THETA LINKAGE MAPS AND A GENERIC ENTAILMENT FOR GSp_4

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ABSTRACT. We construct a new family of mod p weight shifting differential operators on the Siegel threefold. We coin the term theta linkage maps to refer to some operators between automorphic vector bundles with linked weights, which can be thought of as generalizations of the classical theta cycle. In particular there exist such maps within the p-restricted region, whose weight shifts are directly related to the conjectures of Herzig on the weight part of Serre's conjecture. As an application we produce a generic entailment of Serre weights, i.e. a Hecke eigenform with a generic Serre weight in the lowest alcove also has a Serre weight in one of the upper alcoves. We also prove a partial result towards finding a lowest alcove Serre weight for a particular non-ordinary Fontaine-Laffaille $\overline{\rho}$, in the spirit of Faltings-Jordan and Tilouine.

Contents

0.	Introduction	1
1.	Set up	6
2.	Geometric construction of the theta operators on the open strata	18
3.	Extension to the flag Shimura variety	25
4.	Differential operators and Verma modules	35
5.	Applications to the weight part of Serre's conjecture	63
Ret	References	

0. INTRODUCTION

This article deals with weight shifting differential operators on automorphic vector bundles over special fibers of Shimura varieties, often referred to as mod p theta operators. They were first defined on mod p modular forms by Katz [Kat77], and then used by Edixhoven [Edi92] to prove some cases of the weight part of Serre's conjecture. Concretely, given a non-ordinary mod p eigenform of weight

 $2 \le k \le p$, repeated application of the theta operator (the so-called theta cycle, first appearing in [Joc82]) produces a form of weight p - k + 3 with the same eigensystem up to a cyclotomic twist. More general theta operators have been defined by Andreatta-Goren [AG05] for Hilbert modular forms, Yamauchi for GSp₄ [Yam23], Eischen-Mantovan and their collaborators [EFG⁺21] [EM22] for type A and C PEL Shimura varieties, and La Porta [Por23] on strata of some unitary Shimura varieties; by generalizing Katz's construction. In this article we define a more extensive family of operators on the Shimura variety for GSp_4/\mathbb{Q} , generalizing the phenomenon of the theta cycle to allow for a wider range of weight shifts, which enable us to make progress towards the weight part of Serre's conjecture.

Let Sh/\mathbb{Z}_p be the integral model of the Siegel threefold at hyperspecial level, and $\overline{\mathrm{Sh}}$ its special fiber. One novelty is that we work on the flag Shimura variety, which parametrizes flags on the Hodge bundle of the Shimura variety: $\pi : \mathcal{F}l \to \overline{\mathrm{Sh}}$. One can define automorphic line bundles $\mathcal{L}(k,l)$ over $\mathcal{F}l$ such that $\pi_*\mathcal{L}(k,l) = \omega(k,l) = \mathrm{Sym}^{k-l}\omega \otimes \det^l \omega$ are precisely automorphic vector bundles on $\overline{\mathrm{Sh}}$, so that we can construct differential operators on $\overline{\mathrm{Sh}}$ from differential operators on $\mathcal{F}l$. The operators we define come in two different flavours. For the first type we define some *basic theta operators* on $\mathcal{F}l$, whose construction is similar to Katz's approach. First we define them on an open dense subset of $\mathcal{F}l$ characterized by having a canonical splitting of the symplectic flag, so that $\Omega^1_{\mathcal{F}l}$ splits into 4 line bundles, and then we extend them to $\mathcal{F}l$ by multiplying by some appropriate number of Hasse invariants. This produces 4 theta operators $\theta_i : \mathcal{L}(\lambda) \to \mathcal{L}(\lambda + \mu_i)$, where

$$\mu_i = \begin{cases} (p+1, p-1), & i=1\\ (2p, p-1), & i=2\\ (2p, 0), & i=3\\ (p-1, 0), & i=4 \end{cases}$$

with $\pi_*\theta_1$ already appearing in the literature, for instance in [Yam23] and [EFG⁺21]. The weight increase of these operators is of the order of p, which in general is insufficient for applications to the weight part of Serre's conjecture. However, one can construct operators with a smaller weight shift out of these basic operators. Using the property that $\theta_1^p = H_1^p \theta_3$ (where H_1 is the classical Hasse invariant), together with some control on when H_1 divides θ_1 , means that repeated application of θ_1 exhibits a similar behaviour to Jochnowitz's theta cycle. Concretely, it produces an operator

$$\theta_{(k,l)}^1 \coloneqq \theta_1^{p-k+1} / H_1^{p-k+1} : \mathcal{L}(k,l) \to \mathcal{L}(2p-k+2,l) \quad 1 \le k \le p$$

whose weight increase is smaller. We explain the significance of this map to the weight part of Serre's conjecture as outlined in [Her09] and [GHS18]. In characteristic zero for $k \ge l \ge 3$ the non-Eisenstein Hecke eigensystems appearing in $H^0(\operatorname{Sh}_{\overline{\mathbb{Q}}_p}, \omega(k, l))$ are the same as the ones appearing in $H^3_{\operatorname{\acute{e}t}}(\operatorname{Sh}_{\overline{\mathbb{Q}}_p}, V(k-3, l-3))$, where V(a, b) is the $\overline{\mathbb{Q}}_p$ -local system associated to the representation of GSp₄ of highest weight (a, b) with the same name, the so-called Weyl module of weight (a, b). One can define Weyl modules integrally, but in general their mod p reductions will be reducible. Denote by F(a, b) the socle of $V(a, b)_{\overline{\mathbb{F}}_p}$ as a $\operatorname{GSp}_4(\mathbb{F}_p)$ -representation. Then the irreducible $\operatorname{GSp}_4(\mathbb{F}_p)$ -representations over $\overline{\mathbb{F}}_p$, i.e. Serre weights, are all of the form F(a, b) for *p*-restricted weights: $0 \le a - b, b < p$. Given a mod *p* Hecke eigenform *f* the weight part of Serre's conjecture asks for the set $W(\overline{\rho}_f)$ of all Serre weights σ such that the eigensystem of *f* appears in $H^*_{\text{ét}}(\operatorname{Sh}_{\overline{\mathbb{Q}}_p}, \sigma)$. For a slightly more restrictive notion than *p*-restricted weights one can lift mod *p* coherent cohomology to characteristic 0 by a result of [LS13] and [Ale22], and mod *p* étale cohomology to characteristic 0 by a result of [HL23], under some auxiliary genericity conditions on a non-Eisenstein Hecke eigensystem *f*. Then under these hypotheses the Hecke eigensystems appearing in $H^0(\operatorname{Sh}_{\overline{\mathbb{F}}_p}, \omega(k, l))$ and $H^3_{\text{ét}}(\operatorname{Sh}_{\overline{\mathbb{Q}}_p}, V(k-3, l-3)_{\overline{\mathbb{F}}_p})$ are the same. Crucially, we prove that for $0 \le k \le p-2$ the map $\theta^1_{(k,l)}$ is injective on $H^0(\overline{\operatorname{Sh}}, \omega(k, l))$, so that after translating to étale cohomology and using the decomposition of Weyl modules into Serre weights we get the following result.

Theorem 0.1. (Theorem 5.3) Let $\lambda_0 = (a, b) \in X^*(T)$ satisfying $a \ge b \ge 1, a + b and <math>\mathfrak{m} \subset \mathbb{T}$ be a generic non-Eisenstein eigensystem such that $H^3_{\acute{e}t}(\operatorname{Sh}_{\overline{\mathbb{Q}}_p}, F(\lambda_0))_{\mathfrak{m}}$ is non-zero. Then $H^3_{\acute{e}t}(\operatorname{Sh}_{\overline{\mathbb{Q}}_p}, F(\lambda_i))_{\mathfrak{m}} \ne 0$ for either $\lambda_1 = (p - b - 3, p - a - 3) \in C_1$ or $\lambda_2 = (p + b - 1, p - a - 3) \in C_2$. That is, $F(\lambda_1) \in W(\overline{\rho})$ or $F(\lambda_2) \in W(\overline{\rho})$.

The key fact is that $F(\lambda_0)$ is a Jordan-Holder factor of $V(\lambda_1)$, while $V(\lambda_2)$ only has $F(\lambda_1)$ and $F(\lambda_2)$ as factors. In [GHS18] they coined the term $F(\lambda_0)$ entails $F(\lambda_1)$ or $F(\lambda_2)$ to denote this phenomenon. This is the first instance of a generic entailment in a global setting in the literature: generic in the sense that it holds for λ_0 sufficiently away from the walls of the lowest alcove.

We explain the relation of this result to the structure of Breuil-Mezard cycles. The Breuil-Mezard conjecture ([BM02], [EG23] in this formulation) predicts the existence of top dimensional cycles $\mathcal{Z}_V \in \operatorname{Ch}_{\operatorname{top}}(\mathcal{X}_{\operatorname{red}})$ inside the reduced moduli stack of 4 dimensional symplectic (ϕ, Γ) -modules for each $V \in K^0(\operatorname{Rep}_{\mathbb{F}_p}(\operatorname{GSp}_4(\mathbb{F}_p)))$. These should satisfy that $\mathcal{Z}_{V(\lambda)}$ is equal to the class of $\mathcal{X}_{\mathbb{F}_p}^{\lambda+\rho}$: the special fiber of the substack of crystalline Galois representations with Hodge-Tate weights $\lambda + \rho$. Assuming this conjecture [GHS18] formulate a general version of the weight part of Serre's conjecture: $W(\overline{\rho}_f) = \{\sigma : \mathcal{Z}_{\sigma}(\overline{\rho}_f) \neq 0\}$. Assuming the Breuil-Mezard conjecture, this version of the weight part of Serre's conjecture, and strong enough globalization results, then Theorem 0.1 implies that

$$C_{F(\lambda_0)} \subset \mathcal{Z}_{V(\lambda_2)} = \mathcal{Z}_{F(\lambda_2)} + \mathcal{Z}_{F(\lambda_1)},$$

where $C_{F(\lambda_0)}$ is the irreducible component labelled by λ_0 . See Remark 5.1. By some computations on local models of Galois deformation rings one can see that in fact it must be that $C_{F(\lambda_0)} \subset \mathcal{Z}_{F(\lambda_2)}$, and that $F(\lambda_0) \mapsto F(\lambda_2)$ should be the only generic entailment for $\operatorname{GSp}_4/\mathbb{Q}$. See also forthcoming work of Le Hung-Lin [LHL] where they determine generic Breuil-Mezard cycles in terms of their irreducible components for low rank groups including $\operatorname{GSp}_4/\mathbb{Q}_p$, building up on the local approach of [FLH23].

The second kind of differential operator generalizes maps like $\theta_{(k,l)}^1$. In the sense that for some specific combinations of the basic theta operators it might happen that they become highly divisible by Hasse invariants, and their weight increase allows for maps between *p*-restricted weights. We give a more conceptual way of constructing these maps, which we will refer to as *theta linkage maps*.

To do this we construct a functor from maps of Verma modules (over \mathbb{Z}_p) to differential operators between automorphic (line) vector bundles on the integral model of the (flag) Shimura variety, see Theorem 4.16. In characteristic 0 this functor was known since [FC90], where non-zero maps of Verma modules must be between weights which are Weyl reflections of each other. In particular the functor does not produce non-zero maps between dominant weights. This was later modified by Mokrane-Tilouine [MT02] and Lan-Polo [LP18] to integral models of PEL Shimura varieties in order to construct an integral BGG complex for weights in the lowest alcove. Importantly, in characteristic p there are extra maps of Verma modules, which are related to the linkage relation as follows. The interior of the p-restricted region can be divided by coroot walls into four ρ -shifted alcoves C_i . We say that $\lambda \uparrow \mu$ (μ is linked to λ) if μ is obtained from λ by a series of reflections across the walls in the positive direction. For alcoves one gets the same notion by considering any such pair of weights inside them. For GSp₄ one has $C_0 \uparrow C_1 \uparrow C_2 \uparrow C_3$. We prove the existence of non-zero maps between Verma modules in characteristic p with linked weights, whose image under Theorem 4.16 we refer to as theta linkage maps. In particular, we can construct the following theta linkage maps in the p-restricted region.

Theorem 0.2. (Corollary 4.19) Let $\lambda_0 \in C_0$ and let $\lambda_i \in C_i$: i = 1, 2, 3 be their respective linked weights. Then for i = 0, 1, 2 there are non-zero Hecke equivariant away from p maps of sheaves on $\mathcal{Fl}_{\overline{\mathbb{F}}_n}$

$$\mathcal{L}(\lambda_i + (3,3)) \to \mathcal{L}(\lambda_{i+1} + (3,3)).$$

Moreover, these maps are combinations of the basic thetas θ_i that become highly divisible by Hasse invariants. The map from $\lambda_1 + (3,3)$ to $\lambda_2 + (3,3)$ is precisely $\theta_{\lambda_1 + (3,3)}^1$.

We remark that the other two linkage maps would have been almost impossible to find by hand, as they correspond to some complicated polynomial on the θ s, in the same way that maps of Verma modules are complicated polynomials on the roots. The figure below depicts the weights of the theta operators and Hasse invariants, along with some examples of theta linkage maps. The 4 restricted alcoves and the simple roots (twisted by an element of the Weyl group) are also drawn, with $\theta_{(k,l)}^1$ corresponding to the reflection across the long root β . We also prove a BGG decomposition for the



de Rham complex of a lowest alcove representation on the flag Shimura variety, where we identify its differentials as theta linkage maps.

Finally, we obtain a partial result towards proving a case of the weight part of Serre's conjecture in which the Serre weight actually depends on the local Galois representation, which is allowed to be non-semisimple. For this we restrict to $\overline{\rho}$ being Fontaine-Laffaille, so that we can describe $\overline{\rho}$ in terms of the crystalline cohomology of the Shimura variety, which in turn can be described using the BGG complex. This approach is inspired by Faltings and Jordan's [FJ95] proof of the existence of the ordinary companion form for GL_2/\mathbb{Q}_p and the attempt at generalizing it to GSp_4 by Tilouine [Til12]. Using both the basic theta operators and the linkage maps we give a recipe that starting from a Fontaine-Laffaille weight ends up in the other Fontaine-Laffaille Serre weight predicted for some particular non-ordinary $\overline{\rho}$, and we obtain the following result towards proving it.

Theorem 0.3 (Theorem 5.6). Let \mathcal{O}/\mathbb{Z}_p be the ring of integers of a finite extension of \mathbb{Q}_p . Let $\overline{r}: G_{\mathbb{Q}} \to \operatorname{GSp}_4(\overline{\mathbb{F}}_p)$ coming from a non-Eisenstein generic Hecke eigenform in $H^0(\operatorname{Sh}_{\mathcal{O}}, \omega(k, l))$ satisfying $k \geq l \geq 3$ and $k + l \leq p + 1$. Assume that the Fontaine-Laffaille module M of $\overline{\rho} \coloneqq \overline{r}_{G_{\mathbb{Q}_p}}$ satisfies $M^{k-1} = D_{l-2}$. One can find an eigenform $g \in H^1(\operatorname{Sh}_{\mathcal{O}}^{\operatorname{con}}, \omega^{\operatorname{can}}(k, 4 - l))$ such that $\overline{r}_g = \overline{r}$. Assume the following hypotheses

- (1) One has $\theta^1_{(k,4-l)}(\overline{g}) \neq 0$.
- (2) $H^1(\overline{\mathrm{Sh}}^{\mathrm{tor}}, \omega^{\mathrm{can}}(p-k+2, -p(p+l-k-4)-k+2))_{\mathfrak{m}_{\overline{r}}} = 0.$
- (3) $H^1(\operatorname{Sh}^{\operatorname{tor}}, \omega^{\operatorname{can}}(p-k+3, 4-l))_{\mathfrak{m}_{\overline{\tau}}}$ is torsion free and $H^2(\operatorname{\overline{Sh}}^{\operatorname{tor}}, \omega^{\operatorname{can}}(p-k+3, 4-l))_{\mathfrak{m}_{\overline{\tau}}} = 0.$ This holds automatically if $k-l \geq 3$ and $l \geq 5$.

Then there exists another eigenform $h \in H^1(\operatorname{Sh}^{\operatorname{tor}}_{\mathcal{O}}, \omega^{\operatorname{can}}(p-k+3, 4-l))$ such that $\theta^1_{(k,l)}(\overline{g}) = \theta_4(\overline{h})$ and $\overline{r}_h = \overline{r}$ up to cyclotomic twist. In particular $F(p-k, l-3) \in W(\overline{\rho})$.

The condition on the Fontaine-Laffaille module is precisely the one under which $\overline{\rho}$ has a crystalline lift up to cyclotomic twist of Hodge-Tate weights $\{0, l-2, p-k+2, p-k+l\}$, which conjecturally corresponds to having the other Serre weight. In particular this allows for $\overline{\rho}$ to be non-semisimple. Under some Taylor-Wiles assumptions on \overline{r} the result at the level of Serre weights (without our extra hypotheses) was essentially known after possibly working over a larger totally real field F/\mathbb{Q} using the methods of [BLGGT14], since one can find an explicit Fontaine-Laffaille lift of $\overline{\rho}$ with the right Hodge-Tate weights, which is automatically potentially diagonalizable. One advantage of this method is that it doesn't require any Taylor-Wiles assumptions, and it is geometric and constructive in nature.

Remark 0.1. The first assumption is the more serious one, as it seems to be difficult to understand the behaviour of theta operators on higher cohomology by bare hands methods. The second is true in characteristic zero, in fact the weight will be generically far away from the cone that has non-zero degree 1 cohomology [Lan16b]. Based on the results on the vanishing of H^0 in characteristic p of [GK18] we deem it reasonable to conjecture that it vanishes generically.

Given any $\overline{\rho}$ it is in fact possible to give other recipes that end up on any generic weight predicted by [HT13] to be in $W(\overline{\rho})$, by applying and "dividing" by theta (linkage) maps and Hasse invariants. See Section 5.3 for some examples. We currently don't have any good evidence that they can indeed be used to prove cases of the weight part of Serre's conjecture, but we see it as a good sign that the combinatorics of the theta operators are compatible with the weight part of Serre's conjecture.

We make some comments on possible generalizations. The construction of the basic theta operators on the flag Shimura variety would extend without much trouble to a general setting like PEL Shimura varieties or Hilbert-Siegel Shimura varieties, modulo finding the exact power of the Hasse invariants that one needs to use. The relation between Verma modules and differential operators on the Shimura variety is also quite robust, and one can produce more examples of interesting theta linkage maps within the *p*-restricted weights. However, studying the finer properties of these maps is a more difficult question, e.g. determining their (co)kernels or their behaviour in cohomology. These difficulties already appear in the setting of this work, because of that we restrict this paper to GSp_4/\mathbb{Q} , and we leave further study of the more general setting for future work.

The organization of this article is as follows. In Section 1 we recall the geometric and representation theoretic set up for the Siegel threefold. In Section 2 we define the basic thetas on an open dense subset $U \subset \mathcal{F}l$. In Section 3 we extend them to $\mathcal{F}l$ and we prove some key properties about them in the general spirit of previous literature, including the relation to the operators in [EFG⁺21], culminating in Theorem 3.13. In Section 4 we establish the relation between differential operators and Verma modules, and we construct the theta linkage maps. In Section 5 we spell out the two main applications to the weight part of Serre's conjecture.

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1. Set up

Let p be a prime, $G = GSp_4$ as a reductive group over Z. Our convention is that G is the group of symplectic similitudes of \mathbb{Z}^4 with its symplectic form given by the matrix

$$J = \begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix}$$

with $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let \mathbb{A}^{∞} be the finite adeles of \mathbb{Q} , we fix a hyperspecial level $K = K^p K_p \subset G(\mathbb{A}^{\infty})$, in particular its component at p satisfies $K_p = G(\mathbb{Z}_p)$. We also assume that K is neat $(K^p \text{ small enough})$ so that one can define a smooth quasi-projective integral model $\mathrm{Sh}_K/\mathbb{Z}_p$ of the Siegel Shimura variety associated to (G, \mathcal{H}_2, K) as follows.

Definition 1.1. Let $F : \operatorname{Sch}/\mathbb{Z}_p \to \operatorname{Sets}$ be the functor defined by F(S) being the set of tuples (A, λ, η) up to equivalence, where

- A/S is an abelian scheme of relative dimension 2.
- $\lambda : A \to A^{\vee}$ is a prime to p quasi-isogeny such that there exist an integer $N \ge 1$ with $N\lambda$ a polarization, i.e. a prime to p quasi-polarization.
- η is a rational K-level structure of A as in [Lan13, 1.3.8].

Two tuples (A, λ, η) and (A', λ', η') are equivalent if there exists a prime to p quasi-isogeny $\phi : A \to A'$ commuting with λ and λ' up to a $\mathbb{Z}_{(p)}^{\times,>0}$ constant, and such that the pullback of η' under ϕ is η . This is represented by a smooth quasi-projective scheme Sh_K over \mathbb{Z}_p . We will mostly drop the level from the notation. Let $\overline{\mathrm{Sh}} = \mathrm{Sh} \otimes \overline{\mathbb{F}}_p$.

There is also a moduli description of Sh_K in terms of isomorphism classes of abelian varieties, which is more convenient when doing deformation theory.

Proposition 1.1. [Lan13, 1.4.3.3] Sh_K also represents the following moduli problem Sch/ $\mathbb{Z}_p \to$ Sets whose S-points are tuples (A, λ, η) up to isomorphism

- A/S an abelian scheme of relative dimension 2.
- $\lambda : A \to A^{\vee}$ a prime to p polarization.
- η is an integral K-level structure of (A, λ) as in [Lan13, 1.3.7.8].

An isomorphism between (A, λ, η) and (A', λ', η') is an isomorphism $f : A \to A'$ compatible with polarizations, and such that it sends η' to η .

Let A/Sh be the universal abelian variety with identity e, its Hodge bundle is defined by

$$\omega \coloneqq e^* \Omega^1_{A/\mathrm{Sh}}$$

Our notation for weights is as follows: for $(k, l) \in \mathbb{Z}^2$ satisfying $k \geq l$, the automorphic vector bundle on Sh of weight (k, l) is

$$\omega(k,l) \coloneqq \operatorname{Sym}^{k-l} \omega \otimes \operatorname{det}^k \omega.$$

Then the space of mod p Siegel modular forms of genus 2 of weight (k, l) is $H^0(\overline{Sh}, \omega(k, l))$. As a useful auxiliary space we introduce the flag Shimura variety.

Definition 1.2 (Flag Shimura variety). We define $\mathcal{F}l$ to be the flag space of ω , with the convention that

$$\mathcal{F}l = \operatorname{Proj}(\operatorname{Sym}^{\bullet}\omega) \xrightarrow{\pi} \operatorname{Sh}$$

It is smooth proper over Sh, with its fibers isomorphic to \mathbb{P}^1 , and it comes equipped with a line bundle $\mathcal{L} \subset \pi^* \omega$. For $(k, l) \in \mathbb{Z}^2$ we define the line bundle $\mathcal{L}(k, l)$ on $\mathcal{F}l$ by

$$\mathcal{L}(k,l) \coloneqq (\omega/\mathcal{L})^k \otimes \mathcal{L}^l.$$

We record some properties about the flag Shimura variety, let $\pi : \mathcal{F}l \to Sh$.

Lemma 1.2. (1) For $(k,l) \in \mathbb{Z}^2$, we have $\pi_* \mathcal{L}(k,l) = \omega(k,l)$. In particular the space of global sections of $\mathcal{L}(k,l)$ is the space of Siegel modular forms of weight (k,l) [Sta18, Tag 01XX].

- (2) There is a canonical isomorphism $\Omega^1_{\mathcal{F}l/Sh} = \mathcal{L}(-1,1)$.
- (3) det $\omega = \mathcal{L} \otimes (\omega/\mathcal{L})$.
- (4) $R^1\pi_*\mathcal{L}(k,l) = Hom_{\mathcal{O}_{Sh}}(Sym^{l-k-2}\omega, \mathcal{O}_{Sh}) \otimes det^{l-1}\omega$, where a negative symmetric power is equal to 0 by convention [Sta18, Tag 01XX].

Proof. For 2) on each affine $U \subset$ Sh such that ω_U is trivial the isomorphism reduces to the canonical isomorphism $\Omega^1_{\mathbb{P}^1_U/U} = \mathcal{O}(-2)_U = (\mathcal{L} \otimes (\omega/\mathcal{L})^{-1})_U$, and we can check that these isomorphisms glue.

The degeneration of the Hodge-de-Rham spectral sequence for abelian varieties yields the exact sequence

$$0 \to \omega \to H^1_{\mathrm{dR}}(A/\mathrm{Sh}) \to \omega_{A^\vee}^\vee \to 0,$$

and $\omega_{A^{\vee}} \xrightarrow{d\lambda} \omega_A$ is an isomorphism since the λ is prime to p. We will usually make this identification, whose only role will appear when considering equivariance for the Hecke operators. Let (once and for all) H denote $H^1_{dR}(A/Sh)$. This way H is a locally free sheaf, and carries a symplectic pairing \langle, \rangle induced by the quasi-polarization and the canonical pairing $H^1_{dR}(A/Sh) \otimes H^1_{dR}(A^{\vee}/Sh) \to \mathcal{O}_{Sh}$. In characteristic p it also comes with Verschiebung $V : H \to H^{(p)}$ and Frobenius $F : H^{(p)} \to H$ maps induced by the respective maps on A/Sh. Then ω is a maximal isotropic subspace for the symplectic pairing, and V, F satisfy

$$\langle Vx, y \rangle = \langle x, Fy \rangle^{(p)}$$

for $x \in H, y \in H^{(p)}$. It is also equipped with the Gauss-Manin connection.

In general, let Y/S/T be schemes with Y/S and S/T smooth. The Gauss-Manin connection is a natural connection [KO68]

$$\nabla_{Y/S/T}: H^1_{\mathrm{dR}}(Y/S) \to H^1_{\mathrm{dR}}(Y/S) \otimes_{\mathcal{O}_S} \Omega^1_{S/T}$$

It is compatible with base change in the following sense.

Lemma 1.3. (Functoriality of ∇) Let Y/S/T as before, $f: S' \to S$ a morphism of T-schemes, and let $Y' \coloneqq Y \times_S S'$. Let e be a local section of $H^1_{dR}(Y/S)$, we can pull it back to a local section f^*e of $H^1_{dR}(Y'/S') = f^*H^1_{dR}(Y/S)$. Then

$$\nabla_{Y'/S'/T}(f^*e) = \mathrm{id} \otimes \mathrm{df}(f^*\nabla_{Y/S/T}(e))$$

where df: $f^*\Omega^1_{S/T} \to \Omega^1_{S'/T}$ is the differential of f. That is, the diagram

commutes when interpreted at the level of local sections.

In general any connection ∇ on a quasicoherent sheaf E extends to $E^{\otimes k}$ by

$$\nabla(v_1 \otimes v_2 \ldots \otimes v_k) = \nabla(v_1) \otimes v_2 \ldots v_n + \ldots + \dots \otimes \nabla(v_n)$$

where we implicitly use isomorphisms like $E \otimes \Omega^1 \otimes E \cong E \otimes E \otimes \Omega^1$ on the summands so that they land on $E^{\otimes k} \otimes \Omega^1$. The dual connection ∇^{\vee} on E^{\vee} is defined by $\nabla(\nu)(s) = d(\nu(s)) - \nu(\nabla(s))$ for $\nu \in E^{\vee}$ and $s \in E$. Similarly, one can extend it to symmetric powers or wedge powers of E. Consider ∇ with respect to $A/\operatorname{Sh}/\mathbb{Z}_p$. It respects the symplectic structure in the sense that $\langle \nabla v, w \rangle + \langle v, \nabla w \rangle = d \langle v, w \rangle$ for $v, w \in H$. By restricting ∇ to ω and then projecting along $H \to \omega_{A^{\vee}}^{\vee}$ we obtain the Kodaira-Spencer morphism

$$\mathrm{KS}: \omega_A \otimes \omega_{A^{\vee}} \to \Omega^1_{\mathrm{Sh}/\mathbb{Z}_p}.$$

After applying the isomorphism $d\lambda : \omega_{A^{\vee}} \to \omega_A$ one can see the maps factors through an isomorphism ks : $\operatorname{Sym}^2 \omega \cong \Omega^1_{\operatorname{Sh}/\mathbb{Z}_n}$ also denoted by Kodaira-Spencer.

Now we define the two Hasse invariants on $\mathcal{F}l/\overline{\mathbb{F}}_p$. Recall that for L a line bundle on a characteristic p scheme X there is a canonical isomorphism $L^{(p)} := \operatorname{Frob}_X^* L \cong L^{\otimes p}$.

Definition 1.3. (Hasse invariants) On $\mathcal{F}l$ consider the Verschiebung map $V: \omega \to \omega^{(p)}$.

(1) By taking determinants of V we obtain a map det $V : \det \omega \to (\det \omega)^{(p)} = \det^p \omega$, which produces a section

$$H_1 \in H^0(\mathcal{F}l, \mathcal{L}(p-1, p-1))$$

We say that H_1 is the first Hasse invariant of $\mathcal{F}l$.

(2) Consider the composite map $\mathcal{L} \hookrightarrow \omega \xrightarrow{V} \omega^{(p)} \to (\omega/\mathcal{L})^{(p)}$. This produces the second Hasse invariant

$$H_2 \in H^0(\mathcal{F}l, \mathcal{L}(p, -1)).$$

(3) Let D_i be the locus where H_i vanishes, and let U_i be its complement in $\mathcal{F}l$.

The first Hasse invariant is the pullback of the classical Hasse invariant $H \in H^0(\overline{\text{Sh}}, \det^{p-1} \omega)$, while H_2 lives naturally on the flag variety. The D_i are divisors since $\overline{\text{Sh}}$ is smooth and H_i are non-zero, for instance U_1 is the pullback of the ordinary locus. More generally there exists a full stratification of $\mathcal{F}l$ similar to the Ekedahl-Oort stratification of $\overline{\text{Sh}}$, which is cut out by partial

Hasse invariants. It was introduced in the Siegel case in [EvdG09]. We will only briefly need the Ekedahl-Oort stratification of \overline{Sh} itself, which we describe now. There are 4 strata.

- (1) The ordinary locus $\overline{\mathrm{Sh}}^{\mathrm{ord}}$ with the property that $V: \omega \to \omega^{(p)}$ is invertible. It is open and dense.
- (2) The *p*-rank 1 locus $\overline{\text{Sh}}^{=1}$ characterized by *V* having rank 1 and $V^2 \neq 0$. It is locally closed of dimension 2.
- (3) The general supersingular locus $\overline{\text{Sh}}^{\text{gss}}$ characterized by $V^2 = 0$ and $V \neq 0$. It is locally closed of dimension 1.
- (4) The superspecial locus \overline{Sh}^{sss} characterized by V = 0. It has dimension 0.

The closure relations are simple: the closure of the *i*th dimensional strata contains all the other lower dimensional strata. Let also $\overline{\mathrm{Sh}}^{\geq 1} = \overline{\mathrm{Sh}}^{\mathrm{ord}} \cup \overline{\mathrm{Sh}}^{=1}$. It is an open whose complement has codimension 2, which is a useful property by the following "Hartogs lemma".

Lemma 1.4. [Har94, Prop 1.11] Let X be a locally Noetherian normal scheme and \mathcal{F} a finitely generated \mathcal{O}_X -module satisfying Serre's condition S_2 (e.g. a vector bundle). Let $U \subset X$ be an open whose complement has codimension at least 2. Then the restriction map

$$\Gamma(X,\mathcal{F}) \to \Gamma(U,\mathcal{F})$$

is an isomorphism.

Notation 1. We will make use of local computations on $\mathcal{F}l$, so we explain our notation here, which will be used throughout. Let $W \subset \overline{Sh}$ be an open subset of \overline{Sh} such that $\omega | W$ is free with basis $\{e_1, e_2\}$. On the pullback of ω to $\mathcal{F}l$ we use the pullback of this basis. Then on W we can identify $\mathcal{F}l$ with \mathbb{P}^1_W and \mathcal{L} with $\mathcal{O}(-1)$. We denote the coordinates of $\mathbb{P}^1 = \operatorname{Proj}(\mathbb{Z}[x, y])$ by x, y, and we consider \mathcal{L} embedded in $\mathcal{O}^2 \cong \omega$ by

$$\mathcal{L} \cong \mathcal{O}(-1) \xrightarrow{(x,y)} \mathcal{O}^2 \cong \omega.$$

On the affine chart $\{x \neq 0\} \mathcal{L}$ is free with basis $e_1 + \frac{y}{x}e_2$. We will denote $\frac{y}{x}$ by T and refer to the associated chart by $\mathbb{A}^1_W \subset \mathbb{P}^1_W$, where ω/\mathcal{L} is free with basis e_2 . Thus in this chart we will use

$$e_2^k (e_1 + Te_2)^l = e_2^{k-l} (e_1 \wedge e_2)^l$$

as a local basis for $\mathcal{L}(k, l)$. Let

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be the matrix of $V: \omega \to \omega^{(p)}$ with respect to the local basis, e.g. $V(e_1) = ae_1^{(p)} + ce_2^{(p)}$. Then

$$H_1 = (ad - bc)(e_1 \wedge e_2)^{p-1}.$$

A basis for $\mathcal{L}^{(p)}$ on \mathbb{A}^1_W is $e_1^{(p)} + T^p e_2^{(p)}$, and

$$V(e_1 + Te_2) = (a + bT)e_1^{(p)} + (c + dT)e_2^{(p)} = (c + dT - aT^p - bT^{p+1})e_2^p \in (\omega/\mathcal{L})^p$$

so that

$$H_2 = (c + dT - aT^p - bT^{p+1})e_2^p(e_1 + Te_2)^{-1} = (cx^{p+1} + dx^py - ay^px - by^{p+1})e_2^{p-1}(e_2 \wedge e_1)^{-1} \in H^0(\mathbb{P}^1_W, \mathcal{L}(p, -1)).$$

After choosing a basis $\{e_1, e_2\}$ as above we will denote by \tilde{H}_i the local functions defining H_i .

We can also define automorphic vector bundles in a more systematic way. Let $G = GSp_4$, P be the Siegel parabolic

$$P = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix},$$

and B the upper triangular Borel.

Definition 1.4. Let $(\mathbb{Z}_p^4 = L \oplus L^{\vee}, \langle, \rangle)$ be the standard 4-dimensional symplectic space over \mathbb{Z}_p , with $L = \langle (1, 0, 0, 0), (0, 1, 0, 0) \rangle$ as Lagrangian. Further, let $F = \langle (1, 0, 0, 0) \rangle \subset L$ represent the canonical line on the flag variety. Let

$$I_{G} = \underline{\operatorname{Isom}}_{\mathcal{O}_{\operatorname{Sh}}}(H^{1}_{\mathrm{dR}}(A/\operatorname{Sh}), (L \oplus L^{\vee}) \otimes \mathcal{O}_{\operatorname{Sh}})$$

$$I_{P} = \underline{\operatorname{Isom}}_{\mathcal{O}_{\operatorname{Sh}}}((H^{1}_{\mathrm{dR}}(A/\operatorname{Sh}), \omega), (L \oplus L^{\vee}, L) \otimes \mathcal{O}_{\operatorname{Sh}})$$

$$I_{B} = \underline{\operatorname{Isom}}_{\mathcal{O}_{TL}}((H^{1}_{\mathrm{dR}}(A/\mathcal{F}l), \omega, \mathcal{L}), (L \oplus L^{\vee}, L, F) \otimes \mathcal{O}_{\mathcal{F}l}).$$

where all the isomorphisms should respect the symplectic pairing up to a scalar. They are étale G, P and B torsors respectively. For any \mathbb{Z}_p -algebra R define functors

$$F_{G} : \operatorname{Rep}_{R}(G) \to \operatorname{Coh}(\operatorname{Sh}_{R})$$

$$V \mapsto I_{G} \times^{G} V$$

$$F_{P} : \operatorname{Rep}_{R}(P) \to \operatorname{Coh}(\operatorname{Sh}_{R})$$

$$W \mapsto I_{P} \times^{G} W$$

$$F_{B} : \operatorname{Rep}_{R}(B) \to \operatorname{Coh}(\mathcal{F}l_{R})$$

$$V \mapsto I_{B} \times^{G} V$$

which produce automorphic vector bundles.

- **Proposition 1.5.** (1) For $V \in \text{Rep}(G)$, by considering it as a P representation one has $F_P(V) = F_G(V)$, and for $W \in \text{Rep}(P)$ $F_B(W) = \pi^* F_P(W)$.
 - (2) For $V \in \text{Rep}(G)$, $F_G(V)$ is equipped with a canonical connection. When V is the standard representation $F_G(V) = H$, and this connection is identified with the Gauss-Manin connection. Similarly, if V can be obtained as a Schur functor of the standard representation the connection on $F_G(V)$ is induced by the Gauss-Manin connection.

Proof. The first statement follows formally since I_G is the pushout of I_P along $P \subset G$, and $\pi^{-1}I_P$ is the pushout of I_B along $B \subset P$. We describe the connection on $F_G(V)$. Working locally, let

 $\phi \in I_G$ be a trivialization $H \cong (L \oplus L^{\vee}) \otimes_{\mathbb{Z}_p} \mathcal{O}$. By transporting the Gauss-Manin connection on Halong ϕ we get a connection $\tilde{\nabla}_{\phi} : (L \oplus L^{\vee}) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathrm{Sh}} \to (L \oplus L^{\vee}) \otimes_{\mathbb{Z}_p} \Omega^1_{\mathrm{Sh}}$. The restriction to $L \oplus L^{\vee}$ is a \mathbb{Z}_p -linear map, which can be thought of as an element ξ_{ϕ} of $\mathfrak{g} \otimes_{\mathbb{Z}_p} \Omega^1_{\mathrm{Sh}}$ since $\mathrm{End}_{\langle,\rangle}(L \oplus L^{\vee}) = \mathfrak{g}$. The image of ξ_{ϕ} under $\mathfrak{g} \otimes_{\mathbb{Z}_p} \Omega^1_{\mathrm{Sh}} \xrightarrow{\rho_V \otimes \mathrm{id}} \mathrm{End}(V) \otimes_{\mathbb{Z}_p} \Omega^1_{\mathrm{Sh}}$, where ρ_V is the \mathfrak{g} action on V, can be extended to a connection on $V \otimes \mathcal{O}_{\mathrm{Sh}}$ in a unique way, which we denote by ∇_{ϕ} . Define a connection $\nabla : F_G(V) \to F_G(V) \otimes \Omega^1_{\mathrm{Sh}} = I_G \times^G (V \otimes_{\mathbb{Z}_p} \Omega^1_{\mathrm{Sh}})$ (where in the second equality G acts trivially on Ω^1_{Sh}) by

$$abla(\phi, v) = (\phi, \nabla_{\phi}(v)).$$

Checking for compatibility with the G action reduces to the identity $g^{-1} \circ \nabla_{g\phi} \circ g = \nabla_{\phi}$ for all $g \in G$. From the definition of ∇_{ϕ} it follows from the identity $\xi_{g\phi} = g\xi_{\phi}g^{-1}$, which can be easily checked. It is clear that ∇ is a connection since each ∇_{ϕ} is, and that for V the standard representation it agrees with the Gauss-Manin connection. By tensor functoriality of F_G we get the statement for representations that are Schur functors of the standard representation. \Box

1.1. Weights and representation theory. We set up some notation for the weights and roots of $G = GSp_4$. Fix $T \subset B \subset P \subset G$ where T is the diagonal maximal torus, B the upper triangular Borel, and P the Siegel parabolic with Levi quotient M. Weights $X^*(T)$ will be labelled by a triple (a, b, c) of integers such that $a + b = c \mod 2$, representing the character

$$\begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & vt_2^{-1} & \\ & & & vt_1^{-1} \end{pmatrix} \to t_1^a t_2^b v^{(c-a-b)/2}$$

Then c corresponds to the central character, from now on we will largely drop it from the notation and work with weights for Sp₄ instead. **Warning**: with our convention on weights we have $\mathcal{L}(k, l) = F_B(l, k)$.

- Our choice of simple roots will be $\alpha = (1, -1, 0)$ (short) and $\beta = (0, 2, 0)$ (long). Denote by $\rho = (2, 1, 3)$ the half sum of positive roots up to translation by the center. For a root γ let $X_{\gamma} \in \mathfrak{g} = \text{Lie}(G)$ be an element generating its weight space.
- The convention is that the positive roots are $\Phi^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}$ corresponding to *B*. The corresponding coroots are $\alpha^{\vee} = (1, -1), \ \beta^{\vee} = (0, 1), \ (\alpha + \beta)^{\vee} = (1, 1)$ and $(2\alpha + \beta)^{\vee} = (1, 0)$ with the obvious pairing.
- For a linear algebraic group H/\mathbb{Z}_p define the category $\operatorname{Rep}_{\mathbb{Z}_p}(H)$ of algebraic representations of H which are finite free \mathbb{Z}_p -modules.
- For a weight $\lambda \in X^*(T)$, let $V(\lambda) \coloneqq H^0(G/B^-, \mathcal{L}_{\lambda}) \in \operatorname{Rep}_{\mathbb{Z}_p}(G)$ be the Weyl module of G, where B^- is the opposite Borel and $\mathcal{L}_{\lambda} = G \times \lambda/B^-$. Similarly, let $W(\lambda)$ be the Weyl module of M, by considering the lower triangular Borel of M. We will often regard M representations as P representations by inflation.
- The Weyl group of G has a presentation $W = \langle s_0, s_1 : s_0^2, s_1^2, (s_0s_1)^4 \rangle$ with s_0 the reflection corresponding to α and s_1 to β . The action on weights is given by $s_0(a, b, c) = (b, a, c)$,

 $s_1(a, b, c) = (a, -b, c)$. Then $W_M = \langle s_0 \rangle$, and the set W^M of minimal length generators of $W_M \setminus W$ is given by $W^M = \{1, s_1, s_1s_0, s_1s_0s_1\}$, ordered by increasing length. Define the dot action as $\omega \cdot \lambda = \omega(\lambda + \rho) - \rho$.

• We say that $\lambda \in X^*(T)$ is dominant, resp. *M*-dominant if $\langle \lambda, \gamma^{\vee} \rangle \geq 0$ for all simple roots γ in *G*, resp. *M*.

