ASYMPTOTICS OF FUBINI-STUDY CURRENTS FOR SEQUENCES OF LINE BUNDLES

MELODY WOLFF

Abstract. We study the Fubini-Study currents and equilibrium metrics of continuous Hermitian metrics on sequences of holomorphic line bundles over a fixed compact Kähler manifold. We show that the difference between the Fubini-Study currents and the curvature of the equilibrium metric, when appropriately scaled, converges to 0 in the sense of currents. As a consequence, we obtain sufficient conditions for the scaled Fubini-Study currents to converge weakly.

Contents

1. Introduction

In this paper, we will be working with sequences of holomorphic line bundles $\{L_p\}$ with continuous Hermitian metrics h_p, $p \geq 1$, and will be studying asymptotic properties of the Fubini-Study current first explored in [CMM]. We restrict our work to compact Kähler manifolds while allowing the metrics to have non-positive curvature. Although we won't require positivity conditions on our metrics, we will require the existence of positively curved metrics with growth conditions similar to those used in [CMM].

In 1988 Tian explored the case where $(L_p, h_p) = (L^p, h^p)$, with $(L^p, h^p) = (L^{\otimes p}, h^{\otimes p})$ for some holomorphic line bundle L equipped with a smooth metric h (see [T]). He showed that if (X, ω) is a compact Kähler manifold with a line bundle (L, h) such that the curvature $c_1(L, h)$ is positive and h is smooth, then the normalized Fubini-Study forms γ_p/p (see 2.3 for definition) converge to $c_1(L, h)$ in C^2. Later results showed that the

Date: October 6, 2024.

2010 *Mathematics Subject Classification.* Primary 32L10; Secondary 31C10, 32A36, 32L05, 32Q15, 32U05, 32U40.

Key words and phrases. Bergman kernel function, Fubini-Study current, singular Hermitian metric, big line bundle.

convergence was actually in C^∞ [C, R, Z]. We refer to [MM]. As a consequence of this result, Tian showed that particular Kähler-Einstein forms could be approximated by Fubini-Study currents, which answered a question of Yau [Y].

Also included in [MM] is an asymptotic expansion of the Bergman kernel (see [MM, Theorem 5.4.10]). This expansion has been shown to provide information about the underlying Kähler manifold. In particular, the asymptotic expansion can be used to prove the Kodaira embedding theorem (see [MM, section 5.1.2]).

The assumptions of Tian's results on (L, h) were relaxed by Coman and Marinescu in [CM]. They worked in the case $(L_p, h_p) = (L^p, h^p)$ and showed that if $c_1(L, h)$ was an integrable Kähler current, then the aforementioned convergence result holds in the sense of currents. The results were further generalized by Coman, Ma, and Marinescu (see [CMM]). They showed that if $c_1(L_p, h_p) \geq a_p\omega$ where $a_p \to \infty$, and

$$A_p = \int_X c_1(L_p, h_p) \wedge \omega^{n-1},$$

then

$$\frac{\gamma_p - c_1(L_p, h_p)}{A_p} \to 0$$

weakly as currents.

Berman's work in 2009 introduced the notion of an equilibrium metric h^{eq} (See section 2.2 for definition) corresponding to a smooth metric h on a holomorphic line bundle L (see [B]). He worked in the setting where $(L_p, h_p) = (L^p, h^p)$. He showed that for any compact subset Ω of $X \backslash \mathbb{B}_+(L)$, where $\mathbb{B}_+(L)$ is the augmented base locus (definition given in [B]), there exists $C_\Omega \geq 0$ such that

$$-\frac{C_\Omega}{p} \leq \frac{\log P_p}{p} - (\varphi^{eq} - \varphi) \leq \frac{C_\Omega + n\log p}{p},$$

where φ and φ^{eq} are the global weights of h and h^{eq}, respectively (see [B, Theorem 1.5]). Another interesting result of his, is the convergence

$$\limsup_p \frac{\dim H^0(X, L^p)}{p^n} = \int_{U(L)} \frac{c_1(L, h^{eq})^n}{n!},$$

where $n = \dim(X)$ and $U(L)$ is the set where the weights of h^{eq} are locally bounded. He also showed the following weak convergence of measures

$$\frac{P_p\omega^n}{p^n} \to \chi_{U(L)}\left(\frac{c_1(L, h^{eq})^n}{n!}\right),$$

where P_p is the Bergman kernel function (as defined in Section 2.3) and $\chi_{U(L)}$ is the characteristic function.

In 2019, Coman, Marinescu, and Nguyên used the equilibrium metric to generalize Tian's work (see [CMN, Cor. 5.7]). Like Tian and Berman, they worked in the case where $(L_p, h_p) = (L^p, h^p)$ and showed

$$\frac{\gamma_p}{p} \to c_1(L, h^{eq})$$

weakly as currents.

The following conditions will serve as the setting for most of our results in this paper:

(A) (X,ω) is a compact (connected) Kähler manifold of complex dimension n.

(B) L_p, $p \geq 1$, is a holomorphic line bundle on X equipped with a continuous metric h_p and a singular metric g_p verifying

$$c_1(L_p,g_p) \geq a_p\omega \text{ on } X, \text{ where } \lim_{p\to\infty} a_p = \infty. \tag{1.1}$$

Set

$$A_p = \int_X c_1(L_p,g_p) \wedge \omega^{n-1}.$$

(B') L_p, $p \geq 1$, is a pseudo-effective holomorphic line bundle equipped with the continuous metric h_p. There exists an open coordinate polydisc cover $\{V_\alpha\}$ of X, frames e_p^α of L_p on V_α, functions $\phi^\alpha \in C(V_\alpha)$, and constants $A_p > 0$, such that $A_p \to \infty$ and

$$\phi_p^\alpha/A_p \to \phi^\alpha \text{ locally uniformly,}$$

where ϕ_p^α is the local weight of h_p corresponding to e_p^α.

Following Berman, we define h_p^{eq} to be the equilibrium metric of h_p. As well, we recall that γ_p is the Fubini-Study current. For definitions refer to sections 2.2 and 2.3.

For any coordinate polydisc U and $p \geq 1$, let e_p be a local frame of L_p on U (see Section 2.2). Let $\phi_p : U \to \mathbb{R}$ be the continuous function such that

$$h_p(e_p,e_p) = e^{-2\phi_p}.$$

Similarly, we define $\rho_p : U \to [-\infty,\infty)$ by

$$g_p(e_p,e_p) = e^{-2\rho_p}.$$

The functions ϕ_p and ρ_p are called the local weights of h_p and g_p on U.

Our main results are:

Theorem 1.1. *Let (X,ω) and (L_p,h_p) be as in (A) and (B). If every $x \in X$ has a neighborhood U with local frames e_p of L_p, such that the families of local weights $\{\phi_p/A_p\}$ and $\{\rho_p/A_p\}$ are uniformly bounded in $L^1(U)$, and there exists $M > 0$ such that*

$$-M\omega \leq c_1(L_p,h_p)/A_p \leq M\omega, \tag{1.2}$$

then

$$\frac{\gamma_p - c_1(L_p,h_p^{eq})}{A_p} \to 0 \text{ weakly as currents.} \tag{1.3}$$

If h_p verifies (1.1), then $h_p = h_p^{eq}$. In this case, (1.3) becomes

$$\frac{\gamma_p - c_1(L_p,h_p)}{A_p} \to 0$$

weakly as currents. This is a special case of the convergence condition shown in [CMM]. If in addition we assume that $(L_p,h_p) = (L^p,h^p)$ and set $A_p = p$, our assumptions are automatically satisfied, and we obtain the convergence shown in [CM].

Theorem 1.2. *Let (X,ω) and (L_p,h_p) be as in (A) and (B). If every $x \in X$ has a neighborhood U with local frames e_p, such that the collection of scaled local weights $\{\phi_p/A_p\}$ is equicontinuous and uniformly bounded, and $\{\rho_p/A_p\}$ is uniformly bounded in $L^1(U)$, then (1.3) holds.*

When $(L_p,h_p) = (L^p,h^p)$ equicontinuity is trivial, as in that case we can take $A_p = p$, and for particular local frames we have $\phi_p/p = \phi_1$. Like Theorem 1.1, this result can be considered a generalization of the convergence in [CM]. Another case where equicontinuity is automatically satisfied is when (L_p,h_p) is a tensor product of powers of several line bundles. That case is explored further in Section 4. There, as a corollary of this theorem, we show that (1.3) holds with fewer assumptions.

Theorem 1.3. *Let (X,ω) and (L_p,h_p) be as in (A) and (B). If every $x \in X$ has a neighborhood U, with local frames e_p, such that the family $\{\phi_p/A_p\}$ is equicontinuous, and $\{(\phi_p-\rho_p)/A_p\}$ is uniformly bounded, then (1.3) holds.*

When h_p satisfies (1.1), then we may assume $h_p = g_p$, and so $\{(\phi_p-\rho_p)/A_p = 0\}$ is automatically bounded. In general, the difference $(\phi_p-\rho_p)$ defines a function on all X (see Section 2.2). In the case where $(L_p,g_p) = (L^p,g^p)$ and $e_p = e^{\otimes p}$, we can take $A_p = p$, and $\{(\phi_p-\rho_p)/p\}$ is bounded whenever $\{\phi_p/p\}$ is. In this case, Theorem 4.7 shows that γ_p/p converges.

This leads us to the question of under what conditions does γ_p/A_p converge? Due to Coman, Ma, and Marinescu, we know if $h_p = g_p$ and $c_1(L_p,h_p)/A_p$ converges, then so does γ_p/A_p. In the setting of our previous theorems we also know that if $c_1(L_p,h_p^{eq})/A_p$ converges, then so does γ_p/A_p. So to give a partial answer to our question, we assume $c_1(L_p,h_p)/A_p$ converges, and ask under what conditions does $c_1(L_p,h^{eq})/A_p$ converge as well? We will state a theorem with sufficient conditions for such convergence. Before we do, we introduce a necessary proposition.

Proposition 1.4. *Let X, V_α, and ϕ^α be as in (A) and (B'), then for all α,β there exist pluriharmonic functions $\psi^{\alpha\beta}$ on $V_\alpha \cap V_\beta$ such that*

$$\phi^\alpha = \phi^\beta + \psi^{\alpha\beta}.$$

In this case, there exists a real closed $(1,1)$-current T on X defined by

$$T|_{V_\alpha} = dd^c\phi^\alpha, \tag{1.4}$$

where $d^c = \frac{1}{2\pi i}(\partial - \overline{\partial})$. Let $\{T\}$ denote the cohomology class of T, and fix a smooth form $\theta \in \{T\}$. From the definition, we have

$$T = \theta + dd^c\varphi,$$

where $\varphi \in L^1(X)$. Since ϕ^α is continuous, it follows that φ is continuous.

Theorem 1.5. *Let (X,ω), (L_p,h_p), and A_p be as in (A) and (B'). Suppose the following conditions hold:*

(a) There exist $\delta_p \geq 0$ with $\delta_p \to 0$ such that

$$T - \delta_p\omega \leq \frac{c_1(L_p,h_p)}{A_p} \text{ on } V_\alpha \text{ for all } \alpha.$$

(b) There exist $\varrho \in PSH(X,\theta)$ *and some* $c > 0$ *such that*

$$\theta + dd^c\varrho \geq c\omega.$$

Then

$$\frac{\gamma_p}{A_p} \to \theta + dd^c\varphi^{eq} \text{ and } \frac{c_1(L_p, h_p^{eq})}{A_p} \to \theta + dd^c\varphi^{eq} \tag{1.5}$$

weakly as currents, where

$$\varphi^{eq} := \sup\{\psi \in PSH(X,\theta) \mid \psi \leq \varphi\}.$$

Moreover,

$$\varphi^{eq} = \left[\limsup_p \left(\frac{\varphi_p^{eq} - \varphi_p}{A_p}\right)\right]^* + \varphi, \tag{1.6}$$

where φ_p *and* φ_p^{eq} *are the global weights of* h_p *and* h_p^{eq} *(see section 2.3).*

Here $PSH(X,\theta)$ denotes the class of θ–plurisubharmonic functions (see section 2.1) Note that (1.5) and condition (b) are both independent of our choice of θ. The independence of (1.5) is shown in the proof of Theorem 1.5. In the case of (b), if $\widetilde{\theta} \in \{T\}$, then θ and $\widetilde{\theta}$ are in the same cohomology class, so clearly there exists $\widetilde{\varrho} \in L^1(X)$ with $\widetilde{\theta} + dd^c\widetilde{\varrho} = \theta + dd^c\varrho$.

