THE SMALL p-ADIC SIMPSON CORRESPONDENCE IN THE SEMI-STABLE REDUCTION CASE

MAO SHENG AND YUPENG WANG

Abstract. We generalize several known results on small Simpson correspondence for smooth formal schemes over \mathcal{O}_C to the case for semi-stable formal schemes. More precisely, for a liftable semi-stable formal scheme $\mathfrak X$ over $\mathcal O_C$ with generic fiber X, we establish (1) an equivalence between the category of Hitchin-small integral v-bundles on X_v and the category of Hitchin-small Higgs bundles on $\mathfrak{X}_\text{\'et}$, generalizing the previous work of Min–Wang, and (2) an equivalence between the moduli stack of v-bundles on X_v and the moduli stack of rational Higgs bundles on $\mathfrak{X}_{\text{\'et}}$ (equivalently, moduli stack of Higgs bundles on $X_{\text{\'et}}$, generalizing the previous work of Anschütz–Heuer–Le Bras.

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1. INTRODUCTION

1.1. Overview. The small *p*-adic Simpson correspondence was firstly considered by Faltings [\[Fal05](#page-40-1)] and then systematically studied by Abbes–Gros–Tsuji [\[AGT16\]](#page-40-2) and Tsuji [\[Tsu18\]](#page-41-0), which is known as the equivalence between categories of (Faltings-)small generalized representations and (Faltings) small Higgs bundles on X, the generic fiber of a liftable (log-)scheme $\mathfrak X$ over $\mathcal O_C$ with nice singularities. (For example, $\mathfrak X$ could have semi-stable special fiber.) Using this, for curves, Faltings gave an equivalence between the whole category of generalized representations and the whole category of Higgs bundles.

According to the recent development of p-adic geometry, we have several improvements of the previous works on *p*-adic Simpson correspondence. In [\[LZ17\]](#page-41-1), for a smooth rigid variety X over K, a discretely valued complete field over \mathbb{Q}_p with perfect residue field, Liu and Zhu assigned to each \mathbb{Q}_p -local system on $X_{\text{\'et}}$ to a G_K -equivairant Higgs bundle on $X_{\widehat{K},\text{\'et}}$, based on the previous work of Scholze [\[Sch13\]](#page-41-2). Using the decompletion theory in [\[DLLZ23\]](#page-40-3), Min and the second author generalized this assignment to an equivalence between the category of v-bundles on X_v and the category of G_K equivariant Higgs bundles on $X_{\widehat{K},\text{\'et}},$ as the *arithmetic p-adic Simpson correspondence* [\[MW22\]](#page-41-3). More generally, for any *proper* smooth rigid variety X over C , a complete algebraic closed field, Heuer established an equivalence between the whole category of v-bundles on X_v and the whole category of Higgs bundles $X_{\text{\'et}}$ [\[Heu23\]](#page-40-4), generalising the previous work of Faltings [\[Fal05\]](#page-40-1) in the curve case. Heuer also introduced the moduli stack of v-bundles on X_v and the moduli stack of Higgs bundles on $X_{\text{\'et}}$, and proved these two stacks are isomorphic after taking étale sheafifications [\[Heu22\]](#page-40-5). Very recently, Heuer and Xu proved when X is a smooth curve, these two stacks are equivalent to each other up to a choice of extra data [\[HX24\]](#page-41-4).

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On the other hand, the theory of p -adic Simpson correspondence is closely related to the prismatic theory introduced by Bhatt–Scholze [\[BS22\]](#page-40-6) and developed by Bhatt–Lurie [\[BL22a,](#page-40-7) [BL22b\]](#page-40-8). For a smooth formal scheme $\mathfrak X$ over $\mathcal O_C$, in the case $\mathfrak X$ is affine and étale over a formal torus, Tian established an equivalence between the category of Hodge–Tate crystals on the prismatic site associated to $\mathfrak X$ and the category of topologically nilpotent Higgs bundles on $\mathfrak X$ [\[Tia23\]](#page-41-5). In the same setting, Morrow and Tsuji [\[MT20\]](#page-41-6) also obtained a "q-deformation" of Tian's result. When \mathfrak{X} is smooth over \mathcal{O}_K , Min and the second author established an equivalence between the category of rational Hodge– Tate crystals on the absolute prismatic site associated to $\mathfrak X$ and the category of *enhanced Higgs* bundles on $X_{\text{\'et}}$ [\[MW22\]](#page-41-3). This work was generalized by Anschütz–Heuer–Le Bras to the derived case and they also gave a pointwise criterion for a Higgs bundle being enhanced [\[AHLB23a,](#page-40-9) [AHLB23b\]](#page-40-10). When $\mathfrak X$ is smooth and liftable over $\mathcal O_C$ with generic fiber X, Anschütz–Heuer–Le Bras also gave an equivalence between the moduli stack of *Hitchin-small v*-bundles on X_v and the moduli stack of Hitchin-small Higgs bundles $X_{\text{\'et}}$. As (Faltings-)small objects are always Hitchin-small, their result generalized the previous works of Faltings [\[Fal05\]](#page-40-1), Abbes–Gros–Tsuji [\[AGT16\]](#page-40-2) and the second author [\[Wan23\]](#page-41-7). There is another generalisation of Faltings' Simpson correspondence: In [\[MW24\]](#page-41-8), Min and the second author obtain an equivalence between the category of (Faltings-)small integral v-bundles X_v and the category of (Faltings-)small *integral* Higgs bundles on $\mathfrak{X}_{\text{\'et}}$, also generalizing partial results in [\[BMS18\]](#page-40-11).

1.2. Main results. From now on, we freely use the notations in §[1.4.](#page-4-0) In particular, we always let K be a complete discrete valuation field with the ring of integers \mathcal{O}_K and the perfect residue field κ , and let C be the completion of a fixed algebraic closure of K with the ring of integers \mathcal{O}_C and the maximal ideal \mathfrak{m}_C . Let $\mathbf{A}_{\text{inf},K} := \mathbf{A}_{\text{inf}}(\mathcal{O}_C) \otimes_{W(\kappa)} \mathcal{O}_K$ be the ramified infinitesimal period ring of Fontaine (with respect to K), ξ_K be a generator of the natural surjection $\theta_K : \mathbf{A}_{\text{inf},K} \to \mathcal{O}_C$, and $\mathbf{A}_{2,K} := \mathbf{A}_{\text{inf},K}/\xi_K^2$. Equip $\mathbf{A}_{2,K}$ and \mathcal{O}_C with the canonical log structures; that is, the log-structures induced from $\mathcal{O}_C^{\flat} \setminus \{0\} \longrightarrow_{\text{inf},K}$ via the corresponding quotients. In this paper, we always work with semi-stable p-adic formal schemes $\mathfrak X$ (viewed as log-schemes) with the generic fiber X in the sense of [\[CK19\]](#page-40-12).

1.2.1. An integral p-adic Simpson correspondence. Our first result is the following integral p-adic Simpson correspondence for liftable semi-stable formal schemes.

Theorem 1.1 (Theorem [5.4\)](#page-31-0). Let \mathfrak{X} be a semi-stable formal scheme over \mathcal{O}_C of relative dimension d. Suppose that it admits a flat lifting (as a formal log-scheme) $\widetilde{\mathfrak{X}}$ over $\mathbf{A}_{2,K}$. Then there exists a period sheaf $(\mathcal{O}\hat{\mathbb{C}}_{\rm pd}^+, \Theta)$ together with Higgs field Θ (depending on the given lifting $\widetilde{\mathfrak{X}}$) inducing a rank-preserving equivalence

(1.1)
$$
LS^{H\text{-}sm}(\mathfrak{X},\widehat{\mathcal{O}}_X^+) \simeq \text{HIG}^{t\text{-}H\text{-}sm}(\mathfrak{X},\mathcal{O}_\mathfrak{X})
$$

between the category $\text{LS}^{H\text{-}sm}(\mathfrak{X}, \widehat{\mathcal{O}}_X^+)$ of Hitchin-small integral v-bundles on X_v (cf. Definition [5.1\)](#page-31-1) and the category of twisted Hitchin-small Higgs bundles on $\mathfrak{X}_{\text{\'et}}$ (Definition [5.2\)](#page-31-2), which preserves tensor products and dualities. More precisely, let $v : X_v \to \mathfrak{X}_{\text{\'et}}$ be the natural morphism of sites, and then the following assertions are true:

(1) For any $\mathcal{M}^+ \in \text{LS}^{H\text{-}sm}(\mathfrak{X}, \widehat{\mathcal{O}}_X^+)$ of rank r, the complex $\text{R}\nu_*(\mathcal{M}^+ \otimes \mathcal{O}\widehat{\mathbb{C}}_{\text{pd}}^+)$ is concentrated in degree $[0, d]$ such that

$$
L\eta_{\rho_K(\zeta_p-1)}R\nu_*(\mathcal{M}^+\otimes\mathcal{O}\widehat{\mathbb{C}}_{pd}^+)\simeq \left(\nu_*(\mathcal{M}^+\otimes\mathcal{O}\widehat{\mathbb{C}}_{pd}^+)\right)[0],
$$

where $\text{L}\eta_{\rho_K(\zeta_p-1)}$ denotes the décalage functor in [\[BMS18,](#page-40-11) §6]. Moreover, the push-forward

$$
(\mathcal{H}^+(\mathcal{M}^+), \theta) := (\nu_*(\mathcal{M}^+\otimes \mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd}}^+), \nu_*(\mathrm{id}_{\mathcal{M}^+}\otimes \Theta))
$$

defines a twisted Hitchin-small Higgs bundle of rank r on $\mathfrak{X}_{\text{\'et}}$.

(2) For any $(\mathcal{H}^+, \theta) \in \text{HIG}^{t-H\text{-}sm}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ of rank r, the

$$
\mathcal{M}^+(\mathcal{H}^+,\theta) := (\mathcal{H}^+ \otimes \mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd}}^+)^{\theta \otimes \mathrm{id} + \mathrm{id} \otimes \Theta = 0}
$$

defines a Hitchin-small integral v-bundle of rank r on X_v .

(3) The equivalence (1.1) is induced by the functors in Items (1) and (2) .

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Remark 1.2. Theorem [1.1](#page-1-1) generalizes the main result [\[MW24,](#page-41-8) Th. 1.1] in the following sense: In *loc.cit.*, we always work with smooth formal schemes and (Faltings-)small integral v-bundles and Higgs bundles, but here, the result holds for semi-stable formal schemes and for Hitchin-small integral v-bundles and Higgs bundles. We remark that a Faltings-small integral v-bundle (resp. Higgs bundle) is always Hitchin-small but the converse is not true (for example, a nilpotent Higgs bundle is always Hitchin-small but not always Faltings-small). It also upgrades the rational Simpson correspondences of Faltings [\[Fal05\]](#page-40-1), Abbes–Gros–Tsuji [\[AGT16\]](#page-40-2) and Wang [\[Wan23\]](#page-41-7) for Faltings-small objects and Anschütz–Heuer–Le Bras [\[AHLB23b,](#page-40-10) Th. 1.2] for Hitchin-small objects to the integral level (see Theorem [1.9](#page-3-0) for its analogue in semi-stable reduction case).

Given a Hitchin-small integral v-bundle \mathcal{M}^+ with the associated Hitchin-small Higgs bundle (\mathcal{H}^+, θ) in the sense of Theorem [1.1,](#page-1-1) we also want to compare the push-forward R $\nu_* \mathcal{M}^+$ with the Higgs complex $DR(\mathcal{H}^+, \theta)$. Indeed, we are able to prove the following truncated cohomological comparison, generalizing [\[MW24,](#page-41-8) Cor. 1.2 and Th. 1.4] to the semi-stable reduction case.

Theorem 1.3 (Corollary [5.8](#page-34-0) and Theorem [5.9\)](#page-35-0). Keep assumptions in Theorem [1.1](#page-1-1) and let \mathcal{M}^+ be a Hitchin-small integral v-bundle with the corresponding Hitchin-small Higgs bundle (\mathcal{H}^+, θ) . Then there exists a canonical morphism

$$
DR(\mathcal{H}^+, \theta) \to R\nu_*\mathcal{M}^+
$$

whose cofiber is killed by $(\rho_K(\zeta_p-1))^{\max(d+1,2(d-1))}$, and this morphism induces a quasi-isomorphism

 $\tau^{\leq 1} \text{DR}(\mathcal{H}^+, \theta) \simeq \tau^{\leq 1} \text{L} \eta_{\rho_K(\zeta_p - 1)} \text{R} \nu_* \mathcal{M}^+.$

In particular, when $\mathfrak X$ is a curve (i.e. $d=1$), we get a canonical quasi-isomorphism

$$
DR(\mathcal{H}^+, \theta) \simeq L \eta_{\rho_K(\zeta_p - 1)} R \nu_* \mathcal{M}^+.
$$

In particular, by letting $\mathcal{M}^+ = \hat{\mathcal{O}}_X^+$ (or equivalently $(\mathcal{H}^+, \theta) = (\mathcal{O}_X, 0)$), by a standard trick used in the proof of [\[DI87,](#page-40-13) Th. 2.1] and [\[Min21,](#page-41-9) Th. 4.1], we conclude the following analogue of Delighe–Illusie decomposition for semi-stable formal schemes:

Theorem 1.4 (Theorem [5.10\)](#page-35-1). Keep assumptions in Theorem [1.1.](#page-1-1) Then there exists a quasiisomorphism

$$
\oplus_{i=0}^{p-1} \Omega^{i,\log}_\mathfrak{X}\{-i\}[-i] \to \tau^{\leq p-1} \mathrm{L} \eta_{\rho_K(\zeta_p-1)} \mathrm{R} \nu_* \widehat{\mathcal{O}}^+_\mathfrak{X}.
$$

Remark 1.5. To obtain the decomposition in Theorem [1.4,](#page-2-0) it seems that we must assume \mathfrak{X} admits a lifting as formal log-scheme over $\mathbf{A}_{2,K}$ (endowed with the canonical log structure). This phenomenon appears in the classical theory of Deligne–Illusie decomposition for log-schemes in positive characteristic, by a previous work of the first author [\[SS20\]](#page-41-10). See Remark [5.11](#page-35-2) for more discussion.

1.2.2. A stacky p-adic Simpson correspondence. Let $\mathfrak X$ be a semi-stable p-adic formal scheme over \mathcal{O}_C with the generic fiber X as before. Again we assume that X admits a lifting X over $\mathbf{A}_{2,K}$. Denote by Perfd the v-site of affinoid perfectoid spaces over C in the sense of $[Sch17]$. For any $S = \text{Spa}(A, A^+)$, let \mathfrak{X}_S and X_S be the base-change of $\mathfrak X$ abd X to A^+ and A, respectively. Then for any $r > 0$, the following two functors

 $LS_r(X, \widehat{\mathcal{O}}_X) : S \in \text{Perfd} \mapsto \{\text{groupoid of } v\text{-bundles of rank } r \text{ on } X_{S,v}\}\$

and

 $HIG_r(X, \mathcal{O}_X) : S \in \text{Perfd} \mapsto \{\text{groupoid Higgs bundles of rank } r \text{ on } X_{S, \text{\'et}}\}$

are actually small v-stacks [\[Heu22,](#page-40-5) Th. 1.4]. Denote by A_r the following v-sheaf

$$
\mathcal{A}_r : S \in \text{Perfd} \mapsto \bigoplus_{i=1}^r \text{H}^0(X_S, \text{Sym}^i(\Omega^1_{X_S}\{-1\})).
$$

Then we have the Hitchin-fibrations

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where h carries each Higgs bundle (\mathcal{H}, θ) to the characteristic polynomial of θ while \tilde{h} denotes the composite of h with the "HTlog" in [\[Heu22,](#page-40-5) Def. 1.6]. The integral model \mathfrak{X} of X induces a sub-sheaf

$$
\mathcal{A}_r^{\text{H-sm}} : S \in \text{Perfd} \mapsto \bigoplus_{i=1}^r p^{<\frac{i}{p-1}} \text{H}^0(\mathfrak{X}_S, \text{Sym}^i(\Omega_{\mathfrak{X}_S}^{1,\log}\{-1\}))
$$

of \mathcal{A}_r , where $p^{\leq \frac{i}{p-1}}$ denotes the ideal $(\zeta_p - 1)^i \mathfrak{m}_C \subset \mathcal{O}_C$. For any $Z \in \{\text{LS}_r(X, \widehat{\mathcal{O}}_X), \text{HIG}_r(X, \mathcal{O}_X)\},$ denote by $Z^{\text{H-sm}} := Z \times_{\mathcal{A}_r} \mathcal{A}_r^{\text{H-sm}}$ its *Hitchin-small locus.* A Higgs bundle on $X_{S,\text{\'et}}$ (resp. a v-bundle on $X_{S,v}$ of rank r is called Hitchin-small if as a point, it belongs to the Hitchin-small locus of the corresponding moduli stack. Then our second main result is the following equivalence of stacks:

Theorem 1.6 (Theorem [5.22\)](#page-40-14). Let \mathfrak{X} be a semi-stable p-adic formal scheme over \mathcal{O}_C with the generic fiber X which admits a lifting $\widetilde{\mathfrak{X}}$ over $\mathbf{A}_{2,K}$. Then the lifting $\widetilde{\mathfrak{X}}$ induces an equivalence of stacks

$$
\rho_{\widetilde{\mathfrak{X}}} : \operatorname{LS}_r(X, \widehat{\mathcal{O}}_X)^{H\text{-}sm} \xrightarrow{\simeq} \operatorname{HIG}_r(X, \mathcal{O}_X)^{H\text{-}sm}.
$$

Remark 1.7. When \mathfrak{X} is smooth over \mathcal{O}_C , the above equivalence was obtained by Anschütz–Heuer– Le Bras [\[AHLB23b,](#page-40-10) Th. 1.1], based on their previous work [\[AHLB23a\]](#page-40-9) on studying rational Hodge– Tate crystals on the prismatic site associated to \mathfrak{X} . They first constructed a fully faithful functor $S_{\tilde{\tau}}: \text{HIG}_r(X, \mathcal{O}_X) \to \text{LS}_r(X, \mathcal{O}_X)$ via prismatic theory and then showed the essential surjectivity by working locally on \mathfrak{X} . Compared with their construction, we do not need any input of prismatic theory and can give an explicit description of $\rho_{\tilde{\mathbf{x}}}^{-1}$ $\overline{\tilde{\mathfrak{X}}}^1$ (cf. Theorem [1.9\)](#page-3-0).

Remark 1.8. It is still a question if there exists an equivalence of the whole stacks

$$
LS_r(X, \widehat{\mathcal{O}}_X) \xrightarrow{\simeq} \text{HIG}_r(X, \mathcal{O}_X)
$$

for general smooth X over C. Up to now, we only know a few on this question: We only have the desired equivalence when X is either a curve [\[HX24\]](#page-41-4) or the projective space \mathbb{P}^n [\[AHLB23b,](#page-40-10) Cor. 1.3]. It seems the recent announcement of Bhargav Bhatt and Mingjia Zhang on Simpson gerbe may help to solve this question, but up to now, we still do not know if such an equivalence always exists.

We now describe the strategy to prove Theorem [1.6.](#page-3-1) It suffices to show for any $S = \text{Spa}(A, A^+) \in$ Perfd, there is a rank-preserving equivalence

$$
LS(X, \widehat{\mathcal{O}}_X)^{\text{H-sm}}(S) \xrightarrow{\simeq} \text{HIG}(X, \mathcal{O}_X)^{\text{H-sm}}(S)
$$

between the category of Hitchin-small v-bundles on $X_{v,\'{e}t}$ and the category of Hitchin-small Higgs bundles on $X_{S,\text{\'et}}$ which is functorial in S. To do so, we may follow the same argument for the proof of Theorem [1.1:](#page-1-1) Given a lifting $\widetilde{\mathfrak{X}}$ of \mathfrak{X} over $\mathbf{A}_{2,K}$, its base-change $\widetilde{\mathfrak{X}}_S$ along $\mathbf{A}_{2,K} \to \mathbb{A}_{2,K}(S) =$ $\mathbb{A}_{\text{inf}}(S) \otimes_{\mathbf{A}_{\text{inf}}} \mathbf{A}_{2,K}$ is a lifting of \mathfrak{X}_S (the base-change of \mathfrak{X} along $\mathcal{O}_C \to A^+$). Using this, one can still construct a period sheaf ($\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd},S}$, Θ) with Higgs field whose Higgs complex induces a resolution of the structure sheaf \mathcal{O}_{X_S} on $X_{S,v}$. Then one can prove the following rational Simpson correspondence:

Theorem 1.9 (Theorem [5.17](#page-37-0) and Lemma [5.16\)](#page-36-0). Let $\nu : X_{S,\nu} \to X_{S,\text{\'et}}$ be the natural morphism of sites.

(1) For any $\mathcal{M} \in \text{LS}(X, \widehat{\mathcal{O}}_X)^{H\text{-}sm}(S)$ of rank r, we have a quasi-isomorphism

$$
\mathrm{R}\nu_*(\mathcal{M}\otimes\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd},S}^+)\simeq \nu_*(\mathcal{M}\otimes\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd},S}^+)
$$

Moreover, the push-forward

$$
(\mathcal{H}(\mathcal{M}), \theta) := (\nu_*(\mathcal{M} \otimes \mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd},S}), (\zeta_p - 1)\nu_*(\Theta_{\mathrm{id}_\mathcal{M} \otimes \Theta}))
$$

defines a Hitchin-small Higgs bundle of rank r on $X_{S, \text{\'et}}$. (2) For any $(\mathcal{H}, \theta) \in \text{HIG}(X, \mathcal{O}_X)^{H\text{-}sm}(S)$ of rank r, the

$$
\mathcal{M}(\mathcal{H}, \theta) := (\mathcal{H} \otimes \mathcal{O} \widehat{\mathbb{C}}_{\mathrm{pd}, S})^{(\zeta_p - 1)^{-1} \theta \otimes \mathrm{id} + \mathrm{id}_{\mathcal{H}} \otimes \Theta = 0}
$$

defines a Hitchin-small v-bundle of rank r on $X_{S,v}$.

(3) The functors in Items (1) and (2) defines an equivalence of categories

$$
LS(X, \widehat{\mathcal{O}}_X)^{H\text{-}sm}(S) \xrightarrow{\simeq} \text{HIG}(X, \mathcal{O}_X)^{H\text{-}sm}(S).
$$

which preserves ranks, tensor products and dualities. Moreover, for any Hitchin-small vbundle M with associated Hitchin-small Higgs bundle (H, θ) , there exists a quasi-isomorphism

$$
R\nu_*\mathcal{M} \simeq DR(\mathcal{H}, \theta).
$$

Then Theorem [1.6](#page-3-1) follows immediately as the equivalence above is ovbiously functorial in S by the construction.

To obtain the desired period sheaf with Higgs field $(\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd},S},\Theta)$, we need to construct the *integral Faltings' extension* corresponding to the lifting $\tilde{\mathfrak{X}}_S$ of \mathfrak{X}_S using the theory of log-cotangent complex of Olsson (and Gabber) [\[Ols05,](#page-41-12) §8] as we did in [\[Wan23,](#page-41-7) §2]. To do so, we need to endow A^+ with a suitable log-structure, called the canonical log-structure, such that the analogue of [\[BMS18,](#page-40-11) Lem. 3.14] holds true. The most natural log-structure on A^+ is $(A^{\times} \cap A^+ \to A^+)$, and the difficult part is prove it is the correct one (cf. $\S 2.1$).

1.3. **Organization.** The paper is organized as follows: In \S ?, we introduce the canonical logstructure on perfectoid Tate algebra, and the basic set-up on semi-stable formal schemes we will work with. In §[3,](#page-11-0) we construct our period sheaf with Higgs field and prove the corresponding Poincaré's Lemma. In §[4,](#page-18-0) we include the key local calculations and give a local version of Simpson correspondence. Finally, in §[5,](#page-30-0) we prove the integral Simpson correspondence at first and then give the desired equivalence of moduli stacks on Hitchin-small v -bundles and Hitchin-small Higgs bundles for lifable semi-stable formal schemes \mathfrak{X} .

1.4. **Notations.** Throughout this paper, let K be a complete discrete valuation field over \mathbb{Q}_p with the ring of integers \mathcal{O}_K and the residue field κ , which is required to be perfect. Put W := W(κ), let $C = K$ be the completion of a fixed algebraic closure K of K with the ring of integers \mathcal{O}_C and the maximal ideal m_C . Let A_{inf} and B_{dR}^+ be the corresponding infinitesimal and de Rham period rings. Fix an embedding $p^{\mathbb{Q}} \subset C^{\times}$, which induces an embedding $\varpi^{\mathbb{Q}} \subset C^{\flat \times}$, where $\varpi =$ $(p, p^{1/p}, p^{1/p^2}, \ldots) \in C^{\flat}$. Fix a coherent system $\{\zeta_{p^n}\}_{n \geq 0}$ of primitive p^n -th roots of unity in C, and let $\epsilon := (1, \zeta_p, \zeta_p^2, \dots) \in C^{\flat}$. Put $\mathbf{A}_{\text{inf},K} := \mathbf{A}_{\text{inf}} \otimes_{\mathbf{W}} \mathcal{O}_K$ and then we have the canonical surjection θ_K : $\mathbf{A}_{\text{inf},K} \to \mathcal{O}_C$ whose kernel I_K is principally generated and we fix an its generator ξ_K . Define $\mathbf{A}_{2,K} := \mathbf{A}_{\mathrm{inf},K}/\mathrm{I}_K^2.$

For any (sheaf of) $A_{\text{inf},K}$ -module M and any $n \in \mathbb{Z}$, denote by $M\{n\}$ its Breuil–Kisin–Fargues twist

$$
M\{n\}:=M\otimes_{\mathbf{A}_{\mathrm{inf}}}\mathbf{I}_K^{\otimes n},
$$

which can be trivialized by ξ_K^n ; that is, we have the identification $M\{n\} = M \cdot \xi_K^n$. Using this, we may regard M as a sub- $\mathbf{A}_{\text{inf},K}$ -module of $M\{-1\}$ via the identification $M = \xi_K M\{-1\}$.

Let $\mu_{p^{\infty}}$ be the sub-group of \mathcal{O}_C^{\times} \check{C} generated by $\{\zeta_{p^n}\}_{n\geq 1}$ and $\mathbb{Z}_p(1) := T_p(\mu_{p^{\infty}})$ be its Tate module. For any (sheaf of) \mathbb{Z}_p -module M and any $n \in \mathbb{Z}$, denote by $M(n)$ its Tate twist

$$
M(n) := M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)^{\otimes n}.
$$

Let $t = \log([\epsilon]) \in \mathbf{B}_{dR}^+$ be the Fontaine's p-adic analogue of " $2\pi i$ ". Then $M(n)$ can be trivialized by t^n ; that is, we have the identification $M(n) = M \cdot t^n$.

The natural inclusion $\mathbf{A}_{\text{inf},K} \hookrightarrow \mathbf{B}_{\text{dR}}^+$ induces a natural inclusion

$$
\mathcal{O}_C\{1\} \cong \xi_K \mathbf{A}_{\mathrm{inf},K}/\xi_K^2 \mathbf{A}_{\mathrm{inf},K} \hookrightarrow t\mathbf{B}_{\mathrm{dR}}^+/t^2 \mathbf{B}_{\mathrm{dR}}^+ \cong C(1)
$$

identifying $\mathcal{O}_C\{1\}$ with an \mathcal{O}_C -lattice of $C(1)$. Let $\mathcal{O}_C(1) = \mathcal{O}_C \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1) \subset C \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1) = C(1)$ be the standard \mathcal{O}_C -lattice of $C(1)$. Then there exists an element $\rho_K \in \mathcal{O}_C$ with the p-adic valuation $\nu_p(\rho_K) = \nu_p(\mathcal{D}_K) + \frac{1}{p-1}$ such that

$$
\mathcal{O}_C(1) = \rho_K \mathcal{O}_C\{1\},\
$$

where \mathcal{D}_K denotes the ideal of differentials of \mathcal{O}_K . For example, when $\mathcal{O}_K = W$, we have $\mathcal{D}_K = \mathcal{O}_K$ and can choose $\rho_K = \zeta_p - 1$.

Fix a ring R. If an element $x \in R$ admits arbitrary pd-powers, we denote by $x^{[n]}$ its n-th pd-power (i.e. analogue of $\frac{x^n}{n!}$ $\binom{x^n}{n!}$ in R. Put $E_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{N}^d$ with 1 appearing at exactly the *i*-th component. For any $J = (j_1, \ldots, j_d) \in \mathbb{N}^d$ and any $x_1, \ldots, x_d \in R$, we put

$$
\underline{x}^J := x_1^{j_1} \cdots x_d^{j_d}
$$

and if moreover x_i admits arbitrary pd-powers in A for all i, we put

$$
\underline{x}^{[J]} := x_1^{[j_1]} \cdots x_d^{[j_d]}.
$$

Define $|J| := j_1 + \cdots + j_d$. For any $\alpha \in \mathbb{N}[1/p] \cap [0,1)$, we put

$$
\zeta^\alpha=\zeta_{p^n}^m
$$

if $\alpha = \frac{m}{p^n}$ such that p and m are coprime in N. If $x \in A$ admits compatible p^n -th roots $x^{\frac{1}{p^n}}$, we put

$$
x^{\alpha} = x^{\frac{m}{p^n}}
$$

for $\alpha = \frac{m}{p^n}$ as above. In general, for any $\underline{\alpha} := (\alpha_1, \ldots, \alpha_d) \in (\mathbb{N}[1/p] \cap [0, 1))^d$ and any x_1, \ldots, x_d admitting compatible p^n -th roots, we put

$$
\underline{x}^{\underline{\alpha}} := x_1^{\alpha_1} \cdots x_d^{\alpha_d}.
$$

We always denote by Perfd the v-site of affinoid perfectoid spaces over $Spa(C, \mathcal{O}_C)$ in the sense of [\[Sch17\]](#page-41-11).

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2. BASIC SET-UP

2.1. Canonical log structure on perfectoid affinoid algebras. Let (A, A^+) be a perfectoid affinoid algebra with tilting $(A^{\flat}, A^{\flat +})$. Fix $a \underline{\pi} \in A^{\flat,+}$ such that $\pi = \underline{\pi}^{\sharp}$ and $\pi^p = pu$ for some unit $u \in A^{+,\times}$ (cf. [\[BMS18,](#page-40-11) Lem. 3.19]). Then A^+ is π -adically complete and $A = A^+[\frac{1}{\pi}]$ $\frac{1}{\pi}$ while the same holds for $(A^{\flat}, A^{\flat, +}, \underline{\pi})$ (instead of (A, A^+, π)). Let

$$
\sharp : A^{\flat} \to A, \quad \underline{f} = (f_0, f_1, \dots) \in A^{\flat} = \varprojlim_{x \mapsto x^p} A \mapsto f_0
$$

denote the usual sharp map, which is a morphism of multiplicative monoids. Then \sharp coincides with the composite

$$
A^{\flat,+} \xrightarrow{[\cdot]} \mathbb{A}_{\mathrm{inf},K}(A,A^+) \xrightarrow{\theta_K} A^+.
$$

Definition 2.1. For any affinoid perfectoid $U = \text{Spa}(A, A^+)$ over \mathbb{Q}_p , the *canonical log-structure* on $\mathbb{A}_{\text{inf},K}(U) = \mathbb{A}_{\text{inf},K}(A, A^+)$ is the log-structure induced by the pre-log structure

$$
A^{\flat,\times} \cap A^{\flat,\mathcal{+}} \xrightarrow{[\cdot]} \mathbb{A}_{\mathrm{inf},K}(U).
$$

For any $\mathbb{A}_{\text{inf},K}(U)$ -algebra $B^+ \in \{A^{\flat, +}, A^+, \mathbb{A}_{\text{inf},K}(U)/\text{Ker}(\theta_K)^n\}$, the *canonical log-structure* on B^+ is the log-structure induced from the canonical log-structure on $\mathbb{A}_{\text{inf},K}(U)$ via the natural surjection $\mathbb{A}_{\text{inf},K}(U) \to B^+$. In particular, the canonical log-structure on A^+ is the log-structure associated to the pre-log structure $A^{\flat,\times} \cap A^{\flat,\div} \stackrel{\sharp}{\to} A^+$.

