We then infer that $t \rightarrow E(\theta; u_{t,k}, u_{0,k})$ is linear. Moreover, since $u_{t,k}$ is increasing to u_t , $u_{0,k}$ is increasing to u_0 and by (iv) we know that $(u_{t,k} - u_{0,k})$ is uniformly bounded, thanks to Theorem 2.2 we ensure that

$$E(\theta; u_{t,k}, u_{0,k}) = \frac{1}{n+1} \sum_{j=0}^n \int_X (u_{t,k} - u_{0,k}) \theta_{u_{t,k}}^j \wedge \theta_{u_{0,k}}^{n-j} \rightarrow \frac{1}{n+1} \sum_{j=0}^n \int_X (v_t - u_0) \theta_{v_t}^j \wedge \theta_{u_0}^{n-j} = E(\theta; v_t, u_0)$$

as $k \rightarrow +\infty$. Thus $t \rightarrow E(\theta; v_t, u_0)$ is linear. Using the cocycle property (4.5) we deduce that $t \rightarrow E(\theta; v_t, V_\theta)$ is linear. On the other hand $t \rightarrow E(\theta; u_t, V_\theta)$ is linear as well thanks to Proposition 4.1. Thus, for any $t \in [0, 1]$ we have

$$E(\theta; u_t, V_\theta) = (1-t)E(\theta; u_0, V_\theta) + tE(\theta; u_1, V_\theta) = E(\theta; v_t, V_\theta).$$

By maximality of the psh geodesic, we have $u_t \geq v_t$. Hence $u_t = v_t$ by [24, Lemma 2.9]. This proves (vi). \square

Now we want to apply the results of the previous subsection 4.1 to any monotone transcendental Fujita approximation $(\phi_k)_k$. For each k we set $\pi_k : Y_k \rightarrow X$ a modification from Y_k compact Kähler manifold (of complex dimension n) such that

$$\pi_k^* \theta_{\phi_k} = (\eta_k + dd^c \tilde{\phi}_k) + [F_k]$$

where for each k , η_k is a smooth and closed form representing a big and nef class while $\tilde{\phi}_k$ is a η_k -psh function with minimal singularities normalized by $\sup_{Y_k} \tilde{\phi}_k = 0$.

Let $E_{1,k}, \dots, E_{m_k,k}$ be the exceptional divisors of π_k , and $a_{j,k} > 0$ such that $K_{Y_k/X} = \sum_{j=1}^{m_k} a_{j,k} E_{j,k}$.

Definition 4.16. Given a monotone transcendental Fujita approximation $(\phi_k)_k$ we define

$$H(\phi_k) := \liminf_{k \rightarrow +\infty} \{\eta_k^{n-1}\} \cdot K_{Y_k/X}.$$

and

$$H := \inf\{H(\phi_k), (\phi_k)_k \text{ monotone transcendental Fujita approximation}\}.$$

We start by proving that these quantities are well defined.

Lemma 4.17. $H(\phi_k)$ only depend on the choice of the transcendental Fujita approximation $(\phi_k)_k$.

Proof. Let $\phi \in \mathcal{N}_\theta$ and let $\pi_1 : Y_1 \rightarrow X$, $\pi_2 : Y_2 \rightarrow X$ be two modifications where Y_1, Y_2 are compact Kähler manifolds such that for each $i = 1, 2$

$$\pi_i^* \theta_\phi = \eta_{i, \tilde{\phi}_i} + [F_i]$$

where F_i is an effective \mathbb{R} -divisor, η_i is a closed and smooth form representing a big and nef class, and $\tilde{\phi}_i$ is a η_i -psh function with minimal singularities normalized by $\sup_{Y_i} \tilde{\phi}_i = 0$. The goal is to prove that

$$\{\eta_1^{n-1}\} \cdot K_{Y_1/X} = \{\eta_2^{n-1}\} \cdot K_{Y_2/X}.$$

Resolving the graph of the bimeromorphic map $\pi_2^{-1} \circ \pi_1 : Y_1 \dashrightarrow Y_2$, yields modifications $\rho_1 : Z \rightarrow Y_1, \rho_2 : Z \rightarrow Y_2$ such that the diagram

$$\begin{array}{ccc} & Z & \\ \rho_1 \swarrow & & \searrow \rho_2 \\ Y_1 & & Y_2 \\ \pi_1 \searrow & & \swarrow \pi_2 \\ & X & \end{array}$$

is commutative. In particular

$$(4.35) \quad \rho_1^* \eta_{1, \tilde{\phi}_1} + [\rho_1^* F_1] = (\pi_1 \circ \rho_1)^* \theta_\phi = (\pi_2 \circ \rho_2)^* \theta_\phi = \rho_2^* \eta_{2, \tilde{\phi}_2} + [\rho_2^* F_2].$$

It follows from [9, Propositions 3.2, 3.6] that the positive and closed currents $\rho_i^* \eta_{i, \tilde{\phi}_i}$ have zero Lelong numbers everywhere as they are currents with minimal singularities in the big and nef class $\rho_i^* \{\eta_i\}$. Thus, (4.35) gives two decompositions of the same positive and closed current $T := (\pi_1 \circ \rho_1)^* \theta_\phi = (\pi_2 \circ \rho_2)^* \theta_\phi$ into a sum of a current with zero Lelong number and of a current of integration along a divisor. By uniqueness of the Siu's Decomposition [56] (see also [29, §8.(8.16)]) we infer that $\rho_1^* \eta_{1, \tilde{\phi}_1} = \rho_2^* \eta_{2, \tilde{\phi}_2}$. In particular $\{\rho_1^* \eta_1\} = \{\rho_2^* \eta_2\}$.