We said we were interested in cases where $c_1(L_p, h_p)/A_p$ converges, and this is one of them, as if $\phi_p^\alpha/A_p \to \phi^\alpha$ uniformly, then

$$\frac{c_1(L_p, h_p)}{A_p} \to T \text{ weakly as currents, where } T \text{ is as in (1.4).}$$

The function φ_p^{eq} is defined with respect to a smooth metric h_p^0 in section 2.2. One can show using the C^∞ version of the first Cousin Problem, that for particular choices of h_p^0 and θ, equation (1.6) reduces to $\varphi^{eq} = [\limsup_p(\varphi_p^{eq}/A_p)]^*$.

In Lemma 3.4 we will show directly that there exists $\psi \in PSH(X,\theta)$ with $\psi \leq \varphi$, hence $\varphi^{eq} \not\equiv \infty$.

The purpose of conditions (a) and (b) is to allow us to apply Demailly's L^2 estimates for $\overline{\partial}$, which requires some degree of positivity.

In Section 4 we will look at the case where there exists a continuous $(1,1)$–form Φ such that $c_1(L_p, h_p)/A_p \to \Phi$ uniformly. This assumption implies the convergence in (B'). It is examined in Corollaries 4.3 and 4.6. If we make the additional assumption $L_p = L^p$, then (b) is automatically satisfied whenever L is big, a fact which can be used to prove a special case of Theorem 4.7.

This paper has the following structure. In Section 2, we discuss the necessary information about quasi-plurisubharmonic functions, define global weights for our metrics, and give a construction of h_p^{eq}. Section 3 is devoted to proofs of the theorems above. In Section 4, we present applications of our main results.

2. Preliminaries

2.1. **Quasi-plurisubharmonic functions.** Recall that a quasi-plurisubharmonic (qpsh) function on X is a function, $f : X \to [-\infty, \infty)$, that is locally the sum of a plurisubharmonic (psh) function and a smooth function. Given a real closed smooth $(1,1)$–form θ, a θ–plurisubharmonic

(θ–psh) function is a qpsh function $f : X \to [-\infty, \infty)$ such that $\theta + dd^c f \geq 0$. We denote by $PSH(X, \theta)$ the class of all θ–psh functions.

In the setting of Theorem 1.1, note the condition $-M\omega \leq c_1(L_p, h_p)/A_p \leq M\omega$ implies $-M\omega \leq dd^c\phi_p/A_p \leq M\omega$ on any contractible Stein open set V where $dd^c\phi_p = c_1(L_p, h_p)$. In this case $\pm\phi_p/A_p \in PSH(V, M\omega)$. This is one of the reasons we are interested in the notion of qpsh functions. Another is that they are necessary for us to define the equilibrium metrics h_p^{eq}.

One property will we need to note about θ-psh functions, is that two are equal almost everywhere, then they are equal. We will use this fact throughout this paper.

2.2. **Global weights and the equilibrium metric.** We first observe that if Ω is a contractible Stein open set in $\mathbb{C}^n$, then a result of Oka tells us $H^1(\Omega, \mathcal{O}^*) \cong H^2(\Omega, \mathbb{Z})$, hence line bundles over Ω are trivial (see [Hu, p.201] and [O]). In particular, if $V \subset X$ is a coordinate polydisc, then L_p is trivialized on V for all $p \geq 1$.

We fix a finite open cover $\{U_\alpha\}$ of X by coordinate polydiscs, such that for all α, there exists a coordinate polydisc V_α with $U_\alpha \Subset V_\alpha$. Fix local frames e_p^α. Let ϕ_p^α and ρ_p^α denote the local weights of h_p and g_p on V_α. As X is compact, we assume that the V_α are the neighborhoods referred to in our theorems.

In order to define global weights for our given metrics, as well as give a definition of h_p^{eq}, we fix a smooth metric h_p^0 on L_p. Define $\xi_p^\alpha \in C^\infty(V_\alpha)$ to be the local weights of h_p^0, which are given by

$$h_p^0(e_p^\alpha, e_p^\alpha) = e^{-2\xi_p^\alpha}. \tag{2.1}$$

The global weights of h_p and g_p are the functions $\varphi_p : X \to \mathbb{R}$ and $\varrho_p : X \to [-\infty, \infty)$, defined by

$$h_p = h_p^0 e^{-\varphi_p} \text{ and } g_p = h_p^0 e^{-2\varrho_p}. \tag{2.2}$$

Note that $(\phi_p^\alpha - \rho_p^\alpha) = (\varphi_p - \varrho_p)$, which is a global function.

In order to define h_p^{eq}, we first set

$$\theta_p = c_1(L_p, h_p^0).$$

Let $\varphi_p^{eq} : X \to [-\infty, \infty)$ be the θ_p–psh upper envelope

$$\varphi_p^{eq} = \sup\{\psi \in PSH(X, \theta_p) \mid \psi \leq \varphi_p\}. \tag{2.3}$$

Observe that φ_p is continuous and ϱ_p is bounded above (it is psh), so $\varrho_p - C \leq \varphi_p$ for some $C \geq 0$. This tells us

$$\varrho_p - C \in \sup\{\psi \in PSH(X, \theta_p) \mid \psi \leq \varphi_p\},$$

so $\varphi_p^{eq} \in PSH(X, \theta_p)$ and $\varphi_p^{eq} \leq \varphi_p$. We define

$$h_p^{eq} = h_p^0 e^{-2\varphi_p^{eq}}.$$

The global weight of h_p^{eq} is φ_p^{eq}. A local construction of the equilibrium metric is given by Berman in [B]. His definition is clearly equivalent to ours.

2.3. **Fubini-Study currents and Bergman kernels.** Let $H^0_{(2)}(X,L_p)$ denote the Bergman space of square integrable sections of L_p relative to h_p and ω, that is,

$$H^0_{(2)}(X,L_p,h_p) = \left\{S \in H^0(X,L_p) \mid \|S\|^2_{h_p} < \infty\right\},$$

where

$$\|S\|^2_{h_p} := \int_X |S|^2_{h_p} \frac{\omega^n}{n!}.$$

When the metric is clear, the notation is shortened to $H^0_{(2)}(X,L_p)$. This space will be endowed with the inner product

$$\langle S_1, S_2\rangle = \int_X h_p(S_1,S_2)\frac{\omega^n}{n!}.$$

Let P_p be the Bergman kernel function of the space $H^0_{(2)}(X,L_p)$. For all $p \geq 1$, a global definition is given by fixing an orthonormal basis, $\{S^p_j\}$, of $H^0_{(2)}(X,L_p)$. We then define $d(p) := \dim H^0_{(2)}(X,L_p)$ and

$$P_p(x) := \sum_{j=1}^{d(p)} |S^p_j(x)|^2_{h_p}.$$

The following is a well-known variational characterization of the Bergman kernel, which will be useful in our work,

$$P_p = \sup_{S\in H^0_{(2)}(X,L_p)} \left(\frac{|S|^2_{h_p}}{\|S\|^2_{h_p}}\right).$$

We recall that γ_p was the Fubini-Study current of $H^0_{(2)}(X,L_p)$. To define it explicitly, we let $U \subset X$ be a contractible Stein open set. Let $s^p_j \in \mathcal{O}(U)$ be defined by $S^p_j = s^p_j e_p$. Then

$$\gamma_p|_U := \frac{1}{2}\, dd^c \log\left(\sum_{j=1}^{d(p)} |s^p_j|^2\right).$$

Note, that the equation above defines a global current on X. One way to see that is the following equivalent definition. We consider the Kodaira map $\Psi_p : X \dashrightarrow \mathbb{CP}^{k-1}$ defined by

$$x \mapsto [s^p_1(x) : \ldots : s^p_{d(p)}(x)],\ \forall x \in U\setminus V(s^p_1,\ldots s^p_{d(p)}),$$

where $V(s^p_1,\ldots,s^p_k)$ is the analytic variety. It is well known that this map is independent of our choice of U. We then define

$$\gamma_p = \Psi^*_p\omega_{FS},$$

where ω_{FS} is the Fubini-Study form on $\mathbb{CP}^{k-1}$.

Note, the following is a well-known identity which will be helpful in our work:

$$\gamma_p = \frac{dd^c \log P_p}{2} + c_1(L_p,h_p). \tag{2.4}$$

3. Main Results

3.1. **Proofs of Theorems 1.1, 1.2, and 1.3.** We use the notation and definitions introduced in Sections 1 and 2. We start by stating and proving two lemmas necessary for our proof of Theorem 1.1.

Lemma 3.1. *If $V \subseteq \mathbb{C}^n$ is a polydisc and $v \in L^1(V)$ such that for some $M \geq 0$ we have*

$$-M\omega \leq dd^c v \leq M\omega,$$

then there exists $\widetilde{v} \in C(V)$ with $\widetilde{v} = v$ a.e. and $dd^c\widetilde{v} = dd^c v$.

Proof. Let ζ be a smooth potential for $M\omega$ on V. Since $-M\omega \leq dd^c v \leq M\omega$ it follows that $\zeta + v$ and $\zeta - v$ are both equal a.e. to psh functions on V. Let u, w be psh functions with $\zeta + v = u$ and $\zeta - v = w$ a.e. Note that $2\zeta = u + w$ a.e. Since both sides are psh we have $2\zeta = u + w$ everywhere. Then, $2\zeta - u = w$, so as $2\zeta - u$ is lower semicontinuous, and w is as well. We see that w is both upper and lower semicontinuous, so it is continuous, and $v = w$ a.e.

♥

Lemma 3.2. *Let $U, V \subset X$ be open coordinate polydiscs with $U \Subset V$. Assume $v_p \in C(V)$ and $v \in L^1(V)$ such that $v_p \to v$ in L^1,*

$$-M\omega \leq dd^c v_p \leq M\omega.$$

Then, for all $\epsilon > 0$ there exists $p_0 = p_0(\epsilon)$ and $\delta = \delta(\epsilon)$ such that if $z \in U$, $r < \delta$, and $p > p_0$, then

$$\sup_{B(z,r)} v_p(z) - \inf_{B(z,r)} v_p(z) < \epsilon.$$

Proof. By Lemma 3.1 we can and do assume v is C. Let ζ be a smooth real potential for $M\omega$ on V. By [Hö, Theorem 3.2.12], if ζ is a real smooth potential for $M\omega$, then we can and do assume $\zeta \pm v \in PSH(V)$.