The *p*-restricted weights are $X_1(T) = \{(a, b, c) : 0 \le a - b, b < p\}$, and we say that $\lambda \in X^*(T)$ is *p*-small if $|\langle \lambda + \rho, \gamma^{\vee} \rangle| < p$ for all roots γ . We divide $X^*(T) - \rho$ into ρ -shifted alcoves, defined as the interior of regions defined by the hyperplanes $H_{n,\gamma} = \{\langle \lambda + \rho, \gamma^{\vee} \rangle = np\}$ for γ a root and $n \in \mathbb{Z}$. There are four *p*-restricted alcoves defined by the weights $(a, b, c) - \rho$ satisfying

$$\begin{split} C_0 &: a > b > 0, a + b p, b < a < p \\ C_2 &: a - b < p < a, a + b < 2p \\ C_3 &: b < p, a + b > 2p, a - b$$

- For λ dominant, let $L(\lambda)$ be the socle of $V(\lambda)$ as a $G_{\mathbb{F}_p}$ -representation. They are always irreducible.
- A Serre weight is an irreducible representation of $G(\mathbb{F}_p)$ with $\overline{\mathbb{F}}_p$ coefficients. Let $X_0(T) = \mathbb{Z}(0,0,2)$, then Serre weights are in bijection with $F(\lambda) := L(\lambda)_{|G(\mathbb{F}_p)|}$ for $\lambda \in X_1(T)/(p-1)X_0(T)$. In practice we will drop the central character from the notation.
- Let $\lambda \in X^*(T)$. For γ a root and $n \in \mathbb{Z}$ let $s_{\gamma,n} \cdot \lambda \coloneqq \lambda + (np \langle \lambda + \rho, \gamma^{\vee} \rangle)\gamma$. We say that $\mu \uparrow \lambda$ (λ is linked to μ) if there is a chain of weights $\mu_0 = \mu \leq \mu_1 \leq \ldots \leq \mu_k = \lambda$ such that $\mu_{i+1} = s_{\gamma_i,n_i} \cdot \mu_i$ for some γ_i, n_i . This extends to an action of the affine Weyl group $W_{\text{aff}} = p\mathbb{Z}\Phi^+ \rtimes W$ on $X^*(T)$, and $\mu \uparrow \lambda$ implies that $\lambda \in W_{\text{aff}} \cdot \mu$. The affine Weyl group acts simply transitively on ρ -shifted alcoves and the stabilizer of the region of p-small weights is the finite Weyl group W.

For weights in one of the four alcoves we can describe how $V(\lambda)$ decomposes into irreducibles.

Proposition 1.6. [Jan77, §7] Let $\lambda = (a, b, c) \in X_1(T)$. We have the following exact sequences in $\operatorname{Rep}_{\overline{\mathbb{F}}_n}(G)$

$$\begin{array}{l} 0 \to L(\lambda) \to V(\lambda) \to L(2p-b-3,2p-a-3,c) \to 0 \ \ if \ \lambda \in C_3 \\ 0 \to L(\lambda) \to V(\lambda) \to L(2p-a-4,b,c) \to 0 \quad \ if \ \lambda \in C_2 \\ 0 \to L(\lambda) \to V(\lambda) \to L(p-b-3,p-a-3,c) \to 0 \ \ if \ \lambda \in C_1 \\ V(\lambda) = L(\lambda) \ \ if \ \lambda \in C_0 \end{array}$$

If $\lambda \in X_1(T) \setminus \bigcup_0^3 C_i$ then $V(\lambda)$ is irreducible unless a - b = p - 1 and p/2 < b + 1 < p, or p = 2 and $\lambda = (1, 1, c)$. In this range all the $L(\lambda)$ are still irreducible as $G(\mathbb{F}_p)$ -representations.

The theory of Serre weights is applied to étale cohomology of the Shimura variety, and the BGG decomposition tells us that $H^0(\operatorname{Sh}_{\overline{\mathbb{Q}}_p}, \omega(a+3, b+3))$ appears in $H^3_{\operatorname{\acute{e}t}}(\operatorname{Sh}_{\overline{\mathbb{Q}}_p}, V(a, b))$, where $V(\lambda)$ is the local system associated to the corresponding representation of G. Therefore, when talking

about weights (k, l) of automorphic vector bundles we will reparametrize *p*-restricted weights as those (k, l) satisfying $(k-3, l-3) \in X_1(T)$ taking into account this shift. Similarly, the *p*-restricted alcoves correspond to

$$\begin{split} C_0: k-1 > l-2 > 0, k+l < p+3 \\ C_1: k+l > p+3, l-2 < k-1 < p \\ C_2: k-l+1 < p < k-1, k+l < 2p+3 \\ C_3: l < p+2, k+l > 2p+3, k-l+1$$

1.2. Hecke operators away from p. Fix a level $K = K^p K_p \subset G(\mathbb{A}_Q^{\infty})$, and let $\mathbb{A}^{\infty,p}$ be the finite adeles away from p. Consider the abstract Hecke algebra away from $p \mathcal{H}^{\mathrm{ur}} = C_c^{\infty}(G(\mathbb{A}^{\infty,p})//K,\mathbb{Z}_p)$, defined as a convolution algebra of locally constant, compactly supported, bi-K invariant functions on $G(\mathbb{A}^{\infty,p})$ with coefficients in \mathbb{Z}_p . We define the action of $\mathcal{H}^{\mathrm{ur}}$ on coherent cohomology via Hecke correspondences. Let $g \in G(\mathbb{A}^{p,\infty})$, and $K_g = K \cap gKg^{-1}$. There are two finite étale maps over \mathbb{Z}_p p_1, p_2 : $\mathrm{Sh}_{K_g} \to \mathrm{Sh}_K$ defined as follows. The first one is defined on points of Definition 1.1 by $(A, \lambda, \eta_{K_g}) \to (A, \lambda, \eta_K)$ where η_K is obtained by taking any K-orbit of any η in η_{K_g} . The second map is defined by composing $\mathrm{Sh}_{g^{-1}K_gg} \to \mathrm{Sh}_K$ defined as p_1 with an isomorphism $[g] : \mathrm{Sh}_{K_g} \to \mathrm{Sh}_{g^{-1}K_gg}$ sending A to the unique A' prime to p quasi-isogenous to A via $f : A \to A'$ satisfying the conditions in [Lan13, Prop 1.4.3.4]. This correspondence only depends on the coset K^pgK^p , so in particular one can check that it induces an action of $\mathcal{H}^{\mathrm{ur}}$ by correspondences. Moreover, for $Q \in \{G, P\}$, we have natural isomorphisms of torsors $T_g : p_1^*I_Q \cong p_2^*I_Q$. This induces an action in cohomology: for $V \in \mathrm{Rep}_{\mathbb{Z}_p}(Q)$

$$T_g: H^i(\mathrm{Sh}_K, F_Q(V)) \xrightarrow{p_2^*} H^i(\mathrm{Sh}_{K_g}, p_2^*F_Q(V)) \xrightarrow{T_g} H^i(\mathrm{Sh}_{K_g}, p_1^*F_Q(V)) \xrightarrow{\mathrm{Tr}p_1} H^i(\mathrm{Sh}_K, F_Q(V)).$$

We say that a collection of maps $F_Q(V)_K \to F_Q(W)_K$ of automorphic vector bundles, ranging across $K = K^p K_p$ neat with K_p hyperspecial is Hecke equivariant away from p if the associated maps on cohomology are equivariant for $\mathcal{H}^{\mathrm{ur}}$. In practice it is enough to check that the maps are compatible with base change and prime to p quasi-isogenies $f : A \to A'$, in the sense that they are compatible under replacing the torsors I_Q with the ones defined by $H^1_{\mathrm{dR}}(A'/\mathrm{Sh}_K)$.

We explain this in the case of the Kodaira-Spencer isomorphism, which is the only non-trivial step occurring in the case of theta operators. Let $\delta := \det H = \det \omega_A \otimes \det^{-1} \omega_{A^{\vee}}$. It is a line bundle which is non-canonically trivial via the quasi-polarization, and $\delta^{p-1} = \mathcal{O}_{\overline{Sh}}$ canonically by comparing the Hodge and conjugate filtration. Then we can write

$$\mathcal{L}(a,b,c) \coloneqq F_B(b,a,c) = \mathcal{L}(a,b) \otimes \delta^{\frac{c-a-b}{2}}, \quad \omega(a,b,c) \coloneqq F_P(W(a,b,c)) = \omega(a,b) \otimes \delta^{\frac{c-a-b}{2}},$$

so that the central character does not change the automorphic vector bundle, it will only keep track of the Hecke action. Using the quasi-polarization we can write the Kodaira-Spencer isomorphism as a map $k_{S_{A,\lambda}} : \omega(2,0,0) = \text{Sym}^2 \omega_A \otimes \delta_A^{-1} \cong \Omega^1_{\text{Sh}/\mathbb{Z}_p}$. Given a prime to p quasi-isogeny between polarized abelian varieties $f : (A, \lambda) \to (A', \lambda')$ over Sh which preserves λ and λ' up to scalars, the compatibility needed to show that $k_{S_{A,\lambda}}$ is Hecke equivariant is the commutativity of the square $(\text{see} [\text{EFG}^+21, \text{Lem } 4.1.2])$

$$\begin{array}{ccc} \operatorname{Sym}^{2}\omega_{A}\otimes\delta_{A}^{-1} \xrightarrow{\operatorname{ks}_{A,\lambda}} \Omega^{1}_{\operatorname{Sh}/\mathbb{Z}_{p}} \\ & & \\ & & \\ & & \\ & & \\ \operatorname{Sym}^{2}\omega_{A'}\otimes\delta_{A'}^{-1} \xrightarrow{\operatorname{ks}_{A',\lambda'}} \Omega^{1}_{\operatorname{Sh}/\mathbb{Z}_{p}}. \end{array}$$

Given a level K let S be a finite set of places containing p such that $K_{\ell} = G(\mathbb{Z}_{\ell})$ for $\ell \notin S$. Let \mathbb{T}/\mathbb{Z}_p be the spherical Hecke algebra away from S, defined as a restricted tensor product of local Hecke algebras $\bigotimes_{\ell \notin S}^{'} C_c^{\infty}(G(\mathbb{Q}_l)//K_l,\mathbb{Z}_p)$. It is a commutative subalgebra of $\mathcal{H}^{\mathrm{ur}}$, so it also acts on coherent cohomology. Although it depends on the level K we will supress it from the notation. Let $\mathfrak{m} \subset \mathbb{T}$ be a maximal ideal occurring in the coherent cohomology of the toroidal compactification $\overline{\mathrm{Sh}}^{\mathrm{tor}}$, e.g. occurs in $H^0(\overline{\mathrm{Sh}}, \omega(k, l))$ for $k \geq l \geq 0$. Then one can attach a semisimple Galois representation $\overline{r}_{\mathfrak{m}} : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{\mathbb{F}}_p)$ by the construction of Galois representations for automorphic representations of regular weight [Wei05], and the use of generalized Hasse invariants in [Box15] or [GK19]. We say that \mathfrak{m} is non-Eisenstein if $\overline{r}_{\mathfrak{m}}$ is irreducible as a GL₄-valued representation.

1.3. Toroidal compactifications. The Siegel threefold is not proper, so we introduce the standard machinery of toroidal compactifications. By the following proposition essentially all of our set up extends to them.

Proposition 1.7. [LP18, §2.3] Let C be the cone of positive semidefinite symmetric bilinear forms on \mathbb{R}^2 whose radicals are defined over \mathbb{Q} , and Σ a smooth projective K-admissible decomposition of C into polyhedral cones as in [Lan13, Def 6.3.3.4 and 7.3.1.3]. By [Lan13, Thm 6.4.1.1 and 7.3.3.4] there exists an associated toroidal compactification Sh^{tor, Σ} over \mathbb{Z}_p . It is a smooth proper scheme satisfying the following properties.

- (1) The boundary $D \coloneqq Sh^{tor,\sigma} Sh$ with its reduced structure is a Cartier divisor with simple normal crossings.
- (2) The universal abelian scheme extends to a semi-abelian scheme $\pi^{\text{tor}} : A^{\text{tor}} \to \text{Sh}^{\text{tor}}$. The prime to p polarization extends to a prime to p isogeny $\lambda : A^{\text{tor}} \to A^{\text{tor},\vee}$. Define $\omega^{\text{tor}} = e^* \Omega^1_{A^{\text{tor}}/\text{Sh}^{\text{tor}}}$, it extends ω/Sh . In characteristic p the Frobenius and Verchiebung maps extend to ω^{tor} .
- (3) There is a canonical extension H^{tor} of H as a symplectic vector bundle over Sh^{tor} . It fits in the exact sequence

$$0 \to \omega^{\mathrm{tor}} \to H^{\mathrm{tor}} \to \omega_{A^{\vee}}^{\mathrm{tor},\vee} \to 0$$

which makes ω^{tor} into a *P*-torsor.

(4) We extend $\mathcal{F}l \to \mathrm{Sh}$, to $\pi^{\mathrm{tor}} : \mathcal{F}l^{\mathrm{tor}} \to \mathrm{Sh}^{\mathrm{tor}}$ by defining it to be the flag space of ω^{tor} . Let $D_{\mathcal{F}l} = \pi^{\mathrm{tor},-1}(D)$ be its boundary divisor. We still denote $\mathcal{L} \subset \omega^{\mathrm{tor}}$ its canonical line. The functors defining automorphic vector bundles extend to $F_B^{\mathrm{can}} : \mathrm{Rep}_R(Q) \to \mathrm{Coh}(\mathcal{F}l_R^{\mathrm{tor}})$ using the torsors given by $\mathcal{L} \subset \omega^{\mathrm{tor}} \subset H^{\mathrm{tor}}$. We define subcanonical extensions by $F_B^{\mathrm{sub}}(V) =$

 $F_Q^{\operatorname{can}}(V)(-D_{\mathcal{F}l})$, and similarly for $\operatorname{Rep}(P)$. This defines extensions $\mathcal{L}^?(k,l)$, $\omega^?(k,l)$ for $? \in \{\operatorname{can}, \operatorname{sub}\}$, and we still have $\pi_*^{\operatorname{tor}} \mathcal{L}^?(k,l) = \omega^?(k,l)$. From now on we will largely drop the indices for canonical extensions.

(5) There exists a log connection $\nabla^{\text{tor}} : H^{\text{tor}} \to H^{\text{tor}} \otimes \Omega^1_{\mathrm{Sh}^{\text{tor}}/\mathbb{Z}_p}(\log D)$ extending the Gauss-Manin connection. This induces log connections on both $F_G^{\mathrm{can}}(V)$ and $F_G^{\mathrm{sub}}(V)$ for $V \in \mathrm{Rep}(G)$. By the same procedure the Kodaira-Spencer map extends to an isomorphism

$$\omega^{\operatorname{can}}(2,0,0) \cong \Omega^{1}_{\operatorname{Sh^{tor}}/\mathbb{Z}_{p}}(\log D).$$

- (6) The coherent cohomology groups $H^i(Sh^{tor,\Sigma}, \omega^?(k,l))$ for $? \in \{can, sub\}$ are independent of the cone decomposition Σ .
- (7) We have $H^0(\operatorname{Sh}_R^{\operatorname{tor}}, \omega(k, l)) = H^0(\operatorname{Sh}_R, \omega(k, l))$ for $R \in \{\mathbb{Z}_p, \mathbb{F}_p\}$, by [Lan16a]. Thus, we can consider the Hasse invariants as $H_i \in H^0(\mathcal{Fl}_{\mathbb{F}_p}^{\operatorname{tor}}, \mathcal{L}^{\operatorname{can}}(\lambda_i))$, which are defined by considering V on $\omega^{\operatorname{tor}}$. Similarly, one can define an extension of the Ekedahl-Oort strata to the toroidal compactification as in [Box15]. We will use that the extended divisor D_1^{tor} intersects the boundary D at a set of positive codimension.
- (8) The Hecke correspondences away from p extend to (sub)canonical extensions over $\mathrm{Sh}^{\mathrm{tor}}$, and hence they extend to operators $T_g : H^i(\mathrm{Sh}^{\mathrm{tor}}, \omega^?(k, l)) \to H^i(\mathrm{Sh}^{\mathrm{tor}}, \omega^?(k, l))$ for $? \in \{\mathrm{sub}, \mathrm{can}\}$.

We recall some important vanishing theorems for mod p coherent cohomology.

- **Theorem 1.8.** (1) Let $p \ge 5$. For $k \ge l \ge 5$ satisfying (p-1)(k-2) > (p+1)(l-4) (in particular this holds for $l \le p+2$) we have $H^i(\overline{Sh}^{tor}, \omega^{sub}(k,l)) = 0$ for $i \ge 1$. For $k \ge l \ge 4$ with k-l < p-3, we have $H^i(\overline{Sh}^{tor}, \omega^{sub}(k,l)) = 0$ for $i \ge 1$.
 - (2) If $H^0(\overline{Sh}, \omega(k, l)) \neq 0$, then $k \ge l$ and $k + lp \ge 0$ [GK18, Thm 5.1.1].
 - (3) Let $p \ge 5$. For $k \ge l \ge 4$ with $k l , then <math>H^2(\overline{Sh}^{tor}, \omega^{sub}(k, 4 l)) = 0$.

Proof. The first case of (1) follows by the results on the ample cone on $\mathcal{F}l$ in [Ale22, Ex 4.30, Thm 5.10], and the second case by [LS13, Thm 8.13]. Part (3) also follows by [LS13, Thm 8.13].

1.4. **Deformation theory.** It will sometimes be useful to make computations on a formal neighbourhood of a point. Let $q \in \operatorname{Sh}(\overline{\mathbb{F}}_p)$ be a geometric point. Let (A_q, λ, η) be one representative of q, and denote by $\operatorname{Def}(A_q, \lambda, \eta)$ be the moduli space of deformations of q to local Artin algebras with an augmentation $R \to \overline{\mathbb{F}}_p$. Then

Spf
$$\widehat{\mathcal{O}}_{\mathrm{Sh},q} \cong \mathrm{Def}(A_q,\lambda,\eta)$$

as functors on Artin algebras, using the moduli interpretation of Sh in Proposition 1.1. A classical result of Grothendieck then says that

$$\widehat{\mathcal{O}}_{\mathrm{Sh},q} \cong W(\overline{\mathbb{F}}_p)[[T_{11}, T_{12}, T_{22}]]$$

and similarly, if $r \in \mathcal{F}l$ is a geometric point

$$\widehat{\mathcal{O}}_{\mathcal{F}l,r} \cong W(\overline{\mathbb{F}}_p)[[T_{11}, T_{12}, T_{22}, T]]$$

since $\mathcal{F}l$ is locally a \mathbb{P}^1 over Sh.

Let (R, I) be the divided power envelope of $(\mathcal{O}_{\mathrm{Sh}_{\mathbb{Z}_p}}, \mathfrak{m}_q)$, and let $I^{[n]}$ be the ideal of R generated by elements $\prod \gamma_{n_i}(x_i)$ for $x_i \in \mathfrak{m}_q$ and $\sum n_i \geq n$, so that $S_n \coloneqq R/I^{[n]}$ is a divided power thickening of $\kappa(q)$. Grothendieck-Messing theory identifies $H^1_{dR}(A/S_n)$ with the Dieudonne crystal $D(A[p^{\infty}]_{S_n})$ of the p-divisible group $A[p^{\infty}]_{S_n}$. In particular H_{S_n} is equipped with a natural map $\nabla : H_{S_n} \to$ $H_{S_n} \otimes \Omega^1_{S_n/\mathbb{Z}_p,\delta}$ in the sense of [Sta18, Tag 07J6] by virtue of being a crystal, and ∇ is identified with the pullback of the Gauss-Manin connection to S_n . There is a map of PD thickenings $(W(\overline{\mathbb{F}}_p), \overline{\mathbb{F}}_p) \to$ $(S_n \to \kappa(q))$ corresponding to the natural section of $S_n/p \to S_n/I \cong \kappa(q)$. Since the Dieudonne module is a crystal

$$H^{1}_{dR}(A/S_{n}) \cong D(A[p^{\infty}]_{q})(S_{n} \to \kappa(q)) \cong D(A[p^{\infty}]_{q})(W(\overline{\mathbb{F}}_{p}), \overline{\mathbb{F}}_{p}) \otimes_{W} S_{n}$$
(1.1)

Lemma 1.9. [Sta18, Tag 07J6] Let $\{e_i\}$ be a basis of $D(A[p^{\infty}]_q)(W(\overline{\mathbb{F}}_p), \overline{\mathbb{F}}_p)$. Then under the isomorphism (1.1) the natural map $\nabla : H_{S_n} \to H_{S_n} \otimes \Omega^1_{S_n/\mathbb{Z}_n,\delta}$ satisfies $\nabla(e_i) = 0$.

On the formal completion of an ordinary point one has the richer structure of Serre-Tate coordinates. The following theorem is a compatibility result between the Serre-Tate and Grothendieck-Messing coordinates.

Theorem 1.10. [Kat81] Let $s \in \overline{\mathrm{Sh}}^{\mathrm{ord}}(\overline{\mathbb{F}}_p)$. Then there exists a basis $\{e_1, e_2\}$ of $\omega_{\widehat{\mathcal{O}}_{\overline{\mathrm{Sh}},s}}$ such that $Ve_i = e_i^{(p)}$ and $D_{ij} \coloneqq \mathrm{ks}(e_i e_j)^{\vee} = (T_{ij} + 1)\frac{\partial}{\partial T_{ij}}$.

A computation on the formal neighbourhood of various points yields the following result.

Proposition 1.11. Let i = 1, 2. Then H_i vanishes with a simple zero on D_i . Moreover, D_1 does not intersect D_2 on a top dimensional irreducible component.

Proof. The Zariski-Nagata theorem for regular algebras over a perfect field allows us check the first statement on the formal completion of any closed point $q \in D_i$. For H_1 we choose $\tilde{q} \in D_1(\overline{\mathbb{F}}_p)$ mapping to $q \in \overline{\mathrm{Sh}}^{=1}(\overline{\mathbb{F}}_p)$. Write $\widehat{O}_{\overline{\mathrm{Sh}},q} = \overline{\mathbb{F}}_p[[T_{11}, T_{12}, T_{22}]]$. Let $R = \mathcal{O}_{\overline{\mathrm{Sh}},q}/\mathfrak{m}^p$, which is a PD thickening of $\kappa(q)$. Grothendieck-Messing theory tells us that $H^1_{\mathrm{dR}}(A/R) = H^1_{\mathrm{dR}}(A/\kappa(q)) \otimes R$. By the classification of Ekedahl-Oort strata we have that $A[p]_{\kappa(q)}$ is isomorphic to the product of the *p*-torsion of an ordinary elliptic curve and the *p*-torsion of a supersingular elliptic curve. We can

therefore choose a symplectic basis $\{\overline{e}_i\}$ of $H^1_{\kappa(q)}$ so that $V_{\kappa(q)}$ has matrix

$$V_{\kappa(q)} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}$$
(1.2)

with respect to the basis $\{\overline{e}_i\}$ and $\{\overline{e}_i^{(p)}\}$. By Grothendieck-Messing theory lifts of $\omega_{\kappa(q)}$ to H_R^1 parametrize deformations of $A/\kappa(q)$. Since A/R is the universal deformation we can assume without loss of generality that

$$\omega_R = (\overline{e}_1 + T_{12}\overline{e}_3 + T_{11}\overline{e}_4, \overline{e}_2 + T_{22}\overline{e}_3 + T_{12}\overline{e}_4) \subset H_R^1, \tag{1.3}$$

On the other hand $A^{(p)}/R := A_R \times_{R,F_R} R$ is the trivial deformation of $A/\kappa(q)$, since Frobenius factors through $\kappa(q)$. Therefore

$$\omega_R^{(p)} = (\overline{e}_1^{(p)}, \overline{e}_2^{(p)}).$$

We can compute the matrix of $V_R: \omega_R \to \omega_R^{(p)}$ with respect to the basis described above:

$$V_R = \begin{pmatrix} 1 & 0\\ T_{12} & T_{22} \end{pmatrix}. \tag{1.4}$$

Therefore in this basis $\tilde{H}_1 = T_{22}$, which vanishes with a simple zero. For H_2 we choose $\tilde{q} \in D_2(\overline{\mathbb{F}}_q)$ mapping to $q \in \overline{\mathrm{Sh}}^{\mathrm{ord}}(\overline{\mathbb{F}}_p)$. By Theorem 1.10 on the formal completion of q we can write $\tilde{H}_2 = T - T^p$, so that the maximal ideal of $\widehat{\mathcal{O}}_{\mathcal{F}l,\tilde{q}}$ is $(T_{11}, T_{12}, T_{22}, T - \alpha)$ for some $\alpha \in \mathbb{F}_p$, and it is clear that H_2 has a simple zero. For the second statement, any top dimensional irreducible component of D_2 contains an ordinary point, since the fibers of $D_2 \to \overline{\mathrm{Sh}}$ are always zero-dimensional except over the superspecial locus.

2. Geometric construction of the theta operators on the open strata

Let $U = U_1 \cap U_2$ be the open in $\mathcal{F}l$ where $H_1 \cdot H_2$ does not vanish. We first construct the basic theta operators over U, and then we extend them to U_1 .

2.1. The Igusa variety. We describe an Igusa variety Ig/\overline{Sh}^{ord} , which is a finite étale cover. Over \overline{Sh}^{ord} there is a connected-étale sequence for A[p] consisting of finite flat subgroups

$$0 \to A[p]^0 \to A[p] \to A[p]^{\text{\'et}} \to 0.$$

Definition 2.1. Let $\tau : \operatorname{Ig} \to \overline{\operatorname{Sh}}^{\operatorname{ord}}$ be the scheme representing the moduli problem sending $S \in \operatorname{Sch}/\mathbb{F}_p$ to

$$\operatorname{Ig}(S) = \{(A, \lambda, \eta) \in \overline{\operatorname{Sh}}^{\operatorname{ord}}(S) \text{ and } \phi : (\mathbb{Z}/p)^2 \cong A[p]^{\operatorname{\acute{e}t}}\}.$$

It is an étale $\operatorname{GL}_2(\mathbb{F}_p)$ -torsor over $\overline{\operatorname{Sh}}^{\operatorname{ord}}$, with $\operatorname{GL}_2(\mathbb{F}_p)$ acting on ϕ via $(\mathbb{Z}/p)^2$.

18

Taking Cartier duals one also gets the isomorphism $\mu_p^2 \cong A[p]^0$ over Ig, and in fact Ig also parametrizes embeddings $\phi: \mu_p^2 \hookrightarrow A[p]$. This implies that we can trivialize ω over Ig:

$$\tau^*\omega = \omega_{A[p]_{\mathrm{Ig}}} = \omega_{A[p]_{\mathrm{Ig}}^0} = \omega_{\mu_p^2} \cong \mathcal{O}_{\mathrm{Ig}}^2$$

Since we can explicitly describe V on μ_p it follows that there is a basis $\{e_1, e_2\}$ of ω_{Ig} satisfying $Ve_i = e_i^{(p)}$. We will refer to it as a canonical basis of ω_{Ig} , keeping in mind that it is only canonical up to the action of $\text{GL}_2(\mathbb{F}_p)$.

In order to make concrete computations we give some local coordinates for Ig.

Lemma 2.1. There is a natural embedding Ig $\hookrightarrow \underline{\text{Isom}}(\mathcal{O}^2, \omega)$ fitting in



Moreover, let W = Spec R be an open of $\overline{\text{Sh}}$ trivializing ω , and choose a trivialization $\omega = \langle e_1, e_2 \rangle \cong \mathcal{O}_W^2$. Denote the matrix of $V : \omega \to \omega^{(p)}$ with respect to $\{e_1, e_2\}$ and $\{e_1^{(p)}, e_2^{(p)}\}$ by

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then

$$Ig_W \cong Spec R[x_1, x_2, x_3, x_4] [\frac{1}{(x_1 x_4 - x_2 x_3)(ad - bc)}] / I$$
(2.1)

where I is defined by the equations

$$\begin{pmatrix} x_1^p & x_2^p \\ x_3^p & x_4^p \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} V^T.$$

We will denote the matrix of x_i by M, and write these relations as $M^{(p)} = MV^T$.

Proof. The map Ig $\hookrightarrow \underline{\text{Isom}}(\mathcal{O}^2, \omega)$ is given on points over S/\mathbb{F}_p as follows. Given $(A, \lambda, \phi : \mu_p^2 \hookrightarrow A[p])$, consider the map

$$\varphi:\omega\to\omega_{\mu_p}^2$$

given by the differential of ϕ . It is an isomorphism since ϕ induces an isomorphism $\mu_p^2 \cong A[p]^0$. Since $\omega_{\mu_p} \cong \mathcal{O}_S$ canonically, we obtain a well-defined map. We prove that it is a closed embedding. Fix an algebra $S/\overline{\mathbb{F}}_p$, we first construct a map from Ig to the right-hand side of (2.1). Let $(A, \phi) \in \mathrm{Ig}(S)$, and let $\tilde{e}_i \in \omega$ be the images under φ^{-1} of the canonical basis of $\omega_{\mu_p}^2$. We can write the \tilde{e}_i in terms of the basis $\{e_1, e_2\}$ as

$$\begin{pmatrix} \tilde{e}_1\\ \tilde{e}_2 \end{pmatrix} = \begin{pmatrix} y_1 & y_2\\ y_3 & y_4 \end{pmatrix} \begin{pmatrix} e_1\\ e_2 \end{pmatrix}$$
(2.2)

with $y_i \in S$. Since φ is an isomorphism we get that $y_1y_2 - y_3y_4 \in S^{\times}$, and since A is ordinary ad - bc is also a unit. Now, φ commutes with the respective Verschiebungs, and since V acts as the identity with respect to the canonical basis of ω_{μ_p} , we see that $V(\tilde{e}_i) = \tilde{e}_i^{(p)}$, which translates to

$$\begin{pmatrix} y_1^p & y_2^p \\ y_3^p & y_4^p \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} V^{\mathrm{T}}$$

Therefore, the association $(A, \phi) \mapsto (y_i)$ defines such a map. On geometric points Dieudonne theory gives an equivalence between maps $\mu_p^2 \hookrightarrow A[p]$ and isomorphisms $\omega_{\mu_p}^2 \cong \omega_A$ compatible with V. The latter is equivalent to giving tuples $(y_i) = M$ satisfying $M^{(p)} = MV^T$, so the map is an isomorphism on geometric points. We can check that the equations I define an étale cover, which implies that the map constructed is an isomorphism, since an isomorphism of finite locally free modules can be checked on geometric points. \Box

Recall that $\overline{\mathrm{Sh}}^{\mathrm{ord,tor}} \subset \overline{\mathrm{Sh}}^{\mathrm{tor}}$ is the locus where H_1 is non-vanishing.

Corollary 2.2. The $\operatorname{GL}_2(\mathbb{F}_p)$ torsor $\operatorname{Ig} \to \overline{\operatorname{Sh}}^{\operatorname{ord}}$ extends to a $\operatorname{GL}_2(\mathbb{F}_p)$ -torsor $\operatorname{Ig}^{\operatorname{tor}} \to \overline{\operatorname{Sh}}^{\operatorname{ord},\operatorname{tor}}$ with the property that there exists a basis $\{e_1, e_2\}$ of $\omega_{\operatorname{Ig}^{\operatorname{tor}}}$ satisfying $Ve_i = e_i^{(p)}$.

Proof. We define Ig^{tor} locally by the equations of Lemma 2.1, which define an étale torsor since V is invertible over $\overline{Sh}^{ord,tor}$.

Recall that $U_1 = \pi^{-1}(\overline{\mathrm{Sh}}^{\mathrm{ord}})$, so that $\mathcal{F}l_{\mathrm{Ig}} \coloneqq U_1 \times_{\overline{\mathrm{Sh}}^{\mathrm{ord}}} \mathrm{Ig} = \mathbb{P}(\omega_{\mathrm{Ig}}) \cong \mathrm{Ig} \times \mathbb{P}^1$ is a $\mathrm{GL}_2(\mathbb{F}_p)$ -étale torsor of U_1 , with $\mathrm{GL}_2(\mathbb{F}_p)$ acting diagonally on $\mathrm{Ig} \times \mathbb{P}^1$.

Proposition 2.3. The exact sequence

$$0 \to \pi^* \Omega^1_{\overline{\mathrm{Sh}}} \to \Omega^1_{\mathcal{F}l} \to \Omega^1_{\mathcal{F}l/\overline{\mathrm{Sh}}} \to 0$$

canonically splits over U_1 . We denote by $s: \Omega^1_{\mathcal{F}l/\overline{\mathbb{F}}_n} \to \pi^*\Omega^1_{\overline{Sh}}$ the associated section.

Proof. By Galois descent to construct s it is enough to construct a $\operatorname{GL}_2(\mathbb{F}_p)$ -equivariant section $\Omega^1_{\mathcal{F}l_{\mathrm{Ig}}} \to \pi^*\Omega^1_{\mathrm{Ig}}$. Both projections from $\mathcal{F}l_{\mathrm{Ig}} \cong \mathrm{Ig} \times \mathbb{P}^1$ are $\operatorname{GL}_2(\mathbb{F}_p)$ -equivariant, so the splitting of the exact sequence for $\Omega^1_{\mathrm{Ig} \times \mathbb{P}^1}$ is also $\operatorname{GL}_2(\mathbb{F}_p)$ -equivariant.

2.2. Construction on the open strata U. On $U \subset \mathcal{F}l_{\mathbb{F}_p}$ we have both the splittings $H = \mathcal{L} \oplus V^{-1}(\mathcal{L}^{(p)})$ and $H = \omega \oplus \omega^{\vee}$, so that setting $\mathcal{L}' \coloneqq \omega \cap V^{-1}(\mathcal{L}^{(p)})$ we get a full splitting of the symplectic flag on U into line bundles:

$$H = \mathcal{L} \oplus \mathcal{L}' \oplus (\mathcal{L}'^{\perp} \cap \omega^{\vee}) \oplus (\mathcal{L}^{\perp} \cap \omega^{\vee}).$$

Denote by $\pi_{\mathcal{L}} : H \to \mathcal{L}$ and $\pi_{\mathcal{L}'} : H \to \mathcal{L}'$ the corresponding projections, we will use the same notation for the projections restricted to ω . Moreover, $\mathcal{L}(k,l)_U = \mathcal{L}'^k \otimes \mathcal{L}^l$.

Definition 2.2. (Theta operators on U) Let $\lambda = (k, l) \in X^*(T)$. We define the following maps over U.

• The splitting $\omega = \mathcal{L} \oplus \mathcal{L}'$ induces a splitting of $\operatorname{Sym}^2 \omega$. The corresponding projections are

$$\pi_1 : \pi^* \Omega^1_{\overline{\mathrm{Sh}}} \cong \mathrm{Sym}^2 \omega \to \mathcal{L}'^2 = \mathcal{L}(2,0)$$

$$\pi_2 : \pi^* \Omega^1_{\overline{\mathrm{Sh}}} \cong \mathrm{Sym}^2 \omega \to \mathcal{L}' \otimes \mathcal{L} = \mathcal{L}(1,1)$$

$$\pi_3 : \pi^* \Omega^1_{\overline{\mathrm{Sh}}} \cong \mathrm{Sym}^2 \omega \to \mathcal{L}^2 = \mathcal{L}(0,2)$$

• For i = 1, 2, 3 we define operators $\tilde{\theta}_i : \mathcal{L}(\lambda) \to \mathcal{L}(\lambda + \lambda_i)$ with $\lambda_1 = (2, 0), \lambda_2 = (1, 1), \lambda_3 = (0, 2)$. Let L be \mathcal{L} or \mathcal{L}' , then for $\mathcal{L}(\lambda) = L$ it is defined as

$$\tilde{\theta}_i: L \hookrightarrow H \xrightarrow{\nabla} H \otimes \Omega^1_{\mathcal{F}l/\overline{\mathbb{F}}_p} \xrightarrow{\pi_L \otimes s} L \otimes \pi^* \Omega^1_{\overline{\mathrm{Sh}}} \xrightarrow{\mathrm{id} \otimes \pi_i} L \otimes \mathcal{L}(\lambda_i).$$

For a general weight λ , it is defined by prescribing that $\tilde{\theta}_i(fg) = f\tilde{\theta}_i(g) + \tilde{\theta}_i(f)g$ for local sections f, g, and that on $\mathcal{O}_{\mathcal{F}l}$ it is given by composing the differential with the $\pi_i \circ s$. Explicitly: on a local section $fe^{-1} \in L^{-1}$ with $f \in \mathcal{O}$ and $e \in L$ a local basis

$$\tilde{\theta}_i(fe^{-1}) \coloneqq \pi_i \circ s(df) \otimes e^{-1} - f\tilde{\theta}_i(e) \otimes e^{-2}.$$

Then every $\mathcal{L}(k, l)$ is a tensor product of $\mathcal{L}, \mathcal{L}'$ or their duals, so that $\tilde{\theta}_i$ can be defined by the Leibniz rule.

• Let $r: \Omega^1_{\mathcal{F}l} \to \Omega^1_{\mathcal{F}l/\overline{\mathrm{Sh}}}$, and L as before. Define $\tilde{\theta}_4$ on L as

$$\tilde{\theta}_4: L \hookrightarrow H \xrightarrow{\nabla} H \otimes \Omega^1_{\mathcal{F}l} \xrightarrow{\pi_L \otimes r} L \otimes \Omega^1_{\mathcal{F}l/\overline{\mathrm{Sh}}} = L \otimes \mathcal{L}(-1, 1),$$

using Lemma 1.2. We extend it to arbitrary weights as above.

Remark 2.1. Note that in the definition of $\tilde{\theta}_i$ it is natural to use the projection π_L as opposed to the projection to any other line bundle, in which case one would get a linear map (in some cases identically zero). See also Proposition 3.12 ahead for the relation of the extension of $\tilde{\theta}_i$ to $\mathcal{F}l$ to the way [EFG⁺21] define theta operators on the Shimura variety.

Since $\theta_{1,2,3,4}$ are maps of line bundles on U it is a formal consequence that they can be extended to $\mathcal{F}l$ by multiplying by some sufficiently high power of the Hasse invariants ¹. Moreover, it suffices to extend the theta operators on weights (1,0) and (0,1), since all other weights are a combination of tensor products and duals of these.

2.3. Extension to U_1 . Let $L = \mathcal{L}(1,0)$ or $\mathcal{L}(0,1)$. There is a finite étale cover $\tau : U_{\text{Ig}} \to U$ by pulling back under $\text{Ig} \to \overline{\text{Sh}}^{\text{ord}}$, and one can make sense of $\tau^* \tilde{\theta}_i$, defined by the composition of the same maps pulled-back to U_{Ig} , since the pullback of the Gauss-Manin connection under an étale map is again the Gauss-Manin connection. Thus we can prove properties about $\tilde{\theta}_i$ by going to U_{Ig} , where there is a basis $\{e_1, e_2\}$ of ω_{Ig} satisfying $Ve_i = e_i^{(p)}$.

¹One can see this by considering degree 1 differential operators as linear maps $P_{\overline{Sh}}^1 \otimes V \to W$.

Lemma 2.4. Consider the map over U_1

$$\tilde{\theta}: \omega \hookrightarrow H \xrightarrow{\nabla} H \otimes \Omega^1_{\mathcal{F}l} \xrightarrow{\pi_\omega \otimes s} \omega \otimes \pi^* \Omega^1_{\overline{\mathrm{Sh}}},$$

where $\pi_{\omega}: H \to \omega$ is the unit root splitting over U_1 . Then $\tilde{\theta}$ satisfies $\tilde{\theta}(\mathcal{L}) \subset \mathcal{L} \otimes \pi^* \Omega^1_{\overline{Sh}}$, so that it induces maps

$$\theta_L: L \to L \otimes \pi^* \Omega^{\frac{1}{\mathrm{Sb}}}.$$

Proof. It is enough to prove it after pulling back to Ig. The analogous map $\omega \to \omega \otimes \pi^* \Omega^{\underline{1}}_{\underline{Sh}}$ has $\{e_1, e_2\}$ as a horizontal basis, since the kernel of π_{ω} is $\operatorname{Ker}(V : H \to \omega^{(p)})$ and

$$V\nabla(e_i) = \nabla^{(p)}(Ve_i) = \nabla^{(p)}(e_i^{(p)}) = 0$$

by functoriality of ∇ , and the fact that the differential of Frobenius vanishes. Locally in this basis $\tilde{\theta}$ is given by the composition of the trivial connection on \mathcal{O}^2 on $\overline{\mathrm{Sh}}^{\mathrm{ord}} \times \mathbb{P}^1$ and the section s, so it ignores any differentiation along \mathbb{P}^1 . It follows that we can choose a local basis for L that is horizontal for $\tilde{\theta}$.

To extend $\tilde{\theta}_{1,2,3}$ to U_1 , we use the maps $\theta_L : L \to L \otimes \pi^* \Omega^1_{\underline{Sh}}$ from Lemma 2.4. They extend the analogous maps in the definition of $\tilde{\theta}_{1,2,3}$. Then it suffices to extend $\pi_{1,2,3}$ to U_1 . The map $\pi_{(1,0)} : \omega \to \omega/\mathcal{L}$ extends $\pi_{\mathcal{L}'} : \omega \to \mathcal{L}'$ to U_1 , and the map

$$\pi_{(0,1)}:\omega\xrightarrow{V}\omega^{(p)}\to(\omega/\mathcal{L})^{(p)}=\mathcal{L}(p,0)$$

extends $\pi_{\mathcal{L}}: \omega \to \mathcal{L}$, in the sense that $\pi_{(0,1)|U} = H_2 \cdot \pi_{\mathcal{L}}$. Consider the map $\operatorname{Sym}^2 \omega \to \omega^{\otimes 2}$ given by $xy \mapsto x \otimes y + y \otimes x$. Then

$$\Pi_1 : \operatorname{Sym}^2 \omega \xrightarrow{\operatorname{Sym}^2 \pi_{(1,0)}} \mathcal{L}(2,0)$$
(2.3)

$$\Pi_2 : \operatorname{Sym}^2 \omega \to \omega^{\otimes 2} \xrightarrow{\pi_{(1,0)} \otimes \pi_{(0,1)}} \mathcal{L}(p+1,0)$$
(2.4)

$$\Pi_3 : \operatorname{Sym}^2 \omega \xrightarrow{\operatorname{Sym}^2 \pi_{(0,1)}} \mathcal{L}(2p,0)$$
(2.5)

extend the π_i to U_1 , and we have $\Pi_i = H_2^{i-1} \pi_i$ on U.

