

# The Unified Transform Method: beyond circular or convex domains

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## Abstract

A new transform approach that can be used to solve mixed boundary value problems for Laplace's equation in non-convex and other planar domains, is presented. This work is an extension of Crowdy (2015, CMFT, 15, 655–687), where new transform-based techniques were developed for boundary value problems for Laplace's equation in circular domains. The key ingredient of the present method is the exploitation of the properties of the Szegő kernel and its connection with the Cauchy kernel to obtain transform pairs for analytic functions in such domains. Several examples are solved in detail and numerically implemented to illustrate the application of the new transform pairs.

# 1 Introduction

The Unified Transform Method (UTM), introduced by A.S. Fokas in the late 1990s [9], is a technique for analyzing boundary value problems for linear and integrable nonlinear PDEs. Since its inception, the UTM has garnered significant interest within the applied mathematics community. Over time, numerous adaptations of the original method have been developed to address specific classes of equations. For equations such as the Laplace, biharmonic, Helmholtz, and modified Helmholtz equations in convex polygonal domains, the UTM offers integral representations of solutions in the complex Fourier plane [10, 4, 5, 7, 20, 6].

In particular, for Laplace's equation, Fokas & Kapaev [10] formulated a transform method to solve boundary value problems in simply connected polygonal domains. Their approach initially utilized various techniques, including spectral analysis of parameter-dependent ODEs and Riemann–Hilbert methods. Later, Crowdy [2, 3] demonstrated that this method could be reformulated using a complex function-theoretic framework, leading to the development of a new transform method tailored for circular domains (domains bounded by circular arcs, with line segments as a special case). The authors of this paper have recently presented an extension of the original approach of Fokas & Kapaev [10] for convex polygons to arbitrary convex domains [13] constructing so-called *quasi-pairs*.

The focus of the present study is the development of a transform method approach for Laplace's equation in non-convex planar domains, as well as in domains beyond circular. The emphasis is placed on the fact that this new transform approach provides means for solving mixed boundary value problems in non-convex domains, such as in the interior of a Cassini oval. Our method is built upon Crowdy's [2] construction for circular domains and develops a new transform method technique for solving mixed boundary value problems in such domains. We obtain tailor-made transform pairs to represent the value of a given function at any point in the domain of interest by exploiting the properties of the Szegő kernel and its connection with the Cauchy kernel.

The Szegő and Cauchy kernels are important in complex analysis and operator theory. Among their many features, they can be used to produce and reproduce holomorphic or analytic functions in a given simply-connected domain  $D$  from the functions' known boundary values (the classical Cauchy formula is one such case). Both kernels are associated to projections from the Hilbert space of square integrable functions  $L^2(\partial D)$  to the Hardy space  $H^2(\partial D)$  [13]. However while the Cauchy kernel is completely explicit and canonical (it does not change with  $D$ ), the Szegő kernel is in general not known explicitly. On the other hand the Cauchy kernel does not have a good transformation law under conformal maps,  $\Phi : D \rightarrow \mathbb{D}$ , (where  $\mathbb{D}$  is the unit disc), but the Szegő kernel does. Moreover, the Szegő and Cauchy kernels for the unit disc are identical. This means that any time one has enough knowledge of the conformal map for a given domain  $D$ , one can use it to obtain an accurate numerical approximation of the Szegő kernel. This idea makes it possible to extend the strategy of UTM devised by Crowdy [2] for circular domains, to any domain  $D$  for which one has a satisfactory understanding of its conformal map to the unit disc. In this paper, we develop this idea for a general simply-connected domain  $D$ , and then we carry it out explicitly for specific choices of  $D$ , including the case when  $D$  is an ellipse, for which a different approach was developed in our previous paper [13]; the latter serves as a comparison method, as well as when  $D$  is a Cassini oval.

In §2, we present the theoretical background needed to formulate the new transform pairs for analytic functions in non-convex planar domains. The new transform pairs are constructed in §3 for simply-connected domains and in §4 for punctured domains. The next step involves implementing the method for a variety of mixed boundary value problems §5–6. Finally, we conclude and discuss further applications in §7.