From Hartog's lemma (see [Hö, Theorem 3.2.13]) applied to v_p and v it follows that for all $K \Subset V$ we have

$$\limsup_{p\to\infty} \left[\sup_K v_p\right] \leq \sup_K v. \tag{3.1}$$

Let $B_1, \ldots, B_m$ be a finite cover of U by balls such that $2B_j \Subset V$ for all j, where $2B_j$ is the dilation of B_j by a factor of 2. Using (3.1), since there are finitely many B_j, we can choose p_0 large enough so that

$$\sup_{2B_j} v_p \leq \sup_{2B_j} v + \epsilon \tag{3.2}$$

for all j and $p > p_0$. Similarly, we may assume for such p we have

$$\sup_{2B_j}(-v_p) \leq \sup_{2B_j}(-v) + \epsilon,$$

thus

$$-\inf_{2B_j} v_p \leq -\inf_{2B_j} v + \epsilon. \tag{3.3}$$

Let $\delta > 0$ be the minimum of the radii of the B_j. Additionally, assume $r < \delta$ and $z \in U$. Since the sets $B_1, \ldots, B_m$ cover U, we deduce $z \in B_j$ for some j, therefore $B(z, r) \subset 2B_j$, and if $p \geq p_0$, then by (3.2) and (3.3) we have

$$\sup_{B(z,r)} v_p - \inf_{B(z,r)} v_p \le \sup_{2B_j} v_p - \inf_{2B_j} v_p \le \sup_{2B_j} v - \inf_{2B_j} v + 2\epsilon.$$

As v is continuous, by letting the radii of B_j go to zero we have our desired upper bound.

♥

Proof of Theorem 1.1. Recall that $\{U_\alpha\}$ is a finite cover of X by coordinate polydiscs. Moreover, for all α there exists a coordinate polydisc V_α such that $U_\alpha \subseteq V_\alpha$.

Notice that for $p \ge 1$, the sets V_α and local frames e_α satisfy the assumptions of our theorem. That is, the local weights ϕ_p^α and ρ_p^α defined by

$$h_p(e_p^\alpha, e_p^\alpha) = e^{-2\phi_p^\alpha} \text{ and } g_p(e_p^\alpha, e_p^\alpha) = e^{-2\rho_p^\alpha},$$

form a family $\{\phi_p^\alpha/A_p, \rho_p^\alpha/A_p\}$ which is uniformly bounded in $L^1(V_\alpha)$. Additionally, for each $p \ge 1$, we recall that h_p^0 is a smooth metric on L_p with global weights ξ_p^α with respect to e_p^α. The global weights φ_p and ϱ_p are then defined by

$$h_p = h_p^0 e^{-2\varphi_p} \text{ and } g_p = h_p e^{-2\varrho_p}.$$

Furthermore, the equilibrium metric h_p^{eq} is given by $h_p^{eq} = h_p^0 e^{-2\varphi_p^{eq}}$.

For a fixed p, let $\{S_j\}$ be an orthonormal basis of $H^0_{(2)}(X, L_p)$ represented locally by $S_j = s_j^\alpha e_p^\alpha$. Observe that by (2.4) we have

$$\frac{\gamma_p - c_1(L_p, h_p^{eq})}{A_p} = dd^c\left[\left(\frac{(\log P_p)/2 + \varphi_p}{A_p}\right) - \frac{\varphi_p^{eq}}{A_p}\right]. \tag{3.4}$$

Define

$$\beta_p = \frac{(\log P_p)/2 + \varphi_p}{A_p}.$$

It follows that

$$\frac{\gamma_p - c_1(L_p, h_p^{eq})}{A_p} = dd^c\left(\beta_p - \frac{\varphi_p^{eq}}{A_p}\right),$$

so if we show $\beta_p - \varphi_p^{eq}/A_p \to 0$ in L^1 we will prove our claim. This happens if every subsequence has a subsequence where the convergence holds, so it suffices to show a single subsequence where convergence holds.

Let $\zeta^\alpha \in C^\infty(V_\alpha, \mathbb{R})$ be a real smooth potential for $M\omega$. By (1.2), it follows that $\phi_p^\alpha/A_p + \zeta^\alpha \in C(V_\alpha)$ is psh. For all α define

$$\delta = \delta(\alpha) = \operatorname{dist}(\partial V_\alpha, U_\alpha),$$

where dist is the distance induced by ω on X.
For any $\epsilon \ge 0$, let

$$U_\alpha^\epsilon = \{x \in V_\alpha \mid \operatorname{dist}(x, U_\alpha) < \epsilon\}.$$

By the subaveraging property of psh functions and the $L^1(V_\alpha)$ uniform boundedness of $\{\phi_p^\alpha/A_p\}$, it follows that $\{\phi_p^\alpha/A_p + \zeta^\alpha\}$ is uniformly upper bounded on $U_\alpha^{\delta/2}$. The same logic tells us $\{\zeta^\alpha - \phi_p^\alpha/A_p\}$ is uniformly upper bounded on $U_\alpha^{\delta/2}$. Since ζ^α is continuous, putting these two bounds together allows us to conclude that $\{\phi_p^\alpha/A_p + \zeta^\alpha\}$ is a uniformly bounded family of psh functions on $U_\alpha^{\delta/2}$. From [Hö, Theorem 3.2.12], there exists a psh function $\widehat{\phi}^\alpha$ and a subsequence $\{\phi_{p_j}^\alpha/A_{p_j} + \zeta^\alpha\}$ such that

$$\phi^\alpha_{p_j}/A_{p_j} + \zeta^\alpha \to \widehat{\phi}^\alpha \text{ in } L^1(U_\alpha^{\delta/2}).$$

Since there are finitely many α, by passing to a further subsequence, if necessary, we may assume

$$\phi^\alpha_{p_j}/A_{p_j} + \zeta^\alpha \to \widehat{\phi}^\alpha \text{ in } L^1(U_\alpha^{\delta/2}) \text{ and } a.e.$$

for all α simultaneously. This subsequence is the one where we will show convergence holds.

Let $\epsilon > 0$. By Lemma 3.2 it follows that we can choose $r = r(\alpha, \epsilon) \in \mathbb{R}$ satisfying

$$\delta/8 > r > 0$$

small enough and $j_0 > 0$ large enough so that

$$\sup_{B(z,r)} \left(\phi^\alpha_{p_j}/A_{p_j}\right) - \inf_{B(z,r)} \left(\phi^\alpha_{p_j}/A_{p_j}\right) < \epsilon \tag{3.5}$$

for all $j > j_0$ and $z \in U_\alpha$. As there are finitely many α, WLOG we may assume the above holds for all α.

Let $x \in U_\alpha$. By our choice of r, it follows that $V := B(x, 2r) \subset U_\alpha^{\delta/4}$. Let $U = B(x, r)$ and $z \in U$. Suppose $S \in H^p_{(2)}(X, L_p)$ and define $s^\alpha \in \mathcal{O}(V_\alpha)$ by $S = s^\alpha e^\alpha_p$ on V_α. Since s^α is holomorphic, it follows from subaveraging that

$$|S(z)|_p^2 = |s^\alpha(z)|^2 e^{-2\phi^\alpha_p(z)} \le \frac{e^{-2\phi^\alpha_p(z)}}{\lambda(B(z,r))} \int_{B(z,r)} |s^\alpha|^2 d\lambda$$

where λ is the Lebesgue measure. It follows that there exists $C_1 > 0$ such that

$$\begin{aligned} |S(z)|_p^2 &\le \frac{C_1 e^{-2\phi^\alpha_p(z)}}{\lambda(B(z,r))} \int_{B(z,r)} |s^\alpha|^2 \frac{\omega^n}{n!} \\ &\le \frac{C_1 e^{-2\phi^\alpha_p(z) + \max_{\overline{B}(z,r)} 2\phi^\alpha_p}}{\lambda(B(z,r))} \int_{B(z,r)} |s^\alpha|^2 \left(e^{-2\phi^\alpha_p}\right) \frac{\omega^n}{n!} \\ &= \frac{\|S\|^2_{h_p} C_1 e^{-2\phi^\alpha_p(z) + \max_{\overline{B}(z,r)} 2\phi^\alpha_p}}{\lambda(B(z,r))}. \end{aligned}$$

By the variational characterization of the Bergman kernel, we have

$$P_p(z) \le \frac{C_1 e^{-2\phi^\alpha_p(z) + 2\max_{\overline{B}(z,r)} \phi^\alpha_p}}{\lambda(B(z,r))},$$

therefore

$$\frac{\log P_{p_j}(z)}{2A_{p_j}} \le \frac{\log C_1 - 2n\log r - n\log\pi + \log n!}{2A_{p_j}} + \max_{\overline{B}(z,r)} \left(\frac{\phi^\alpha_{p_j}}{A_{p_j}}\right) - \frac{\phi^\alpha_{p_j}(z)}{A_{p_j}}.$$

By (3.5) and by enlarging j_0, if necessary, it follows that if $j \ge j_0$, then on $B(x, r)$ we have

$$\frac{\log P_{p_j}}{2A_{p_j}} \le \frac{-2n\log r}{2A_{p_j}} + \epsilon.$$

By covering X with finitely many such $B(x, r)$, we may assume WLOG that the above inequality holds on X.

Note that

$$\frac{\log P_{p_j}}{2} + \varphi_{p_j} - A_{p_j}\epsilon + \frac{2n\log r}{2} \le \varphi_{p_j}, \tag{3.6}$$

and

$$dd^c\left(\frac{\log P_{p_j}}{2}+\varphi_{p_j}-\epsilon A_{p_j}+\frac{2n\log r}{2}\right)=\gamma_{p_j}-\theta_{p_j},$$

so the L.H.S. of (3.6) is θ_{p_j}–psh. By the definition of φ_p^{eq}, we have

$$\frac{\log P_{p_j}}{2}+\varphi_{p_j}-\epsilon A_{p_j}+\frac{2n\log r}{2}\le\varphi_{p_j}^{eq},$$

so

$$\frac{\log P_{p_j}}{2A_{p_j}}+\frac{\varphi_{p_j}}{A_{p_j}}\le\frac{-2n\log r}{2A_{p_j}}+\epsilon+\frac{\varphi_{p_j}^{eq}}{A_{p_j}},$$

and

$$\beta_{p_j}-\frac{\varphi_{p_j}^{eq}}{A_{p_j}}\le\frac{-2n\log r}{2A_{p_j}}+\epsilon. \tag{3.7}$$

This is the upper bound we shall use.

To show the lower bound, we first define $t_p=1/\sqrt{a_p}$ and set

$$\widetilde{h_p}=h_p^0e^{-2(1-t_p)\varphi_p^{eq}-2t_p\varrho_p}.$$

We compute

$$c_1(L_p,\widetilde{h_p})=\theta_p+(1-t_p)dd^c\varphi_p^{eq}+t_pdd^c\varrho_p$$

$$=(1-t_p)\left(\theta_p+dd^c\varphi_p^{eq}\right)+t_p\left(\theta_p+dd^c\varrho_p\right)\ge t_p\left(\theta_p+dd^c\varrho_p\right)$$

$$\ge\frac{a_p}{\sqrt{a_p}}\omega=\sqrt{a_p}\omega.$$

Here, $\sqrt{a_p}\to\infty$ as $p\to\infty$. This shows that $\widetilde{h}_p$ satisfies the assumptions of Demailly's L^2 estimates for $\overline{\partial}$ (see [CMM, Theorem 2.5]). As in the proof of [CMM, Theorem 1.1], for all p large enough, we use the Ohsawa-Takegoshi Extension Theorem and Demailly's estimates for $\overline{\partial}$ (see [OT] and [D], resp.) to show the following. There exists $C_2>0$ such that for all $z\in U_\alpha$ with $\rho_p^\alpha(z)\ne-\infty$, there exists $S_{z,p}\in H^0(X,L_p)$ such that $S_{z,p}(z)\ne0$ and

$$\|S_{z,p}\|^2_{\widetilde{h}_p}\le C_2|S_{z,p}(z)|^2_{\widetilde{h}_p}. \tag{3.8}$$

We compute

$$\widetilde{h}_p=h_p^0e^{-2(1-t_p)\varphi_p^{eq}-2t_p\varrho_p}=h_pe^{2\varphi_p-2(1-t_p)\varphi_p^{eq}-2t_p\varrho_p}$$

$$=h_pe^{2(1-t_p)(\varphi_p-\varphi_p^{eq})}e^{2t_p(\varphi_p-\varrho_p)}\ge h_pe^{2t_p(\varphi_p-\varrho_p)}=h_pe^{2t_p\left(\phi_p^\alpha-\rho_p^\alpha\right)}.$$

Since $\{\rho_p^\alpha/A_p\}$ is a family of psh functions uniformly bounded in $L^1(V_\alpha)$ it is locally uniformly bounded above. Then, as $\{\phi_p^\alpha/A_p\}$ is locally uniformly bounded, it follows that $\{(\phi_p^\alpha-\rho_p^\alpha)/A_p\}$ is bounded below on U_α for all α. Since $(\phi_p^\alpha-\rho_p^\alpha)/A_p$ forms a global function, there exists $D\in\mathbb{R}$ with

$$(\phi_p^\alpha-\rho_p^\alpha)/A_p\ge D,$$

for all p,α. Therefore,

$$\widetilde{h}_p\ge h_pe^{2t_p\left(\phi_p^\alpha-\rho_p^\alpha\right)}\ge h_pe^{2t_pDA_p}. \tag{3.9}$$

From above and (3.8), we deduce that $S_{z,p}\in H^0_{(2)}(X,L_p)$ for all p large enough.