Definition 2.3. (Extending $\tilde{\theta}_{1,2,3}$ to U_1) Let $\lambda \in X^*(T)$, define the maps of sheaves on U_1

$$\Theta_1 : \mathcal{L}(\lambda) \to \mathcal{L}(\lambda + (2, 0))$$

$$\Theta_2 : \mathcal{L}(\lambda) \to \mathcal{L}(\lambda + (p + 1, 0))$$

$$\Theta_3 : \mathcal{L}(\lambda) \to \mathcal{L}(\lambda + (2p, 0))$$

as follows. For $\lambda = (1,0)$ or (0,1), $L = \mathcal{L}(\lambda)$ they are defined as

$$\begin{split} \Theta_1 : L &\xrightarrow{\theta_L} L \otimes \pi^* \Omega_{\mathrm{Sh}}^1 \xrightarrow{\mathrm{id} \otimes \Pi_1} L \otimes \mathcal{L}(2,0) \\ \Theta_2 : L &\xrightarrow{\theta_L} L \otimes \pi^* \Omega_{\mathrm{Sh}}^1 \xrightarrow{\mathrm{id} \otimes \Pi_2} L \otimes \mathcal{L}(p+1,0) \\ \Theta_3 : L &\xrightarrow{\theta_L} L \otimes \pi^* \Omega_{\mathrm{Sh}}^1 \xrightarrow{\mathrm{id} \otimes \Pi_3} L \otimes \mathcal{L}(2p,0), \end{split}$$

and they are extended to arbitrary λ as in Definition 2.2. When restricted to U they satisfy $\Theta_{i,U} = H_2^{i-1} \tilde{\theta}_i$.

Now we extend $\tilde{\theta}_4$ to $\mathcal{F}l$ (the map has no poles along D_1). Importantly, on $\mathcal{F}l$

$$\omega \hookrightarrow H \xrightarrow{\nabla} H \otimes \Omega^1_{\mathcal{F}l/\overline{\mathrm{Sh}}}$$

factors through $\omega \otimes \Omega^1_{\mathcal{F}l/\overline{Sh}}$, since by functoriality of ∇ , every section that is a pullback from \overline{Sh} is horizontal for $\nabla_{\mathcal{F}l/\overline{Sh}}$. This allows us to extend $\tilde{\theta}_4$ on $\mathcal{L}(0,1)$ as

$$\theta_4: \mathcal{L}(0,1) \to H \xrightarrow{\nabla} H \otimes \Omega^1_{\mathcal{F}l/\overline{\mathrm{Sh}}} \xrightarrow{\pi_{(0,1)} \otimes r} \mathcal{L}(p-1,1).$$

For $\mathcal{L}(1,0)$ we prove that the map over U

$$\theta_4: \mathcal{L}(1,0) = \omega/\mathcal{L} \cong \mathcal{L}' \xrightarrow{\theta_4} \mathcal{L}' \otimes \mathcal{L}(-1,1) \xrightarrow{H_2} \mathcal{L}(1,0) \otimes \mathcal{L}(p-1,0)$$
(2.6)

extends to $\mathcal{F}l$, by computing a local expression. We use the notation of Notation 1. When restricting the canonical isomorphism $\Omega^1_{\mathcal{F}l/Sh} = \mathcal{L}(-1,1)$ to W and using the trivialization given by e_1, e_2 we have

$$\Omega^{1}_{\mathcal{F}l/\overline{\mathrm{Sh}}}|W \cong \Omega^{1}_{\mathbb{P}^{1}_{W}/W} = \mathcal{O}(-2)_{W} \cong \mathcal{L}(-1,1)_{W},$$

sending dT to $e_2^{-1}(e_1 + Te_2)$. On U a basis for \mathcal{L}' is $e_2 - \lambda(e_1 + Te_2)$, where $\lambda \coloneqq \frac{1}{\tilde{H}_2} \frac{d\tilde{H}_2}{dT}$. Then $fe_2 \in \omega/\mathcal{L}$ for $f \in \mathcal{O}_{\mathbb{A}^1_W}$, is sent by θ_4 to

$$fe_{2} \mapsto f(e_{2} - \lambda(e_{1} + Te_{2})) \xrightarrow{H_{2}\pi_{\omega} \circ \nabla_{\mathcal{F}l/\overline{Sh}}} H_{2}(e_{2} - \lambda(e_{1} + Te_{2})) \otimes df$$
$$-H_{2}f(e_{1} \otimes d(\lambda) + e_{2} \otimes d(\lambda T)) \xrightarrow{\pi_{(1,0)}} (\tilde{H}_{2}\frac{df}{dT} - f\frac{d\tilde{H}_{2}}{dT})e_{2}^{p-1}(e_{1} + Te_{2})$$

We have used that $V \circ \nabla_{\mathcal{F}l/\overline{\mathrm{Sh}}}(e_i) = 0$. Similarly, $f(e_1 + Te_2) \in \mathcal{L}(0, 1)$ is sent to

$$f(e_1 + Te_2) \mapsto (\tilde{H}_2 e_2^p \otimes df + f \frac{dH_2}{dT} e_2^p \otimes dT) = (\tilde{H}_2 \frac{df}{dT} + f \frac{dH_2}{dT}) e_2^{p-1} (e_1 + Te_2).$$

Definition 2.4. (Extending $\tilde{\theta}_4$ to $\mathcal{F}l$) Define maps on $\mathcal{F}l$

$$\theta_4 : \mathcal{L}(0,1) \to H^1 \xrightarrow{\nabla} H^1 \otimes \Omega^1_{\mathcal{F}l/\overline{\mathrm{Sh}}} \xrightarrow{\pi_{(0,1)} \otimes r} \mathcal{L}(p-1,1)$$

$$\theta_4 : \mathcal{L}(1,0) = \omega/\mathcal{L} \cong \mathcal{L}' \xrightarrow{\tilde{\theta}_4} \mathcal{L}' \otimes \mathcal{L}(-1,1) \xrightarrow{H_2} \mathcal{L}(1,0) \otimes \mathcal{L}(p-1,0).$$

The latter is defined on U, but it extends to $\mathcal{F}l$ as explained before. For arbitrary $\lambda \in X^*(T)$ they are defined by the Leibniz rule. This gives maps $\theta_4 : \mathcal{L}(\lambda) \to \mathcal{L}(\lambda + (p-1,0))$ on $\mathcal{F}l$ satisfying $\theta_{4|U} = H_2 \tilde{\theta}_4$, and they are Hecke equivariant away from p. By using the extension of ∇ and V on canonical extensions they extend to maps on (sub)canonical extensions on $\mathcal{F}l^{\text{tor}}$.

Putting together the local expressions from before we obtain the following. Lemma 2.5. Let $fe_2^k(e_1 + Te_2)^l$ be a local section of $\mathcal{L}(k, l)$ with $f \in \mathcal{O}_{\mathbb{A}^1_W}$. Then

$$\theta_4(fe_2^k(e_1+Te_2)^l) = (\tilde{H}_2\frac{df}{dT} + (l-k)f\frac{dH_2}{dT})e_2^{k+p-1}(e_1+Te_2)^l.$$

The same expression holds over $\mathcal{L}(k,l)^{\operatorname{can}}$ by extending H_2 .

Remark 2.2. If f comes from a section of \mathbb{P}^1_W , i.e. has degree at most k - l, the polynomial $\tilde{H}_2 \frac{df}{dT} + (l-k)f\frac{d\tilde{H}_2}{dT}$ is of degree at most k - l + p - 1, even if the two summands might not.

We now prove some properties about the theta operators Θ_i that can be checked on an open dense subset, and hence they will automatically extend to the operators on $\mathcal{F}l$. We use the convention $\Theta_4 = \theta_4$.

Proposition 2.6. Let $i = 1, 2, 3, 4, \lambda, \mu \in X^*(T)$.

(1)
$$\Theta_i(fg) = f\Theta_i(g) + \Theta_i(f)g$$
 for f, g local sections of $\mathcal{L}(\lambda)$ and $\mathcal{L}(\mu)$.
(2) $\Theta_i(H_j) = 0$ for $j = 1, 2$.

Proof. The first one is automatic from the construction of Θ_i . We check the second over $U_{1,\text{Ig}}$, let $\{e_1, e_2\}$ be a canonical basis of ω_{Ig} . In this basis

$$H_1 = e_2^{p-1} (e_1 + Te_2)^{p-1}$$
$$H_2 = (T - T^p) e_2^p (e_1 + Te_2)^{-1}.$$

Using that $V\nabla(e_i) = \nabla^{(p)}(e_i^{(p)}) = 0$ we see that both $e_2 \in \mathcal{L}(1,0)$ and $e_1 + Te_2 \in \mathcal{L}(0,1)$ are horizontal for the map θ_L in the definition of $\Theta_{1,2,3}$. It is then clear that $\Theta_{1,2,3}(H_i) = 0$ for i = 1, 2, and $\Theta_4(H_1) = 0$. Finally, $\Theta_4(H_2) = 0$ from a direct computation with Lemma 2.5.

Let $s \in \overline{\mathrm{Sh}}^{\mathrm{ord}}(\overline{\mathbb{F}}_p)$, and let $\{e_1, e_2\}$ be a Serre-Tate basis of ω over $\widehat{\mathcal{O}}_{\overline{\mathrm{Sh}},s}$, so that $Ve_i = e_i^{(p)}$. Following the recipe for their construction gives the following expressions for Θ_i on the fiber of $\mathcal{F}l$ over $\widehat{\mathcal{O}}_{\overline{\mathrm{Sh}},s}$

$$\begin{split} \Theta_1(fe_2^k(e_1+Te_2)^l) &= \\ (T^2D_{11}(f) - TD_{12}(f) + D_{22}(f))e_2^{k+2}(e_1+Te_2)^l \\ \Theta_2(fe_2^k(e_1+Te_2)^l) &= \\ (2T^{p+1}D_{11}(f) - (T+T^p)D_{12}(f) + 2D_{22}(f))e_2^{k+p+1}(e_1+Te_2)^l \\ \Theta_3(fe_2^k(e_1+Te_2)^l) &= \\ (T^{2p}D_{11}(f) - T^pD_{12}(f) + D_{22}(f))e_2^{k+2p}(e_1+Te_2)^l, \end{split}$$

where $D_{ij} \coloneqq \mathrm{ks}(e_i e_j)^{\vee} = (T_{ij} + 1) \frac{\partial}{\partial T_{ij}}$. Using these, we can prove that the following relations hold among the Θ_i .

Proposition 2.7. We have the following relations as operators on U_1 .

 $\begin{array}{l} (1) \ [\Theta_1, \theta_4] \coloneqq \Theta_1 \circ \theta_4 - \theta_4 \circ \Theta_1 = \Theta_2. \\ (2) \ [\Theta_2, \theta_4] = 2\Theta_3. \\ (3) \ [\Theta_3, \theta_4] = 0. \\ (4) \ [\Theta_i, \Theta_j] = 0 \ for \ i, j = 1, 2, 3. \\ (5) \ \Theta_1^p = \Theta_3. \\ (6) \ The \ operator \ \Theta \coloneqq \frac{1}{H_2^2} (4\Theta_1\Theta_3 - \Theta_2^2) \ is \ well-defined \ and \ satisfies \ \Theta^p = H_1^2\Theta. \\ (7) \ [\Theta_2^p, \theta_4] = 0, \ [\Theta, \theta_4] = 0 \end{array}$

Proof. All of them can be checked locally on fibers $\mathbb{P}^1_{\mathcal{O}_{\overline{Sh},s}}$ for $s \in \overline{Sh}^{\mathrm{ord}}(\overline{\mathbb{F}}_p)$. We can further base change to $\widehat{\mathcal{O}}_{\overline{Sh},s}$, since the map to the formal completion is faithfully flat. There they follow from a simple computation with the local expressions above, and Lemma 2.5 with $\widetilde{H}_2 = T - T^p$. \Box

3. EXTENSION TO THE FLAG SHIMURA VARIETY

We extend $\Theta_{1,2,3,4}$ to maps on $\mathcal{F}l_{\mathbb{F}_p}$, by studying their formal local expressions at non-ordinary points. We then prove some finer properties regarding divisibility by Hasse invariants, and we give alternative descriptions for $\pi_*\theta_{1,2,3}$ as maps on the Shimura variety. Before that we start with a careful study of θ_4 , which is simpler than the other 3 since it is linear with respect to Sh.

3.1. Properties of θ_4 . We prove that θ_4 satisfies similar properties to Katz's theta operator, except for the fact that "it does not cycle".

Proposition 3.1. (1) We have $\theta_4^p = 0$ as a map on $\mathcal{F}l$, hence also as a map on $\mathcal{F}l^{\text{tor}}$. (2) Let $f \in \mathcal{L}(k, l)^{\text{can}}$ be a local section such that $H_2 \nmid f$. Then $H_2 \mid \theta_4(f)$ if and only if $p \mid k-l$.

Proof. For 1) we use Lemma 2.5 to locally identify θ_4 with a function on $\mathcal{O}_{\mathbb{A}^1_W} \times \mathbb{Z}/p$, where the second factor keeps track of l - k modulo p. Under this identification

$$\theta_4(f,N) = (\tilde{H}_2 \frac{df}{dT} + N \frac{dH_2}{dT} f, N+1).$$

Since $(\frac{d}{dT})^2 \tilde{H}_2 = 0$ we see that $D := \tilde{H}_2 \frac{d}{dT}$ and $\frac{d\tilde{H}_2}{dT}$ id commute as operators, so that

$$\theta_4^p(f,N) = (\prod_{i=0}^{p-1} (D+i\frac{d\tilde{H}_2}{dT})f,N) = ([D^p - D(\frac{d\tilde{H}_2}{dT})^{p-1}]f,N).$$

Now, $D^p \in \text{Der}_{\mathcal{O}_W}(\mathcal{O}_{\mathbb{A}^1_W}, \mathcal{O}_{\mathbb{A}^1_W})$ which is generated by $\frac{d}{dT}$. Moreover, the set $\{(\frac{d}{dT})^i : i = 0, 1, \dots, p-1\}$ is linearly independent as elements of $\text{Hom}_{\mathcal{O}_W}(\mathcal{O}_{\mathbb{A}^1_W}, \mathcal{O}_{\mathbb{A}^1_W})$. Since $\frac{d}{dT}\tilde{H}_2 = \frac{d\tilde{H}_2}{dT} + \tilde{H}_2\frac{d}{dT}$ we compute that

$$D^p = \tilde{H}_2 \left(\frac{d\tilde{H}_2}{dT} + \tilde{H}_2 \frac{d}{dT}\right)^{p-1} \frac{d}{dT} = D\left(\frac{d\tilde{H}_2}{dT}\right)^{p-1}$$

as elements of $\operatorname{Hom}_{\mathcal{O}_U}(\mathcal{O}_{\mathbb{A}^1_W}, \mathcal{O}_{\mathbb{A}^1_W})$, by expanding in terms of $\{(\frac{d}{dT})^i : i = 0, 1, \ldots, p-1\}$. This proves that $\theta_4^p = 0$.

For 2) we work locally on $W \subset \overline{\mathrm{Sh}}^{\mathrm{tor}}$, and write f as $ge_2^k(e_1 + Te_2)^l$, for $g \in \Gamma(W, \mathbb{A}_W^1)$. The divisor D_2^{tor} is given by the vanishing of \tilde{H}_2 . If $p \mid k - l$ it is easy to see from the local expression of Lemma 2.5 that $H_2 \mid \theta_4(f)$. Conversely suppose $H_2 \nmid f$ and $H_2 \mid \theta_4(f)$. Then we can find some open $W \subset \overline{\mathrm{Sh}}^{\mathrm{tor}}$ such that $\tilde{H}_2 \nmid g$ and $\tilde{H}_2 \mid (k-l)\frac{d\tilde{H}_2}{dT}g$. This still holds true over $W \cap U_1^{\mathrm{tor}}$, otherwise g/\tilde{H}_2 would extend to a section on $W \cap (U_1^{\mathrm{tor}} \cup U_2^{\mathrm{tor}})$, whose complement has codimension 2 in W, so by Hartogs lemma it would extend to W. We still have $H_2 \mid f$ over $\mathrm{Ig}^{\mathrm{tor}}$, since it is a finite étale cover, and H_2 has a simple zero on D_2 . There $\tilde{H}_2 = T - T^p$, so that $\tilde{H}_2 \mid (k-l)g$, which implies that $p \mid k - l$.

Even though $\theta_4^p = 0$ we still obtain an intermediate map by cycling θ_4 . This is an example of a theta linkage map.

Proposition 3.2. Let $(k,l) \in X^*(T)$ and write k-l = ap+b with $0 \le b \le p-1$. Then the map $\theta_4^{b+1} : \mathcal{L}(k,l) \to \mathcal{L}(k+(b+1)(p-1),l)$ factors through $H_2^{b+1} : \mathcal{L}(k-b-1,l+b+1) \to \mathcal{L}(k+(b+1)(p-1),l)$ and a map

$$\theta_{(k,l)}^4 : \mathcal{L}(k,l) \to \mathcal{L}(k-b-1,l+b+1)$$

which is Hecke equivariant away from p. It extends to a map on (sub)canonical extensions on $\mathcal{F}l^{\text{tor}}$. As a consequence, for $0 \leq k - l \leq p - 1$, $\pi_* \theta_4^{k-l+1} = 0$. Proof. This can be rephrased as saying that θ_4^{b+1} is divisible by H_2^{b+1} . We prove it by induction on b: if b = 0 it follows by Proposition 3.1(2). Suppose $b \ge 1$, and let $f \in \mathcal{L}(k, l)^{\operatorname{can}}$ be a local section, we want to prove $H_2^{b+1} \mid \theta_4^{b+1}(f)$. If $H_2 \mid f$, write $f = H_2g$ with $g \in \mathcal{L}(k - p, l + 1)$, then we can apply the induction hypothesis to g. So suppose $H_2 \nmid f$, then by Proposition 3.1(2) $H_2 \nmid \theta_4^b(f)$ but $H_2^N \mid \theta_4^{b+1}(f)$ for some $N \ge 1$, which we can assume is at most p - 1. Write $H_2^N h = \theta_4^{b+1}(f)$ for some $H_2 \nmid h$, since $\theta_4^p = 0$ we have $\theta_4^{p-b-1}(h) = 0$. For $H_2 \mid \theta_4^{p-b-1}(h)$ to be possible the b in the weight of one of $\{h, \theta_4(h), \ldots, \theta_4^{p-b-2}(h)\}$ has to be 0. This implies that $N \ge b + 1$ as desired. \Box

Using Proposition 3.1(2) we can also describe the kernel of θ_4 and $\theta_{(k,l)}^4$. Let $F : \mathcal{F}l \to \mathcal{F}l^{(p)}$ be the relative Frobenius with respect to \overline{Sh} . The twist $\mathcal{F}l^{(p)}$ can be identified with the flag variety of $\omega^{(p)}$ over \overline{Sh} , let $\mathcal{L}^{(p)} \subset \omega^{(p)}$ be its tautological line bundle. It can be identified with $\sigma^*(\mathcal{L} \subset \omega)$, where $\sigma : \mathcal{F}l^{(p)} \to \mathcal{F}l$ is the base change morphism. The projection $\pi^{(p)} : \mathcal{F}l^{(p)} \to \overline{Sh}$ satisfies $R\pi_*^{(p)} \circ F_* \cong R\pi_*$. We will write $\pi^{(p)*}\omega$ as ω for convenience. Over \overline{Sh} there is a map $\omega^{(p)} \hookrightarrow \mathrm{Sym}^p \omega$ of sheaves which is induced by the map of representations of $M = \mathrm{GL}_2/\overline{\mathbb{F}}_p$

$$\overline{\mathbb{F}}_p^{2,(p)} \hookrightarrow \mathrm{Sym}^p \overline{\mathbb{F}}_p^2$$

Proposition 3.3. (Kernel of θ_4) Let $(k, l) \in X^*(T)$, write k - l = ap + b with $0 \le b \le p - 1$. Then

(1)

$$F_* \operatorname{Ker}(\theta_4 : \mathcal{L}(k, l) \to \mathcal{L}(k+p-1, l)) = (\omega^{(p)} / \mathcal{L}^{(p)})^{a-b} \otimes \det^{l+b} \omega$$

It is embedded into $F_*\mathcal{L}(k,l)$ as follows

$$(\omega^{(p)}/\mathcal{L}^{(p)})^{a-b} \otimes \det^{l+b} \omega \to F_*((\omega/\mathcal{L})^{(a-b)p} \otimes \det^{l+b} \omega) \xrightarrow{F_*H_2^b} F_*((\omega/\mathcal{L})^{k-l} \otimes \det^l \omega),$$

where the first map is obtained by applying the unit $id \to F_*F^*$ of the adjunction together with the identification $F^*(\omega^{(p)}/\mathcal{L}^{(p)}) \cong (\omega/\mathcal{L})^p$. The same formula holds for the map on canonical extensions on $\mathcal{F}l^{\text{tor}}$. In particular for 0 < k - l < p, $\pi_*\theta_4$ is injective.

(2) Moreover,

$$F_* \operatorname{Ker} \theta^4_{(k,l)} = (\omega^{(p)} / \mathcal{L}^{(p)})^a \otimes \operatorname{Sym}^b \omega \otimes \operatorname{det}^l \omega$$

The embedding into $F_*\mathcal{L}(k,l)$ is induced by first applying the unit $id \to F_*F^*$, and then the projection $\operatorname{Sym}^b \omega \to (\omega/\mathcal{L})^b$. For $0 \le k - l \le p^2 - 1$, this implies

$$\operatorname{Ker} \pi_* \theta^4_{(k,l)} = \operatorname{Sym}^a \omega^{(p)} \otimes \operatorname{Sym}^b \omega \otimes \operatorname{det}^l \omega = F_P(L_M(k,l))$$

where $L_M(k,l)$ is the irreducible representation of M of highest weight (k,l).

Proof. For 1) first assume b = 0. Then by Lemma 2.5 a local section $f \in \mathcal{L}(k,l)^{\operatorname{can}}$ in $\operatorname{Ker}\theta_4$ satisfies $\frac{df}{dT} = 0$, since \tilde{H}_2 doesn't vanish identically on an open subset. This implies $f = g(T^p)$ for some g, and one can check that the gluing data of g precisely corresponds to a section of $(\omega^{(p)}/\mathcal{L}^{(p)})^a \otimes \det^l \omega$. Now assume $b \ge 1$. Let $f \in \mathcal{L}(k,l)^{\operatorname{can}}$ be a local section such that $\theta_4(f) = 0$. By the case b = 0 we have $\tilde{H}_2^{p-b}f = g(T^p)$ for some g. Let $Y \to \overline{\operatorname{Sh}}^{\operatorname{tor}}$ be a finite type map

obtained by extracting sufficiently high pth power roots, so that we can find $\tilde{g}(T) \in \mathcal{O}_Y[T]$ such that $\tilde{g}(T)^p = g(T^p)$. Over the base change to $\mathrm{Ig^{tor}}$ we see that $T^p - T = \tilde{H}_2 \mid \tilde{g}(T)$, so that $\tilde{g}(T)/\tilde{H}_2$ extends to $(U_1^{\mathrm{tor}} \cup U_2^{\mathrm{tor}})_Y$. Using étale coordinates we can check that Y is smooth, so that by Hartogs lemma it further extends to Y. Therefore we can write $\tilde{g}(T) = \tilde{H}_2\tilde{h}(T)$ for some $\tilde{h} \in \mathcal{O}_Y[T]$. We claim that $h(T^p) \coloneqq \tilde{h}(T)^p \in \mathcal{O}_{\overline{\mathrm{Sh}}^{\mathrm{tor}}}[T]$. Since it lies on $\mathcal{O}_Y[T]$ it satisfies $h(T^p)^{p^n} \in \mathcal{O}_{\overline{\mathrm{Sh}}^{\mathrm{tor}}}[T]$ for some n. It also lies on $\mathcal{O}_{\overline{\mathrm{Sh}}^{\mathrm{tor}}}[T][1/\tilde{H}_2]$, so by normality $h(T^p) \in \mathcal{O}_{\overline{\mathrm{Sh}}^{\mathrm{tor}}}[T]$. Thus, we can write $\tilde{H}_2^{p-b}f = \tilde{H}_2^ph(T^p)$, and cancelling the \tilde{H}_2 s gives $f = \tilde{H}_2^bh(T^p)$. By translating the transition cocycles of f to $h(T^p)$ we see that the coefficients of $h(T^p)$ precisely contain the information of a section of $(\omega^{(p)}/\mathcal{L}^{(p)})^{a-b} \otimes \det^{l+b}\omega$ embedded in the required way in $F_*\mathcal{L}(k, l)^{\mathrm{can}}$.

For 2), since H_2 is injective as a map of sheaves it is enough to consider the kernel of θ_4^{b+1} . We prove it by induction on b, with b = 0 being part 1). Assume $b \ge 1$, by the induction hypothesis we know $F_* \text{Ker}(\theta_4^b : \mathcal{L}(k + p - 1, l) \to \mathcal{L}(k + (b + 1)(p - 1), l)) = (\omega^p / \mathcal{L}^{(p)})^{a+1} \otimes \text{Sym}^{b-1} \omega \otimes \det^l \omega =: E_{(a+1,b-1)}$, and carefully going through the embedding into $F_* \mathcal{L}(k + p - 1, l)$ we see that its local sections are precisely the ones that can be written as $g(T)f(T^p)e_2^{k-l+p-1}(e_1 \wedge e_2)^l$ with g a polynomial of degree at most b - 1. For $h = g_1(T)f_1(T^p)$ a local section of $\mathcal{L}(k, l)$,

$$\theta_4(h) = f_1(T^p)(\tilde{H}_2 \frac{dg_1}{dT} - b \frac{d\tilde{H}_2}{dT}g_1),$$

so if g_1 has degree at most b one can check that $\theta_4(h) \in E_{(a+1,b-1)}$, so that $E_{(a,b)} \subseteq \operatorname{Ker} \theta_{(k,l)}^4$. We prove that this inclusion is an isomorphism over U_1 . For that we prove that the preimage of $E_{(a+1,b-1)}$ under θ_4 is $E_{(a,b)}$. Consider a section $g(T)f(T^p)$ of $E_{(a+1,b-1)}$ over U_1 . We claim that it can be written as $\theta_4(g_1(T)f_1(T^p))$ with g_1 of degree at most b: it is enough to check it for $g = T^i$, $i = 0, \ldots, b-1$. We can base change to Ig, so that $\tilde{H}_2 = T^p - T$. There

$$\theta_4((i-b)^{-1}T + \sum_{k=1}^{i} T^{i+k(p-1)}(i-b)^{-1} \prod_{j=1}^{k} \frac{i-j+1}{i-j-b}) = T^i$$

for $0 \leq i \leq b-1$. Moreover, from the local expression it is easy to check that $\theta_4^{-1}E_{(a+1,b-1)} \subseteq E_{(a,b)}$ over U_{Ig} . This shows that $E_{(a,b)} \cong \text{Ker}\theta_{(k,l)}^4$ over U_1 . Furthermore, the cokernel of $E_{(a,b)} \to F_*\mathcal{L}(k,l)$ is free, so in particular the cokernel of $E_{(a,b)} \hookrightarrow \text{Ker}\theta_{(k,l)}^4$ is torsion-free. This implies that $E_{(a,b)} \cong$ $\text{Ker}\theta_{(k,l)}^4$.

Remark 3.1. (1) In the case $0 \le k-l \le p^2-1$, the target of $\pi_*\theta_{(k,l)}^4$ is $\omega(k-b-1, l+b+1)$, which contains the other Jordan-Holder factor of $\omega(k, l)$: it is precisely the kernel of $\theta_{(k-b-1,l+b+1)}^4$. The identity $\theta_4^p = 0$ is then consistent with the fact that W(k, l) and W(k-p, l+p) have no common Jordan-Holder factor. The representation W(k, l) is irreducible if $k-l = -1 \mod p$, where $\theta_{(k,l)}^4$ is just 0, since $\theta_4^p = 0$. This is no coincidence, in fact $\pi_*\theta_{(k,l)}^4$ comes from a map of *M*-representations, see Corollary 4.19 for a more conceptual explanation of this phenomenon.

(2) If $k - l \ge p^2$ one can again apply the analogue map to $\theta_{(k,l)}^4$ on $F_* \text{Ker} \pi_* \theta_4$, where now the automorphic vector bundles are taken with respect to $\omega^{(p)}$. This obtains a subsheaf of $F_*^2 \text{Ker} \pi_* \theta_4$. Repeating this enough times the resulting sheaf will correspond to $L_M(k,l)$ again. E.g. for $k - l = p^2$, $\text{Ker} F_* \pi_* \theta_4 = \text{Sym}^p \omega^{(p)}$, and the kernel of $\theta_4^{(p)}$ on it is $\omega^{(p^2)}$.

3.2. Extending $\Theta_{1,2,3}$ to $\mathcal{F}l$. We extend $\Theta_{1,2,3}$ to $\mathcal{F}l$ by computing their poles along $D_1 = \pi^{-1}(\overline{\mathrm{Sh}}^{\mathrm{n-ord}})$. We first prove that the following common piece of $\Theta_{1,2,3}$ only has a simple pole.

Lemma 3.4. The map on U_1

$$\theta := H_1 \tilde{\theta} : \omega \xrightarrow{\nabla} H \otimes \Omega^1_{\mathcal{F}l} \xrightarrow{H_1 \pi_\omega \otimes s} \omega \otimes \mathcal{L}(p-1, p-1) \otimes \pi^* \Omega^1_{\overline{Sh}},$$

where $\tilde{\theta}$ is as in Lemma 2.4, extends to (sub)canonical extensions on $\mathcal{F}l^{\text{tor}}$. Moreover, on sections that come from $\omega/\overline{\text{Sh}}$ via pullback it can be identified with the map on $\overline{\text{Sh}}$

$$\omega \xrightarrow{\nabla} H \otimes \Omega^{1}_{\overline{\mathrm{Sh}}} \xrightarrow{H_{1}\pi_{\omega} \otimes id} \omega \otimes \mathrm{det}^{p-1} \omega \otimes \Omega^{1}_{\overline{\mathrm{Sh}}}$$

Proof. The second statement is immediate from the first by functoriality of ∇ . We prove that θ extends to $\mathcal{F}l$ by computing a local expression, we use the presentation of Ig from Lemma 2.1. Let $W = \operatorname{Spec}(R) \subset \overline{\operatorname{Sh}}$ such that ω is trivial with basis $\{e_1, e_2\}$, and work over $\mathbb{A}^1_W \subset \pi^{-1}(W)$, then U_1 is given by R[1/(ad - bc)][T]. Let d denote the differential $d : \mathcal{O}_{\pi^{-1}(W)} \to \Omega^1_{\pi^{-1}(W)}$. Let $f \in R[T]$, we compute that $H_1\tilde{\theta}(fe_i)$ has no poles. On Ig_W we have a preferred basis $\{\tilde{e}_1, \tilde{e}_2\}$ satisfying $M(e_1, e_2)^T = (\tilde{e}_1, \tilde{e}_2)^T$, which is horizontal for $\tilde{\theta}$. Differentiating the relation $M^{(p)} = MV^T$ we obtain $dM = -M \circ d(V^T) \circ (V^T)^{-1}$, so that

$$(\pi_{\omega} \otimes 1) \circ \nabla(e_1, e_2)^T = d(M^{-1})M(e_1, e_2)^T = d(V^T) \circ (V^T)^{-1}(e_1, e_2)^T.$$
(3.1)

This shows that the map $(H_1\pi_\omega \otimes 1) \circ \nabla$ extends to $\mathcal{F}l$. Concretely it sends fe_1 to

$$fe_1 \mapsto [(ad - bc)df + f(dd(a) - bd(c))]e_1(e_1 \wedge e_2)^{p-1} + f(ad(c) - cd(a))e_2(e_1 \wedge e_2)^{p-1},$$

and similarly for fe_2 . Let $W' = W \cap \overline{\mathrm{Sh}}^{\mathrm{ord}}$, the last ingredient of θ involves the section $s : \Omega^1_{\pi^{-1}(W')} \to \pi^* \Omega^1_{W'}$ which is defined by the diagram



where the isomorphisms are given étale locally by the basis $\{\tilde{e}_1, \tilde{e}_2\}$, and r is the natural projection. Since the first square commutes for $g \in \pi^{-1}\mathcal{O}_W$ we have $s \circ d(g) = d_{W'}(g) \in \pi^*\Omega^1_{W'}$, where $d_{W'}: \pi^{-1}\mathcal{O}_{W'} \to \pi^{-1}\Omega^1_{W'}$. Thus, to compute θ we replace d by d_W in (3.1). Explicitly,

$$\theta(fe_1) = [(ad - bc)df + f(dd(a) - bd(c))]e_1(e_1 \wedge e_2)^{p-1} + f(ad(c) - cd(a))e_2(e_1 \wedge e_2)^{p-1}, \quad (3.2)$$

where now the differentials correspond to $d: \mathcal{O}_W \to \Omega^1_W$. This proves that the map extends to $\mathcal{F}l$. The same computation shows it extends to canonical extensions, and for subcanonical extensions one uses that $d\log(f) \in \Omega^1_{\overline{Sh}^{tor}}(\log D)$.

Remark 3.2. The local expression on (3.2) is easier to compute using the map on \overline{Sh} . This lemma is just a careful check that the construction using the section s is compatible with the map on \overline{Sh} .

Corollary 3.5. The maps $H_1 \cdot \Theta_{1,2,3}$ extend to maps of sheaves on $\mathcal{F}l$. They further extend to maps of (sub)canonical extensions on $\mathcal{F}l^{\text{tor}}$.

Proof. By Lemma 2.4 the map θ from Lemma 3.4 sends \mathcal{L} to $\mathcal{L} \otimes \mathcal{L}(p-1, p-1) \otimes \pi^* \Omega^1_{Sh}$, being a condition that can be checked on an open dense subset. Then $H_1 \cdot \Theta_{1,2,3}$ can be constructed on $\mathcal{F}l$ using the same definition as for $\Theta_{1,2,3}$ but replacing $\tilde{\theta}$ by θ .

However, it might be the case that multiplying by H_1 is not necessary, in this case we will show that Θ_3 already extends to $\mathcal{F}l$. To do that we compute a local expression for Θ_3 on the formal neighbourhood of a *p*-rank 1 point. For the purposes of checking poles along D_1 it is enough to check them inside $\overline{\mathrm{Sh}}^{\geq 1} = \overline{\mathrm{Sh}}^{\mathrm{ord}} \cup \overline{\mathrm{Sh}}^{=1}$, since by Hartogs lemma a function on U_1 that extends to $\mathcal{F}l_{\overline{\mathrm{Sh}}^{\geq 1}}$ already extends to $\mathcal{F}l$. Being a local problem, we will work on formal completions $\widehat{\mathcal{O}}_{\overline{\mathrm{Sh}},q}$ (or their pullbacks to $\mathcal{F}l$) for a fixed $q \in \overline{\mathrm{Sh}}^{=1}(\overline{\mathbb{F}}_p)$, and use Grothendieck-Messsing theory to get a nice basis for ω . Write $\widehat{\mathcal{O}}_{\overline{\mathrm{Sh}},q} = \overline{\mathbb{F}}_p[[T_{11}, T_{12}, T_{22}]]$, and let $R = \mathcal{O}_{\overline{\mathrm{Sh}},q}/\mathfrak{m}^p$. As in the proof of Proposition 1.11 we can choose a basis of ω_R such that V is represented by the matrix

$$V_R = \begin{pmatrix} 1 & 0\\ T_{12} & T_{22} \end{pmatrix}. \tag{3.3}$$

By Hensel's lemma we can choose a basis $\{e_1, e_2\}$ of $\omega_{\widehat{\mathcal{O}}_{\overline{Sh},q}}$ lifting the one above such that V is still given by the same matrix.

Let $q \in \overline{\mathrm{Sh}}^{-1}(\overline{\mathbb{F}}_p)$, and fix a basis $\{e_1, e_2\}$ of ω on $\widehat{O}_{\overline{\mathrm{Sh}},q}$ such that V has matrix (3.3). Consider $D_{ij} \coloneqq \mathrm{ks}(e_i e_j)^{\vee}$ as continuous derivations on $\widehat{\mathcal{O}}_{\overline{\mathrm{Sh}},q}$.

Lemma 3.6. Let $q \in \overline{\mathrm{Sh}}^{-1}(\overline{\mathbb{F}}_p)$. Then

$$D_{ij} = \frac{\partial}{\partial T_{ij}} \mod \mathfrak{m}_{\overline{\mathrm{Sh}},q}^{p-1}.$$

Proof. Dually, we prove that $\langle \nabla(e_i), e_j \rangle = dT_{ij} \mod \mathfrak{m}_{\overline{\mathrm{Sh}},q}^{p-1}$. Let $R = \mathcal{O}_{\overline{\mathrm{Sh}},q}/\mathfrak{m}_{\overline{\mathrm{Sh}},q}^p$, then

$$\langle \nabla(e_i), e_j \rangle = \langle \nabla_R(e_{R,i}), e_{R,j} \rangle \mod \mathfrak{m}_{\overline{\mathrm{Sh}},q}^{p-1}$$

since $\nabla_R = \nabla \mod \mathfrak{m}_{\overline{Sh},q}^p$ determines ks mod $\mathfrak{m}_{\overline{Sh},q}^{p-1}$. In the notation of (1.3) $e_{R,1} = \overline{e}_1 + T_{12}\overline{e}_3 + T_{11}\overline{e}_4$

$$e_{R,2} = \overline{e}_2 + T_{22}\overline{e}_3 + T_{12}\overline{e}_4,$$

where $\{\overline{e}_i\}$ is a symplectic basis on $H^1_{\kappa(q)}$ coming from a trivialization of $A[p]_{\kappa(q)}$. By Lemma 1.9 $\nabla_R(\overline{e}_i) = 0$. Then

$$\langle \nabla(e_1), e_1 \rangle = \langle dT_{12}\overline{e}_3 + dT_{11}\overline{e}_4, \overline{e}_1 \rangle = dT_{11} \mod \mathfrak{m}_{\overline{\mathrm{Sh}},q}^{p-1},$$

and similarly for the others.

Using the lemma above and the local expression (3.2) we can compute the following local expressions for $H_1\Theta_1$ and $H_1\Theta_3$.

Proposition 3.7. Let $q \in \overline{\mathrm{Sh}}^{-1}(\overline{\mathbb{F}}_p)$, and $\{e_1, e_2\}$ be a basis of ω on $\widehat{O}_{\overline{\mathrm{Sh}},q}$ satisfying (3.3). We have the following local expressions at $\mathbb{A}_T^1 \subset \mathbb{P}^1_{\widehat{O}_{\overline{\mathrm{Sh}},q}}$ with respect to the local sections $fe_2^{k-l}(e_1 \wedge e_2)^l \in \mathcal{L}(k,l)$ for $f \in \widehat{\mathcal{O}}_{\overline{\mathrm{Sh}},q}[T]$, and $ge_1^i e_2^{k-l-i}(e_1 \wedge e_2)^l$ for $g \in \widehat{\mathcal{O}}_{\overline{\mathrm{Sh}},q}$, $i \in \mathbb{Z}$. $H_1\Theta_1(f) = kf + T^2T_{22}\frac{\partial f}{\partial T_{11}} - TT_{22}\frac{\partial f}{\partial T_{12}} + T_{22}\frac{\partial f}{\partial T_{22}} \mod \mathfrak{m}^{p-1}$ $H_1\Theta_3(g) = T_{22}e_1^i e_2^{k-l-i}(e_1 \wedge e_2)^{l+p-1}\Pi_3 \mathrm{KS}^{-1}dg$ $+ (k-i)gT_{22}^2 e_1^i e_2^{k-l-i+2p}(e_1 \wedge e_2)^{l+p-1}$ $+ igT_{22}T_{12}e_1^{i-1}e_2^{k-l-i+2p+1}(e_1 \wedge e_2)^{l+p-1}$ $- T_{22}ige_1^{i+p-1}e_2^{k-l-i+p+1}(e_1 \wedge e_2)^{l+p-1} \mod \mathfrak{m}^{p-1}.$

Proof. This is a combination of using the explicit matrix of V in (3.3) on the local expressions for θ in Lemma 3.4, as well as in (2.3); and the knowledge of Kodaira-Spencer modulo \mathfrak{m}_q^{p-1} of Lemma 3.6.

It is then clear that Θ_3 already extends to $\pi^{-1}(\overline{\mathrm{Sh}}^{\geq 1})$, since it does pointwise, and hence to $\mathcal{F}l$. Since D_1^{tor} only intersects the boundary on a space of dimension at most 2 by Proposition 1.7(7) Θ_3 also extends to a map on (sub)canonical extensions on $\mathcal{F}l^{\mathrm{tor}}$. Including the central character in the Kodaira-Spencer isomorphism we get Hecke equivariant maps.

Proposition 3.8. Define $\theta_{1,2,3}$ on $\mathcal{F}l$ by

$$\begin{aligned} \theta_1 &\coloneqq H_1 \Theta_1 : \mathcal{L}(k, l, c) \to \mathcal{L}(k + p + 1, l + p - 1, c) \\ \theta_2 &\coloneqq H_1 \Theta_2 : \mathcal{L}(k, l, c) \to \mathcal{L}(k + 2p, l + p - 1, c) \\ \theta_3 &\coloneqq \Theta_3 : \mathcal{L}(k, l, c) \to \mathcal{L}(k + 2p, l, c). \end{aligned}$$

They are Hecke equivariant away from p and furthermore they extend to maps on (sub)canonical extensions on $\mathcal{F}l^{\text{tor}}$. They satisfy relations as in Proposition 2.7, namely

 $\begin{array}{l} (1) \ [\theta_1, \theta_4] = \theta_2. \\ (2) \ [\theta_2, \theta_4] = 2H_1\theta_3. \\ (3) \ [\theta_3, \theta_4] = 0. \\ (4) \ [\theta_i, \theta_j] = 0 \ for \ i, j = 1, 2, 3. \\ (5) \ \theta_1^p = H_1^p\theta_3. \\ (6) \ The \ operator \ \Theta \coloneqq \frac{1}{H_2^2}(4H_1\theta_1\theta_3 - \theta_2^2) \ is \ well-defined \ and \ satisfies \ \Theta^p = H_1^{2p}\Theta. \\ (7) \ [\theta_2^p, \theta_4] = 0, \ [\Theta, \theta_4] = 0. \end{array}$

Looking at the local expression for θ_1 we immediately obtain a criterion for the divisibility of θ_1 by H_1 .

Proposition 3.9. Let $f \in \mathcal{L}(k,l)$ be a local section such that $H_1 \nmid f$. Then $H_1 \mid \theta_1(f)$ if and only if $p \mid k$. The same holds for a local section of the (sub)canonical extension on $\mathcal{F}l^{\text{tor}}$.