Note that A^+ admits another log-structure $(A^{\times} \cap A^+ \hookrightarrow A^+)$. Our purpose in this section is to prove the following result:

Proposition 2.2. For any perfectoid affinoid algebra $(A, A⁺)$, the canonical log-structure on $A⁺$ coincides with the log-structure $(A^{\times} \cap A^+ \hookrightarrow A^+).$

The key ingredient for proving Proposition [2.2](#page-5-3) is the following result, whose proof is similar to that of [\[DH23,](#page-40-15) Prop. 2.9].

Proposition 2.3. Let (A, A^+) be a perfectoid affinoid algebra over \mathbb{Q}_p with tilting $(A^{\flat}, A^{\flat,+})$. Then for any $f \in A^{\times} \cap A^+$, there exists $a \underline{g} \in A^{\flat,\times} \cap A^{\flat+}$ and an $h \in A^+$ such that $f = \underline{g}^{\sharp} \cdot (1 + ph)$.

Proof. Put $U = \text{Spa}(A, A^+)$ and $U^{\flat} = \text{Spa}(A^{\flat}, A^{\flat+})$. By tilting equivalence, we can an homeomorphism of underlying topological spaces of $|U| \cong |U^{\flat}|$. For any $x \in |U|$, we denote by $x^{\flat} \in |U^{\flat}|$ the image of x via this identification.

Fix an $f \in A^{\times} \cap A^{+}$. By [\[Hub93,](#page-40-16) Lem. 3.3(i)] and [\[Hub94,](#page-40-17) Lem. 1.4], for any $x \in |U|$, we always have $0 < |f(x)| \leq 1$. As π is a pesudo-uniformizer and |U| is quasi-compact, there exists an integer $N \geq 0$ such that for any $x \in |U|$,

(2.1)
$$
|\underline{\pi}(x^{\flat})|^{pN} = |\pi(x)|^{pN} \le |f(x)| \le 0.
$$

By the approximation lemma [\[CS24,](#page-40-18) Lem. 2.3.1], there exists a $g \in A^{\flat, +}$ such that for any $x \in |U|$,

(2.2)
$$
|f(x) - \underline{g}^{\sharp}(x)| \leq |p(x)| \cdot \max(|\underline{g}(x^{\flat})|, |\underline{\pi}(x^{\flat})|^{pN}).
$$

As $|g^{\sharp}(x)| = |g(x^{\flat})|$ and $|p(x)| < 1$, by strong triangular inequality, we deduce from (2.1) and (2.2) that for any $x \in |U|$,

$$
|f(x)| = |\underline{g}^{\sharp}(x)| = |\underline{g}(x^{\flat})|.
$$

In particular, using [\(2.1\)](#page-6-0) again, for any $x^{\flat} \in |U^{\flat}|$, we have

$$
0 < |\underline{\pi}(x^{\flat})|^{pN} \le |\underline{g}(x^{\flat})| \le 1.
$$

This forces that $g \in A^{\flat \times} \cap A^{\flat,+}$ and that $g^{-1}\underline{\pi}^{pN} \in A^{\flat,+}$, by [\[Hub93,](#page-40-16) Lem. 3.3(i)] and [\[Hub94,](#page-40-17) Lem. 1.4] again. A similar argument also shows that $(g^{\sharp})^{-1} f \in A^{+, \times}$ is a unit in A^+ . So we can rewrite (2.2) as

$$
|((\underline{g}^{\sharp})^{-1}f)(x) - 1| \le |p(x)| \cdot \max(1, |(\underline{g}^{-1}\underline{\pi}^{pN})(x^{\flat})|) = |p(x)|.
$$

As p is invertible in A, the above argument implies that $h := p^{-1}((\underline{g}^{\sharp})^{-1}f - 1)$ is a well-defined element in A^+ . By construction, we conclude that $f = g^{\sharp}(1 + ph)$ as desired.

Corollary 2.4. Let (A, A^+) be a perfectoid affinoid algebra over \mathbb{Q}_p with tilting $(A^{\flat}, A^{\flat,+})$. Then for any ? $\in \{+, \emptyset\}$, we have

$$
A^{\times} \cap A^? = (A^{\flat,\times} \cap A^{\flat,?})^{\sharp} \cdot (1 + pA^+).
$$

Proof. For $? = \emptyset$, we have to show $A^{\times} = (A^{\flat,\times})^{\sharp} \cdot (1 + pA^{+})$. By recalling that $A = A^{+}[\frac{1}{\pi}]$ $\frac{1}{\pi}$ and $A^{\flat} = A^{\flat,+}[\frac{1}{\pi}]$ $\frac{1}{\pi}$ with $\pi^{\sharp} = \pi$, we are reduced to the case for ? = +. That is, we have to show

$$
A^{\times} \cap A^{+} = (A^{\flat,\times} \cap A^{\flat,\div})^{\sharp} \cdot (1 + pA^{+}).
$$

But this follows from Proposition [2.3](#page-6-2) immediately.

Proof of Proposition [2.2:](#page-5-3) Let $G \subset A^{\flat,\times}$ be the kernel of the homomorphism $\sharp: A^{\flat,\times} \to A^{\times}$. Keep the notations in the proof of Proposition [2.3.](#page-6-2) As for any $g \in G$ and any $x \in |U|$, we have

$$
|g(x^{\flat})| = |g^{\sharp}(x)| = 1,
$$

by [\[Hub93,](#page-40-16) Lem. 3.3(i)] and [\[Hub94,](#page-40-17) Lem. 1.4], G is a sub-group of $A^{\flat, +, \times}$ and hence the kernel of the homomorphism $\sharp : A^{\flat, +, \times} \to A^{+, \times}$. It is also the kernel of the homomorphism of monoids

$$
\sharp : A^{\flat,\times} \cap A^{\flat,+} \to A^{\times} \cap A^+.
$$

So the canonical log-structure on A^+ is the log-structure associated to the pre-log structure

$$
(A^{\flat,\times} \cap A^{\flat,+})^{\sharp} \hookrightarrow A^+.
$$

So the result follows from Corollary [2.4](#page-6-3) because $1 + pA^+ \subset A^{+,\times}$.

7

8

Remark 2.5 (Surjectivity of \sharp for sympathetic algebras). Let Λ be a sympathetic algebra over C in the sense of $[Col02, §5]$; that is, it is a connected, p-closed and spectral Banach C-algebra Λ (with respect to the spectral norm $|\cdot|_{\Lambda}$). Put $\mathcal{O}_{\Lambda} = \{\lambda \in \Lambda \mid |\lambda|_{\Lambda} \leq 1\}$ and then by [\[Col02,](#page-40-19) Lem. 2.15(iii)], $(\Lambda, \mathcal{O}_\Lambda)$ is a perfectoid affinoid algebra. The p-closeness of Λ [\[Col02,](#page-40-19) §2.8] together with Corollary [2.4](#page-6-3) implies that $\Lambda^{\times} = (\Lambda^{\flat,\times})^{\sharp}$, that $\Lambda^{\times} \cap \mathcal{O}_{\Lambda} = (\Lambda^{\flat,\times} \cap \mathcal{O}_{\Lambda}^{\flat})^{\sharp}$ and that $\mathcal{O}_{\Lambda}^{\times} = (\mathcal{O}_{\Lambda}^{\flat,\times})^{\sharp}$ $_{\Lambda}^{\flat,\times}$ ^{\sharp}. We point out that the sympathetic algebras usually form a basis for the pro-étale topology of a rigid space (cf. [\[Sch13,](#page-41-2) Prop. 4.8] and the proof therein).

As a consequence, we have an analogue of [\[BMS18,](#page-40-11) Lem. 3.14] in the logarithmic setting.

Definition 2.6. Let A be a perfect \mathbb{F}_p -algebra (resp. a perfectoid algebra over \mathbb{Z}_p). A log-structure $M_A \to A$ on A is called *perfect*, if it is associated to a pre-log-structure $N \to A$ with N a uniquely p-divisible monoid; that is, the map $N \xrightarrow{n \to pn} N$ is bijective.

The following lemma is well-known to experts:

- **Lemma 2.7.** (1) Let $(M_A \to A) \to (M_B \to B)$ be a morphism of perfect \mathbb{F}_p -algebras with perfect log-structures. Then the corresponding cotangent complex $L_{(M_B\to B)/(M_A\to A)}=0$.
	- (2) Let $(M_A \to A) \to (M_B \to B)$ be a morphism of perfectoid algebras over \mathbb{Z}_p with perfect log-structures. Then the corresponding p-complete cotangent complex $\widehat{L}_{(M_B\to B)/(M_A\to A)} = 0$.

Proof. For Item (1): Considering the morphisms of log-rings

 $(A^{\times} \hookrightarrow A) \to (M_A \to A) \to (M_B \to B)$

and the associated exact triangle (cf. [\[Ols05,](#page-41-12) Th. 8.18])

$$
L_{(M_A \to A)/(A^{\times} \to A)} \otimes_A^L B \to L_{(M_B \to B)/(A^{\times} \to A)} \to L_{(M_B \to B)/(M_A \to A)},
$$

we are reduced to the case for $(M_A \to A) = (A^* \leftrightarrow A)$. Assume the log-structure $M_B \to B$ is associated to the pre-log-structure $N \rightarrow B$ with N uniquely p-divisible. Then the morphism $(A^{\times} \to A) \to (M_B \to B)$ of log-structures is associated to the morphism $(0 \xrightarrow{0 \to 1} A) \to (N \to B)$ of pre-log-structures. By [\[Ols05,](#page-41-12) Th. 8.20], it is enough to show that

$$
L_{(N\to B)/(0\to A)} = 0.
$$

As $L_{(0\rightarrow B)/(0\rightarrow A)} \simeq L_{B/A} = 0$ (cf. [\[Ols05,](#page-41-12) Lem. 8.22]), by considering the exact triangle

 $L_{(0\rightarrow B)/(0\rightarrow A)} \rightarrow L_{(N\rightarrow B)/(0\rightarrow A)} \rightarrow L_{(N\rightarrow B)/(0\rightarrow B)},$

we are reduced to showing that

$$
L_{(N\to B)/(0\to B)} = 0.
$$

By [\[Ols05,](#page-41-12) Lem. 8.28] and [\[Ols05,](#page-41-12) Lem. 8.23(ii)], we have quasi-isomorphisms

$$
L_{(N\to B)/(0\to B)} \simeq L_{(N\to \mathbb{Z}[N])/(0\to \mathbb{Z})} \otimes^L_{\mathbb{Z}[N]} B \simeq N^{\rm gp} \otimes^L_{\mathbb{Z}} B,
$$

where N^{gp} denotes the group associated to the monoid N. It remains to show that

$$
N^{\rm gp} \otimes_{\mathbb{Z}}^{\rm L} B = 0.
$$

As N is uniquely p-divisible, we see that p acts isomorphically on N^{gp} and thus on $N^{gp} \otimes_{\mathbb{Z}}^{\mathbf{L}} B$. This forces $N^{\text{gp}} \otimes^{\mathbf{L}}_{\mathbb{Z}} B = 0$ because $pB = 0$.

For Item (2): Assume the log-structure $M_B \to B$ is associated to the pre-log-structure $N \to B$ with N uniquely p-divisible. Similar to the proof of Item (1) , we are reduced to the case to show that the derived p-adic completion of $N^{\text{gp}} \otimes_{\mathbb{Z}}^{\mathbf{L}} B$ vanishes. By derived Nakayama's Lemma, this amounts to that

$$
N^{\rm gp} \otimes_{\mathbb{Z}}^{\rm L} B \otimes_{\mathbb{Z}}^{\rm L} \mathbb{F}_p = 0.
$$

But this is trivial because N is uniquely p-divisible.

Combining Proposition [2.2](#page-5-3) together with Lemma [2.7\(](#page-7-0)2), we can conclude the following analogue of [\[BMS18,](#page-40-11) Lem. 3.14] immediately.

Proposition 2.8. For any morphism $(A, A^+) \rightarrow (B, B^+)$ of perfectoid affinoid algebras, the pcomplete cotangent complex $\widehat{L}_{(B^{\times} \cap B^+ \to B^+)/(A^{\times} \cap A^+ \to A^+)} = 0.$

Convention 2.9. From now on, without other clarification, for any $n \geq 1$, we always regard

$$
\mathrm{Spf}(\mathbb{A}_{\mathrm{inf},K}(A,A^+)/\mathrm{Ker}(\theta_K)^n)
$$

as a p-adic log formal scheme with the canonical log-structure.

Although Proposition [2.2](#page-5-3) is enough for our use, to complete the theory, we deal with a general case at the end of this subsection. In what follows, let A^+ be any perfectoid ring with $A = A^+[\frac{1}{n}]$ $\frac{1}{p}$. By [\[BMS18,](#page-40-11) Lem. 3.9], one can still choose a $\pi \in A^+$ and a $\underline{\pi} \in A^{+,b}$ such that $\pi = \underline{\pi}^{\sharp}$ and $\pi^p = pu$ for some $u \in A^{+,\times}$. In this case, we also have $A = A^+[\frac{1}{\pi}]$ $\frac{1}{\pi}$] and $A^{\flat} = A^{+,\flat}[\frac{1}{\pi}]$ $\frac{1}{\pi}$ as well. Also, for any $\alpha \in \mathbb{N}[1/p]$, the π^{α} is well-defined.

Proposition 2.10. Let A^+ be a perfectoid ring with $A = A^+[\frac{1}{n}]$ $\frac{1}{p}$]. Then for any $x \in A^+$ whose image in A is invertible, it always factors as

$$
x = \underline{x}^{\sharp}(1 + \pi^{p\alpha}y_{\alpha}),
$$

where $\underline{x} \in A^{+, \flat}$ whose image in A^{\flat} is invertible, $\alpha \in \mathbb{N}[\frac{1}{p}]$ $\frac{1}{p}$ \cap $(0,1)$ and $y_{\alpha} \in A^+$.

Proof. Fix an $\alpha \in \mathbb{N}[\frac{1}{p}]$ $\frac{1}{p}$ \cap (0, 1). We first assume A^+ is *p*-torsion free. In this case, we have $A^+ \subset A^{\circ}$ (and $A^{+,b} \subset A^{b,\circ}$). As (A, A°) is perfectoid affinoid, by Proposition [2.3,](#page-6-2) there exists an $\underline{x} \in A^{b,\circ}$ and an $y \in A^{\circ}$ such that $x = \underline{x}^{\sharp}(1 + py)$. As $\sqrt{pA^{\circ}} \subset A^{+}$, we have $y_{\alpha} := \frac{p}{\pi^{p\alpha}}y = u^{-1}\pi^{p(1-\alpha)}y \in A^{+}$ such that

$$
x = \underline{x}^{\sharp}(1 + \pi^{p\alpha}y_{\alpha}).
$$

As $1 + \pi^{p\alpha}y_{\alpha} \in A^{+, \times}$, we have $\underline{x}^{\sharp} \in A^{+}$, yielding that $\underline{x} \in A^{+, \flat}$. Clearly, we have $\underline{x} \in A^{\flat, \times}$ as desired. Now we move to the general case. Put $B^+ := A^+ / A^+ [\sqrt{pA^+}]$, $A^+ := A^+ / \sqrt{pA^+}$ and $\overline{B}^+ :=$ $B^+/\sqrt{pB^+}$. Then we have a commutative diagram of morphisms of perfectoid rings

which is both a fiber square and a cofiber square (cf. [\[Bha19,](#page-40-20) Prop. 3.2]). Note that B^+ is p-torsion free with $B^+[\frac{1}{n}]$ $\frac{1}{p}$ = A. As $A^+ \to B^+$ is surjective, so is $A^{+, \flat} \to B^{+, \flat}$. By what we have proved, there exists a $w_{\alpha} \in A^+$ such that the image of

$$
y := (1 + \pi^{p\alpha} w_{\alpha})^{-1} x
$$

in B^+ is of the form \underline{z}^{\sharp} for some $\underline{z} \in B^{+, \flat} \cap B^{\flat, \times}$. It remains to show $y = \underline{y}^{\sharp}$ for some $\underline{y} \in A^{+, \flat}$. Granting this, the image of y in $A^{\flat} = B^{\flat}$ coincides with the image of \underline{z} , which is invertible as desired.

Write $\underline{z} = (z_0, z_1, \dots) \in B^{+, \flat} = \varprojlim_{\overline{D}^+} B^+$. Denote by \overline{y} the image of y in \overline{A}^+ . As [\(2.3\)](#page-8-0) is commutative, \overline{y} and $z_0 = \underline{z}^{\sharp}$ coincide in \overline{B}^+ . As both \overline{A}^+ and \overline{B}^+ are perfectoid in characteristic p (and thus perfect), we have $\overline{y}^{p^{-n}}$ is well-defined in \overline{A}^+ and coincides with z_n in \overline{B}^+ for any $n \geq 0$. So the $y_n := (\overline{y}^{p^{-n}}, z_n) \in \overline{A}^+ \times B^+$ defines an element in A^+ . By construction, we have $y_0 = y$ and $y_{n+1}^p = y_n$ for any $n \geq 0$. Put

$$
\underline{y} = (y_0, y_1, \dots) \in \varprojlim_{x \mapsto x^p} A^+ = A^{+,b}
$$

and then we have $y = y^{\sharp}$ as desired.

Corollary 2.11. Let A^+ be any perfectoid ring with $A = A^+ \left[\frac{1}{n} \right]$ $\frac{1}{p}$]. For any ? $\in \{\emptyset, \flat\}$, put

$$
A^{+,?} \cap A^{?,\times} := \{ x \in A^{+,?} \mid \text{the image of } x \text{ in } A^? \text{ is invertible.} \}.
$$

Then for any $\alpha \in \mathbb{N}[\frac{1}{p}]$ $\frac{1}{p}]\cap (0,1)$, we have

$$
A^+ \cap A^\times = (A^{+,b} \cap A^{b,\times})^\sharp \cdot (1 + \pi^{p\alpha} A^+).
$$

Proof. This follows from Proposition [2.10](#page-8-1) immediately. \square

2.2. Semi-stable formal schemes over A^+ . This subsection is closely related with the work of Cesnavičius and Koshikawa [\[CK19\]](#page-40-12). Fix an affinoid perfectoid $S = \text{Spa}(A, A^+) \in \text{Perfd}$. By a semi-stable formal scheme \mathfrak{X}_S of relative dimension d over A^+ , we mean a p-adic formal scheme \mathfrak{X}_S together with the log-structure $\mathcal{M}_{\mathfrak{X}_S}$ over A^+ (cf. Convention [2.9\)](#page-8-2), which is étale locally of the form $Spf(R_S^+$ $S⁺$) with $R⁺$ S_S small semi-stable of relative dimension d over $A⁺$ as defined below. We remark that the generic fiber X_S of a semi-stable \mathfrak{X}_S over A^+ is always smooth over S.

Definition 2.12. Fix an affinoid perfectoid $S = \text{Spa}(A, A^+) \in \text{Perfd}$. A p-complete A^+ -algebra R_S^+ $\frac{1}{S}$ is called *small semi-stable* of relative dimension d over A^+ , if there exists an étale morphism of p-adic formal schemes

$$
\psi : \mathrm{Spf}(R_S^+) \to \mathrm{Spf}(A^+\langle T_0, \ldots, T_r, T_{r+1}^{\pm 1}, \ldots, T_d^{\pm 1} \rangle / (T_0 \cdots T_r - p^a))
$$

for some $a \in \mathbb{Q}$ and $0 \le r \le d$. Equip R_S^+ with the log-structure associated to the pre-log-structure

$$
(2.4) \qquad M_{r,a}(A^+) := (\bigoplus_{i=0}^r \mathbb{N} \cdot e_i) \bigoplus_{\mathbb{N} \cdot (e_1 + \dots + e_r)} (A^\times \cap A^+) \xrightarrow{(\sum_{i=0}^r n_i e_i, x) \mapsto x \prod_{i=0}^r T_i^{n_i}} R_S^+,
$$

where $(\bigoplus_{i=0}^r \mathbb{N} \cdot e_i) \bigoplus_{\mathbb{N} \cdot (e_1 + \dots + e_r)} (A^{\times} \cap A^+)$ denotes the push-out of monoids

$$
\mathbb{N} \xrightarrow{1 \mapsto e_0 + \dots + e_r} \mathbb{N} \cdot e_1 \oplus \dots \oplus \mathbb{N} \cdot e_r
$$

$$
\downarrow_{1 \mapsto p^a}
$$

$$
(A^{\times} \cap A^+).
$$

In this case, we also say the semi-stable formal scheme $Spf(R_S^+)$ S_S) is small affine. We call such a ψ (resp. T_0, \ldots, T_d) a *chart* (resp. *coordinates*) on R_S^+ ^+_S or on $\text{Spf}(R_S^+)$ ⁺_S). The generic fiber $Spa(R_S, R_S^+)$ of $Spf(R_S^+$ $S(S)$ is then smooth over S and endowed with the induced chart ψ .

Denote by $\Omega_{\mathfrak{X}_S}^{1,\log}$ the module of (continuous) log-differentials of \mathfrak{X}_S over A^+ and for any $n \geq 1$, define $\Omega_{\mathfrak{X}_S}^{n,\log} = \wedge^n \Omega_{\mathfrak{X}_S}^{1,\log}$. Then $\Omega_{\mathfrak{X}_S}^{n,\log}$ is a locally finite free $\mathcal{O}_{\mathfrak{X}}$ -module for any $n \geq 0$. When $\mathfrak{X}_S = \mathrm{Spf}(R_S^+)$ ⁺_S) is small affine, the $\Omega_{\mathfrak{X}_S}^{1,\log}$ is associated to a finite free R_S^+ S_S -module $\Omega_{R_S^+}^{1,\text{log}}$ of rank d, and the chart ψ on R_S^+ $_{S}^{+}$ induces an identification

$$
(2.5) \quad \begin{aligned} \Omega_{R_S^+}^{1,\log} &= \left((\bigoplus_{i=0}^d \mathbb{Z} \cdot e_i) / \mathbb{Z} \cdot (e_0 + \dots + e_r) \right) \otimes_{\mathbb{Z}} R_S^+ \oplus \left(R_S^+ \cdot \text{dlog} T_{r+1} \oplus \dots \oplus R^+ \cdot \text{dlog} T_d \right) \\ &= \left((\bigoplus_{i=0}^d R_S^+ \cdot e_i) / R_S^+ \cdot (e_0 + \dots + e_r) \right) \oplus \left(R_S^+ \cdot \text{dlog} T_{r+1} \oplus \dots \oplus R_S^+ \cdot \text{dlog} T_d \right). \end{aligned}
$$

A semi-stable \mathfrak{X}_S over A^+ is called *liftable*, if there exists a flat log p-adic formal scheme $\widetilde{\mathfrak{X}}_S$ with the log-structure $\mathcal{M}_{\widetilde{\mathfrak{X}}_S}$ over $\mathbb{A}_{2,K}(S) := \mathbb{A}_{\text{inf},K}(S)/\text{Ker}(\theta_K)^2$ (as a log formal scheme with the canonical log-structure, cf. Convention [2.9\)](#page-8-2), such that \mathfrak{X}_S is the reduction of \mathfrak{X}_S modulo ξ_K ; that is, the underlying scheme \mathfrak{X}_S is the base-change of \mathfrak{X}_S along the canonical surjection $\mathbb{A}_{2,K}(S) \to A^+$ while the log-structure $\mathcal{M}_{\mathfrak{X}_S}$ is induced by the composite $\mathcal{M}_{\widetilde{\mathfrak{X}}_S}\to\mathcal{O}_{\widetilde{\mathfrak{X}}_S}\to\mathcal{O}_{\mathfrak{X}_S}$.

Given a semi-stable \mathfrak{X}_S over A^+ , its lifting $\widetilde{\mathfrak{X}}_S$ over $\mathbb{A}_{2,K}(S)$ may not always exist. However, when $\mathfrak{X}_S = \mathrm{Spf}(R_S^+)$ \mathcal{F}_S^+) is small affine, by (log-)smoothness of \mathfrak{X}_S , the lifting $\widetilde{\mathfrak{X}}_S$ always exists and is unique up to isomorphisms. More precisely, the

$$
\mathbb{A}_{2,K}(S)\langle T_0,\ldots,T_r,T_{r+1}^{\pm 1},\ldots,T_d^{\pm 1}\rangle/(T_0\cdots T_r-[\varpi^a])
$$

with the log-structure associated to the pre-log-structure

$$
(2.6) \qquad M_{r,a}(A^{\flat,+}) := (\bigoplus_{i=0}^r \mathbb{N} \cdot e_i) \bigoplus_{\mathbb{N} \cdot (e_1 + \dots + e_r)} (A^{\flat, \times} \cap A^{\flat, +})
$$
\n
$$
\xrightarrow{(\sum_{i=0}^r n_i e_i, x) \mapsto [x] \prod_{i=0}^r T_i^{n_i}} \mathbb{A}_{2,K}(S) \langle T_0, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_d^{\pm 1} \rangle / (T_0 \cdots T_r - [\varpi^a]),
$$

where $(\bigoplus_{i=0}^r \mathbb{N} \cdot e_i) \bigoplus_{\mathbb{N} \cdot (e_1 + \dots + e_r)} (A^{\flat, \times} \cap A^{\flat, +})$ denotes the push-out of monoids

$$
\mathbb{N} \xrightarrow{1 \mapsto e_0 + \dots + e_r} \mathbb{N} \cdot e_1 \oplus \dots \oplus \mathbb{N} \cdot e_r
$$

$$
(A^{\flat,\times} \cap A^{\flat,\div}),
$$

is a lifting of the log-structure associated to

$$
(M_{r,a}(A^+) \to A^+\langle T_0, \ldots, T_r, T_{r+1}^{\pm 1}, \ldots, T_d^{\pm 1} \rangle/(T_0 \cdots T_r - p^a)).
$$

By the étaleness of the chart ψ , there exists a unique \tilde{R}_{S}^{+} together with a unique homomorphism of $A_{2,K}(S)$ -algebras

$$
\widetilde{\psi}: \mathbb{A}_{2,K}(S)\langle T_0,\ldots,T_r,T_{r+1}^{\pm 1},\ldots,T_d^{\pm 1}\rangle/(T_0\cdots T_r-[\varpi^a])\to \widetilde{R}_S^+
$$

lifting ψ . Then we have $\widetilde{\mathfrak{X}}_S = \text{Spf}(\widetilde{R}^+_S)$ with the log-structure $\mathcal{M}_{\widetilde{\mathfrak{X}}_S}$ induced by the composite

$$
(2.7) \tM_{r,a}(A^{\flat,+}) \to \mathbb{A}_{2,K}(S)\langle T_0, \ldots, T_r, T_{r+1}^{\pm 1}, \ldots, T_d^{\pm 1} \rangle / (T_0 \cdots T_r - [\varpi^a]) \stackrel{\widetilde{\psi}}{\to} \widetilde{R}_S^+.
$$

Using this, it is clear that $\widetilde{\mathfrak{X}}_S$ is (log-)smooth over $\mathbb{A}_{2,K}(S)$.

By definition, any smooth $\mathfrak X$ over A^+ endowed with the induced log-structure from A^+ is always semi-stable over A^+ . We now give another typical example of semi-stable formal schemes over A^+ .

Example 2.13. The semi-stable formal scheme over \mathcal{O}_C defined above is exactly the semi-stable formal scheme $\mathfrak X$ with the log-structure $(\mathcal M_{\mathfrak X} = \mathcal O_X^{\times} \cap \mathcal O_{\mathfrak X} \hookrightarrow \mathcal O_{\mathfrak X})$ considered in [\[CK19,](#page-40-12) §1.5 and §1.6], where X is the generic fiber of \mathfrak{X} . For any $S = \text{Spa}(A, A^+)$, denote by \mathfrak{X}_S the base-change of $\mathfrak X$ along $\text{Spf}(A^+) \to \text{Spf}(\mathcal O_C)$ (viewed log formal schemes, Convention [2.9\)](#page-8-2) with the fiber product log-structure $\mathcal{M}_{\mathfrak{X}_S}$. Then \mathfrak{X}_S is a semi-stable formal scheme over A^+ . Moreover, if \mathfrak{X} is liftable, then so is \mathfrak{X}_S . Indeed, let $\widetilde{\mathfrak{X}}$ (with the log-structure $\mathcal{M}_\mathfrak{X}$) is a lifting of \mathfrak{X} over $\mathbf{A}_{2,K}$, then its base-change $\widetilde{\mathfrak{X}}_S$ along $\mathbf{A}_{2,K} \to \mathbb{A}_{2,K}(S)$ with the log-structure $\mathcal{M}_{\widetilde{\mathfrak{X}}_S}$ induced from the fiber product gives rise to a lifting of \mathfrak{X}_S . This is the typical case we shall work in.

Now, we are going to introduce some notations, which will be used in local calculations, as in [\[CK19,](#page-40-12) §3.2] for small semi-stable $\mathfrak{X}_S = \text{Spf}(R_S^+)$ $S(S)$ over A^+ with the chart ψ as in Definition [2.12.](#page-9-0) Denote its generic fiber by $X_S = \text{Spa}(R_S, R_S^+)$.

For any $n \geq 0$, put

$$
A_{r,a,n}^{+} := A^{+} \langle T_0^{\frac{1}{p^{n}}}, \ldots, T_r^{\frac{1}{p^{n}}}, T_{r+1}^{\pm \frac{1}{p^{n}}}, \ldots, T_d^{\pm \frac{1}{p^{n}}}\rangle / (T_0^{\frac{1}{p^{n}}} \cdots T_r^{\frac{1}{p^{n}}} - p^{\frac{a}{p^{n}}})
$$

and

$$
M_{r,a,n}(A^+) := (\bigoplus_{i=0}^r \frac{1}{p^n} \mathbb{N} \cdot e_i) \oplus_{\frac{1}{p^n} \mathbb{N} \cdot (e_1 + \dots + e_r)} (A^\times \cap A^+) \xrightarrow{(\sum_{i=0}^r \frac{n_i}{p^n} e_i, x) \mapsto x \prod_{i=0}^r T_i^{n_i}} A_{r,a,n},
$$

where $(\bigoplus_{i=0}^r \frac{1}{p^n} \mathbb{N} \cdot e_i) \oplus_{\frac{1}{p^n} \mathbb{N} \cdot (e_1 + \dots + e_r)} (A^{\times} \cap A^+)$ denotes the push-out of monoids

$$
\frac{1}{p^n} \mathbb{N} \xrightarrow{\frac{1}{p^n} \mapsto \sum_{i=0}^r \frac{1}{p^n} e_i} \frac{1}{p^n} \mathbb{N} \cdot e_1 \oplus \dots \oplus \frac{1}{p^n} \mathbb{N} \cdot e_r
$$

$$
\downarrow^{\frac{1}{p^n} \mapsto p^{\frac{a}{p^n}}}
$$

$$
(A^{\times} \cap A^+).
$$

Put $A^+_{r,a,\infty} := (\mathrm{colim}_n A^+_{r,a,n})^\wedge$ and $M_{r,a,\infty}(A^+) := \mathrm{colim}_n M_{r,a,n}(A^+)$. Then $A^+_{r,a,\infty}$ is perfectoid over A^+ and the natural map $M_{r,a,\infty} \to A_{r,a,\infty}^+$ induces a perfect log-structure on $A_{r,a,\infty}^+$ (cf. Definition [2.6\)](#page-7-1). Indeed, we have

$$
M_{r,a,\infty}(A^+) := (\bigoplus_{i=0}^r \mathbb{N}[\frac{1}{p}] \cdot e_i) \oplus_{\mathbb{N}[\frac{1}{p}]\cdot(e_1+\cdots+e_r)} (A^\times \cap A^+) \xrightarrow{(\sum_{i=0}^r \frac{n_i}{p^n} e_i, x) \mapsto x \prod_{i=0}^r T_i^{n_i}} A_{r,a,\infty}^+,
$$

where $(\bigoplus_{i=0}^r \frac{1}{p^n} \mathbb{N} \cdot e_i) \oplus_{\frac{1}{p^n} \mathbb{N} \cdot (e_1 + \dots + e_r)} (A^{\times} \cap A^+)$ denotes the push-out of monoids

$$
\mathbb{N}[\frac{1}{p}] \xrightarrow{\frac{1}{p^{n}} \mapsto \sum_{i=0}^{r} \frac{1}{p^{n}} e_i} \mathbb{N}[\frac{1}{p}] \cdot e_1 \oplus \cdots \oplus \mathbb{N}[\frac{1}{p}] \cdot e_r
$$

$$
\downarrow^{\frac{1}{p^{n}} \mapsto p^{\frac{a}{p^{n}}}}
$$

$$
(A^{\times} \cap A^{+}).
$$

Note that $Spa(A_{r,a,\infty} = A_{r,a,\infty}^+[\frac{1}{p}]$ $\frac{1}{p}], A^{+}_{r,a,\infty} \rightarrow \text{Spa}(A_{r,a,0} = A^{+}_{r,a,0}[\frac{1}{p}]$ $\frac{1}{p}$, $A^{+}_{r,a,0}$) is a pro-étale Galois covering with the Galois group

(2.8) $\Gamma \cong {\delta} = \delta_0^{n_0} \cdots \delta_r^{n_r} \delta_{r+1}^{n_{r+1}} \cdots \delta_d^{n_d}$ $\mathbb{Z}_d^{n_d} \mid n_i \in \mathbb{Z}_p, \ \forall \ 0 \leq i \leq d, \text{ such that } n_0 + \cdots + n_r = 0 \} \cong \mathbb{Z}_p^{\oplus d},$ where the action of Γ on $A^+_{r,a,\infty}$ is uniquely determined such that for any $0 \leq i \leq d$, and $n \geq 0$ and any $\delta = \delta_0^{n_0} \cdots \delta_d^{n_d} \in \Gamma$, we have

(2.9)
$$
\delta(T_i^{\frac{1}{p^n}}) = \zeta_{p^n}^{n_i} T_i^{\frac{1}{p^n}}.
$$

Put $\gamma_i := \delta_0^{-1} \delta_i$ when $1 \leq i \leq r$ and $\gamma_j = \delta_j$ when $r + 1 \leq j \leq d$. Then we have an isomorphism (2.10) $\Gamma \cong \mathbb{Z}_n \cdot \gamma_1 \oplus \cdots \oplus \mathbb{Z}_n \cdot \gamma_d.$

This is useful in some calculations.