From (3.9) and (3.8) we obtain

$$\|S_{z,p}\|^2_{h_p} e^{2t_pDA_p} \leq C_2 |S_{z,p}(z)|^2_{h_p} \exp\left[2\varphi_p(z) - 2(1-t_p)\varphi_p^{eq}(z) - 2t_p\varrho_p(z)\right],$$

so

$$\frac{\exp\left[2t_pDA_p - 2\varphi_p(z) + 2(1-t_p)\varphi_p^{eq}(z) + 2t_p\varrho_p(z)\right]}{C_2} \leq \frac{|S_{z,p}(z)|^2_{h_p}}{\|S_{z,p}\|^2_{h_p}} \leq P_p(z).$$

We compute

$$t_pDA_p + (1-t_p)\varphi_p^{eq}(z) + t_p\varrho_p(z) - \frac{\log(C_2)}{2} \leq \frac{\log P_p(z)}{2} + \varphi_p(z)$$

$$t_pD + \frac{-t_p\varphi_p^{eq}(z) + t_p\varrho_p(z) - \log(C_2)/2}{A_p} \leq \left(\beta_p - \frac{\varphi_p^{eq}}{A_p}\right)(z).$$

As $\varphi_p \geq \varphi_p^{eq}$ we get

$$t_pD + t_p\frac{(\varrho_p - \varphi_p)(z)}{A_p} - \frac{\log C_2}{2A_p} \leq \left(\beta_p - \frac{\varphi_p^{eq}}{A_p}\right)(z).$$

This serves as our lower bound.

Combining the above inequality with that of (3.7), it follows that for j large enough, we have

$$t_pD + t_p\frac{(\varrho_p - \varphi_p)(z)}{A_p} - \frac{\log C_2}{2A_p} \leq \left(\beta_{p_j} - \frac{\varphi_{p_j}^{eq}}{A_{p_j}}\right)(z) \leq -\frac{2n\log r}{2A_{p_j}} + \epsilon$$

for a.e. $z \in U$. By compactness it holds for a.e. $z \in X$. Then,

$$\left|\beta_{p_j} - \frac{\varphi_{p_j}^{eq}}{A_{p_j}}\right|(z) \leq \left|t_pD + t_p\frac{(\varrho_p - \varphi_p)(z)}{A_p} - \frac{\log C_2}{2A_p}\right| + \left|\frac{2n\log r}{2A_{p_j}}\right| + \epsilon,$$

Recall that $(\phi_p^\alpha - \rho_p^\alpha) = (\varphi_p - \varrho_p)|_{V_\alpha}$ is bounded in $L^1(U_\alpha)$ for all α. So, $(\varphi_p - \varrho_p)$ is bounded in $L^1(X)$. As well, $t_p \to 0$, so by integrating and letting $j \to \infty$, we find

$$0 \leq \limsup_j \int_X \left|\beta_{p_j} - \frac{\varphi_{p_j}^{eq}}{A_{p_j}}\right| \frac{\omega^n}{n!} \leq \epsilon \int_U \frac{\omega^n}{n!}.$$

By letting $\epsilon \to 0$ we see that

$$\beta_{p_j} - \frac{\varphi_{p_j}^{eq}}{A_{p_j}} \to 0$$

in $L^1(X)$.

♥

Proof of Theorem 1.2 Like in Theorem 1.1, the family $\{U_\alpha\}$ forms a finite cover of X by coordinate polydiscs, V_α is a coordinate polydisc containing U_α, and e_p^α is a local frame of L_p on V_α. Here the global weights ϕ_p^α and ρ_p^α of h_p and g_p satisfy the property that $\{\phi_p^\alpha/A_p\}$ is uniformly bounded and equicontinuous, and $\{\rho_p^\alpha/A_p\}$ is uniformly bounded in L^1.

Define

$$\beta_p = \frac{(\log P_p)/2 + \varphi_p}{A_p}.$$

As we established in the preceding proof, if we show $\beta_p - \varphi_p^{eq}/A_p \to 0$ in L^1, then the proof is completed.

Let $\epsilon > 0$ and $x \in X$. Using our subaveraging argument from the proof of Theorem 1.1 and the equicontinuity argument in the proof of [CMM, Proposition 4.4], it follows that

$$A_p\beta_p - \frac{\log C_1 - 2n\log r}{2} - A_p\epsilon \le \varphi_p, \tag{3.10}$$

for some $C_1 \in \mathbb{R}$. The L.H.S. is θ_p−psh, hence by our arguments in the proof of Theorem 1.1, the upper bound φ_p may be replaced by φ_p^{eq}. Therefore,

$$\left(\beta_p - \frac{\varphi_p^{eq}}{A_p}\right) \le \frac{\log C_1 - 2n\log r}{2A_p} + \epsilon.$$

This serves as our upper bound.

For the lower bound, we apply the same argument as in the proof of Theorem 1.1, which only required $\{\phi_p^\alpha/A_p\}$ to be locally uniformly bounded and $\{\rho_p^\alpha/A_p\}$ to be bounded in L^1. These methods allow us to deduce that for some neighborhood U of x, if $z \in U$, then

$$t_pC_2 + \frac{t_p(\varrho_p - \varphi_p)(z) - \log(C_3)/2}{A_p} \le \left(\beta_p - \frac{\varphi_p^{eq}}{A_p}\right)(z) \le \frac{\log C_1 - 2n\log r}{2A_p} + \epsilon$$

for some $C_2, C_3 > 0$. This is the same situation we had at the end of our proof of Theorem 1.1, so we deduce $\beta_p - \varphi_p^{eq}/A_p \to 0$ in L^1.

♥

Proof of Theorem 1.3. Here $U_\alpha, V_\alpha, e_p^\alpha$ and the global weights $\phi_p^\alpha, \rho_p^\alpha$ are defined as in the previous two theorems. The conditions imposed on the weights by our assumptions are that $\{\phi_p/A_p\}$ is equicontinuous and $\{(\phi_p - \rho_p)/A_p\}$ is uniformly bounded. We define β_p as in the previous two proofs. As before, it suffices to show $\beta_p - \varphi_p^{eq}/A_p \to 0$ in L^1.

As in the proof of Theorem 1.2, (3.10) holds. For the lower bound, we first fix $z \in X$, then proceed as in our proof of Theorem 1.1. Set $t_p = 1/\sqrt{A_p}$ and

$$\widetilde{h}_p = h_p^0 e^{-2(1-t_p)\varphi_p^{eq} - 2t_p\varrho_p}.$$

This definition is the same as in Theorem 1.1, hence it satisfies

$$c_1(L_p, \widetilde{h_p}) \ge \sqrt{a_p}\omega.$$

In exactly the same fashion as in our proof of Theorem 1.1, for p large enough, we conclude there exists $C_2 > 0$ such that for all $z \in U_\alpha$ with $\rho_p^\alpha(z) \neq -\infty$, there exists $S_{z,p} \in H^0(X, L_p)$ such that $S_{z,p}(z) \neq 0$ and

$$\|S_{z,p}\|^2_{\widetilde{h}_p} \le C_1|S_{z,p}|^2_{\widetilde{h}_p}(z). \tag{3.11}$$

As we have computed in our previous proofs, we have

$$\widetilde{h}_p \ge h_p e^{2t_p\varphi_p - 2t_p\varrho_p}.$$

Since $\{(\varphi_p - \varrho_p)/A_p\}$ is uniformly bounded, it follows that there exists D independent of p such that

$$\widetilde{h}_p \ge h_p e^{2A_pt_pD},$$

and so $S_{z,p} \in H^0_{(2)}(X, L_p, h_p)$. From this and (3.11), we deduce

$$\frac{\exp\left[2A_pt_pD - 2\varphi_p(z) + 2(1-t_p)\varphi_p^{eq}(z) + 2t_p\varrho_p(z)\right]}{C_1} \le P_p(z).$$

Here $z \in X$ was arbitrary, so the above holds on all X, and we deduce

$$t_p D + \frac{-t_p\varphi_p^{eq} + t_p\varrho_p - \log C_1}{A_p} \le \beta_p - \frac{\varphi_p^{eq}}{A_p}.$$

Since $\varphi_p \ge \varphi_p^{eq}$, we find

$$t_p D + \frac{-t_p\varphi_p + t_p\varrho_p - \log C_1}{A_p} \le \beta_p - \frac{\varphi_p^{eq}}{A_p}.$$

By the uniform boundedness of $\{(\varphi_p - \varrho_p)/A_p\}$, it follows that there exists $D_2 > 0$ such that

$$t_p D_2 + \frac{-\log C_1}{A_p} \le \beta_p - \frac{\varphi_p^{eq}}{A_p}.$$

Using our lower bound and (3.10), one concludes as in the proof of Theorem 1.1 that

$$\beta_p - \frac{\varphi_p^{eq}}{A_p} \to 0 \text{ in } L^1(X).$$

♥

3.2. **Proofs of Proposition 1.4 and Theorems 1.5.** We start with the proof of Proposition 1.4.

Proof. Let $k_p^{\alpha\beta}$ be the transition functions for L_p with respect to the open cover $\{V_\alpha\}$. Set $\psi_p^{\alpha\beta} = \log|k_p^{\alpha\beta}|$. Then, we have the formula $\psi_p^{\alpha\beta} = \phi_p^\alpha - \phi_p^\beta$. Since ϕ_p^α/A_p converges uniformly for all p, so does $\psi_p^{\alpha\beta}/A_p$. Let the limit be $\psi^{\alpha\beta}$. It follows that

$$\psi^{\alpha\beta} = \lim_p \left(\frac{\phi_p^\alpha - \phi_p^\beta}{A_p} \right) = \phi^\alpha - \phi^\beta. \tag{3.12}$$

Moreover, $k_p^{\alpha\beta}$ is holomorphic and non-vanishing, hence $\psi_p^{\alpha\beta} = e^{\log|k_p^{\alpha\beta}|}$ is pluriharmonic, and so is the uniform limit $\psi^{\alpha\beta} = \lim_p(\psi_p^{\alpha\beta}/A_p)$. This shows that $\psi^{\alpha\beta}$ satisfies the desired conclusions.

♥

The following lemmas will be used to prove Theorem 1.5.