Moreover, by formula 4.13 applied to ρ_i, π_i and their compositions, we have $K_{Z/X} = K_{Z/Y_i} + \rho_i^* K_{Y_i/X}$ and K_{Z/Y_i} is ρ_i -exceptional. It then follows that

$$\begin{aligned} \{\eta_1\}^{n-1} \cdot K_{Y_1/X} &= (\rho_1^* \{\eta_1\})^{n-1} \cdot \rho_1^* K_{Y_1/X} \\ &= \{\rho_1^* \eta_1\}^{n-1} \cdot (K_{Z/X} - K_{Z/Y_1}) \\ &= \{\rho_1^* \eta_1\}^{n-1} \cdot K_{Z/X}, \end{aligned}$$

since K_{Z/Y_1} is ρ_1 -exceptional.

An analogous formula holds for ρ_2, η_2, Y_2 . Since $\{\rho_1^* \eta_1\} = \{\rho_2^* \eta_2\}$, we are done. \square

In the following we will work under one of these two conditions:

$$\text{(Condition A)} \quad H < +\infty$$

$$\text{(Condition B)} \quad H = 0.$$

We refer to subsection 4.4 for some examples when these conditions hold and for a digression on how they are related to the uniform version of Yau-Tian-Donaldson conjecture in the algebraic case.

Our main theorem states as follows:

Theorem 4.18. *Let $u_0, u_1 \in \text{PSH}(X, \theta)$ with minimal singularities and let $(u_t)_{t \in [0,1]}$ be the psh geodesic connecting u_0 and u_1 . Let also $\varphi \in \mathcal{E}(X, \theta)$ be such that $\theta_\varphi^n = \text{Vol}(\theta)\omega^n$, $\sup_X \varphi = 0$. Then u_t has minimal singularities and the function $t \mapsto \mathcal{M}_{\theta, \varphi}(u_t)$ is almost convex in $[0, 1]$, i.e.*

$$(4.36) \quad \mathcal{M}_{\theta, \varphi}(u_t) \leq (1-t)\mathcal{M}_{\theta, \varphi}(u_0) + t\mathcal{M}_{\theta, \varphi}(u_1) + H\|u_0 - u_1\|_\infty.$$

In particular, if **(Condition B)** holds, then $\mathcal{M}_{\theta, \varphi}$ is convex along u_t .

Proof. We can assume that $\text{Ent}_\theta(u_i, \varphi)$ is finite. Indeed otherwise either $\mathcal{M}_{\theta, \varphi}(u_0)$ or $\mathcal{M}_{\theta, \varphi}(u_1)$ would be equal to $+\infty$ and the requested inequality would be trivial. Without loss of generality, we can consider $(\phi_k)_k$ be a monotone transcendental Fujita approximation with $H(\phi_k) < +\infty$. Consider

$$u_{0,k} := P_\theta[\phi_k](u_0), \quad u_{1,k} := P_\theta[\phi_k](u_1).$$

By Lemma 4.15, $u_{i,k}$ has ϕ_k -relative minimal singularities with a constant independent of k and $u_{i,k}$ converges to u_i for $i = 0, 1$.

For $k \in \mathbb{N}$, let φ_k be the unique solution in $\mathcal{E}(X, \theta, \phi_k)$ of

$$(\theta + dd^c \varphi_k)^n = \left(\int_X \theta_{\phi_k}^n \right) \omega^n, \quad \sup_X \varphi_k = 0.$$

Thanks to the stability result in [24, Theorem 1.4] (that can be applied thanks to [24, Lemma 4.1]) we have that φ_k converges in capacity to φ .

Also, we claim that $|\varphi_k - \phi_k| \leq C$, for $C > 0$ independent of k . Indeed, by [22, Theorem 4.7] $|\varphi_k - \phi_k| \leq C_k$, where $C_k = \frac{A(\theta, \omega, n)}{V_k^2}$. Since V_k is increasing we get that $C_k \leq C_1$. The claim is then proved.