Let $X_{\infty,S} = \text{Spa}(\widehat{R}_{\infty,S}, \widehat{R}_{\infty,S}^+)$ be the base-change of X_S along $\text{Spa}(A_{r,a,\infty}, A_{r,a,\infty}^+) \to \text{Spa}(A_{r,a,0}, A_{r,a,0}^+)$ with respect to the chart $\psi: X_S \to \text{Spa}(A_{r,a,0}, A_{r,a,0}^+)$. Then $X_{\infty,S}$ is affinoid perfectoid such that $X_{\infty,S} \to X_S$ is a pro-étale Galois covering with Galois group Γ whose action on $\widehat{R}^+_{\infty,S}$ is determined by [\(2.9\)](#page-11-1). Consider the set of indices

(2.11)
$$
J_r := \{ \underline{\alpha} = (\alpha_0, \dots, \alpha_r, \alpha_{r+1}, \dots, \alpha_d) \in (\mathbb{N}[\frac{1}{p}] \cap [0, 1))^{d+1} \mid \prod_{i=0}^r \alpha_i = 0 \}.
$$

Then $\widehat{R}^+_{\infty, S}$ admits a Γ -equivariant decomposition

(2.12)
$$
\widehat{R}_{\infty,S}^+ = \widehat{\bigoplus}_{\underline{\alpha} \in J_r} R_S^+ \cdot \underline{T}^{\underline{\alpha}},
$$

where " $\widehat{\oplus}$ " denotes the *p*-adic topological direct sum. Note that the composite

$$
M_{r,a,\infty}(A^+) \to A^+_{r,a,\infty} \to \widehat{R}^+_{\infty,S}
$$

defines a perfect log-structure on $\widehat{R}^+_{\infty,s}$, which factors through the canonical log-structure on $\widehat{R}^+_{\infty,s}$ because $T_i \in \widehat{R}_{\infty,S}^{\times} \cap \widehat{R}_{\infty,S}^+$ for any $0 \leq i \leq r$.

For any $0 \leq i \leq d$, let $T_i^{\flat} := (T_i, T_i^{\frac{1}{p}}, \dots) \in \widehat{R}_{\infty,S}^{\flat,\times} \cap \widehat{R}_{\infty,S}^{\flat,\pm}$. Then the map

$$
\iota_{\psi}: \mathbb{A}_{2,K}(S)\langle T_0,\ldots,T_r,T_{r+1}^{\pm 1},\ldots,T_d^{\pm 1}\rangle/(T_0\cdots T_r-[\varpi^a])\to \mathbb{A}_{2,K}(X_{\infty})
$$

carrying each T_i to $[T_i]^{\flat}$ is a well-defined morphism of $\mathbb{A}_{2,K}(S)$ -algebras compatible with log-structures on the source and target. By the étaleness of ψ above, the ι_{ψ} uniquely extends to a morphism of $\mathbb{A}_{2,K}(S)$ -algebras (still denoted by)

(2.13)
$$
\iota_{\psi}: (M_{r,a}(A^{\flat,+}) \to \widetilde{R}_S^+) \to (\widehat{R}_{\infty,S}^{\flat,\times} \cap \widehat{R}_{\infty,S}^{\flat,+} \xrightarrow{[\cdot]} \mathbb{A}_{2,K}(X_{\infty})),
$$

lifting the natural morphism $(M_{r,a}(A^+) \to R_S^+$ $(S^{\dagger}) \to (\widehat{R}^{\times}_{\infty,S} \cap \widehat{R}^+_{\infty,S} \to \widehat{R}^+_{\infty,S}).$ The Γ -action on X_{∞} induces a Γ-action on $R^b_{\infty, S}$ and thus on $\mathbb{A}_{2,K}(X_\infty)$ such that for any $0 \le i \le d$, any $n \ge 0$ and any $\delta = \delta_0^{n_0} \cdots \delta_d^{n_d} \in \Gamma$, we have

(2.14)
$$
\delta((T_i^{\flat})^{\frac{1}{p^n}}) = \epsilon^{\frac{n_i}{p^n}}(T_i^{\flat})^{\frac{1}{p^n}}.
$$

Note that the ι_{ψ} above is not Γ-equivariant!

3. Period sheaves

Fix an $S = \text{Spa}(A, A^+) \in \text{Perfd}$. Throughout this section, we always assume \mathfrak{X}_S is a liftable semi-stable formal scheme over A^+ with the generic fiber X_S and a fix an its lifting $\widetilde{\mathfrak{X}}_S$ over $\mathbb{A}_{2,K}(S)$. Denote by $X_{S,v}$ the v-site associated to X_S in the sense of [\[Sch17\]](#page-41-11), and by $\widehat{\mathcal{O}}_{X_S}$ (resp. $\widehat{\mathcal{O}}_{X_S}^+$, $\widehat{\mathcal{O}}_{X_S}^{\flat}$ and $\widehat{\mathcal{O}}_{X_S}^{\flat,+}$ the sheaf sending each affinoid perfectoid $U = \text{Spa}(B, B^+) \in X_{S,v}$ to B^{\flat} (resp. B^+, B^{\flat} and $B^{\flat,+}$). For any $n \geq 1$, the *canonical log-structure* on $\mathbb{A}_{\text{inf},K}(\widehat{\mathcal{O}}_{X_S}^+)/\text{Ker}(\theta_K)^n$ is the log-structure associated to the morphism of monoids

$$
\widehat{\mathcal{O}}_{X_S}^{\flat,\times} \cap \widehat{\mathcal{O}}_{X_S}^{\flat,\div} \xrightarrow{[\cdot]} \mathbb{A}_{\mathrm{inf},K}(\widehat{\mathcal{O}}_{X_S}^+)/\mathrm{Ker}(\theta_K)^n.
$$

It follows from Proposition [2.2](#page-5-3) that the canonical log-structure on $\widehat{\mathcal{O}}_{X_S}^+$ is exactly the log-structure

$$
\widehat{\mathcal{O}}_{X_S}^{\times} \cap \widehat{\mathcal{O}}_{X_S}^+ =: \mathcal{M}_{X_S} \hookrightarrow \widehat{\mathcal{O}}_{X_S}^+.
$$

3.1. Integral Faltings' extension. In this part, we follow the argument in [\[Wan23,](#page-41-7) §2] to construct an analogue of integral Faltings' extension in loc.cit. on $X_{S,v}$ with respect to the given lifting $\tilde{\mathfrak{X}}_S$.

Denote by $\mathcal{M}_{\tilde{\mathfrak{X}}_S}$ and $\mathcal{M}_{\tilde{\mathfrak{X}}_S}$ the log-structures on \mathfrak{X}_S and $\tilde{\mathfrak{X}}_S$, respectively. Then we have the morphisms of log-ringed topoi over $\mathbb{A}_{2,K}(S)$:

$$
(\mathcal{O}_{\widetilde{\mathfrak{X}}_S}, \mathcal{M}_{\widetilde{\mathfrak{X}}_S}) \to (\mathcal{O}_{\mathfrak{X}_S}, \mathcal{M}_{\mathfrak{X}_S}) \to (\widetilde{\mathcal{O}}_{X_S}^+, \mathcal{M}_{X_S}).
$$

This gives rise to an exact triangle of p-complete cotangent complexes

$$
(3.1) \qquad \widehat{\mathcal{L}}_{(\mathcal{O}_{\tilde{\mathfrak{X}}_S},\mathcal{M}_{\tilde{\mathfrak{X}}_S})/(\mathcal{O}_{\tilde{\mathfrak{X}}_S},\mathcal{M}_{\tilde{\mathfrak{X}}_S})} \widehat{\otimes}_{\mathcal{O}_{\tilde{\mathfrak{X}}_S}}^{\mathbf{L}} \widehat{\mathcal{O}}_{X_S}^+ \to \widehat{\mathcal{L}}_{(\widehat{\mathcal{O}}_{X_S}^+, \mathcal{M}_{X_S})/(\mathcal{O}_{\tilde{\mathfrak{X}}_S},\mathcal{M}_{\tilde{\mathfrak{X}}_S})} \to \widehat{\mathcal{L}}_{(\widehat{\mathcal{O}}_{X_S}^+, \mathcal{M}_{X_S})/(\mathcal{O}_{\tilde{\mathfrak{X}}_S},\mathcal{M}_{\tilde{\mathfrak{X}}_S})}.
$$

The first term is easy to handle with: As the log-structure $\mathcal{M}_{\mathfrak{X}_S}$ is induced from $\mathcal{M}_{\widetilde{\mathfrak{X}}_S}$ via the composite $\mathcal{M}_{\widetilde{\mathfrak{X}}_S} \to \mathcal{O}_{\widetilde{\mathfrak{X}}_S} \to \mathcal{O}_{\mathfrak{X}_S}$, it follows from [\[Ols05,](#page-41-12) Lem. 8.22] that

$$
\widehat{L}_{(\mathcal{O}_{\mathfrak{X}_S},\mathcal{M}_{\mathfrak{X}_S})/(\mathcal{O}_{\widetilde{\mathfrak{X}}_S},\mathcal{M}_{\widetilde{\mathfrak{X}}_S})} \simeq \widehat{L}_{\mathcal{O}_{\mathfrak{X}_S}/\mathcal{O}_{\widetilde{\mathfrak{X}}_S}}.
$$

As $\widetilde{\mathfrak{X}}_S$ is flat over $\mathbb{A}_{2,K}(S)$, using [\[Wan23,](#page-41-7) Cor. 2.3], we have quasi-isomorphisms

$$
\widehat{\mathcal{L}}_{\mathcal{O}_{\mathfrak{X}_S}/\mathcal{O}_{\widetilde{\mathfrak{X}}_S}} \simeq \widehat{\mathcal{L}}_{A^+/\mathbb{A}_{2,K}(S)} \widehat{\otimes}^{\mathcal{L}}_{A^+} \mathcal{O}_{\mathfrak{X}_S} \simeq \mathcal{O}_{\mathfrak{X}_S} \{1\}[1] \oplus \mathcal{O}_{\mathfrak{X}_S} \{2\}[2].
$$

So we finally conclude that

(3.2)
$$
\widehat{\mathcal{L}}_{(\mathcal{O}_{\mathfrak{X}_S},\mathcal{M}_{\mathfrak{X}_S})/(\mathcal{O}_{\widetilde{\mathfrak{X}}_S},\mathcal{M}_{\widetilde{\mathfrak{X}}_S})} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}_S}}^{\mathcal{L}} \widehat{\mathcal{O}}_{X_S}^+ \simeq \widehat{\mathcal{O}}_{X_S}^+ \{1\}[1] \oplus \widehat{\mathcal{O}}_{X_S}^+ \{2\}[2].
$$

The last term of (3.1) is also easy to handle with: Consider the morphisms of log-rings

$$
(A^{\times} \cap A^{+} =: M_A \to A^{+}) \to (\mathcal{O}_{\mathfrak{X}_{S}}, \mathcal{M}_{\mathfrak{X}_{S}}) \to (\widehat{\mathcal{O}}_{X_{S}}^{+}, \mathcal{M}_{X_{S}})
$$

and the induced exact triangle

$$
\widehat{\mathcal{L}}_{(\mathcal{O}_{\mathfrak{X}_S},\mathcal{M}_{\mathfrak{X}_S})/(A^+,M_A)} \widehat{\otimes}^{\mathcal{L}}_{\mathcal{O}_{\mathfrak{X}_S}} \widehat{\mathcal{O}}^+_{X_S} \to \widehat{\mathcal{L}}_{(\widehat{\mathcal{O}}^+_{X_S},\mathcal{M}_{X_S})/(A^+,M_A)} \to \widehat{\mathcal{L}}_{(\widehat{\mathcal{O}}^+_{X_S},\mathcal{M}_{X_S})/(\mathcal{O}_{\mathfrak{X}_S},\mathcal{M}_{\mathfrak{X}_S})}.
$$

By Proposition [2.8\(](#page-7-2)2), the middle term $\mathrm{L}_{(\widehat{\mathcal{O}}_{X_S}^+, \mathcal{M}_{X_S})/(A^+,M_A)}$ above vanishes, yielding a quasi-isomorphism

$$
\widehat{\mathcal{L}}_{(\widehat{\mathcal{O}}_{X_S}^+, \mathcal{M}_{X_S})/(\mathcal{O}_{\mathfrak{X}_S}, \mathcal{M}_{\mathfrak{X}_S})} \simeq (\widehat{\mathcal{L}}_{(\mathcal{O}_{\mathfrak{X}_S}, \mathcal{M}_{\mathfrak{X}_S})/(A^+, M_A)} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}_S}}^{\mathcal{L}} \widehat{\mathcal{O}}_{X_S}^+)[1].
$$

If \mathfrak{X}_S is not only semi-stable but smooth over A^+ , then we have a quasi-isomorphism

$$
\widehat{\mathcal{L}}_{(\mathcal{O}_{\mathfrak{X}_S},\mathcal{M}_{\mathfrak{X}_S})/(A^+,M_A)} \simeq \Omega_{\mathfrak{X}_S}^{1,\mathrm{log}}[0] = \Omega_{\mathfrak{X}_S}^1[0].
$$

However, in the logarithmic case, it is not straightforward that $L_{(\mathcal{O}_{\mathfrak{X}_S},\mathcal{M}_{\mathfrak{X}_S})/(A^+,M_A)}$ is discrete. So we have to exhibit the discreteness of $L_{(\mathcal{O}_{\mathfrak{X}_S},\mathcal{M}_{\mathfrak{X}_S})/(A^+,M_A)}$ directly.

Lemma 3.1. We have $\widehat{L}_{(\mathcal{O}_{\mathfrak{X}_S},\mathcal{M}_{\mathfrak{X}_S})/(A^+,M_A)} \simeq \Omega_{\mathfrak{X}_S}^{1,\log}[0].$

Proof. It suffices to show the complex $L_{(\mathcal{O}_{\mathfrak{X}_S},\mathcal{M}_{\mathfrak{X}_S})/(A^+,M_A)}$ is discrete. Since the problem is local on $\mathfrak{X}_{S,\text{\'et}},$ we may assume $\mathfrak{X}_S = \text{Spf}(R_S^+)$ \mathcal{S}) is small affine such that the log-structure $\mathcal{M}_{\mathfrak{X}_S}$ is induced by the pre-log-structure $M_{r,a}(A^+) \oplus (\oplus_{j=r+1}^d \mathbb{Z} \cdot e_j) \to R_S^+$ where $M_{r,a}(A^+)$ is defined in [\(2.4\)](#page-9-1) and e_j is mapped to T_j for any $r + 1 \le j \le d$. To conclude, we have to show

$$
\widehat{\mathcal{L}}_{(M_{r,a}(A^+)\oplus(\oplus_{j=r+1}^d \mathbb{Z}\cdot e_j)\rightarrow R_S^+)/(M_A\rightarrow A^+)} \simeq \Omega_{R_S^+}^{1,\text{log}}[0].
$$

By the étaleness of $A_{r,a}^+ := A^+\langle T_0, \ldots, T_r, T_{r+1}^{\pm 1}, \ldots, T_d^{\pm 1} \rangle / (T_0 \cdots T_r - p^a) \to R_S^+$ ^+_S , using [\[Ols05,](#page-41-12) Lem. 8.22], we are reduced to the case to show

$$
\widehat{\mathcal{L}}_{M_{r,a}(A^+) \oplus (\oplus_{j=r+1}^d \mathbb{Z} e_j) \to A_{r,a}^+)/(M_A \to A^+)} \simeq \Omega_{A_{r,a}^+}^{1, \log} [0].
$$
\nPut $A_{r,a}^{+,nc} = A^+[T_0, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_d^{\pm 1}]/(T_0 \cdots T_r - p^a)$. Then we have\n
$$
A_{r,a}^{+,nc} \cong A^+ \otimes_{\mathbb{Z}[M_A]} \mathbb{Z}[M_{r,a}(A^+) \oplus (\oplus_{j=r+1}^d \mathbb{Z} \cdot e_j)].
$$

As $\mathbb{Z}[M_{r,a}(A^+) \oplus (\oplus_{j=r+1}^d \mathbb{Z} \cdot e_j)]$ is flat over $\mathbb{Z}[M_A]$, we see that $A_{r,a}^{+,nc}$ is flat over A^+ and thus flat over \mathbb{Z}_p . So we have a quasi-isomorphism

$$
\widehat{L}_{(M_{r,a}(A^+)\oplus(\oplus_{j=r+1}^d \mathbb{Z} \cdot e_j) \to A_{r,a}^+)/(M_A \to A^+)} \simeq \widehat{L}_{(M_{r,a}(A^+)\oplus(\oplus_{j=r+1}^d \mathbb{Z} \cdot e_j) \to A_{r,a}^{+,nc})/(M_A \to A^+)}.
$$

By the flat base-change theorem [\[Ols05,](#page-41-12) Cor. 8.13], we then have a quasi-isomorphism

 $\mathcal{L}_{(M_{r,a}(A^+)\oplus (\oplus_{j=r+1}^d \mathbb{Z}\cdot e_j) \rightarrow A_{r,a}^+)^{nc})/(M_A \rightarrow A^+)} \simeq \mathcal{L}_{(M_{r,a}(A^+)\oplus (\oplus_{j=r+1}^d \mathbb{Z}\cdot e_j) \rightarrow \mathbb{Z}[M_{r,a}(A^+)\oplus (\oplus_{j=r+1}^d \mathbb{Z}\cdot e_j)])/(M_A \rightarrow \mathbb{Z}[M_A])} \otimes_{\mathbb{Z}[M_A]}^{\mathcal{L}} A^+.$ According to $[Ols05, Lem. 8.23(ii)]$, we have a quasi-isomorphism

$$
L_{(M_{r,a}(A^+)\oplus(\oplus_{j=r+1}^d \mathbb{Z} \cdot e_j) \to A_{r,a}^{+,nc})/(M_A \to A^+)} \simeq (M_{r,a}(A^+)^{gp}/M_A^{gp} \oplus (\oplus_{j=r+1}^d \mathbb{Z})) \otimes_{\mathbb{Z}}^L A^+ \simeq ((\oplus_{i=0}^d A^+ \cdot e_i)/A^+ \cdot (e_0 + \cdots + e_r)) \oplus (\oplus_{j=r+1}^d A^+ \cdot e_j))[0].
$$

So we finally conclude a quasi-isomorphism

$$
\widehat{\mathcal{L}}_{(M_{r,a}(A^+)\oplus(\oplus_{j=r+1}^d \mathbb{Z} \cdot e_j) \to A_{r,a}^+)/(M_A \to A^+)} \simeq \left(\left((\oplus_{i=0}^d A^+ \cdot e_i) / A^+ \cdot (e_0 + \dots + e_r) \right) \oplus (\oplus_{j=r+1}^d A^+ \cdot e_j) \right) [0],
$$

yielding the desired discreteness of $\widehat{\mathcal{L}}_{(M_{r,a}(A^+)\oplus(\oplus_{j=r+1}^d \mathbb{Z} \cdot e_j) \to A_{r,a}^+)/(M_A \to A^+)}.$

Thanks to the above lemma, the third term in (3.1) reads

(3.3)
$$
\widehat{\mathcal{L}}_{(\widehat{\mathcal{O}}_{X_S}^+, \mathcal{M}_{X_S})/(\mathcal{O}_{\mathfrak{X}_S}, \mathcal{M}_{\mathfrak{X}_S})} \simeq (\widehat{\mathcal{O}}_{X_S}^+ \otimes_{\mathcal{O}_{\mathfrak{X}_S}} \Omega_{\mathfrak{X}_S}^{1, \log})[1].
$$

Definition 3.2 (Integral Faltings' extension). Let \mathfrak{X}_S be a liftable semi-stable formal scheme over A^+ with the generic fiber X_S . For a lifting $\widetilde{\mathfrak{X}}_S$ of \mathfrak{X}_S over $\mathbb{A}_{2,K}(S)$ (with the log-structure $\mathcal{M}_{\widetilde{\mathfrak{X}}_S}$), we call

$$
\mathcal{E}^+_{\widetilde{\mathfrak{X}}_S} := \mathrm{H}^0(\widehat{\mathrm{L}}_{(\widehat{\mathcal{O}}^+_{X_S},\mathcal{M}_{X_S})/(\mathcal{O}_{\widetilde{\mathfrak{X}}_S},\mathcal{M}_{\widetilde{\mathfrak{X}}_S})}[-1])
$$

the *integral Faltings' extension* associated to the lifting \mathfrak{X}_S .

Proposition 3.3. The integral Faltings' extension $\mathcal{E}^{\pm}_{\tilde{\mathbf{x}}_k}$ $\hat{\mathfrak{X}}_S^+$ is a locally finite free $\hat{\mathcal{O}}^+_{X_S}$ -module of rank $d+1$ and fits into the following short exact sequence

(3.4)
$$
0 \to \widehat{\mathcal{O}}_{X_S}^+ \{1\} \to \mathcal{E}_{\widetilde{\mathfrak{X}}_S}^+ \to \widehat{\mathcal{O}}_{X_S}^+ \otimes_{\mathcal{O}_{\mathfrak{X}_S}} \Omega_{\mathfrak{X}_S}^{1,\log} \to 0.
$$

Moreover, the cohomological class $[\mathcal{E}^+_{\tilde{\mathfrak{X}}_2}]$ $[\widetilde{\mathfrak{X}}_S^+]\in \text{Ext}^1(\widehat{\mathcal{O}}_{X_S}^+\otimes_{\mathcal{O}_{\mathfrak{X}_S}}\Omega^{1,\log}_{X_S},\widehat{\mathcal{O}}_{X_S}^+\{1\})$ is exactly the obstruction class for lifting the natural morphism of A^+ -algebras $(\mathcal{O}_{\mathfrak{X}_S}, \mathcal{M}_{\mathfrak{X}_S}) \to (\widehat{\mathcal{O}}_{X_S}^+, \mathcal{M}_{X_S})$ to a morphism of $\mathbb{A}_{2,K}(S)$ -algebras $(\mathcal{O}_{\widetilde{\mathfrak{X}}_S},\mathcal{M}_{\widetilde{\mathfrak{X}}_S})$ --> $(\mathbb{A}_{2,K}(\widehat{\mathcal{O}}^+_{X_S}),\mathcal{M}_{\widetilde{X}_S})$, where $\mathcal{M}_{\widetilde{X}_S}$ denotes the canonical logstructure on $\mathbb{A}_{2,K}(\widehat{\mathcal{O}}^+_{X_S}).$

Proof. The first statement follows from Equations (3.1) , (3.2) and (3.3) immediately. It remains to prove the "moreover" part. By the construction of $\mathcal{E}^+_{\tilde{\mathfrak{X}}_+}$ $\tilde{\mathfrak{X}}_s$, as argued in [\[Wan23,](#page-41-7) Lem. 2.10], the class $[\mathcal{E}^+_{\widetilde{\mathfrak{X}}_t}]$ $(\hat{\mathcal{O}}_{X_S}^+, \mathcal{M}_{X_S})$ is exactly the obstruction class for lifting $(\hat{\mathcal{O}}_{X_S}^+, \mathcal{M}_{X_S})$ to a log-ring over $(\mathcal{O}_{\tilde{X}_S}, \mathcal{M}_{\tilde{X}_S})$. As $\widehat{\mathcal{L}}_{(\widehat{\mathcal{O}}_{X_S}^+, \mathcal{M}_{X_S})/(A^+, M_A)} = 0$ by Proposition [2.8\(](#page-7-2)2), the lifting of $(\widehat{\mathcal{O}}_{X_S}^+, \mathcal{M}_{X_S})$ over $(\mathbb{A}_{2,K}(S), M_{\mathbb{A}_{2,K}(S)})$ is unique (up to the unique isomorphism), where $M_{A_{2,K}(S)}$ denotes the canonical log-structure on $\mathbb{A}_{2,K}(S)$. So it must be $(\mathbb{A}_{2,K}(\hat{\mathcal{O}}_{X_S}^+),\mathcal{M}_{\tilde{X}_S})$. Thus one can conclude by noting that lifting the morphism

$$
(\mathcal{O}_{\mathfrak{X}_S},\mathcal{M}_{\mathfrak{X}_S}) \to (\widehat{\mathcal{O}}_{X_S}^+, \mathcal{M}_{X_S})
$$

over $(\mathcal{O}_{\widetilde{\mathfrak{X}}_S},\mathcal{M}_{\widetilde{\mathfrak{X}}_S})$ amounts to lifting the morphism of A^+ -algebras

$$
(\mathcal{O}_{\mathfrak{X}_S}, \mathcal{M}_{\mathfrak{X}_S}) \to (\widehat{\mathcal{O}}_{X_S}^+, \mathcal{M}_{X_S})
$$

to a morphism of $\mathbb{A}_{2,K}(S)$ -algebras

$$
(\mathcal{O}_{\widetilde{\mathfrak{X}}_S}, \mathcal{M}_{\widetilde{\mathfrak{X}}_S}) \dashrightarrow (\mathbb{A}_{2,K}(\widehat{\mathcal{O}}^+_{X_S}), \mathcal{M}_{\widetilde{X}_S}).
$$

Using Proposition [2.8\(](#page-7-2)2) instead of $[BMS18, Lem. 3.14]$ in *loc.cit.*, the result follows from the same proof for [\[Wan23,](#page-41-7) Prop. 2.14].

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When the lifting $\widetilde{\mathfrak{X}}_S$ of \mathfrak{X}_S is fixed (as we did at the very beginning of this section), we also use \mathcal{E}^+ to stand for the Breuil–Kisin–Fargues twist of the corresponding integral Faltings' extension

$$
\mathcal{E}^+ := \mathcal{E}^+_{\widetilde{\mathfrak{X}}_S} \{-1\}
$$

for short. The following result is obvious from Proposition [3.3.](#page-13-1)

Corollary 3.4. The \mathcal{E}^+ is a locally finite free $\widehat{\mathcal{O}}^+_{X_S}$ -module of rank $d+1$ and fits into the following short exact sequence

$$
0 \to \widehat{\mathcal{O}}_{X_S}^+ \to \mathcal{E}^+ \to \widehat{\mathcal{O}}_{X_S}^+ \otimes_{\mathcal{O}_{\mathfrak{X}_S}} \Omega_{\mathfrak{X}_S}^{1,\log} \{-1\} \to 0.
$$

In the rest of this subsection, we want to describe of \mathcal{E}^+ in the case where $\mathfrak{X}_S = \text{Spf}(R_S^+)$ $_{S}^{+}$) is small affine with the chart ψ (cf. Definition [2.12\)](#page-9-0). We adapt the notations below Example [2.13.](#page-10-0) Put $E^+_{\tilde{P}}$ $\tilde{R}_{S}^{+}:=\mathcal{E}^{+}_{\widetilde{\mathfrak{X}}_{S}}$ $\frac{1}{\tilde{x}_S}(X_\infty)$ and then it admits a continuous action of Γ fitting into the following Γ -equivariant short exact sequence

$$
0 \to \widehat{R}_{\infty,S}^+\{1\} \to E_{\widetilde{R}_S^+}^+ \to \widehat{R}_{\infty,S}^+ \otimes_{R_S^+} \Omega_{R_S^+}^{1,\log} \to 0.
$$

As an $\widehat{R}_{\infty,S}^+$ -module, we have

$$
E_{\tilde{R}_{S}^{+}}^{+} \cong \widehat{R}_{\infty,S}^{+}\{1\} \oplus \widehat{R}_{\infty,S}^{+} \otimes_{R_{S}^{+}} \Omega_{R_{S}^{+}}^{1,\log}.
$$

So the $E_{\tilde{p}}^{+}$ \tilde{R}^+_{S} is uniquely determined by its Γ-action; that is, it is determined by an 1-cocycle

$$
c \in \mathrm{H}^1(\Gamma, \mathrm{Hom}_{\widehat{R}^+_{\infty,S}}(\widehat{R}^+_{\infty,S} \otimes_{R_S^+} \Omega^{1,\log}_{R_S^+}, \widehat{R}^+_{\infty,S}\{1\})) \cong \mathrm{H}^1(\Gamma, \mathrm{Hom}_{R_S^+}(\Omega^{1,\log}_{R_S^+}, \widehat{R}^+_{\infty,S}\{1\})).
$$

Now, we are going to calculate the above 1-cocycle c by using that $E_{\tilde{p}}^{+}$ \tilde{R}_{S}^{+} stands for the obstruction class for lifting the Γ-equivariant morphism

$$
(M_{r,a}(A^+) \to R_S^+) \to (\widehat{R}_{\infty,S}^\times \cap \widehat{R}_{\infty,S}^+ \to \widehat{R}_{\infty,S}^+)
$$

to a Γ-equivariant morphism

$$
(M_{r,a}(A^{\flat,+})\oplus (\oplus_{j=r+1}^d \mathbb{Z} \cdot e_j) \xrightarrow{\alpha} \widetilde{R}_S^+) \dashrightarrow (\widehat{R}_{\infty,S}^{\flat,\times} \cap \widehat{R}_{\infty,S}^{\flat,+} \to \mathbb{A}_{2,K}(X_{\infty,S})).
$$

Here, the map α : $M_{r,a}(A^{b,+}) \oplus (\oplus_{j=r+1}^d \mathbb{Z} \cdot e_j) \to \widetilde{R}_S^+$ is determined by [\(2.7\)](#page-10-1) together with that $\alpha(e_j) = T_j$ for all $r + 1 \leq j \leq d$. As in the proof of Lemma [3.1,](#page-12-2) we have an isomorphism of R_S^+ $_S⁺$ -modules</sub>

(3.6)
$$
\Omega_{R_S^+}^{1,\log} \cong (\bigoplus_{i=0}^d R_S^+ \cdot e_i)/R_S^+ \cdot (e_0 + \cdots + e_r),
$$

where e_j stands for $dlogT_j$ for any $r + 1 \leq j \leq d$ via the identification [\(2.5\)](#page-9-2). Let

$$
\iota_{\psi}: (M_{r,a}(A^{\flat,+})\oplus (\oplus_{j=r+1}^d \mathbb{Z} \cdot e_j) \xrightarrow{\alpha} \widetilde{R}_S^+) \to (\widehat{R}_{\infty,S}^{\flat,\times} \cap \widehat{R}_{\infty,S}^{\flat,\div} \xrightarrow{[\cdot]} \mathbb{A}_{2,K}((X_{\infty,S}))
$$

be the morphism of $\mathbb{A}_{2,K}(S)$ -algebras introduced in [\(2.13\)](#page-11-2). Then one can describe the 1-cocycle c above by using ι_{ψ} : For any $\delta = \prod_{i=0}^{d} \delta_i^{n_i} \in \Gamma(2.8)$, the map

$$
c(\delta):\Omega^{1,\log}_{R^+_S}\to \widehat{R}^+_{\infty,S}
$$

is determined by that for any $0 \le j \le d$, in $\xi \mathbb{A}_{2,K}(X_{\infty,S}) \cong \widehat{R}^+_{\infty,S}\{1\}$,

$$
c(\delta)(e_j) \cdot \alpha(e_j) = \delta(\iota_{\psi}(\alpha(e_j))) - \iota_{\psi}(\alpha(e_j)).
$$

As $\alpha(e_j) = T_j$ for all j, it then follows from (2.14) that

$$
c(\delta)(e_j) = [\epsilon^{n_j}] - 1 = \frac{[\epsilon^{n_j}] - 1}{[\epsilon] - 1} \cdot ([\epsilon] - 1) \in \xi_K \mathbb{A}_{2,K}(X_{\infty,S}).
$$

As $\frac{[\epsilon]-1}{t} \in \mathbf{B}_{\text{dR}}^{+}$ goes to 1 modulo t, via the identification

$$
\mathcal{O}_C \cdot \xi_K = \mathcal{O}_C\{1\} = \rho_K^{-1} \mathcal{O}_C(1) = \mathcal{O}_C \cdot \rho_K^{-1} t \text{ (cf. §1.4)},
$$

for any $\delta = \prod_{i=0}^d \delta_i^{n_i} \in \Gamma$ and any $0 \le j \le d$, we have

$$
c(\delta)(e_j) = n_j \rho_K \xi_K.
$$

In summary, we have proved the following proposition:

Proposition 3.5. There exists an isomorphism of $\widehat{R}^+_{\infty,S}$ -modules

$$
\mathcal{E}^+_{\widetilde{\mathfrak{X}}_S}(X_{\infty,S})=:E^+_{\widetilde{R}^+_S}\cong \widehat{R}^+_{\infty,S}\cdot \xi_K\oplus \left((\oplus_{i=0}^d\widehat{R}^+_{\infty,S}\cdot e_i)/\widehat{R}^+_{\infty,S}\cdot (e_0+\cdots+e_r)\right)
$$

such that via this isomorphism, the Γ -action on $E^+_{\tilde{p}}$ $\tilde{R}_{\tilde{S}}^+$ is given by that for any $\delta = \prod_{i=0}^d \delta_i^{n_i} \in \Gamma$,

$$
\delta(a\xi_K + \sum_{j=0}^d b_j e_j) = (\delta(a) + \rho_K \sum_{j=0}^d \delta(b_j) n_j) \xi_K + \sum_{j=0}^d \delta(b_j) e_j.
$$

Put $E^+ := \mathcal{E}^+(X_{\infty,S})$ where \mathcal{E}^+ is the Breuil–Kisin–Fargues twist of the integral Faltings' extension (cf. (3.5)). Then it admits a continuous action of Γ.