Lemma 3.3. *If (X,ω) and (L_p,h_p) satisfy (A) and (B'), and $\psi^{\alpha\beta}$ are functions such that $\phi^\alpha = \phi^\beta + \psi^{\alpha\beta}$, then there exists $\xi^\alpha \in C^\infty$ with*

$$\xi^\alpha = \xi^\beta + \psi^{\alpha\beta}.$$

Proof. Note that (3.12) implies

$$\psi^{\alpha\beta} = -\psi^{\beta\alpha} \text{ and } \psi^{\alpha\beta} + \psi^{\beta\gamma} + \psi^{\gamma\alpha} \text{ for all } \alpha, \beta, \gamma.$$

By the C^∞ version of the first Cousin Problem [Kr, Proposition 6.1.7] there exist functions $\xi^\alpha \in C^\infty(V_\alpha)$ for all α such that

$$\xi^\alpha = \xi^\beta + \psi^{\alpha\beta} \text{ for all } \alpha, \beta.$$

This is what we desired to show.

♥

Recall that the local weights ϱ_p, φ_p of g_p and h_p were defined in Section 2.2 (see (2.2)). As were φ_p^{eq} and ξ_p^α (see (2.3) and (2.1)). We also recall that $\varphi_p = \xi_p^\alpha + \phi_p^\alpha$. In general, similar relationships exist for the local and global weights of any metric.

In the following lemmas and the proof of Theorem 1.5, we let θ be the $(1,1)$–form such that $\theta|_{V_\alpha} = dd^c\xi^\alpha$, where ξ^α are as in the previous lemma. Also, we will set $\varphi = \phi^\alpha - \xi^\alpha$.

Lemma 3.4. *Let (X,ω) and (L_p,h_p) satisfy (A) and (B'), and suppose the following conditions hold.*

(a) There exist $\delta_p \geq 0$ with $\delta_p \to 0$ such that

$$dd^c\phi^\alpha - \delta_p\omega \leq \frac{c_1(L_p,h_p)}{A_p} \text{ on } V_\alpha.$$

(b) There exist $\varrho \in PSH(X,\theta)$ and some $c>0$ such that

$$\theta + dd^c\varrho \geq c\omega.$$

Then there exists a subsequence $\{(\varphi_{p_j}^{eq} - \varphi_{p_j})/A_{p_j}\}$ such that

$$\left(\frac{\varphi_{p_j}^{eq} - \varphi_{p_j}}{A_{p_j}}\right) \to \widetilde{\varphi^{eq}} - \varphi \text{ in } L^1(X),$$

where

$$\widetilde{\varphi^{eq}} := \left[\limsup_{j\to\infty}\left(\frac{\varphi_{p_j}^{eq} - \varphi_{p_j}}{A_{p_j}}\right)\right]^* + \varphi \in \{\psi \in PSH(X,\theta) \mid \psi \leq \varphi\}. \tag{3.13}$$

Proof. First, since φ_p is continuous we may assume, by subtracting a constant, that $A_p(\varrho - \varphi) \leq 0$. We compute

$$dd^c(A_p(\varrho-\varphi) + \phi_p^\alpha) = A_p\left(dd^c\varrho + \theta\right) + dd^c(\phi_p^\alpha - A_p\phi^\alpha) \geq A_pc\omega - A_p\delta_p\omega,$$

hence for p large enough it follows that $[A_p(\varrho-\varphi) + \phi_p^\alpha]$ is psh.

Let p be large enough such that $[A_p(\varrho-\varphi)+\phi_p^\alpha]$ is psh. Since $A_p(\varrho-\varphi) + \varphi_p \in PSH(X,\theta_p)$ and $A_p(\varrho-\varphi)+\varphi_p \leq \varphi_p$, it follows from the definition of φ_p^{eq} that $A_p(\varrho-\varphi) + \phi_p^\alpha \leq \xi_p^\alpha + \varphi_p^{eq} \leq \phi_p^\alpha$, therefore

$$(\varrho - \varphi) + \phi_p^\alpha/A_p \leq (\xi_p^\alpha + \varphi_p^{eq})/A_p \leq \phi_p^\alpha/A_p.$$

By the local uniform convergence $\phi_p^\alpha/A_p \to \phi^\alpha$, the L.H.S. and R.H.S. of the previous equation are uniformly bounded in $L^1_{loc}(V_\alpha)$, thus $\{(\xi_p^\alpha + \varphi_p^{eq})/A_p\}$ is as well. As a consequence, there exists a subsequence $\{(\xi_{p_j}^\alpha + \varphi_{p_j}^{eq})/A_{p_j}\}$ which converges in $L^1(U_\alpha)$ and a.e. to a function $\widetilde{\phi}^{eq,\alpha} \in PSH(V_\alpha)$ (see [Hö, Theorem 3.2.12]). As there are finitely many α, we may assume the convergence holds for all α simultaneously.

We claim that

$$\widetilde{\phi}^{eq,\alpha} - \phi^\alpha = \left[\limsup_{j\to\infty}\left(\frac{\varphi_{p_j}^{eq} - \varphi_{p_j}}{A_{p_j}}\right)\right]^*, \tag{3.14}$$

To prove the claim, note that from [Kl, Theorem 2.6.3] we can deduce

$$\widetilde{\phi}^{eq,\alpha} = \left[\limsup_{j\to\infty}\left(\frac{\xi_{p_j}^\alpha + \varphi_{p_j}^{eq}}{A_{p_j}}\right)\right]^*.$$

Due to the uniform convergence $\phi_p^\alpha/A_p \to \phi^\alpha$ and the continuity of ϕ^α, it follows that

$$\widetilde{\phi}^{eq,\alpha} - \phi^\alpha = \left[\limsup_{j\to\infty}\left(\frac{\xi^\alpha_{p_j} + \varphi^{eq}_{p_j}}{A_{p_j}}\right)\right]^* - \phi^\alpha = \left[\limsup_{j\to\infty}\left(\frac{\xi^\alpha_{p_j} + \varphi^{eq}_{p_j}}{A_{p_j}} - \phi^\alpha\right)\right]^*$$
$$= \left[\limsup_{j\to\infty}\left(\frac{\xi^\alpha_{p_j} + \varphi^{eq}_{p_j} - \phi^\alpha_{p_j}}{A_{p_j}}\right)\right]^* = \left[\limsup_{j\to\infty}\left(\frac{\varphi^{eq}_{p_j} - \varphi_{p_j}}{A_{p_j}}\right)\right]^*.$$

Which gives us (3.14).

By definition

$$\widetilde{\varphi^{eq}} = \widetilde{\phi}^{eq,\alpha} - \xi^\alpha$$

is θ−psh. Moreover, by (3.14), since $\varphi^{eq}_p \le \varphi_p$ for all p, we have $\widetilde{\varphi^{eq}} \le \varphi$ which gives us (3.13). To finish off the computation we note that

$$\frac{\varphi^{eq}_{p_j} - \varphi_{p_j}}{A_{p_j}} = \frac{(\xi^\alpha_{p_j} + \varphi^{eq}_{p_j}) - \phi^\alpha_{p_j}}{A_{p_j}} \to \widetilde{\phi}^{eq,\alpha} - \phi^\alpha = \widetilde{\varphi^{eq}} - \varphi,$$

where the convergence is in $L^1(U_\alpha)$. Since this holds on all U_α, it follows that $(\varphi^{eq}_{p_j} - \varphi_{p_j})/A_{p_j} \to \widetilde{\varphi^{eq}} - \varphi$ in $L^1(X)$. This completes the proof.

For the following proof we will use the definition of $\widetilde{\varphi^{eq}}$ from the previous Lemma.

Proof of Theorem 1.5. Let $\epsilon > 0$ and $x \in X$. Although we are working with currents, it still suffices to show the claim holds for a subsequence.

Since $\{\phi^\alpha_p/A_p\}$ is uniformly convergent on V_α it is equicontinuous on U_α, hence by our arguments in the proof of Theorem 1.2, (3.10) holds. That is, there exists a $C_1 \in \mathbb{R}$ such that

$$\frac{\log P_p}{2A_p} - \left(\frac{\varphi^{eq}_p - \varphi_p}{A_p}\right) \le \frac{\log C_1 - 2n\log r}{2A_p} + \epsilon. \tag{3.15}$$

From the above and Lemma 3.4, it follows that there exists a subsequence $\{p_j\}$ such that for all j, we have

$$\frac{\log P_{p_j}}{2A_{p_j}} - (\widetilde{\varphi^{eq}} - \varphi) \le \frac{\log C_1 - 2n\log r}{2A_{p_j}} + \epsilon + \left[\left(\frac{\varphi^{eq}_{p_j} - \varphi_{p_j}}{A_{p_j}}\right) - (\widetilde{\varphi^{eq}} - \varphi)\right].$$

By the definition of φ^{eq}, it follows that

$$\frac{\log P_{p_j}}{2A_{p_j}} - (\varphi^{eq} - \varphi) \le \frac{\log C_1 - 2n\log r}{2A_{p_j}} + \epsilon + \left[\left(\frac{\varphi^{eq}_{p_j} - \varphi_{p_j}}{A_{p_j}}\right) - (\widetilde{\varphi^{eq}} - \varphi)\right]. \tag{3.16}$$

This shall serve as our upper bound.

By continuity of φ_p, we assume WLOG that $\varrho \in PSH(X,\theta)$ and $(\varrho - \varphi) \le 0$. From (b), we have

$$-\delta_p\omega \le c_1(L_p, h_p)/A_p - dd^c\phi^\alpha = dd^c(\phi^\alpha_p/A_p - \phi^\alpha).$$

Define $t_{p_j} = \sqrt{\delta_{p_j}} + 1/\sqrt{A_{p_j}}$, and set $\widetilde{h_{p_j}} = h_{p_j}e^{-2(1-t_{p_j})A_{p_j}(\varphi^{eq}-\varphi) - 2t_{p_j}A_{p_j}(\varrho-\varphi)}$. We compute

$$c_1(L_{p_j}, \widetilde{h_{p_j}}) = dd^c\phi^\alpha_{p_j} + (1-t_{p_j})A_{p_j}dd^c(\varphi^{eq} - \varphi) + t_{p_j}A_{p_j}dd^c(\varrho - \varphi)$$
$$= dd^c(\phi^\alpha_{p_j} - A_{p_j}\phi^\alpha) + (1-t_{p_j})A_{p_j}dd^c(\varphi^{eq} + \xi^\alpha) + t_{p_j}A_{p_j}dd^c(\varrho + \xi^\alpha)$$
$$\ge -\delta_{p_j}A_{p_j}\omega + \sqrt{\delta_{p_j}}A_{p_j}c\omega + c\sqrt{A_{p_j}}\omega \ge c\sqrt{A_{p_j}}\omega.$$

Since $c\sqrt{A_{p_j}} \to \infty$ as $j \to \infty$, $\widetilde{h_{p_j}}$ satisfies the assumptions of Demailly's L^2 estimates for $\overline{\partial}$. By our arguments from Theorem 1.1 using Ohsawa-Takegoshi extension and Demailly's estimate for $\overline{\partial}$, there is a $C_2 > 0$ such that for all $z \in U_\alpha$ with $\rho^\alpha_p(z) \neq -\infty$, there exists $S_{z,j} \in H^0(X, L_{p_j})$ satisfying $S_{z,j}(z) \neq 0$ and

$$\|S_{z,j}\|^2_{\widetilde{h_{p_j}}} \leq C_2 |S_{z,j}(x)|^2_{\widetilde{h_{p_j}}}. \tag{3.17}$$

Since $(\varphi^{eq} - \varphi)$ and $(\varrho - \varphi)$ are negative, we have $\widetilde{h_{p_j}} \geq h_{p_j}$, and so $S_{z,j} \in H^0_{(2)}(X, L_{p_j}, h_{p_j})$. From this and (3.17) it follows that

$$\|S_{z,j}\|_{h_{p_j}} \leq C_2 |S_{z,j}(x)|_{h_{p_j}} e^{-2(1-t_{p_j})A_{p_j}(\varphi^{eq}-\varphi)(x) - 2t_{p_j}A_{p_j}(\varrho-\varphi)(x)},$$

thus

$$\frac{e^{2(1-t_{p_j})A_{p_j}(\varphi^{eq}-\varphi)(x) + 2t_{p_j}A_{p_j}(\varrho-\varphi)(x)}}{C_2} \leq \frac{|S_{z,j}(x)|^2_{h_{p_j}}}{\|S_{z,j}\|^2_{h_{p_j}}}.$$