Thanks to Theorem 4.6 we have the almost convexity of the Mabuchi functional $\mathcal{M}_{\theta, \varphi_k}$ along psh geodesic segments joining functions with ϕ_k -relative minimal singularities:

$$\begin{aligned} \mathcal{M}_{\theta, \varphi_k}(u_t) &\leq t\mathcal{M}_{\theta, \varphi_k}(u_{1,k}) + (1-t)\mathcal{M}_{\theta, \varphi_k}(u_{0,k}) + \frac{n\|u_{0,k} - u_{1,k}\|_\infty}{2V_k} \{\eta_k^{n-1}\} \cdot K_{Y_k/X} \\ &\leq t\mathcal{M}_{\theta, \varphi_k}(u_{1,k}) + (1-t)\mathcal{M}_{\theta, \varphi_k}(u_{0,k}) + \frac{n\|u_0 - u_1\|_\infty}{2V_k} \{\eta_k^{n-1}\} \cdot K_{Y_k/X} \end{aligned}$$

where the last inequality follows from $\|u_{0,k} - u_{1,k}\|_\infty \leq \|u_0 - u_1\|_\infty$.

By assumption we know that $\theta_{u_i}^n = f_i \theta_\varphi^n = f_i V \omega^n$. By Lemma 4.15(v) $\theta_{u_{i,k}}^n$ has finite entropy w.r.t. $\theta_{\varphi_k}^n$. Let $t \rightarrow u_{t,k}$ denote the psh geodesic segment joining $u_{0,k}, u_{1,k}$. Then Corollary 4.11 together with $H(\phi_k) < +\infty$ and Lemma 4.15(v) ensure that $\theta_{u_{t,k}}^n$ has finite entropy as well for any $t \in [0, 1]$ and

$$\theta_{u_{t,k}}^n = V_k f_{t,k} \omega^n = f_{t,k} \theta_{\varphi_k}^n, \quad \int_X f_{t,k} \log f_{t,k} \omega^n \leq C,$$

for some C independent of t and k .

As mentioned above, $u_{i,k}$ has ϕ_k -relative minimal singularities with uniform constants, thus by the Lipschitz property of psh geodesics we have that

$$(4.37) \quad \phi_k \geq u_{t,k} \geq \phi_k - C$$

with $C > 0$ independent of k and of t .

We now claim that to get (4.36) it is enough to show that

$$(4.38) \quad \mathcal{M}_{\theta, \varphi_k}(u_{i,k}) \rightarrow \mathcal{M}_{\theta, \varphi}(u_i), \quad i = 0, 1$$

and

$$(4.39) \quad \liminf_{k \rightarrow +\infty} \mathcal{M}_{\theta, \varphi_k}(u_{t,k}) \geq \mathcal{M}_{\theta, \varphi}(u_t), \quad \forall t \in (0, 1).$$

Indeed, (4.38) and (4.39), together with the almost convexity of $\mathcal{M}_{\theta, \varphi_k}$, imply

$$\begin{aligned} \mathcal{M}_{\theta, \varphi}(u_t) &\leq (1-t)\mathcal{M}_{\theta, \varphi}(u_0) + t\mathcal{M}_{\theta, \varphi}(u_1) + \frac{n\|u_0 - u_1\|_\infty}{2V} \liminf_{k \rightarrow +\infty} \{\eta_k^{n-1}\} \cdot K_{Y_k/X} \\ &= (1-t)\mathcal{M}_{\theta, \varphi}(u_0) + t\mathcal{M}_{\theta, \varphi}(u_1) + \frac{n\|u_0 - u_1\|_\infty}{2V} H(\phi_k). \end{aligned}$$

Taking the infimum over all monotone transcendental Fujita approximations we conclude.

We now prove (4.38) and (4.39).

Step 1: Convergence of the energies. Since $u_{t,k} - \varphi_k$ is uniformly bounded, $u_{t,k} \nearrow u_t$, $\varphi_k \rightarrow \varphi$ in capacity, $u_{t,k}$ and φ_k are more singular than u_t and φ , respectively Remark 2.6 and Theorem 2.2 ensure that for any $j = 0, \dots, n$, we have

$$\int_X (u_{t,k} - \varphi_k) \theta_{u_{t,k}}^j \wedge \theta_{\varphi_k}^{n-j} \longrightarrow \int_X (u_t - \varphi) \theta_{u_t}^j \wedge \theta_\varphi^{n-j}$$

and

$$\int_X (u_{t,k} - \varphi_k) \theta_{u_{t,k}}^j \wedge \theta_{\varphi_k}^{n-j-1} \wedge \text{Ric}(\omega) \longrightarrow \int_X (u_t - \varphi) \theta_{u_t}^j \wedge \theta_\varphi^{n-j-1} \wedge \text{Ric}(\omega).$$

We then deduce that $E(\theta; u_{t,k}, \varphi_k) \rightarrow E(\theta; u_t, \varphi)$ and $E_{\text{Ric}(\omega)}(\theta; u_{t,k}, \varphi_k) \rightarrow E_{\text{Ric}(\omega)}(\theta; u_t, \varphi)$, for any $t \in [0, 1]$,