Corollary 3.6. (1) The E^+ is a free $\widehat{R}^+_{\infty,S}$ -module of rank $d+1$ fitting into the short exact sequence

$$
0 \to \widehat{R}_{\infty, S}^+ \xrightarrow{i} E^+ \xrightarrow{\text{pr}} \widehat{R}_{\infty, S}^+ \otimes_{R_S^+} \Omega_{R_S^+}^{1, \log} \{-1\} \to 0.
$$

(2) There exists an isomorphism of $\widehat{R}^+_{\infty, S}$ -modules

$$
E^+ \cong \widehat{R}^+_{\infty,S} \cdot e \oplus \left((\bigoplus_{i=0}^d \widehat{R}^+_{\infty,S} \cdot y_i) / \widehat{R}^+_{\infty,S} \cdot (y_0 + \cdots + y_r) \right)
$$

such that the following statements hold true:

- (a) The $\widehat{R}^+_{\infty,S} \cdot e$ is identified with $\widehat{R}^+_{\infty,S}$ via the injection i above and $e = i(1)$.
- (b) Via the isomorphism $(cf. (3.6))$ $(cf. (3.6))$ $(cf. (3.6))$

$$
\Omega_{R_S^+}^{1,\log}\{-1\} \cong (\oplus_{i=0}^d R_S^+ \cdot \frac{e_i}{\xi_K})/R_S^+ \cdot (\frac{e_0}{\xi_K} + \cdots + \frac{e_r}{\xi_K}),
$$

the image of y_i via the projection pr above is $pr(y_i) = \frac{e_i}{\xi_K}$ for any $0 \le i \le d$.

(c) The Γ -action on E^+ is given by that for any $\delta = \prod_{i=0}^d \delta_i^{n_i} \in \Gamma$,

$$
\delta(ae + \sum_{j=0}^{d} b_j y_j) = (\delta(a) + \rho_K \sum_{j=0}^{d} \delta(b_j) n_j) e + \sum_{j=0}^{d} \delta(b_j) y_j.
$$

Proof. The Item (1) follows from Corollary [3.4.](#page-14-2) The Item (2) follows from Proposition [3.5](#page-15-0) by letting $y_i = \frac{e_i}{\varepsilon_k}$ $\frac{e_i}{\xi_K}$ with e_i 's appearing there.

3.2. Period sheaves. Now, we are going to follow the strategy in [\[MW24,](#page-41-8) §2] to construct the period sheaves with Higgs fields $(\mathcal{O} \hat{\mathbb{C}}_{\text{pd},S}, \Theta)$ and $(\mathcal{O} \hat{\mathbb{C}}_{\text{pd},S}, \Theta)$ by using the twisted integral Faltings' extension $\mathcal{E}^+ = \mathcal{E}^+_{\widetilde{\mathfrak{X}}_t}$ $\widetilde{\mathfrak{X}}_S$ {-1} associated to the lifting $\widetilde{\mathfrak{X}}_S$ of \mathfrak{X}_S over $\mathbb{A}_{2,K}(S)$.

Recall the following well-known lemma due to Quillen and Illusie:

Lemma 3.7 ($[SZ18, \text{ Lem. A.28}]$ $[SZ18, \text{ Lem. A.28}]$). Let B be a commutative ring. For any short exact sequence of flat B-modules

$$
0 \to E \xrightarrow{u} F \xrightarrow{v} G \to 0
$$

and any $n > 0$, there exists an exact sequence of B-modules:

$$
(3.7) \qquad 0 \to \Gamma^{n}(E) \to \Gamma^{n}(F) \xrightarrow{\partial} \Gamma^{n-1}(F) \otimes G \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Gamma^{n-i}(F) \otimes \wedge^{i}G \xrightarrow{\partial} \cdots \xrightarrow{\partial} \wedge^{n}G \to 0,
$$

where the differentials ∂ are induced by sending each

$$
f_1^{[m_1]}\cdots f_r^{[m_r]}\otimes \omega\in \Gamma^m(F)\otimes \wedge^l G
$$

with $f_i \in F$, $m_i \geq 1$ satisfying $m_1 + \cdots + m_r = m$ and $\omega \in \wedge^l G$ to

$$
\sum_{i=1}^r f_1^{[m_1]} \cdots f_i^{[m_i-1]} \cdots f_r^{[m_r]} \otimes v(f_i) \wedge \omega \in \Gamma^{m-1}(F) \otimes \wedge^{l+1} G.
$$

Moreover, there exists an exact sequence

(3.8)
$$
0 \to \Gamma(E) \to \Gamma(F) \xrightarrow{\partial} \Gamma(F) \otimes G \xrightarrow{\partial} \Gamma(F) \otimes \wedge^2 G \xrightarrow{\partial} \cdots,
$$

where the differentials ∂ are all $\Gamma(E)$ -linear.

Applying the above lemma to the short exact sequence in Corollary [3.4,](#page-14-2) we get an exact sequence

(3.9)
$$
0 \to \Gamma(\widehat{\mathcal{O}}_{X_S}^+) \to \Gamma(\mathcal{E}^+) \stackrel{\partial}{\to} \Gamma(\mathcal{E}^+) \otimes_{\mathcal{O}_{\mathfrak{X}_S}} \Omega_{\mathfrak{X}_S}^{1,\log} \{-1\} \to \cdots
$$

where ∂ is $\Gamma(\widehat{\mathcal{O}}_{X_S}^+)$ -linear. Note that $\Gamma(\mathcal{E}^+) \cong \widehat{\mathcal{O}}_{X_S}^+[e]_{\text{pd}}$ is the free pd-algebra generated by e over $\widehat{\mathcal{O}}_{X_S}^+$, where e stands for the basis $1 \in \widehat{\mathcal{O}}_{X_S}^+$. As $\zeta_p - 1$ admits arbitrary pd-powers in $\widehat{\mathcal{O}}_{X_S}^+$, the $e - (\zeta_p - 1)$ generates a pd-ideal \mathcal{I}_{pd} of $\widehat{\mathcal{O}}_{X_S}^+[e]_{\text{pd}}$ such that we have $\Gamma(\widehat{\mathcal{O}}_{X_S}^+)/\mathcal{I}_{\text{pd}} \cong \widehat{\mathcal{O}}_{X_S}^+$.

Definition 3.8. Let $\mathcal{O}^{\mathbb{C}^+}_{\mathrm{pd},S}$ be the *p*-adic completion of

$$
\mathcal{O}\mathbb{C}_{\mathrm{pd},S}^+ := \Gamma(\mathcal{E}^+)/\mathcal{I}_{\mathrm{pd}}\cdot\Gamma(\mathcal{E}^+)
$$

and let $\Theta: \mathcal{O} \widehat{\mathbb{C}}_{\mathrm{pd},S}^+ \to \mathcal{O} \widehat{\mathbb{C}}_{\mathrm{pd},S}^+ \otimes_{\mathcal{O}_{\mathfrak{X}_S}} \Omega_{\mathfrak{X}_S}^{1,\mathrm{log}}\{-1\}$ be the morphism induced by ∂ in [\(3.9\)](#page-16-0). Define

$$
(\mathcal O\widehat{\mathbb C}_{\mathrm{pd},S},\Theta:\mathcal O\widehat{\mathbb C}_{\mathrm{pd},S}\to\mathcal O\widehat{\mathbb C}_{\mathrm{pd},S}\otimes_{\mathcal O_{\mathfrak X_S}}\Omega^{1,\log}_{\mathfrak X_S}\{-1\}):=(\mathcal O\widehat{\mathbb C}_{\mathrm{pd},S}^+,\Theta)[\frac{1}{p}].
$$

Theorem 3.9 (Poincaré's Lemma). For any ? $\in \{\emptyset, +\}$, the following sequence

$$
0 \to \widehat{\mathcal{O}}_{X_S}^? \to \widehat{\mathcal{O}}_{\mathrm{pd},S}^{\widehat{\mathcal{C}}} \xrightarrow{\Theta} \widehat{\mathcal{O}}_{\mathrm{pd},S}^{\widehat{\mathcal{C}}} \otimes_{\mathcal{O}_{\mathfrak{X}_S}} \Omega_{\mathfrak{X}_S}^{1,\mathrm{log}} \{-1\} \xrightarrow{\Theta} \widehat{\mathcal{O}}_{\mathrm{pd},S}^{\widehat{\mathcal{C}}} \otimes_{\mathcal{O}_{\mathfrak{X}_S}} \Omega_{\mathfrak{X}_S}^{2,\mathrm{log}} \{-2\} \to \cdots
$$

is exact on X_v . In particular, Θ defines a Higgs field on $\mathcal{O} \hat{\mathbb{C}}_{\mathrm{pd},S}^+$.

Proof. It suffices to prove the case for ? = +. By Corollary [3.4,](#page-14-2) locally on X_v , the \mathcal{E}^+ is a direct sum of $\widehat{\mathcal{O}}_{X_S}^+$ and $\widehat{\mathcal{O}}_{X_S}^+ \otimes_{\mathcal{O}_{X_S}} \Omega_{\mathcal{X}_S}^{1,\log} \{-1\}$. So locally on X_v , the $\Gamma(\mathcal{E}^+)$ is a free pd-polynomial ring over $\Gamma(\hat{\mathcal{O}}_{X_S}^+)$. Modulo \mathcal{I}_{pd} , by the exactness of [\(3.9\)](#page-16-0), we have the following exact sequence

$$
(3.10) \t 0 \to \widehat{\mathcal{O}}_{X_S}^+ \to \mathcal{O}\mathbb{C}_{\mathrm{pd},S}^+ \xrightarrow{\Theta} \mathcal{O}\mathbb{C}_{\mathrm{pd},S}^+ \otimes_{\mathcal{O}_{\mathfrak{X}_S}} \Omega_{\mathfrak{X}_S}^{1,\mathrm{log}} \{-1\} \xrightarrow{\Theta} \mathcal{O}\mathbb{C}_{\mathrm{pd},S}^+ \otimes_{\mathcal{O}_{\mathfrak{X}_S}} \Omega_{\mathfrak{X}_S}^{2,\mathrm{log}} \{-2\} \to \cdots
$$

where Θ denotes the reduction of ∂ in [\(3.9\)](#page-16-0). Moreover, locally on X_v , the $\mathcal{O}\mathbb{C}^+_{\mathrm{pd},S}$ is a free pdpolynomial ring and thus faithfully flat over $\hat{\mathcal{O}}_{X_S}^+$. In particular, taking *p*-adic completion preserves the exactness of [\(3.10\)](#page-16-1), yielding the desired exactness in the case $? = +$.

Now, we give the local description of $(\mathcal{O} \widehat{\mathbb{C}}_{\mathrm{pd},X_S}^+,\Theta)$ in the case $\mathfrak{X}_S = \mathrm{Spf}(R_S^+)$ $\binom{+}{S}$ is small affine. We first introduce some notations. Let

 $\widehat{R}^+_{\infty,S}[Y_0,\ldots,Y_r,Y_{r+1},\ldots,Y_d]_{\rm pd}^{\wedge}$

be the *p*-complete free pd-algebra over $\widehat{R}^+_{\infty, S}$ generated by Y_0, \ldots, Y_d and let

$$
\Theta: \widehat{R}_{\infty,S}^+[Y_0,\ldots,Y_r,Y_{r+1},\ldots,Y_d]_{\rm pd}^\wedge \to \oplus_{i=0}^d \widehat{R}_{\infty,S}^+[Y_0,\ldots,Y_r,Y_{r+1},\ldots,Y_d]_{\rm pd}^\wedge \cdot \frac{e_i}{\xi_K}
$$

be the map sending each $f \in \widehat{R}^+_{\infty, S}[Y_0, \ldots, Y_r, Y_{r+1}, \ldots, Y_d]_{\text{pd}}^{\wedge}$ to

$$
\Theta(f) = \sum_{i=0}^{d} \frac{\partial f}{\partial Y_i} \cdot \frac{e_i}{\xi_K}.
$$

Noting that for any $n \geq 0$ and for any $f \in \widehat{R}^+_{\infty, S}[Y_0, \ldots, Y_r, Y_{r+1}, \ldots, Y_d]_{\text{pd}}^{\wedge}$, we have

$$
\Theta((Y_0 + \dots + Y_r)^{[n]}f) = (Y_0 + \dots + Y_r)^{[n]} \Theta(f) + (Y_0 + \dots + Y_r)^{[n-1]}f \cdot \sum_{i=0}^r \frac{e_i}{\xi_K}
$$

So we get a well-defined map

$$
(3.11) \qquad \Theta: P_{\infty,S}^+ \to P_{\infty,S}^+ \otimes_{R_S^+} \Omega_{R_S^+}^{1,\log} \{-1\} = (\oplus_{i=0}^d P_{\infty,S}^+ \cdot \frac{e_i}{\xi_K})/P_{\infty,S}^+ \cdot (\frac{e_0}{\xi_K} + \dots + \frac{e_r}{\xi_K})
$$

via the identification $\Omega_{R_S^+}^{1,\log}\{-1\} \cong (\oplus_{i=0}^d R_S^+)$ $\frac{+}{S} \cdot \frac{e_i}{\xi_K}$ $\frac{e_i}{\xi_K}$)/ $R_S^+ \cdot (\frac{e_0}{\xi_K})$ $\frac{e_0}{\xi_K} + \cdots + \frac{e_r}{\xi_K}$ $\frac{e_r}{\xi_K}$) (cf. [\(3.6\)](#page-14-1)), where

$$
P_{\infty,S}^+ := \widehat{R}_{\infty,S}^+[Y_0,\ldots,Y_r,Y_{r+1},\ldots,Y_d]_{\rm pd}^\wedge/(Y_0+\cdots+Y_r)_{\rm pd}
$$

.

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denotes the quotient of $\widehat{R}^+_{\infty, S}[Y_0, \ldots, Y_r, Y_{r+1}, \ldots, Y_d]_{\text{pd}}^{\wedge}$ by the pd-ideal generated by $Y_0 + \cdots + Y_r$. Clearly, via the isomorphisms

$$
\Omega_{R_S^+}^{1,\log}\{-1\} \cong \bigoplus_{i=1}^d R_S^+ \cdot \frac{e_i}{\xi_K}
$$

and

$$
P_{\infty,S}^+ \cong \widehat{R}_{\infty,S}^+[Y_1,\ldots,Y_d]_{\text{pd}}^{\wedge},
$$

the Θ is exactly

$$
\Theta = \sum_{i=1}^d \frac{\partial}{\partial Y_i} \otimes \frac{e_i}{\xi_K}.
$$

When context is clear, we also express Θ as

$$
\Theta = \sum_{i=0}^d \frac{\partial}{\partial Y_i} \otimes \frac{e_i}{\xi_K} : P_{\infty,S}^+ \to P_{\infty,S}^+ \otimes_{R_S^+} \Omega_{R_S^+}^{1,\log}\{-1\}.
$$

By construction of $\mathcal{O}^{\mathbb{C}^+}_{\mathrm{pd},S}$, we see that $\mathcal{O}^{\mathbb{C}^+}_{\mathrm{pd},S}(X_\infty)$ is the p-adic completion of $\Gamma(E^+)/(e-(\zeta_p-\zeta_p)$ 1))_{pd}, where E^+ and e are described in Corollary [3.6](#page-15-1) and $(e - (\zeta_p - 1))_{pd}$ denotes the pd-ideal generated by $e - (\zeta_p - 1)$. Let y_i be elements in E^+ described in Corollary [3.6](#page-15-1) as well. By abuse of notations, we will not distinguish y_i with its image in $\mathcal{O} \widehat{\mathbb{C}}_{\mathrm{pd},S}^+(X_\infty)$.

Proposition 3.10. Keep notations above. The morphism of $\widehat{R}^+_{\infty,S}$ -algebras

$$
\widehat{R}_{\infty,S}^+[Y_0,\ldots,Y_r,Y_{r+1},\ldots,Y_d]_{\rm pd}^\wedge\to\mathcal{O}\widehat{\mathbb{C}}_{\rm pd,S}^+(X_\infty)
$$

sending each Y_i to y_i induces an isomorphism

$$
\iota: P_{\infty,S}^+ \to \mathcal O \widehat{\mathbb C}^+_{\mathrm{pd},S}(X_\infty)
$$

compatible with Higgs fields. Via the isomorphism ι , the Γ -action on $\mathcal{O} \hat{C}^+_{\text{pd},X_S}(X_\infty)$ is given by that for any $\delta = \prod_{i=0}^d \delta_i^{n_i} \in \Gamma$ and any $0 \le j \le d$, we have

$$
\delta(Y_j) = Y_j + n_j \rho_K(\zeta_p - 1).
$$

Proof. Consider the isomorphisms

$$
E^+ \cong \widehat{R}^+_{\infty, S} \cdot e \oplus (\oplus_{i=1}^d \widehat{R}^+_{\infty, S} \cdot y_i) \text{ and } \Omega^1_{R_S^+}\{-1\} \cong \oplus_{i=1}^d R_S^+ \cdot \frac{e_i}{\xi_K}.
$$

Via the projection pr : $E^+ \to \widehat{R}^+_{\infty, S} \otimes_{R_S^+} \Omega^1_{\widehat{F}}$ $\frac{1}{R_S^+}\{-1\}$ in Corollary [3.6,](#page-15-1) the image of y_i is $\frac{e_i}{\xi_K}$. By Lemma [3.7,](#page-15-2) we have $\Gamma(E^+) = \widehat{R}^+_{\infty, S}[e, y_1, \ldots, y_d]_{\text{pd}}$ is the free pd-polynomial ring over $\widehat{R}^+_{\infty, S}$ generated by e, y_1, \ldots, y_d while the differential map $\partial : \Gamma(E^+) \to \Gamma(E^+) \otimes_{R_S^+} \Omega^1$ $\frac{1}{R_S^+} \{-1\}$ reads

$$
\partial = \sum_{i=1}^d \frac{\partial}{\partial y_i} \otimes \frac{e_i}{\xi}.
$$

Modulo $(e - (\zeta_p - 1))_{\text{pd}}$ and after taking *p*-adic completion, we see that

$$
(\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd},X_S}^+(X_\infty),\Theta)=(\widehat{R}_{\infty,S}^+[y_1,\ldots,y_d]_{\mathrm{pd}}^\wedge,\sum_{i=1}^d\frac{\partial}{\partial y_i}\otimes\frac{e_i}{\xi_K}).
$$

Using the isomorphism $(P_{\infty,S}^+, \Theta) \cong (\widehat{R}_{\infty,S}^+[Y_1,\ldots,Y_d]_{\text{pd}}^{\wedge}, \sum_{i=1}^d$ ∂ $\frac{\partial}{\partial Y_i} \otimes \frac{e_i}{\xi_K}$ $\frac{e_i}{\xi_K}$) mentioned above, we conclude that ι is an isomorphism. As for any $\delta = \prod_{i=0}^d \delta_i^{n_i} \in \Gamma$ and any $0 \le j \le d$, we have $\delta(y_j) = y_j + n_j \rho_K e$ and e is mapped to $\zeta_p - 1$ via the map $E^+ \to \mathcal{O}^{\mathbb{C}^+}_{\mathrm{pd},X_S}(X_\infty)$, the desired Γ -action follows directly. \Box

Note that for any ? $\in \{\emptyset, +\}$, the sheaf $\mathcal{O}\mathbb{C}_{\mathrm{pd},S}^?$ is an $\mathcal{O}_{X_S}^?$ -algebra and the $\mathcal{O}\mathbb{C}_{\mathrm{pd},S}^? \otimes_{\mathcal{O}_{X_S}^+} \mathcal{O}\mathbb{C}_{\mathrm{pd},S}^?$ is endowed with the tensor product Higgs field $\Theta \otimes id + id \otimes \Theta$.

Proposition 3.11. For any ? $\in \{\emptyset, +\}$, the multiplication map

$$
\mathcal O\widehat{\mathbb C}_{\mathrm{pd},S}^? \otimes_{\widehat{\mathcal O}_{X_S}^+} \mathcal O\widehat{\mathbb C}_{\mathrm{pd},S}^? \to \mathcal O\widehat{\mathbb C}_{\mathrm{pd},S}^?
$$

on $\mathcal{O} \hat{\mathbb{C}}^2_{\text{pd},S}$ is compatible with Higgs fields; that is, for any local sections $f, g \in \mathcal{O} \hat{\mathbb{C}}^2_{\text{pd},S}$, we have $\Theta(fq) = \Theta(f)q + f\Theta(q).$

Proof. Since the problem is local on both $\mathfrak{X}_{S,\text{\'et}}$ and $X_{S,v}$, we may assume $\mathfrak{X}_S = Spf(R_S^+)$ $_{S}^{+}$) is small affine and are reduced to showing that for any $f, g \in \mathcal{O}(\widehat{C}_{\text{pd},S}^+(X_\infty))$, we have

$$
\Theta(fg) = \Theta(f)g + f\Theta(g).
$$

But this follows from Proposition [3.10](#page-17-0) immediately.

We remark that Proposition [3.11](#page-18-1) amounts to that the Higgs complex $\text{HIG}(\mathcal{O} \hat{\mathbb{C}}_{\text{pd},S}^? \Theta)$ is a commutative differential graded algebra over $\hat{\mathcal{O}}_{X_S}^?$. Using this Theorem [3.9](#page-16-2) amounts to that the morphism

$$
\widehat{\mathcal{O}}_{X_S}^?\to {\mathrm{HIG}}(\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd},S}^?)
$$

is actually an isomorphism of algebras in the derived category of $\hat{\mathcal{O}}_{X_S}^+$ -modules.

We end this section with the following remarks.

Remark 3.12. Provided \mathcal{E}^+ in Corollary [3.4,](#page-14-2) for any $r \geq 0$, one can define \mathcal{E}_r^+ as the pull-back of \mathcal{E}^+ along the natural inclusion

$$
p^r \widehat{\mathcal{O}}_X^+ \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^{1,\log}_\mathfrak{X} \{-1\} \hookrightarrow \widehat{\mathcal{O}}_X^+ \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^{1,\log}_\mathfrak{X} \{-1\}.
$$

So it fits into the short exact sequence

$$
0 \to \widehat{\mathcal{O}}_X^+ \to \mathcal{E}_r^+ \to p^r \widehat{\mathcal{O}}_X^+ \otimes_{\mathcal{O}_{\mathfrak{X}_S}} \Omega^{1,\log}_\mathfrak{X} \{-1\} \to 0.
$$

As in [\[Wan23,](#page-41-7) §2], one can construct period sheaves $\mathcal{O}\mathbb{C}_r^2$ with Higgs fields Θ for $? \in \{\emptyset, +\}.$ (Recall that $\mathcal{O}_C\{-1\} = \rho_K \mathcal{O}_C(-1)$, the $\mathcal{O}(\mathbb{C}_r^2)$ coincides with $\mathcal{O}(\mathbb{C}_{p^r \rho_K}^2)$ in loc.cit..) Put

$$
(\mathcal{O}\widehat{\mathbb{C}}^{\dagger,?}, \widetilde{\Theta}) := \mathrm{colim}_{r \to 0^+}(\mathcal{O}\widehat{\mathbb{C}}_r^?, \widetilde{\Theta}).
$$

The one can show that the Higgs complex $DR(\mathcal{O}\mathbb{C}^{\dagger}, \Theta)$ is a resolution of \mathcal{O}_X (cf. [\[Wan23,](#page-41-7) Th. 2.28]). Similar to [\[MW24,](#page-41-8) Consruction 2.9], there exists a natural injection

$$
\iota_{\rm PS}: \mathcal{O}\widehat{\mathbb{C}}_{\rm pd}^+ \rightarrow \mathcal{O}\widehat{\mathbb{C}}_0^+
$$

identifying $(\zeta_p - 1)\Theta$ with Θ . As [[MW24,](#page-41-8) Prop. 2.10], the natural inclusion $\mathcal{O}\mathbb{C}^{\dagger} \hookrightarrow \mathcal{O}\mathbb{C}_0$ factors through the image of ι_{PS} , yielding the following inclusions of period sheaves with Higgs fields

$$
(\mathcal{O}\widehat{\mathbb{C}}^{\dagger}, \widetilde{\Theta}) \to (\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd}}, (\zeta_p - 1)\Theta) \to (\mathcal{O}\widehat{\mathbb{C}}_0, \widetilde{\Theta}).
$$

Remark 3.13. Recall for smooth \mathfrak{X} over \mathcal{O}_C , the log-structure on \mathfrak{X} is induced from the canonical log-structure on \mathcal{O}_C . By [\[Ols05,](#page-41-12) Lem. 8.22], all (log)-contangent complexes in [\(3.1\)](#page-12-0) reduces to the usual (p-complete) cotangent complexes as in [\[Wan23,](#page-41-7) Eq.(2-4)]. So all constructions in this section is compatible with those in [\[Wan23\]](#page-41-7) and [\[MW24\]](#page-41-8) (ans thus so are our main results in §[1\)](#page-0-0).

4. A local Simpson correspondence

Fix an affinoid perfectoid $S = \text{Spa}(A, A^+) \in \text{Perfd}$. In this section, we always assume R_S^+ $\frac{+}{S}$ is small semi-stable and keep the notations at the end of §[2.2.](#page-9-3) We first make some definitions.

Definition 4.1. Let $B \in \{R_S^+\}$ S^+ , R_S , $\widehat{R}^+_{\infty,S}, \widehat{R}_{\infty,S}$.

(1) By a Γ -representation of rank r over B, we mean a finite projective B-module M of rank r which is endowed with a continuous Γ-action.

A Γ-representation M^+ over R_S^+ S_S of rank r is called *Hitchin-small* if it admits an R_S^+ ζ^+ -basis e_1, \ldots, e_d such that for any $\delta \in \Gamma$, the matrix of δ with respect to the given basis is of the form $\exp(-(\zeta_p - 1)\rho_K A_\delta)$ for some (*p*-adically) topologically nilpotent matrix $A_\delta \in \text{Mat}_r(R^+_S)$ $\binom{+}{S}$.

By a Hitchin-small Γ-representation of rank r over B, we mean a finite free B-module M which is endowed with a (continuous) Γ -action which is of the form $M = M^+ \otimes_{R^+_S} B$ for some Hitchin-small Γ -representation M^+ over rank r over R_S^+ $^+_S.$

(2) By a Higgs module of rank r over R_S^+ $_S⁺$, we mean a pair</sub>

$$
(H^+,\theta: H^+ \to H^+ \otimes_{R_S^+} \Omega_{R_S^+}^{1,\log} \{-1\})
$$

consisting of a finite free R_S^+ S_S -module $H⁺$ of rank r and an Higgs field θ , i.e., an $R_S⁺$ $\frac{1}{S}$ -linear morphism θ satisfying $\theta \wedge \theta = 0$. For any Higgs module $(H^?, \theta)$, we denote by $DR(H^?, \theta)$ the induced Higgs complex.

A Higgs module (H^+,θ) over R_S^+ $_{S}^{+}$ is called

- (a) twisted Hitchin-small if the Higgs field θ is topologically nilpotent;
- (b) Hitchin-small if it is of the form

$$
(H^+,\theta)=(H^+,(\zeta_p-1)\theta')
$$

for some twisted Hitchin-small Higgs module (H^+,θ') . By a *(twisted)* Hitchin-small Higgs module of rank r over R_S , we mean a pair

$$
(H, \theta : H \to H \otimes_{R_S^+} \Omega_{R_S^+}^{1, \log} \{-1\})
$$

consisting of a finite free R_S -module H of rank r and an Higgs field θ , which is of the form $(H, \theta) = (H^+[\frac{1}{n}]$ $(\frac{1}{p}], \theta)$ for some (twisted) Hitchin-small Higgs module (H^+, θ) of rank r over R_S^+ $^+_S.$

(3) For any $? \in \{\emptyset, +\}$, we denote by $\text{Rep}_{\Gamma}^{\text{H-sm}}(B^?)$ the category of Hitchin-small Γ -representations over $B^?$, and by $HIG^{(t)}^{\text{H-sm}}(R_S^?)$ the category of (twisted) Hitchin-small Higgs modules over $R_S^?$.

Roughly, the purpose of this section is to establish the equivalences of categories

$$
\operatorname{Rep}_{\Gamma}^{\operatorname{H-sm}}(\widehat{R}_{\infty,S}^?)\simeq \operatorname{Rep}_{\Gamma}^{\operatorname{H-sm}}(R_S^?)\simeq \operatorname{HIG}^{\operatorname{t-H-sm}}(R_S^?)\simeq \operatorname{HIG}^{\operatorname{H-sm}}(R_S^?).
$$

Clearly, it suffices to deal with the case for $? = +$, which will be handled with in the next three subsections.

4.1. **Γ-representations over** R_S^+ **v.s. Γ-representations over** $\widehat{R}^+_{\infty,S}$. Note that the base-change $M^+ \mapsto M^+_{\infty} := M^+ \otimes_{R^+_S} \hat{R}^+_{\infty, S}$ induces a well-defined functor

$$
\operatorname{Rep}_{\Gamma}^{\operatorname{H-sm}}(R_S^+) \to \operatorname{Rep}_{\Gamma}^{\operatorname{H-sm}}(\widehat{R}_{\infty,S}^+).
$$

We now show this functor is exactly an equivalence of categories. More precisely, we shall prove the following result:

Proposition 4.2. The base-change along $R_S^+ \to \widehat{R}_{\infty,S}^+$ induces an equivalence of categories

$$
\text{Rep}_{\Gamma}^{H\text{-}sm}(R_S^+) \xrightarrow{\simeq} \text{Rep}_{\Gamma}^{H\text{-}sm}(\widehat{R}_{\infty,S}^+)
$$

such that for any $M^+ \in \text{Rep}_{\Gamma}^{H\text{-}sm}(R^+_S)$ ⁺/_S) with the induced $M^+_{\infty} \in \text{Rep}_{\Gamma}^{H\text{-}sm}(\widehat{R}^+_{\infty, S})$, the natural morphism $\mathrm{R}\Gamma(\Gamma,M^+) \to \mathrm{R}\Gamma(\Gamma,M_{\infty}^+)$

identifies the former with a direct summand of the latter whose complementary is concentrated in degree
$$
\geq 1
$$
 and killed by $\zeta_p - 1$.