By the variational characterization of the Bergman kernel, it follows that

$$(1 - t_{p_j})(\varphi^{eq} - \varphi)(x) + t_{p_j}(\varrho - \varphi)(x) - \frac{\log C_2}{2A_{p_j}} \leq \frac{\log P_{p_j}(x)}{2A_{p_j}},$$

hence

$$-t_{p_j}(\varphi^{eq} - \varphi)(x) + t_{p_j}(\varrho - \varphi)(x) - \frac{\log C_2}{2A_{p_j}} \leq \frac{\log P_{p_j}(x)}{2A_{p_j}} - (\varphi^{eq} - \varphi)(x).$$

As $(\varphi^{eq} - \varphi)$ is negative, we have

$$t_{p_j}(\varrho - \varphi)(x) - \frac{\log C_2}{2A_{p_j}} \leq \frac{\log P_{p_j}(x)}{2A_{p_j}} - (\varphi^{eq} - \varphi)(x).$$

Combining with (3.16) yields

$$\begin{aligned} t_{p_j}(\varrho - \varphi)(x) - \frac{\log C_2}{2A_{p_j}} &\leq \frac{\log P_{p_j}(x)}{2A_{p_j}} - (\varphi^{eq} - \varphi)(x) \\ &\leq \frac{\log C_1 - 2n \log r}{2A_{p_j}} + \epsilon + \left[\left(\frac{\varphi^{eq}_{p_j} - \varphi_{p_j}}{A_{p_j}} \right) - (\widetilde{\varphi^{eq}} - \varphi) \right]. \end{aligned} \tag{3.18}$$

Note, by Lemma 3.4 we have $(\varphi^{eq}_{p_j} - \varphi_{p_j})/A_{p_j} \to (\widetilde{\varphi^{eq}} - \varphi)$ in $L^1(X)$ and by definition $t_{p_j} \to 0$, so from computations similar to those in the proof of Theorem 1.1 we have

$$\limsup_j \left\| \frac{\log P_{p_j}}{2A_{p_j}} - (\varphi^{eq} - \varphi) \right\|_{L^1(X)} \leq \epsilon \int_X \frac{\omega^n}{n!}.$$

Letting $\epsilon \to 0$, we find that

$$\frac{\log P_{p_j}}{2A_{p_j}} - (\varphi^{eq} - \varphi) \to 0$$

in $L^1(X)$. Therefore

$$\frac{\gamma_{p_j}}{A_{p_j}} \to T + dd^c\varphi^{eq} - dd^c\varphi = \theta + dd^c\varphi^{eq} \tag{3.19}$$

weakly as currents.

Now, we address the second half of (1.5). By the definition of φ^{eq} and Lemma 3.4 we have

$$\left(\frac{\varphi^{eq}_{p_j}-\varphi_{p_j}}{A_{p_j}}\right)-\left[\left(\frac{\varphi^{eq}_{p_j}-\varphi_{p_j}}{A_{p_j}}\right)-(\widetilde{\varphi^{eq}}-\varphi)\right]=(\widetilde{\varphi^{eq}}-\varphi)\leq(\varphi^{eq}-\varphi).$$

With this and (3.18) we can deduce

$$t_{p_j}(\varrho-\varphi)(x)-\frac{\log C_2}{A_{p_j}}-\left[\left(\frac{\varphi^{eq}_{p_j}-\varphi_{p_j}}{A_{p_j}}\right)-(\widetilde{\varphi^{eq}}-\varphi)\right](x)-\epsilon\leq\frac{\log P_{p_j}(x)}{2A_{p_j}}-\left(\frac{\varphi^{eq}_{p_j}-\varphi_{p_j}}{A_{p_j}}\right)(x).$$

Note $t_{p_j}(\varrho-\varphi)\to 0$ by definition and $[(\varphi^{eq}_{p_j}-\varphi_{p_j})/A_{p_j}-(\varphi^{eq}-\varphi)]\to 0$ in $L^1(X)$ by Lemma 3.4. So, it follows from (3.15) and the inequality above that

$$\limsup_j\left\|\frac{\log P_{p_j}}{2A_{p_j}}-\left(\frac{\varphi^{eq}_{p_j}-\varphi_{p_j}}{A_{p_j}}\right)\right\|_{L^1(X)}\leq\epsilon\int_X\frac{\omega^n}{n!}.$$

Letting $\epsilon\to 0$ tells us that

$$\frac{\log P_{p_j}}{2A_{p_j}}-\left(\frac{\varphi^{eq}_{p_j}-\varphi_{p_j}}{A_{p_j}}\right)\to 0$$

in $L^1(X)$, hence by (3.4) in the proof of Theorem 1.1 we have

$$\frac{\gamma_{p_j}-c_1(L_{p_j},h^{eq}_{p_j})}{A_{p_j}}\to 0 \text{ as currents.}$$

From this and (3.19) it follows that

$$\frac{c_1(L_{p_j},h^{eq}_{p_j})}{A_{p_j}}\to\theta+dd^c\varphi^{eq},$$

as desired.

Next, we prove (1.6). Since

$$\left(\frac{\log P_{p_j}}{2A_{p_j}}-(\varphi^{eq}-\varphi)\right),\left(\frac{\log P_{p_j}}{2A_{p_j}}-\frac{\varphi^{eq}_{p_j}-\varphi_{p_j}}{A_{p_j}}\right)\to 0$$

in $L^1(X)$, we also have

$$\left(\frac{\varphi^{eq}_{p_j}-\varphi_{p_j}}{A_{p_j}}\right)\to(\varphi^{eq}-\varphi)$$

in $L^1(X)$. Then, from Lemma 3.4 we deduce that $(\varphi^{eq}-\varphi)=(\widetilde{\varphi}^{eq}-\varphi)$ a.e., and $\widetilde{\varphi}^{eq}=\varphi^{eq}$ a.e. Since φ^{eq} and $\widetilde{\varphi}^{eq}$ are qpsh and equal a.e., they are equal everywhere. Since φ^{eq} is independent of the subsequence $\{p_j\}$, it follows from Lemma 3.4 that every subsequence of $(\varphi^{eq}_p-\varphi_p)/A_p+\varphi$ has a subsequence converging to φ^{eq} almost everywhere. Then $(\varphi^{eq}_p-\varphi_p)/A_p+\varphi\to\varphi^{eq}$ a.e., and (1.6) holds, that is,

$$\varphi^{eq}=\left[\limsup_p\left(\frac{\varphi^{eq}_p-\varphi_p}{A_p}\right)\right]^*+\varphi.$$

Lastly, we only proved the theorem for a particularly smooth $\theta\in\{T\}$, and we must show it holds for all. Let $\widetilde{\theta}\in\{T\}$ be smooth and choose $\widetilde{\varphi}\in C(X)$ such that $\widetilde{\theta}+dd^c\widetilde{\varphi}=T$. Set $\widetilde{\varphi^{eq}}=\sup\{\xi\in PSH(X,\widetilde{\theta})\mid\psi\leq\widetilde{\varphi}\}$. Define $\sigma=\varphi-\widetilde{\varphi}$. We see that $\sigma\in C(X)$ and $dd^c\sigma=dd^c(\varphi-\widetilde{\varphi})=\widetilde{\theta}-\theta$, hence $\varphi^{eq}=\widetilde{\varphi^{eq}}+\sigma$, and

$$\theta+dd^c\varphi^{eq}=(\widetilde{\theta}-dd^c\sigma)+dd^c(\widetilde{\varphi^{eq}}+\sigma)=\widetilde{\theta}+dd^c\widetilde{\varphi^{eq}}.$$

It follows that

$$\frac{\gamma_{p_j}}{A_{p_j}} \to \widetilde{\theta} + dd^c \widetilde{\varphi^{eq}},$$

which shows the result is independent of our choice of θ as desired.

♥

4. Applications

Applying Theorem 1.2 to tensor products of powers of line bundles yields the following:

Corollary 4.1. *Assume F_j, $1 \leq j \leq k$, are holomorphic line bundles on X equipped with continuous metrics h^{F_j} and singular metrics g^{F_j} such that $c_1(F_j, g^{F_j}) \geq 0$ and $c_1(F_1, g^{F_1}) \geq \epsilon\omega$, for some $\epsilon > 0$. Let $\{m_{j,p}\}$ be a sequence of natural numbers with $m_{1,p} \to \infty$, as $p \to \infty$. Let γ_p, $p \geq 1$, be the Fubini-Study currents associated with $H^0_{(2)}(X, L_p)$ where*

$$L_p = F_1^{m_{1,p}} \otimes \ldots \otimes F_k^{m_{k,p}}, \; h_p = (h^{F_1})^{m_{1,p}} \otimes \ldots \otimes (h^{F_k})^{m_{k,p}}.$$

Let h_p^{eq} be the equilibrium metric of h_p. Then

$$\frac{\gamma_p - c_1(L_p, h_p^{eq})}{\sum_{j=1}^k m_{j,p}} \to 0$$

weakly as currents.

Proof. Set

$$g_p = (g^{F_1})^{m_{1,p}} \otimes \ldots \otimes (g^{F_k})^{m_{k,p}}.$$

Let $x \in X$ and suppose $U, V \subset X$ are open polydisc neighborhoods of x with $U \Subset V$. For $1 \leq j \leq k$, let $\phi_p^{F_j}$ and $\rho_p^{F_j}$ denote the local weights of h^{F_j} and g^{F_j} on V, respectively. Then the local weights of h_p and g_p are given by

$$\phi_p := \sum_{j=1}^k m_{j,p} \phi^{F_j} \text{ and } \rho_p := \sum_{j=1}^k m_{j,p} \rho^{F_j}.$$

Set

$$A_p := \int_X c_1(L_p, g_p) \wedge \omega^{n-1} = \sum_{j=1}^k m_{j,p} \int_X c_1(F_j, g^{F_j}) \wedge \omega^{n-1}.$$

Note there exists $c > 0$, such that $cA_p \leq \sum_{j=1}^k m_{j,p}$ and

$$c_1(L_p, g_p) \geq m_{1,p} c_1(F_1, g^{F_j}) \geq m_{1,p} \epsilon\omega.$$

By Theorem 1.2, it suffices to show that $\{\phi_p / \sum_{j=1}^k m_{j,p}\}$ is equicontinuous and uniformly bounded in $L^1(U)$, and $\{\rho_p / \sum_{j=1}^k m_{j,p}\}$ is uniformly bounded in $L^1(U)$.

Observe that

$$\frac{\phi_p}{\sum_{j=1}^k m_{j,p}} = \frac{\sum_{j=1}^k m_{j,p} \phi^{F_j}}{\sum_{j=1}^k m_{j,p}} = \sum_{j=1}^k \left(\frac{m_{j,p}}{\sum_{\ell=1}^k m_{\ell,p}} \right) \phi^{F_j},$$

so as $\{m_{j,p} / \sum_{\ell=1}^k m_{\ell,p}\}$ is bounded for all j, and ϕ^{F_j} is continuous on $\overline{U}$, it follows that $\{\phi_p / \sum_{j=1}^k m_{j,p}\}$ is equicontinuous and uniformly bounded in $L^1(U)$. Similarly,

$$\frac{\rho_p}{\sum_{j=1}^k m_{j,p}} = \sum_{j=1}^k \left(\frac{m_{j,p}}{\sum_{\ell=1}^k m_{\ell,p}} \right) \rho^{F_j},$$

hence $\{\rho_p/\sum_{j=1}^k m_{j,p}\}$ is uniformly bounded in $L^1(U)$. This completes the proof.