Proof. It follows from Proposition 3.7, since if $H_1 \nmid f$ we can take $q \in \overline{\mathrm{Sh}}^{=1}(\overline{\mathbb{F}}_p)$ so that $\tilde{H}_1 = T_{22}$ doesn't divide f in these coordinates. Then in these coordinates $H_1 \mid \theta_1(f)$ implies $p \mid k$. Conversely, if $T_{22} \mid \theta_1(f)$ for all $q \in \overline{\mathrm{Sh}}^{=1}(\overline{\mathbb{F}}_p)$ then $H_1 \mid \theta_1(f)$.

Combining Proposition 3.9 with the identity $\theta_1^p = H_1^p \theta_3$ we obtain one of the theta linkage maps, similarly to the way one determines the theta cycle for the modular curve.

Theorem 3.10. Let $(k,l) \in X^*(T)$, and write k = pb + a with $1 \le a \le p$. Then the map

$$\theta_{(k,l)}^{1} \coloneqq \frac{1}{H_{1}^{p-a+1}} \theta_{1}^{p-a+1} : \mathcal{L}(k,l) \to \mathcal{L}(2p-2a+k+2,l)$$

exists as a map of sheaves on $\mathcal{F}l$, and it is Hecke equivariant away from p. It further extends to a map on (sub)canonical extensions on $\mathcal{F}l^{\text{tor}}$.

Proof. We prove the statement by induction on a starting at p, then p-1, p-2 and so on. For $f \in \mathcal{L}(k,l)$ we write a(f) to emphasize that it depends on f. Let $V \subset \mathcal{F}l$ an open small enough to be contained in $\pi^{-1}(W)$ for $W \subset \overline{Sh}$ such that ω_W is trivial, and let $f \in \mathcal{L}(k,l)(V)$. We want to prove that $H_1^{p-a+1} \mid \theta_1^{p-a+1}(f)$. If $V \subset U_1$ then the assertion is trivial, suppose it's not. First assume a = p, if $H_1 \mid f$ the statement is obvious, otherwise it follows from Proposition 3.9. Now assume $a \neq p$. If $H_1 \mid f$ write $f = H_1 f_0$, then the statement $H_1^{p-a+1} \mid \theta_1^{p-a+1}(f)$ follows from $H_1^{p-a(f_0)+1} \mid \theta_1^{p-a(f_0)+1}(f_0)$ for $a(f_0) = a(f) + 1$, which appears earlier in the induction process. Therefore we may assume $H_1 \nmid f$.

By Proposition 3.9 we see that $H_1 \nmid \theta_1^{p-a}(f)$ but $H_1 \mid \theta_1^{p-a+1}(f)$. Let N be the exact power of H_1 that divides $\theta^{p-a+1}(f)$. We can assume that a > 1 and $1 \le N \le p-1$, otherwise the statement follows from $\theta_1^p = H_1^p \theta_3$. Then $h \coloneqq \frac{1}{H_1^N} \theta_1^{p-a+1}(f)$ satisfies a(h) = N + 1. We know that $H_1^{p-N} \mid \theta_1^{a-1}(h)$ by the relation $\theta_1^p = H_1^p \theta_3$, so for some $0 \le i \le a-2$, the $a(\theta^i(h))$ has to be

32

divisible by p again. This implies that $(N + 1) + a - 2 \ge p$, i.e. $N \ge p - a + 1$. The proof readily extends to the toroidal compactification.

Note that the weight increase of $\theta_{(k,l)}^1$ corresponds to a ρ -shifted affine Weyl reflection after shifting by -(3,3), in particular it maps the C_1 alcove to the C_2 alcove. We will prove that for generic *p*-restricted weights the map is injective on global sections in Theorem 3.13.

3.3. $\pi_*\theta_{1,2,3}$ as maps on the Shimura variety. Here we describe $\pi_*\theta_{1,2,3}$ without referring to the flag Shimura variety. First we recall the theta operator on $\overline{\text{Sh}}$ already defined in the literature. On $\overline{\text{Sh}}^{\text{ord}}$ we have the projection $\pi_{\omega} : H \to \omega$, which can be extended to a map $\Pi_{\omega} : H \to \omega \otimes \det^{p-1} \omega$ on $\overline{\text{Sh}}$ by multiplying by the Hasse invariant. We use H(k,l) to denote the sheaf $\operatorname{Sym}^{k-l} H \otimes (\bigwedge^2 H)^{\otimes l}$ over $\overline{\text{Sh}}$, the Gauss-Manin connection ∇ extends to it. The Hodge filtration on H defines a decreasing filtration, and by Griffiths transversality ∇ factors through the first nontrivial step of the filtration $F^1H(k,l)$. Then Π_{ω} extends to a map $\Pi : F^1H(k,l) \to \omega(k,l) \otimes \det^{p-1} \omega$ such that on a simple tensor one only applies Π_{ω} to the component which doesn't lie in ω in case there is any, otherwise Π does nothing.

Definition 3.1. Let $(k,l) \in X^*(T)$ be a *M*-dominant weight. Define the differential operator $\theta_{\overline{Sh}} : \omega(k,l) \to \omega(k+p-1,l+p-1) \otimes \operatorname{Sym}^2 \omega$ as the composite of the following maps

$$\theta_{\overline{\mathrm{Sh}}} \coloneqq \omega(k,l) \xrightarrow{\nabla} F^1 H(k,l) \otimes \Omega^1_{\overline{\mathrm{Sh}}/\mathbb{F}_p} \xrightarrow{\Pi \otimes \mathrm{ks}^{-1}} \omega(k+p-1,l+p-1) \otimes \mathrm{Sym}^2 \omega.$$

Recall the 3 projections from $\text{Sym}^2 \omega$ on $\mathcal{F}l$ in (2.3). Similarly, we can also define 3 projections from $\text{Sym}^2 \omega$ over $\overline{\text{Sh}}$:

$$\begin{split} \Pi_{\overline{\mathrm{Sh}},1} &: \mathrm{Sym}^{2}\omega \xrightarrow{\mathrm{id}} \mathrm{Sym}^{2}\omega \\ \Pi_{\overline{\mathrm{Sh}},2} &: \mathrm{Sym}^{2}\omega \to \omega^{\otimes 2} \xrightarrow{\mathrm{id}\otimes V} \omega \otimes \omega^{(p)} \to \omega \otimes \mathrm{Sym}^{p}\omega \to \mathrm{Sym}^{p+1}\omega \\ \Pi_{\overline{\mathrm{Sh}},3} &: \mathrm{Sym}^{2}\omega \xrightarrow{\mathrm{Sym}^{2}V} \mathrm{Sym}^{2}\omega^{(p)} \to \mathrm{Sym}^{2p}\omega, \end{split}$$

where we are using the natural embedding $\omega^{(p)} \hookrightarrow \operatorname{Sym}^p \omega$.

Lemma 3.11. Under the identifications $\pi_*\mathcal{L}(k,l) = \omega(k,l)$ we have $\pi_*\Pi_i = \Pi_{\overline{Sh},i}$. Moreover, the pushforward of

$$\mathcal{L}(k,l) \otimes \operatorname{Sym}^2 \omega \xrightarrow{id \otimes \Pi_i} \mathcal{L}(k+k_i,l+l_i)$$

is

$$\omega(k,l) \otimes \operatorname{Sym}^2 \omega \xrightarrow{id \otimes \Pi_{\overline{\operatorname{Sh}},i}} \omega(k,l) \otimes \omega(k_i,l_i) \xrightarrow{r} \omega(k+k_i,l+l_i),$$

where r comes from the canonical projection $\operatorname{Sym}^n \otimes \operatorname{Sym}^m \to \operatorname{Sym}^{n+m}$.

Proof. For i = 1 the property that we need is that the map $\mathcal{L}(k, l) \otimes \text{Sym}^n \omega \to \mathcal{L}(k+n, l)$ corresponds to r under π_* . This can be checked on local coordinates, reducing to the case of \mathbb{P}^1 . We also have

that $\omega^{(p)} \to \mathcal{L}(p,0)$ corresponds to $\omega^{(p)} \to \operatorname{Sym}^p \omega$, also by reducing to the case of \mathbb{P}^1 . The statement for i = 2, 3 follows by combining these two properties and the fact that $\pi_*(V : \omega \to \omega^{(p)})$ is again the Verschiebung on $\overline{\operatorname{Sh}}$ by the projection formula.

We can use these to describe $\pi_*\theta_{1,2,3}$ as maps on Sh.

Proposition 3.12. Let $(k, l) \in X^*(T)$ *M*-dominant. We define 3 different theta operators as maps of sheaves on \overline{Sh} :

$$\begin{split} \theta_{\overline{\mathrm{Sh}},1} &:= r \circ \Pi_{\overline{\mathrm{Sh}},1} \circ \theta_{\overline{\mathrm{Sh}}} : \omega(k,l) \to \omega(k+p+1,l+p-1) \\ \theta_{\overline{\mathrm{Sh}},2} &:= r \circ \Pi_{\overline{\mathrm{Sh}},2} \circ \theta_{\overline{\mathrm{Sh}}} : \omega(k,l) \to \omega(k+2p,l+p-1) \\ \theta_{\overline{\mathrm{Sh}},3} &:= r \circ \Pi_{\overline{\mathrm{Sh}},3} \circ \theta_{\overline{\mathrm{Sh}}} : \omega(k,l) \to \omega(k+3p-1,l+p-1) \end{split}$$

where r is the natural map $\operatorname{Sym}^n \otimes \operatorname{Sym}^m \to \operatorname{Sym}^{n+m}$. Then $\pi_* H_1 \Theta_i = \theta_{\overline{\operatorname{Sh}} i}$.

Proof. By a Leibniz rule type of argument we can reduce to weights (1,0) and (1,1). Lemma 3.4 shows that $\pi_*\theta = \theta_{\overline{Sh}}$, which combined with Lemma 3.11 proves it for weight (1,0). For (1,1) it follows similarly from Lemma 3.4 by following the recipe for $H_1\Theta_i$ in the case det $\omega = \mathcal{L} \otimes \omega/\mathcal{L}$. \Box

The operator $\theta_{\overline{Sh},1}$ is the theta operator already considered by $[EFG^+21]$ in much greater generality, and by [Yam23] (denoted by θ_3 in his paper) for GSp_4 . Since Θ_3 already extends to $\mathcal{F}l$ we see that $H_1 \mid \theta_{\overline{Sh},3}$. One could have proved this via a local expression directly on \overline{Sh} .

With this description we can prove the following important result.

Theorem 3.13. Let $(k,l) \in X^*(T)$ such that $p-1 \ge k \ge l \ge 0$, then $\operatorname{Ker} \pi_* \theta^1_{(k,l)} = \operatorname{Ker} \pi_* \theta_1$. If furthermore $(k,l) \ne (p-1,p-1)$

$$\theta^1_{(k,l)}: H^0(\overline{\operatorname{Sh}}, \omega(k,l)) \to H^0(\overline{\operatorname{Sh}}, \omega(2p-k+2,l))$$

is injective. The map $\theta_{(k,l)}^1$ is still injective for (k,l) = (p-1, p-1) after localizing at a non-Eisenstein maximal ideal.

Proof. We have an inclusion $\operatorname{Ker} \pi_* \theta_1 \subset \operatorname{Ker} \pi_* \theta_{(k,l)}^1$ since $H_1^{p-k+1} \theta_{(k,l)}^1 = \theta_1^{p-k+1}$ and H_1 is injective. Also, $\operatorname{Ker} \pi_* \theta_{(k,l)}^1 \subset \operatorname{Ker} \pi_* \theta_3$ since $\theta_3 = \theta_{(2p-k+2,l)}^1 \circ \theta_{(k,l)}^1$. We prove first that whenever $0 \leq k-l \leq p-1$ the map $f: \omega(k,l) \otimes \operatorname{Sym}^2 \omega \xrightarrow{\operatorname{id} \otimes \Pi_{\overline{Sh},3}} \omega(k,l) \otimes \operatorname{Sym}^{2p} \omega \to \omega(2p+k,l)$ is injective as a map of sheaves. It is enough to do this on $\overline{\operatorname{Sh}}^{\operatorname{ord}}$, which is open dense, and one can further pass to the finite étale cover Ig. The canonical basis $\{e_1, e_2\}$ of ω satisfies $Ve_i = e_i^{(p)}$, so f is identified with the map $\operatorname{Sym}^{k-l} \mathcal{O}^2 \otimes \operatorname{Sym}^2 \mathcal{O}^2 \to \operatorname{Sym}^{2p+k-l} \mathcal{O}^2$ sending basis elements $e_1^n e_2^{k-l-n} \otimes e_i e_j$ to $e_1^n e_2^{k-l-n} e_i^p e_j^p$, which is clearly injective when $k-l \leq p-1$. Therefore $\operatorname{Ker} \pi_* \theta_3 = \operatorname{Ker} (\theta_{\overline{Sh},3} = \operatorname{Ker} (\theta_{\overline{Sh}} : \omega(k,l) \to \theta_1)$. $\omega(k+p-1,l+p-1)\otimes \text{Sym}^2\omega)$. Since the map $\theta_{\overline{\text{Sh}}}$ is a composition factor of $\pi_*\theta_1 = \theta_{\overline{\text{Sh}},1}$ we have $\text{Ker}\pi_*\theta_3 \subset \text{Ker}\pi_*\theta_1$, so that for $0 \leq k-l \leq p-1$

$$\operatorname{Ker} \pi_* \theta^1_{(k,l)} = \operatorname{Ker} \pi_* \theta_1.$$

Thus, to prove injectivity on H^0 it is enough to prove that θ_1 is injective on global sections. Assume first $(k,l) \neq (p-1, p-1)$. Let $f \in H^0(\overline{\operatorname{Sh}}, \omega(k,l))$ non-zero. It can't be divisible by H_1 since $H^0(\overline{\operatorname{Sh}}, \omega(k-p+1, l-p+1)) = 0$ by Theorem 1.8(2). Therefore $\theta_1(f) \neq 0$ by Proposition 3.9. For (k,l) = (p-1, p-1), note that $H^0(\overline{\operatorname{Sh}}, \mathcal{O}_{\overline{\operatorname{Sh}}})$ has no non-Eisenstein Hecke eigensystems, and H_1 is not non-Eisenstein either, so by the same argument $\theta_{(k,l)}^1$ is injective after localizing at a non-Eisenstein maximal ideal \mathfrak{m} .

- **Remark 3.3.** (1) As opposed to Proposition 3.3 where in some cases $\pi_*\theta_4$ is injective one can't hope to obtain such a result for operators involving $\theta_{1,2,3}$, since these are genuine differential operators on \overline{Sh} . Therefore, one can find non-trivial elements of the kernel by considering sections of $\mathcal{O}_{\overline{Sh}}$ that are *p*th powers.
 - (2) Following the proof of Theorem 3.13 one can see that it also says that $\theta_{(k,l)}^1$ is injective on global sections for sufficiently generic *p*-restricted weights (k, l).

4. Differential operators and Verma modules

In this section we relate some differential operators on the (flag) Shimura variety to maps of Verma modules. In characteristic p this will produce the theta linkage maps. This relation was well-known in characteristic 0 [FC90], and used over \mathbb{Z}_p in the Siegel case in [PT02] and [MT02, §4-5], and [LP18] in the PEL case. Here we emphasize the importance of using crystalline differential operators and the right kind of Verma modules, and we extend the results to include the flag Shimura variety. The main technical novelty is the proof of Theorem 4.6 via the Grothendieck-Messing period map, which allows to deduce many of its properties by working on the flag variety.

We also prove a reduction that occurs over $U \subset \mathcal{F}l$, which allows to characterize all the theta operators on U in terms of roots of GSp_4 . We use this to prove that theta linkage maps are combinations of these basic theta operators, which become highly divisible by Hasse invariants. The methods of this section are quite robust, and most of it can be generalized without much trouble to a more general setting like PEL Shimura varieties, but we've chosen to stick to GSp_4 for conciseness.

4.1. Sheaves of differential operators and Lie algebras. In this section let $G = GSp_4$ over a ring R, which will be either \mathbb{Z}_p or \mathbb{F}_p . We start by reviewing various definitions of differential operators, and the relation to Verma modules in the case of flag varieties.

Definition 4.1. Let \mathfrak{g} be the Lie algebra of G.

- (Universal enveloping algebra) Let $U\mathfrak{g}$ be the *R*-algebra $\bigoplus \mathfrak{g}^{\otimes n}/\langle x \otimes y y \otimes x [x, y] \rangle$ with x, y, running along \mathfrak{g} .
- (Algebra of distributions) Let J be the kernel of the identity $e : \mathcal{O}_G \to R$. Let $U(G) = \bigcup_N \operatorname{Hom}_R(\mathcal{O}_G/J^N, R)$ be the algebra of G distributions over R. The algebra structure is induced by comultiplication on \mathcal{O}_G .
- (Restricted universal enveloping algebra) Over \mathbb{F}_p the Lie algebra has a p operation $x \mapsto x^{[p]}$ corresponding to considering x as a left invariant derivation and composing it p times. The image of the map

$$\operatorname{Sym}^{\bullet}\mathfrak{g}^{(p)} \hookrightarrow U\mathfrak{g}$$

sending X to $X^p - X^{[p]}$ lands in the center of $U\mathfrak{g}$, denote it Z^{Fr} . Define $U^0\mathfrak{g} = U\mathfrak{g} \otimes_{Z^{\mathrm{Fr}}} \mathbb{F}_p$, where the character $Z^{\mathrm{Fr}} \to \mathbb{F}_p$ is induced by $\mathrm{Sym}^{\bullet}\mathfrak{g}^{(p)} \to \mathbb{F}_p$.

The first two are filtered algebras, for $m \ge 0$ we denote $U^{\le m}\mathfrak{g}$ and $U^{\le m}(G)$ the pieces of degree at most m.

There is a map of algebras $U\mathfrak{g} \to U(G)$ induced by the isomorphism $\mathcal{O}_G \otimes \mathcal{O}_G/I^2 \to \mathcal{O}_G/J^2$ sending $x \otimes 1 - 1 \otimes x$ to x. For $R = \mathbb{Z}_p$ this is an injection that becomes an isomorphism over \mathbb{Q}_p , but it is not injective over \mathbb{F}_p . In fact the surjection $U\mathfrak{g} \to U^0\mathfrak{g}$ can be identified with the image of $U\mathfrak{g} \to U(G)$. One can see this explicitly: let $\{H_i : i = 1, \ldots k\}$ be some generators of the Cartan algebra for the diagonal maximal torus \mathfrak{h} , and extend it to a basis $\{H_i, X_j\}$ of \mathfrak{g} . Then $U(G)_{\mathbb{Z}_p}$ is freely generated by elements of the form $\binom{H_i}{k} = \frac{H_i(H_i-1)\dots(H_i-k+1)}{k!}$ and $\frac{X_i^n}{n!}$. Since $X_i^{[p]} = 0$ and $H_i^{[p]} = H_i$ we see that $U\mathfrak{g} \to U(G)$ factors through $U^0\mathfrak{g}$.

Let $T \subset B \subset Q \subset G$ be a choice of maximal torus, Borel, and some parabolic Q.

Definition 4.2. (Verma modules) Let V be an algebraic representation of Q over R. A $(U\mathfrak{g}, Q)$ module is a Q-module over R together with a $U\mathfrak{g}$ action such that the derivative of the Q-action
agrees with the $U\mathfrak{q}$ -action. For the derivative of Q to be defined we assume V is a filtered union of
finite free modules. Similarly, for a (U(G), Q)-module and a $(U^0\mathfrak{g}, Q)$ -module.

- Let $\operatorname{Ver}_Q(V) \coloneqq U\mathfrak{g} \otimes_{U\mathfrak{q}} V$ as a $(U\mathfrak{g}, Q)$ -module, with $U\mathfrak{g}$ acting on the left and Q by the adjoint action on the left, and by its action on V on the right.
- Similarly, we define $\operatorname{Ver}_Q^{PD}(V) \coloneqq U(G) \otimes_{U(Q)} V$ as a (U(G), Q)-module.
- (Baby Verma modules) For $R = \mathbb{F}_p$ let $\operatorname{Ver}^0_Q(V) \coloneqq U^0 \mathfrak{g} \otimes_{U^0 \mathfrak{q}} V$ as a $(U^0 \mathfrak{g}, Q)$ -module.
- For $V \in \operatorname{Rep}(B)$ we can define the variants $\operatorname{Ver}_{P/B}(V) \coloneqq U\mathfrak{p} \otimes_{U\mathfrak{b}} V$ and $\operatorname{Ver}^0_{P/B}(V) \coloneqq U^0\mathfrak{p} \otimes_{U^0\mathfrak{b}} V$.

The first two types have a filtration induced by the one on $U\mathfrak{g}$ or U(G), e.g. $\operatorname{Ver}_Q(V)^{\leq m}$ is generated by simple tensors $x \otimes v$ with $x \in U^{\leq m}\mathfrak{g}$. These filtered pieces are preserved by the action of Q.

36
In characteristic 0 the first two Verma modules are isomorphic, but they are different over \mathbb{Z}_p . For instance $\operatorname{gr}^{\bullet}\operatorname{Ver}_Q(V) \cong \operatorname{Sym}^{\bullet}\mathfrak{g}/\mathfrak{q} \otimes V$ as Q-representations, but the same is not true for $\operatorname{Ver}_Q^{PD}(V)$. In general the graded pieces don't split, except in the case

$$\operatorname{Ver}_{O}^{\leq 1}(1) \cong 1 \oplus \mathfrak{g}/\mathfrak{q}.$$

Definition 4.3. (Sheaves of differential operators) Let Y/S be a smooth map of schemes. Let I be the ideal sheaf for the diagonal $\Delta : Y \to Y \times_S Y$, and P_Y^{crys} its divided power envelope. Let J be the kernel of $P^{\text{crys}} \to \mathcal{O}_Y$. Denote by $J^{[n]}$ the ideal of P_Y^{crys} generated by elements $\prod \gamma_{n_i}(x_i)$ for $x_i \in I$ and $\sum n_i \geq n$.

- (Crystalline differential operators) For an integer $m \ge 0$, let $P_Y^{\text{crys},m} = P^{\text{crys}}/J^{[m+1]}$ as a \mathcal{O}_Y -bimodule via each projection. The sheaf of crystalline differential operators of degree at most m is $D_Y^{\text{crys},m} = \text{Hom}_{\mathcal{O}_Y}(P_Y^{\text{crys},m}, \mathcal{O}_Y)$, where the Hom is taken as left \mathcal{O}_Y -modules. Then $D_Y^{\text{crys}} = \bigcup_{m\ge 0} D_Y^{\text{crys},m}$. More explicitly, D_Y^{crys} is the sheaf of rings generated by \mathcal{O}_Y and $T_{Y/S}$, under the relations AB BA = [A, B], Af = fA + A(f) for all $A, B \in T_{Y/S}$ (seen as derivations), and $f \in \mathcal{O}_Y$.
- (Differential operators) For an integer $m \ge 0$, let $P_Y^m = \mathcal{O}_Y \otimes \mathcal{O}_Y / I^{m+1}$ seen as a \mathcal{O}_Y bimodule. The ring of differential operators of degree at most m is defined as $D_Y^m = \text{Hom}_{\mathcal{O}_Y}(P_Y^m, \mathcal{O}_Y)$, and $D_Y = \bigcup_{m \ge 0} D_Y^m$, where the Hom is as left \mathcal{O}_Y -modules.
- (Log crystalline differential operators) Let $D \hookrightarrow Y$ be a relative Cartier divisor with normal crossings. Define $D_{(Y,D)}^{\text{crys,log}}$ as the sheaf generated by $T_{Y/S}(-\log D)$ and \mathcal{O}_Y together with the relations described above.

Both P_Y and P_Y^{crys} are cofiltered \mathcal{O}_Y bi-modules, and one has $\operatorname{gr}^{\bullet} P_Y^{\text{crys}} = \operatorname{Sym}^{\bullet} \Omega_Y^1$. They also have a coalgebra structure. On P_Y it is induced by the maps $\epsilon : P_Y^{n+m} \to P_Y^n \otimes_{\mathcal{O}_Y} P_Y^m$ of \mathcal{O}_Y -bimodules, defined on local coordinates by

$$a \otimes_R b \mapsto a \otimes_R 1 \otimes_{\mathcal{O}_Y} 1 \otimes_R b.$$

For $x \in \mathcal{O}_Y$ let $\xi_x = x \otimes 1 - 1 \otimes x \in I$, if Y/S is smooth P_Y is locally freely generated by elements like this, using étale coordinates. Then $\epsilon : \prod_i \xi_{x_i} \mapsto \prod_i (\xi_{x_i} \otimes 1 + 1 \otimes \xi_{x_i})$, and one sees that P_Y^{n+m} lands in $P_Y^n \otimes P_Y^m$. Similarly, ϵ can be extended to a map on P_Y^{crys} , which is étale locally freely generated by divided powers of ξ_{x_i} .

Definition 4.4. Let Y/S be a scheme, and $\mathcal{F}_{1,2}$ sheaves of \mathcal{O}_Y -modules. Define the sheaf of (crystalline) differential operators of degree at most m from \mathcal{F}_1 to \mathcal{F}_2 as

DiffOp^{crys,m}(
$$\mathcal{F}_1, \mathcal{F}_2$$
) := Hom _{\mathcal{O}_Y} ($P_{Y/S}^{crys,m} \otimes_{\mathcal{O}_Y} \mathcal{F}_1, \mathcal{F}_2$)
DiffOp^m($\mathcal{F}_1, \mathcal{F}_2$) := Hom _{\mathcal{O}_Y} ($P_{Y/S}^m \otimes_{\mathcal{O}_Y} \mathcal{F}_1, \mathcal{F}_2$),

where in the tensor product both P_Y^m and $P_Y^{\text{crys},m}$ are \mathcal{O}_Y -modules on the right, and the \mathcal{O}_Y -module structure is given by the left \mathcal{O}_Y action on P_Y^m . Then DiffOp^{crys} $(\mathcal{F}_1, \mathcal{F}_2) = \bigcup_m \text{DiffOp}^{\text{crys},m}(\mathcal{F}_1, \mathcal{F}_2)$. There are maps

$$\operatorname{DiffOp}^{\operatorname{crys}}(\mathcal{F}_1, \mathcal{F}_2) \to \operatorname{DiffOp}(\mathcal{F}_1, \mathcal{F}_2) \hookrightarrow \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{F}_1, \mathcal{F}_2), \tag{4.1}$$

the second one defined by precomposing with the (not \mathcal{O} -linear) map $\mathcal{F}_1 \to P_Y^m \otimes \mathcal{F}_1$ given by $v \mapsto 1 \otimes 1 \otimes v$. Given $\phi \in \text{DiffOp}^m(\mathcal{F}_1, \mathcal{F}_2), \psi \in \text{DiffOp}^n(\mathcal{F}_2, \mathcal{F}_3)$ its composition is defined by

$$\psi \circ \phi \coloneqq P_Y^{n+m} \otimes_{\mathcal{O}_Y} \mathcal{F}_1 \xrightarrow{\epsilon \otimes \mathrm{id}} P^n \otimes_{\mathcal{O}_Y} P^m \otimes \mathcal{F}_1 \xrightarrow{\mathrm{id} \otimes \phi} P^n \otimes_{\mathcal{O}_Y} \mathcal{F}_2 \xrightarrow{\psi} \mathcal{F}_3$$

and similarly for crystalline differential operators. This makes (4.1) compatible with composition on $\operatorname{Hom}_{\mathcal{O}_S}(\mathcal{F}_1, \mathcal{F}_2) \otimes \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{F}_2, \mathcal{F}_3) \to \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{F}_1, \mathcal{F}_3).$

Notation 2. For a left \mathcal{O} -module \mathcal{F} and \mathcal{G} a bimodule we will always use $\mathcal{G} \otimes \mathcal{F}$ for the tensor product with the right \mathcal{O} structure of \mathcal{G} , and $\mathcal{F} \otimes \mathcal{G}$ when the tensor product is taken with respect to the left \mathcal{O} structure on \mathcal{G} .

Definition 4.5. (Stratifications) For a quasi-coherent sheaf E on a scheme Y, a connection on E can be seen as a \mathcal{O} -linear isomorphism $P_Y^1 \otimes E \cong E \otimes P_Y^1$ such that the associated projection map $E \to E$ is the identity, and the connection is flat if the isomorphism satisfies some cocycle conditions [BO78, §2]. An extension to a collection of compatible isomorphisms of left \mathcal{O}_Y -modules $(P_Y^n \otimes E \cong E \otimes P_Y^n)_{n\geq 1}$ or to isomorphisms $(P_Y^{\operatorname{crys},n} \otimes E \cong E \otimes P_Y^{\operatorname{crys},n})_{n\geq 1}$ is said to be a stratification on E, respectively a PD stratification on E. A flat connection always extends uniquely to a PD stratification, while in characteristic p a connection rarely extends to a stratification.

In characteristic p crystalline differential operators interact nicely with the Frobenius.

Proposition 4.1. (Frobenius differentials) Let Y/S a smooth map of schemes over \mathbb{F}_p and E a \mathcal{O}_Y -module. Let $F: Y \to Y^{(p)}$ be the relative Frobenius. Then for all $m \ge 1$ there is a natural map

$$F^*F_*E \to P_{Y/S}^{\mathrm{crys},m} \otimes E \tag{4.2}$$

which is injective for sufficiently large m. For $\mathcal{V}_{1,2}$ sheaves of \mathcal{O}_Y -modules define the sheaf of Frobenius differential operators as

$$D^{[p]}(\mathcal{V}_1, \mathcal{V}_2) \coloneqq \operatorname{Hom}_{\mathcal{O}_V}(F^*F_*\mathcal{V}_1, \mathcal{V}_2)$$

The composition of two maps $f: F^*F_*\mathcal{V}_1 \to \mathcal{V}_2$ and $g: F^*F_*\mathcal{V}_2 \to \mathcal{V}_3$ is defined as $F^*F_*\mathcal{V}_1 \xrightarrow{\text{idosoid}} F^*F_*F_*\mathcal{V}_1 \xrightarrow{F^*F_*f} F^*F_*\mathcal{V}_2 \xrightarrow{g} \mathcal{V}_3$, where $s: 1 \to F_*F^*$ is the unit of the adjunction. Then (4.2) induces a surjection $\text{DiffOp}^{\text{crys}}(\mathcal{V}_1, \mathcal{V}_2) \to D^{[p]}(\mathcal{V}_1, \mathcal{V}_2)$ compatible with composition which can be identified with the image of $\text{DiffOp}^{\text{crys}}(\mathcal{V}_1, \mathcal{V}_2) \to D^{[fOp}(\mathcal{V}_1, \mathcal{V}_2)$.

Proof. The associated Frobenius linear map $E \to P_{Y/S}^{\operatorname{crys},m} \otimes E$ is given by $v \mapsto 1 \otimes 1 \otimes v$, which is Frobenius linear since $(f \otimes 1 - 1 \otimes f)^p = 0$ on $P_{Y/S}^{\operatorname{crys}}$ for all $f \in \mathcal{O}_Y$. The rest of the statements can easily be checked on étale local coordinates.

Thus, the map from crystalline differential operators to differential operators is far from injective in characteristic p. The above also says that crystalline differential operators are always Frobenius linear in characteristic p.

4.1.1. Differential operators on flag varieties. Let $Q \subset G$ be a split reductive group and a parabolic subgroup, defined over \mathbb{Z}_p . Differential operators on the flag variety G/Q are intimately related to Verma modules.

Lemma 4.2. There is an equivalence of categories between finite free algebraic representations of Qover \mathbb{Z}_p and G-equivariant coherent $\mathcal{O}_{G/Q}$ modules. The functor in one direction $F_{G/Q}$: $\operatorname{Rep}(Q) \to \operatorname{Coh}_G(\mathcal{O}_{G/Q})$ is given by $V \mapsto (V \times G)/Q$ with Q acting as $h \cdot (v, g) = (hv, gh^{-1})$. The functor in the other direction is taking the fiber at $\infty = [Q] \in G/Q$. Both are exact and tensor functors.

This equivalence readily extends to infinite dimensional representations like $\operatorname{Ver}_Q(V)$ which have an exhaustive filtration by finite free pieces. It also naturally extends to intermediate flag varieties like P/B.

Proposition 4.3. Let $V \in \operatorname{Rep}_R(Q)$. Under the equivalence of Lemma 4.2 we have the following identifications

(1) There exist canonical isomorphisms

$$F_{G/Q}(\operatorname{Ver}_Q(V)) = D_{G/Q}^{\operatorname{crys}} \otimes F_{G/Q}(V) \text{ and } F_{G/Q}(\operatorname{Ver}_Q^{PD}(V)) = D_{G/Q} \otimes F_{G/Q}(V).$$

(2) If V is defined over \mathbb{F}_p , then $F_{G/Q}(\operatorname{Ver}^0_Q(V))^{\vee} = F^*F_*F_{G/Q}(V)^{\vee}$, where $F : G/Q \to (G/Q)^{(p)}$ is the Frobenius. The surjection

$$D_{G/Q}^{\operatorname{crys}} \otimes F(V) = \operatorname{DiffOp}_{G/Q}^{\operatorname{crys}}(F(V)^{\vee}, \mathcal{O}) \twoheadrightarrow D_{G/Q}^{[p]}(F(V)^{\vee}, \mathcal{O})$$

from Proposition 4.1 corresponds under the equivalence of Lemma 1.2 to the surjection

$$\operatorname{Ver}_Q(V) \twoheadrightarrow \operatorname{Ver}_Q^0(V)$$
 (4.3)

of Q-modules induced by $U\mathfrak{g} \to U^0\mathfrak{g}$.

(3) Let $V, W \in \operatorname{Rep}_{\mathbb{F}_p}(Q)$, and $f : \operatorname{Ver}_Q(V^{\vee}) \to \operatorname{Ver}_Q(W^{\vee})$ a map of $(U\mathfrak{g}, Q)$ -modules. It descends to a map $\overline{f} : \operatorname{Ver}_Q^0(V^{\vee}) \to \operatorname{Ver}_Q^0(W^{\vee})$. By 1) the dual of f induces a map $\phi(f) \in \operatorname{DiffOp}^{\operatorname{crys}}(F_{G/Q}(W), F_{G/Q}(V))$ and the dual of \overline{f} induces a map $g : F^*F_*F_{G/Q}(W) \to F^*F_*F_{G/Q}(V)$ by part 2). Then $g = F^*F_*\phi(f)$ where $\phi(f) : F_{G/Q}(W) \to F_{G/Q}(V)$ is the map of sheaves associated to $\phi(f)$. In particular

$$\operatorname{Ker} F^* F_* \phi(f) = F_{G/Q} (\operatorname{coker} \overline{f})^{\vee}.$$

Proof. Part 1) is a classical result, which can be proved using that $T_{G/Q} = F_{G/Q}(\mathfrak{g}/\mathfrak{q})$, and computing the fiber at ∞ of $D_{G/Q}^{\text{crys}} \otimes F_{G/Q}(V)$ on the coordinates of an open Bruhat cell. Part 2) was originally proven in [Haa87], and one can easily deduce part 3) from the compatibility of crystalline and Frobenius differentials.

If V is a G-representation then $F_{G/Q}(V)$ is canonically isomorphic to $V \otimes_R \mathcal{O}_{G/Q}$ by sending (v, g) to (gv, \overline{g}) . Thus it is endowed with a trivial connection which extends to the dual of a stratification

$$\nabla: D_{G/Q} \otimes F_{G/Q}(V) \cong F_{G/Q}(V) \otimes D_{G/Q}, \tag{4.4}$$

and similarly for D^{crys} .

Corollary 4.4. (Tensor identity) Let $V \in \operatorname{Rep}_R(G)$. Then there are canonical isomorphisms $\phi_V : \operatorname{Ver}_Q(V) \cong V \otimes_R \operatorname{Ver}_Q(1)$ and $\phi_V^{PD} : \operatorname{Ver}_Q^{PD}(V) \cong V \otimes_R \operatorname{Ver}_Q^{PD}(1)$ fitting in the diagram

where the vertical map (and the horizontal isomorphism) is the isomorphism of Proposition 4.3(1). The analogous statement holds for ϕ_V .

Proof. Since the vertical and diagonal maps are isomorphisms we define ϕ_V and ϕ_V^{PD} via the equivalence of Lemma 4.2.

By tracking down the equivalences ϕ_V is given by $x \otimes v \mapsto xv \otimes 1 + v \otimes x$ for $x \in \mathfrak{g}$ and $1 \otimes v \mapsto 1 \otimes v$.

4.2. From Verma modules to differential operators on the (flag) Shimura variety. Now let $G = \operatorname{GSp}_4$ over \mathbb{Z}_p and P be the Siegel parabolic. Recall that there are functors $F_P : \operatorname{Rep}_R(P) \to \operatorname{Coh}(\operatorname{Sh}_R)$ and $F_B : \operatorname{Rep}_R(B) \to \operatorname{Coh}(\mathcal{F}l_R)$ defining automorphic vector bundles on the (flag) Shimura variety. We will use the notation $F_Q(V) = \mathcal{V}$ when it is clear what F_Q is.

Lemma 4.5. There are canonical Hecke equivariant away from p isomorphisms $F_P(\mathfrak{g}/\mathfrak{p}) = T_{\mathrm{Sh}}$ and $F_B(\mathfrak{g}/\mathfrak{b}) = T_{\mathcal{F}l}$ over \mathbb{Z}_p . The exact sequence $0 \to \mathfrak{p}/\mathfrak{b} \to \mathfrak{g}/\mathfrak{b} \to \mathfrak{g}/\mathfrak{p} \to 0$ of *B*-modules is sent under these isomorphisms to the natural exact sequence

$$0 \to T_{\mathcal{F}l/Sh} \to T_{\mathcal{F}l} \to \pi^* T_{Sh} \to 0.$$

Proof. We have that $\mathfrak{g}/\mathfrak{p} \cong W(2,0,0)^{\vee}$ as *P*-representations, so the first canonical isomorphism is induced by the Kodaira-Spencer isomorphism. We construct the isomorphism $F_B(\mathfrak{g}/\mathfrak{b}) = T_{\mathcal{F}l}$. Fix a local section $\phi \in I_B$ which trivializes $\mathcal{L} \subset \omega \subset H$. Under this trivialization $\mathcal{F}l$ is isomorphic to $\mathbb{P}^1_{\mathrm{Sh}}$, and hence it defines a splitting of $0 \to T_{\mathcal{F}l/\mathrm{Sh}} \to T_{\mathcal{F}l} \to \pi^*T_{\mathrm{Sh}} \to 0$. Then $T_{\mathcal{F}l/\mathrm{Sh}} = \mathcal{L}(1,-1)$ naturally corresponds to $\mathfrak{p}/\mathfrak{b} \cong (-1,1) \hookrightarrow \mathfrak{g}/\mathfrak{b}$. Identify the sub-vector space of $\mathfrak{g}/\mathfrak{b}$ generated by $X_{-\beta}, X_{-\alpha-\beta}, X_{-2\alpha-\beta}$ with $\mathfrak{u}_P^- \cong \mathrm{Hom}_G(L, L^{\vee})$ in the obvious way. Then $(\phi, X) \in F_B(\mathfrak{g}/\mathfrak{b})$ for $X \in \mathfrak{u}_P^-$ is sent to the class of π^*T_{Sh} defined via Kodaira-Spencer through the map

$$\omega \xrightarrow{\phi} L \otimes \mathcal{O} \xrightarrow{X} L^{\vee} \otimes \mathcal{O} \xrightarrow{\phi^{-1}} \omega^{\vee},$$

and then embedded into $T_{\mathcal{F}l}$ by the splitting induced by ϕ . We can check this construction is independent of the choice of ϕ .

We remark that after fixing the Borel and working with de Rham cohomology (as opposed to homology), the convention that $F_B(\lambda) = \mathcal{L}(s_0\lambda)$ comes from wanting $F_B(\mathfrak{g}/\mathfrak{b}) = T_{\mathcal{F}l}$, while still having $\pi_*\mathcal{L}(\lambda) = \omega(\lambda)$. The next theorem is the main technical tool of this section, which directly relates differential operators on the Shimura variety with Verma modules.

Theorem 4.6. Let $Q \in \{B, P\}$, $V \in \operatorname{Rep}_{\mathbb{Z}_p}(Q)$. Then there are canonical Hecke equivariant away from p isomorphisms

$$e_V: F_Q(\operatorname{Ver}_Q(V)) \cong D_{\mathcal{F}l_Q/\mathbb{Z}_p}^{\operatorname{crys}} \otimes \mathcal{V}$$

and

$$e_V^{P/B}: F_B(\operatorname{Ver}_{P/B}(V)) \cong D_{\mathcal{F}l/\operatorname{Sh}}^{\operatorname{crys}} \otimes \mathcal{V},$$

where $\mathcal{F}l_Q$ is $\mathcal{F}l$ if Q = B and Sh if Q = P. If $V \in \operatorname{Rep}_{\mathbb{F}_p}(B)$ has a lift to $\operatorname{Rep}_{\mathbb{Z}_p}(B)$ there are canonical isomorphisms

$$e_V^0: F_B(\operatorname{Ver}^0_{P/B}(V)) \cong (F^*F_*\mathcal{V}^{\vee})^{\vee},$$

where $F : \mathcal{F}l \to \mathcal{F}l^{(p)}$ is the relative Frobenius with respect to \overline{Sh} . They satisfy the following properties

- (1) The map $\operatorname{Ver}_{P/B}(V) \to \operatorname{Ver}_B(V)$ is identified with the map $D_{\mathcal{F}l/\operatorname{Sh}}^{\operatorname{crys}} \otimes \mathcal{V} \to D_{\mathcal{F}l}^{\operatorname{crys}} \otimes \mathcal{V}$ induced by $T_{\mathcal{F}l/\operatorname{Sh}} \to T_{\mathcal{F}l/\mathbb{Z}_p}$.
- (2) e_V^0 is uniquely determined by the commutative square

$$F_B(\operatorname{Ver}_{P/B}(V)) \xrightarrow{F_B(\pi)} F_B(\operatorname{Ver}_{P/B}^0(V))$$
$$\downarrow e_V^{P/B} \qquad \qquad \qquad \downarrow e_V^0$$
$$D_{\mathcal{F}l/Sh}^{\operatorname{crys}} \otimes \mathcal{V} \xrightarrow{\Pi} (F^*F_*\mathcal{V}^{\vee})^{\vee}$$

where $\pi : \operatorname{Ver}_{P/B}(V) \to \operatorname{Ver}_{P/B}^{0}(V)$ is the map (4.3), and Π is defined in Proposition 4.1.