We will prove Proposition [4.2](#page-19-0) later.

Recall [\[BMS18,](#page-40-11) Lem. 7.3] that for any p-complete \mathbb{Z}_p -module N equipped with a continuous Γ -action, $R\Gamma(\Gamma, N)$ can be calculated by the Koszul complex

$$
K(\gamma_1-1,\ldots,\gamma_d-1;N): N \xrightarrow{\gamma_1-1,\ldots,\gamma_d-1} N^d \to \cdots,
$$

where $\gamma_i \in \Gamma$ is defined in [\(2.10\)](#page-11-5). Recall we have the Γ-equivariant decomposition [\(2.12\)](#page-11-6)

$$
\widehat{R}_{\infty,S}^+ = \widehat{\bigoplus}_{\underline{\alpha} \in J_r} R_S^+ \cdot \underline{T}^{\underline{\alpha}} = R_S^+ \oplus \widehat{\bigoplus}_{\underline{\alpha} \in J_r \setminus \{0\}} R_S^+ \cdot \underline{T}^{\underline{\alpha}},
$$

where J_r is the set of indices defined in (2.11) .

Lemma 4.3. Let M^+ be a finite free Γ -representation over R_S^+ $_{S}^{+}$ of rank r such that it admits an R_{S}^{+} $\frac{+}{S}$ basis e_1, \ldots, e_r such that for any $\delta \in \Gamma$, its action on M^+ is given by the matrix $\exp(-(\zeta_p-1)A_\delta)$ for some topologically nilpotent $A_{\delta} \in Mat_r(R^+_S)$ ⁺_S). Then the R $\Gamma(\Gamma, M^+ \otimes \widehat{\bigoplus}_{\underline{\alpha} \in J_r \setminus \{0\}} R_S^+$ $(\frac{1}{S} \cdot \underline{T^{\alpha}})$ is concentrated in degree ≥ 1 and killed by $\zeta_p - 1$.

Proof. We claim that for any $\underline{\alpha} = (\alpha_0, \dots, \alpha_d) \in J_r \setminus \{0\}$, the RΓ(Γ, $M^+ \cdot \underline{T^{\alpha}}$) is concentrated in degree ≥ 1 and killed by $\zeta_p - 1$. Granting this, we can conclude by noting that

$$
\mathrm{R}\Gamma(\Gamma,M^+\otimes_{R_S^+}(\widehat{\bigoplus}_{\underline{\alpha}\in J_r\setminus\{0\}}R_S^+\cdot \underline{T}^{\underline{\alpha}}))=\bigoplus_{\underline{\alpha}\in J_r\setminus\{0\}}\mathrm{R}\Gamma(\Gamma,M^+\cdot \underline{T}^{\underline{\alpha}})
$$

because the right hand side above is killed by $\zeta_p - 1$ and thus already p-complete.

It remains to prove the claim. Without loss of generality, we may assume $\alpha_0 = 0$. Then $\alpha_1, \ldots, \alpha_d$ are not all zero. Without loss of generality, we may assume $\alpha_1 \neq 1$. By Hochschild–Serre spectral sequence, it suffices to show $\mathrm{R}\Gamma(\mathbb{Z}_p \cdot \gamma_1, M^+ \cdot \underline{T^{\alpha}})$ is concentrated in degree ≥ 1 and killed by $\zeta_p - 1$. We now check this by working with the Koszul complex

$$
K(\gamma_1 - 1; M^+ \cdot \underline{T^{\underline{\alpha}}}) : M^+ \cdot \underline{T^{\underline{\alpha}}} \xrightarrow{\gamma_1 - 1} M^+ \cdot \underline{T^{\underline{\alpha}}}.
$$

Put $\theta_i := A_{\gamma_i}$ for any $1 \leq i \leq d$ and

(4.1)
$$
F(\theta) := \frac{1 - \exp(-(\zeta_p - 1) \theta)}{(\zeta_p - 1) \theta} := 1 + \sum_{n \ge 1} (-1)^n (\zeta_p - 1)^{[n]} \theta^n \in \mathcal{O}_C[[\theta]].
$$

Then we have $\theta_i \in \text{Mat}_r(R^+_S)$ $S(S)$ which is topologically nilpotent and thus $F(\theta_i)$ is a well-defined matrix in $\mathrm{GL}_r(R_S^+)$ $S(S)$ such that γ_1 acts on M^+ via $1 - (\zeta_p - 1)\theta_i F(\theta_i)$. Thus, for any $m \in M^+$, we have

$$
(\gamma_1 - 1)(m\underline{T}^{\underline{\alpha}}) = (\zeta^{\alpha_1} (1 - (\zeta_p - 1)\theta_i F(\theta_i)) - 1)(m\underline{T}^{\underline{\alpha}}) = (\zeta^{\alpha_1} - 1) \left(1 - \zeta^{\alpha_1} \frac{\zeta_p - 1}{\zeta^{\alpha_1} - 1} \theta_1 F(\theta_1)\right) (m\underline{T}^{\underline{\alpha}}).
$$

As $\alpha_1 \neq 0$, we have $\frac{\zeta_p - 1}{\zeta^{\alpha_1} - 1} \in \mathcal{O}_C$ and thus $1 - \zeta^{\alpha_1} \frac{\zeta_p - 1}{\zeta^{\alpha_1} - 1}$ $\frac{\zeta_p-1}{\zeta^{\alpha_1}-1}\theta_1 F(\theta_1) \in \mathrm{GL}_r(R_S^+$ S_S) because θ_1 is topologically nilpotent, yielding that

$$
\mathrm{H}^n(\mathbb{Z}_p \cdot \gamma_1, M \cdot \underline{T}^{\alpha}) = \begin{cases} 0, & n = 0 \\ (M \cdot \underline{T}^{\alpha})/(\zeta_p - 1), & n = 1 \end{cases}
$$

as desired. This completes the proof.

Proof of Proposition [4.2](#page-19-0). As the essential surjectivity is a part of definition, it suffices to show that full faithfulness of the base-change functor. As the functor preserves tensor products and dualitis, it suffices to show the expected cohomological comparison. That is, we have to show that for any $M^+ \in \text{Rep}_{\Gamma}^{\text{H-sm}}(R_S^+)$ $\binom{+}{S}$, the natural map

$$
\mathrm{R}\Gamma(\Gamma,M^+) \to \mathrm{R}\Gamma(\Gamma,M^+\otimes_{R_S^+} \widehat{R}_{\infty,S}^+)
$$

identifies the former as a direct summand of the latter whose complementary is concentrated in degree ≥ 1 and killed by $\zeta_p - 1$. But this follows from Lemma [4.3](#page-20-0) immediately.

4.2. Hitchin-small Higgs modules v.s. Hitchin-small twisted Higgs modules. Clearly, for any ? $\in \{\emptyset, +\}$ the rule $(H^+, \theta) \mapsto (H^+, (\zeta_p - 1)\theta)$ defines a well-defined functor

$$
\text{HIG}^{\text{t-H-sm}}(R_S^? \to \text{HIG}^{\text{H-sm}}(R_S^?).
$$

One can prove this functor is indeed an equivalence of categories.

Proposition 4.4. The twist functor
$$
(H^+, \theta) \mapsto (H^+, (\zeta_p - 1)\theta)
$$
 induces an equivalence of categories

$$
\text{HIG}^{t-H\text{-sm}}(R_S^+) \to \text{HIG}^{H\text{-sm}}(R_S^+)
$$

such that for any $(H^+,\theta) \in \text{HIG}^{t-H\text{-}sm}(R^+_S)$ ⁺_S) with induced $(H^+,\theta') \in \text{HIG}^{H\text{-}sm}(R^+_S)$ $\binom{+}{S}$, there exists a quasi-isomorphism

$$
L\eta_{\zeta_p-1}DR(H^+,\theta') \simeq DR(H^+,\theta).
$$

where $L_{\eta_{\zeta_p-1}}$ denotes the décalage functor in [\[BMS18,](#page-40-11) §6].

Proof. Via the isomorphism [\(3.6\)](#page-14-1), one can write

$$
\theta = \sum_{i=1}^d \theta_i \otimes \frac{e_i}{\xi_K}
$$
 and $\theta'_i = \sum_{i=1}^d \theta'_i \otimes \frac{e_i}{\xi_K}$.

Using this, we have $\theta_i' = (\zeta_p - 1)\theta_i$ for all i. Note that $DR(H^+, \theta')$ and $DR(H^+, \theta)$ can be computed by the Koszul complexes

$$
K(\theta'_1, \ldots, \theta'_d; H^+)
$$
 and $K(\theta_1, \ldots, \theta_d; H^+),$

respectively. By [\[BMS18,](#page-40-11) Lem. 7.9], we have a quasi-isomorphism

$$
\eta_{\zeta_p-1}K(\theta'_1,\ldots,\theta'_d;H^+)\simeq K(\theta_1,\ldots,\theta_d;H^+),
$$

yielding the desired quasi-isomorphism

$$
L\eta_{\zeta_p-1}DR(H^+,\theta') \simeq DR(H^+,\theta).
$$

In particular, taking H^0 , we have

$$
H^0(\text{DR}(H^+,\theta)) \cong H^0(\text{L}\eta_{\zeta_p-1}\text{DR}(H^+,\theta')) \cong H^0(\text{DR}(H^+,\theta'))
$$

where the last isomorphism follows from [\[BMS18,](#page-40-11) Lem. 6.4] because $H^0(DR(H^+,\theta'))$ is a sub- R^+_S $\frac{+}{S}$ module of H^+ and thus $(\zeta_p - 1)$ -torsion free. As the twist functor preserves tensor product and dualities, the above cohomological comparison implies the full faithfulness of the twist functor. One can conclude as the essential surjectivity is a part of the definition. \Box

4.3. Local Simpson correspondence. Let $(P_{\infty,S}^+, \Theta = \sum_{i=0}^d$ ∂ $\frac{\partial}{\partial Y_i} \otimes \frac{e_i}{\xi_K}$ $\frac{e_i}{\xi_K}$) be as in Proposition [3.10.](#page-17-0) Let $P_S^+ = R_S^+$ $S[S(Y_0,\ldots,Y_d]_{\text{pd}}^{\wedge}/(Y_0+\cdots+Y_r)_{\text{pd}}]$ be the quotient of R_S^+ $S^+ [Y_0, \ldots, Y_d]_{\text{pd}}^{\wedge},$ the free pdpolynomial ring over R_S^+ $S_{\mathcal{S}}^+$ generated by Y_0, \ldots, Y_d , by the closed pd-ideal generated by $Y_0 + \cdots + Y_r$. Then P_S^+ S^+ is a sub- R_S^+ S_{S}^{+} -algebra of $P_{\infty,S}^{+}$ which is Θ -preserving and stable under the action of Γ such that via the decomposition (2.12) , there is a Γ-equivariant isomorphism

(4.2)
$$
P_{S,\infty}^+ = P_S^+ \widehat{\otimes}_{R_S^+} \widehat{R}_{\infty,S}^+ = \widehat{\bigoplus}_{\underline{\alpha} \in J_r} P_S \cdot \underline{T}^{\underline{\alpha}}.
$$

Now, we are going to prove the following local Simpson correspondence.

Theorem 4.5 (Local Simpson correspondence). Let $? \in \{\emptyset, \infty\}$

(1) For any $M_?^+ \in \text{Rep}_{\Gamma}^{H\text{-}sm}(\widehat{R}^+_{?,S})$ of rank r, define

$$
\Theta_{M_?^+} = \mathrm{id} \otimes \Theta : M_?^+ \otimes_{\widehat{R}^+_{?,S}} P^+_{?,S} \to M_?^+ \otimes_{\widehat{R}^+_{?,S}} P^+_{?,S} \otimes_{R^+_{S}} \Omega^{1,\log}_{R^+_{S}} \{-1\}.
$$

Then we have

$$
\mathrm{H}^n(\Gamma, M_?^+ \otimes_{\widehat{R}^+_{?,S}} P^+_{?,S}) = \left\{ \begin{array}{ll} H^+(M_?^+), & n = 0\\ (\zeta_p - 1)\rho_K\text{-}torsion, & n \ge 1 \end{array} \right.
$$

where $H^+(M_?^+)$ is a finite free R^+_S ^+_S -module of rank r such that the restriction of $\Theta_{M^+_7}$ to $H^+(M_?^+)$ induces a Higgs field θ making $(H^+(M_?^+), \theta) \in \text{HIG}^{t-H\text{-}sm}(R_S^+)$ $\binom{+}{S}$.

(2) For any $(H^+,\theta) \in \text{HIG}^{t-H\text{-}sm}(R^+_S)$ $\binom{+}{S}$ of rank r, define

$$
\Theta_{H^+} := \theta \otimes \mathrm{id} + \mathrm{id} \otimes \Theta : H \otimes_{R_S^+} P_{?,S}^+ \to H^+ \otimes_{\widehat{R}_{?,S}^+} P_{?,S}^+ \otimes_{R_S^+} \Omega_{R_S^+}^{1,\log} \{-1\}.
$$

Then the $M_?^+(H^+,\theta) := (H \otimes_{R^+_S} P^+_{?,S})^{\Theta_{H^+}=0}$ together with the induced Γ -action from $P^+_{?,S}$ is a well-defined object in $\text{Rep}_{\Gamma}^{H\text{-}sm}(\widehat{R}_{?,S}^{+})$ of rank r.

(3) The functors $M_?^+ \mapsto (H^+(M_?^+), \theta)$ and $(H^+, \theta) \mapsto M_?^+(H^+, \theta)$ above define an equivalence of categories

$$
\mathrm{Rep}^{H\text{-}sm}_{\Gamma}(\widehat{R}^+_{?,S})\simeq \mathrm{HIG}^{t\text{-}H\text{-}sm}(R^+_S)
$$

such that for any $M_?^+ \in \text{Rep}_{\Gamma}^{H\text{-}sm}(\widehat{R}^+_{?,S})$ with the induced (H^+,θ) , there exists an isomorphism of Higgs modules

(4.3)
$$
(M_?^+ \otimes_{\widehat{R}^+_{?,S}} P^+_{?,S}, \Theta_{M^+_{?}}) \simeq (H^+ \otimes_{\widehat{R}^+_{?,S}} P^+_{?,S}, \Theta_{H^+})
$$

and a quasi-isomorphism

(4.4)
$$
\mathrm{R}\Gamma(\Gamma,M_?^+[\frac{1}{p}])\simeq \mathrm{DR}(H^+[\frac{1}{p}],\theta).
$$

(4) The following diagram is commutative

(4.5) RepH-sm Γ (R + S) −⊗^R + S Rb⁺ ∞,S /RepH-sm Γ (Rb⁺ [∞],S) HIGt-H-sm(R + S). M⁺ ^g❖❖❖❖❖❖❖❖❖❖❖❖ ^M+[∞] ♥7 ♥♥ ♥♥ ♥♥ ♥♥ ♥♥ ♥

Before we establish the above local Simpson correspondence, let us do some preparations.

Construction 4.6. Let B be a p-complete p-torsion free \mathcal{O}_C -algebra and $G = \mathbb{Z}_p \cdot \gamma$ such that G acts on B trivially. Let $B[Y]_{\text{pd}}^{\wedge}$ be the free pd-polynomial ring over B generated by Y which is equipped with an action of G such that $\gamma(Y) = Y + (\zeta_p - 1)\rho_K$. For any $\alpha \in \mathbb{N}[\frac{1}{p}]$ $\frac{1}{p}$ \cap $[0, 1)$, let $B \cdot e_{\alpha}$ be the 1-dimensional representation of G over B with the basis e_α such that $\gamma(e_\alpha) = \zeta^\alpha e_\alpha$. For any finite free B-module V of rank r together with a fixed B-basis e_1, \ldots, e_r which is endowed with a G-action satisfying the condition:

(*) With respect to the given basis, the action of γ on V is given by the matrix $\exp(-(\zeta_p-1)\rho_K\theta)$

for some topologically nilpotent matrix $\theta \in Mat_r(B)$.

we define

$$
M_{\alpha}(V) := V \otimes_{B} B \cdot e_{\alpha} \otimes_{B} B[Y]_{\mathrm{pd}}^{\wedge}
$$

and equip it with the diagonal G-action.

We remark that the G-representation V satisfying the condition $(*)$ is exactly the *log-nilpotent* B-representation of G in the sense of $[MW24, Def. 3.2]$. The following lemma plays the key role in this section.

Lemma 4.7. Keep notations in Construction $\angle 4.6$.

(1) Suppose that $\alpha \neq 0$. We have

$$
H^{n}(G, M_{\alpha}(V)) = \begin{cases} 0, & n = 0\\ M_{\alpha}(V)/(\zeta^{\alpha} - 1), & n = 1. \end{cases}
$$

(2) Suppose that $\alpha = 0$. We have

$$
\mathrm{H}^n(G, M_0(V)) = \begin{cases} \exp(\theta Y)(V), & n = 0\\ M_\alpha(V)/\rho_K(\zeta^\alpha - 1), & n = 1, \end{cases}
$$

where

$$
\exp(\theta Y)(V) := \{ \sum_{i \ge 0} \theta^i(v) Y^{[i]} \in M_0(V) \mid v \in V \}.
$$

Moreover, the natural inclusion $M_0(V)^G \to M_0(V)$ induces a G-equivariant isomorphism

$$
M_0(V)^G \otimes_B B[Y]_{\text{pd}}^{\wedge} \xrightarrow{\cong} M_0(V).
$$

Proof. Comparing Construction [4.6](#page-22-0) with [\[MW24,](#page-41-8) Not. 3.1 and Def. 3.2], one can conclude by applying [\[MW24,](#page-41-8) Prop. 3.4].

Corollary 4.8. Keep notations in Construction [4.6.](#page-22-0) Let $G^d = \bigoplus_{i=1}^d \mathbb{Z}_p \cdot \gamma_i$ which acts on B trivially and $B[Y_1,\ldots,Y_d]_{\text{pd}}^{\wedge}$ be the p-complete free pd-polynomial ring over B generated by Y_1,\ldots,Y_d with the a G^d -action such that $\gamma_i(Y_j) = Y_j + \delta_{ij}(\zeta_p - 1)\rho_K$ for any $1 \leq i, j \leq d$. Let V be a finite free B-module of rank r which is equipped with a G-action such that for some B-basis e_1, \ldots, e_r of V and for any $1 \leq i \leq d$, the action of γ_i on V satisfies the condition $(*)$ in Construction [4.6](#page-22-0) for some topologically nilpotent matrix $\theta_i \in \text{Mat}_r(B)$. For any $\underline{\alpha} = (\alpha_1, \dots, \alpha_d) \in (\mathbb{N}[\frac{1}{p}$ $\frac{1}{p}]\cap [0,1))^{d}, let B \cdot e_{\underline{\alpha}}$ be the 1-dimensional representation of G^d over B such that $\gamma_i(e_{\underline{\alpha}}) = \zeta^{\alpha_i} e_{\underline{\alpha}}$ for any $1 \leq i \leq d$, and define

$$
M_{\underline{\alpha}}(V) := V \otimes_B B \cdot e_{\underline{\alpha}} \otimes_B B[Y_1, \dots, Y_d]_{\mathrm{pd}}^{\wedge}.
$$

Then the following assertions are true:

(1) Suppose that $\alpha \neq 0$. Then we have

$$
\mathrm{H}^n(G^d, M_{{\underline{\alpha}}}(V)) = \begin{cases} 0, & n = 0\\ (\zeta_p - 1) \text{-}torson, & n \ge 1. \end{cases}
$$

(2) Suppose that $\alpha = 0$. Then we have

$$
\mathrm{H}^n(G^d, M_0(V)) = \begin{cases} \exp(\sum_{i=1}^d \theta_i Y_i)(V), & n = 0\\ (\zeta_p - 1)\rho_K \text{-torson}, & n \ge 1, \end{cases}
$$

where

$$
\exp(\sum_{i=1}^d \theta_i Y_i)(V) := \{ \sum_{J \in \mathbb{N}^d} \underline{\theta}^J(v) \underline{Y}^{[J]} \mid v \in V \}.
$$

Moreover, the natural inclusion $M_0(V)^G \to M_0(V)$ induces a G-equivariant isomorphism

$$
M_0(V)^G \otimes_B B[Y_1,\ldots,Y_d]_{\text{pd}}^{\wedge} \xrightarrow{\cong} M_0(V).
$$

Proof. Note that as G^d is commutative, the θ_i 's commute with each others. In particular, the

$$
\exp(\sum_{i=1}^d \theta_i Y_i) := \sum_{J \in \mathbb{N}^d} \underline{\theta}^J \underline{Y}^{[J]}
$$

is a well-defined matrix in $GL_r(B[Y_1,\ldots,Y_d]^{\wedge}_{\text{pd}})$. So $\exp(\rho_K \sum_{i=1}^d \theta_i Y_i)(V)$ is also well-defined and is a finite free R_S^+ $⁺_S$ -module of rank r.</sup>

For Item (1): Without loss of generality, we may assume $\alpha_d \neq 0$. By Serre–Hochschild spectral sequence, it suffices to show $H^n(\mathbb{Z}_p \cdot \gamma_d, M_{\underline{\alpha}}(V))$ is killed by $\zeta_p - 1$ for $n = 1$ and vanishes for $n = 0$. But this follows from Lemma [4.7\(](#page-22-1)1) (by working with $B[Y_1, \dots, Y_{d-1}]_{\text{pd}}^{\wedge}$ instead of B there).

For Item (2): Using the same spectral sequence argument as in Item (1), one can deduce from Lemma [4.7\(](#page-22-1)2) that for any $n \geq 1$, the $H^n(G^d, M_0(V))$ is killed by $(\zeta_p - 1)\rho_K$. Using Lemma 4.7(2) again, by iteration, we have

$$
H^{0}(G^{d}, M_{0}(V)) = M_{0}(V)^{\gamma_{1}=\cdots=\gamma_{d}=1}
$$

= $(\exp(\theta_{d}Y_{d})(V \otimes_{B} B[Y_{1}, \ldots, Y_{d-1}]^{\wedge}_{\text{pd}}))^{\gamma_{1}=\cdots=\gamma_{d-1}=1}$
= \cdots
= $\exp(\sum_{i=1}^{d} \theta_{i}Y_{i})(V).$

Now, the final isomorphism

$$
M_0(V)^G \otimes_B B[Y_1,\ldots,Y_d]_{\text{pd}}^{\wedge} \xrightarrow{\cong} M_0(V)
$$

follows as $\exp(\sum_{i=1}^d \theta_i Y_i) \in \mathrm{GL}_r(B[Y_1,\ldots,Y_d]_p^{\wedge}$ \bigcap_{pd}).

Now, we are able to establish the local Simpson correspondence.

Proof of Theorem [4.5](#page-21-0). We first prove Items (1) , (2) and (3) for $? = \emptyset$.

For Item (1): Fix an $M^+ \in \text{Rep}_{\Gamma}^{\text{H-sm}}(R^+_S)$ $\binom{+}{s}$ of rank r such that via the isomorphism (2.10) , the γ_i -action on M^+ is given by $\exp(-(\zeta_p - 1) \rho_K \theta_i)$ for some topologically nilpotent $\theta_i \in \text{End}_{R_S^+}(M^+)$ for any $1 \leq i \leq d$. Consider the isomorphisms

(4.6)
$$
\Omega_{R_S^+}^{1,\log} \cong \bigoplus_{i=1}^d R_S^+ \cdot e_i \text{ and } (P_S^+, \Theta) \cong (R_S^+[Y_1,\ldots,Y_d]_{\text{pd}}^{\wedge}, \sum_{i=1}^d \frac{\partial}{\partial Y_i} \otimes \frac{e_i}{\xi_K}).
$$

It follows from Corollary [4.8](#page-22-2) that $\mathrm{H}^n(\Gamma,M^+\otimes_{R_S^+}P_S^+)$ $S(S)$ is killed by $\rho_K(\zeta_p-1)$ for $n \geq 1$ and gives rise to a finie free R_S^+ $_{S}^{+}$ -module

(4.7)
$$
H^+(M^+) = \exp(\sum_{i=1}^d \theta_i Y_i)(M^+).
$$

Then the restriction of Θ_{M^+} to $H^+(M^+)$ is given by

(4.8)
$$
\theta = \sum_{i=1}^{d} \theta_i \otimes \frac{e_i}{\xi_K} : H^+(M^+) \to H^+(M^+) \otimes_{R_S^+} \Omega_{R_S^+}^{1, \text{log}} \{-1\}.
$$

In particular, we have θ is topologically nilpotent as each θ_i does. This completes the proof for Item (1).

For Item (2): Again we use the isomorphisms [\(4.6\)](#page-23-0). Fix an $(H^+,\theta) \in \text{HIG}^{\text{t-H-sm}}(R^+_S)$ $\binom{+}{S}$ of rank r and write $\theta = \sum_{i=1}^{d} \theta_i \otimes \frac{e_i}{\xi_k}$ $\frac{e_i}{\xi_K}$ with $\theta_i \in \text{End}_{R_S^+}(H^+)$ topologically nilpotent for all *i*. Fix an

$$
x = \sum_{J \in \mathbb{N}^d} h_J \underline{Y}^J \in H^+ \otimes_{R_S^+} P_S^+.
$$

Then we have

$$
\Theta_{H^+}(x) = \sum_{i=1}^d \left(\sum_{J \in \mathbb{N}^d} (\theta_i(x_J) + x_{J+E_i}) \underline{Y}^J \right) \otimes \frac{e_i}{\xi_K}.
$$

In particular, $\Theta_{H^+}(x) = 0$ if and only if for any $1 \leq i \leq d$, we have $x_{J+E_i} = -\theta_i(x_j)$. By iteration, this amounts to that for any $J \in \mathbb{N}^d$,

$$
x_J = (-1)^{|J|} \underline{\theta}^J(x_0),
$$

yielding that

(4.9)
$$
M^+(H^+,\theta) = \exp(-\sum_{i=1}^d \theta_i Y_i)(H^+) := \{\sum_{J \in \mathbb{N}^d} (-1)^{|J|} \underline{\theta}^J(h) \underline{Y}^{[J]} \mid h \in H^+\}.
$$

As $\gamma_i(Y_j) = Y_j + (\zeta_p - 1)\rho_K$ via the isomorphism [\(2.10\)](#page-11-5) by Corollary [3.6,](#page-15-1) we see that the γ_i -action on $M^+(H^+,\theta)$ is given by

(4.10)
$$
\gamma_i = \exp(-\rho_K(\zeta_p - 1)\theta_i).
$$

This forces $M^+(H^+,\theta) \in \text{Rep}_{\Gamma}^{\text{H-sm}}(R^+_{S_{\Gamma}})$ $\frac{1}{S}$ as desired.

For Item (3): For any $M^+ \in \text{Rep}_{\Gamma}^{\text{H-sm}}(R^+_{S})$ $\frac{1}{S}$, it follows from the "moreover" part of Corollary [4.8](#page-22-2) that the natural morphism $H^+(M^+) \to M^{\tilde{+}} \otimes_{R^+_S} P^+_S$ S^+ induces a Γ -equivariant isomorphism

$$
H^+(M^+) \otimes_{R_S^+} P_S^+ \xrightarrow{\cong} M^+ \otimes_{R_S^+} P_S^+.
$$

It follows from the construction of θ on $H^+(M^+)$ that the above isomorphism is compatible with Higgs fields. By Poincaré's Lemma 3.9 , we have quasi-isomorphisms

$$
R\Gamma(\Gamma, M^+[\frac{1}{p}]) \xrightarrow{\simeq} R\Gamma(\Gamma, \text{DR}(M^+ \otimes_{R_S^+} P_S^+[\frac{1}{p}]), \Theta_{M^+})
$$

$$
\xleftarrow{\simeq} R\Gamma(\Gamma, \text{DR}(H^+ \otimes_{R_S^+} P_S^+[\frac{1}{p}]), \Theta_{H^+})
$$

$$
\xleftarrow{\simeq} \text{DR}(H^+[\frac{1}{p}], \theta),
$$

where the last quasi-isomorphism follows from $\mathrm{R}\Gamma(\Gamma,P_S^+[\frac{1}{p}$ $\left(\frac{1}{p}\right]$ = 0, by Item (1). Thus, one only need to show the functors in Items (1) and (2) are quasi-inverses of each other. We divide the proof into three steps:

Step 1: Fix an $M^+ \in \text{Rep}_{\Gamma}^{\text{H-sm}}(R_S^+)$ $(S⁺)$ and put $(H⁺, \theta) = (H⁺(M⁺), \theta)$. We denote by

$$
\iota_{M^+(H^+,\theta)}: M^+(H^+(M^+),\theta) = M^+(H^+,\theta) \to M^+
$$

the natural morphism induced by the composites

$$
M^+(H^+,\theta) = (H^+ \otimes_{R_S^+} P_S^+)^{\Theta_{H^+} = \theta \otimes id + id \otimes \Theta = 0}
$$

=
$$
\left((M^+ \otimes_{R_S^+} P_S^+)^{\Gamma} \otimes_{R_S^+} P_S^+ \right)^{\Theta_{M^+} \otimes id + (id \otimes id) \otimes \Theta = 0}
$$

$$
\hookrightarrow (M^+ \otimes_{R_S^+} P_S^+ \otimes_{R_S^+} P_S^+)^{id \otimes \Theta \otimes id + id \otimes id \otimes \Theta = 0}
$$

$$
\longrightarrow (M^+ \otimes_{R_S^+} P_S^+)^{id \otimes \Theta = 0}
$$

=
$$
M^+,
$$

where the last equality follows from the Poincaré's Lemma 3.9 while the last arrow is induced by the multiplication on P_S^+ S^+ (cf. Proposition [3.11\)](#page-18-1).

Step 2: Fix an $(H^+,\theta) \in \text{HIG}^{\text{t-H-sm}}(R^+_S)$ $S(S)$ and put $M^+ = M^+(H^+, \theta)$. We denote by

$$
\iota_{(H^+,\theta)}:H^+(M^+)\to H^+
$$

the natural morphism compatible with Higgs fields induced by the composites

$$
H^+(M^+) = (M^+ \otimes_{R_S^+} P_S^+)^{\Gamma}
$$

=
$$
((H^+ \otimes_{R_S^+} P_S^+)^{\Theta_{H^+}=0} \otimes_{R_S^+} P_S^+)^{\Gamma}
$$

$$
\hookrightarrow (H^+ \otimes_{R_S^+} P_S^+ \otimes_{R_S^+} P_S^+)^{\Gamma}
$$

$$
\rightarrow (H^+ \otimes_{R_S^+} P_S^+)^{\Gamma}
$$

=
$$
H^+,
$$

where the last equality follows as $P_S^{\Gamma} = R_S^+$ S_S (by Item (1)) while the last arrow is again induced by the multiplication on P_S^+ s^+ .

Step 3: It remains to show the morphism ι_{M^+} and $\iota_{(H^+,\theta)}$ are both isomorphisms. But this can be deduced from their constructions together with Equations [\(4.7\)](#page-23-1), [\(4.8\)](#page-24-0), [\(4.9\)](#page-24-1) and [\(4.10\)](#page-24-2) immediately. We have finished the proof in the ? = case. Now, we move to the case for ? = ∞ .