Step 2: Lower semicontinuity of the entropy. As $V_k^{-1} \theta_{\varphi_k}^n = \omega^n$, it follows from [4, Proposition 2.10] that for any θ -psh function v we have

$$\text{Ent}_\theta(v, \varphi_k) = \text{Ent}(m_v^{-1} \theta_v^n, V_k^{-1} \theta_{\varphi_k}^n) = \text{Ent}(m_v^{-1} \theta_v^n, \omega^n) = \sup_{g \in C^0(X)} \left(\int_X g \frac{\theta_v^n}{m_v} - \log \int_X e^g \omega^n \right).$$

In particular the functional $\text{Ent}_\theta(\cdot, \varphi_k) = \text{Ent}(\cdot, \omega^n)$ is lower semicontinuous on the space of probability measures on X with respect to the weak convergence. Since $u_{t,k} \nearrow u_t$, $\theta_{u_{t,k}}^n$ converges weakly to $\theta_{u_t}^n$ and

$$\liminf_{k \rightarrow +\infty} \text{Ent}_\theta(u_{t,k}, \varphi_k) = \liminf_{k \rightarrow +\infty} \text{Ent}(V_k^{-1} \theta_{u_{t,k}}^n, \omega^n) \geq \text{Ent}_\theta(V^{-1} \theta_{u_t}^n, \omega^n) = \text{Ent}_\theta(u_t, \varphi).$$

Next, for $i = 0, 1$, we write $\theta_{u_{i,k}}^n = g_{i,k} \theta_{u_i}^n$ where $0 \leq g_{i,k} \leq 1$ and $g_{i,k} \nearrow 1$ almost everywhere with respect to $\theta_{u_i}^n$ by Lemma 4.15. Since

$$V_k = \int_X \theta_{u_{i,k}}^n = \int_X g_{i,k} \theta_{u_i}^n \longrightarrow V = \int_X \theta_{u_i}^n$$

and $\theta_{u_i}^n = V f_i \omega^n$ we obtain that $\|f_i(1 - g_{i,j})\|_{L^1(\omega^n)} \rightarrow 0$ as $j \rightarrow +\infty$. By Lemma 4.15 and by the dominated convergence theorem we deduce that

$$\text{Ent}_\theta(u_{i,k}, \varphi_k) = \frac{V}{V_k^2} \int_X g_{i,k} f_i \log \left(\frac{V}{V_k} g_{i,k} f_i \right) \theta_{\varphi_k}^n = \frac{V}{V_k} \int_X g_{i,k} f_i \log \left(\frac{V}{V_k} g_{i,k} f_i \right) \omega^n$$

converges as $k \rightarrow +\infty$ to

$$\int_X f_i \log f_i \omega^n = \frac{1}{V} \int_X f_i \log f_i \theta_\varphi^n = \text{Ent}_\theta(u_i, \varphi).$$

Step 3: Conclusion of the proof. Since $\varphi_k \simeq \phi_k$ and $\varphi \simeq V_\theta$, after re-writing $\text{Ric}(\omega) = T_2 - T_1$ for some smooth Kähler forms, [20, Proposition 2.1] ensures that

$$\bar{S}_{\varphi_k} = \frac{n}{V_k} \int_X \text{Ric}(\omega) \wedge \theta_{\varphi_k}^{n-1} = \frac{n}{V_k} \int_X \text{Ric}(\omega) \wedge \theta_{\phi_k}^{n-1}, \quad \bar{S}_\varphi = \frac{n}{V} \int_X \text{Ric}(\omega) \wedge \theta_\varphi^{n-1} = \frac{n}{V} \int_X \text{Ric}(\omega) \wedge \theta_{V_\theta}^{n-1}.$$

Also, since ϕ_k increases to V_θ , Theorem 2.2 gives that $T_i \wedge \theta_{\phi_k}^{n-1}$ weakly converges to $T_i \wedge \theta_{V_\theta}^{n-1}$ for $i = 1, 2$. As $V_k \nearrow V$, we have that \bar{S}_{φ_k} converges to \bar{S}_φ . Then Step 2 and 3 give the required convergences (4.38), (4.39). \square

As a corollary we obtain:

Corollary 4.19. *Let $C_1 > 0$ and let $u_0, u_1 \in \text{PSH}(X, \theta)$ such that $|u_0 - u_1| \leq C$, $C > 0$. Assume $\text{Ent}_\theta(u_0), \text{Ent}_\theta(u_1) \leq C_1$. Then there exists positive constants C_2, C_3 such that*

$$\text{Ent}_\theta(u_t) \leq C_1 + C_2 + C_3 H$$

for any $t \in [0, 1]$. Moreover C_2, C_3 only depends on $n, X, \{\omega\}, \{\theta\}, \|u_0 - u_1\|_\infty$, and on a lower bound of $\text{Vol}(\theta)$.