- (3) If $\operatorname{Ver}_Q(V) \to \operatorname{Ver}_Q(W)$ is induced by a map $f: V \to W$ of Q-representations, then the associated map $D^{\operatorname{crys}} \otimes \mathcal{V} \to D^{\operatorname{crys}} \otimes \mathcal{W}$ is $\operatorname{id} \otimes F_Q(f)$.
- (4) The map $\operatorname{Ver}(\operatorname{Ver}(V)) \to \operatorname{Ver}(V)$ given by $x \otimes (y \otimes v) \mapsto xy \otimes v$ is sent to $D^{\operatorname{crys}} \otimes (D^{\operatorname{crys}} \otimes \mathcal{V}) \to D^{\operatorname{crys}} \otimes \mathcal{V}$ induced by composition of differential operators.
- (5) On graded pieces of $\operatorname{Ver}_Q(V)$ and $D^{\operatorname{crys}} \otimes \mathcal{V}$, e_V induces the map $F(\operatorname{gr}^n \operatorname{Ver}(V)) \cong F(\operatorname{Sym}^n \mathfrak{g}/\mathfrak{b} \otimes V) \xrightarrow{\operatorname{Sym}^{\bullet} s \otimes id} \operatorname{Sym}^n T \otimes \mathcal{V} \cong \operatorname{gr}^n D^{\operatorname{crys}} \otimes \mathcal{V}$, where s is defined in Lemma 4.5.

(6) For $V \in \operatorname{Rep}_{\mathbb{Z}_p}(G)$, consider the isomorphism $\phi_V : \operatorname{Ver}(V) \cong V \otimes_{\mathbb{Z}_p} \operatorname{Ver}(1)$ from Corollary 4.4. It fits into the following commutative diagram



where the diagonal map is the isomorphism corresponding the dual of the PD stratification given by the Gauss-Manin connection on \mathcal{V}^{\vee} .

(7) All the isomorphisms extend to (sub)canonical extensions on toroidal compactifications by replacing D^{crys} with D^{crys,log}. In 2) one just considers D^{crys}_{Fl^{tor}/Sh^{tor}}. They satisfy all the previous properties.

The proof of Theorem 4.6 will take the rest of the subsection. Here we sketch the strategy of the proof. First we construct local isomorphisms on divided power formal completions (over \mathbb{Z}_p) of points by using the Grothendieck-Messing period map to a flag variety, which allows to transport Proposition 4.3 to the Shimura variety. Then we construct candidate maps e_V and show that they agree with the local ones on divided power formal completions, so that we can check all of its properties there. We could only manage to construct e_V in an indirect way. First reduce to the case of $\operatorname{Ver}^{\leq 1}(V)$ by an inductive process relating the maps $\operatorname{Ver}^{\leq 1}(\operatorname{Ver}^{\leq n}(V)) \to \operatorname{Ver}^{\leq n+1}(V)$ to the algebra structure on D^{crys} . If V is a G-representation we can naturally construct $e_V^{\leq 1}$ via property 6). For the rest of V we prove that they are subquotients of G-representations, so that we can indirectly construct e_V by exactness of F and $\operatorname{Ver}_Q(-)$. A priori the inductive process is not well-defined, but compatibility with the local isomorphisms ensures it is. We remark that in order to be able to check all their properties on PD formal completions one has to work over \mathbb{Z}_p .

Remark 4.1. It is not true that $F_P(\operatorname{Ver}_P^{PD}(V)) \cong D_{\operatorname{Sh}_{\mathbb{Z}_p}} \otimes F_P(V)$ for all V in a functorial way. If we had such isomorphisms extending e_V along the embeddings $\operatorname{Ver}_P(V) \hookrightarrow \operatorname{Ver}_P^{PD}(V)$ and $D_{\operatorname{Sh}}^{\operatorname{crys}} \hookrightarrow D_{\operatorname{Sh}}$ for both V the trivial and the standard representation, using the diagram of part 6) together with Corollary 4.4 would imply that the Gauss-Manin connection on H over $\operatorname{Sh}_{\mathbb{Z}_p}$ extends to a stratification $(P_{\operatorname{Sh}}^n \otimes H \cong H \otimes P_{\operatorname{Sh}}^n)_n$. Over the ordinary locus $W \subset \operatorname{Sh}_{\mathbb{Z}_p}$ there is a lift of the Frobenius isogeny $F: A \to A'$, and by functoriality the existing PD stratification respects this morphism. Since the cokernel of $P_{\operatorname{Sh}}^n \to P_{\operatorname{Sh}}^{\operatorname{crys},n}$ is p-torsion, and all the sheaves are locally free, we see that the potential stratification on W would have to respect F too. Therefore it would also be compatible with Frobenius on $\overline{\operatorname{Sh}}^{\operatorname{ord}}$, and hence in all of $\overline{\operatorname{Sh}}$ being an open condition. Such an extension of ∇ to a stratification on $\overline{\operatorname{Sh}}$ compatible with the Frobenius is well-known not to exist, see [BO78, Ex 2.18].

4.2.1. The Grothendieck-Messing period map. Let $W = W(\overline{\mathbb{F}}_p)$, $\mathcal{F}l/\mathrm{Sh}/W$ be the integral model of the (flag) Shimura variety, and $y \in \mathcal{F}l(\overline{\mathbb{F}}_p)$ mapping to $x \in \mathrm{Sh}(\overline{\mathbb{F}}_p)$. Let Sh_x^{\sharp} be the divided power formal completion of Sh at x considered as a formal scheme, and \mathcal{F}_y^{\sharp} be the divided power

42

completion of $\mathcal{O}_{\mathcal{F}l,y}^{\wedge}$ along the ideal generated by \mathfrak{m}_x . There is a map $\mathcal{F}_y^{\sharp} \to \operatorname{Sh}_x^{\sharp}$ whose fibers are formal discs without divided powers. Similarly, let G/Q_{∞}^{\wedge} be the formal completion of G/Q over W at $\infty = [Q] \in (G/Q)(\overline{\mathbb{F}}_p).$

Definition 4.6. Let (R, I, γ_i) be a ring with divided powers over \mathbb{Z}_p .

- Define the sheaf of PD differential forms $\Omega^1_{(R,I)}$ as the quotient of the sheaf of differentials
- $\Omega^1_{R/\mathbb{Z}_p}$ by imposing the relations $d\gamma_n(x) = \gamma_{n-1}(x)dx$ for all $x \in I$ and $n \ge 1$. Let $I_\Delta \subset D_{I_\Delta}(R \otimes R)$ be the ideal of the diagonal inside its divided power envelope. Then the sheaf of divided power differentials can be identified with $I_{\Delta}/(I_{\Delta}^{[2]} + K^{[2]})$, where K is generated by elements of the form $\gamma_m(x) \otimes 1 - 1 \otimes \gamma_m(x)$ for $x \in I$, $m \ge 1$ [Sta18, Tag 07HT]. • Define $P_{(R,I)}^{\operatorname{crys},n} \coloneqq D_{I_{\Delta}}(R \otimes R)/(I_{\Delta}^{[n+1]} + K^{[n+1]})$ with the natural bimodule structure and coalgebra structure, and let $D_{(R,I)}^{\operatorname{crys},n}$ be its dual. Then $\operatorname{gr}^{\circ}P_{(R,I)}^{\operatorname{crys}} = \operatorname{Sym}^{\circ}\Omega_{(R,I)}^{1}$.
- If $f: (R, I) \to (S, J)$ is a map of PD pairs, the usual differential extends to a map $df: f^*\Omega^1_{(R,I)} \to \Omega^1_{(S,J)}$, and a map $f^*: f^*P^{\text{crys}}_{(R,I)} \to P^{\text{crys}}_{(S,J)}$ which on graded pieces is induced by df.
- For $\mathfrak{X} = \operatorname{Spf}(\varprojlim R/I_n)$ a formal scheme with an implicit divided power ideal $J \subset I_0$, let $D_{\mathfrak{X}}^{\operatorname{crys}} \coloneqq \varprojlim D_{(R/I_n, J+I_n)}^{\operatorname{crys}}$. We will only use this for $\operatorname{Sh}_x^{\sharp}$ and $\mathcal{F}l_y^{\sharp}$.

We claim that for $i : \operatorname{Sh}_x^{\sharp} \to \operatorname{Sh}$ and $j : \mathcal{F}l_y^{\sharp} \to \mathcal{F}l$, $i^* D_{\operatorname{Sh}}^{\operatorname{crys}} \cong D_{\operatorname{Sh}_x^{\sharp}}^{\operatorname{crys}}$ and $j^* D_{\mathcal{F}l}^{\operatorname{crys}} \cong D_{\mathcal{F}l_y^{\sharp}}^{\operatorname{crys}}$. Globalizing the constructions above there are natural maps $i^* P_{\operatorname{Sh}}^{\operatorname{crys}} \to P_{\operatorname{Sh}_x^{\sharp}}^{\operatorname{crys}}$ and $j^* P_{\mathcal{F}l}^{\operatorname{crys}} \to P_{\mathcal{F}l_y^{\sharp}}^{\operatorname{crys}}$. On graded pieces they are induced by the maps

$$i^*\Omega^1_{\mathrm{Sh}} \cong \Omega^1_{\mathrm{Sh}_x^{\sharp}}, \ j^*\Omega^1_{\mathcal{F}l} \cong \Omega^1_{\mathcal{F}l_y^{\sharp}}$$

$$(4.5)$$

which are isomorphisms, this can be checked étale locally.

Let $\operatorname{Art}_{\overline{\mathbb{F}}_p}^{\sharp}$ be the category of local Artin algebras (R, \mathfrak{m}) such that $R/\mathfrak{m} \cong \overline{\mathbb{F}}_p$, together with an ideal $J \subset \mathfrak{m}$ which has divided powers. By Serre-Tate theory we can interpret the formal completion $\operatorname{Sh}_x^{\wedge}$ as a moduli of deformations of p-divisible groups up to isomorphism. Let p-Div^G be the category of p-divisible groups with a G structure (a symplectic pairing). Then for $(R, \mathfrak{m}, J) \in \operatorname{Art}_{\overline{\mathbb{F}}_n}^{\sharp}$ their functor of points as formal schemes are

$$\operatorname{Sh}_{x}^{\wedge}(R) = \{ \mathbb{X} \in \operatorname{p-Div}_{R}^{G} \text{ and } \alpha : \mathbb{X} \times R/\mathfrak{m} \cong A[p^{\infty}]_{x} \} / \cong$$

$$(4.6)$$

$$\mathcal{F}l_{y}^{\wedge}(R) = \{ \mathbb{X} \in \text{p-Div}_{R}^{G}, L \subset \omega_{\mathbb{X}} \text{ and } \alpha : (\mathbb{X}, L) \times R/\mathfrak{m} \cong (A[p^{\infty}]_{y}, \mathcal{L}_{y}) \} / \cong .$$

$$(4.7)$$

The same formula holds for $\operatorname{Sh}_{x}^{\sharp}(R)$ and $\mathcal{F}l_{y}^{\sharp}(R)$, with the convention that we only consider points compatible with the divided power structure on both sides (by the universal property of the divided power envelope). This will be enough for us, since we can define maps of formal schemes like $\operatorname{Sh}_x^{\sharp} \to G/P_{\infty}^{\wedge}$ and $\mathcal{F}l_y^{\sharp} \to G/B_{\infty}^{\wedge}$ on points belonging to $\operatorname{Art}_{\mathbb{F}_n}^{\sharp}$. Now let R be a ring such that p

is locally nilpotent, $S \to R$ a divided power thickening, and \mathbb{X}_0 a p-divisible group over R. The Dieudonne crystal of \mathbb{X}_0 is a crystal $D(\mathbb{X}_0)$ over the crystalline site of R. We denote $D(\mathbb{X}_0)(S \to R)$ its value on a PD thickening $S \to R$ (by definition it is a finite locally free S-module), which is canonically isomorphic to $D(\mathbb{X})(S)$ for any lift \mathbb{X} of \mathbb{X}_0 .

For the rest of this subsection fix $y \in \mathcal{F}l(\overline{\mathbb{F}}_p)$, and let $x \in \mathrm{Sh}(\overline{\mathbb{F}}_p)$ be its projection. Fix also a symplectic trivialization $\phi : D(A[p^{\infty}]_y)(W) \cong W^4$ that sends $\mathcal{L}_y \subset \omega_{A_x} \subset H^1_{\mathrm{dR}}(A_x) \cong D(A[p^{\infty}]_x) \otimes \overline{\mathbb{F}}_p$ to $\infty \in G/B(\overline{\mathbb{F}}_p)$.

Proposition 4.7. Let $y \in \mathcal{F}l(\overline{\mathbb{F}}_p)$, $x \in Sh(\overline{\mathbb{F}}_p)$ and ϕ as above. There are maps π_P , π_B of formal schemes over SpfW (depending on ϕ) fitting in a cartesian diagram

$$\begin{array}{ccc} \mathcal{F}l_y^{\sharp} & \stackrel{\pi_B}{\longrightarrow} & G/B_{\infty}^{\wedge} \\ & & & \downarrow \\ \pi & & & \downarrow \\ \mathrm{Sh}_x^{\sharp} & \stackrel{\pi_P}{\longrightarrow} & G/P_{\infty}^{\wedge}. \end{array}$$

Furthermore, the P-torsor $\pi_P^*[G_P^{\wedge} \to G/P_{\infty}^{\wedge}]$ is isomorphic to the pullback of I_P to $\operatorname{Sh}_x^{\sharp}$. Similarly, $\pi_B^*[G_B^{\wedge} \to G/B_{\infty}^{\wedge}]$ is the pullback of I_B to \mathcal{Fl}_y^{\sharp} . Therefore for $V \in \operatorname{Rep}_{\mathbb{Z}_p}(P), W \in \operatorname{Rep}_{\mathbb{Z}_p}(B)$ there are isomorphisms (depending on ϕ)

 $i^*F_P(V) \cong \pi_P^*F_{G/P}(V), \quad j^*F_B(W) \cong \pi_B^*F_{G/B}(W)$

where $F_{G/P}$, $F_{G/B}$ are the functors of Lemma 4.2 pulled back to the formal completions.

Proof. Let $R \in \operatorname{Art}_{\overline{\mathbb{F}}_p}^{\sharp}$, and $(\mathbb{X}, \alpha) \in \operatorname{Sh}_x^{\sharp}(R)$. There is a map of PD thickenings $(W, \overline{\mathbb{F}}_p) \to (R \to R/\mathfrak{m})$ corresponding to the natural section of $R/p \to R/\mathfrak{m}$. Since the Dieudonne module is a crystal

$$D(\mathbb{X})(R) \cong D(\mathbb{X} \times R/\mathfrak{m})(W, \overline{\mathbb{F}}_p) \otimes_W R \xrightarrow{\alpha} D(A[p^{\infty}]_x)(W) \otimes_W R,$$
(4.8)

where the last map is an isomorphism. Define $\pi_P(G, \alpha)$ as the Lagrangian $\omega_{\mathbb{X}} \subset D(\mathbb{X})(R) \cong W^4 \otimes R$, where the isomorphism is the composition of (4.8) and ϕ . By the choice of ϕ this defines a point of $G/P_{\infty}^{\wedge}(R)$. Similarly, let $(\mathbb{X}, L, \alpha) \in \mathcal{F}l_y^{\sharp}(R)$. Then $\pi_B(\mathbb{X}, L, \alpha)$ is defined by $(L \subset \omega_{\mathbb{X}} \subset W^4 \otimes R)$, where the second inclusion is as before. It is clear that the diagram is cartesian, using (4.7).

The fiber of $[G_P^{\wedge} \to G/P_{\infty}^{\wedge}]$ over some $M \subset R^4$ is naturally identified the set of *P*-isomorphisms of pairs $(M \subset R^4) \cong (R^2 \subset R^4)$, and π_P is defined precisely by $A \mapsto \omega_A \subset H^1_{dR}(A) \cong R^4$. From this it follows that $\pi_P^*[G_{\infty}^{\wedge} \to G/P_{\infty}^{\wedge}] \cong i^*I_P$, which depends on ϕ since the trivialization $H^1_{dR}(A) \cong R^4$ depends on it. The same analysis holds for π_B .

Let y and x points as before, consider $i: \operatorname{Sh}_x^{\sharp} \to \operatorname{Sh}$ and $j: \mathcal{F}l_y^{\sharp} \to \mathcal{F}l$. By Proposition 4.7 there is an isomorphism $\pi_P^*\Omega^1_{G/P_{\infty}^{\wedge}} \cong \pi_P^*F_{G/P}(W(2,0)) \cong i^*\operatorname{Sym}^2\omega$, depending on ϕ .

Proposition 4.8. The Grothendieck-Messing period maps fit in the commutative diagrams



where the vertical isomorphism comes from the identification of torsors in Proposition 4.7, and

In particular $d\pi_P$ and $d\pi_B$ are isomorphisms, and $d\pi_B$ can be identified with the pullback of the canonical isomorphism $F_B(\mathfrak{g}/\mathfrak{b})^{\vee} \cong \Omega^1_{\mathcal{F}l}$ of Lemma 4.5.

Proof. The commutativity of the first diagram is essentially formal. Let $\{e_1, e_2, e_3, e_4\}$ be a symplectic basis of $D(A[p^{\infty}]_x)$ given by ϕ . Let $\omega_{A_R} = \langle \tilde{e}_1, \tilde{e}_2 \rangle \coloneqq \langle e_1 + \tau_{12}e_3 + \tau_{11}e_4, e_2 + \tau_{22}e_3 + \tau_{12}e_4 \rangle$, where τ_{ij} are functions in $\mathcal{O}_{\mathrm{Sh}_x^{\sharp}}$. Let $\{x_{ij}\}$ be the coordinates on G/P that parametrize the Lagrangian subspace in a similar manner, then π_P sends x_{ij} to τ_{ij} . We claim that the vertical isomorphism sends dx_{ij} to $\tilde{e}_i \tilde{e}_j$. One can check this fiberwise, by identifying $(\mathfrak{g}/\mathfrak{p})^{\vee}$ with $\mathrm{Hom}_G(L^{\vee}, L) = \mathrm{Sym}^2 L$, and keeping track of the isomorphism $\pi_P^*[G_P^{\wedge} \to G/P_{\infty}^{\wedge}] \cong i^*I_P$. By Lemma 1.9 $\nabla(e_i) = 0$, so that $\mathrm{ks}(\tilde{e}_i \tilde{e}_j) = d\tau_{ij}$ as desired. The existence of the second diagram follows from the cartesian diagram (4.7). Since by construction the pullback of the isomorphism $F_B(\mathfrak{g}/\mathfrak{b})^{\vee} \cong \Omega_{\mathcal{F}l}^1$ fits in the same diagram as $d\pi_B$ it means they agree on $\mathcal{F}l_y^{\sharp}$, and in particular $d\pi_B$ is an isomorphism.

Let $f: X \to Y$ be a map of schemes (or PD formal schemes) such that $df: D_X^{\text{crys}} \to f^* D_Y^{\text{crys}}$ is an isomorphism, and let V a quasicoherent sheaf on Y. Define the isomorphism

$$df_V: D_X^{\text{crys}} \otimes f^*V \to f^*(D_V^{\text{crys}} \otimes V) \tag{4.9}$$

as follows. Locally, let $f: A \to C$ be a map of rings, B an A-bimodule, V a left A-module, D a Cbimodule and $\psi: C \otimes_A B \cong D$ an isomorphism of left C modules which respects the right A module structure on both sides. Then ψ extends to an isomorphism $\varphi: C \otimes_A (B \otimes_A V) \to D \otimes_C (C \otimes_A V)$ given by $c \otimes b \otimes v \mapsto \psi(c \otimes b) \otimes (1 \otimes v)$. Define df_V to be the inverse of φ . On graded pieces df_V is given by $\text{Sym}^{\bullet} df$ on the left and the identity on f^*V . One can check the following compatibility with composition: with f and V as above the diagram

$$D_X^{\operatorname{crys}} \otimes f^*V \xrightarrow{df_V} f^*(D_Y^{\operatorname{crys}} \otimes V)$$

$$\mu \otimes \operatorname{id} \uparrow f^*(\mu \otimes \operatorname{id}) \uparrow$$

$$D_X^{\operatorname{crys}} \otimes D_X^{\operatorname{crys}} \otimes f^*V \xrightarrow{\operatorname{id} \otimes df_V} D_X^{\operatorname{crys}} \otimes f^*(D_Y^{\operatorname{crys}} \otimes V) \xrightarrow{df_{D_Y^{\operatorname{crys}} \otimes V}} f^*(D_Y^{\operatorname{crys}} \otimes D_Y^{\operatorname{crys}} \otimes V)$$

commutes. The next result defines the canonical isomorphisms of Theorem 4.6 on divided power formal completions of points.

Proposition 4.9. Let $y \in \mathcal{F}l(\overline{\mathbb{F}}_p)$ and $x \in Sh(\overline{\mathbb{F}}_p)$ its image, define $\pi_{B/P}$ for some fixed choice of ϕ . Let $V \in \operatorname{Rep}_{\mathbb{Z}_p}(P), W \in \operatorname{Rep}_{\mathbb{Z}_p}(B)$. Define the following composition of isomorphisms,

$$e_{V,x}: i^*(D_{\mathrm{Sh}}^{\mathrm{crys}} \otimes F_P(V)) \cong D_{\mathrm{Sh}_x^{\sharp}}^{\mathrm{crys}} \otimes \pi_P^* F_{G/P}(V) \xrightarrow{d\pi_{P,V}} \pi_P^*(D_{G/P_{\infty}}^{\mathrm{crys}} \otimes F_{G/P}(V)) \cong i^* F_P(\mathrm{Ver}_P(V))$$

$$e_{W,y}: j^*(D_{\mathcal{F}l}^{\operatorname{crys}} \otimes F_B(W)) \cong D_{\mathcal{F}l_y}^{\operatorname{crys}} \otimes \pi_B^* F_{G/B}(W) \xrightarrow{d\pi_{B,W}} \pi_B^*(D_{G/B_{\infty}}^{\operatorname{crys}} \otimes F_{G/B}(W)) \cong j^* F_B(\operatorname{Ver}_B(W)),$$
$$e_{P/B,W,y}: j^*(D_{\mathcal{F}l/Sh}^{\operatorname{crys}} \otimes F_B(W)) \cong D_{\mathcal{F}l_y}^{\operatorname{crys}} \otimes \pi_B^* F_{G/B}(W) \to j^* F_B(\operatorname{Ver}_{P/B}(W))$$

where the isomorphisms on the sides come from Proposition 4.7 and Proposition 4.3, and the ones in the middle are defined by (4.9). Then $e_{V,x}$, $e_{V,y}$, $e_{P/B,W,y}$ do not depend on the choice of ϕ . If $W \in \operatorname{Rep}_{\mathbb{F}_n}(B)$ there is also a canonical isomorphism

$$e^{0}_{W,y}: j^{*}F_{B}(\operatorname{Ver}^{0}_{P/B}(W)) \cong j^{*}(F^{*}F_{*}F_{B}(W^{\vee}))^{\vee},$$

where $F : \mathcal{F}l \to \mathcal{F}l^{(p)}$ is the relative Frobenius with respect to \overline{Sh} . Furthermore, these local isomorphisms satisfy the same properties 1) - 6) as in Theorem 4.6.

Proof. By the definition of D^{crys} on divided power formal schemes we get maps $d\pi_P : D_{\text{Sh}_x^{\sharp}}^{\text{crys}} \rightarrow \pi_P^* D_{G/P_{\infty}^{\wedge}}^{\text{crys}}$ and $d\pi_B : D_{\mathcal{Fl}_y^{\sharp}}^{\text{crys}} \rightarrow \pi_B^* D_{G/B_{\infty}^{\wedge}}^{\text{crys}}$ which on graded sheaves are induced by the differential map on tangent bundles. Thus by Proposition 4.8 these maps are isomorphisms. Moreover, they respect the natural $\mathcal{O}_{G/Q_{\infty}^{\wedge}}$ -bimodule structure on both sides. We then define the maps $d\pi_{P,V}$ as in (4.9). Using the first diagram of Proposition 4.8 we see that on graded pieces $e_{V,x}$ is induced by the symmetric powers of the isomorphism on Lemma 4.5. The map $e_{P/B,y}$ is induced by the cartesian diagram on Proposition 4.7 and Proposition 4.3.

Let $W \in \operatorname{Rep}_{\mathbb{F}_p}(B)$. There is an isomorphism $j^*F_B(\operatorname{Ver}_{P/B}^0(W)) \cong (G^*G_*j^*F_B(W^{\vee}))^{\vee}$, where in the right-hand side G is the relative Frobenius with respect to $\mathcal{F}l_y^{\sharp} \to \overline{\operatorname{Sh}}_x^{\sharp}$, which follows from the relative version of the isomorphism on the flag variety Proposition 4.3(2), and the cartesian diagram in Proposition 4.7 that identifies both relative Frobenius. To obtain $e_{W,y}^0$ we claim that there is a canonical isomorphism $G^*G_*j^*E \cong j^*F^*F_*E$ for E a vector bundle. After localizing on $\mathcal{F}l$ we may assume that $E = \mathcal{O}_{\mathcal{F}l}$. Write $\mathcal{F}l \to \overline{\operatorname{Sh}}$ locally as $A \to B$, $\mathcal{F}l_y^{\sharp} \to \overline{\operatorname{Sh}}_x^{\sharp}$ as $\tilde{A} \to \tilde{B}$ and $j: B \to \tilde{B}$. By smoothness we may assume that $\tilde{A} = \overline{\mathbb{F}}_p[[\frac{x_i^n}{n!}]]$ and $\tilde{B} = \overline{\mathbb{F}}_p[[\frac{x_i^n}{n!}, y_j]]$. The sheaf $G^*G_*\mathcal{O}_{\mathcal{F}l_x^{\sharp}}$. is finite locally free, since one can check that G is flat and of finite presentation. There is a map $\phi: j^*F^*F_*\mathcal{O}_{\mathcal{F}l} \to G^*G_*\mathcal{O}_{\mathcal{F}l_y^{\sharp}}$ as follows. Locally it is given by $\tilde{B} \otimes_B (B \otimes_{B \otimes_{A,F}A} B) \to \tilde{B} \otimes_{\tilde{B} \otimes_{\tilde{A},G}\tilde{A}} \tilde{B}$ sending $1 \otimes x \otimes y \mapsto j(x) \otimes j(y)$. Since both the domain and target are finite locally free of the same rank it is enough to check whether the determinant of ϕ is a unit in \tilde{B} . The map $\tilde{B} \to (\tilde{B}/\mathfrak{m}_{\tilde{A}}\tilde{B})$ sends non-units to non-units, so we can check it after replacing \tilde{B} by $\tilde{B}/\mathfrak{m}_A \tilde{B}$, and B by $B/\mathfrak{m}_A B$. Relabelling everything, $\tilde{B} \to \tilde{A} = A = \kappa(x)$ becomes the formal completion of B at $\kappa(y)$, F is the Frobenius for $B/\kappa(y)$ and G the Frobenius for $\tilde{B}/\kappa(y)$. We can work étale locally since the formal completions don't change and the Frobenius pushforward commutes with étale base change. Thus we can reduce to affine space, where it can checked by hand.

Properties 1) – 2) follow by construction of $e_{P/B,V}$ and e_V^0 , and the second diagram in Proposition 4.8. Property 3) follows directly from the associated statement on the flag variety, and $d\pi$ being functorial. Property 4) follows from the statement on the flag variety and diagram (4.2.1). Part 5) follows from Proposition 4.8 and the statements in Definition 4.6. For 6) it is sufficient to prove it restricted to the degree at most 1 filtered piece, since the Gauss-Manin connection extends uniquely to a PD-stratification. The commutativity of the diagram is equivalent to ∇ on Sh_x^{\sharp} being the pullback connection along π_P of the trivial connection on $F_{G/P}(V) \cong V \otimes \mathcal{O}_{G/P_{\infty}^{\wedge}}$ over G/P_{∞}^{\wedge} , since the tensor identity is precisely the one inducing the trivial connection on $F_{G/P}(V)$. We can reduce to the case when V is the standard representation from the definition of ∇ , where it is equivalent to the compatibility between the Gauss-Manin and Grothendieck-Messing connections in Lemma 1.9.

Remark 4.2. The differentials $d\pi$ also extend to a map from $D_{\mathrm{Sh}^{\sharp}} \to \pi^* D_{G/P_{\infty}^{\wedge}}$, but since usual differential operators over \mathbb{Z}_p are not generated by the tangent bundle this map is not automatically an isomorphism. One can see this as the local obstruction for why $F_P(\mathrm{Ver}_P^{PD}(1))$ is not equal to D_{Sh} .

4.2.2. Definition of the canonical isomorphisms. First we construct the canonical isomorphism e_V for $\operatorname{Ver}_{Q}^{\leq 1}(V)$ for any $V \in \operatorname{Rep}(Q)$.

Lemma 4.10. Every finite free \mathbb{Z}_p representation of $Q \in \{P, B\}$ is a subquotient of (the restriction to Q of) a finite free G-representation.

Proof. Let V be a finite free \mathbb{Z}_p representation of Q. Let $\{v_i \mid i = 1, 2, ..., n\}$ be a basis of V, and $\{v_i^*\}$ its dual basis. The map $V \to \mathcal{O}_Q^n$ sending $v \to (g \mapsto \langle gv, v_i^* \rangle)$ defines an injective map of Q-representations, where Q acts on \mathcal{O}_Q by $g \cdot f(x) = f(g^{-1}x)$. The restriction map $\phi : \mathcal{O}_G^n \to \mathcal{O}_Q^n$ is a surjection of Q-representations. By [Ser68, §1.5 Prop 2] there exists a finitely generated (hence free) $W \in \operatorname{Rep}_{\mathbb{Z}_p}(Q)$ which contains some choice of elements $w_i \in \phi^{-1}(v_i)$, and which is contained in $\phi^{-1}(V)$, so that W maps surjectively to V. Similarly, there exists $M \in \operatorname{Rep}_{\mathbb{Z}_p}(G)$ finite free containing W and contained in \mathcal{O}_G^n . Then $W \subset M$ gives a presentation of V as a subquotient. \Box

We prove a small lemma that will allow us to check all the properties of the canonical isomorphisms on PD formal completions. To apply it in our case we will use that one can cover $\operatorname{Sh}_{\mathbb{Z}_p}$ by connected affine open subsets that have at least one $\overline{\mathbb{F}}_p$ -point.

Lemma 4.11. Let R/\mathbb{Z}_p be a smooth connected algebra of finite type having at least one $\overline{\mathbb{F}}_p$ -point. Let M, N be R-modules which are the sum of a finite free part and finitely many summands of the form R/p^n , and let $\psi : M \to N$ be a map between them. Then ψ is 0 if and only if it is 0 after base change to every PD formal completion of $\overline{\mathbb{F}}_p$ -points.

Proof. The map ψ decomposes into maps between free modules and maps of the form $R \to R/p^n$ or $R/p^m \to R/p^n$. For the former the the lemma follows since under the assumptions on R the composition $R \to R_{\mathfrak{m}} \to R_{\mathfrak{m}}^{\wedge} \to R_{\mathfrak{m}}^{\sharp}$ is injective for any $\overline{\mathbb{F}}_p$ -point \mathfrak{m} . The last injection can be checked using étale coordinates, and will be of the form $\mathbb{Z}_p[[x_i]] \hookrightarrow \mathbb{Z}_p[[\frac{x_i^n}{n!}]]$. For the latter $R/p^n \to R_{\mathfrak{m}}^{\sharp}/p^n$ is not necessarily injective for a particular \mathfrak{m} , but any element in the kernel will be contained in \mathfrak{m} . Therefore an element x that is killed for all maximal ideals \mathfrak{m} above p will be contained in pR, since $R \otimes \mathbb{F}_p$ is a Jacobson ring. Writing x = py, since $R_{\mathfrak{m}}^{\sharp}$ is p-torsion free, we see that n = 1 or yis also in the kernel for every \mathfrak{m} , so that by induction x = 0.

We define $e_V^{\leq 1}$ using its relation to the Gauss-Manin connection prescribed by Theorem 4.6(6). We remark that the functors F_Q restricted to $\mathcal{F}l_y^{\sharp}$ are still exact over $\operatorname{Rep}_{\mathbb{Z}_p}(Q)$, since divided power formal completions are \mathbb{Z}_p -flat.

Proposition 4.12. Define $e_1^{\leq 1} : F_Q(\operatorname{Ver}^{\leq 1}(1)) \cong F_Q(1 \oplus \mathfrak{g}/\mathfrak{q}) \cong \mathcal{O} \oplus T_{\mathcal{F}l_{\operatorname{Sh},Q}} \cong D_{\mathcal{F}l_{\operatorname{Sh},Q}}^{\leq 1}$ where the isomorphism of the middle is given by Lemma 4.5.

(1) For $V \in \operatorname{Rep}_{\mathbb{Z}_n}(G)$ define $e_V^{\leq 1}$ by the commutative diagram



where ϕ_V is the tensor identity of Corollary 4.4, and ∇ is the dual of the Gauss-Manin connection of Proposition 1.5. The definition makes sense since the horizontal and diagonal maps are isomorphisms.

(2) Let $i: V \hookrightarrow W$ be a Q-equivariant embedding, where $W \in \operatorname{Rep}_{\mathbb{Z}_p}(G)$ and $V \in \operatorname{Rep}_{\mathbb{Z}_p}(Q)$. Then $e_W^{\leq 1}$ sends $F_Q(\operatorname{Ver}_Q^{\leq 1}(V))$ to $D^{\leq 1} \otimes F_Q(V) \subseteq D^{\leq 1} \otimes F_Q(W)$. We define $e_V^{\leq 1}$ as the restriction of $e_W^{\leq 1}$, by exactness of F_Q and $\operatorname{Ver}_Q(-)$. It is an isomorphism independent of the embedding i.

- (3) Let $\pi : W \to V$ be a surjection in $\operatorname{Rep}_{\mathbb{Z}_p}(Q)$ with W a submodule of an element of $\operatorname{Rep}_{\mathbb{Z}_p}(G)$. Then $e_W^{\leq 1}$ from point (2) sends $F_Q(\operatorname{Ver}_Q^{\leq 1}(\operatorname{Ker}\pi))$ to $D^{\leq 1} \otimes F_Q(\operatorname{Ker}\pi)$, so that we can define $e_V^{\leq 1}$ as the induced map. It is an isomorphism independent of the surjection.
- (4) Let $V \in \operatorname{Rep}_{\mathbb{Z}_p}(Q)$. By Lemma 4.10 it is a subquotient of a G-representation, so we define $e_V^{\leq 1}$ via the previous two points. It is an isomorphism independent of the presentation as a subquotient.

Proof. We prove 2) and 3). The independence of embedding/quotient follows by comparing any two with their sum, since the construction of $e_V^{\leq 1}$ on 1) is clearly functorial. By Lemma 4.11 we can check well-definedness on PD formal completions of $\overline{\mathbb{F}}_p$ points, e.g. for 2) we have to prove that $F(\operatorname{Ver}_Q^{\leq 1}(V)) \to D^{\leq 1} \otimes F_Q(W/V)$ is 0. The key fact is that by Proposition 4.9 the diagram in 1) defining $e_W^{\leq 1}$ agrees with the local map $e_{W,x}^{\leq 1}$ at every PD formal completions. Then by naturality of the local isomorphisms parts 2) and 3) hold on PD formal completions. This shows that the maps $e^{\leq 1}$ are well-defined, and they are isomorphisms since one can construct an inverse by the same procedure.

We define $e_V^{\leq n}$ by induction on n, by leveraging the algebra structure on the sheaf of differential operators.

Proposition 4.13. Let $V \in \operatorname{Rep}_{\mathbb{Z}_p}(Q)$. For n = 1 $e_V^{\leq n}$ is defined in Proposition 4.12. For $n \geq 2$ the surjection $\operatorname{Ver}_Q^{\leq 1}(\operatorname{Ver}_Q^{\leq n-1}V) \twoheadrightarrow \operatorname{Ver}_Q^{\leq n}(V)$ induces by Proposition 4.12 a surjection

$$\mu: D^{\leq 1} \otimes F_Q(\operatorname{Ver}_Q^{\leq n-1}(V)) \twoheadrightarrow F_Q(\operatorname{Ver}_Q^{\leq n}(V)).$$

Let $v \in F_Q(\operatorname{Ver}^{\leq n}(V))$ be a local section on a small enough open so that it lies in the image of μ , and let $g = \sum D_i \otimes v_i$ be any element in $\mu^{-1}(v)$. Define $e_V^{\leq n}(v) \coloneqq \sum D_i \cdot e_V^{\leq n-1}(v_i)$, where $\cdot : D^{\leq 1} \otimes D^{\operatorname{crys}, \leq n-1} \otimes F(V) \to D^{\operatorname{crys}, \leq n} \otimes F(V)$ is given by composition of differential operators. Then $e_V^{\leq n}$ is a well-defined isomorphism independent of the choice of g.

Proof. We can check well-definedness on PD formal completions, since for R/\mathbb{Z}_p smooth connected, the map to a PD formal completion of a $\overline{\mathbb{F}}_p$ -point is injective, and all the maps are between vector bundles. It follows from property 4) of Proposition 4.9 that e_V and $e_{V,x}$ agree on PD-formal completions of points, so in particular e_V is well-defined and it is independent of the choice of g. By its inductive construction the map e_V is surjective. Let N be its kernel, and suppose it is non-zero. For \mathfrak{m} a $\overline{\mathbb{F}}_p$ -point, the base change of N to $R^{\wedge}_{\mathfrak{m}}[1/p] = R^{\sharp}_{\mathfrak{m}}[1/p]$ is non-zero since N is torsion-free, and $R_{\mathfrak{m}} \to R^{\wedge}_{\mathfrak{m}}$ is faithfully flat. Since $N_{R^{\sharp}_{\mathfrak{m}}} = 0$ by Proposition 4.9 we conclude that N = 0, and the map is an isomorphism.

We put all the ingredients together to prove Theorem 4.6.

Proof of Theorem 4.6. The isomorphisms $e_{Q,V}$ for $Q \in \{P,B\}$ are defined in Proposition 4.13, and they agree on PD formal completions with the ones defined on Proposition 4.9, by all the functoriality and compatibility properties that the local isomorphisms satisfy. For $e_V^{P/B}$, let E be the cokernel of $D_{\mathcal{F}l/Sh}^{crys} \otimes F(V) \to D_{\mathcal{F}l}^{crys} \otimes F(V)$. We claim that the composition $F_B(\operatorname{Ver}_{P/B}(V)) \to$ $F_B(\operatorname{Ver}_B(V)) \xrightarrow{e_V} D_{\mathcal{F}l}^{crys} \otimes F(V) \to E$ is 0. By Lemma 4.11 it is enough to check it on PD formal completions, where it holds by Proposition 4.9 (1). By exactness of F_B this defines the isomorphism $e_V^{P/B}$, and it automatically satisfies property 1). To define e_V^0 we use the same strategy with the square of property 2): now using that we can check if a map of vector bundles over $\mathcal{F}l_{\mathbb{F}_p}$ is zero on closed points, so in particular on PD completions of points. This automatically proves property 2). To check 3) - 5) we can use Lemma 4.11, by considering the difference of the expected map and the actual map (to be strict we consider each of these maps on $\operatorname{Ver}^{\leq n}$), and then it holds over PD formal completions by Proposition 4.9. Part 6) holds by definition and the fact that the Gauss-Manin connection extends uniquely to a PD stratification. The isomorphisms $e^{\leq 1}$ are Hecke equivariant since the Gauss-Manin connection and the isomorphisms of Lemma 4.5 are, so the maps $e^{\leq n}$ are also Hecke equivariant by their inductive definition.

Finally, we construct e_V on toroidal compactifications. For $V \in \operatorname{Rep}_{\mathbb{Z}_p}(G)$, $e_V^{\leq 1}$ is constructed as in Proposition 4.12 using the (sub)canonical extension of Gauss-Manin to a log connection. To construct $e_V^{\leq 1}$ for general $V \in \operatorname{Rep}_{\mathbb{Z}_p}(Q)$ we use the same procedure and the fact that one can check whether a map of sheaves as in Lemma 4.11 vanishes on the interior $\operatorname{Sh} \hookrightarrow \operatorname{Sh}^{\operatorname{tor}}$. It is easy to see that they are isomorphisms, since the ones for $V \in \operatorname{Rep}_{\mathbb{Z}_p}(G)$ are. For $e^{\leq n}$ we follow the procedure of Proposition 4.13, which is well-defined since it can be checked on the interior. From that construction we immediately see that e_V is a surjection, and it is injective since it is so in the interior. One can see that all the required properties can be checked on an open dense subset of $\operatorname{Sh}^{\operatorname{tor}}$, so they follow from the properties on Sh. \Box

Remark 4.3. Going through the proof of Theorem 4.6 one could also have proved it for some $V \in \operatorname{Rep}_{\mathbb{F}_p}(B)$ that don't lift to \mathbb{Z}_p . For instance $V = L(\lambda)$ embeds into $V(\lambda)$, so by devissage one could define e_V . We won't use this in this article, so we decide not to prove it full generality.