For Item (1): Fix an $M^+_{\infty} \in \text{Rep}_{\Gamma}^{\text{H-sm}}(\widehat{R}^+_{\infty,S})$. By Proposition [4.2,](#page-19-0) there exists an $M^+ \in \text{Rep}_{\Gamma}^{\text{H-sm}}(\widehat{R}^+_{S})$ such that $M^{\pm}_{\infty} \cong M^+ \otimes_{R^{\pm}_S} \widehat{R}^{\pm}_{\infty, S}$. Then the decomposition [\(4.2\)](#page-21-1) induces a Γ -equivariant decomposition

$$
M^+_{\infty} \otimes_{\widehat{R}^+_{\infty,S}} P^+_{\infty,S} = \widehat{\bigoplus}_{\underline{\alpha} \in J_r} M^+ \otimes_{R^+_S} P^+_S \cdot \underline{T}^{\underline{\alpha}}.
$$

Fix an $\underline{\alpha} = (\alpha_0, \ldots, \alpha_r, \alpha_{r+1}, \ldots, \alpha_d) \in J_r \setminus \{0\}$. We claim that $\mathrm{R}\Gamma(\Gamma, M^+ \otimes_{R_S^+} P_S^+$ $g_S^{\prime +} \cdot \underline{T^{\alpha}}$ vanishes for $n = 0$ and is killed by $\zeta_p - 1$ for any $n \ge 1$. Without loss of generality, we may assume $\alpha_0 = 0$ and then $\alpha_1, \ldots, \alpha_d$ are not all zero. Then the claim follows from the isomorphism (2.10) together with Corollary $4.8(1)$ $4.8(1)$.

Thanks to the claim, the Item (1) follows from the ? = \emptyset case and moreover we have the identity

$$
(H^+(M^+_{\infty}), \theta) = (H^+(M^+), \theta).
$$

The Items (2) and (3) follows from the same arguments in the proof for the ? = \emptyset case. In particular, for any $(H^+,\theta = \sum_{i=1}^d \theta_i \otimes \frac{e_i}{\xi_K})$ $\frac{e_i}{\xi_K}$) \in HIG^{t-H-sm} (R_S^+) $_{S}^{+}$), we have (4.11)

$$
M^+_{\infty}(H^+,\theta)=\exp(-\sum_{i=1}^d\theta_iY_i)(H^+\otimes_{R_S^+}\widehat{R}^+_{\infty,S})=\exp(-\sum_{i=1}^d\theta_iY_i)(H^+)\otimes_{R_S^+}\widehat{R}^+_{\infty,S}=M^+(H^+,\theta)\otimes_{R_S^+}\widehat{R}^+_{\infty,S}
$$

on which γ_i acts via the formulae [\(4.10\)](#page-24-2). This implies the Item (4), and then completes the proof. \Box

Finally, we obtain the following equivalence of categories.

Corollary 4.9. For any $? \in \{\emptyset, +\}$, we have the following equivalences of categories

$$
\text{Rep}_{\Gamma}^{H\text{-}sm}(\widehat{R}_{\infty,S}^?)\simeq \text{Rep}_{\Gamma}^{H\text{-}sm}(R_S^?)\simeq \text{HIG}^{t\text{-}H\text{-}sm}(R_S^?)\simeq \text{HIG}^{H\text{-}sm}(R_S^?).
$$

Moreover, for any $M_{\infty} \in \text{Rep}_{\Gamma}^{H\text{-}sm}(\widehat{R}_{\infty,S})$ with corresponding $M \in \text{Rep}_{\Gamma}^{H\text{-}sm}(\widehat{R}_S)$, $(H, \theta) \in \text{HIG}^{t-H\text{-}sm}(R_S)$ and $(H', \theta') \in \text{HIG}^{H\text{-}sm}(R_S)$, we have quasi-isomorphisms

$$
R\Gamma(\Gamma, M_{\infty}) \simeq R\Gamma(\Gamma, M) \simeq DR(H, \theta) \simeq DR(H', \theta').
$$

Proof. This follows from Proposition [4.2,](#page-19-0) Proposition [4.4](#page-20-1) and Theorem [4.5](#page-21-0) immediately.

It is worth showing the explicit form of the equivalence

$$
\operatorname{Rep}_{\Gamma}^{\operatorname{H-sm}}(\widehat{R}_{\infty,S}) \simeq \operatorname{HIG}^{\operatorname{H-sm}}(R_S).
$$

Let M_{∞} be a Hitchin-small Γ -representation of rank r over $\widehat{R}_{\infty,S}$ with associated $M \in \text{Rep}_{\Gamma}^{\text{H-sm}}(R_S);$ that is, we have $M_{\infty} = M \otimes_{R_S} \widehat{R}_{\infty,S}$. Fix an R_S -basis e_1, \ldots, e_d of M such that for any $1 \leq i \leq d$, the action of $\gamma_i \in \Gamma$ on M is given by $\exp(-(z_p-1)\rho_K\theta_i)$ for some topolohically nilpotent $\theta_i \in Mat_r(R_S^+)$ $\binom{+}{S}$ with respect to the given basis. Let (H, θ) be the Hitchin-small Higgs module associated to M_{∞} over R_S . Then we have

(4.12)
$$
(H = M, \theta = \sum_{i=1}^{d} (\zeta_p - 1)\theta_i \otimes \frac{\mathrm{dlog} T_i}{\zeta_K}),
$$

where we use the isomorphism $\Omega_{R_S}^1 = \Omega_{R_S^+}^{1, \log}[\frac{1}{p}]$ $\frac{1}{p}$ and identify $e_i \in \Omega_{R_{\n\zeta}}^{1,\log}$ ^{1,log} (cf. [\(3.6\)](#page-14-1)) with dlog $T_i \in \Omega_{R_S}^1$. Conversely, if we start with a Hitchin-small Higgs module

$$
(H, \theta = \sum_{i=1}^{d} (\zeta_p - 1)\theta_i \otimes \frac{\text{dlog} T_i}{\zeta_K}) \in \text{HIG}^{\text{H-sm}}(R_S)
$$

with $\theta_i \in \text{Mat}_r(R_S^+)$ $S(S)$ topologically nilpotent, then its associated Hitchin-small Γ -representation M over R_S is given by $M = H$ together with the Γ-action such that for any $1 \leq i \leq d$, the γ_i acts via $\exp(-(\zeta_p - 1) \rho_K \theta_i).$

4.4. An integral comparison theorem. Let $(H^+,\theta) \in \text{HIG}^{\text{t-H-sm}}(R^+_S)$ S_S be a twisted Hitchin-small Higgs module of rank r with associated Hitchin-small Γ -representation $M^+ \in \text{Rep}_{\Gamma}^{\text{H-sm}}(R^+_{\mathcal{S}})$ $_{S}^{+}$) (resp. $M^+_{\infty} \in \text{Rep}_{\Gamma}^{\text{H-sm}}(\widehat{R}^+_{\infty,S})$). As $\text{DR}(P^+_S)$ $(S^+$, $\Theta)$ (resp. $DR(P_{\infty,S}^+$, $\Theta)$) is a resolution of R_S^+ S^+ (resp. $\widehat{R}^+_{\infty,S}$), we have the left part of the following commutative diagram:

(4.13)
$$
R\Gamma(\Gamma, M^+) \xrightarrow{\simeq} R\Gamma(\Gamma, DR(M^+ \otimes_{R_S^+} P_S^+, \Theta_{M^+})) \xrightarrow{\simeq} DR(H^+, \theta)
$$

\n
$$
R\Gamma(\Gamma, M^+_{\infty}) \xrightarrow{\simeq} R\Gamma(\Gamma, DR(M^+_{\infty} \otimes_{\widehat{R}^+_{\infty, S}} P^+_{\infty, S}, \Theta_{M^+_{\infty}})) \xrightarrow{\simeq} DR(H^+, \theta)
$$

while the right part follows from the Theorem [4.5.](#page-21-0) On the other hand, combining Proposition [4.2](#page-19-0) with [\[BMS18,](#page-40-11) Lem. 6.4 and Lem. 6.10], we have the following commutative diagram

(4.14)
\n
$$
L\eta_{\rho_K(\zeta_p-1)}R\Gamma(\Gamma,M^+) \longrightarrow R\Gamma(\Gamma,M^+)
$$
\n
$$
\downarrow \simeq \qquad \qquad \downarrow
$$
\n
$$
L\eta_{\rho_K(\zeta_p-1)}R\Gamma(\Gamma,M^+_{\infty}) \longrightarrow R\Gamma(\Gamma,M^+_{\infty}).
$$

Then we have the following result:

Proposition 4.10. Keep notations as above. For any $? \in \{0, \infty\}$, the composite

 $\tau^{\leq 1} \text{DR}(H^+, \theta) \to \text{DR}(H^+, \theta) \xrightarrow{(4.13)} \text{R}\Gamma(\Gamma, M_?^+)$ $\tau^{\leq 1} \text{DR}(H^+, \theta) \to \text{DR}(H^+, \theta) \xrightarrow{(4.13)} \text{R}\Gamma(\Gamma, M_?^+)$ $\tau^{\leq 1} \text{DR}(H^+, \theta) \to \text{DR}(H^+, \theta) \xrightarrow{(4.13)} \text{R}\Gamma(\Gamma, M_?^+)$

uniquely factors through the composite

$$
\tau^{\leq 1} \mathcal{L} \eta_{\rho_K(\zeta_p - 1)} \mathcal{R} \Gamma(\Gamma, M_?^+) \to \mathcal{L} \eta_{\rho_K(\zeta_p - 1)} \mathcal{R} \Gamma(\Gamma, M_?^+) \xrightarrow{(4.14)} \mathcal{R} \Gamma(\Gamma, M^+)
$$

and induces a quasi-isomorphism

$$
\tau^{\leq 1} \mathcal{D}\mathcal{R}(H^+, \theta) \xrightarrow{\simeq} \tau^{\leq 1} \mathcal{L}\eta_{\rho_K(\zeta_p - 1)} \mathcal{R}\Gamma(\Gamma, M_?^+).
$$

Proof. By the commutativity of diagrams [\(4.13\)](#page-26-0) and [\(4.14\)](#page-26-1), it suffices to deal with the case for $? = \emptyset$. Then one can conclude by the same argument used in the proof of [\[MW24,](#page-41-8) Prop. 5.6]. But for the convenience of readers, we exhibit details here.

We claim the morphism $DR(H^+,\theta) \to R\Gamma(\Gamma,M^+)$ induces an isomorphism

(4.15)
$$
H_{dR}^1(H^+,\theta) := H^1(DR(H^+,\theta)) \cong \rho_K(\zeta_p - 1)H^1(\Gamma,M^+).
$$

Granting this, one can conclude the result as follows: By [\[BMS18,](#page-40-11) Lem. 8.16], the morphism $\tau^{\leq 1} \text{DR}(H^+, \theta) \to \tau^{\leq 1} \text{R}\Gamma(\Gamma, M^+) \to \text{R}\Gamma(\Gamma, M^+)$ uniquely factors as

$$
\tau^{\leq 1} \mathcal{D}\mathcal{R}(H^+, \theta) \to \tau^{\leq 1} \mathcal{L}\eta_{\rho_K(\zeta_p - 1)} \mathcal{R}\Gamma(\Gamma, M^+) \to \tau^{\leq 1} \mathcal{R}\Gamma(\Gamma, M^+) \to \mathcal{R}\Gamma(\Gamma, M^+).
$$

Here, we implicitly commute $\tau^{\leq 1}$ and $\text{L}\eta_{\rho_K(\zeta_p-1)}$ (by [\[BMS18,](#page-40-11) Cor. 6.5]). By the proof of [BMS18, Lem. 6.10], the H^{*i*} of the morphism $\tau^{\leq 1} \mathcal{L} \eta_{\rho_K(\zeta_{p}-1)} \mathcal{R} \Gamma(\Gamma, M^+) \to \tau^{\leq 1} \mathcal{R} \Gamma(\Gamma, M^+)$ is given by

$$
\mathrm{H}^i(\tau^{\leq 1}\mathrm{L}\eta_{\rho_K(\zeta_p-1)}\mathrm{R}\Gamma(\Gamma,M^+))=\mathrm{H}^i(\Gamma,M^+)/\mathrm{H}^i(\Gamma,M^+)[\rho_K(\zeta_p-1)]\xrightarrow{\times(\rho_K(\zeta_p-1))^i}\mathrm{H}^i(\Gamma,M^+)
$$

for $i \in \{0, 1\}$. In particular, this induces an isomorphism

$$
H^{i}(\tau^{\leq 1} \mathrm{L} \eta_{\rho_K(\zeta_p - 1)} \mathrm{R} \Gamma(\Gamma, M^+)) \cong (\rho_K(\zeta_p - 1))^{i} H^{i}(\Gamma, M^+)
$$

for $i \in \{0,1\}$. Then it follows from the claim above that the morphism

$$
\tau^{\leq 1} \mathcal{D} \mathcal{R}(H^+, \theta) \to \tau^{\leq 1} \mathcal{L} \eta_{\rho_K(\zeta_p - 1)} \mathcal{R} \Gamma(\Gamma, M_?^+)
$$

is a quasi-isomorphism as desired.

Now, we focus on the proof of the claim above. Put $P(M^+) := M^+ \otimes_{R^+_S} P^+_S$ $S_S⁺$ for short and consider the following commutative diagram: (4.16)

$$
H^{+} \longrightarrow H^{+} \otimes \Omega_{R_{S}^{+}}^{1, \log} \{-1\} \longrightarrow H^{+} \otimes \rho^{2} \Omega^{2} (-2) \longrightarrow \dots
$$
\n
$$
\downarrow \qquad \qquad \downarrow
$$
\n
$$
M^{+} \longrightarrow P(M^{+}) \longrightarrow P(M^{+}) \otimes \Omega_{R_{S}^{+}}^{1, \log} \{-1\} \longrightarrow P(M^{+}) \otimes \Omega_{R_{S}^{+}}^{2, \log} \{-2\} \longrightarrow \dots
$$
\n
$$
\downarrow \qquad \qquad \downarrow
$$
\n
$$
\wedge^{1} (M^{+})^{d} \longrightarrow \wedge^{1} P(M^{+})^{d} \longrightarrow \wedge^{1} P(M^{+})^{d} \otimes \Omega_{R_{S}^{+}}^{1, \log} \{-1\} \longrightarrow \wedge^{1} P(M^{+})^{d} \otimes \Omega_{R_{S}^{+}}^{2, \log} \{-2\} \longrightarrow \dots
$$
\n
$$
\downarrow \qquad \qquad \downarrow
$$
\n
$$
\wedge^{2} (M^{+})^{d} \longrightarrow \wedge^{2} P(M^{+})^{d} \longrightarrow \wedge^{2} P(M^{+})^{d} \otimes \Omega_{R_{S}^{+}}^{1, \log} \{-1\} \longrightarrow \wedge^{2} P(M^{+})^{d} \otimes \Omega_{R_{S}^{+}}^{2, \log} \{-2\} \longrightarrow \dots
$$
\n
$$
\downarrow \qquad \qquad \downarrow
$$
\n
$$
\vdots \qquad \vdots
$$

where the horizontal arrows are induced by Higgs fields and the vertical arrows are induced by Koszul complexes associated to Γ-actions. Then we have the following commutative diagram

(4.17)
\n
$$
H^+ \longrightarrow H^+ \otimes_{R_S^+} \Omega_{R_S^+}^{1,\log}\{-1\} \longrightarrow \dots
$$
\n
$$
P(M^+) \longrightarrow P(M^+) \otimes_{R_S^+} \Omega_{R_S^+}^{1,\log}\{-1\} \oplus \wedge^1 P(M^+)^d \longrightarrow \dots
$$
\n
$$
M^+ \longrightarrow \wedge^1 (M^+)^d \longrightarrow \dots
$$

such that the arrows from the bottom to the middle induce the quasi-isomorphism

 $R\Gamma(\Gamma, M^+) \simeq R\Gamma(\Gamma, \text{HIG}(P(M^+), \Theta_{M^+})).$

We have to deduce the relation between $H^1_{dR}(H^+,\theta)$ and $H^1(\Gamma,M^+)$ from the diagram [\(4.17\)](#page-27-0).

Let $x_1, \ldots, x_d \in H^+$ such that $\omega = \sum_{i=1}^d x_i \otimes \frac{e_i}{\xi_K}$ $\frac{e_i}{\xi_K}$ represents an element in $H^1_{dR}(H^+,\theta)$ via the isomorphism $\Omega_{R_S^+}^{1,\text{log}} \cong \bigoplus_{i=1}^d R_S^+$ $\frac{1}{S} \cdot e_i$, cf. [\(3.6\)](#page-14-1). Equivalently, we have that for any $1 \leq i, j \leq d$, $\theta_i(x_j) = \theta_j(x_i)$, where we write $\theta = \sum_{i=1}^d \theta_i \otimes \frac{e_i}{\xi_k}$ $\frac{e_i}{\xi_K}$. We want to determine the element in $\mathrm{H}^1(\Gamma,M^+)$ induced by ω . To do so, we have to solve the equation

(4.18)
$$
\Theta_{M^{+}}(\sum_{\underline{n}}h_{\underline{n}}\underline{Y}^{[\underline{n}]})=\omega,
$$

where $\sum_{n} h_{\underline{n}} \underline{Y}^{[\underline{n}]} =: m \in M$. Note that

$$
\Theta_{M^+}(\sum_{\underline{n}} h_{\underline{n}} \underline{Y}^{[\underline{n}]}) = \sum_{i=1}^d \left(\sum_{\underline{n}} \theta_i (h_{\underline{n}}) \underline{Y}^{[\underline{n}]} + \sum_{\underline{n}} h_{\underline{n}} \underline{Y}^{[\underline{n}-\underline{1}_i]} \right) \otimes \frac{e_i}{\xi_K}
$$

$$
= \sum_{i=1}^d \sum_{\underline{n}} (\theta_i (h_{\underline{n}}) + h_{\underline{n}+\underline{1}_i}) \underline{Y}^{[\underline{n}]} \otimes \frac{\text{dlog} T_i}{\xi_K}.
$$

So [\(4.18\)](#page-28-0) holds true if and only if for any $1 \le i \le d$ and $\underline{n} \in \mathbb{N}^d$ satisfying $|\underline{n}| \ge 1$,

(4.19) hn+1ⁱ = −θi(hn)

and

(4.20)
$$
h_{\underline{1}_i} = -\theta_i(h_0) + x_i.
$$

As $\theta_i(x_j) = \theta_j(x_i)$ for any $1 \leq i, j \leq d$, it is easy to see that for any $h \in H^+$, one can put $h_0 = h$ and use [\(4.19\)](#page-28-1) and [\(4.20\)](#page-28-2) to achieve an element $m(h) \in M$ satisfying $\Theta_{M^+}(m(h)) = \omega$. Moreover, if we put $m(\omega) := m(0)$, then

$$
m(h) = m(\omega) + \exp(-\sum_{k=1}^{d} \theta_k Y_k)h.
$$

As a consequence, the image of ω in $H^1(\Gamma, M^+) = H^1(K(\gamma_1 - 1, \ldots, \gamma_d - 1; M^+))$ is represented by

$$
v(\omega) := (\gamma_1(m(\omega)) - m(\omega), \dots, \gamma_d(m(\omega)) - m(\omega)) \in \wedge^1 (M^+)^d.
$$

On the other hand, as γ_1 acts on Y_2, \ldots, Y_d trivially (cf. Proposition [3.10\)](#page-17-0), we have $\gamma_1(m(\omega)) - m(\omega)$

$$
= \gamma_{1}\left(\sum_{n_{1}\geq 1, n_{2}, \dots, n_{d}\geq 0} (-\theta_{1})^{n_{1}-1}(-\theta_{2})^{n_{2}} \cdots (-\theta_{d})^{n_{d}} x_{1} Y_{1}^{[n_{1}]} \cdots Y_{d}^{[n_{d}]}\right) - \sum_{n_{1}\geq 1, n_{2}, \dots, n_{d}\geq 0} (-\theta_{1})^{n_{1}-1}(-\theta_{2})^{n_{2}} \cdots (-\theta_{d})^{n_{d}} x_{1} Y_{1}^{[n_{1}]} \cdots Y_{d}^{[n_{d}]} = \sum_{n_{1}\geq 1, n_{2}, \dots, n_{d}\geq 0} (-\theta_{1})^{n_{1}-1}(-\theta_{2})^{n_{2}} \cdots (-\theta_{d})^{n_{d}} x_{1} (Y_{1} + \rho_{K}(\zeta_{p}-1))^{[n_{1}]} Y_{2}^{[n_{2}]} \cdots Y_{d}^{[n_{d}]} - \sum_{n_{1}\geq 1, n_{2}, \dots, n_{d}\geq 0} (-\theta_{1})^{n_{1}-1}(-\theta_{2})^{n_{2}} \cdots (-\theta_{d})^{n_{d}} x_{1} Y_{1}^{[n_{1}]} \cdots Y_{d}^{[n_{d}]} = \sum_{n_{1}\geq 1, n_{2}, \dots, n_{d}\geq 0} (-\theta_{1})^{n_{1}-1}(-\theta_{2})^{n_{2}} \cdots (-\theta_{d})^{n_{d}} x_{1} Y_{1}^{[n_{1}]} \cdots Y_{d}^{[n_{d}]} - \sum_{n_{1}\geq 1, n_{2}, \dots, n_{d}\geq 0, 0\leq l \leq n_{1}} (-\theta_{1})^{n_{1}-1}(-\theta_{2})^{n_{2}} \cdots (-\theta_{d})^{n_{d}} x_{1} Y_{1}^{[n_{1}]} \cdots Y_{d}^{[n_{d}]} = \sum_{n_{1}\geq 1, n_{2}, \dots, n_{d}\geq 0, 0\leq l \leq n_{1}-1} (-\theta_{1})^{n_{1}-1}(-\theta_{2})^{n_{2}} \cdots (-\theta_{d})^{n_{d}} x_{1} Y_{1}^{[n_{1}]} \cdots Y_{d}^{[n_{d}]} = \sum_{n_{1}\geq
$$

where $F(\theta) = \frac{1-\exp(-(z_p-1)\theta)}{(\zeta_p-1)\theta}$ was defined in [\(4.1\)](#page-20-2). Similarly, for any $1 \leq i \leq d$, we have

$$
(\gamma_i - 1)(m(\omega)) = \rho(\zeta_p - 1)F(\theta_i) \exp(-\sum_{k=1}^d \theta_k Y_k)x_i.
$$

As a consequence, the image of ω in $\mathrm{H}^1(\Gamma, M^+)$ is represented by

(4.21)
$$
v(\omega) = (\zeta_p - 1)\rho_K(F(\rho_K \theta_1) \exp(-\sum_{k=1}^d \theta_k Y_k)x_1, \dots, F(\rho_K \theta_d) \exp(-\sum_{k=1}^d \theta_k Y_k)x_d).
$$

Since for any $1 \leq i, j \leq d$,

$$
(\gamma_j - 1)(F(\rho_K \theta_i) \exp(-\sum_{k=1}^d \theta_k Y_k) x_i) = F(\rho_K \theta_i) \exp(-\sum_{k=1}^d \theta_k Y_k) (\exp(-(\zeta_p - 1)\rho_K \theta_j) - 1) x_i
$$

$$
= (\zeta_p - 1)\rho_K F(\rho_K \theta_i) F(\rho_K \theta_j) \exp(-\sum_{k=1}^d \theta_k Y_k) \theta_j(x_i),
$$

and $\theta_j(x_i) = \theta_i(x_j)$, we deduce that

$$
(\gamma_j - 1)(F(\rho_K \theta_i) \exp(-\sum_{k=1}^d \Theta_k Y_k)x_i) = (\gamma_i - 1)(F(\rho_K \theta_i) \exp(-\sum_{k=1}^d \theta_k Y_k)x_j)
$$

and hence that

$$
v'(\omega) := (F(\rho_K \theta_1) \exp(-\sum_{k=1}^d \theta_k Y_k) x_1, \dots, F(\rho_K \theta_d) \exp(-\sum_{k=1}^d \theta_k Y_k) x_d) \in \wedge^1 (M^+)^d
$$

represents an element in $H^1(\Gamma, M^+)$.

Therefore, as a cohomological class, we have

$$
v(\omega) = \rho_K(\zeta_p - 1)v'(\omega) \in \rho_K(\zeta_p - 1)H^1(\Gamma, V_0).
$$

In other words, the map $DR(H, \theta_H) \to RT(\Gamma, M^+)$ carries $H^1_{dR}(H^+, \theta_H)$ into $\rho_K(\zeta_p - 1)H^1(\Gamma, M^+)$. We have to show it induces an isomorphism

$$
H_{\mathrm{dR}}^1(H^+,\theta_H) \cong \rho_K(\zeta_p - 1)H^1(\Gamma,M^+).
$$

The injectivity is obvious. Indeed, let T be the total complex of the double complex in (4.16) representing $R\Gamma(\Gamma, DR(P(M^+), \Theta_M))$. By the spectral sequence argument, we have

$$
H_{\text{dR}}^1(H^+,\Theta) = E_2^{1,0} = E_\infty^{1,0} \subset H^1(T) \cong H^1(\Gamma, M^+)
$$

and the desired injectivity follows.

It remains to prove the surjectivity. For this, let $y_1, \ldots, y_d \in H$ such that

$$
v' = (F(\rho_K \theta_1) \exp(-\sum_{j=1}^d \theta_j Y_j) y_1, \dots, F(\rho_K \theta_d) \exp(-\sum_{j=1}^d \theta_j Y_j) y_d) \in \wedge^1 (M^+)^d
$$

represents an element in $H^1(K(\gamma_1 - 1, \ldots, \gamma_d - 1; M^+)) \cong H^1(\Gamma, M^+)$. Equivalently, we have that for any $1 \leq i, j \leq d$,

$$
(\gamma_j-1)(F(\rho_K\theta_i)\exp(-\sum_{k=1}^d\theta_kY_k)y_i)=(\gamma_i-1)(F(\rho_K\theta_j)\exp(-\sum_{k=1}^d\theta_kY_k)y_j).
$$

By [\(4.22\)](#page-29-0), this amounts to that

$$
(\zeta_p - 1)\rho_K F(\rho_K \theta_i) F(\rho_K \theta_j) \exp(-\sum_{k=1}^d \theta_k Y_k) \theta_j(y_i) = \rho_K(\zeta_p - 1) F(\rho_K \theta_i) F(\rho_K \theta_j) \exp(-\sum_{k=1}^d \theta_k Y_k) \theta_i(y_j).
$$

By noting that $F(\rho_K \theta_i)$'s are invertible (as θ_i 's are topologically nilpotent and $F(\theta) \equiv 1 \mod \theta$), we conclude that v' represents an element in $H^1(\Gamma, M^+)$ if and only if $\theta_i(y_j) = \theta_j(y_i)$ for any $1 \leq i, j \leq d$. As a consequence, $\omega' = \sum_{i=1}^{d} y_i \otimes \frac{\text{dlog} T_i}{\xi_K}$ $\frac{\log T_i}{\xi_K}$ represents an element in $H^1_{\rm dR}(H,\theta)$. Now for any $v' \in H^1(\Gamma, M^+)$, by [\(4.21\)](#page-29-1), we know that $v(\omega') = \rho_K(\zeta_p - 1)v'$. That is, $\rho_K(\zeta_p - 1)v'$ is contained in the image of

$$
H_{dR}^1(H,\theta) \to \rho_K(\zeta_p - 1)H^1(\Gamma, M^+).
$$

As v' is arbitrary in $H^1(\Gamma, M^+)$, this proves the desired surjectivity and thus the desired claim. \Box

5. The p-adic Simpson correspondence on Hitchin-small objects

In this section, we aim to generalize the integral Simpson correspondence [\[MW24,](#page-41-8) Th. 1.1] and the geometric stacky Simpson correspondence [\[AHLB23b,](#page-40-10) Th. 1.1] to the case for semi-stable \mathfrak{X} . From now on, we always assume $\mathfrak X$ is a liftable semi-stable formal scheme over $\mathcal O_C$ of relative dimension d over \mathcal{O}_C with the generic fiber X in the sense of [\[CK19\]](#page-40-12) and fix an its lifting \mathfrak{X} over $\mathbf{A}_{2,K}$. For any $S = \text{Spa}(A, A^+) \in \text{Perfd}$, denote by \mathfrak{X}_S , X_S and $\widetilde{\mathfrak{X}}_S$ the base-changes of \mathfrak{X}, X and $\widetilde{\mathfrak{X}}$ to A^+, A and $\mathbb{A}_{2,K}(S)$, respectively. Then \mathfrak{X}_S is a liftable semi-stable formal scheme over A^+ with the generic fiber X_S and the induced lifting $\widetilde{\mathfrak{X}}_S$ from $\widetilde{\mathfrak{X}}$. When $\mathfrak{X} = Spf(R^+)$ with the lifting $\widetilde{\mathfrak{X}} = Spf(\widetilde{R}^+)$ and the generic fiber $X = \text{Spa}(R, R^+),$ we also denote by $\text{Spf}(R^+_S)$ (S^+S) , Spa (R_S, R_S^+) and \widetilde{R}_S^+ for the corresponding base-changes. For any $? \in \{\emptyset, +\}$, let $\mathcal{O} \mathbb{C}^?_{\text{pd},S}$ be the period sheaf with Higgs fields constructed in §[3.2](#page-15-3) corresponding to the lifting \mathfrak{X}_S . By construction, it is also functorial in S and so is the notation $P_{\infty,S}^?$ in Proposition [3.10.](#page-17-0)

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5.1. The integral p-adic Simpson correspondence. Fix an $S = \text{Spa}(A, A^+) \in \text{Perfd}$ and let \mathfrak{X}_S , \mathfrak{X}_S , X_S , and etc. be as before.

Definition 5.1. By a *Hitchin-small integral v-bundle of rank r* on $X_{S,v}$, we mean a sheaf of locally finite free $\widehat{\mathcal{O}}^+_{X_S}$ -modules \mathcal{M}^+ on $X_{S,v}$ such that there exists an étale covering $\{\mathfrak{X}_{i,S}\to \mathfrak{X}_S\}_{i\in I}$ by small affine $\mathfrak{X}_{i,S} = \text{Spf}(R_{i,S}^+)$ such that for any $i \in I$, the $\mathcal{M}^+(X_{i,\infty,S})$ is a Hitchin-small Γ -representation of rank r over $\widehat{R}^+_{i,\infty,S}$ in the sense of Definition [4.1.](#page-18-2) Denote by $\text{LS}^{\text{H-sm}}(\mathfrak{X}_S, \widehat{\mathcal{O}}^+_{X_S})$ the category of Hitchin-small integral v-bundles on $X_{S,v}$.

Definition 5.2. By an *Higgs bundle of rank r* on $\mathfrak{X}_{S,4}$, we mean a pair (\mathcal{H}^+, θ) consisting of a sheaf \mathcal{H}^+ of locally finite free $\mathcal{O}_{\mathfrak{X}_S}$ -modules of rank r on $X_{S,\text{\'et}}$ and a Higgs field θ on \mathcal{H}^+ , i.e., an $\mathcal{O}_{\mathfrak{X}_S}$ -linear homomorphism

$$
\theta: \mathcal{H}^+ \to \mathcal{H}^+ \otimes_{\mathcal{O}_{\mathfrak{X}_S}} \Omega^{1,\log}_{\mathfrak{X}_S}\{-1\}
$$

satisfying the condition $\theta \wedge \theta = 0$. For any Higgs bundle (\mathcal{H}^+, θ) , denote by $DR(\mathcal{H}^+, \theta)$ the induced Higgs complex. A Higgs bundle (\mathcal{H}^+, θ) is called

- (1) twisted Hitchin-small if θ is topologically nilpotent;
- (2) Hitchin-small if it is of the form

$$
(\mathcal{H}^+, \theta) = (\mathcal{H}^+, (\zeta_p - 1)\theta')
$$

for some twisted Hitchin-small integral Higgs bundles.

Denote by $HIG^{(t)} H\text{-sm}(\mathfrak{X}_S, \mathcal{O}_{\mathfrak{X}_S})$ the category of (twisted) integral Hitchin-small Higgs bundles on $\mathfrak{X}_{S, \text{\'et}}$.