Proof. When u_0, u_1 have minimal singularities, the same arguments in the proof of Corollary 4.11 give the conclusion. In the general case we can adapt the arguments in [36, Proposition 4.3]. We observe indeed that in the proof [36, Proposition 4.3] the authors use the distance d_1 on the space $\mathcal{E}^1(X, \theta)$ when θ is big and nef. However this distance has been defined in the big case as well and all the relevant properties have been proved [19]. Another key ingredient in the proof of [36, Proposition 4.3] is the convexity of the Mabuchi functional, but the almost convexity in our case (Theorem 4.18) suffices. \square

4.4. On the Condition B and the Yau-Tian-Donaldson Conjecture. In this subsection we prove that if $V_\theta \in \mathcal{N}_\theta$ then (Condition B) is satisfied. In particular, thanks to Theorem 4.18 the Mabuchi functional $\mathcal{M}_{\theta, \varphi}$ is convex along geodesic segments joining potentials with minimal singularities.

We recall that, by [9], any pseudoeffective cohomology class $\alpha \in H^{1,1}(X, \mathbb{R})$ admits a *divisorial Zariski decomposition*. When the class α is big such decomposition can be described as the Siu Decomposition of a positive and closed current with minimal singularities.

Proposition 4.20. *Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a big class, θ be a smooth closed $(1, 1)$ -form in α and let $T_{\min} := \theta_{V_\theta}$. Then the divisorial Zariski decomposition of α is given as*

$$\alpha = \mathcal{P}(\alpha) + \mathcal{N}(\alpha)$$

where the negative and positive parts $\mathcal{N}(\alpha), \mathcal{P}(\alpha)$ are given as follows:

- i) *The negative part $\mathcal{N}(\alpha)$ is the cohomology class of $\sum_D \nu(T_{\min}, D)[D]$ where the sum is over all prime divisors on X and where there is only a finite number of prime divisors D for which $\nu(T_{\min}, D) > 0$.*
- ii) *The positive part $\mathcal{P}(\alpha)$ is defined as the difference $\alpha - \mathcal{N}(\alpha)$, and the current $T_{\min} - \sum_D \nu(T_{\min}, D)[D]$ has minimal singularities in the class $\mathcal{P}(\alpha)$. In particular $\text{Vol}(\mathcal{P}(\alpha)) = \text{Vol}(\alpha)$.*

Proof. (i) follows from [9, Proposition 3.6, Definition 3.7, Theorem 3.12]. To prove (ii) we observe that if S_{\min} is a positive current with minimal singularities in $\mathcal{P}(\alpha)$, then $S_{\min} + \sum_D \nu(T_{\min}, D)[D]$ is more singular than T_{\min} . Thus $T_{\min} - \sum_D \nu(T_{\min}, D)[D]$ is a positive and closed current less singular than S_{\min} , so they have the same singularities. \square

Since $\alpha = \{T_{\min}\}$, as direct consequence of Siu's Decomposition we have that $\mathcal{P}(\alpha)$ is the cohomology class of a positive and closed current with zero Lelong numbers along any prime divisor on X . The class $\mathcal{P}(\alpha)$ is said to be *nef in codimension 1* or also *modified nef*.

Definition 4.21. ¹ Let α be a big cohomology class. We say that α admits a *Zariski decomposition* if $\mathcal{P}(\alpha)$ is nef. Moreover we say that α admits a *bimeromorphic Zariski decomposition* if there exists a modification $\mu : Y \rightarrow X$ from Y compact Kähler manifold of complex dimension n , such that $\mu^*\alpha$ admits a Zariski decomposition.

If $\dim X = 2$ then it is well-known that any big class (actually any pseudoeffective class) admits a Zariski decomposition, generalizing the pioneering work of Zariski [64]. However there are example of big classes that do not admit a bimeromorphic Zariski decomposition: see for instance [9, Section A.2].

We are now ready to observe the following:

Proposition 4.22. $V_\theta \in \mathcal{N}_\theta$ if and only if $\{\theta\}$ admits a bimeromorphic Zariski decomposition.

Proof. Set $\alpha = \{\theta\}$. Assume that $\mu : Y \rightarrow X$ is a modification from Y compact Kähler manifold such that $\mu^*\alpha$ admits a Zariski decomposition $\mu^*\alpha = \mathcal{P}(\alpha) + \mathcal{N}(\alpha)$. Again from [8] we know that any positive closed real $(1, 1)$ current in $\mu^*(\alpha)$ is the pull-back of a positive closed real $(1, 1)$ current in α , so $\mu^*(\theta_{V_\theta})$ is a current with minimal singularities in $\mu^*(\alpha)$. From Proposition 4.20 we know that

$$(4.40) \quad \mu^*\theta_{V_\theta} = S + [F]$$

where S is a current with minimal singularities in $\mathcal{P}(\alpha)$ while F is an effective \mathbb{R} -divisor in $\mathcal{N}(\alpha)$. Since by assumption $\mathcal{P}(\alpha)$ is nef we conclude that $V_\theta \in \mathcal{N}_\theta$.