♥

Remark 4.2. If $h^{F_j} = g^{F_j}$, $1 \le j \le k$, then $h_p^{eq} = h_p$, and

$$\frac{c_1(L_p, h_p)}{\sum_{\ell=1}^k m_{\ell,p}} = \sum_{j=1}^k \left(\frac{m_{j,p}}{\sum_{\ell=1}^k m_{\ell,p}} \right) c_1(F_j, h^{F_j}).$$

If additionally $m_{j,p}/\sum_{\ell=1}^k m_{\ell,p}$ converges for all j, then

$$\frac{\gamma_p}{\sum_\ell m_{\ell,p}} \to \sum_{j=1}^k \left(\lim_p \frac{m_{j,p}}{\sum_\ell m_{\ell,p}} \right) c_1(F_j, h^{F_j}).$$

The following results are corollaries of Theorem 1.5:

Corollary 4.3. *Let (X, ω) be as in (A) and for $p \ge 1$, let L_p be a holomorphic line bundle on X equipped with the metric h_p with C^2 local weights ϕ_p^α for all α. As well, let $A_p > 0$ with $A_p \to \infty$. Suppose the following conditions hold.*

(a) There exists a continuous $(1,1)$–form Φ such that

$$\frac{c_1(L_p, h_p)}{A_p} \to \Phi$$

uniformly.

(b) There exists $\varrho \in L^1(X)$ such that

$$dd^c\varrho + \Phi \ge c\omega$$

Then for any $\theta \in \{\Phi\}$, and $\varphi \in C(X)$ such that $dd^c\varphi = \Phi - \theta$, we have

$$\frac{\gamma_p}{A_p} \to \theta + dd^c\varphi^{eq} \text{ and } \frac{c_1(L_p, h_p^{eq})}{A_p} \to \theta + dd^c\varphi^{eq}$$

weakly as currents, where $\varphi^{eq} = \sup\{\psi \in PSH(X, \theta) \mid \psi \le \varphi\}$.

To prove the above Corollary, we will show that the given conditions imply the conditions stated in Lemma 1.5. To do this, we first prove the following lemma.

Lemma 4.4. *Let U be a coordinate polydisc. Assume $\{\phi_p : U \to \mathbb{R}\}$ is a family of C^2 functions bounded in L^1, and there exists a continuous $(1,1)$–form Φ such that*

$$dd^c\phi_p \to \Phi \text{ uniformly.}$$

Then, there exists $\phi \in C^1(U)$ and a subsequence $\{\phi_{p_k}\}$ such that

$$\phi_{p_k} \to \phi \text{ locally uniformly.}$$

Proof. Let $K \Subset U$ and $\epsilon > 0$. Choose $r_0 > 0$ such that

$$\widehat{K} = \{x \in U \mid \text{dist}(x, K) \le r_0\} \Subset U.$$

Since $dd^c\phi_p \to \Phi$ uniformly it follows that there exists $M > 0$ such that

$$-M\omega \le dd^c\phi_p \le M\omega.$$

Let $\zeta : U \to \mathbb{R}$ be a smooth real potential for ω. As $\{\phi_p + M\zeta\}$ is a family of psh functions bounded in L^1, it follows from [Hö, Theorem 3.2.12] that there exists $\widetilde{\phi}_U \in PSH(U)$ and a subsequence $\{\phi_{p_k} + M\zeta\}$ such that $\phi_{p_k} + M\zeta \to \widetilde{\phi}_U$ in L^1_{loc}. We define $\phi_U = \widetilde{\phi}_U - M\zeta$. We claim ϕ_U is continuous differentiable.

To see that ϕ_U is C^1 smooth we note that $\Delta\phi_U = \mathrm{Tr}(\Phi)$ is continuous. If $K \Subset U$ is a ball, then $\Delta\phi_U$ is bounded on K, so by [GT, Lemma 4.1] there exists $\psi_K \in C^1(K)$ with $\psi_K = \phi_U$ a.e. on K. Then, $\psi_K + M\zeta = \phi_U + M\zeta$ a.e. and both sides are psh, so they're equal everywhere. It follows that $\phi_U \in C^1(K)$. As $K \subseteq U$ was an arbitrary ball we have $\phi_U \in C^1(U)$.

Next, as $dd^c\phi_{p_k} \to \Phi$ uniformly, it follows that for k large enough, we have

$$-\omega \leq dd^c(\phi_{p_k} - \phi_U) \leq \omega.$$

From this and subaveraging, we find

$$(\phi_{p_k} - \phi_U + \zeta)(x) < \frac{n!}{\pi^n r^{2n}} \int_{B(x,r)} (\phi_{p_k} - \phi_U + \zeta) d\lambda,$$

for all $x \in K$ and $0 < r < r_0$, where $d\lambda$ is the Lebesgue measure. It follows that

$$(\phi_{p_k} - \phi_U)(x) \leq \frac{n!}{\pi^n r^{2n}} \int_{B(x,r)} (\phi_{p_k} - \phi_U) d\lambda + \frac{n!}{\pi^n r^{2n}} \int_{B(x,r)} (\zeta - \zeta(x))\, d\lambda.$$

By the uniform continuity of ζ on $\widehat{K}$, it follows that we can choose r small enough so that

$$(\phi_{p_k} - \phi_U)(x) \leq \frac{n!}{\pi^n r^{2n}} \int_{B(x,r)} (\phi_{p_k} - \phi_U) d\lambda + \epsilon$$

for all $x \in K$. By applying similar arguments to $(\phi_U - \phi_{p_k})$, it follows that by shrinking r, if necessary, we may assume

$$(\phi_U - \phi_{p_k})(x) \leq \frac{n!}{\pi^n r^{2n}} \int_{B(x,r)} (\phi_U - \phi_{p_k}) d\lambda + \epsilon.$$

Combining these inequalities gives

$$\frac{n!}{\pi^n r^{2n}} \int_{B(x,r)} (\phi_{p_k} - \phi_U) d\lambda - \epsilon \leq (\phi_{p_k} - \phi_U)(x) \leq \frac{n!}{\pi^n r^{2n}} \int_{B(x,r)} (\phi_{p_k} - \phi_U) d\lambda + \epsilon,$$

so

$$|\phi_{p_k} - \phi_U|(x) \leq \frac{n!}{\pi^n r^{2n}} \int_{B(x,r)} |\phi_{p_k} - \phi_U| d\lambda + \epsilon \leq \frac{n!}{\pi^n r^{2n}} \int_{\widehat{K}} |\phi_{p_k} - \phi_U| d\lambda + \epsilon.$$

As $\phi_{p_k} \to \phi_U$ in $L^1(\widehat{K})$, it follows that if k is large enough, then

$$|\phi_{p_k} - \phi_U|(x) \leq \frac{n!\epsilon}{\pi^n r^{2n}} + \epsilon.$$

As x was arbitrary, we see $\phi_{p_k} \to \phi_U$ uniformly on $\widehat{K}$, therefore $\{\phi_p\}$ has a subsequence converging uniformly to ϕ_U on K.

To see that $\{\phi_p\}$ has a subsequence converging locally uniformly on U, we let $\{K_n\}$ be an exhaustion of U by compact sets, and then apply a diagonalization argument to construct a sequence converging uniformly on each K_n, and hence, locally uniformly on K.

♥

Proof of Corollary 4.3: As $dd^c\phi_p^\alpha/A_p \to \Phi$ uniformly on V_α it follows that there exists $M > 0$ such that

$$-M\omega \leq dd^c\phi_p^\alpha/A_p \leq M\omega.$$

Hence, there exists a smooth function ζ such that $\{\phi_p^\alpha/A_p + \zeta\}$ is plurisubharmonic for all p, α. By [DS, Proposition A.16], it follows that there exists a $c > 0$ and $\widehat{\phi_p^\alpha} \in PSH(U_\alpha)$ for all p, α such that $dd^c\widehat{\phi_p^\alpha} = dd^c(\phi_p^\alpha + A_p\zeta)$ and

$$\|\widehat{\phi_p^\alpha}/A_p\|_{L^1(U_\alpha)} \leq c \int_{V_\alpha} dd^c(\phi_p^\alpha/A_p + \zeta) \wedge \omega^{n-1}.$$

Define $\widetilde{\phi_p^\alpha}/A_p = \widehat{\phi_p^\alpha}/A_p - \zeta$. From above it is clear that $\{\widetilde{\phi_p^\alpha}/A_p\}$ is bounded in $L^1(U_\alpha)$. As $dd^c\widetilde{\phi_p^\alpha} = dd^c\phi_p^\alpha$, it follows that $\widetilde{\phi_p^\alpha} = \phi_p^\alpha + k_p^\alpha$ almost everywhere for some pluriharmonic function $k_p^\alpha \in C^\infty(U_\alpha)$. Hence, $\widetilde{\phi_p^\alpha}$ is C^2, and $\widetilde{\phi_p^\alpha} = \phi_p^\alpha + k_p^\alpha$ everywhere.

Since k_p^α is pluriharmonic, it is the real part of a holomorphic function f_p^α. Let $\widetilde{e_p^\alpha} = e_p^\alpha e^{-f_p^\alpha}$. We find

$$h_p(\widetilde{e_p^\alpha}, \widetilde{e_p^\alpha}) = e^{-f_p^\alpha}\overline{e^{-f_p^\alpha}}h_p(e_p^\alpha, e_p^\alpha) = e^{-2\mathrm{Re}f_p^\alpha}e^{-2\phi_p^\alpha} = e^{-2(\phi_p^\alpha + k_p^\alpha)} = e^{-2\widetilde{\phi_p^\alpha}},$$

so $\widetilde{\phi_p^\alpha}$ are C^2 local weights of h_p with respect to the local frames $\widetilde{e_p^\alpha}$, and the family $\{\widetilde{\phi_p^\alpha}/A_p\}$ is bounded in $L^1(U_\alpha)$.

Suppose $\{p_j\}$ is a subsequence. By condition (a) and Lemma 4.4, it follows that for all α, there exists $\phi^\alpha \in C^1(U_\alpha)$ and a subsequence $\{p_{j_k}\}$ such that $\widetilde{\phi_{p_{j_k}}^\alpha}/A_{p_{j_k}} \to \phi^\alpha$ locally uniformly on U_α. We will start by showing that the claim holds for this subsequence.

We shall apply Theorem 1.5 to the subsequence $\{p_{j_k}\}$ with open cover $\{U_\alpha\}$ and local frames $\widetilde{e_p^\alpha}$. Given $\widetilde{\phi_{p_{j_k}}^\alpha}/A_{p_{j_k}} \to \phi^\alpha$, conditions (A) and (B') are satisfied for our subsequence. To see that (b) of Theorem 1.5 holds, we first note that $dd^c(\varrho + \phi^\alpha) = dd^c\varrho + \Phi|_{V_\alpha} = dd^c\varrho + T|_{V_\alpha}$. Next, we define ϱ' by $\varrho' = \varrho + \varphi$. Then,

$$dd^c(\varrho + \phi^\alpha) = dd^c\varrho + T|_{V_\alpha} = dd^c\varrho + \theta + dd^c\varphi = dd^c\varrho' + \theta,$$

so $dd^c\varrho' + \theta \geq c\omega$, and (b) is satisfied. It remains to show that condition (a) of Theorem 1.5 holds for the subsequence.

As $c_1(L_p, h_p)/A_p$ is uniformly convergent to $\Phi = dd^c\phi^\alpha$, it follows that there exists $\delta_p \geq 0$ with $\delta_p \to 0$ such that

$$dd^c\phi^\alpha - \delta_p\omega \leq \frac{c_1(L_p, h_p)}{A_p},$$

which gives us (a) of Theorem 1.5.