Lemma 5.3. The twist functor $(\mathcal{H}^+, \theta) \mapsto (\mathcal{H}^+, (\zeta_p - 1)\theta)$ induces an equivalence of categories $\mathrm{HIG}^{t\text{-}H\text{-}sm}(\mathfrak{X}_S,\mathcal{O}_{\mathfrak{X}_S})\xrightarrow{\simeq}\mathrm{HIG}^{H\text{-}sm}(\mathfrak{X}_S,\mathcal{O}_{\mathfrak{X}_S})$

such that for any $(\mathcal{H}^+,\theta) \in \text{HIG}^{t-H\text{-}sm}(\mathfrak{X}_S,\mathcal{O}_{\mathfrak{X}_S})$ with the induced $(\mathcal{H}^+,\theta') \in \text{HIG}^{H\text{-}sm}(\mathfrak{X}_S,\mathcal{O}_{\mathfrak{X}_S})$, there exists a quasi-isomorphism

$$
L\eta_{\zeta_p-1}DR(\mathcal{H}^+,\theta')\simeq DR(\mathcal{H}^+,\theta).
$$

Proof. This indeed follows from the definition (of Lη). By [\[Sta24,](#page-41-14) [Tag 06YQ\]](https://stacks.math.columbia.edu/tag/06YQ), the DR(\mathcal{H}^+ , θ') is strongly K-flat in $D(\mathcal{O}_{\mathfrak{X}_S})$ in the sense of [\[BMS18,](#page-40-11) §6] (as it is a bounded complex of vector bundles). So $\text{L}_{\eta_{\zeta_p-1}}\text{DR}(\mathcal{H}^+,\theta')$ is represented by the complex $\eta_{\zeta_p-1}\text{DR}(\mathcal{H}^+,\theta')$ defined in [\[BMS18,](#page-40-11) Def. 6.2]. Denote its differentials by d for simplicity. As $\theta' = (\zeta_p - 1)\theta$, for any $n \ge 0$, we have

$$
(\eta_{\zeta_p-1} \mathrm{DR}(\mathcal{H}^+, \theta'))^n = \mathcal{H}^+ \otimes_{\mathcal{O}_{\mathfrak{X}_S}} \Omega_{\mathfrak{X}_S}^{n, \log} \{-n\} \otimes (\zeta_p - 1)^n.
$$

By definition of d (cf. the commutative diagram in [\[BMS18,](#page-40-11) Def. 6.2]), for any local section $x \in \mathcal{H}^+ \otimes_{\mathcal{O}_{\mathfrak{X}_S}} \Omega_{\mathfrak{X}_S}^{n,\log} \{-n\}$, we have

$$
d(x) \otimes (\zeta_p - 1)^{n+1} = \theta'(x) \otimes (\zeta_p - 1)^n = \theta(x) \otimes (\zeta_p - 1)^{n+1}.
$$

Via the isomorphism $(\eta_{\zeta_p-1}DR(\mathcal{H}^+,\theta'))^n \cong \mathcal{H}^+ \otimes_{\mathcal{O}_{\mathfrak{X}_S}} \Omega_{\mathfrak{X}_S}^{n,\log} \{-n\}$, we have an isomorphism of complexes

$$
(\eta_{\zeta_p-1} \mathrm{DR}(\mathcal{H}^+, \theta'), d) \simeq \mathrm{DR}(\mathcal{H}^+, \theta),
$$

yielding the desired quasi-isomorphism.

The main theorem in this section is the following generalisation of [\[MW24,](#page-41-8) Th. 1.1].

Theorem 5.4. Let $(\mathcal{O}\hat{\mathbb{C}}_{\mathrm{pd},S}^+,\Theta)$ be the period sheaf with Higgs field associated to $\widetilde{\mathfrak{X}}_S$. Let $\nu:X_{S,v}\to\mathbb{C}$ $\mathfrak{X}_{S,\text{\'et}}$ be the natural morphism of sites.

(1) For any $\mathcal{M}^+ \in \text{LS}^{H\text{-}sm}(\mathfrak{X}_S, \widehat{\mathcal{O}}^+_{X_S})$ of rank r, put

$$
\Theta_{\mathcal{M}^+} := \mathrm{id} \otimes \Theta : \mathcal{M}^+ \otimes \mathcal{O} \widehat{\mathbb{C}}_{\mathrm{pd},S}^+ \to \mathcal{M}^+ \otimes \mathcal{O} \widehat{\mathbb{C}}_{\mathrm{pd},S}^+ \otimes_{\mathcal{O}_{\mathfrak{X}_S}} \Omega_{\mathfrak{X}_S}^{1,\mathrm{log}} \{-1\}.
$$

Then the complex $\mathbb{R}\nu_{*}(\mathcal{M}^+\otimes \mathcal{O}\mathbb{C}^+_{\mathrm{pd},S})$ is concentrated in degree $[0,d]$ such that

$$
L\eta_{\rho_K(\zeta_p-1)}R\nu_*(\mathcal{M}^+\otimes \mathcal{O}\widehat{\mathbb{C}}^+_{\mathrm{pd},S})\simeq \left(\nu_*(\mathcal{M}^+\otimes \mathcal{O}\widehat{\mathbb{C}}^+_{\mathrm{pd},S})\right)[0].
$$

Moreover, the push-forward

$$
(\mathcal{H}^+(\mathcal{M}^+), \theta) := (\nu_*(\mathcal{M}^+\otimes \mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd},S}^+), \nu_*(\Theta_{\mathcal{M}^+}))
$$

defines a twisted Hitchin-small Higgs bundle of rank r on $\mathfrak{X}_{S,\mathrm{\acute{e}t}}$. (2) For any $(\mathcal{H}^+, \theta) \in \text{HIG}^{t-H\text{-}sm}(\mathfrak{X}_S, \mathcal{O}_{\mathfrak{X}_S})$ of rank r, put

$$
\Theta_{\mathcal{H}^+} = \theta \otimes id + id \otimes \Theta : \mathcal{H}^+ \otimes \mathcal{O} \widehat{\mathbb{C}}_{\mathrm{pd},S}^+ \to \mathcal{H}^+ \otimes \mathcal{O} \widehat{\mathbb{C}}_{\mathrm{pd},S}^+ \otimes_{\mathcal{O}_{\mathfrak{X}_S}} \Omega_{\mathfrak{X}_S}^{1,\mathrm{log}} \{-1\}.
$$

Then the

$$
\mathcal{M}^+(\mathcal{H}^+,\theta):=(\mathcal{H}^+\otimes\mathcal{O}\widehat{\mathbb{C}}^+_{\mathrm{pd},S})^{\Theta_{\mathcal{H}^+}=0}
$$

defines a Hitchin-small integral v-bundle of rank r on $X_{S,v}$. (3) The functors in Items (1) and (2) defines an equivalence of categories

$$
LS^{H\text{-}sm}(\mathfrak{X}_S,\widehat{\mathcal{O}}^+_{X_S}) \xrightarrow{\simeq} \text{HIG}^{t\text{-}H\text{-}sm}(\mathfrak{X}_S,\mathcal{O}_{\mathfrak{X}_S}).
$$

which preserves ranks, tensor products and dualities.

Combining this with Lemma [5.3,](#page-31-3) we obtain

Corollary 5.5. The composite

$$
LS^{H\text{-}sm}(\mathfrak{X}_S, \widehat{\mathcal{O}}^+_{X_S}) \to \text{HIG}^{t\text{-}H\text{-}sm}(\mathfrak{X}_S, \mathcal{O}_{\mathfrak{X}_S}) \xrightarrow{\times (\zeta_p-1)} \text{HIG}^{H\text{-}sm}(\mathfrak{X}_S, \mathcal{O}_{\mathfrak{X}_S})
$$

defines an equivalence of categories

$$
LS^{H\text{-}sm}(\mathfrak{X}_S,\widehat{\mathcal{O}}^+_{X_S}) \xrightarrow{\simeq} \text{HIG}^{H\text{-}sm}(\mathfrak{X}_S,\mathcal{O}_{\mathfrak{X}_S})
$$

preserving ranks, tensor products and dualities.

5.1.1. Proof of Theorem [5.4.](#page-31-0) We follow the strategy in [\[MW24,](#page-41-8) §4.2].

Lemma 5.6. Suppose $\mathfrak{X}_S = \text{Spf}(R_S^+)$ $S⁺$) is small affine with the generic fiber $X_S = \text{Spa}(R_S, R_S^+)$ Let \mathcal{M}^+ be a Hitchin-small integral v-bundle of rank r on $X_{S,v}$ such that $M^+_{\infty} := \mathcal{M}^+(X_{\infty,S})$ is a Hitchin-small Γ -representation of rank r over $\widehat{R}^+_{\infty,S}$. Then there exists a natural morphism

$$
\mathrm{R}\Gamma(\Gamma,(\mathcal{M}^+\otimes_{\widehat{\mathcal{O}}^+_{X_S}}\mathcal{O}\widehat{\mathbb{C}}^+_{\mathrm{pd},S})(X_{\infty,S}))\to \mathrm{R}\Gamma(X_{S,v},\mathcal{M}^+\otimes_{\widehat{\mathcal{O}}^+_{X_S}}\mathcal{O}\widehat{\mathbb{C}}^+_{\mathrm{pd},S})
$$

which is an almost quasi-isomorphism and is an isomorphism in degree 0.

Proof. It suffices to show that there exists some $c > 0$ such that there is an almost isomorphism of sheaves of $\widehat{\mathcal{O}}_{X_S}^+$ -modules

$$
\mathcal{M}^+/p^c \cong (\widehat{\mathcal{O}}_{X_S}^+/p^c)^r.
$$

Granting this, one can conclude by the same argument in the proof of [\[MW24,](#page-41-8) Lem. 4.15].

Denote by $X_{\infty,S}^{\bullet}$ the Čech nerve associated to the Γ -torsor $X_{\infty,S} \to X_S$. Then we have

$$
X_{\infty,S}^{\bullet} = \text{Spa}(\mathcal{C}(\Gamma^{\bullet}, \widehat{R}_{\infty,S}), \mathcal{C}(\Gamma^{\bullet}, \widehat{R}_{\infty,S}^{+}))
$$

where $C(\Gamma^{\bullet}, N)$ denotes the ring of continuous functions from Γ^{\bullet} into N for any \mathbb{Z}_p -module N equipped with a continuous Γ -action. The Hitchin-smallness of M^{\pm}_{∞} yields a Γ -equivariant isomorphism

$$
M_{\infty}^+/\rho_K(\zeta_p-1) \cong (\widehat{R}_{\infty,S}^+/\rho_K(\zeta_p-1))^r.
$$

Note that the continuous Γ -action M^+_{∞} induces a cosimplicial $C(\Gamma^{\bullet}, \hat{R}^+_{\infty, S})$ -module $C(\Gamma^{\bullet}, M^+_{\infty})$, yielding an isomorphism

$$
\mathcal{C}(\Gamma^{\bullet}, M_{\infty}^{+}/\rho_{K}(\zeta_{p}-1)) \cong \mathcal{C}(\Gamma^{\bullet}, \widehat{R}_{\infty, S}^{+}/\rho_{K}(\zeta_{p}-1))^{r}
$$

of cosimplicial $C(\Gamma^{\bullet}, \widehat{R}_{\infty, S}^+)$ -modules. As $X_{\infty, S} \to X_S$ is a Γ -torsor, by the proof of [\[Sch13,](#page-41-2) Lem. 4.10(i)], we have the desired almost isomorphism

$$
\mathcal{M}^+/p^c \cong (\widehat{\mathcal{O}}_{X_S}^+/p^c)^r
$$

for any $0 < c < \nu_p((\zeta_p - 1)\rho_K)$.

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Corollary 5.7. Keep assumption and notations in Lemma [5.6.](#page-32-0) Then the above natural morphism induces a quasi-isomorphism

$$
L\eta_{(\zeta_p-1)\rho_K}R\Gamma(\Gamma,(\mathcal{M}^+\otimes_{\widehat{\mathcal{O}}^+_{X_S}}\mathcal{O}\widehat{\mathbb{C}}^+_{\mathrm{pd},S})(X_{\infty,S}))\xrightarrow{\simeq} L\eta_{(\zeta_p-1)\rho_K}R\Gamma(X_{S,v},\mathcal{M}^+\otimes_{\widehat{\mathcal{O}}^+_{X_S}}\mathcal{O}\widehat{\mathbb{C}}^+_{\mathrm{pd},S}).
$$

Moreover, we have the following quasi-isomorphisms

$$
H^{0}(X_{v,S},(\mathcal{M}^{+}\otimes_{\widehat{\mathcal{O}}_{X_{S}}^{+}}\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd},S}^{+}))[0] \cong H^{0}(\Gamma,(\mathcal{M}^{+}\otimes_{\widehat{\mathcal{O}}_{X_{S}}^{+}}\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd},S}^{+})(X_{\infty,S}))[0]
$$

$$
\cong L\eta_{\rho_{K}(\zeta_{p}-1)}\mathrm{R}\Gamma(\Gamma,(\mathcal{M}^{+}\otimes_{\widehat{\mathcal{O}}_{X_{S}}^{+}}\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd},S}^{+})(X_{\infty}))
$$

$$
\cong L\eta_{\rho_{K}(\zeta_{p}-1)}\mathrm{R}\Gamma(X_{v,S},(\mathcal{M}^{+}\otimes_{\widehat{\mathcal{O}}_{X_{S}}^{+}}\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd},S}^{+})).
$$

Proof. For the first desired quasi-isomorphism, by [\[BMS18,](#page-40-11) Lem. 8.11(2)] and Lemma [5.6,](#page-32-0) it suffices to show for any $k \geq 0$, the

$$
H^k(\Gamma, (\mathcal{M}^+ \otimes \hat{\sigma}_{X_S}^+ \mathcal{O} \widehat{\mathbb{C}}_{\mathrm{pd},S}^+)(X_{\infty,S})) = H^k(\Gamma, M^+_{\infty} \otimes_{\widehat{R}^+_{\infty,S}} P^+_{\infty,S}).
$$

has no \mathfrak{m}_C -torsion. Let (H^+,θ) be the twisted Hitchin-small Higgs module over R_S^+ S_S associated to M^{\dagger}_{∞} via Theorem [5.4.](#page-31-0) Then we have a quasi-isomorphism

$$
\mathrm{R}\Gamma(\Gamma,M^+_{\infty}\otimes_{\widehat{R}^+_{\infty,S}}P^+_{\infty,S})\simeq \mathrm{R}\Gamma(\Gamma,H^+\otimes_{R^+_S}P^+_{\infty,S})\simeq H^+\otimes_{R^+_S}\mathrm{R}\Gamma(\Gamma,P^+_{\infty,S}),
$$

where the second follows as H^+ is finite free over R^+_S with trivial Γ -action. So we have an isomorphism

$$
H^k(\mathrm{R}\Gamma(\Gamma,M^+_{\infty}\otimes_{\widehat{R}^+_{\infty,S}}P^+_{\infty,S}))\cong H^+\otimes_{R^+_S}\mathrm{H}^k(\mathrm{R}\Gamma(\Gamma,P^+_{\infty,S}))
$$

and thus are reduced to showing that $H^k(\Gamma, P^+_{\infty, S})$ has no \mathfrak{m}_C -torsion. By decomposition [\(4.2\)](#page-21-1), it suffices to show for any $\underline{\alpha} \in J_r$, the $H^k(\Gamma, P_S^+ \cdot \underline{T^{\alpha}})$ has no \mathfrak{m}_C -torsion. For $\underline{\alpha} = 0$, the R $\Gamma(\Gamma, P_S^+)$ is computed by the Koszul complex

$$
\mathbf{K}(\gamma_1 - 1, \dots, \gamma_d - 1; P_S^+) \simeq \widehat{\bigotimes}_{1 \leq i \leq d}^{\mathbf{L}} \mathbf{K}(\gamma_i - 1; R_S^+[Y_i]_{\mathrm{pd}}^\wedge)
$$

where $\widehat{\otimes}^{\mathcal{L}}$ denotes the *p*-complete derived tensor product. By Künneth formulae, we are reduced to the case for $d = 1$; that is, we just need to show

$$
R_S^+[Y]^\wedge_{\mathrm{pd}} \xrightarrow{\gamma-1} R_S^+[Y]^\wedge_{\mathrm{pd}}
$$

has cohomologies with no \mathfrak{m}_C -torsion, where the Γ-action on $R_S[Y]^{\wedge}_{\mathrm{pd}}$ is given by Construction [4.6](#page-22-0) (for $B=R_S^+$ ⁺_S). It follows from Lemma [4.7\(](#page-22-1)2) that $H^{i}(\Gamma, R_S^+[Y]^{\wedge}_{\text{pd}})$ has no \mathfrak{m}_C -torsion for any $i \geq 0$. For $\alpha = (\alpha_0, \ldots, \alpha_d) \neq 0$, without loss of generality, we may assume $\alpha_0 = 0$ and then the RΓ(Γ, $P_S^+ \cdot \underline{T}^{\alpha}$) is computed by the Koszul complex

$$
K(\gamma_1 - 1, \ldots, \gamma_d - 1; P_S^+ \cdot \underline{T}^{\underline{\alpha}}) \simeq \widehat{\bigotimes}_{1 \leq i \leq d}^{L} K(\gamma_i - 1; R_S^+[Y_i]_{\text{pd}}^{\wedge} \cdot T_i^{\alpha_i}).
$$

Using Lemma [4.7](#page-22-1) (1) and (2) again, we may conclude the result from the same argument as above. For the "moreover" part, by [\[BMS18,](#page-40-11) Cor. 6.5], we have to show that for any $k \ge 1$, the

$$
\mathrm{H}^k(\Gamma, (\mathcal{M}^+ \otimes_{\widehat{\mathcal{O}}^+_{X_S}} \mathcal{O} \widehat{\mathbb{C}}^{+}_{\mathrm{pd}, S})(X_\infty))=\mathrm{H}^k(\Gamma, M_\infty \otimes_{\widehat{R}^+_{\infty, S}} P^+_{\infty, S})
$$

is killed by $\rho_K(\zeta_p - 1)$. But this follows from Theorem [4.5\(](#page-21-0)1) immediately.

Proof of Theorem [5.4](#page-31-0). We proceed as in the proof of of [\[MW24,](#page-41-8) Th. 4.4].

For Item (1): Let \mathcal{M}^+ be a Hitchin-small integral v-bundle on $X_{S,v}$ of rank r with the corresponding covering $\{\mathfrak{X}_{i,S} \to \mathfrak{X}_S\}_{i \in I}$. As argued in the proof of [\[MW24,](#page-41-8) Th. 4.4], the $\text{L}\eta_{\rho_K(\zeta_{p}-1)}\text{R}\nu_*(\mathcal{M}^+\otimes_{\widehat{\mathcal{O}}_{X_S}^+})$

 $\mathcal{O} \widehat{\mathbb{C}}_{\mathrm{pd},S}^+$ is the sheafification of the presheaf

$$
\mathfrak{U}_S \in \mathfrak{X}_{S, \text{\'et}} \mapsto \mathcal{L}\eta_{\rho_K(\zeta_p-1)}\mathrm{R}\Gamma(U_v, \mathcal{M}^+ \otimes_{\widehat{\mathcal{O}}_{X_S}^+} \mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd}, S}^+)
$$

while $\nu_*(\mathcal{M}^+ \otimes_{\widehat{\mathcal{O}}_{X_S}^+} \widehat{\mathcal{O}}_{\mathrm{pd},S}^{\mathbb{C}^+})$ is the sheafification of the presheaf

$$
\mathfrak{U}_S \in \mathfrak{X}_{S, \text{\'et}} \mapsto \mathrm{H}^0(U_v, \mathcal{M}^+ \otimes_{\widehat{\mathcal{O}}_{X_S}^+} \mathcal{O} \widehat{\mathbb{C}}_{\mathrm{pd}, S}^+).
$$

To conclude the desired quasi-isomorphism

$$
L\eta_{\rho_K(\zeta_p-1)}R\nu_*(\mathcal{M}^+\otimes_{\widehat{\mathcal{O}}_{X_S}^+}\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd},S}^+) \simeq \nu_*(\mathcal{M}^+\otimes_{\widehat{\mathcal{O}}_{X_S}^+}\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd},S}^+)[0],
$$

as this is local problem on $\mathfrak{X}_{S,\text{\'et}}$, we are reduced to showing that for any small semi-stable $\mathfrak{U}_S =$ $Spf(R_S^+$ $S(S) \in \mathfrak{X}_{S, \text{\'et}}$ lying over some $\mathfrak{X}_{i, S}$, the followings are true:

(i) There is a quasi-isomorphism

$$
\mathcal{L}\eta_{\rho_K(\zeta_p-1)}\mathrm{R}\Gamma(U_v,\mathcal{M}^+\otimes_{\widehat{\mathcal{O}}^+_{X_S}}\mathcal{O}\widehat{\mathbb{C}}^+_{\mathrm{pd},S})\simeq \mathrm{H}^0(U_v,\mathcal{M}^+\otimes_{\widehat{\mathcal{O}}^+_{X_S}}\mathcal{O}\widehat{\mathbb{C}}^+_{\mathrm{pd},S})[0].
$$

(ii) The $H^+ := H^0(U_v, \mathcal{M}^+ \otimes_{\widehat{\mathcal{O}}_{X_S}^+} \mathcal{O} \widehat{\mathbb{C}}_{\text{pd},S}^+)$ is a finite free R_S^+ $⁺_S$ -module of rank r such that the</sup> induced Higgs field θ from $\mathcal{V}_*(\Theta_{\mathcal{M}^+})$ on H^+ makes (H^+,θ) a Hitchin-small twisted Higgs module over R_S^+ $^+_S.$

As $\mathcal{M}^+(U_{\infty,S})$ is a Hitchin-small Γ -representation over $\widehat{R}^+_{\infty,S}$ of rank r, the Item (i) above follows from Corollary [5.7](#page-33-0) while the Item (ii) follows from Theorem [4.5.](#page-21-0)

For Item (2): Let (\mathcal{H}^+, θ) be a twisted Hitchin-small Higgs bundle on $X_{S, \text{\'{e}t}}$ of rank r. Then there exists an étale covering $\{\mathfrak{X}_{i,S} = \text{Spf}(R^+_{i,S}) \to \mathfrak{X}_S\}_{i \in I}$ by small semi-stable \mathfrak{X}_i 's such that the evaluation (H_i^+) (i, θ_i) of (\mathcal{H}^+, θ) at $\mathfrak{X}_{i,S}$ is a twisted Hitchin-small Higgs module over $R_{i,S}^+$ of rank r. Put $\mathcal{M}^+ := \mathcal{M}^+(\mathcal{H}^+, \theta)$ for short. By Theorem [4.5,](#page-21-0) we have $\mathcal{M}(X_{i,S,\infty})$ is a Hitchin-small Γ representation of rank r over $\widehat{R}^+_{i,\infty,S}$, yielding that \mathcal{M}^+ is a Hitchin-small integral v-bundle of rank r as desired.

For Item (3): Using Proposition [3.11,](#page-18-1) similar to the proof of Theorem [4.5\(](#page-21-0)3), for any $\mathcal{M}^+ \in$ $\text{LS}^{\text{H-sm}}(\mathfrak{X}_S, \widehat{\mathcal{O}}^+_{X_S})$ and any $(\mathcal{H}^+, \theta) \in \text{HIG}^{\text{t-H-sm}}(\mathfrak{X}_S, \mathcal{O}_{\mathfrak{X}_S})$, one can construct canonical morphisms

$$
\iota_{\mathcal{M}^+} : \mathcal{M}^+(\mathcal{H}^+(\mathcal{M}^+), \theta) \to \mathcal{M}^+
$$

and

 $\iota_{(\mathcal{H}^+,\theta)}: (\mathcal{H}^+(\mathcal{M}^+(\mathcal{H}^+,\theta)),\theta) \to (\mathcal{H}^+,\theta)$

of integral v-bundles and Higgs bundles respectively. To conclude, it suffices to show these two maps are both isomorphism. But this is again a local problem, and thus we are reduced to Theorem [4.5\(](#page-21-0)3) and the proof therein. By standard linear algebra, the equivalence preserves tensor products and dualities. This completes the proof.

5.1.2. Cohomological comparison. Let \mathcal{M}^+ be a Hitchin-small integral v-bundle with the associated twisted Hitchin-small Higgs bundle (\mathcal{H}^+, θ) in the sense of Theorem [5.4.](#page-31-0) It remains to compare the complexes

 $DR(\mathcal{H}^+,\theta)$ and $R\nu_*(\mathcal{M}^+).$

By Poincaré's Lemma [3.9,](#page-16-2) we have a quasi-isomorphism

$$
R\nu_{*}(\mathcal{M}^{+}) \xrightarrow{\simeq} R\nu_{*}(DR(\mathcal{M}^{+} \otimes_{\widehat{\mathcal{O}}_{X_{S}}^{+}} \mathcal{O}\widehat{\mathbb{C}}_{pd,S}^{+},\Theta_{\mathcal{M}^{+}})).
$$

By Theorem $5.4(1)$ $5.4(1)$, we have

$$
DR(\mathcal{H}^+, \theta) = \nu_* (DR(\mathcal{M}^+ \otimes_{\widehat{\mathcal{O}}_{X_S}^+} \mathcal{O} \widehat{\mathbb{C}}_{\mathrm{pd},S}^+, \Theta_{\mathcal{M}^+})).
$$

Thus, we get a canonical morphism

(5.1)
$$
DR(\mathcal{H}^+, \theta) \to R\nu_*(\mathcal{M}^+).
$$

Corollary 5.8. Recall $\mathfrak X$ has the relative dimension d over $\mathcal O_C$. The canonical map [\(5.1\)](#page-34-1) has cofiber killed by $(\rho_K(\zeta_p-1))^D$ where $D = \max(d+1, 2(d-1)).$

Proof. This follows from the same argument in the proof of $[MW24, Cor. 4.5]$.

On the other hand, as $\nu_*(\mathcal{M}^+) \simeq H^0(DR(\mathcal{H}^+, \theta)) \subset \mathcal{H}^+$ is p-torsionfree, by [\[BMS18,](#page-40-11) Lem. 6.10], there exists a canonical map

(5.2)
$$
L\eta_{\rho_K(\zeta_p-1)}R\nu_*\mathcal{M}^+\to R\nu_*\mathcal{M}^+.
$$

Similar to [\[MW24,](#page-41-8) Th. 5.4], we have the following cohomological comparison theorem.

Theorem 5.9. The natural morphism

$$
\tau^{\leq 1} \mathcal{D} \mathcal{R}(\mathcal{H}^+, \theta) \to \mathcal{D} \mathcal{R}(\mathcal{H}^+, \theta) \xrightarrow{(5.1)} \mathcal{R} \nu_* (\mathcal{M}^+)
$$

uniquely factors over the composite

$$
\tau^{\leq 1} \mathcal{L} \eta_{\rho_K(\zeta_p - 1)} \mathcal{R} \nu_* \mathcal{M}^+ \to \mathcal{L} \eta_{\rho_K(\zeta_p - 1)} \mathcal{R} \nu_* \mathcal{M}^+ \xrightarrow{(5.2)} \mathcal{R} \nu_* \mathcal{M}^+
$$

and induces a quasi-isomorphism

$$
\tau^{\leq 1} \mathcal{D} \mathcal{R}(\mathcal{H}^+, \theta) \xrightarrow{\simeq} \tau^{\leq 1} \mathcal{L} \eta_{\rho_K(\zeta_p - 1)} \mathcal{R} \nu_* \mathcal{M}^+.
$$

Proof. To show the map $\tau^{\leq 1} \text{DR}(\mathcal{H}^+, \theta) \to \text{R}_{\nu_*}(\mathcal{M}^+)$ uniquely factors through $\tau^{\leq 1} \text{L}_{\eta_{\rho_K}(\zeta_{p}-1)} \text{R}_{\nu_*} \mathcal{M}^+,$ by [\[BMS18,](#page-40-11) Lem. 8.16], we have to show the induced map

$$
H^1(DR(\mathcal{H}^+,\theta)) \to R^1 \nu_*(\mathcal{M}^+)
$$

factors through $\rho_K(\zeta_p - 1)R^1 \nu_*(\mathcal{M}^+)$. Since the problem is local on $\mathfrak{X}_{S,\text{\'et}}$, we may assume $\mathfrak{X}_S =$ $Spf(R_S^+$ $S(S)$ is small semi-stable and then apply Proposition [4.10.](#page-26-2) It remains to show this map

$$
\leq^1 \text{DR}(\mathcal{H}^+, \theta) \to \tau^{\leq 1} \text{L} \eta_{\rho_K(\zeta_p - 1)} \text{R} \nu_* \mathcal{M}^+
$$

is a quasi-isomorphism. But this is again a local problem on $\mathfrak{X}_{S,\text{\'et}}$ and thus we can conclude by using Proposition [4.10](#page-26-2) again.

By letting $\mathcal{M}^+ = \widehat{\mathcal{O}}_{X_S}^+$ in Theorem [5.9,](#page-35-0) we obtain a quasi-isomorphism

τ

(5.3)
$$
\gamma_1: \mathcal{O}_{\mathfrak{X}_S} \oplus \Omega_{\mathfrak{X}_S}^{1,\log} \{-1\}[-1] = \tau^{\leq 1} \mathrm{DR}(\mathcal{O}_{\mathfrak{X}_S}, 0) \xrightarrow{\simeq} \tau^{\leq 1} \mathrm{L} \eta_{\rho_K(\zeta_p - 1)} R \nu_* \widehat{\mathcal{O}}_{X_S}^+.
$$

Using this, we have the following analogue of Deligne–Illusie decomposition [\[DI87,](#page-40-13) Th. 2.1] in mixed characteristic case for semi-stable formal schemes.

Theorem 5.10. The quasi-isomorphism γ_1 above extends to a quasi-isomorphism

$$
\gamma: \bigoplus_{i=0}^{p-1} \Omega_{\mathfrak{X}_S}^{i,\log}\{-i\} = \tau^{\leq p-1} \mathrm{DR}(\mathcal{O}_{\mathfrak{X}_S}, 0) \xrightarrow{\simeq} \tau^{\leq 1} \mathrm{L} \eta_{\rho_K(\zeta_{p}-1)} \mathrm{R} \nu_* \widehat{\mathcal{O}}_{X_S}^+.
$$

Proof. This follows from a standard trick of Deligne–Illusie. See for example $[\text{Min21}, \text{Th. 4.1}]$. \square

Remark 5.11. Recall to obtain Theorem [5.10,](#page-35-1) we need \mathfrak{X}_S admits a lifting \mathfrak{X}_S over $\mathbb{A}_{2,K}(S)$ as a log-scheme (as this is used to construct the period sheaf $(\mathcal{O}^{\mathfrak{S}^+}_{\mathrm{pd},S},\Theta)$). It seems that such a decomposition γ in Theorem [5.10](#page-35-1) never exists if we only assume \mathfrak{X}_S lifts to $\mathbb{A}_{2,K}(S)$ as a scheme without lifting the log-structure at the same time. The phenomenon also appears in [\[SS20\]](#page-41-10). In loc.cit., for a semi-stable formal scheme \mathfrak{X}_0 over $W(\kappa)$ with the canonical log-structure $\mathcal{M}_{\mathfrak{X}_0}$ and the special fiber X over κ with the induced log-structure \mathcal{M}_{χ} . Let $F : \mathcal{X} \to \mathcal{X}^{(1)}$ be the relative Frobenius map associated to X . Then there exists a quasi-isomorphism

$$
\bigoplus_{i=0}^{p-1} \Omega^{i,\log}_{\mathcal{X}^{(1)}}[-i] \to \tau^{\leq p-1} \mathcal{F}_*(\text{DR}(\mathcal{O}_{\mathcal{X}},\mathrm{d}))
$$

if and only if $(\mathcal{X},\mathcal{M}_{\mathcal{X}})$ admits a lifting over $\mathfrak{S} := W(\kappa)[u]$ with the log-structure associated to $\mathbb{N} \longrightarrow \mathbb{S}$, which lifts the usual log-structure on $W(\kappa)$ induced by $(\mathbb{N} \longrightarrow^{\mathbb{N} \to \mathbb{N}} W(\kappa))$ via the surjection $\mathfrak{S} \xrightarrow{u \mapsto p} W(\kappa)$ (cf. [\[SS20,](#page-41-10) Th. 2.9]). Recall that the map $\mathfrak{G} \xrightarrow{u \mapsto [\varpi]} \mathbf{A}_{2,K}$ lifts the natural morphism $W(\kappa) \hookrightarrow \mathcal{O}_C$ which is compatible with log-structures as well. This also suggests that the the canonical log-structure on $\mathbb{A}_{2K}(S)$ should be the *only* reasonable one to obtain Theorem [5.10.](#page-35-1)

In general, we make the following conjecture.