Vice-versa suppose that $V_\theta \in \mathcal{N}_\theta$, i.e. there exist a modification $\mu : Y \rightarrow X$ from Y compact Kähler manifold, a current with minimal singularities S representing a big and nef class and an effective \mathbb{R} -divisor F such that (4.40) holds. Then since, $\mu^*(\theta_{V_\theta})$ is with minimal singularities and the non-pluripolar product does not charge pluripolar sets, we obtain that

$$\text{Vol}(\mu^*\alpha) = \int_Y \mu^*\theta_{V_\theta} = \int_Y S^n = \text{Vol}(S).$$

Hence it follows from [31, Main Theorem] that $\{S\}$ is the positive part in the Zariski decomposition of $\mu^*\alpha$, i.e. $\mu^*\alpha$ admits a Zariski decomposition as $\{S\}$ is nef. \square

We are now ready to prove the main result of this subsection.

Theorem 4.23. Let $\alpha = \{\theta\}$ be a big cohomology class that admits a bimeromorphic Zariski decomposition. Assume $\varphi \in \mathcal{E}(X, \theta)$ such that $\theta_\varphi^n = \text{Vol}(\theta)\omega^n$. Then $\mathcal{M}_{\theta, \varphi}$ is convex along psh geodesics joining potentials with minimal singularities.

It then follows that $\mathcal{M}_{\theta, \varphi}$ is convex when $\dim_{\mathbb{C}} X = 2$.

Proof. By Proposition 4.22 we know that $V_\theta \in \mathcal{N}_\theta$. Let $\mu : Y \rightarrow X$ be a modification from Y compact Kähler manifold such that

$$\mu^*\theta_{V_\theta} = \eta_{\tilde{\varphi}} + [F]$$

where $\eta_{\tilde{\varphi}}$ is a positive and closed current with minimal singularities representing a big and nef class and F is an effective \mathbb{R} -divisor. By Theorem 4.6 it is enough to check that

$$\{\eta^{n-1}\} \cdot K_{Y/X} = 0.$$

However as observed in the proof of Proposition 4.22, $\{\eta\} = \mathcal{P}(\mu^*\alpha)$ and by [17, Lemmas 4.3, 6.1] it follows that $E_{nK}(\mu^*\alpha) = E_{nK}(\mathcal{P}(\mu^*\alpha))$. Thus by the main result in [16], $\text{Vol}(\eta|_E) = 0$ for any exceptional divisor E . As $K_{Y/X} = \sum_{j=1}^m a_j E_j$, we infer that $\{\eta^{n-1}\} \cdot K_{Y/X} = 0$, which concludes the proof. \square

¹Such definition can be given for pseudoeffective classes but for the purposes of the paper we only consider big classes.

Remark 4.24. We claim that if X is a projective manifold, α is the cohomology class of a big \mathbb{Q} -divisor and there exists a birational morphism $\pi : Y \rightarrow X$ such that $\pi^*\alpha$ admits a Zariski decomposition then it is possible to produce a Fujita approximation in the sense of [40], and such that $\{\eta_k^{n-1}\} \cdot K_{Y_k/X} \rightarrow 0$.

Indeed, as explained in [50, Lemma 4.13], we can perturb the class $\{\eta\} = \mathcal{P}(\pi^*\alpha)$ and construct a sequence of ample classes $\{\eta_k\}_{k \in \mathbb{N}}$ on $Y_k = Y$ such that $\pi^*\alpha - \{\eta_k\} = \{D_k\}$ for a decreasing sequence of effective \mathbb{Q} -divisors D_k and such that $\{\eta_k\}_k$ converges to $\mathcal{P}(\pi^*\alpha)$.

In this situation, the proof of Lemma 4.13 then ensures that we have a sequence of model potentials $\phi_k \in \mathcal{N}_\theta$. One can check that ϕ_k is an increasing sequence. Moreover, its limit is a model potential as well thanks to [24, Corollary 4.7] and $\int_X \theta_{\phi_k}^n = \text{Vol}_Y(\eta_k) \rightarrow \text{Vol}_X(\theta) = \int_X \theta_{V_\theta}^n$. We can then infer that $\phi_k \nearrow V_\theta$.

We thus get $\{\eta_k^{n-1}\} \cdot K_{Y/X} \rightarrow \{\eta^{n-1}\} \cdot K_{Y/X} = 0$ where the latter equality follows as in the proof of Theorem 4.23. As observed in [50, Lemma 4.5] (and before in [12, Conjecture 2.5]), producing a Fujita approximation (for the so-called *big models*) such that (Condition B) holds for semiample classes $\{\eta_k\}$ implies a resolution to the Yau-Tian-Donaldson Conjecture.

In the general case of big classes on compact Kähler manifolds, clearly (Condition B) can be seen as a transcendental extension of [50, Conjecture 4.7].

5. MONGE-AMPÈRE MEASURES ON CONTACT SETS

The following result extends [36, Theorem 1.2] to big classes.