It follows from Theorem 1.5 that for any $\theta \in \{\Phi\}$ and $\varphi \in C(X)$ such that $dd^c\varphi = \Phi - \theta$, we have

$$\frac{\gamma_{p_{j_k}}}{A_{p_{j_k}}} \to \theta + dd^c\varphi^{eq} \text{ and } \frac{c_1(L_{p_{j_k}}, h_{p_{j_k}}^{eq})}{A_{p_{j_k}}} \to \theta + dd^c\varphi^{eq},$$

where

$$\varphi^{eq} = \sup\{\psi \in PSH(X, \theta) \mid \psi \leq \varphi\}.$$

Since this convergence holds for an arbitrary subsequence, it holds for the entire sequence.

♥

Remark 4.5. In the above scenario, we do get for all α that ϕ_p^α/A_p has a subsequence converging uniformly to some $\phi^\alpha \in C^1(U_\alpha)$. Unlike the φ^{eq} in our proof above, the function ϕ^α is dependent on the subsequence. However, in the case where $L_p = L^p$, the function ϕ^α is unique up to constant addition.

Before stating our next corollary, we let $\{V_\alpha\}$ be an open cover of X as in Section 2.2, and define $\phi_p^\alpha, \rho_p^\alpha$ to be the local weights of h_p and g_p on V_α.

Corollary 4.6. *Let (X,ω) and L_p be as in (A) and (B). Assume the local weights ϕ_p^α of h_p are C^2 and $\{\phi_p^\alpha/A_p\}$ is bounded in $L^1(V_\alpha)$. Suppose the following conditions hold.*

(a) There exists a continuous $(1,1)$–form Φ such that

$$\frac{c_1(L_p,h_p)}{A_p} \to \Phi$$

uniformly.

(b) For all α the family $\{\rho_p^\alpha/A_p\}$ is uniformly bounded in $L^1(V_\alpha)$, and there exists $C \geq 0$ such that $A_p/a_p \leq C$.

Then for any real smooth closed $\theta \in \{\Phi\}$, and $\varphi \in C(X)$ such that $dd^c\varphi = \Phi - \theta$, we have

$$\frac{\gamma_p}{A_p} \to \theta + dd^c\varphi^{eq} \text{ and } \frac{c_1(L_p, h_p^{eq})}{A_p} \to \theta + dd^c\varphi^{eq}$$

weakly as currents, where $\varphi^{eq} = \sup\{\psi \in PSH(X,\theta) \mid \psi \leq \varphi\}$.

Proof: Note that, since (B) is assumed, our setup is stronger than the one in Corollary 4.3, so (a) implies condition (a) of Corollary 4.3. It suffices to show that condition (b) of Corollary 4.3 holds, i.e., there exists a $\varrho \in L^1(X)$ such that $\Phi + dd^c\varrho > c\omega$ for some $c > 0$.

By Lemma 4.4, there exists $\phi^\alpha \in C^1(V_\alpha)$ and a subsequence $\{\phi_{p_j}^\alpha\}$ such that $\phi_{p_j}^\alpha/A_p \to \phi^\alpha$ locally uniformly. Let ζ be a real smooth potential for ω on V_α. As A_p/a_p is bounded, it follows that for some $c > 0$ we have

$$dd^c(\rho_p^\alpha/A_p - c\zeta) = c_1(L_p, g_p)/A_p - c\omega \geq (a_p/A_p)\omega - c\omega \geq 0,$$

so $(\rho_p^\alpha/A_p - c\zeta)$ is psh for all α. Since $(\rho_p^\alpha/A_p - c\zeta)$ is uniformly bounded in $L^1_{loc}(V_\alpha)$, it follows from [Hö, Theorem 3.2.12] that $(\rho_{p_j}^\alpha/A_{p_j} - c\zeta)$ has a subsequence converging to a function $\widetilde{\rho}^\alpha \in PSH(U_\alpha)$ in $L^1(U_\alpha)$. By refining our original subsequence, we may assume WLOG that the convergence holds for all α simultaneously.

We define $\rho^\alpha = \widetilde{\rho}^\alpha + c\zeta$, and set $\varrho^\alpha = \rho^\alpha - \phi^\alpha$. We compute

$$\Phi + dd^c\varrho^\alpha = \Phi + dd^c\rho^\alpha - dd^c\phi^\alpha = dd^c(\widetilde{\rho}^\alpha) + dd^c c\zeta \geq c\omega.$$

Next, notice that

$$\frac{\rho_{p_j}^\alpha - \phi_{p_j}^\alpha}{A_{p_j}} \to \rho^\alpha - \phi^\alpha = \varrho^\alpha$$

in $L^1(U_\alpha)$ for all α. The left-most function glues to a global function, hence $\varrho^\alpha = \varrho^\beta$ almost everywhere for all α, β. Since ϱ^α is qpsh for all α, the equality holds everywhere. Then, the function $\varrho : X \to [-\infty, \infty)$ defined by $\varrho|_{V_\alpha} = \varrho^\alpha$ is well defined and satisfies

$$\Phi + dd^c\varrho \geq c\omega.$$

Clearly, $\varrho \in L^1(x)$, so condition (b) of Corollary 4.3 holds.

♥

Before stating the next theorem we recall than in (B'), the functions ϕ^α were the local uniform limits of the scaled local weights ϕ_p^α/A_p.

Theorem 4.7. *Let (X, ω) be as in (A) and L be a big line bundle on X. Suppose (L_p, h_p) are as in (B′), where $L_p = L^p$, $e_p^\alpha = (e_1^\alpha)^{\otimes p}$, and $A_p = p$. Then, there is a metric h on L with local weights ϕ^α on V_α, and*

$$\frac{\gamma_p}{p} \to c_1(L, h^{eq}) \text{ and } \frac{c_1(L^p, h_p^{eq})}{p} \to c_1(L, h^{eq}) \text{ weakly as currents,} \tag{4.1}$$

where h^{eq} is the equilibrium metric determined by h, as defined in Section 2.2.

The convergence (4.1) is a special case of (1.5), and one can formulate a statement similar to (4.1) whenever the ϕ^α are local weights of a metric. This isn't the only similarity to Theorem 1.5. In fact, the only condition from Theorem 1.5 which we can't show holds here is the assumption that $c_1(L_p, h_p)/A_p \geq dd^c\phi^\alpha - \delta_p\omega$. This property isn't needed here, as g handles the positivity requirement. Despite the similarities, Theorem 4.7 is proved using Theorem 1.2 and not Theorem 1.5.

The following lemma will be needed in our proof of Theorem 4.7.

Lemma 4.8. *If θ is a real smooth closed $(1,1)$–form on X with $PSH(X, \theta) \neq \varnothing$, and $\sigma_p \in C(X)$ converges uniformly to $\sigma \in C(X)$, then $\tau_p, \tau \in PSH(X)$ with $\tau_p \to \tau$ uniformly, where*

$$\tau_p := \sup\{\psi \in PSH(X, \theta) \mid \psi \leq \sigma_p\} \text{ and } \tau := \sup\{\psi \in PSH(X, \theta) \mid \psi \leq \sigma\}.$$

Proof. Define

$$F_p = \{\psi \in PSH(X, \theta) \mid \psi \leq \sigma_p\} \text{ and } F = \{\psi \in PSH(X, \theta) \mid \psi \leq \sigma\}.$$

Let $\psi \in F_p$. We see

$$\psi - \|\sigma_p - \sigma\|_{L^\infty} \leq \sigma \text{ and } \psi - \|\sigma_p - \sigma\|_{L^\infty} \in PSH(X, \theta),$$

so $\psi - \|\sigma_p - \sigma\|_{L^\infty} \in F$. Taking the supremum over all such ψ we obtain $\tau_p - \|\sigma_p - \sigma\|_{L^\infty} \leq \tau$. Similarly, $\tau - \|\sigma - \sigma_p\|_{L^\infty} \leq \tau_p$, thus $\|\tau - \tau_p\|_{L^\infty} \leq \|\sigma_p - \sigma\|_{L^\infty}$, and so $\tau_p \to \tau$ uniformly as desired.

♥

For the following proof, we fix a smooth metric h_0 on L with curvature $\theta = c_1(L, h_0)$. Recall definition (1.4), that is, T is the current satisfying $T|_{V_\alpha} = dd^c\phi^\alpha$. By definition, the global weight of h^{eq} is given by $\varphi^{eq} = \sup\{\psi \in PSH(X, \theta) \mid \psi \leq \varphi\}$.

Proof of Theorem 4.7. As in the proof of Proposition 1.4, there exist pluriharmonic functions $\psi_p^{\alpha\beta} \in C^\infty(V_\alpha \cap V_\beta)$ for all α, β, and p, which are defined by $\psi_p^{\alpha\beta} = \log|k_p^{\alpha\beta}|$, where $k_p^{\alpha\beta}$ denotes the transition function of L_p. These have the property that

$$\phi_p^\alpha = \phi_p^\beta + \psi_p^{\alpha\beta}.$$

Since $e_p^\alpha = (e^\alpha)^{\otimes p}$, it follows that $\psi_p^{\beta\alpha} = p\psi_1^{\beta\alpha}$, thus

$$\phi^\alpha = \lim_p \left(\frac{\phi_p^\alpha}{p}\right) = \lim_p \left(\frac{\phi_p^\beta + p\psi_1^{\beta\alpha}}{p}\right) = \phi^\beta + \psi_1^{\beta\alpha},$$

hence h is well defined, and $\phi_p^\alpha/p - \phi^\alpha$ glue to a global potential for $c_1(L_p, h_p)/p - T$.

Let g be a metric on L with strictly positive curvature current and local weights ρ^α. The family of bounded local weights of g^p is $\{p \cdot \rho^\alpha / A_P\} = \{\rho^\alpha\}$, which is clearly bounded in $L^1(V_\alpha)$. Since $\phi_p^\alpha / p \to \phi^\alpha$ locally uniformly on V_α, it follows that $\{\phi_p^\alpha / p\}$ is equicontinuous and uniformly bounded on U_α. This shows the hypothesis of Theorem 1.2 hold with $g_p = g^p$, hence

$$\frac{\gamma_p - c_1(L_p, h_p^{eq})}{p} \to 0.$$

Since $c_1(L, h^{eq}) = \theta + dd^c \varphi^{eq}$, we'll be done if we show

$$\frac{c_1(L_p, h_p^{eq})}{p} \to \theta + dd^c \varphi^{eq}.$$

As

$$\frac{c_1(L_p, h_p^{eq})}{p} = \theta + \frac{dd^c \varphi_p^{eq}}{p},$$

our desired convergence will hold provided

$$\frac{\varphi_p^{eq}}{p} \to \varphi^{eq} \text{ locally uniformly.}$$

We show this now.

Let φ and φ_p denote the global weights of h and h_p with respect to h_0 and h_0^p, respectively. If ψ^α denotes the local weight of h_0 on V_α, then

$$\frac{\varphi_p}{p} = \frac{\phi_p^\alpha - p\psi^\alpha}{p} = \frac{\phi_p^\alpha}{p} - \psi^\alpha.$$

Then $\varphi_p / p \to \phi^\alpha - \psi^\alpha = \varphi$ uniformly. By definition we have

$$\frac{\varphi_p^{eq}}{p} = \sup\{\psi \in PSH(X, \theta) \in \psi \le \varphi_p / p\} \text{ and } \varphi^{eq} = \sup\{\psi \in PSH(X, \theta) \in \psi \le \varphi\},$$

hence by Lemma 4.8 it follows that $\varphi_p^{eq}/p \to \varphi^{eq}$ locally uniformly, which completes the proof

♥

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Department of Mathematics, Syracuse University, Syracuse, NY 13244-1150, USA
Email address: mawolff@syr.edu