4.3. Reduction to the open strata. We now work on the special fiber. Over $U \subset \mathcal{F}l$ the symplectic flag $\mathcal{L} \subset \omega \subset H$ splits by definition, so we can define a *T*-torsor I_T parametrizing trivializations of *H* respecting its canonical splitting. There is a *T*-equivariant embedding $I_T \subset I_B$ and I_B is the pushout of I_T along $T \to B$. The torsor I_T defines a functor $F_T : \operatorname{Rep}_{\mathbb{F}_n}(T) \to \operatorname{Coh}(U)$,

and by the above remark the square

$$\operatorname{Rep}_{\mathbb{F}_p}(B) \xrightarrow{F_B} \operatorname{Coh}(\mathcal{F}l_{\mathbb{F}_p})$$

$$\downarrow^{\operatorname{res}} \qquad \qquad \qquad \downarrow^{\operatorname{res}}$$

$$\operatorname{Rep}_{\mathbb{F}_n}(T) \xrightarrow{F_T} \operatorname{Coh}(U)$$

commutes. In particular for $V \in \operatorname{Rep}_{\mathbb{F}_p}(B)$ lifting to $\operatorname{Rep}_{\mathbb{Z}_p}(B)$, $F_T(\operatorname{Ver}_B(V)) \cong D_U^{\operatorname{crys}} \otimes \mathcal{V}$ canonically. Let $\tau : U_{\operatorname{Ig}} \to U$ be the base change along $\operatorname{Ig} \to \overline{\operatorname{Sh}}^{\operatorname{ord}}$. Then in fact $\tau^* I_T \cong U_{\operatorname{Ig}} \times T$ with the isomorphism being canonical up to the action of $\operatorname{GL}_2(\mathbb{F}_p)$. We can construct a global section ϕ_0 of $\tau^* I_T$ as follows. Let $\{e_1, e_2\}$ be a canonical basis of $\omega_{\operatorname{Ig}}$, and $\{e_3, e_4\}$ the corresponding basis of $\omega^{\vee} \hookrightarrow H$, so that $\{e_i\}$ is a symplectic basis of H over Ig . Then ϕ_0 is given by the symplectic basis

$$\{f_i\} \coloneqq \{\frac{1}{T - T^p}(e_1 + Te_2), e_1 + T^p e_2, \frac{1}{T^p - T}(e_3 - Te_4), e_3 - T^p e_4\}$$
(4.10)

of $H_{U_{Ig}}$, which is ordered according to the canonical flag of $L \oplus L^{\vee}$. In this basis we can give a very explicit description of the canonical isomorphisms of Theorem 4.6 over U. Let $\{x_i\}$ be an ordered basis of $\mathfrak{g}/\mathfrak{b}$, and let $D_i \in \tau^* T_U \cong T_{U_{Ig}} \cong \tau^* F_T(\mathfrak{g}/\mathfrak{b})$ correspond to (ϕ_0, x_i) . Explicitly we have $D_{-\beta} = \mathrm{ks}(f_2^2)^{\vee}, D_{-\alpha-\beta} = \mathrm{ks}(f_1f_2)^{\vee}, D_{-2\alpha-\beta} = \mathrm{ks}(f_1^2)^{\vee}, D_{-\alpha} = -(T^p - T)^2 \frac{\partial}{\partial T} = f_1^{-1}f_2.$

Proposition 4.14. Let $V \in \operatorname{Rep}_{\mathbb{F}_p}(B)$ lifting to \mathbb{Z}_p . Then $\tau^* e_V : \tau^* F_T(\operatorname{Ver}_B(V)) \cong \tau^*(D_U^{\operatorname{crys}} \otimes \mathcal{V})$ is given by

$$(\phi_0, \prod_{i=1}^4 x_i^{n_i} \otimes v) \mapsto D_1^{n_1} \circ \ldots \circ D_4^{n_4} \otimes (\phi_0, v)$$

for $v \in V$, and extended \mathcal{O} -linearly.

Proof. We can reduce to the case of $e_V^{\leq 1}$ by induction on the degree, using Theorem 4.6(4) on the compatibility of composition of differential operators and the surjection $\operatorname{Ver}_B(\operatorname{Ver}_B(V)) \twoheadrightarrow$ $\operatorname{Ver}_B(V)$. Now assume that $V \in \operatorname{Rep}_{\mathbb{F}_p}(G)$, to prove the proposition we will use the commutativity of the diagram from Theorem 4.6(6)



Here ∇ is the stratification associated to the Gauss-Manin connection on \mathcal{V}^{\vee} . Recall $\psi_V : \operatorname{Ver}_B^{\leq 1}(V) \cong V \otimes \operatorname{Ver}^{\leq 1}(1)$ is given by $x \otimes v \mapsto xv \otimes 1 + v \otimes x$ for $x \in \mathfrak{g}$, and $1 \otimes v \mapsto v \otimes 1$. Let $\{u_i\}$ be a basis of V. Then ψ_V^{\vee} is given by $(x_j \otimes u_i)^{\vee} \mapsto u_i^{\vee} \otimes x_j^{\vee}$ and $(1 \otimes u_i)^{\vee} \mapsto u_i^{\vee} \otimes 1 - \sum_{x_j u_k = \lambda_{jk} u_i} \lambda_{jk} u_k^{\vee} \otimes x_j^{\vee} = u_i^{\vee} \otimes 1 + \sum_j x_j u_i^{\vee} \otimes x_j^{\vee}$. From the construction of the Gauss-Manin connection in Proposition 1.5

we see that the proposition we want to prove is equivalent to having

$$\tilde{\nabla}_{\phi_0} = \sum_j x_j \otimes D_j^{\vee} \in \mathfrak{g} \otimes \Omega^1_{U_{\mathrm{Ig}}}.$$

Recall that $\tilde{\nabla}_{\phi_0}$ is defined by transporting the Gauss-Manin connection on H to $L \oplus L^{\vee}$ via ϕ_0 , so it reduces to a computation of ∇ on $H_{U_{\text{Ig}}}$ with respect to the basis $\{f_i\}$ in (4.10). The key information is that $V \circ \nabla(e_{1,2}) = 0$ by functoriality and $\nabla(e_{3,4}) = 0$ since $V \circ \nabla(e_{3,4}) = 0$ and $\langle \nabla e_{1,2}, e_{3,4} \rangle + \langle e_{1,2}, \nabla e_{3,4} \rangle = 0$. For instance

$$\nabla(f_2) = f_3 \otimes \operatorname{ks}(f_2^2)^{\vee} + f_4 \otimes \operatorname{ks}(f_1 f_2)^{\vee} = x_{-\beta} f_2 \otimes D_{-\beta}^{\vee} + x_{-\alpha-\beta} f_2 \otimes D_{-\alpha-\beta}^{\vee},$$
$$\nabla(f_4) = \nabla(e_3) - T^p \nabla(e_4) = 0 = \sum_{\gamma \in \Phi^-} x_{\gamma} f_4 \otimes D_{\gamma}^{\vee}$$

as desired. Finally, since any $V \in \operatorname{Rep}_{\mathbb{Z}_p}(B)$ is a subquotient of a *G*-representation by Lemma 4.10, we can easily deduce the proposition for all $\operatorname{Ver}_B^{\leq 1}(V)$.

Remark 4.4. Since e_V is independent of the order of the basis $\{x_i\}$ this shows in particular that the $D_i \in \tau^* T_U$ corresponding to $\mathfrak{g}/\mathfrak{p}$ commute among themselves. One could have also checked it on Serre-Tate coordinates, since the coordinates on Ig and on a Serre-Tate disk agree up to the $\operatorname{GL}_2(\mathbb{F}_p)$ action.

Fix an ordered basis $\{x_i\}$ of $\mathfrak{g}/\mathfrak{b}$ as before. Then for $V \in \operatorname{Rep}_{\mathbb{F}_p}(B)$ lifting to $\operatorname{Rep}_{\mathbb{Z}_p}(B)$ there is an isomorphism $i_V : \operatorname{Ver}_B(V) \cong V \otimes \operatorname{Sym}^{\bullet}\mathfrak{g}/\mathfrak{b}$ as *T*-modules, given by $\prod x_i^{n_i} \otimes_{U\mathfrak{b}} v \mapsto v \otimes \prod x_i^{n_i}$. Let $i: U \to \mathcal{F}l$ be the natural embedding. The following commutative diagram defines a canonical isomorphism β_V

Note that together β_1 and β_V define a PD stratification on \mathcal{V}^{\vee} . Let $\nabla_{T,V}$ denote the underlying connection.

Corollary 4.15. Over U_{Ig} the isomorphism β_V sends $v \otimes \prod D_i^{n_i}$ to $D_1^{n_1} \circ \ldots D_4^{n_4} \otimes v$. In particular, under the trivialization of $\tau^* F_T(V)$ the underlying connection $\nabla_{T,V}$ is the trivial connection.

Now we can prove the main theorem of the section, which produces differential operators both on U and $\mathcal{F}l$ out of maps of Verma modules.

Theorem 4.16. Let $R \in \{\mathbb{Z}_p, \mathbb{F}_p\}$, $V_{1,2} \in \operatorname{Rep}_R(P)$, and $W_{1,2} \in \operatorname{Rep}_R(B)$ (lifting to \mathbb{Z}_p if $R = \mathbb{F}_p$) then there are functorial embeddings

$$\Phi_P : \operatorname{Hom}_{(U\mathfrak{g},P)_R}(\operatorname{Ver}_P(V_1), \operatorname{Ver}_P(V_2)) \hookrightarrow \operatorname{DiffOp}_{\operatorname{Sh}_R}^{\operatorname{crys}}(\mathcal{V}_2^{\vee}, \mathcal{V}_1^{\vee}),$$

$$\Phi_B : \operatorname{Hom}_{(U\mathfrak{g},B)_R}(\operatorname{Ver}_B(W_1), \operatorname{Ver}_B(W_2)) \hookrightarrow \operatorname{DiffOp}_{\mathcal{Fl}_R}^{\operatorname{crys}}(\mathcal{W}_2^{\vee}, \mathcal{W}_1^{\vee}),$$

52

 $\Phi_{P/B} : \operatorname{Hom}_{(U\mathfrak{p},B)_R}(\operatorname{Ver}_{P/B}(W_1), \operatorname{Ver}_{P/B}(W_2)) \hookrightarrow \operatorname{DiffOp}_{\mathcal{F}l/\operatorname{Sh}_R}^{\operatorname{crys}}(\mathcal{W}_2^{\vee}, \mathcal{W}_1^{\vee}).$ If $W_i \in \operatorname{Rep}_{\mathbb{F}_p}(B)$ there are also functorial embeddings

 $\Phi_{P/B}^{Fr}: \operatorname{Hom}_{(U^0\mathfrak{p},B)}(\operatorname{Ver}^0_{P/B}(V_1), \operatorname{Ver}^0_{P/B}(V_2)) \hookrightarrow D^{[p]}_{\mathcal{F}l/\overline{\operatorname{Sh}}}(\mathcal{V}_2^{\vee}, \mathcal{V}_1^{\vee}),$

and a functor Φ_T fitting in the diagram

All the functors extend to functors $\Phi^{\text{can,sub}}$ on (sub)canonical extensions on toroidal compactifications, by using log crystalline differential operators. All the differential operators produced in this way are Hecke equivariant away from p. Moreover, they satisfy the following properties.

(1) For $f \in \operatorname{Hom}_{(U\mathfrak{p},B)}(\operatorname{Ver}_{P/B}(V_1), \operatorname{Ver}_{P/B}(V_2))$, let \overline{f} be the induced map on baby Verma modules. Then $\Phi_{P/B}(f): \mathcal{V}_2^{\vee} \to \mathcal{V}_1^{\vee}$ is Frobenius linear with respect to $\overline{\operatorname{Sh}}$, and it satisfies

$$F^*F_*\Phi_{P/B}(f) = F_B(\overline{f}^\vee)$$

under the canonical isomorphism $e_{P/B}^0$, where $F : \mathcal{F}l \to \mathcal{F}l^{(p)}$ is the relative Frobenius with respect to \overline{Sh} .

- (2) For $\Phi_B, \Phi_{P,B}$ and $\Phi_{P/B}^{Fr}$ composition as maps on $(U\mathfrak{g}, Q)$ -modules on the left matches with composition of differential operators on the right.
- (3) For Φ_T , in the case that W_1 and W_2 are characters, after choosing an ordered basis $\{x_i\}$ for $\mathfrak{g}/\mathfrak{b}$, elements of $\operatorname{Hom}_T(W_1, \operatorname{Ver}_B(W_2))$ are given by certain elements $\sum x_i^{n_i}$ of $U\mathfrak{u}_B^-$. The composition of $f_1 = \sum x_i^{n_i} \in \operatorname{Hom}_T(W_1, \operatorname{Ver}_B(W_2)), f_2 = \sum x_i^{m_i} \in \operatorname{Hom}_T(W_2, \operatorname{Ver}_B(W_3))$ is defined as $f_2 \circ f_1 \coloneqq (\sum x_i^{n_i})(\sum x_i^{m_i})$. This agrees with composition of differential operators on U under Φ_T .

Proof. Note that $\operatorname{Hom}_{(U_{\mathfrak{g}},Q)}(\operatorname{Ver}_Q(V_1), \operatorname{Ver}_Q(V_2)) = \operatorname{Hom}_Q(V_1, \operatorname{Ver}_Q(V_2))$ and similarly for the Ver_B^0 . Applying F_P , F_B and F_T respectively, and then dualizing gives the desired differential operators as in Definition 4.3, by Theorem 4.6. The maps are embeddings since on divided power formal completions of points there is an isomorphism between crystalline differential operators on the (flag) Shimura variety and the ones on the flag variety, which are equivalent to maps of Verma modules by Proposition 4.3. In characteristic p one should just consider the (p-1)-th formal neighbourhood of a point. The functors extend to toroidal compactifications since Theorem 4.6 does, and they are Hecke equivariant away from p. Properties 1)-3 can be checked on the interior, being closed conditions.

Part 1) follows from Proposition 4.1 and Theorem 4.6(2), by checking that the two maps are the same on relative formal completions of points. For part 2) given two Q-equivariant maps

 $f_i: V_i \to \operatorname{Ver}_Q(V_{i+1})$ its composition is given by $f_2 \circ f_1: V_1 \xrightarrow{f_1} \operatorname{Ver}_Q(V_2) \xrightarrow{\operatorname{Ver}(f_2)} \operatorname{Ver}_Q(\operatorname{Ver}_Q(V_3)) \xrightarrow{j} \operatorname{Ver}_Q(V_3)$, where j is induced from multiplication on $U\mathfrak{g}$. By part 3) and 4) of Theorem 4.6 after applying F_Q and the canonical isomorphisms the previous map is identified with $F(V_1) \xrightarrow{\Phi_Q(f_1)^{\vee}} D^{\operatorname{crys}} \otimes F(V_2) \xrightarrow{\operatorname{id} \otimes \Phi_P(f_2)^{\vee}} D^{\operatorname{crys}} \otimes D^{\operatorname{crys}} \otimes F(V_3) \to D^{\operatorname{crys}} \otimes F(V_3)$, where the last arrow is induced by composition. By taking duals this is precisely how composition is defined for elements of DiffOp^{crys}.

For part 3), from the proof of part 2) it is enough to prove that for a *T*-equivariant map f: $\lambda_1 \to \operatorname{Ver}_B(\lambda_2)$ the map $g: \operatorname{Ver}_B(\lambda_1) \to \operatorname{Ver}_B(\operatorname{Ver}_B(\lambda_2))$ defined by $\prod x_i^{n_i} \otimes \lambda_1 \mapsto \prod x_i^{n_i} \otimes f(\lambda_1)$ is identified with $D^{\operatorname{crys}} \otimes \mathcal{L}(s_0\lambda_1) \xrightarrow{\operatorname{id} \otimes F(f)} D^{\operatorname{crys}} \otimes D^{\operatorname{crys}} \otimes \mathcal{L}(s_0\lambda_2)$ under F_T and the canonical isomorphisms from Theorem 4.6. Let $W \in \operatorname{Rep}_{\mathbb{F}_p}(B)$ lifting to \mathbb{Z}_p . The basis $\{x_i\}$ of $\mathfrak{g}/\mathfrak{b}$, defines a splitting $i_W: \operatorname{Ver}_B(W) \cong \operatorname{Sym}^{\bullet}\mathfrak{g}/\mathfrak{b} \otimes W$ as *T*-modules as before, which induces an isomorphism $\alpha_W: \operatorname{Sym}^{\bullet} T_U \otimes \mathcal{W} \cong F_T(\operatorname{Ver}(W))$. Consider the following diagram

$$F_{T}(\operatorname{Ver}(\lambda_{1})) \xrightarrow{F_{T}(g)} F_{T}(\operatorname{Ver}_{B}(\operatorname{Ver}_{B}(\lambda_{2})))$$

$$\downarrow^{\alpha_{\lambda_{1}}} \qquad \qquad \downarrow^{\alpha_{\operatorname{Ver}(\lambda_{2})}}$$

$$\operatorname{Sym}^{\bullet} T_{U} \otimes \mathcal{L}(s_{0}\lambda_{1}) \xrightarrow{\operatorname{id} \otimes F(f)} \operatorname{Sym}^{\bullet} T_{U} \otimes F(\operatorname{Ver}_{B}(\lambda_{2}))$$

$$\downarrow^{\beta_{\lambda_{1}}} \qquad \qquad \downarrow^{\beta_{\operatorname{Ver}_{B}(\lambda_{2})}}$$

$$D_{U}^{\operatorname{crys}} \otimes \mathcal{L}(s_{0}\lambda_{2}) \xrightarrow{\operatorname{id} \otimes F(f)} D_{U}^{\operatorname{crys}} \otimes F(\operatorname{Ver}_{B}(\lambda_{2})).$$

The top square commutes by the definition of α_W and g. The bottom square commutes using the description of β_W of Corollary 4.15 (the commutativity descends along $U_{\text{Ig}} \to U$). The composition of the vertical maps are by definition the canonical isomorphisms e_{λ_1} and $e_{\text{Ver}(\lambda_2)}$, so the commutativity of the big square proves what we wanted.

Remark 4.5. In characteristic zero $\operatorname{Hom}_{(U\mathfrak{g},B)}(\operatorname{Ver}_B(\lambda_1), \operatorname{Ver}_B(\lambda_2))$ is non-zero precisely when there is a sequence $\lambda_1 \leq \mu_1 \ldots \leq \mu_n = \lambda_2$ such that μ_{i+1} is the reflection of μ_i by a positive root, and in that case the Hom space is 1-dimensional [Ver68], [BGG71]. In particular there is no nonzero map between Vermas of weights that are *G*-dominant. In characteristic *p* one can construct more maps, see Proposition 4.18.

Finally, we relate the restriction to U of the functors of Theorem 4.16 to the basic theta operators of Section 2.

Theorem 4.17. Let $\lambda \in X^*(T)$. For $\gamma \in \mathfrak{u}_B^-$ define the map of T-modules

$$\phi_{\gamma} : \lambda \to \operatorname{Ver}_{B}^{\leq 1}(\lambda - \gamma)$$
$$v_{\lambda} \mapsto x_{\gamma} \otimes (\lambda - \gamma).$$

The functor Φ_T of Theorem 4.16 defines a map

$$\Phi:\mathfrak{u}_B^-\hookrightarrow \mathrm{DiffOp}_U^{\mathrm{crys}}(\bigoplus_{\lambda\in X^*(T)}\mathcal{L}(\lambda),\bigoplus_{\mu\in X^*(T)}\mathcal{L}(\mu))$$

sending x_{γ} to $\Phi_T(\phi_{\gamma})$. This is a map of Lie algebras, with the right-hand side having the commutator of operators as bracket, so it extends to a map of algebras

$$U\mathfrak{u}_B^- \hookrightarrow \mathrm{DiffOp}_U^{\mathrm{crys}}(\bigoplus_{\lambda \in X^*(T)} \mathcal{L}(\lambda), \bigoplus_{\mu \in X^*(T)} \mathcal{L}(\mu)).$$

Further, there are identifications with the operators defined in Definition 2.2:

$$\Phi(x_{-\beta}) = \hat{\theta}_1, \ \Phi(x_{-\alpha-\beta}) = \hat{\theta}_2, \ \Phi(x_{-2\alpha-\beta}) = \hat{\theta}_3, \ \Phi(x_{-\alpha}) = \hat{\theta}_4.$$

Proof. We only need to prove the identifications with $\tilde{\theta}_i$, the rest is proved by the compatibility of Φ_T with composition. Let $\lambda \in X^*(T)$, and fix a basis $\{x_i\}$ of $\mathfrak{g}/\mathfrak{b}$ inducing an isomorphism of *T*-modules $i_{\lambda} : \operatorname{Ver}_B(\lambda)^{\leq 1} \cong \lambda \otimes_{\mathbb{F}_p} U^{\leq 1}(\mathfrak{g}/\mathfrak{b})$ as before. We claim that this fits in the diagram

where the diagonal map corresponds to the dual of the connection on $\mathcal{L}(-s_0\lambda)$ defined as follows. For \mathcal{L} and \mathcal{L}' it is defined by applying the Gauss-Manin connection on H and then projecting back via the splitting on U. For a general weight λ it is defined by tensor/dual functoriality. The claim follows from Corollary 4.15 and the observation that under the trivialization of the flag of $H_{U_{\text{Ig}}}$ given by $\{f_i\}$ the described connection is the trivial one. The maps ϕ_{γ} are identified under the dual of the horizontal map (id $\otimes e_1$) $\circ F(i_\lambda)$ with the composition $\mathcal{L}(-s_0\lambda) \otimes P_U^1 \rightarrow$ $\mathcal{L}(-s_0\lambda) \otimes \Omega_U^1 \xrightarrow{\text{id} \otimes \pi_{-s_0\gamma}} \mathcal{L}(-s_0\lambda - s_0\gamma)$. By the claim above, $\Phi_T(\phi_{\gamma})$ is given by precomposing this with the map $\pi \circ \nabla : P_U^1 \otimes \mathcal{L}(-s_0\lambda) \to \mathcal{L}(-s_0\lambda) \otimes P_U^1$. This precisely corresponds to the definition of the $\tilde{\theta}_i$.

Remark 4.6. It is not true that $F_B(\operatorname{Ver}^0_B(\lambda)) = D_{\mathcal{F}l}^{[p]} \otimes \mathcal{L}(s_0\lambda)$. Otherwise, we would have the analogue of Theorem 4.16(1) for $\operatorname{Ver}^0_B(\lambda)$, and together with part 3) it would imply that $\theta_1^p = 0$. Another way to see this is to use Serre-Tate coordinates. By Proposition 4.14 on the formal completion of a point in U, $F_B(\operatorname{Ver}^0_P(1))$ is the quotient sheaf of $\pi^* D_{\overline{Sh}}^{\operatorname{crys}}$ generated by $\prod_{i=1}^{3} D_i^{n_i}$ for $D_i \in \pi^* T_{\overline{Sh}|_{U_{\mathrm{Ig}}}}$ the canonical basis, and $0 \leq n_i \leq p-1$. On the other hand $D_{\overline{Sh}}^{[p]}$ is generated by $\prod_{i=1}^{3} (\frac{\partial}{\partial T_{ij}})^{n_{ij}}$ for $0 \leq n_{ij} \leq p-1$. One can explicitly see that they are not isomorphic as quotients of $\pi^* D_{\overline{Sh}}^{\operatorname{crys}}$.

In particular, under the labelling above, the theta operators θ_i commute as the opposite of \mathfrak{u}_B . This proves the commutation relations of Proposition 2.7 in a more robust way. Using Theorem 4.17

and Theorem 4.16 whenever we have some map on U which is a combination of θ_i s we can test if it extends to $\mathcal{F}l$ by checking if the associated map of Verma modules as T-modules is in fact a map of B-modules. Although this won't give if and only if statements it recovers many of the results proved in Section 3. See Corollary 4.19 for a comprehensive list of examples.

4.4. Theta linkage maps. We construct non-zero maps between Verma modules in characteristic p whose weights are linked, hence giving name to their associated theta linkage maps. We have already seen two examples in Theorem 3.10 and Proposition 3.2, which correspond to reflections by the simple roots. We use the following notation: for $\gamma \in \Phi^+$ and $n \in \mathbb{Z}$ let $s_{\gamma,n} \cdot \lambda = \lambda + (pn - \langle \lambda + \rho, \gamma^{\vee} \rangle)\gamma$. We say that $\lambda \uparrow_{\gamma} \mu$ if there exists $n \in \mathbb{Z}$ such that $\mu = s_{\gamma,n} \cdot \lambda, \lambda \leq \mu$, and n is minimal with this property. Geometrically $\lambda \uparrow_{\gamma} \mu$ if and only if μ is the reflection of λ in the positive direction across the closest wall defined by γ^{\vee} . Then $\lambda \uparrow \mu$ if there exists a chain $\lambda \uparrow_{\gamma_1} \lambda_1 \dots \uparrow_{\gamma_n} \mu$.

Proposition 4.18. Let $\lambda, \mu \in X^*(T)$. Assume there exists $\gamma \in \Phi^+$ such that $\lambda \uparrow_{\gamma} \mu$. Then we can construct an explicit 1-dimensional subspace of

 $\operatorname{Hom}_{(U\mathfrak{g},B)_{\mathbb{F}_n}}(\operatorname{Ver}_B(\lambda),\operatorname{Ver}_B(\mu))\neq 0.$

In general if $\lambda \uparrow \mu$, then $\operatorname{Hom}_{(U\mathfrak{g},B)_{\mathbb{F}_n}}(\operatorname{Ver}_B(\lambda),\operatorname{Ver}_B(\mu)) \neq 0$.

Proof. Choose $\nu \in X^*(T)$ such that $\langle \nu, \gamma^{\vee} \rangle = 1$, and let $n \in \mathbb{Z}$ such that $\mu = s_{\gamma,n} \cdot \lambda$. Consider $\tilde{\lambda} \coloneqq \lambda - pn\nu$, then there exists a unique up to scalar non-zero map $\phi : \tilde{\lambda} \to \operatorname{Ver}_B(s_{\gamma,0} \cdot \tilde{\lambda})$ over \mathbb{Q}_p , given by an element $\tilde{f}(X) \in U\mathfrak{u}_{B,\mathbb{Q}_p}^-$ in some PBW basis. After rescaling $\tilde{f}(X)$ choose f(X) over \mathbb{Z}_p which is not 0 mod p, this is unique up to a unit. Then f(X) defines a map $\tilde{\lambda} \to \operatorname{Ver}_B(s_{\gamma,0} \cdot \tilde{\lambda})$ of B-modules over \mathbb{Z}_p since the Vermas are p-torsion free. We claim that the map $\lambda \to \operatorname{Ver}_B(\mu)_{\mathbb{F}_p}$ defined by $\lambda \mapsto \overline{f}(X) \otimes \mu$ is B-equivariant. This is the same as being U(B)-equivariant, which reduces to being equivariant for elements of the form $\frac{Y^n}{n!}$ for positive roots Y, and $\binom{H}{n}$ for $H \in \mathfrak{h}$. For the latter it suffices to observe that the weight increases are the same: $\mu - \lambda = s_{\gamma,0} \cdot \tilde{\lambda} - \tilde{\lambda}$. For the former, write $\frac{Y^n}{n!}f(X)$ as g(X)h(X) for $g \in U\mathfrak{u}_B^-$ and $h \in U(B)$ by the following procedure. By induction on the weight of Y^n and the degree of f it suffices to prove that $\frac{Y^n}{n!}X \in X\frac{Y^n}{n!} + U(B)U\mathfrak{u}_B^-$ for $X \in \mathfrak{u}_B^-$, where the terms in U(B) have smaller weight that Y^n . This follows from the PBW theorem on U(G), and noticing that in the expression of $[Y^n, X]$ in a PBW basis the degree of any element of \mathfrak{u}_B^- appearing can be at most 1. Since ϕ is B-equivariant it implies that $g(X) \otimes h(X) \tilde{\lambda} = 0$ over \mathbb{Z}_p , we can assume that $h(X)\tilde{\lambda} = 0$. Then

$$\frac{Y^n}{n!}\overline{f}(X)\otimes\mu=\overline{g}(X)\otimes\overline{h}(X)(\tilde{\lambda}+pn\nu)=0.$$

We claim that the one-dimensional subspace generated by this map is independent of the choice of ν . This amounts to the fact that in characteristic 0 the maps $\operatorname{Ver}_B(\lambda) \to \operatorname{Ver}_B(s_{\gamma} \cdot \lambda)$ are given by an element of $U\mathfrak{b}^-$ for all λ lying in a hyperplane $\langle \lambda, \gamma^{\vee} \rangle = r$ [Hum08, 4.12]. For two valid $\nu_{1,2}$ the maps in characteristic zero $\operatorname{Ver}_B(\lambda - pn\nu_i) \to \operatorname{Ver}_B(s_{\gamma} \cdot (\lambda - pn\nu_i))$ are then given by the same element $h = h_1 \cdot h_2 \in U\mathfrak{b}^- = U\mathfrak{u}^- \cdot U\mathfrak{h}$. After rescaling to get a non-zero map over \mathbb{F}_p , we have that h_2 acts by the same scalar on both $s_{\gamma}(\lambda - pn\nu_i)$, since the difference of their weights is a multiple of p. Thus, we end up with the same elements in $U\mathfrak{u}_{B,\mathbb{F}_p}^-$. Finally, the last statement follows from the first since $U\mathfrak{u}_{B,\mathbb{F}_p}^-$ has no nontrivial zero-divisors, so we can compose the previously defined maps.

- **Remark 4.7.** (1) If there is a non-zero map $\operatorname{Ver}_B(\lambda) \to \operatorname{Ver}_B(\mu)$ over \mathbb{F}_p then $\mu \in W_{\operatorname{aff}} \cdot \lambda$ by considering action of the Harish-Chandra center of $U\mathfrak{g}$. Moreover, if the projection to $\operatorname{Ver}_B^0(\lambda) \to \operatorname{Ver}_B^0(\mu)$ is non-zero, then $\lambda \uparrow \mu$ by the linkage principle on G_1B -representations [Jan03, Cor 9.12] (the simple $L(\lambda)$ is the head of $\operatorname{Ver}_B^0(\lambda)$).
 - (2) Proposition 4.18 doesn't prove that for two linked weights $\lambda \uparrow \mu$ the map constructed above is independent of a choice of sequence $\lambda \uparrow_{\gamma_1} \lambda_1 \uparrow_{\gamma_2} \ldots \uparrow_{\gamma_n} \mu$. In general we don't know if the Hom space is at most one-dimensional as in the characteristic 0 case. A key difference in characteristic p is that Verma modules are not of finite length as $(U\mathfrak{g}, B)$ -modules.

We will say that differential operators which come from maps in Proposition 4.18 via Theorem 4.16 are theta linkage maps. We remark that since $F_B(\lambda) = \mathcal{L}(s_0\lambda)$ a map of Verma modules with $\lambda \uparrow_{\gamma} \mu$ induces a map $\mathcal{L}(-s_0\mu) \to \mathcal{L}(-s_0\mu + ns_0\gamma)$ for $n \ge 0$. For γ not in Φ_M^+ this shifts the weight by a dominant weight, but for $\gamma = \alpha$ the shift is by a non-dominant weight. This produces two more theta linkage maps in the *p*-restricted region, and it also conceptually reproves many of the results of Section 3.

- **Corollary 4.19.** (1) For k = bp + a with $1 \le a \le p$ the map $\theta_{(k,l)}^1 : \mathcal{L}(k,l) \to \mathcal{L}(2p-2a+k+2,l)$ comes from the map of *B*-modules $(-l, 2a-k-2-2p) \to \operatorname{Ver}_B(-l, -k)$ defined as $(-l, 2a-k-2-2p) \mapsto x_{-\beta}^{p-a+1} \otimes (-l, -k)$. In particular when $p \mid k, \theta_1 : \mathcal{L}(k,l) \to \mathcal{L}(k+p+1, l+p-1)$ factors through H_1 .
 - (2) For k-l = ap + b with $0 \le b \le p-1$, the map on $\theta_{(k,l)}^4 : \mathcal{L}(k,l) \to \mathcal{L}(k-b-1,l+b+1)$ comes from the map of B-modules $(-l-b-1,b+1-k) \to \operatorname{Ver}_B(-l,-k)$ defined as (-l-b-1,b+1-k) to $X_{-\alpha}^{b+1} \otimes (-l,-k)$. In particular when $p \mid k-l, \theta_4 : \mathcal{L}(k,l) \to \mathcal{L}(k+p-1,l)$ factors through H_2 .
 - (3) The map $\frac{1}{H_1^p} \dot{\theta}_1^p$ extends to the whole of $\mathcal{F}l$, since it can be written as the composition of two affine Weyl reflections: $\frac{1}{H_1^p} \theta_1^p = \theta_{(2p-k+2,l)}^1 \circ \theta_{(k,l)}^1$. Together with the relations over the ordinary locus in Proposition 2.7 this shows that θ_3 does indeed have weight (2p,0).
 - (4) $(C_0 \mapsto C_1 \text{ linkage map})$. For $(k-3, l-3) \in C_0$ there is a non-zero map $\theta_{(k,l)}^{\alpha+\beta} : \mathcal{L}(k,l) \to \mathcal{L}(p-l+3, p-k+3)$.
 - (5) $(C_2 \mapsto C_3 \text{ linkage map})$. For $(k-3, l-3) \in C_2$ satisfying k-l < p-1 there is a non-zero map $\theta_{(k,l)}^{\alpha+\beta} : \mathcal{L}(k,l) \to \mathcal{L}(2p-l+3, 2p-k+3)$.
 - (6) The compatibility with baby Verma modules $\operatorname{Ver}_{P/B}^{0}$ shows that $\theta_{4}^{p} = 0$.
 - (7) The map $\pi_*\theta^4_{(k,l)}$ comes from a map of *M*-representations, and we can reprove the results on its kernel from Proposition 3.3 from the representation theory of baby Verma modules.

Proof. For 1) - 3) and 6) we use Theorem 4.16 and Theorem 4.17 together with the explicit description of $\theta_{(k,l)}^1$ and $\theta_{(k,l)}^4$ in terms of θ_1 and θ_4 . For 4) we follow the recipe of Proposition 4.18 for $\nu = (0, 1)$. Let the map $(k - 3 - p, l - 3) \rightarrow \operatorname{Ver}_B(-l, p - k)_{\mathbb{Q}_p}$ be given by $f(X) \in U\mathfrak{u}_{B,\mathbb{Z}_p}$ not vanishing mod p. We have the following commutative diagram in characteristic 0

up to a unit, since the Hom of *B*-Verma modules is 1-dimensional in characteristic zero. Since $U\mathfrak{u}_{B,\mathbb{F}_p}^-$ has no nontrivial zero-divisors we can take f so that $f(X)X_{-\beta}^{l-2} = X_{-\beta}^{p-k+1}X_{-\alpha}^{p-k-l+3}$ over \mathbb{Z}_p . We define $\theta_{(k,l)}^{\alpha+\beta}$ as the operator corresponding to the map $(k-p-3, l-p-3) \to \operatorname{Ver}_B(-l, -k)_{\mathbb{F}_p}$ given by $\overline{f(X)}$. By the diagram above it satisfies $\theta_{(p-l+3,p-k+3)}^1 \circ \theta_{(k,l)}^{\alpha+\beta} = \theta_{(2p-k+2,l)}^4 \circ \theta_{(k,l)}^1$. One can check on Serre-Tate coordinates that the right-hand side is not identically 0. For instance $T_{22}T^{p-1}e_1^{k-l}(e_1 \wedge e_2)^l$ is not sent to 0, by Proposition 3.3. Therefore $\theta_{(k,l)}^{\alpha+\beta}$ is non-zero.

For 5) we construct both $\theta_{(k,l)}^{\alpha+\beta}$ and another linkage map $\theta_{(2p-l+3,2p-k+3)}^{2\alpha+\beta}$ using $\nu = (1,1)$ in the proo of Proposition 4.18. For $\theta_{(k,l)}^{\alpha+\beta}$ even though $\langle \nu, (\alpha+\beta)^{\vee} \rangle = 2$ the procedure still works since the weights are linked by $s_{\alpha+\beta,-2}$. As before let $f(X), g(X) \in U\mathfrak{u}_{\mathbb{Z}_p}$ non-zero mod p fitting in the commutative diagram

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Define $\theta_{(k,l)}^{\alpha+\beta}$ by the map $(k-3-2p, l-3-2p) \to \operatorname{Ver}_B(-l, -k)_{\mathbb{F}_p}$ given by $\overline{f(X)}$, and $\theta_{(2p-l+3,2p-k+3)}^{2\alpha+\beta}$ by the map $(-1-k, l-3-2p) \to \operatorname{Ver}_B(k-3-2p, l-3-2p)_{\mathbb{F}_p}$ given by $\overline{g(X)}$. Then by the diagram above we have $\theta_{(l-1,k+1)}^1 \circ \theta_{(k,l)}^4 = \theta_{(2p-l+3,2p-k+3)}^{2\alpha+\beta} \circ \theta_{(k,l)}^{\alpha+\beta}$. We can also prove that the left-hand side is not identically 0 by doing a computation on Serre-Tate coordinates, e.g. $T_{22}T^{k-l+1}e_1^{k-l}(e_1 \wedge e_2)^l$ is not sent to 0. Therefore $\theta_{(k,l)}^{\alpha+\beta}$ is non-zero.

For 7), the map of $\operatorname{Ver}_{P/B}$ modules inducing $\theta_{(k,l)}^4$ also induces a map of line bundles on P/B, which on global sections gives a map of M-representations. We claim that $\pi_*\theta_{(k,l)}^4$ comes from this map. This can be checked on relative formal completions of points (in our set up these don't have divided powers), where it follows by Theorem 4.6. To compute the kernel of $F^*F_*\theta_{(k,l)}^4$ we use Theorem 4.16(1). The key fact is that the cokernel of $\operatorname{Ver}_{P/B}^0(b+1-k, -b-l-1) \to \operatorname{Ver}_{P/B}^0(-l, -k)$ is $[(ap, 0) \otimes W(l+b, l)]^{\vee} = (-ap, 0) \otimes W(-l, -l-b)$ as a B-representation. Baby Vermas have an action of \mathfrak{p} , which equips the dual of $F(\operatorname{Ver}^{0}_{P/B}(V))$ with a connection relative to $\overline{\operatorname{Sh}}$ by using Corollary 4.4 for P/B. This connection is identified with the connection given by Cartier descent on the Frobenius pullback of a sheaf. Taking horizontal sections of this connection recovers $F_*\theta^4_{(k,l)}$, and we can check that the kernel of $F_*\theta^4_{(k,l)}$ is the expected one.

The map on *M*-representations corresponding to $\pi_*\theta^4_{(k,l)}$ also has a classical description. Consider Serre's operator $D = x^p \frac{d}{dx} + y^p \frac{d}{dy}$ on $\operatorname{GL}_2(\mathbb{F}_p)$ -representations over $\overline{\mathbb{F}}_p$, and Dickson's invariant $H = x^p y - x y^p$. Then for k - l = ap + b, D^{b+1} acting on W(k, l) becomes divisible by H^{b+1} , and it is in fact GL_2 -equivariant. This procedure is analogous to the definition as $\theta^4_{(k,l)} = \frac{1}{H_2^{b+1}} \theta^{b+1}_4$ on the flag Shimura variety, and over $\mathcal{F}l_{\mathrm{Ig}}$ both can be identified.

Remark 4.8. We emphasize that to prove 1) with the methods of Section 3 we had to compute a local expression along the formal neighbourhood of a point in $\overline{\text{Sh}}^{=1}$, and use a Grothendieck-Messing basis there, it doesn't follow formally by choosing an arbitrary basis for ω . Therefore Theorem 4.16 does contain some non-trivial information about the lower strata. Another instance of this is that on some locally closed Ekedahl-Oort strata of $\overline{\text{Sh}}$ one has various reductions of the torsor I_P given by the interaction of the Hodge and conjugate filtration. Using these one gets partial filtrations on the pullback of $D_{\overline{\text{Sh}}}^{\text{crys}}$ to the strata, which are not immediate to obtain otherwise.

4.5. The BGG complex. Fix $\lambda \in C_0$. Then the Weyl module $V(\lambda)$ over \mathbb{F}_p is irreducible. Consider the standard complexes $\operatorname{Std}_P(V(\lambda))$ for $V(\lambda) \in \operatorname{Rep}_{\mathbb{Z}_p}(P)$

$$0 \to \operatorname{Ver}_P(\bigwedge^3(\mathfrak{g}/\mathfrak{p}) \otimes_{\mathbb{Z}_p} V(\lambda)) \to \ldots \to \operatorname{Ver}_P((\mathfrak{g}/\mathfrak{p}) \otimes V(\lambda)) \to \operatorname{Ver}_P V(\lambda) \to 0,$$

and $\operatorname{Std}_B(V(\lambda))$ for $V(\lambda) \in \operatorname{Rep}_{\mathbb{Z}_p}(B)$

$$0 \to \operatorname{Ver}_B(\bigwedge^4(\mathfrak{g}/\mathfrak{b}) \otimes_{\mathbb{Z}_p} V(\lambda)) \to \ldots \to \operatorname{Ver}_B((\mathfrak{g}/\mathfrak{b}) \otimes V(\lambda)) \to \operatorname{Ver}_B V(\lambda) \to 0,$$

defined as in [PT02, 2.2]. They are complexes of $(U\mathfrak{g}, Q)$ -modules, exact except at the term $\operatorname{Ver}_{Q}V(\lambda)$, which has homology $V(\lambda)$. Let $W^{M} = \{w_{0}, w_{1}, w_{2}, w_{3}\}$ with the elements ordered by length, and let $W(a) \subset W$ be the elements of length a. Recall that a weight $\mu \in X^{*}(T)$ is p-small if $|\langle \mu + \rho, \gamma^{\vee} \rangle| < p$ for all roots γ .

Theorem 4.20. Let $\lambda \in X^*(T)$ in C_0 . There exists a complex of $(U\mathfrak{g}, P)$ -modules $BGG_{P,\lambda}$, quasiisomorphically embedded as a summand $BGG_{P,\lambda} \subset Std_P(V(\lambda))$, of the form

$$0 \to \operatorname{Ver}_P W(w_3 \cdot \lambda) \to \operatorname{Ver}_P W(w_2 \cdot \lambda) \to \operatorname{Ver}_P W(w_1 \cdot \lambda) \to \operatorname{Ver}_P W(\lambda) \to 0.$$

If we further assume that $\lambda = (a, b)$ satisfies a + 4 < p (this is only relevant for b = 0), there exists a complex of $(U\mathfrak{g}, B)$ modules $BGG_{B,\lambda}$ quasi-isomorphically embedded as a summand $BGG_{B,\lambda} \subset$

 $\operatorname{Std}_B(V(\lambda))$ such that

$$\mathrm{BGG}^d_{B,\lambda} = \oplus_{w \in W(d)} \mathrm{Ver}_B(w \cdot \lambda)$$

and the differentials go in the direction $BGG^d_{B\lambda} \to BGG^{d-1}_{B\lambda}$.