Conjecture 5.12. Let \mathfrak{X}_S be a liftable semi-stable formal scheme over A^+ with a fixed lifting $\widetilde{\mathfrak{X}}_S$ over $\mathbb{A}_{2,K}(S)$. For any Hitchin-small integral v-bundle M on $X_{S,v}$ with the associated twisted Hitchinsmall Higgs bundle (\mathcal{H}^+, θ) in the sense of Theorem [5.4,](#page-31-0) if we denote by r the nilpotency length of $(\zeta_p - 1)\theta$ modulo p, then the canonical morphism

$$
DR(\mathcal{H}^+, \theta) \to R\nu_*\mathcal{M}^+
$$

induces a quasi-isomorphism

$$
\tau^{\leq p-r} \mathcal{D}\mathcal{R}(\mathcal{H}^+,\theta) \simeq \tau^{\leq p-r} \mathcal{L}\eta_{\rho_K(\zeta_p-1)} \mathcal{R}\nu_* \mathcal{M}^+.
$$

We remark that Theorem [5.10](#page-35-1) tells us Conjecture [5.12](#page-35-3) holds true when $r = 1$. Another special case of Conjecture [5.12](#page-35-3) we are able to prove is the following result:

Theorem 5.13. Let \mathfrak{X}_S be a liftable semi-stable curve over A^+ with a fixed lifting $\widetilde{\mathfrak{X}}_S$ over $\mathbb{A}_{2,K}(S)$. For any Hitchin-small integral v-bundle \cal{M} on $X_{S,v}$ with the associated twisted Hitchin-small Higgs bundle (\mathcal{H}^+, θ) in the sense of Theorem [5.4,](#page-31-0) the canonical morphism [\(5.1\)](#page-34-1) induces a quasi-isomorphism

$$
DR(\mathcal{H}^+, \theta) \simeq L \eta_{\rho_K(\zeta_p - 1)} R \nu_* \mathcal{M}^+.
$$

Proof. When \mathfrak{X}_S is a curve, both DR(\mathcal{H}^+ , θ) and $L\eta_{\rho_K(\zeta_p-1)}R\nu_*\mathcal{M}^+$ are concentrated in degree [0, 1].
So the result follows from Theorem 5.9 immediately. So the result follows from Theorem [5.9](#page-35-0) immediately.

5.2. The stacky Simpson correspondence. This part is devoted to establishing an equivalence between the stacks of Hitchin-small v-bundles on $X_{S,v}$ and Hitchin-small rational Higgs bundles on $\mathfrak{X}_{S,\text{\'et}}$, generalizing the previous work of Anschütz–Heuer–Le Bras [\[AHLB23b,](#page-40-10) Th. 1.1] to the semi-stable reduction case.

We first make the following definitions.

Definition 5.14. By a *Hitchin-small v-bundle of rank r* on $X_{S,v}$, we mean a sheaf of locally finite free $\widehat{\mathcal{O}}_{X_S}$ -modules M on $X_{S,v}$ such that there exists an étale covering $\{\mathfrak{X}_{i,S} \to \mathfrak{X}_S\}_{i \in I}$ by small affine $\mathfrak{X}_{i,S} = \text{Spf}(R_{i,S}^+)$ such that for any $i \in I$, the $\mathcal{M}(X_{i,\infty,S})$ is a Hitchin-small Γ -representation of rank r over $\widehat{R}_{i,\infty,S}$ in the sense of Definition [4.1.](#page-18-2) Denote by $\mathrm{LS^{H-sm}}(\mathfrak{X}_S, \widehat{\mathcal{O}}_{X_S})$ the category of Hitchin-small v-bundles on $X_{S,v}$.

Definition 5.15. By an rational Higgs bundle of rank r on $\mathfrak{X}_{S,\text{\'et}}$, we mean a pair (\mathcal{H}, θ) consisting of a sheaf $\mathcal H$ of locally finite free $\mathcal O_{\mathfrak X_S}[\frac{1}{p}]$ $\frac{1}{p}$ -modules of rank r on $\mathfrak{X}_{S,\text{\'et}}$ and a Higgs field θ on \mathcal{H} . For any rational Higgs bundle (\mathcal{H}, θ) , denote by $DR(\mathcal{H}, \theta)$ the induced Higgs complex. A rational Higgs bundle (H, θ) is called *(twisted) Hitchin-small* if there exists an étale covering ${\{\mathfrak{X}_{i,S} \to \mathfrak{X}_{S}\}_{i \in I}}$ by small affine $\mathfrak{X}_{i,S} = \mathrm{Spf}(R^+_{i,S})$ such that for any $i \in I$, the evaluation $(\mathcal{H}, \theta)(\mathfrak{X}_{i,S})$ is a (twisted) Hitchinsmall Higgs modules over $R_{i,S}$ in the sense of Definition [4.1\(](#page-18-2)2). Denote by $HIG^{(t-)H-sm}(\mathfrak{X}_S, \mathcal{O}_{\mathfrak{X}_S}[\frac{1}{p}])$ $\frac{1}{p}]$ the category of (twisted) Hitchin-small rational Higgs bundles on $\mathfrak{X}_{S,\text{\'et}}$.

Lemma 5.16. The functor $(\mathcal{H}, \theta) \mapsto (\mathcal{H}, (\zeta_p - 1)\theta)$ induces an equivalence of categories

$$
\text{HIG}^{H\text{-}sm}(\mathfrak{X}_S, \mathcal{O}_{\mathfrak{X}_S}[\frac{1}{p}]) \xrightarrow{\simeq} \text{HIG}^{t\text{-}H\text{-}sm}(\mathfrak{X}_S, \mathcal{O}_{\mathfrak{X}_S}[\frac{1}{p}])
$$

such that for any twisted Hitchin-small (H, θ) with the induced $(H, \theta' = (\zeta_p - 1)\theta)$, there exists a quasi-isomorphism

$$
DR(\mathcal{H}, \theta) \simeq DR(\mathcal{H}, \theta').
$$

Proof. By unwinding definitions, the result follows from Lemma [5.3](#page-31-3) immediately. \square

We give a remark on rational Higgs bundles on $\mathfrak{X}_{S,\text{\'et}}$. Let $i : X_{S,\text{\'et}} \to \mathfrak{X}_{S,\text{\'et}}$ be the natural morphism of sites. Clearly, for any locally finite free $\mathcal{O}_{X,S}$ -module $\mathcal E$ of rank r on $X_{\text{\'et},S}$, we have

$$
\mathrm{R}i_*\mathcal{E}=i_*\mathcal{E}
$$

which is a locally finite free $\mathcal{O}_{\mathfrak{X}_S}[\frac{1}{p}]$ $\frac{1}{p}$]-module of rank r and $i_*\mathcal{O}_{X_S} = \mathcal{O}_{frak X_S}[\frac{1}{p}]$ $\frac{1}{p}$. So the functors i_* and i^{-1} induces an equivalence between the category of locally finite free \mathcal{O}_{X_S} -module on $X_{S,\text{\'et}}$ and the category of locally finite free $\mathcal{O}_{\mathfrak{X}_S}$ -module on $\mathfrak{X}_{S,\'{e}t}$, yielding an equivalence of categories

(5.4)
$$
\mathrm{HIG}(X_S, \mathcal{O}_{X_S}) \simeq \mathrm{HIG}(\mathfrak{X}_S, \mathcal{O}_{\mathfrak{X}_S}[\frac{1}{p}]).
$$

Here, $\text{HIG}(X_S, \mathcal{O}_{X_S})$ denotes the category of Higgs bundles on $X_{S, \text{\'et}}$. Denote by

 $\text{\rm HIG}^{(\text{\rm t-})\text{\rm H-sm}}({\mathfrak X}_S, {\mathcal O}_{X_S}) \subset \text{\rm HIG}(X_S, {\mathcal O}_{X_S})$

the full sub-category corresponding to $\text{HIG}^{(\text{t-})\text{H-sm}}(\mathfrak{X}_S, \mathcal{O}_{\mathfrak{X}_S}[\frac{1}{p})$ $\frac{1}{p}$) via the above equivalence [\(5.4\)](#page-36-1); that is, we have an equivalence of categories induced by i_* and i^{-1} :

(5.5)
$$
\mathrm{HIG}^{(t\text{-})H\text{-sm}}(\mathfrak{X}_S, \mathcal{O}_{X_S}) \simeq \mathrm{HIG}^{(t\text{-})H\text{-sm}}(\mathfrak{X}_S, \mathcal{O}_{\mathfrak{X}_S}[\frac{1}{p}]).
$$

The following theorem is the analogue of Theorem [5.4](#page-31-0) on the rational level.

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Theorem 5.17. Let $(\hat{OC}_{\text{pd},S}, \Theta)$ be the period sheaf with Higgs field associated to $\widetilde{\mathfrak{X}}_S$. Let $\nu : X_{S,v} \to$ $\mathfrak{X}_{S,\text{\'et}}$ be the natural morphism of sites.

(1) For any $\mathcal{M} \in \mathcal{LS}^{H\text{-}sm}(\mathfrak{X}_S, \widehat{\mathcal{O}}_{X_S})$ of rank r, put

$$
\Theta_{\mathcal{M}} := id \otimes \Theta : \mathcal{M} \otimes \mathcal{O} \widehat{\mathbb{C}}_{\mathrm{pd},S} \to \mathcal{M} \otimes \mathcal{O} \widehat{\mathbb{C}}_{\mathrm{pd},S} \otimes_{\mathcal{O}_{\mathfrak{X}_S}} \Omega^{1,\log}_{\mathfrak{X}_S} \{-1\}.
$$

Then we have a quasi-isomorphism

$$
R\nu_{*}(\mathcal{M}\otimes\mathcal{O}\widehat{\mathbb{C}}^{+}_{\mathrm{pd},S})\simeq\nu_{*}(\mathcal{M}\otimes\mathcal{O}\widehat{\mathbb{C}}^{+}_{\mathrm{pd},S})[0]
$$

Moreover, the push-forward

$$
(\mathcal{H}(\mathcal{M}), \theta) := (\nu_*(\mathcal{M} \otimes \mathcal{O} \widehat{\mathbb{C}}_{\mathrm{pd},S}), \nu_*(\Theta_{\mathcal{M}}))
$$

defines a twisted Hitchin-small rational Higgs bundle of rank r on $\mathfrak{X}_{S, \text{\'et}}$.

(2) For any $(\mathcal{H}, \theta) \in \text{HIG}^{t-H\text{-}sm}(\mathfrak{X}_S, \mathcal{O}_{\mathfrak{X}_S}[\frac{1}{p})$ $\frac{1}{p}$) of rank r, put

$$
\Theta_{\mathcal{H}} = \theta \otimes id + id \otimes \Theta : \mathcal{H} \otimes \mathcal{O} \widehat{\mathbb{C}}_{\mathrm{pd},S} \to \mathcal{H} \otimes \mathcal{O} \widehat{\mathbb{C}}_{\mathrm{pd},S} \otimes_{\mathcal{O}_{\mathfrak{X}_S}} \Omega_{\mathfrak{X}_S}^{1,\mathrm{log}} \{-1\}.
$$

Then the

$$
\mathcal{M}(\mathcal{H}, \theta) := (\mathcal{H} \otimes \mathcal{O} \widehat{\mathbb{C}}_{\mathrm{pd}, S})^{\Theta_{\mathcal{H}} = 0}
$$

defines a Hitchin-small integral v-bundle of rank r on $X_{S,v}$.

(3) The functors in Items (1) and (2) defines an equivalence of categories

$$
\mathrm{LS}^{H\text{-}sm}({\mathfrak X}_S,\widehat{\mathcal O}_{X_S})\xrightarrow{\simeq}\mathrm{HIG}^{t\text{-}H\text{-}sm}({\mathfrak X}_S,{\mathcal O}_{{\mathfrak X}_S}[\frac{1}{p}]).
$$

which preserves ranks, tensor products and dualities. Moreover, for any $M \in \text{LS}^{H\text{-}sm}(\mathfrak{X}_S, \widehat{\mathcal{O}}_{X_S})$ with associated (\mathcal{H}, θ) , there exists a quasi-isomorphism

$$
R\nu_*\mathcal{M} \simeq DR(\mathcal{H}, \theta).
$$

Proof. For Item (1): Let M be a Hitchin-small v-bundle of rank r with the associated étale covering $\{\mathfrak{X}_{i,S} \to \mathfrak{X}_{S}\}_{i \in I}$ by small semi-stable \mathfrak{X}_{i} 's as in Definition [5.14.](#page-36-2) Then by definition [5.1,](#page-31-1) the restriction $\mathcal{M}_{|_{X_{i,S}}}$ is of the form $\mathcal{M}_{|_{X_{i,S}}} = \mathcal{M}_i^+[\frac{1}{p}]$ $\frac{1}{p}$ for some Hitchin-small integral v-bundle on $X_{i,S,v}$.

To see $\mathbb{R}\nu_*(\mathcal{M}\otimes\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd},S})\simeq \nu_*(\mathcal{M}\otimes\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd},S}),$ it is enough to show that for any $k\geq 1$, we have

$$
\mathrm{R}^k\nu_\ast(\mathcal{M}\otimes\mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd},S})=0.
$$

As $R^k \nu_*(\mathcal{M} \otimes \mathcal{O} \widehat{\mathbb{C}}_{\mathrm{pd},S})$ is the sheafification of the presheaf

$$
\mathfrak{U}\in\mathfrak{X}_{\mathrm{\acute{e}t}}\mapsto \mathrm{H}^{k}(U_v,(\mathcal{M}\otimes \mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd},S})_{|_U}),
$$

to see $\mathbb{R}^k \nu_*(\mathcal{M} \otimes \mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd},S}) = 0$, we may work locally on $\mathfrak{X}_{\mathrm{\acute{e}t}}$. So one can check this on each \mathfrak{X}_i . But this follows from Theorem [5.4](#page-31-0) (1) immediately.

Let $(\mathcal{H}_i^+, \theta_i)$ be the twisted Hitchin-small integral Higgs bundle on $\mathfrak{X}_{i,S,\text{\'et}}$ associated to \mathcal{M}_i^+ in the sense of Theorem [5.4.](#page-31-0) It is clearly that

$$
(\mathcal{H}(\mathcal{M}),\theta)_{|x_{i,S}}=(\mathcal{H}_i^+,\theta_i)[\frac{1}{p}].
$$

This implies $(\mathcal{H}(\mathcal{M}), \theta)$ is a twisted Hitchin-small rational Higgs bundle of rank r as desired.

For Item (2): It follows from a similar argument as above by using Theorem [5.4\(](#page-31-0)2) directly. For Item (3): One can conclude the functors in Items (1) and (2) are the quasi-inverses of each other by the same argument for the proof of Theorem [5.4\(](#page-31-0)3). So we get the desired equivalence of categories, which preserves ranks by construction. By standard linear algebra, this equivalence preserves tensor products and dualities. Finally, by Poincaré's Lemma, we have the quasi-isomorphisms

$$
R\nu_\ast(\mathcal{M}) \xrightarrow{\simeq} R\nu_\ast(DR(\mathcal{M} \otimes \mathcal{O}\widehat{\mathbb{C}}_{pd,S}, \Theta_\mathcal{M})).
$$

On the other hand, by Item (1), we have a quasi-isomorphism

$$
\mathrm{DR}(\mathcal{H}, \theta) \xrightarrow{\simeq} \mathrm{R}\nu_*(\mathrm{DR}(\mathcal{M} \otimes \mathcal{O}\widehat{\mathbb{C}}_{\mathrm{pd},S}, \Theta_{\mathcal{M}})),
$$

yielding the quasi-isomorphism

$$
R\nu_{*}(\mathcal{M})\simeq DR(\mathcal{H},\theta)
$$

as desired. This completes the proof.

Remark 5.18. Denote by $\pi : X_{S,v} \to X_{S,\text{\'et}}$ the natural morphism of sites, and then we have $\nu = i \circ \pi$. Via the equivalence (5.5) , it is easy to see that Theorem [5.17](#page-37-0) still holds true if one replaces ν and $\mathrm{HIG}^{\text{t-H-sm}}(\mathfrak{X}_S, \mathcal{O}_{\mathfrak{X}_S}[\frac{1}{p}$ $\frac{1}{p}$) by π and HIG^{t-H-sm}($\mathfrak{X}_S, \mathcal{O}_{X_S}$), respectively.

Corollary 5.19. The following composite

$$
\mathrm{LS}^{H\text{-}sm}(\mathfrak{X}_S,\widehat{\mathcal{O}}_{X_S})\xrightarrow{\mathrm{Th. 5.17}} \mathrm{HIG}^{t\text{-}H\text{-}sm}(\mathfrak{X}_S,\mathcal{O}_{\mathfrak{X}_S}[\frac{1}{p}])\xrightarrow{\mathrm{Lem. 5.16}} \mathrm{HIG}^{H\text{-}sm}(\mathfrak{X}_S,\mathcal{O}_{\mathfrak{X}_S}[\frac{1}{p}])
$$

defines an equivalence of categories

$$
\rho_{\widetilde{\mathfrak{X}}_S} : \mathrm{LS}^{H\text{-}sm}(\mathfrak{X}_S, \widehat{\mathcal{O}}_{X_S}) \xrightarrow{\simeq} \mathrm{HIG}^{H\text{-}sm}(\mathfrak{X}_S, \mathcal{O}_{\mathfrak{X}_S}[\frac{1}{p}]) \simeq \mathrm{HIG}^{H\text{-}sm}(\mathfrak{X}_S, \mathcal{O}_{X_S})
$$

which is functorial in S such that for any Hitchin-small v-bundle $\mathcal M$ on $X_{S,v}$ with associated Hitchinsmall (rational) Higgs bundle (H, θ) on $X_{S, \text{\'et}}$ (resp. $\mathfrak{X}_{S, \text{\'et}}$), there exists a quasi-isomorphism

$$
R\nu_*\mathcal{M} \simeq DR(\mathcal{H}, \theta).
$$

Proof. This follows from Lemma [5.16](#page-36-0) and Theorem [5.17](#page-37-0) directly.

For any r, we denote by $(R^1\pi_* GL_r)(X_S)$ and $((Mat_r \otimes \Omega^1_{X_S}(-1)) // GL_r)(X_S)$ the sheafifications of the presheaves

 $U_S \in X_{S,\text{\'et}} \mapsto \{\text{the set of isomorphic classes of } v\text{-bundles on } U_{S,v} \text{ of rank } r\}$

and

 $U_S \in X_{S,\text{\'et}} \mapsto \{\text{the set of isomorphic classes of Higgs bundles on } U_{S,\text{\'et}} \text{ of rank } r\},\$

respectively. By [\[Heu22,](#page-40-5) Th. 1.2], there exists an isomorphism

(5.6)
$$
\mathrm{HTlog} : (\mathrm{R}^1\pi_*\mathrm{GL}_r)(X_S) \xrightarrow{\cong} \big((\mathrm{Mat}_r \otimes \Omega^1_{X_S}(-1))/\!/ \mathrm{GL}_r \big)(X_S).
$$

See [\[Heu22,](#page-40-5) §5] for its construction and [Heu22, §4] for the construction of its inverse HTlog⁻¹ (denoted by Ψ in *loc.cit.*).

Lemma 5.20. The equivalence $\rho_{\widetilde{\mathfrak{X}}_S} : \text{LS}^{H\text{-}sm}(\mathfrak{X}_S, \widehat{\mathcal{O}}_{X_S}) \xrightarrow{\simeq} \text{HIG}^{H\text{-}sm}(\mathfrak{X}_S, \mathcal{O}_{X_S})$ is compatible with the isomorphism HTlog above in the sense that the following diagram commutes

$$
(5.7)
$$
\n
$$
LS_r^{H\text{-}sm}(\mathfrak{X}_S, \widehat{\mathcal{O}}_{X_S}) \xrightarrow{\rho_{\widetilde{\mathfrak{X}}_S}} \text{HIG}_r^{H\text{-}sm}(\mathfrak{X}_S, \mathcal{O}_{X_S})
$$
\n
$$
\downarrow \qquad \qquad \downarrow
$$
\n
$$
(R^1 \pi_* GL_r)(X_S) \xrightarrow{\text{HTlog}} ((\text{Mat}_r \otimes \Omega^1_{X_S}(-1)) // GL_r)(X_S)
$$

where $\mathrm{LS}_{r}^{H\text{-}sm}(\mathfrak{X}_S,\widehat{\mathcal{O}}_{X_S})$ and $\mathrm{HIG}_r^{H\text{-}sm}(\mathfrak{X}_S,\mathcal{O}_{X_S})$ denotes the corresponding full sub-categories of rankr objects.

Proof. We follow the strategy used in the proof of [\[AHLB23b,](#page-40-10) Cor. 3.12]. For any $M \in \text{LS}_r^{\text{H-sm}}(\mathfrak{X}_S, \widehat{\mathcal{O}}_{X_S})$ with associated $(\mathcal{H}, \theta) \in \text{HIG}_r^{\text{H-sm}}(\mathfrak{X}_S, \mathcal{O}_{X_S})$, we have to show that the (local) isomorphic class $c(\mathcal{M})$ (resp. $c(\mathcal{H}, \theta)$) associated to M (resp. (\mathcal{H}, θ)) satisfies the condition HTlog($c(\mathcal{M})$) = $c(\mathcal{H}, \theta)$. Clearly one can check this locally on $X_{S, \text{\'et}}$. So we may assume $\mathfrak{X}_S = \text{Spf}(R_S^+)$ S_S) is small semi-stable such that $M_{\infty} := \mathcal{M}(X_{\infty,S})$ is a Hitchin-small Γ-representation of rank r over $\widehat{R}_{\infty,S}$ with the associated Hicthin-small Higgs module (H, θ) over R_S . Let $M \in \text{Rep}_{\Gamma}^{\text{H-sm}}(R_S)$ such that $M_{\infty} \cong M \otimes_{R_S} \widehat{R}_{\infty, S}$. Let $\theta_i \in \text{Mat}_r(R_S^+)$ s⁺) be the topologically nilpotent matrix such that γ_i acts on M via $\exp(-(\zeta_p - 1) \rho_K \theta_i)$ with respect to a fixed R_S -basis e_1, \ldots, e_r of M. As argued in the paragraph around [\(4.12\)](#page-26-3), we have

(5.8)
$$
(H, \theta) \cong (M, \sum_{i=1}^{d} (\zeta_p - 1)\theta_i \otimes \frac{\mathrm{dlog} T_i}{\zeta_K}).
$$

Let $Spa(R_{1,S}, R_{1,S}^+) =: X_{1,S} \to X_S$ be the finite étale Galois covering associated to the Galois group Γ/Γ^p , where $\Gamma^p = \bigoplus_{i=1}^d \mathbb{Z}_p \cdot \gamma_i^p \subset \Gamma$ via the ismorphism [\(2.10\)](#page-11-5). As the problem is local on $X_{S, \text{\'et}}$ as well, we are reduced to showing that

$$
\mathrm{HTlog}(\mathrm{c}(\mathcal{M}_{|_{\mathrm{X}_{1,\mathrm{S}}}})) = \mathrm{c}((\mathcal{H}, \theta)_{|_{\mathrm{X}_{1,\mathrm{S}}}}).
$$

By [\(5.8\)](#page-38-0), the Higgs module over $R_{1,S}$ induced by $(\mathcal{H}, \theta)_{|_{X_{1,S}}}$ is

$$
(5.9) \quad (H \otimes_{R_S} R_{1,S}, \theta \otimes \mathrm{id}) \cong (M, \sum_{i=1}^d p(\zeta_p - 1)\theta_i \otimes \frac{\mathrm{dlog} T_i^{\frac{1}{p}}}{\zeta_K}) = (M, \sum_{i=1}^d p(\zeta_p - 1)\rho_K \theta_i \otimes \frac{\mathrm{dlog} T_i^{\frac{1}{p}}}{t})
$$

while the Γ^p -representation over $\widehat{R}_{\infty,S}$ from $\mathcal{M}_{|X_{1,S}}$ is the restriction of M_∞ to Γ^p . In [\(5.9\)](#page-39-0), we use the identification $\mathcal{O}_C\{-1\} = \rho_K \mathcal{O}_C(-1)$ (cf. §[1.4\)](#page-4-0). In particular, the γ_i^p i^p acts on M_{∞} (with respect to the choose basis e_1, \ldots, e_r) via the matrix

(5.10)
$$
\gamma_i^p = \exp(-p(\zeta_p - 1)\rho_K \theta_i), \ \forall \ 1 \leq i \leq d.
$$

Comparing [\(5.9\)](#page-39-0) with [\(5.10\)](#page-39-1), it then follows from the construction of HTlog (cf. [\[Heu22,](#page-40-5) Prop. 5.3 and its proof]) that $HTlog(c(\mathcal{M}_{|X_{1,S}})) = c((\mathcal{H}, \theta)_{|X_{1,S}})$. This completes the proof.

Now, we are going to give a geometric interpretion of Hitchin-smallness. For any $r \geq 0$, we consider two functors on Perfd:

(5.11)
$$
LS_r(X, \mathcal{O}_X) : S \in \text{Perfd} \mapsto \{\text{the groupoid of } v\text{-bundles of rank } r \text{ on } X_{S,v}\}
$$

and

 $(H, 12)$ HIG_r (X, \mathcal{O}_X) : $S \in \text{Perfd} \mapsto \{\text{the groupoid of Higgs bundles of rank } r \text{ on } X_{S, \text{\'et}}\}.$

By $[Heu22, Th. 1.4]$, these two functors are both small v-stacks (on Perfd). Consider the v-sheaf

(5.13)
$$
\mathcal{A}_r : S \in \text{Perfd} \mapsto \mathcal{A}_r(S) := \bigoplus_{i=1}^r H^0(X_S, \text{Sym}^i(\Omega^1_{X_S}\{-1\})),
$$

which is referred as *Hitchin-base* in *loc.cit.*. Then there are morphisms of v-sheaves, called *Hitchin* fibrations,

(5.14)
$$
h: \text{HIG}_r(X, \mathcal{O}_X) \to \mathcal{A}_r \text{ and } \widetilde{h}: \text{LS}_r(X, \widehat{\mathcal{O}}_X) \to \mathcal{A}_r
$$

where h is defined by sending each Higgs bundle (H, θ) to the characteristic polynomial of θ and h is defined by the composite of $h \circ \text{HTlog}$ (cf. [\[Heu22,](#page-40-5) §1.3]). In our setting, one can define an open subset $A_r^{\text{H-sm}} \subset A_r$ such that

(5.15)
$$
\mathcal{A}_r^{\mathrm{H\text{-}sm}}(S) = \bigoplus_{i=1}^r p^{<\frac{i}{p-1}} \mathrm{H}^0(\mathfrak{X}_S, \mathrm{Sym}^i(\Omega_{\mathfrak{X}_S}^{1,\log}\{-1\})),
$$

where $p^{\leq \frac{i}{p-1}}$ denotes the ideal $(\zeta_p-1)^i \mathfrak{m}_C \subset \mathcal{O}_C$. For any v-stack Z over \mathcal{A}_r , define its *Hitchin-small* locus by

(5.16)
$$
Z^{\text{H-sm}} := Z \times_{\mathcal{A}_r} \mathcal{A}_r^{\text{H-sm}}.
$$

In particular, we have $LS_r(X, \mathcal{O}_X)^{\text{H-sm}}$ and $\text{HIG}_r(X, \mathcal{O}_X)^{\text{H-sm}}$.

Proposition 5.21. Keep notations as above. Then we have

- (1) HIG^{H-sm} $(\mathfrak{X}_S, \mathcal{O}_{X_S}) = \cup_{r \geq 0} HIG_r(X, \mathcal{O}_X)^{H\text{-}sm}(S)$, and
	- (2) LS^{H-sm} $(\mathfrak{X}_S, \widehat{\mathcal{O}}_{X_S}) = \cup_{r \geq 0} \text{LS}_r(X, \widehat{\mathcal{O}}_X)^{H\text{-}sm}(S).$

More precisely, a Higgs bundle (H, θ) (resp. v-bundle M) of rank r on $X_{S, \text{\'et}}$ (resp. $X_{S,v}$) is Hitchinsmall if and only if as an S-point of $\text{HIG}_r(X, \mathcal{O}_X)$ (resp. $\text{LS}(X, \widehat{\mathcal{O}}_X)$),

$$
(\mathcal{H}, \theta) \in \text{HIG}_r(X, \mathcal{O}_X)^{H\text{-}sm}(S)
$$
 (resp. $\mathcal{M} \in \text{LS}(X, \widehat{\mathcal{O}}_X)^{H\text{-}sm}(S)$).

Proof. It suffcies to prove Item (1) while Item (2) follows by Lemma [5.20.](#page-38-1) For any Hitchin-small Higgs bundle $(\mathcal{H}, \theta) \in \text{HIG}^{\text{H-sm}}(\mathfrak{X}_S, \mathcal{O}_{X_S})$ of rank r, we regard it as a Hitchin-small rational Higgs bundle on $\mathfrak{X}_{S,\text{\'et}}$. It follows from Definition [5.15](#page-36-4) that as S-point of HIG_r(X, \mathcal{O}_X),

$$
(\mathcal{H}, \theta) \in \text{HIG}_r(X, \mathcal{O}_X)^{\text{H-sm}}(S).
$$

It remains to show for any Higgs bundle $(\mathcal{H}, \theta) \in \text{HIG}_r(X, \mathcal{O}_X)^{\text{H-sm}}(S)$ (again viewed as a rational Higgs bundle in $\text{HIG}(\mathfrak{X}_S, \mathcal{O}_{\mathfrak{X}_S}[\frac{1}{p}$ $(\frac{1}{p}]))$, étale locally on $\mathfrak{X}_{S,\text{\'et}}$, it is of the form $(\mathcal{H}^+, \theta)[\frac{1}{p}]$ for some Hitchinsmall integral Higgs bundle $(\hat{\mathcal{H}}^+, \theta)$. To do so, we may assume $\mathfrak{X}_S = Spf(R_S^+)$ $\binom{+}{S}$ is small semi-stable such that (\mathcal{H}, θ) is induced by a Higgs module $(H, \theta = \sum_{i=1}^{d} \theta_i \otimes \frac{e_i}{\xi_K})$ $\frac{e_i}{\xi_K}$) of rank r over R_S (such that $\theta_i \in \text{Mat}_r(R_S)$ with respect to a fixed R_S -basis of H). As $(\mathcal{H}, \theta) \in \text{HIG}_r(X, \mathcal{O}_X)^{\text{H-sm}}(S)$, for any $1 \leq$ $i \leq d$, the θ_i has eigenvalues in $(\zeta_p - 1) \mathfrak{m}_C R_S^+$ $\frac{1}{S}$. As θ_i 's commute with each others, by standard linear algebra, there exists a matrix $X \in GL_r(\overline{\text{Frac}(R_S)})$ such that $\theta'_i := X\theta_iX^{-1} \in (\zeta_p - 1)\mathfrak{m}_C\text{Mat}_r(R_S^+)$ $\binom{+}{S}$ for all i, where $Frac(R_S)$ denotes the algebraic closure of the fractional field of R_S . Thus X induces an isomorphism

$$
(H,\theta) \cong (H',\theta' = \sum_{i=1}^d \theta_i' \otimes \frac{e_i}{\xi_K})
$$

for some Hitchin-small Higgs module (H', θ') over R_S .

Now, we are able to give the following equivalence of stacks, generalizing the previous work of [\[AHLB23b,](#page-40-10) Th. 1.1].

Theorem 5.22. Let $\mathfrak X$ be liftable a semi-stable formal scheme over $\mathcal O_C$ with a fixed lifting $\widetilde{\mathfrak X}$ over $A_{2,K}$. Then for any $r > 0$, there exists an equivalence of stacks

$$
\rho_{\widetilde{\mathfrak{X}}} : \operatorname{LS}_r(X, \widehat{\mathcal{O}}_X)^{H\text{-}sm} \xrightarrow{\simeq} \operatorname{HIG}_r(X, \mathcal{O}_X)^{H\text{-}sm}.
$$

Proof. For our purpose, by Proposition [5.21,](#page-39-2) it is enough to assign to each $S \in$ Perfd a rankpreserving equivalence of categories

$$
\rho_{\widetilde{\mathfrak{X}}_S} : \text{LS}^{\text{H-sm}}(X_S, \widehat{\mathcal{O}}_{X_S}) \simeq \text{HIG}^{\text{H-sm}}(X_S, \mathcal{O}_{X_S})
$$

which is functorial in S. But this follows from Corollary [5.19](#page-38-2) directly. \square

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