Theorem 5.1. *Assume (Condition A). Let $u_0, u_1 \in \mathcal{E}(X, \theta)$ such that $u_0, u_1 \in \text{Ent}(X, \theta)$. Let u_t be the psh geodesic connecting u_0 and u_1 . Fix $p \geq 1$. If $u_0 - u_1$ is bounded then \dot{u}_t^+ , and \dot{u}_t^- and $|u_t - u_0|$ are uniformly bounded, and $\dot{u}_t^+ = \dot{u}_t^- := \dot{u}_t$ almost everywhere with respect to $\theta_{u_t}^n$. Moreover*

$$\int_X |\dot{u}_t|^p \theta_{u_t}^n$$

is independent of $0 \leq t \leq 1$.

Proof. First of all recall that each geodesic is convex in t , moreover as $|u_0 - u_1|$ is uniformly bounded by a positive constant C , so is each derivative \dot{u}_t^+ and \dot{u}_t^- as well as each incremental ratio $\frac{u_{t+h} - u_t}{h}$ and $\frac{u_{t-h} - u_t}{-h}$ with $h > 0$ (see (2.13)). In particular $|u_t - u_0| \leq Ct \leq C$.

By Lemma 4.15 (items (iv) and (v)) we know that $|u_{1,k} - u_{0,k}| \leq \sup_X |u_1 - u_0| \leq C$, and $u_{0,k}, u_{1,k} \in \text{Ent}(X, \theta)$ for any $k \in \mathbb{N}$. Let $u_{t,k}$ be the psh geodesic connecting $u_{0,k}$ and $u_{1,k}$. It then follows that $|u_{k,t}^+|, |u_{k,t}^-|, |\dot{u}_t^+|, |\dot{u}_t^-|$ are uniformly bounded independently of $k \in \mathbb{N}, t \in [0, 1]$, and $x \in X$.

We now let ϕ_k be a monotone transcendental Fujita approximation of $\{\theta\}$ (whose existence is ensured by Lemma 4.14) and consider $\mathbf{L}_k : \text{PSH}(X, \theta, \phi_k) \rightarrow \text{PSH}(Y_k, \eta_k)$ be the map given by Lemma 4.7. We denote $v_{0,k} := \mathbf{L}_k(u_{0,k}), v_{1,k} := \mathbf{L}_k(u_{1,k})$. Then by Lemma 4.7 we know that $v_{t,k} := \mathbf{L}_k(u_{t,k})$ is the psh geodesic segment joining $v_{0,k}$ and $v_{1,k}$.

Since, for any $t \in [0, 1]$, the sets $\text{Exc}(\pi_k), \pi_k(\text{Exc}(\pi_k)), \{u_{t,k} = -\infty\}, \{u_{t,k} \circ \pi_k = -\infty\}$, are pluripolar we obtain that

$$\dot{u}_{t,k}^+ \circ \pi_k = \dot{u}_t^+, \quad \text{outside a pluripolar set}$$

and

$$(5.1) \quad \int_X |\dot{u}_{t,k}^+|^p \theta_{u_{t,k}}^n = \int_{Y_k} |\dot{v}_{t,k}^+|^p (\eta_k + dd^c v_{t,k})^n.$$

The same identity holds for the left derivatives for any $t \in (0, 1]$.

By Lemma 3.3 we have $u_{0,k} \circ \pi_k, u_{1,k} \circ \pi_k \in \text{Ent}(Y_k, \pi_k^*\theta)$. Hence, thanks to Lemma 4.7(v) we infer that $v_{0,k}, v_{1,k} \in \text{Ent}(Y_k, \eta_k)$.

Now, [36, Theorem 4.4] ensures that $\dot{v}_{t,k}^+ = \dot{v}_{t,k}^- =: \dot{v}_{t,k}$ almost everywhere with respect to $(\eta_k + dd^c v_{t,k})^n$ and that $\int_{Y_k} |\dot{v}_{t,k}|^p (\eta_k + dd^c v_{t,k})^n$ is constant in $t \in [0, 1]$. We then deduce that $\dot{u}_{t,k}^+ = \dot{u}_{t,k}^-$ almost everywhere with respect to $\theta_{u_{t,k}}^n$ and from (5.1) that

$$\int_X |\dot{u}_{t,k}|^p \theta_{u_{t,k}}^n$$

is constant in $t \in [0, 1]$.

On the other hand, by [36, Lemma 3.1] we can conclude that $\dot{u}_t^+ = \dot{u}_t^- =: \dot{u}_t$ almost everywhere with respect to $\theta_{u_t}^n$. The latter lemma is proved when the reference form is Kähler. Nevertheless, the arguments in [36, Lemma 3.1] work in the big setting as well since:

- by [39, Theorem 2.8] we know that $u_0, u_1 \in \mathcal{E}^1(X, \theta)$. Hence $u_t \geq P_\theta(u_0, u_1) \in \mathcal{E}^1(X, \theta)$ thanks to [21, Theorem 2.13];
- the Monge-Ampère energy $E(\theta; \cdot, V_\theta)$ is linear along psh geodesic by Theorem 4.1 and concave along affine paths [23, Corollary 5.9];
- $\text{Ent}(\text{Vol}(\theta)^{-1} \theta_{u_t}^n, \omega^n)$ is uniformly bounded in t thanks to Corollary 4.19.