Proof. Let χ_{λ} be the character by which the center $Z(U\mathfrak{g})_{\mathbb{Z}_p}$ acts on $V(\lambda)_{\mathbb{F}_p}$, since it is an irreducible representation. Then for $Q \in \{P, B\}$ define

$$\mathrm{BGG}_{Q,\lambda} \coloneqq \mathrm{Std}_Q(V(\lambda))_{\chi_\lambda}$$

as the isotypic component of χ_{λ} (modulo p). These are summands of $\operatorname{Std}_{\mathcal{O}}(V(\lambda))$ since the latter is a direct sum over all the mod p characters of $Z(U\mathfrak{g})$. The homology of $\operatorname{Std}_Q(V(\lambda))$ is $V(\lambda)$ concentrated in one degree, which is killed by every other mod p character $\chi \neq \chi_{\lambda}$ of $Z(U\mathfrak{g})_{\mathbb{Z}_p}$. Therefore $BGG_{Q,\lambda}$ is quasi-isomorphic to $Std_Q(V(\lambda))$. The rest of the proposition for $BGG_{P,\lambda}$ is proved in [PT02, Thm D] and [LP18, Thm 5.2] in great generality. For the Borel we adapt a proof in characteristic 0 to this setting. By the assumption on λ for any $d \leq 4$, the weights of $\wedge^d(\mathfrak{g}/\mathfrak{b}) \otimes V(\lambda)$ are *p*-small, so it admits an exhaustive filtration M^{\bullet} by finite-free B modules over \mathbb{Z}_p such that $\operatorname{gr}^i M^{\bullet} = \lambda_i$ for some *p*-small weights λ_i (as 1-dimensional \mathbb{Z}_p -modules). Since $\operatorname{Ver}_B(-)$ is exact we get a filtration $\operatorname{Ver}_B(M^{\bullet})_{\chi_{\lambda}} \subset \operatorname{BGG}_{B,\lambda}^d$ with graded pieces $\operatorname{Ver}_B(\lambda_i)_{\chi_{\lambda}}$. The latter is non-zero precisely when $\lambda_i \in W_{aff} \cdot \lambda$. Since both weights are *p*-small this happens if and only if $\lambda_i = \omega \cdot \lambda$ for some $w \in W$. For $w \cdot \lambda$ to be a weight of $\wedge^d(\mathfrak{g}/\mathfrak{b}) \otimes V(\lambda)$ one needs l(w) = d in which case its multiplicity is 1: this can be checked over $\mathbb C$ where it is a classical result. Therefore, we get a filtration of $BGG_{B,\lambda}^d$ whose graded pieces are precisely $Ver_B(w \cdot \lambda)$ for $w \in W(d)$. Let $w \in W(d)$. We claim that we can choose two different filtrations of $\wedge^d(\mathfrak{g}/\mathfrak{b}) \otimes V(\lambda)$ as a *B*-module such that in one of them the weight $w \cdot \lambda$ appears before all the other weights $\{w' \cdot \lambda : w' \neq w \in W(d)\}$ in the filtration, and another one where the opposite happens. Let N be subquotient of the original filtration such that $w \cdot \lambda$ is a submodule of N and $w' \cdot \lambda$ is a quotient. We may assume that N contains no highest weight vectors apart from possibly $w \cdot \lambda$ or $w' \cdot \lambda$, otherwise such a vector would be a B-submodule (using that for p-small weights B-representations are equivalent to $U\mathfrak{b}$ -modules) of N, and we can move it in front of $w \cdot \lambda$. Thus by induction the weights of the graded pieces of N, except maybe the last one, are in $w \cdot \lambda + \mathbb{Z}^{\leq 0} \Phi^+$. Since $w' \cdot \lambda$ is not in that set we conclude that it must be a highest weight vector in N, and thus it can be moved in front of $w \cdot \lambda$. Repeating this process proves the claim.

Therefore there are maps $\operatorname{Ver}_B(w \cdot \lambda) \hookrightarrow \operatorname{BGG}_{B,\lambda}^d \twoheadrightarrow \operatorname{Ver}_B(w \cdot \lambda)$ over \mathbb{Z}_p . The composition must be given by a scalar in \mathbb{Z}_p , and it cannot be identically zero, since the weight $w \cdot \lambda$ has multiplicity one in $\operatorname{BGG}_{B,\lambda}^d$. Therefore, by rescaling we can assume the map is the identity. This defines $\operatorname{Ver}_B(w \cdot \lambda)$ as a summand of $\operatorname{BGG}_{B,\lambda}^d$ over \mathbb{Z}_p . Finally, any such two summands for different $w \in W(d)$ are disjoint, since there are no non-zero maps $\operatorname{Ver}_B(w \cdot \lambda) \to \operatorname{Ver}_B(w' \cdot \lambda)$ for $w \neq w' \in W(d)$. \Box

For $\lambda \in C_0$ we have the identity $V(\lambda)^{\vee} = V(-w_0\lambda)$, and $W(\lambda)^{\vee} = W(-s_0\lambda)$. Denote $\lambda^{\vee} = -w_0\lambda$ so that $(a, b, c)^{\vee} = (a, b, -c)$.

Lemma 4.21. The image of $\operatorname{Std}_P V(\lambda^{\vee})$ under the functor Φ_P of Theorem 4.16 is the complex $dR_{\operatorname{Sh}}(\lambda) \coloneqq (F_P(V(\lambda)) \otimes \Omega_{\operatorname{Sh}}^{\bullet})$ over $\operatorname{Sh}_{\mathbb{Z}_p}$ endowed with the Gauss-Manin connection as differentials. Similarly, the image of $\operatorname{Std}_B V(\lambda^{\vee})$ under Φ_B is $dR_{\mathcal{Fl}}(\lambda) \coloneqq (F_B V(\lambda) \otimes \Omega_{\mathcal{Fl}}^{\bullet})$ over $\mathcal{Fl}_{\mathbb{Z}_p}$ endowed with the Gauss-Manin connection. The same result holds for the (sub)canonical extensions defined by $\Phi_Q^{\operatorname{can,sub}}$.

Proof. We prove it on the interior Sh ⊂ Sh^{tor}, since one can check whether a map of vector bundles is zero on an open dense subset. It is then enough to check it on divided power completions of points by Lemma 4.11. The differentials are all of degree 1, so it is enough to prove that the differentials are the same considered as crystalline differential operators. Since the Gauss-Manin connection on Sh[#]_x is the pullback of the trivial connection on G/P^{\wedge}_{∞} , it is enough to prove that applying $F_{G/Q}$ to Std_Q(V(λ)[∨]) yields the de Rham complex on G/Q. This is a standard computation. For instance we can check that the composition $\mathfrak{g}/\mathfrak{q} \otimes V(\lambda^{\vee}) \xrightarrow{d} \operatorname{Ver}_{Q}^{\leq 1}(V(\lambda^{\vee})) \xrightarrow{\phi_V} V(\lambda^{\vee}) \otimes \operatorname{Ver}^{\leq 1}(1)$ is the natural inclusion, from the explicit description of the tensor identity in Corollary 4.4.

Let $BGG_{Sh}(\lambda)$ be the image of $BGG_{P,\lambda^{\vee}}$ under the functor Φ_P , and $BGG_{\mathcal{F}l}(\lambda)$ the image of $BGG_{B,\lambda^{\vee}}$ under Φ_B , defined in Theorem 4.16. By Lemma 4.21 and Theorem 4.20 they come equipped with maps $BGG_{Sh}(\lambda) \to dR_{Sh}(\lambda)$ and $BGG_{\mathcal{F}l}(\lambda) \to dR_{\mathcal{F}l}(\lambda)$. Similarly, let $BGG_{Sh}^{?}(\lambda)$ and $dR_{Sh}^{?}(\lambda)$ for $? \in \{\text{can}, \text{sub}\}$ be their (sub)canonical extensions to Sh^{tor}. Both the de Rham and BGG complexes are equipped with a natural Hodge filtration, as in [LP18, Def 3.10].

Theorem 4.22. Let $\lambda = (a, b) \in C_0$ satisfying a + 4 < p, and $? \in \{\text{can}, \text{sub}\}$.

- (1) The maps $\mathrm{BGG}^?_{\mathrm{Sh}^{\mathrm{tor}}}(\lambda) \to \mathrm{dR}^?_{\mathrm{Sh}^{\mathrm{tor}}}(\lambda)$ and $\mathrm{BGG}^?_{\mathcal{F}l^{\mathrm{tor}}}(\lambda) \to \mathrm{dR}^?_{\mathcal{F}l^{\mathrm{tor}}}(\lambda)$ are quasi-isomorphisms of filtered complexes. They also admit a section.
- (2) Moreover, there is a quasi-isomorphism of filtered complexes

$$\pi_* \mathrm{BGG}^{!}_{\mathcal{F}l^{\mathrm{tor}}}(\lambda) \cong \mathrm{BGG}^{!}_{\mathrm{Sh}^{\mathrm{tor}}}(\lambda).$$

Proof. The first point is essentially explained in [MT02, §4-5], we give a brief summary here. For any smooth scheme $X/\overline{\mathbb{F}}_p$ there is an exact functor L from the category of \mathcal{O}_X -modules with morphisms given by crystalline differential operators to the category crystals in the crystalline site $(X/\overline{\mathbb{F}}_p)_{\text{crys}}$ (which is equivalent to the category of \mathcal{O}_X -modules with a PD stratification) given by $E \mapsto P_X^{\text{crys}} \otimes E$. Moreover, composing with the derived pushforward $Ru_* : (X/\overline{\mathbb{F}}_p)_{\text{crys}} \to (X/\overline{\mathbb{F}}_p)_{\text{zar}}$ satisfies $Ru_* \circ L(E^{\bullet}) = [E^{\bullet}]$. Then by Theorem 4.20 and Lemma 4.21 the map $L(\text{BGG}_{\text{Sh}}(\lambda)) =$ $F_P(\text{BGG}_{P,\lambda^{\vee}})^{\vee} \to F_P(\text{Std}_P(V(\lambda^{\vee}))^{\vee} = L(\text{dR}_{\text{Sh}}(\lambda))$ is a quasi-isomorphic summand. Pushing this quasi-isomorphism forward along Ru_* recovers the quasi-isomorphisms $\text{BGG}_{\text{Sh}}(\lambda) \to \text{dR}_{\text{Sh}}(\lambda)$, and the former is a summand of the latter in the abelian category of complexes by the compatibility with composition of the functors in Theorem 4.16. Using log-crystals and log-crystalline differential operators we obtain the same result on toroidal compactifications. For 2), we have $\pi_* \text{dR}_{\mathcal{F}l}^{\text{can}}(\lambda) =$ $\text{dR}_{\text{Sh}}^{\text{can}}(\lambda)$ as complexes using that $\pi_*\Omega_{\mathcal{F}l^{\text{tor}}/\text{Sh}^{\text{tor}}} = 0$, and that the Gauss-Manin connection on $\mathcal{F}l^{\text{tor}}$

is the pullback of the Gauss-Manin connection on Sh^{tor}. In fact one has

$$R\pi_* dR_{\mathcal{F}l}^{\operatorname{can}}(\lambda) = dR_{\operatorname{Sh}}^{\operatorname{can}}(\lambda) \oplus dR_{\operatorname{Sh}}^{\operatorname{can}}(\lambda)[-2]$$
(4.11)

in the derived category. For trivial coefficients (on any projective bundle) this follows from [Sta18, Tag 0FUN]. For $\lambda = (1,0)$ corresponding to the standard representation consider the pullback $\tilde{g}: \tilde{A} \to \mathcal{F}l$ of $g: A \to Sh$, and let $\tilde{\pi}: \tilde{A} \to A$ be the base change map. The Koszul filtration on $\Omega^{\bullet}_{\tilde{A}/\mathbb{Z}_p}$ with respect to $\tilde{A}/\mathcal{F}l/\mathbb{Z}_p$ induces a filtration on $R\tilde{g}_*\Omega^{\bullet}_{\tilde{A}/\mathbb{Z}_p}$ whose graded pieces are $H^i_{\mathrm{dR}}(\tilde{A}/\mathcal{F}l) \otimes \Omega^{\bullet}_{\mathcal{F}l}$, by definition of the Gauss-Manin connection. Applying $R\pi_*$ we then get a filtration on $R(\pi \circ \tilde{g})_* \Omega^{\bullet}_{\tilde{A}/\mathbb{Z}_p}$. The latter is also equal to $Rg_*(\Omega^{\bullet}_{A/\mathbb{Z}_p} \oplus \Omega^{\bullet}_{A/\mathbb{Z}_p}[-2])$ by the case of trivial coefficients, and each summand has a Koszul filtration induced from the one on $\Omega^{\bullet}_{A/\mathbb{Z}_p}$ with respect to $A/\mathrm{Sh}/\mathbb{Z}_p$. One can check that the two filtrations on $R(\pi \circ \tilde{g})_*\Omega^{\bullet}_{\tilde{A}/\mathbb{Z}_p}$ are the same, it essentially boils down to the fact that $\pi_* \Omega^1_{\mathcal{F}l/Sh} = 0$. By comparing their graded pieces we obtain (4.11). For general λ one follows the same strategy as above for powers of the universal abelian varieties and then uses [LS13, Prop 4.8] (base changed to the flag Shimura variety too) to cut down $\mathcal{V}(\lambda)$ out of $H^1_{\mathrm{dR}}(A^n/\mathrm{Sh})$. We claim that $\pi_*\mathrm{BGG}^{\mathrm{can}}_{\mathcal{F}l}(\lambda)$ is quasi-isomorphic to $\pi_*\mathrm{dR}^{\mathrm{can}}_{\mathcal{F}l}(\lambda) = \mathrm{dR}^{\mathrm{can}}_{\mathrm{Sh}}(\lambda)$, which proves 2) by part 1). The cohomology of $R\pi_* dR_{\mathcal{F}l}^{can} = R\pi_* BGG_{\mathcal{F}l}^{can}$ can be computed by two E_2 spectral sequences $\mathcal{H}^i R \pi^j_* BGG^{can}_{\mathcal{F}l} \hookrightarrow \mathcal{H}^i R \pi^j_* dR^{can}_{\mathcal{F}l}$, where e.g. $R \pi^j_* BGG^{can}_{\mathcal{F}l}$ means that we apply $R\pi_*^j$ to each element of the complex. By part 1) this inclusion is a summand of spectral sequences. By (4.11) the second spectral sequence degenerates at the E_2 page, so the first one also degenerates at the E_2 page, and the inclusion must be an isomorphism. This proves the claim.

Theorem 4.22 together with Theorem 4.17 shows that the differentials on $BGG_{\overline{Sh}}(\lambda)$ are built from basic theta operators.

Corollary 4.23. For $\lambda = (a, b) \in C_0$ satisfying a + 4 < p the differentials of the BGG complexes $\operatorname{BGG}_{\overline{\operatorname{Sh}}^{\operatorname{tor}}}(\lambda)$ and $\operatorname{BGG}_{\mathcal{Fl}_{\overline{\operatorname{Fp}}}^{\operatorname{tor}}}(\lambda)$ are given by particular combinations of theta operators θ_i which become highly divisible by Hasse invariants. More precisely, if $d : \mathcal{L}(\lambda_1) \to \mathcal{L}(\lambda_2)$ is a differential corresponding to the map of Verma modules $-s_0\lambda_2 \to \sum X_{-\alpha}^{n_4} X_{-2\alpha-\beta}^{n_3} X_{-\alpha-\beta}^{n_1} X_{-\beta}^{n_1} \otimes -s_0\lambda_1$ then $d = \sum \frac{1}{H_1^{n_1+n_2} H_2^{n_2+2n_3+n_4}} \theta_1^{n_1} \circ \theta_2^{n_2} \circ \theta_3^{n_3} \circ \theta_4^{n_4}.$

Explicitly, for $\lambda = (k - 3, l - 3)$ the two complexes have the following form

$$\mathcal{L}(3-l,3-k) \bigoplus \mathcal{L}(2-k,4-l) \longrightarrow \mathcal{L}(2-k,l) \longrightarrow \mathcal{L}(3-l,k+1)$$

$$\bigoplus \bigoplus \mathcal{L}(l-1,3-k) \longrightarrow \mathcal{L}(k,4-l) \longrightarrow \mathcal{L}(k,l)$$

$$\omega(3-l,3-k) \xrightarrow{d_0} \omega(l-1,3-k) \xrightarrow{d_1} \omega(k,4-l) \xrightarrow{d_2} \omega(k,l) \longrightarrow 0,$$

where the complex on top represents $BGG_{\mathcal{F}l}(\lambda)$ and the one at the bottom $BGG_{Sh}(\lambda)$. The differentials in red are the ones that survive after pushing forward to the Shimura variety. The Hodge filtration on Sh is given by the "stupid truncation" with weights 0, l-2, k-1, k+l-3.

Importantly, the Hodge-de-Rham spectral sequence degenerates on its first page over \mathbb{Z}_p for $\lambda \in C_0$, which allows to describe de Rham cohomology in terms of coherent cohomology.

Theorem 4.24. [LS13, Thm 8.2] Let $\lambda \in X^*(T)$ and $? \in \{\text{can, sub}\}$. Then the spectral sequence associated to the Hodge filtration of $\mathrm{dR}^?_{\mathrm{Sh}^{\mathrm{tor}}_{\mathbb{Q}_p}}(\lambda)$ degenerates on its first page. The same is true for $\mathrm{dR}^?_{\mathrm{Sh}^{\mathrm{tor}}_{\mathbb{Z}_p}}(\lambda)$, if $\lambda \in C_0$. Thus, by Theorem 4.22 the graded pieces of $H^3_{\mathrm{dR}}(\mathrm{Sh}_{\mathbb{Z}_p}, V(\lambda)) := H^3(\mathrm{dR}^{\mathrm{can}}_{\mathrm{Sh}^{\mathrm{tor}}}(\lambda))$ are given by $H^{l(w)}(\mathrm{Sh}^{\mathrm{tor}}_{\mathbb{Z}_p}, \omega(w \cdot \lambda + (3, 3)))$ for $w \in W^M$, and the ones of $H^3_{\mathrm{dR},c}(\mathrm{Sh}_{\mathbb{Z}_p}, V(\lambda))$ by $H^{l(w)}(\mathrm{Sh}^{\mathrm{tor}}_{\mathbb{Z}_p}, \omega^{\mathrm{sub}}(w \cdot \lambda + (3, 3)))$.

5. Applications to the weight part of Serre's conjecture

5.1. Generalities about the weight part of Serre's conjecture. We briefly recall the weight part of Serre's conjecture for $\operatorname{GSp}_4/\mathbb{Q}$ and how its combinatorics relates to the weight shifting of our operators. Given an odd Galois representation $\overline{r}: G_{\mathbb{Q}} \to \operatorname{GSp}_4(\overline{\mathbb{F}}_p)$ denote by $W(\overline{r})$ the set of Serre weights σ such that the Hecke eigensystem associated to \overline{r} (if there is any) appears in $H^*_{\operatorname{\acute{e}t}}(\operatorname{Sh}_{\overline{\mathbb{Q}}_p}, \sigma)$ for some level away from p. Conjecturally this only depends on $\overline{\rho} := \overline{r}_{G_{\mathbb{Q}_p}}$, so we will also use $W(\overline{\rho})$ to denote it. We explain Herzig's [Her09] conjectural recipe for $W(\overline{\rho})$ in the case when $\overline{\rho}$ is semisimple, adapted to the case of $\operatorname{GSp}_4/\mathbb{Q}$ in [HT13], and now proved in the latter case under some Taylor-Wiles and genericity conditions in [Lee23]². In general one expects that $W(\overline{\rho}) \subseteq W(\overline{\rho}^{ss})$, so that the latter would exhaust all possible Serre weights. Given $\overline{\rho}: G_{\mathbb{Q}_p} \to \operatorname{GSp}_4(\overline{\mathbb{F}}_p)$ we can write the semisimplification of the restriction to inertia $\tau := \overline{\rho}_{|I_p}^{ss}$ as

$$\tau = \tau(\mu, w) \coloneqq (\overline{\mu} + pw^{-1}\overline{\mu} + \dots p^{t-1}w^{-t}\overline{\mu})\omega_t$$

 $^{^{2}}$ In [Lee23] they use a different global setting, but their results can be translated to our setting using the vanishing results of [HL23].

where $t \in \{1, 2, 3, 4\}$, ω_t is the fundamental character of niveau $t, w \in W$ is such that $\check{w}\tau \sim \tau^p$, $\mu = (a, b, c) \in X^*(T)$, and $\overline{\mu} = (\frac{a+b+c}{2}, \frac{a-b+c}{2}, c) \in X_*(\check{T})$. We are fixing the isomorphism of GSp_4 with its dual group which on Weyl groups we denote by $w \to \check{w}$, that switches the actions of s_0 and s_1 . Then any semisimple $\tau : I_{\mathbb{Q}_p} \to \operatorname{GSp}_4(\overline{\mathbb{F}}_p)$ that lifts to $G_{\mathbb{Q}_p}$ (a tame inertial parameter) can always be written as $\tau(\mu, w)$, and for generic τ we can assume $\mu \in X_1(T)$. Let $\delta > 0$, we say that a weight $\lambda \in X^*(T)$ is δ -generic if $|\langle \lambda, \gamma^{\vee} \rangle - pn | < \delta$ for each root γ and some $n \in \mathbb{Z}$ depending on γ . A tame inertial parameter $\tau = \tau(\mu + \rho, w)$ is δ -generic if μ is. We say that a statement holds for sufficiently generic weights/tame parameters if there exists some δ independent of p such that the statement holds for all δ -generic weights/tame parameters. The set of regular weights $X_{\operatorname{reg}}(T) \subset X_1(T)$ consists of all $(a, b) \in X_1(T)$ such that a - b, b . Based on $the reduction of the Deligne-Lusztig representation associated to <math>\tau$ Herzig constructs a set $W^?(\overline{\rho})$ of Serre weights, and conjectures that it should equal the set of regular Serre weights when $\overline{\rho}$ is semisimple. Restricting to some set of sufficiently generic weights (or to sufficiently generic tame parameters) $W^?(\overline{\rho})$ can also be described as

$$W^{?}(\overline{\rho}) = \{\mu \in X_{\mathrm{reg}}(T) \text{ such that } \exists \ \mu' \uparrow \mu \text{ with } \mu' + \rho \text{ dominant and} \ w \in W, \text{ such that } \overline{\rho}_{\mathrm{L}}^{\mathrm{ss}} = \tau(\mu' + \rho, w)\}$$

[Her09, Prop 6.28]. For $\overline{\rho}$ ordinary this description holds for all regular weights and in general 2-generic is enough [HT13, Prop 4.11]. In the general case $W^?(\overline{\rho})$ contains 20 Serre weights, which can be classified into 8 *obvious* weights and 12 *shadow* weights, using the terminology of [GHS18]. The 8 obvious weights correspond to taking $\mu' = \mu$ in the description of $W^?(\overline{\rho})$, and the 12 remaining shadow weights correspond to the cases where $\mu' \neq \mu$. The obvious weights match up with the 8 "obvious" crystalline lifts ³ of $\overline{\rho}$ with Hodge-Tate weights prescribed by $\mu + \rho$, obtained by applying an element of W to $\overline{\rho}$ and then using the congruences of the fundamental characters. When considering μ as a character of Sp₄ one instead considers crystalline lifts only corresponds to having one of the Jordan-Holder factors of $V(\mu)$ as a Serre weight. In fact, restricting to sufficiently generic weights $W^?(\overline{\rho})$ is also the smallest set containing all the obvious weights, and such that if some Jordan-Holder factor of $V(\lambda)$ for $\lambda \in X_1(T)$ is in the set so is $F(\lambda)$. This follows from Proposition 1.6 or by [GHS18, Thm 10.2.11] in much more generality.

Definition 5.1. We say that $\overline{\rho}: G_{\mathbb{Q}_p} \to \mathrm{GSp}_4(\overline{\mathbb{F}}_p)$ is Fontaine-Laffaille if it has a crystalline lift of Hodge-Tate weights in the range [0, p-2]. We can describe all such $\overline{\rho}$ in terms of their associated Fontaine-Laffaille module $M_{\overline{\rho}}$. We fix the Hodge-Tate weights of the lift to be $\{0, l-2, k-1, k+l-3\}$. Then $M_{\overline{\rho}}$ is a 4-dimensional symplectic $\overline{\mathbb{F}}_p$ -vector space, equipped with a descending filtration M^{\bullet} with jumps at the Hodge-Tate weights, and a symplectic isomorphism $\phi: \operatorname{gr} M^{\bullet} \to M$. Define its conjugate filtration by $D_i = \sum_{j \leq i} \phi(\operatorname{gr}^j M)$.

We can classify $\overline{\rho}$ according to the conjugacy class of $w \in W$ appearing in $\tau(\mu, w) = \overline{\rho}_{|I}^{ss}$: $\{s_0s_1, s_1s_0\}, \{w_0\}, \{s_0, s_1s_0s_1\}, \{s_1, s_0s_1s_0\}, \{1\}$. We describe the different $\overline{\rho}$ with $\tau = \tau(\mu, w)$ for some $\mu = (k - 1, l - 2, k + l - 3) \in X_1(T)$. We ignore unramified characters throughout.

³The lift only has to agree with $\overline{\rho}$ restricted to inertia.

(1) (Irreducible)

$$\overline{\rho} = \operatorname{Ind}_{\mathbb{Q}_p^4}^{\mathbb{Q}_p} \omega_4^a$$

for $a = a_0 + a_1 p + a_2 p^2 + a_3 p^3$ with $0 \le a_i \le p - 1$ and $a_0 + a_2 = a_1 + a_3$. There are two options:

Irred
$$(s_0s_1)$$
: $a = (k + l - 3, l - 2, 0, k - 1)$, Irred (s_1s_0) : $a = (k + l - 3, k - 1, 0, l - 2)$,
with $\overline{\rho}_{|I_p} = \text{Diag}(\omega_4^a, \omega_4^{ap}, \omega_4^{ap^3}, \omega_4^{ap^2})$. Their respective $\tau = (\overline{\rho}_{|I_p})^{\text{ss}}$ are
 $\tau(\mu, s_0s_1)$ and $\tau(\mu, s_1s_0)$.

Here $\overline{\rho}$ is always semisimple. If $\mu \in C_0$ all of its Fontaine-Laffaille lifts are irreducible as a GL₄-valued representation, i.e. totally non-ordinary.

(2) (Endoscopic) $\overline{\rho} = \overline{\rho}_1 \oplus \overline{\rho}_2$ where the sum is with respect the endoscopic embedding $\operatorname{GL}_2 \times \operatorname{GL}_2/\mathbb{G}_m = \{A, B : \det(A) = \det(B)\} \subseteq \operatorname{GSp}_4$

$$\mathrm{GL}_2 \times \mathrm{GL}_2 / \mathbb{G}_m = \begin{pmatrix} * & 0 & 0 & * \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ * & 0 & 0 & * \end{pmatrix},$$

and $\overline{\rho}_i = \operatorname{Ind}_{\mathbb{Q}_{p^2}}^{\mathbb{Q}_p} \omega_2^{a_i+b_ip}$ for $0 \leq a_i < b_i \leq p-1$ and $a_1 + b_1 = a_2 + b_2 \mod p-1$. There is only one option: $((a_1, b_1), (a_2, b_2)) = ((l-2, k-1), (0, k+l-3))$ since swapping $\overline{\rho}_i$ around corresponds to the action of s_0 by conjugation. Thus

$$\tau = \tau(\mu, w_0).$$

Note that $\overline{\rho}$ is irreducible as a GSp₄ valued representation but not as a GL₄ valued representation. For $\mu \in C_0$ all of its Fontaine-Laffaille lifts are totally non-ordinary. (3) (Siegel parabolic)

$$\overline{\rho} = \begin{pmatrix} \overline{\rho}_1 & * \\ 0 & \omega^c \overline{\rho}_1^{\vee} \end{pmatrix} \text{ where } \overline{\rho}_1 = \operatorname{Ind}_{\mathbb{Q}_{p^2}}^{\mathbb{Q}_p} \omega_2^{a+bp},$$

 $0 \le a < b \le p - 1$. There are 2 options.

(a) (Siegel non-ordinary, $w = s_1 s_0 s_1$) Here $\tau = \tau(\mu, s_1 s_0 s_1)$ and

$$\overline{\rho} = \begin{pmatrix} \operatorname{Ind}\omega_2^{k-1} & * \\ 0 & \operatorname{Ind}\omega_2^{l-2+(k+l-3)p} \end{pmatrix}$$

or the matrix swapping around $\operatorname{Ind} \omega_2^{k-1}$ and $\operatorname{Ind} \omega_2^{l-2+(k+l-3)p}$. For $\mu \in C_0$ all of its Fontaine-Laffaille lifts are totally non-ordinary.

(b) (Siegel ordinary, $w = s_0$) Here $a = k - 1, b = k + l - 3, c = k + l - 3, \tau = \tau(\mu, s_0)$,

$$\overline{\rho} = \begin{pmatrix} \operatorname{Ind}\omega_2^{k-1+(k+l-3)p} & * \\ 0 & \operatorname{Ind}\omega_2^{l-2} \end{pmatrix}$$

For $\mu \in C_0$ all of its Fontaine-Laffaille lifts are Siegel ordinary, i.e. ordinary for the Siegel parabolic.

65

(4) (Klingen parabolic)

$$\overline{\rho} = \omega^c \begin{pmatrix} \omega^{a+b} & * & * \\ 0 & \overline{\rho}_1 & * \\ 0 & 0 & 1 \end{pmatrix} \text{ where } \overline{\rho_1} = \operatorname{Ind}_{\mathbb{Q}_{p^2}}^{\mathbb{Q}_p} \omega_2^{a+bp}$$

where $0 \le a < b \le p - 1$. There are 2 options.

(a) (Klingen ordinary, $w = s_1$) Here $c = 0, a = l - 2, b = k - 1, \tau = \tau(\mu, s_1)$,

$$\overline{\rho} = \omega^c \begin{pmatrix} \omega^{k+l-3} & * & * \\ 0 & \text{Ind}\omega_2^{(l-2)+(k-1)p} & * \\ 0 & 0 & 1 \end{pmatrix}$$

For $\mu \in C_0$ all of its Fontaine-Laffaille lifts are Klingen ordinary.

(b) (Klingen non-ordinary, $w = s_0 s_1 s_0$) Here $\tau = \tau(\mu, s_0 s_1 s_0)$, and

$$\overline{\rho} = \begin{pmatrix} \omega^{k-1} & * & * \\ 0 & \text{Ind}\omega_2^{k+l-3} & * \\ 0 & 0 & \omega^{l-2} \end{pmatrix}$$

or the matrix swapping ω^{k-1} and ω^{l-2} around. For $\mu \in C_0$ all of its Fontaine-Laffaille lifts are totally non-ordinary.

(5) (Borel parabolic) In this case we have $\tau = \tau(\mu, 1)$. In the Fontaine-Laffaille range we can always arrange it so that

$$\overline{\rho} = \begin{pmatrix} \omega^{k+l-3} & * & * & * \\ 0 & \omega^{k-1} & * & * \\ 0 & 0 & \omega^{l-1} & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and all of its Fontaine-Laffaille lifts are Borel ordinary.

Starting with $\overline{\rho}$ Fontaine-Laffaille we list the 4 possibilities for the other Serre weight in the C_0 alcove. If $\tau = \tau(\mu, w)$, the other C_0 weight corresponds to applying $s_0 s_1 s_0$ to a cyclotomic twist of τ and then using the congruences of the fundamental characters to get a $\mu' \in C_0$. Case 4) will be the relevant one in Theorem 5.6.

Proposition 5.1. For $\overline{\rho}$ Fontaine-Laffaille of weights $\{0, l-2, k-1, k+l-3\}$, there exists at most one (a, b) with $a \ge b \ge 3$, $a + b \le p + 1$ and $(a, b) \ne (k, l)$ such that a cyclotomic twist of $\overline{\rho}$ has a crystalline lift of weights $\{0, a-2, b-1, a+b-3\}$. We describe the 4 different options for (a, b) and the condition for $\overline{\rho}$ to have such a crystalline lift. Let $M = M_{\overline{\rho}}$.

- (1) (a,b) = (p-k+1,l) if and only if $\phi(\operatorname{gr}^{k-1}M) \oplus \phi(\operatorname{gr}^{k+l-3}M) = M^{k-1}$. In particular $\overline{\rho}$ is Borel ordinary (a one-dimensional family on inertia) or Siegel ordinary split.
- (2) (a,b) = (p-k+2,l+1) if and only if $\phi(\operatorname{gr}^{k+l-3}M) \subset M^{k-1}$ and $\dim(D_{l-2} \cap M^{k-1}) = 1$. In particular $\overline{\rho}$ is Klingen ordinary (a one-dimensional family on inertia) or $\operatorname{Irred}(s_0s_1)$.
- (3) (a,b) = (p-k+2,l-1) if and only if $M^{k-1} \subset D_{k-1}$ and $\phi(\operatorname{gr}^{k-1}M) \subset M^{k-1}$. In particular $\overline{\rho}$ is Klingen non-ordinary (a one-dimensional family on inertia) or $\operatorname{Irred}(s_1s_0)$.

66

(4) (a,b) = (p-k+3,l) if and only if $D_{l-2} = M^{k-1}$. In particular $\overline{\rho}$ is Endoscopic or Siegel non-ordinary (a one-dimensional family on inertia).

For these weights to be Fontaine-Laffaille we might need extra genericity conditions on (k, l), which we assume in each case.

Proof. We prove 4), which is the only case that will be relevant later, following the notation of [HLM17, Appendix A]. Let $\overline{\rho}$ be Fontaine-Laffaille and Siegel non-ordinary; in the Endoscopic case $D_{l-2} = M^{k-1}$ is automatically satisfied, and one only needs to find an explicit lift. Let $\{e_0, e_{l-2}, e_{k-1}, e_{k+l-3}\}$ be a symplectic basis of $M_{\overline{\rho}}$, with the numbering indicating that they generate the respective graded parts of the Hodge filtration. Then with respect to this basis ϕ has the form (where we are using the contravariant functor from Fontaine-Laffaile modules to $G_{\mathbb{Q}_p}$ -representations, and we are ignoring unramified characters)

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ x & 0 & -y & 1 \\ 1 & 0 & 0 & 0 \\ y & -1 & 0 & 0 \end{pmatrix}$$

for $x, y \in \overline{\mathbb{F}}_p$. Then x = 0 is equivalent to $M^{k-1} = D_{l-2}$. We want to check that $\overline{\rho} \otimes \omega^{2-k}$ is the only cyclotomic twist that can potentially have a Fontaine-Laffaille lift with weights $\{0, a - 2, b - 1, a + b - 3\}$, and that in that case x = 0. Consider the functors

$$D: \overline{\mathbb{F}}_p - \mathcal{FL}^{[0,p-2]} \xrightarrow{F} \overline{\mathbb{F}}_p - \operatorname{BrMod}_0^{p-2} \xrightarrow{G} \overline{\mathbb{F}}_p - \mathfrak{Mod}_{\overline{\mathbb{F}}_p[[T]]}^{p-2}$$

and

$$\tilde{D}: \operatorname{Rep}_{\overline{\mathbb{F}}_p} G_{\mathbb{Q}_p} \xrightarrow{\operatorname{Res}} \operatorname{Rep}_{\overline{\mathbb{F}}_p} G_{\mathbb{Q}_{p,\infty}} \xrightarrow{H} \overline{\mathbb{F}}_p \operatorname{-\mathfrak{Mod}}_{\overline{\mathbb{F}}_p((T))}$$

defined in [HLM17, A.1] and [Bre99]. The functors G and H are equivalences. The two are fit in a commutative diagram together with $T: \overline{\mathbb{F}}_p - \mathcal{FL}^{[0,p-2]} \to \operatorname{Rep}_{\overline{\mathbb{F}}_p} G_{\mathbb{Q}_p}$ and the fully faithful localization $\overline{\mathbb{F}}_p - \mathfrak{Mod}_{\overline{\mathbb{F}}_p[[T]]}^{p-2} \to \overline{\mathbb{F}}_p - \mathfrak{Mod}_{\overline{\mathbb{F}}_p((T))}$. Briefly, $\overline{\mathbb{F}}_p - \mathfrak{Mod}_{\overline{\mathbb{F}}_p[[T]]}^{p-2}$ is the category of finite projective modules over $\overline{\mathbb{F}}_p[[T]]$ with a semilinear φ whose cokernel is killed by T^{p-2} , and $\overline{\mathbb{F}}_p$ -BrMod $_0^{p-2}$ consists of modules N over $S = \overline{\mathbb{F}}_p[T]/T^p$ together with a submodule $\operatorname{Fil}^{p-2}N \subset N$ and a semilinear $\varphi: \operatorname{Fil}^{p-2}N \to N$ whose image generates N (and a few other smaller conditions). By following the definition of D we see that $D(M_{\overline{\rho}})$ has a basis $\{v_0, v_{l-2}, v_{k-1}, v_{k+l-3}\}$ such that the matrix of φ has the form

$$\begin{pmatrix} 0 & 0 & T^{k-1} & 0 \\ a_1 & 0 & -T^{k-1}a_2 & T^{k+l-3} \\ 1 & 0 & 0 & 0 \\ a_2 & T^{l-2} & a_4 & 0 \end{pmatrix}$$

where the *T*-adic valuation of a_4 satisfies $v(a_4) \ge k$ and $v(a_1) > 0$ is equivalent to x = 0. By compatibility of *D* and \tilde{D} and tensor functoriality, $\tilde{D}(\overline{\rho} \otimes \omega^{2-k})$ has a basis such that ϕ has the form

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ T^{p-k+1}a_1 & 0 & -a_2 & T^{l-2} \\ T^{p-k+2} & 0 & 0 & 0 \\ a_2T^{p-k+2} & T^{p+l-k} & T^{2-k}a_4 & 0 \end{pmatrix}.$$

For $k-l \geq 2$ and $v(a_1) = 0$, $\tilde{D}(\overline{\rho} \otimes \omega^{2-k})$ comes from a Breuil module N with $\operatorname{Fil}^{p-2}N$ generated by $\{T^{k-3}v_0, T^{k-l-2}v_{l-2}, T^{p-2}v_{k-1}, T^{k-4}v_0 - \lambda T^{p-1-l}v_{k+l-3}\}$ for some non-zero λ depending on $a_1(0)$. The Breuil modules coming from Fontaine-Laffaille modules have the property that $\operatorname{Fil}^{p-2}N \subset N$ is represented by a diagonal matrix in some basis. Since the submodule above is given by a non-semisimple matrix we see that for $\overline{\rho} \otimes \omega^{2-k}$ to have a Fontaine-Laffaille lift one needs $v(a_1) > 0$, and hence x = 0. By a similar computation one sees that $\overline{\rho} \otimes \omega^{2-k}$ is the only cyclotomic twist that can have a Fontaine-Laffaille lift of the required weights. Finally, in the case when $v(a_1) > 0$ we can check that $\tilde{D}(\overline{\rho} \otimes \omega^{2-k})$ comes from applying D to the Fontaine-Laffaille module with the right weights and ϕ given by

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -a_2(0) & 1 \\ 1 & 0 & 0 & 0 \\ a_2(0) & -1 & 0 & 0 \end{pmatrix}$$

as desired.

5.2. The generic entailment of Serre weights. We first explain how to translate between coherent and étale cohomology. For $V \in \operatorname{Rep}_{\overline{\mathbb{F}}_p} G$ there is an associated $\operatorname{GSp}_4(\mathbb{F}_p)$ -local system on $\operatorname{Sh}_{\mathbb{Q}_p}$ with the same name, by composing the local system A[p] with $V_{|G(\mathbb{F}_p)}$. Generically for *p*-restricted weights we can compare the Hecke eigensystems appearing in coherent and étale cohomology, but only when the coefficients for étale cohomology are Weyl modules. Recall that the Hecke algebra away from the places dividing the level \mathbb{T}/\mathbb{Z}_p acts both on coherent and étale cohomology, and we say that a maximal ideal $\mathfrak{m} \subset \mathbb{T}$ appearing in either cohomology is non-Eisenstein if the associated residual Galois representation is irreducible as a GL_4 valued representation. We will also use the notion of genericity in [HL23, Def 1.1], which is a mild condition on the local Galois representation at an auxiliary prime $l \neq p$.

Proposition 5.2. Let $\lambda = (a, b) \in X^*(T)$ be a dominant weight. Let $\mathfrak{m} \subset \mathbb{T}$ be a non-Eisenstein generic maximal ideal. Then

$$H^{3}_{\acute{e}t}(\mathrm{Sh}_{\overline{\mathbb{Q}}_{p}}, V(a, b) \otimes \overline{\mathbb{F}}_{p})_{\mathfrak{m}} = H^{3}_{\acute{e}t,c}(\mathrm{Sh}_{\overline{\mathbb{Q}}_{p}}, V(a, b) \otimes \overline{\mathbb{F}}_{p})_{\mathfrak{m}},$$
$$H^{0}(\mathrm{Sh}_{\overline{\mathbb{F}}_{p}}^{\mathrm{tor}}, \omega^{\mathrm{can}}(a+3, b+3))_{\mathfrak{m}} = H^{0}(\mathrm{Sh}_{\overline{\mathbb{F}}_{p}}^{\mathrm{tor}}, \omega^{\mathrm{sub}}(a+3, b+3))_{\mathfrak{m}},$$

and

$$H^3_{\acute{e}t}(\mathrm{Sh}_{\mathbb{Q}_p}, V(a, b) \otimes \overline{\mathbb{F}}_p)_{\mathfrak{m}} \neq 0 \implies H^0(\mathrm{Sh}^{\mathrm{tor}}_{\overline{\mathbb{F}}_p}, \omega^{\mathrm{can}}(a+3, b+3))_{\mathfrak{m}} \neq 0,$$

where for M a \mathbb{T} -module $M_{\mathfrak{m}}$ is the localization. Moreover, $H^*_{\acute{e}t}(\operatorname{Sh}_{\mathbb{Q}_p}, V(a, b) \otimes \overline{\mathbb{F}}_p)_{\mathfrak{m}}$ is concentrated in degree 3. Further, if $\lambda = (a, b) \in X_1(T)$ satisfies $b \geq 2$ or b = 1 and $a \leq p - 3$ then the reverse implication to the one above is also true.

68

Proof. The first equality follows from a similar (but simpler) argument to the one in [NT16, Thm 4.2]. To adapt their method one needs to prove that the cohomology of the boundary of the Borel-Serre compactification of Sh_K is Eisenstein. The boundary is stratified by locally symmetric spaces associated to rational parabolics of G, whose Levi quotients in this case are just products of GL_2/\mathbb{Q} with a rational torus. Since one can attach Galois representations to cohomological automorphic representations for these Levi subgroups one can show that the Hecke eigenclasses appearing in the boundary are always Eisenstein: they are either the sum of 4 characters (Borel), the sum of two 2-dimensional representations (Siegel), or the sum of 2 characters and a 2-dimensional representation (Klingen). The second equality is proved in [AH21, Thm 24.iii)].

For the implication about existence of nontrivial eigensystems, the vanishing theorem of [HL23] and the first equality imply that $H^*_{\text{ét}}(\operatorname{Sh}_{\overline{\mathbb{Q}}_p}, V(a, b) \otimes \overline{\mathbb{F}}_p)_{\mathfrak{m}}$ is concentrated in degree 3. This means that we can lift any non-trivial mod p eigensystem to $H^3_{\text{\acute{e}t}}(\operatorname{Sh}_{\overline{\mathbb{Q}}_p}, V(a, b) \otimes \overline{\mathbb{Z}}_p)_{\mathfrak{m}}$. The latter is also torsion-free, so that there exists an ideal $\tilde{\mathfrak{m}} \subset \mathbb{T}$ corresponding to an integral Hecke eigensystem in $H^*_{\text{\acute{e}t}}(\operatorname{Sh}_{\overline{\mathbb{Q}}_p}, V(a, b) \otimes \overline{\mathbb{Z}}_p)_{\mathfrak{m}}$. Then

$$H^{3}_{\text{\acute{e}t}}(\operatorname{Sh}_{\overline{\mathbb{Q}}_{p}}, V(a, b)_{\overline{\mathbb{Q}}_{p}})_{\tilde{\mathfrak{m}}} = \bigoplus_{w \in W^{M}} H^{l(w)}(\operatorname{Sh}_{\overline{\mathbb{Q}}_{p}}^{\operatorname{tor}}, \omega(w \cdot (a, b) + (3, 3)))_{\tilde{\mathfrak{m}}}$$
(5.1)

by Theorem 4.24, and the rational étale-crystalline comparison theorem. Moreover, since the Galois representation attached to $\tilde{\mathfrak{m}}$ is irreducible we have that the non-vanishing of the left-hand side implies the non-vanishing of $H^0(\operatorname{Sh}_{\overline{\mathbb{Q}}_p}^{\operatorname{tor}}, \omega(a+3, b+3))_{\tilde{\mathfrak{m}}}$. This follows from the fact in the étale cohomology localized at such $\tilde{\mathfrak{m}}$ (concretely in the case where the underlying automorphic representations is not CAP, which have reducible Galois representations [Wei05, Thm 2]) one only sees contributions from automorphic representations whose components at infinity belong to the discrete series [Wei09, §4.1]. Then [Wei05, Prop 1.5] (using that weakly endoscopic representations are not non-Eisenstein) implies that the multiplicity of the (anti)holomorphic discrete series, corresponding to coherent cohomology in degree 0 and 3 [Wei09, §1.1], equals the one of the nonholomorphic discrete series. By finding an appropriate lattice of $H^0(\operatorname{Sh}_{\overline{\mathbb{Q}}_p}^{\operatorname{tor}}, \omega(a+3, b+3))_{\tilde{\mathbb{m}}}$ and reducing mod p we get one implication. For the other direction, by Theorem 1.8(1) $H^1(\operatorname{Sh}_{\overline{\mathbb{F}}_p}^{\operatorname{tor}}, \omega^{\operatorname{sub}}(a+3, b+3)) = 0$ under the restrictions on the weights, so by the second equality we can lift eigensystems in $H^0(\operatorname{Sh}_{\overline{\mathbb{F}}_p}^{\operatorname{tor}}, \omega^{\operatorname{can}}(a+3, b+3))_{\mathfrak{m}}$ to characteristic 0. Using (5.1) and reducing mod p implies what we wart.

We can now explain how Theorem 3.13 proves a generic entailment of Serre weights.

Theorem 5.3. Let $\lambda_0 = (a, b) \in C_0$ satisfying $b \geq 1$, and $\mathfrak{m} \subset \mathbb{T}$ a non-Eisenstein generic eigensystem appearing in $H^3_{\acute{e}t}(\operatorname{Sh}_{\overline{\mathbb{Q}}_p}, F(\lambda_0))$. Then \mathfrak{m} has either $\lambda_1 = (p - b - 3, p - a - 3) \in C_1$ or $\lambda_2 = (p + b - 1, p - a - 3) \in C_2$ as Serre weights, i.e. $H^3_{\acute{e}t}(\operatorname{Sh}_{\overline{\mathbb{Q}}_p}, F(\lambda_i))_{\mathfrak{m}}$ is non-zero for i = 1 or i = 2.

Proof. By Proposition 1.6 we have the exact sequence

$$0 \to F(\lambda_1) \to V(\lambda_1) \to F(\lambda_0) \to 0$$

in $\operatorname{Rep}_{\overline{\mathbb{F}}_p}\operatorname{GSp}_4(\mathbb{F}_p)$. By Proposition 5.2 we can also find \mathfrak{m} in $H^3_{\operatorname{\acute{e}t}}(\operatorname{Sh}, V(\lambda_1)_{\overline{\mathbb{F}}_p})$ and $H^0(\overline{\operatorname{Sh}}, \omega(p-b,p-a))$. Assuming that $b \geq 1$ the map $\theta^1_{(p-b,p-a)}$ is injective on global sections localized at \mathfrak{m} by Theorem 3.13, so it transports the eigensystem to $H^0(\overline{\operatorname{Sh}}, \omega(p+b+2, p-a))$. By applying Proposition 5.2 again we can also see it on $H^3_{\operatorname{\acute{e}t}}(\operatorname{Sh}, V(\lambda_2)_{\overline{\mathbb{F}}_p})$. By Proposition 1.6

$$0 \to F(\lambda_2) \to V(\lambda_2) \to F(\lambda_1) \to 0$$

is exact, so we conclude again by the concentration of étale cohomology in Proposition 5.2. \Box

Remark 5.1. We give a sketch proof of the implication of Theorem 5.3 to the irreducible components of some Breuil-Mezard cycles alluded to in the introduction. We assume that the Breuil-Mezard conjecture is true, and that the cycles are effective, i.e. the multiplicity of each irreducible component is non-negative. We also assume the weight part of Serre's conjecture in the sense that $W(\overline{\rho}) = \{\sigma : \mathcal{Z}_{\sigma}(\overline{\rho}) \neq 0\}$, where $\mathcal{Z}_{\sigma}(\overline{\rho})$ denotes the base change to the versal ring $R_{\overline{\rho}}$. Further, we assume strong enough globalization results as we explain below. We want to prove that for sufficiently generic $\lambda_0 \in C_0$ (and $p \geq 7$), then $\mathcal{Z}_{V(\lambda_2)}$ contains the irreducible component $C_{F(\lambda_0)}$. We have that $\mathcal{Z}_{F(\lambda_0)}$ contains $C_{F(\lambda_0)}$ by the weight part of Serre's conjecture and the following globalization result. By [EL23, Lem 4.4.5] we can choose some $\overline{\rho} \in C_{F(\lambda_0)}$ maximally non-split ordinary which only lies in that component, and such that it has an automorphic globalization \overline{r} over a totally real field F/\mathbb{Q} where p is totally split. Moreover, its Hodge-Tate weights are parallel of weight $\lambda_0 + \rho$. Therefore we can assume \overline{r} comes from a Hilbert-Siegel modular form at hyperspecial level of parallel weight $\lambda_0 + (3,3)$. In this setting all the theta operators and Hasse invariants behave as several copies of the case GSp_4/\mathbb{Q}_p , so one could prove the analogue of Theorem 3.13 for each place of F almost verbatim. However, to prove the analogue of Theorem 5.3 one would need the analogue of Proposition 5.2, where one needs to lift mod p coherent cohomology to characteristic 0, and this relies on variants of the vanishing results of [EV92], which require the dimension of the Hilbert-Siegel modular variety to be less than p. The results of [EL23] don't give a good enough bound on the degree of F, therefore we assume that one can take $F = \mathbb{Q}$ (or of small degree with respect to p) and \overline{r} to be generic. Then from $F(\lambda_0) \in W(\overline{\rho})$ and the analogue of Theorem 5.3 (and the generalization of the weight part Serre's conjecture in terms of Breuil-Mezard cycles by considering the restriction to each place) we get that $\mathcal{Z}_{V(\lambda_2)}(\overline{\rho}) \neq 0$. The effectivity of the cycles, and the fact that $\overline{\rho}$ only lies in one irreducible component implies the result.

In this strategy to produce an entailment there is a delicate relation to the story on the flag variety G/B. Similar maps of baby Verma modules $\operatorname{Ver}_{B^-}^0$ to the ones in Proposition 4.18 produce maps of Weyl modules $V(\lambda_2) \to V(\lambda_1) \to V(\lambda_0)$ by looking at global sections on the flag variety. However crucially this composition is 0 by Proposition 1.6, unlike the corresponding composition of maps on $\mathcal{F}l$ (as maps of sheaves).

In fact, starting from $f \in H^0(\overline{\operatorname{Sh}}, \omega(a+3, b+3))_{\mathfrak{m}}$ as in Theorem 5.3, applying another theta linkage map yields $\theta_{(a+3,b+3)}^{\alpha+\beta}(f) \in H^0(\overline{\operatorname{Sh}}, \omega(p-b, p-a))$, which would potentially give another way to construct the entailment, without first lifting to étale cohomology. However we don't know if $\theta_{(a+3,b+3)}^{\alpha+\beta}(f)$ is necessarily non-zero, although one might hope the map is injective after localizing at a maximally nonsplit ordinary $\overline{\rho} \in C_{F(\lambda_0)}$, which would still imply the inclusion of Breuil-Mezard cycles. On the other hand, for $\lambda_2 \in C_2$ we know that $\theta_{\lambda_2}^{\alpha+\beta}$ cannot be injective on $H^0(\overline{Sh}, -)$ even after localizing at a maximally non-split $\overline{\rho} \in C_{F(\lambda_1)}$, otherwise the same argument as in Theorem 5.3 together with Proposition 1.6 would imply that $F(\lambda_1)$ entails $F(\lambda_2)$ or $F(\lambda_3)$, where λ_i are the linked weights in the respective alcoves. However, from weight elimination results, or knowledge about the Breuil-Mezard cycles, one knows that this not the case. This suggests that in general kernels of theta linkage maps should cut out some locus in the stack of Galois representations corresponding to $\overline{\rho}$ having a particular Serre weight, i.e. a union of Breuil-Mezard cycles.

5.3. A companion form approach. We now explain a partial result towards producing a C_0 Serre weight for a particular non-ordinary Fontaine-Laffaille $\overline{\rho}$, in the spirit of [FJ95] and [Til12]. Although their approach concerns an ordinary $\overline{\rho}$, we follow the same basic strategy of using the BGG complex and Fontaine-Laffaille theory to geometrically extract information about $\overline{\rho}$.

Let \mathcal{O}/\mathbb{Z}_p be the ring of integers of a finite extension F/\mathbb{Q}_p , with uniformizer π and residue field κ . Let $f \in H^0(Sh^{tor}_{\mathcal{O}}, \omega(k, l))$ be a non-Eisenstein generic eigenform where $(k - 3, l - 3) \in C_0$, with associated Galois representation $r_f: G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathcal{O})$. Let $\mathfrak{m} \subset \mathbb{T}_{\mathcal{O}}$ be the maximal ideal associated to (π, f) . It is known that under the hypothesis on the maximal ideal $r_f^{\vee} \otimes F$ is a subrepresentation of $H^3_{\text{\acute{e}t}}(\operatorname{Sh}_{\overline{\mathbb{O}}}, V(k-3, l-3)_F)_{\mathfrak{m}}$ [Wei05], and since \overline{r}_f is irreducible all lattices of $r_f^{\vee} \otimes F$ are homothetic to each other. By Proposition 5.2 we know that $H^3_{\text{ét}}(\operatorname{Sh}_{\overline{\mathbb{O}}}, V(k-3, l-3)_{\mathcal{O}})_{\mathfrak{m}}$ is free as a \mathcal{O} module, so we may assume it contains r_f^{\vee} as a subrepresentation. The local Galois representation $\rho \coloneqq r_{f,G_{\mathbb{Q}_p}}$ is a lattice of a crystalline representation of Hodge-Tate weights in [0, p-2], and these are equivalent to strongly divisible modules. By the integral étale-crystalline comparison theorem [LS13, §5.2] the strongly divisible module M_{ρ} (here we are using the contravariant Fontaine-Laffaille functor) associated to ρ is a submodule of $H^3_{dR}(\operatorname{Sh}^{\operatorname{tor}}_{\mathcal{O}}, V(k-3, l-3))_{\mathfrak{m}}$. Its reduction mod p is a Fontaine-Laffaille module $M_{\overline{\rho}}$, and it is a subobject of $H^3_{dR}(Sh^{tor}_{\kappa}, V(k-3, l-3))_{\mathfrak{m}}$ by the torsion freeness of the integral crystalline cohomology. The Fontaine-Laffaille module is a 4-dimensional symplectic κ -vector space equipped with a descending Hodge filtration $M_{\overline{\rho}}^{\bullet} \subset M_{\overline{\rho}}$ with weights $\{0, l-2, k-1, k+l-3\}$, and an isomorphism $\phi : \operatorname{gr}^{\bullet} M_{\overline{\rho}} \cong M_{\overline{\rho}}$. By Theorem 4.24 we have that $\operatorname{gr}^{\bullet} M_{\overline{\rho}}$ is a submodule of $\oplus H^{l(w)}(\operatorname{Sh}_{\kappa}^{\operatorname{tor}}, \omega^{\operatorname{can}}(w \cdot (k-3, l-3) + (3, 3)))_{\mathfrak{m}}$, and ϕ is constructed by dividing the crystalline Frobenius by appropriate powers of p. For a Fontaine-Laffaille module M^{\bullet} its conjugate filtration is an ascending filtration defined by $D_i := \bigoplus_{j \leq i} \phi(\operatorname{gr}^j M)$.

Proposition 5.4. Let $M = H^3_{dR}(Sh^{tor}_{\kappa}, V(k-3, l-3))_{\mathfrak{m}}$ with $(k-3, l-3) \in C_0$, considered as a Fontaine-Laffaille module. The conjugate filtration on $BGG^{can}_{Sh^{tor}_{\kappa}}(k-3, l-3)$ obtained by taking the truncation that preserves cohomology (with the same weights as the Hodge filtration) induces an ascending filtration on $M = H^3(BGG^{can}_{Sh^{tor}_{\kappa}}(k-3, l-3))_{\mathfrak{m}}$. The associated spectral sequence degenerates on its first page, and this filtration agrees with the conjugate filtration D_{\bullet} of M as a

Fontaine-Laffaille module. In particular the Hodge/conjugate filtration on $M_{\overline{\rho}}$ is a submodule of the Hodge/conjugate filtration induced by the BGG complex.

Proof. The quasi-isomorphism of Theorem 4.22 respects the Hodge filtrations, and since the Hodgede Rham spectral sequence degenerates so does the conjugate one, since its graded pieces are isomorphic by the Cartier isomorphism. The comparison between the two filtrations on the Fontaine-Laffaille module is a result of Mazur [Maz73, Thm 7.6] in the case of trivial coefficients, which can be adapted to work with coefficients as in [LS12, Prop 4.8/12] and [LS13, §9.2]. The statement about the Hodge/conjugate filtration of $M_{\overline{\rho}}$ follows from the fact that if $N \subset M$ is an embedding of Fontaine-Laffaille modules, the Hodge and conjugate filtrations on N are the intersection of the ones of M with N. For the Hodge filtration this is [FL82, Prop 1.10(b)], which implies that for every i there are embeddings $\phi(\operatorname{gr}^i N) \hookrightarrow \phi(\operatorname{gr}^i M) \cap N$. The sum of these embeddings over all iis an equality since ϕ is an isomorphism, so for every integer m their sum for $i \leq m$ must be an equality.

We prove a commutation relation between several theta linkage maps, some of them appearing in the BGG complex.

Lemma 5.5. Let $(k,l) \in X^*(T)$ satisfying $k \ge l \ge 3$, k+l < p+3 and k < p-1. The two top rows of the following diagram describe a part of BGG^{can}_{Fl}(k-3,l-3) up to a scalar in $\overline{\mathbb{F}}_p^{\times}$, and the lower square commutes up to $\overline{\mathbb{F}}_p^{\times}$

Proof. All the maps in the diagram come from maps of B-Verma modules by Corollary 4.19, so we prove everything at the level of maps of Verma modules. The maps in the BGG complex shown are easy to determine up to a scalar since the weight increase of those maps are multiples of simple roots. We argue that none of them is identically zero. For this we can work on the flag variety, since the BGG complex also computes de Rham cohomology of the flag variety. If $\operatorname{Ver}_B(k-3, 1-l) \to \operatorname{Ver}_B(k-3, l-3)$ were zero, then on G/P the kernel of the trivial connection on $\mathcal{V}(k-3, l-3)$ would be equal to $\mathcal{W}(3-l, 3-k)$. However, their respective ranks after Frobenius pushforward are $\frac{1}{6}(k-l+1)(l-2)(k-1)(k+l-3)$, which is not divisible by p, and $p^3(k-l+1)$. A similar dimension count shows that $\operatorname{Ver}_B(l-4, -k) \to \operatorname{Ver}_B(k-3, 1-l)$ is non-zero, hence the remaining two maps must be non-zero too. For the lower square, by anticommutativity of the piece of the BGG complex we have $\theta^1_{(k,4-l)} \circ d_1 \circ \theta^1_{(3-l,3-k)} = \frac{-1}{H_1^p} \theta_1^p \circ \theta_{(2p-l+3,3-k)}^4 = -\theta_{(2p-l+3,3-k)}^4 \circ \frac{1}{H_1^p} \theta_1^p$
up to $\overline{\mathbb{F}}_p^{\times}$, the last equality since $X_{-\beta}^p$ is in the center of $U\mathfrak{g}$. Therefore the square commutes after precomposing with $\theta_{(3-l,3-k)}^1$. Since the map $\operatorname{Ver}_B(k-3,1-l) \to \operatorname{Ver}_B(k-3,l-3)$ corresponding to $\theta_{(3-l,3-k)}^1$ is injective, we see that the lower square also commutes.

Remark 5.2. The commutativity of the lower square would follow immediately if one knew that the Hom space of *B*-Verma modules is at most 1-dimensional.

We now give a recipe that starting with $\lambda \in C_0$, potentially produces the other C_0 Serre weight in the case 4) of Proposition 5.1. Explicitly: starting with $(k-3, l-3) \in C_0$ the following sequence of maps

$$\mathcal{L}(k,l) \xrightarrow{\theta_{(k,l)}^1} \mathcal{L}(2p-k+2,l) \xleftarrow{\theta_4} \mathcal{L}(p-k+3,l)$$

ends up in such a weight. In the sense that if for some eigenform $f \in \mathcal{L}(k, l)$ we have that $\theta_{(k,l)}^1(f)$ is in the image of θ_4 , then one gets an eigenform of the correct weight. For the $\overline{\rho}$ that conjecturally have this Serre weight according to Proposition 5.1 we prove a partial result suggesting that one should be able to "divide" by θ_4 when applying the recipe above twisted by s_1

$$\mathcal{L}(k,4-l) \xrightarrow{\theta_{(k,4-l)}^{1}} \mathcal{L}(2p-k+2,4-l) \xleftarrow{\theta_{4}} \mathcal{L}(p-k+3,4-l)$$

to $H^1(\overline{\mathrm{Sh}}^{\mathrm{tor}}, -)$, where the condition on $\overline{\rho}$ can be naturally described by comparing the Hodge and conjugate filtration of the BGG complex. Recall Definition 5.1 for the notation.

Theorem 5.6. Let $(k,l) \in X^*(T)$ satisfying $k \ge l \ge 3$, $k+l \le p+1$, and $f \in H^0(\operatorname{Sh}_{\mathcal{O}}, \omega(k, l))$ a non-Eisenstein generic eigenform such that the associated Fontaine-Laffaille module $M_{\overline{\rho}}$ satisfies $M_{\overline{\rho}}^{k-1} = D_{\overline{\rho},l-2}$. There exists an eigenform $g \in H^1(\operatorname{Sh}_{\mathcal{O}}^{\operatorname{con}}, \omega^{\operatorname{can}}(k, 4-l))$ with the same mod peigensystem as f. Assume the following conditions.

- (1) $\theta^1_{(k,4-l)}(\overline{g}) \neq 0.$
- (2) $H^1(\overline{\mathrm{Sh}}^{\mathrm{tor}}, \omega^{\mathrm{can}}(p-k+2, -p(p+l-k-4)-k+2))_{\mathfrak{m}_f} = 0.$
- (3) $H^1(\operatorname{Sh}^{\operatorname{tor}}, \omega^{\operatorname{can}}(p-k+3, 4-l))_{\mathfrak{m}_f}$ is torsion free and $H^2(\operatorname{Sh}^{\operatorname{tor}}, \omega^{\operatorname{can}}(p-k+3, 4-l))_{\mathfrak{m}_f} = 0.$ This holds automatically if $k-l \geq 3$ and $l \geq 5$ by Theorem 1.8(2,3) and Proposition 5.2.

Then there exists another eigenform $h \in H^1(\operatorname{Sh}^{\operatorname{tor}}_{\mathcal{O}}, \omega^{\operatorname{can}}(p-k+3, 4-l))$ such that $\theta^1_{(k,l)}(\overline{g}) = \theta_4(\overline{h})$ and $\overline{r}_h = \overline{r}_f$ up to cyclotomic twist. In particular $F(p-k, l-3) \in W(\overline{p}_f)$.

Proof. By the setup at the beginning of this subsection we can find such a $g \in H^1(\operatorname{Sh}_{\mathcal{O}}^{\operatorname{tor}}, \omega^{\operatorname{can}}(k, 4-l))$, in fact under our assumptions we can take its mod p reduction \overline{g} to be a generator of one of the graded pieces of $M_{\overline{\rho}}$. We can describe the condition on $M_{\overline{\rho}}$ in terms of the BGG complex. By Proposition 5.4 the composition $M^{k-1} \to M \to M/D_{l-2}$ is induced by the following map of complexes

$$[0 \to 0 \to \omega(k, 4-l) \xrightarrow{a_2} \omega(k, l)] \to [0 \to 0 \to \omega(k, 4-l)/\mathrm{Im}d_1 \to \omega(k, l)]$$

A computation with Cech cohomology (after taking the Frobenius pushforward) shows that $M_{\overline{\rho}}^{k-1} = D_{\overline{\rho},l-2}$ implies that \overline{g} lies in

 $\operatorname{Ker}(H^{1}(\overline{\operatorname{Sh}}^{\operatorname{tor}}, \omega^{\operatorname{can}}(k, 4-l)) \to H^{1}(\overline{\operatorname{Sh}}^{\operatorname{tor}}, \operatorname{coker} d_{1})) = \operatorname{Im}(H^{1}(\overline{\operatorname{Sh}}^{\operatorname{tor}}, \operatorname{Im}(d_{1})) \to H^{1}(\overline{\operatorname{Sh}}^{\operatorname{tor}}, \omega(k, 4-l)),$ where $d_{1} : \omega(l-1, 3-k) \to \omega(k, 4-l)$ is a differential of $\operatorname{BGG}_{\operatorname{Sh}}^{\operatorname{can}}(k-3, l-3).$ Let $\tilde{d}_{1} : \mathcal{L}(l-1, 3-k) \to \mathcal{L}(k, 4-l)$ be the differential of $\operatorname{BGG}_{\mathcal{F}l}^{\operatorname{can}}(k-3, l-3)$ satisfying $\pi_{*}\tilde{d}_{1} = d_{1}$, which exists by Theorem 4.22. By Lemma 1.2 $H^{1}(\mathcal{F}l^{\operatorname{tor}}, \mathcal{L}(k, 4-l)) = H^{1}(\overline{\operatorname{Sh}}^{\operatorname{tor}}, \omega(k, 4-l)).$ Similarly, applying Lemma 1.2(3) to the exact sequence $0 \to \operatorname{Im}\tilde{d}_{1} \to \mathcal{L}(k, 4-l) \to \operatorname{coker}\tilde{d}_{1} \to 0$, together with $R^{2}\pi_{*}\operatorname{Im}\tilde{d}_{1} = 0$ (even though $\operatorname{Im}\tilde{d}_{1}$ is not coherent, its Frobenius pushforward is) we obtain $H^{1}(\mathcal{F}l^{\operatorname{tor}}, \operatorname{coker}\tilde{d}_{1}) = H^{1}(\overline{\operatorname{Sh}}^{\operatorname{tor}}, \pi_{*}\operatorname{coker}\tilde{d}_{1}).$ There is a natural map $\operatorname{coker} d_{1} \to \pi_{*}\operatorname{coker}\tilde{d}_{1}$, therefore $\operatorname{Ker}(H^{1}(\overline{\operatorname{Sh}}^{\operatorname{tor}}, \omega(k, 4-l)) \to H^{1}(\overline{\operatorname{Sh}}^{\operatorname{tor}}, \operatorname{coker} d_{1})) \subseteq \operatorname{Ker}(H^{1}(\mathcal{F}l^{\operatorname{tor}}, \mathcal{L}(k, 4-l)) \to H^{1}(\mathcal{F}l^{\operatorname{tor}}, \operatorname{coker}\tilde{d}_{1})),$ so we can find $\overline{s} \in H^{1}(\mathcal{F}l^{\operatorname{tor}}, \operatorname{Im}\tilde{d}_{1})$ mapping to \overline{g} Hecke equivariantly.

We claim that $\theta_{(k,4-l)}^1(\overline{s})$ lies in the image of $\theta_4 : H^1(\mathcal{F}l^{\text{tor}}, \mathcal{L}(p-k+3, 4-l)) \to H^1(\mathcal{F}l^{\text{tor}}, \mathcal{L}(2p-k+2, 4-l))$. First we prove that it lies in the image of $H^1(\mathcal{F}l^{\text{tor}}, \text{Im}(\theta_4)) \to H^1(\mathcal{F}l^{\text{tor}}, \mathcal{L}(2p-k+2, 4-l))$. For that it is enough to prove the existence of a map $r : \text{Im}\tilde{d}_1 \to \text{Im}\theta_4$ fitting in the commutative diagram

$$\mathcal{L}(l-1,3-k) \xrightarrow{\tilde{d}_1} \mathcal{L}(k,4-l) \xrightarrow{\theta_{(k,4-l)}^1} \mathcal{L}(2p-k+2,4-l)$$

This is equivalent to checking that the composition $t : \mathcal{L}(l-1,3-k) \xrightarrow{\tilde{d}_1} \mathcal{L}(k,4-l) \xrightarrow{\theta_{(k,4-l)}^l} \mathcal{L}(2p-k+2,4-l) \to \operatorname{coker}_{\theta_4} \operatorname{vanishes}$ on $\mathcal{F}l^{\operatorname{tor}}$. We do this in several stages.

(1). First we prove the vanishing of t restricted to U_2^{tor} . By Lemma 5.5 we have $\theta_{(k,4-l)}^1 \circ \tilde{d}_1 = \theta_{(2p-l+3,3-k)}^4 \circ \theta_{(l-1,3-k)}^1$. Since $\theta_{(2p-l+3,3-k)}^4 = \theta_4^{k-l+1}/H_2^{k-l+1}$ factors through $\theta_4 : \mathcal{L}(p-k+3,4-l) \to \mathcal{L}(2p-k+2,4-l)$ over U_2^{tor} , t vanishes.

Lemma 5.7. Let $F : \mathcal{F}l_{\mathbb{F}_p}^{\mathrm{tor}} \to \mathcal{F}l_{\mathbb{F}_p}^{\mathrm{tor}}$ be the absolute Frobenius, and $\mathcal{J}_{D_2^{\mathrm{tor}}}$ the ideal sheaf of the divisor D_2^{tor} . Consider $\theta_4 : \mathcal{L}(k,l) \to \mathcal{L}(k+p-1,l)$. Then $F_* \operatorname{coker} \theta_4$ restricted to U_1^{tor} is coherent, and it is $\mathcal{J}_{D_2^{\mathrm{tor}}}$ -torsion free.

Proof. Write k - l = pa + b for $0 \le b \le p - 1$. The sheaf $F_* \operatorname{coker} \theta_4$ is coherent since θ_4 is a degree 1 differential operator, which are Frobenius linear. We work locally on some chart \mathbb{A}^1_W over $W \subset U_1^{\operatorname{tor}}$ sufficiently small, so that $(\operatorname{coker} \theta_4) | \mathbb{A}^1_W = \mathcal{L}(k + p - 1, l)(\mathbb{A}^1_W)/\operatorname{Im}(\theta_4(\mathbb{A}^1_W))$. Let $f \in \mathcal{L}(k + p - 1, l)^{\operatorname{can}}$ be a local section which is \tilde{H}_2 -torsion on $F_* \operatorname{coker} \theta_4$, so that there exists $g \in \mathcal{L}(k, l)^{\operatorname{can}}$ such that $\tilde{H}_2^p f = \theta_4(g)$. We want to prove f is 0 in $F_* \operatorname{coker} \theta_4$. By étale descent we can check it after pulling back via $\operatorname{Ig}^{\operatorname{tor}} \to \operatorname{Sh}^{\operatorname{ord}, \operatorname{tor}}$, where $\tilde{H}_2 = T - T^p$. Assume first $b \ge 1$.

Using Lemma 2.5 we see that $\tilde{H}_2 \mid -b\frac{d\tilde{H}_2}{dT}g = -bg$, so that $g \in (\tilde{H}_2)$. By cancelling the \tilde{H}_2 and repeating the process of checking divisibility by \tilde{H}_2 we can write $g = \tilde{H}_2^b h$ for some local section $h \in \mathcal{L}(k,l)^{\text{can}}$ satisfying $\tilde{H}_2^{p-b-1}f = \frac{dh}{dT}$. Without loss of generality we can assume that $\tilde{H}_2 \mid h$: we can modify g by an element in the kernel of θ_4 of the form $\tilde{H}_2^b h_1(T^p)$ to arrange that h vanishes on all the roots of $\tilde{H}_2 = T - T^p$. Repeating the argument of before shows that $g = \tilde{H}_2^p s$ for some s. This implies that $f = \theta_4(s)$ as desired. The argument for b = 0 is almost identical.

(2). By Lemma 5.7 above the vanishing of F_*t on U_1^{tor} can be checked on $U_2^{\text{tor}} \cap U_1^{\text{tor}}$, where we know that it vanishes by point (1). Since the absolute Frobenius is an isomorphism of topological spaces t vanishes as a map of abelian sheaves over $U_1^{\text{tor}} \cup U_2^{\text{tor}}$.

Lemma 5.8. Let $(k,l) \in X^*(T)$, consider $\theta_4 : \mathcal{L}(k,l) \to \mathcal{L}(k+p-1,l)$. Let $F : \mathcal{F}l^{\text{tor}} \to \mathcal{F}l^{\text{tor},(p)}$ be the relative Frobenius with respect to $\overline{\text{Sh}}^{\text{tor}}$. Then $F_*\text{Im}\theta_4$ is a coherent sheaf that satisfies Serre's S_2 property.

Proof. Write k - l = ap + b for $0 \le b \le p - 1$. We use the presentation given by Proposition 3.3

where the top row is exact. Let $\mathcal{F} = F_* \operatorname{Im} \theta_4$ and $W = \operatorname{Spec}(R) \subset \overline{\operatorname{Sh}}^{\operatorname{tor}}$ be an affine open, we use the notation of Notation 1. Let \mathfrak{p} a prime ideal of R[T] (or $R[T^{-1}]$) of height at least 3, then by [Sta18, Tag 00LX] and the presentation of \mathcal{F}

$$\operatorname{depth}(\mathcal{F}_{\mathfrak{p}}) \geq \operatorname{depth}((\omega^{(p)}/\mathcal{L}^{(p)})^{a-b} \otimes \operatorname{det}^{l+b}\omega_{\mathfrak{p}}) - 1 \geq 2,$$

so $\mathcal{F}_{\mathfrak{p}}$ satisfies S_2 . Therefore we only need to check S_2 for ideals of height at most 2. First we prove that \mathcal{F} is a vector bundle (hence S_2) on $U_1^{\mathrm{tor}} \cup U_2^{\mathrm{tor}}$. We use that the property of being a vector bundle descends along faithfully flat maps [Sta18, Tag 03C4]. On \mathbb{A}^1_W we have $F_*\mathcal{L}(\lambda) = \bigoplus_{i=0}^{p-1} R[T^p]T^i$ as a free R[T]-module via $R[T] \to R[T^p]$. After picking the usual basis for $\mathcal{L}(\lambda)$ the map j is given by sending T^i to $T^{pi} \cdot 1$, so that the image of j is free of rank 1. This immediately shows that \mathcal{F} is a vector bundle of rank p-1 on U_2^{tor} . For U_1^{tor} it is sufficient to prove it after pullback via the faithfully flat map $\mathrm{Ig^{tor}} \to \overline{\mathrm{Sh}}^{\mathrm{ord,tor}}$, where $\tilde{H}_2 = T - T^p$. Then $F_*\tilde{H}_2^b \circ \mathrm{Im}j$ is generated by the vector $(T - T^p)^b$ in $\bigoplus_{i=0}^{p-1}R[T^p]T^i$, whose coordinate at T^b is 1. Thus its quotient is free of rank p-1. The same holds over the other chart $R[T^{-1}]$, so we have proved that \mathcal{F} is vector bundle over $U_1^{\mathrm{tor}} \cup U_2^{\mathrm{tor}}$. Let $S \coloneqq D_1^{\mathrm{tor}} \cap D_2^{\mathrm{tor}} \cap \pi^{-1}(\overline{\mathrm{Sh}}^{=1})$. Its complement in $D_1^{\mathrm{tor}} \cap D_2^{\mathrm{tor}}$ has dimension 1, since the intersection of $D_1^{\mathrm{tor}} \cap D_2^{\mathrm{tor}}$ are only 1-dimensional on the locus where V = 0, i.e. on $\overline{\mathrm{Sh}}^{\mathrm{ess,tor}}$ of dimension 0.

It is then enough to prove that \mathcal{F} is a vector bundle on all height 2 primes of $\mathcal{O}_{\mathcal{F}l,q}$ for all geometric points $q \in S$. The map $\mathcal{O}_{\mathcal{F}l,q} \to \widehat{\mathcal{O}}_{\mathcal{F}l,q}$ is faithfully flat and preserves the height of primes, so we can check the statement on formal completions. Let $\tilde{q} \in \overline{\mathrm{Sh}}$ be the image of q. By the description of V on $\kappa(\tilde{q})$ of (1.2) the point q has coordinate T = 0 or $T^{-1} = 0$. For T = 0 by repeating the process after (1.2) we can assume that $\widehat{\mathcal{O}}_{\mathcal{F}l,q} = \overline{\mathbb{F}}_p[[T_{11}, T_{12}, T_{22}, T]]$ with $\tilde{H}_1 = T_{22}$ and $\tilde{H}_2 = T_{12} + T_{22}T - T^p$. The line $F_*\tilde{H}_2^b \circ \mathrm{Im}j$ is then given by a vector whose coordinate for 1 is $(T_{12} - T)^b$. Thus $\mathcal{F}_{\widehat{\mathcal{O}}_{\mathcal{F}l,q}}$ is a vector bundle outside the locus cut out by the ideal $(\tilde{H}_1, \tilde{H}_2, T_{12} - T) = (T_{22}, T_{12} - T, T_{12} - T^p)$ of height 3. In particular it is a vector bundle on height 2 prime ideals. For $T^{-1} = 0$ now $\widehat{\mathcal{O}}_{\mathcal{F}l,q} = \overline{\mathbb{F}}_p[[T_{11}, T_{12}, T_{22}, T^{-1}]]$ and $\tilde{H}_2 = -T^{-1} + T_{22}T^{-p} + T_{12}T^{-p-1}$. The coefficient of T^{-b} in the line $F_*\tilde{H}_2^b \circ \mathrm{Im}j$ is $(T_{12}T^{-1} - 1)^b + bT_{22}T^{-1}(T_{12}T^{-1} - 1)^{b-1} + \dots$ which is invertible, so \mathcal{F} is a vector bundle.

(3). To further prove that t vanishes on $\mathcal{F}l^{\text{tor}}$ it is enough to extend the composition $\mathcal{L}(l - 1, 3 - k) \to \text{Im}\tilde{d}_1 \xrightarrow{r} \text{Im}\theta_4$ on $U_1^{\text{tor}} \cup U_2^{\text{tor}}$ to $\mathcal{F}l^{\text{tor}}$, since any such extension would have to fit in the commutative diagram, being a condition that can be checked on an open dense subset. Let $F : \mathcal{F}l^{\text{tor}} \to \mathcal{F}l^{\text{tor},(p)}$ be the relative Frobenius with respect to $\overline{\text{Sh}}$. By Lemma 1.4 and Lemma 5.8 $F_*(\mathcal{L}(l-1,3-k) \to \text{Im}\tilde{d}_1 \xrightarrow{r} \text{Im}\theta_4)$ over $U_1^{\text{tor}} \cup U_2^{\text{tor}}$ extends to $\mathcal{F}l^{\text{tor},(p)}$, being defined outside a subspace of codimension 2. Since the base change map $\mathcal{F}l^{(p)} \to \mathcal{F}l$ is an isomorphism of topological spaces we obtain the desired extension of $\mathcal{L}(l-1,3-k) \to \text{Im}\theta_4$ to $\mathcal{F}l^{\text{tor}}$.

Recall that we want to prove that $\theta^1_{(k,4-l)}(\overline{s})$ lies in the image of $\theta_4 : H^1(\mathcal{F}l^{\text{tor}}, \mathcal{L}(p-k+3, 4-l)) \to H^1(\mathcal{F}l^{\text{tor}}, \mathcal{L}(2p-k+2, 4-l))$. By the previous discussion, and the exact sequence

$$H^1(\mathcal{F}l^{\mathrm{tor}}, \mathcal{L}(p-k+3, 4-l)) \to H^1(\mathcal{F}l^{\mathrm{tor}}, \mathrm{Im}(\theta_4)) \to H^2(\mathcal{F}l^{\mathrm{tor}}, \ker \theta_4)$$

it is enough to prove the vanishing of $H^2(\mathcal{F}l^{\text{tor}}, \text{Ker}\theta_4)_{\mathfrak{m}}$. To do that there is no harm in taking Frobenius pushforward with respect the relative Frobenius $F : \mathcal{F}l^{\text{tor}} \to \mathcal{F}l^{\text{tor},(p)}$. By Proposition 3.3

$$F_* \operatorname{Ker} \theta_4 = (\omega^{(p)} / \mathcal{L}^{(p)})^{-(p+l-k-1)} \otimes \operatorname{det}^{p-k+3} \omega,$$

which by Lemma 1.2 has no $\pi_*^{(p)}$, and $R^1 \pi_*^{(p)} F_* \operatorname{Ker} \theta_4 = \operatorname{Sym}^{p+l-k-3} \omega^{(p)} \otimes \operatorname{det}^{-p(p+l-k-3)+p-k+2} \omega$, so that $H^2(\mathcal{F}l^{\operatorname{tor}}, \operatorname{Ker} \theta_4) = H^1(\operatorname{Sh}^{\operatorname{tor}}, \operatorname{Sym}^{p+l-k-3} \omega^{(p)} \otimes \operatorname{det}^{-p(p+l-k-3)+p-k+2} \omega)$. By the description of the Jordan Holder factors of algebraic representations of $\operatorname{GL}_2[\operatorname{Der} 81]$ we have the following relation as vector bundles

$$\frac{\omega(p-k+2,-p(p+l-k-4)-k+2)}{\operatorname{Sym}^{p+l-k-3}\omega^{(p)}\otimes\operatorname{det}^{-p(p+l-k-3)+p-k+2}\omega} = \operatorname{Sym}^{p+l-k-4}\omega^{(p)}\otimes\operatorname{Sym}^{p-2}\omega\otimes\operatorname{det}^{-p(p+l-k-3)+p-k+3}\omega.$$

The term in the middle has vanishing H^0 by Theorem 1.8(2), and vanishing canonical $H^1_{\mathfrak{m}}$ by assumption. Therefore

$$H^{2}(\mathcal{F}l^{\mathrm{tor}},\mathrm{Ker}\theta_{4})_{\mathfrak{m}} = H^{0}(\overline{\mathrm{Sh}}^{\mathrm{tor}},\mathrm{Sym}^{p+l-k-4}\omega^{(p)}\otimes\mathrm{Sym}^{p-2}\omega\otimes\mathrm{det}^{-p(p+l-k-3)+p-k+3}\omega)_{\mathfrak{m}}.$$

The latter embeds into $H^0(\overline{Sh}^{tor}, \omega(p-k+1, -p(p+l-k-3)+p-k+3))_{\mathfrak{m}}$, which also vanishes by Theorem 1.8(2).

We then take \overline{h} to be any element in $H^1(\mathcal{F}l_{\kappa}^{\text{tor}}, \mathcal{L}^{\text{can}}(p-k+3, 4-l))_{\mathfrak{m}}$ such that $\theta_4(\overline{h}) = \theta_{(k,4-l)}^1(\overline{s}) = \theta_{(k,4-k)}^1(\overline{g})$, and $h \in H^1(\mathcal{F}l_{\mathcal{O}}^{\text{tor}}, \mathcal{L}^{\text{can}}(p-k+3, 4-l))_{\mathfrak{m}}$ a Hecke eigenform lifting the eigensystem of \overline{h} , where we use assumption 3) to ensure its existence. By the Hecke equivariance of all the operators h satisfies $\overline{r}_f = \overline{r}_h$ up to a cyclotomic twist as long as $\theta_{(k,4-l)}^1(\overline{g}) \neq 0$, which we assume.

Remark 5.3. One might hope that $\theta_{(k,l)}^1(\overline{f}) \neq 0$, which we know to be true, implies $\theta_{(k,4-l)}^1(\overline{g}) \neq 0$. However, we currently don't have a good way of studying how the theta operators interact with the Fontaine-Laffaille module given by crystalline cohomology. Similarly, one might ask if it is always the case that $\theta_{(k,l)}^1(\overline{f})$ is in the image of θ_4 .

To end this subsection we give 3 other possible recipes for the rest of the C_0 obvious weights.

- (1) On H^0 the path $\mathcal{L}(k,l) \xrightarrow{\theta_{(k,l)}^1} \mathcal{L}(2p-k+2,l) \xrightarrow{H_1} \mathcal{L}(3p-k+1,p+l-1) \xleftarrow{\theta_2} \mathcal{L}(p-k+1,l)$ ends at the obvious weight 1) of Proposition 5.1.
- (2) On H^0 the path $\mathcal{L}(k,l) \xrightarrow{\theta_{(k,l)}^1} \mathcal{L}(2p-k+2,l) \xleftarrow{H_2} \mathcal{L}(p-k+2,l+1)$ ends at the obvious weight 2) of Proposition 5.1.
- (3) On H^1 the path $\mathcal{L}(k, 4-l) \xrightarrow{\theta^1_{(k,4-l)}} \mathcal{L}(2p-k+2, 4-l) \xleftarrow{H_2} \mathcal{L}(p-k+2, 5-l)$ ends at the obvious weight 3) of Proposition 5.1.

We make the easy observation that in Herzig's recipe, for any $\lambda, \mu \in W^{?}(\overline{\rho})$, then $\lambda - \mu$ is in the lattice spanned by the weights of the Hasse invariants and the theta operators, so that in principle one can give similar recipes that end up at each of the possible 8 obvious weights. The shadow weights should then be understood through their relation to the theta linkage maps. However, we currently don't have a good understanding of how to translate the reducibility of Weyl modules occuring in étale cohomology to coherent cohomology, so outside the weights in the lowest alcove, reaching a weight in coherent cohomology is not enough to obtain the Serre weight with the same label. We will address this point in future work, by considering de Rham cohomology with arbitrary coefficients.

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80