Next, we use the sequence $u_{t,k}$ (which has uniformly bounded entropy and weakly converges to $\theta_{u_t}^n$ thanks to Lemma 4.15 and Remark 2.6) to implement the arguments in [36, pages 14-15]. This gives that for $t \in [0, 1]$

$$(5.2) \quad \int_X |\dot{u}_{t,k}|^p \theta_{u_{t,k}}^n \rightarrow \int_X |\dot{u}_t|^p \theta_{u_t}^n$$

as $k \rightarrow +\infty$. Since, by the above, the quantity at the left hand side is constant in t , so are the quantities at the right hand side. The proof is then complete. \square

Repeating word by word the arguments in [36, Theorem 5.1] we have the following result which generalizes [36, Theorem 5.1] and the main theorem in [38].

Proposition 5.2. *Assume (Condition A). Assume $u \in \text{PSH}(X, \theta)$, $v \in \text{Ent}(X, \theta)$ and $u \leq v$. Then*

$$\mathbf{1}_{\{u=v\}} \theta_u^n = \mathbf{1}_{\{u=v\}} \theta_v^n.$$

Following again word by word the proof of [36, Corollary 5.2] we obtain:

Corollary 5.3. *Assume (Condition A). Assume $u \in \text{PSH}(X, \theta)$, $v \in \text{Ent}(X, \theta)$ and $u = v$ θ_v^n -almost everywhere, then $u = v$.*

6. GEODESIC DISTANCE

In [45], Gupta showed how $\mathcal{E}^p(X, \theta)$ is naturally endowed with a complete distance d_p . We briefly recall how such distance is constructed. Let ϕ_k be a monotone transcendental Fujita approximation of $\{\theta\}$ (Lemma 4.14) and consider $\mathbf{L}_k : \text{PSH}(X, \theta, \phi_k) \rightarrow \text{PSH}(Y_k, \eta_k)$ be the map given by Lemma 4.7. Then each $\mathcal{E}^p(X, \theta, \phi_k)$ can be endowed with a metric d_p . More precisely,

$$(6.1) \quad d_p(u_0, u_1) := d_p(\mathbf{L}_k(u_0), \mathbf{L}_k(u_1))$$

for any $u_0, u_1 \in \mathcal{E}^p(X, \theta, \phi_k)$ where the d_p -distance on $\mathcal{E}^p(Y_k, \eta_k)$ is defined in [35]. Moreover, as proved in [45, Theorem 7.4] the d_p distance on $\mathcal{E}^p(X, \theta)$ satisfies

$$d_p(u_0, u_1) = \lim_{k \rightarrow +\infty} d_p(P_\theta[\phi_k](u_0), P_\theta[\phi_k](u_1))$$

for any $u_0, u_1 \in \mathcal{E}^p(X, \theta)$.

We prove here that the d_p distance on $\mathcal{E}^p(X, \theta)$ has an explicit expression when the potentials have the same singularity type and finite entropy. The following result extends [36, Theorem 1.2] to the case of big cohomology classes.

Theorem 6.1. *Assume (Condition A). Fix $p \geq 1$. Let $u_0, u_1 \in \text{Ent}(X, \theta) \cap \mathcal{E}^p(X, \theta)$ and u_t be the psh geodesic connecting u_0 and u_1 . If $u_0 - u_1$ is bounded then*

$$(6.2) \quad d_p^p(u_0, u_1) = \int_X |\dot{u}_t|^p \theta_{u_t}^n$$

for any $t \in [0, 1]$.

We recall that the right term in (6.2) is independent of $0 \leq t \leq 1$ thanks to Theorem 5.1.

Proof. Using the same notations of above we set $u_{i,k} := P_\theta[\phi_k](u_i)$ for $i = 0, 1$ and $v_{i,k} := \mathbf{L}_k(u_{i,k})$. We also denote by $u_{t,k}, v_{t,k}$ the psh geodesics joining $u_{0,k}, u_{1,k}$ and $v_{0,k}, v_{1,k}$ respectively. Combining [36, Theorem 1.2] with (5.1) we have

$$d_p^p(u_{0,k}, u_{1,k}) = d_p^p(v_{0,k}, v_{1,k}) = \int_{Y_k} |\dot{v}_{t,k}|^p (\eta_k + dd^c v_{t,k})^n = \int_X |\dot{u}_{t,k}|^p \theta_{u_{t,k}}^n,$$

where we observe that the first identity follows from the definition given in (6.1). It follows from (5.2) that

$$d_p^p(u_0, u_1) = \lim_{k \rightarrow +\infty} d_p^p(u_{0,k}, u_{1,k}) = \lim_{k \rightarrow +\infty} \int_X |\dot{u}_{t,k}|^p \theta_{u_{t,k}}^n = \int_X |\dot{u}_t|^p \theta_{u_t}^n.$$

This concludes the proof. \square

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