## 2 Background

Let  $D \subset \mathbb{C}$  denote a bounded domain in  $\mathbb{C}$  whose boundary is denoted by  $\partial D$  and closure by  $\overline{D}$ , that is  $\overline{D} = D \cup \partial D$ . In what follows, interior points are denoted by  $z \in D$  and boundary points by  $\zeta \in \partial D$ . We assume that  $D$  is a sufficiently smooth domain, say  $D$  is of class  $C^2$ , so that the unit normal and tangent vectors to  $\partial D$  are continuously differentiable functions of  $\zeta \in \partial D$ . The unit tangent vector to  $\zeta \in \partial D$  is denoted by  $T_D(\zeta)$  and is given by

$$T_D(\zeta) = \frac{\zeta'(t)}{|\zeta'(t)|}, \quad (2.1)$$

where  $\zeta = \zeta(t)$  denotes any parametrization for  $\partial D$ . Finally, the arc-length measure for  $\partial D$  is denoted by  $\sigma$  and hence

$$d\sigma(\zeta) = |\zeta'(t)|dt. \quad (2.2)$$

### 2.1 The Cauchy integral formula

#### 2.1.1 Definition

For a function  $f(z)$  analytic in domain  $D$  and continuous on  $\overline{D}$ , Cauchy's integral formula states that for  $z \in D$ :

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta. \quad (2.3)$$

#### 2.1.2 The Cauchy integral formula as an inner product

In the sequel it will be convenient to express the Cauchy integral formula (2.3) in terms of the inner product in  $L^2(\partial D, \sigma)$ . To this end, recall that the quantity  $d\zeta$  in (2.3) is the complex differential  $d\zeta = \zeta'(t)dt$ , hence

$$\overline{T_D(\zeta)}d\zeta = d\sigma(\zeta), \quad \text{or, equivalently,} \quad d\zeta = T_D(\zeta)d\sigma(\zeta), \quad (2.4)$$

so that (2.3) can be rewritten as

$$\begin{aligned} f(z) &= \int_{\partial D} f(\zeta) \frac{T_D(\zeta)}{2\pi i(\zeta - z)} d\sigma(\zeta) = \int_{\partial D} f(\zeta) \left[ \frac{\overline{T_D(\zeta)}}{-2\pi i(\overline{\zeta} - \overline{z})} \right] d\sigma(\zeta) \\ &= \left\langle f, \frac{\overline{T_D(\cdot)}}{-2\pi i(\overline{\cdot} - \overline{z})} \right\rangle_{L^2(\partial D, d\sigma)}. \end{aligned} \quad (2.5)$$

Thus Cauchy's integral formula (2.3) may be reinterpreted as follows:

**Theorem 1.** (Cauchy formula) For any  $f$  analytic in  $D$  and continuous on  $\overline{D}$ , we have

$$f(z) = \int_{\partial D} f(\zeta) \overline{C_D(\zeta, z)} d\sigma(\zeta) = \langle f, C_D(\cdot, z) \rangle_{L^2(\partial D, d\sigma)}, \quad z \in D, \quad (2.6)$$

where  $C_D(\zeta, z)$  is the **Cauchy kernel** for  $D$  [1], which is given by

$$C_D(\zeta, z) := -\frac{\overline{T_D(\zeta)}}{2\pi i(\overline{\zeta} - \overline{z})}. \quad (2.7)$$

## 2.2 The Szegő kernel as an alternate for the Cauchy kernel

### 2.2.1 Definition and examples

**Theorem 2.** (*Szegő formula*) For any  $f$  analytic in  $D$  and continuous on  $\overline{D}$ , we have

$$f(z) = \int_{\partial D} f(\zeta) \overline{S_D(\zeta, z)} d\sigma(\zeta) = \langle f, S_D(\cdot, z) \rangle_{L^2(\partial D, d\sigma)}, \quad z \in D, \quad (2.8)$$

where  $S_D(\zeta, z)$  is the **Szegő kernel for  $D$**  [1].

The Szegő kernel depends on the domain  $D$  while the Cauchy kernel does not. The Szegő kernel is defined on  $H^2(D)$ , of which analytic function extending continuously to the boundary are a proper subset. There are a few specific domains whose Szegő kernel can be computed explicitly, and a number of examples are listed below:

(a) If  $D = \mathbb{D} \equiv \{z \in \mathbb{C} \mid |z| < 1\}$ , then

$$S_{\mathbb{D}}(\zeta, z) = C_{\mathbb{D}}(\zeta, z) = -\frac{\overline{T_{\mathbb{D}}(\zeta)}}{2\pi i(\overline{\zeta} - \overline{z})} = \frac{1}{2\pi} \frac{1}{(1 - \zeta\overline{z})}, \quad |\zeta| = 1, \quad |z| < 1, \quad (2.9)$$

where we have used that  $T_{\mathbb{D}}(\zeta) = i\zeta$ . Hence, if  $D = \mathbb{D}$ , then the Szegő kernel for  $\mathbb{D}$  equals the Cauchy kernel. The disc is the only domain for which the two kernels are identical [1].

(b) If  $D = \mathbb{D}^* \equiv \{z \in \mathbb{C} \mid 0 < |z| < 1\}$  (the punctured disc) and  $f$  has a pole of order  $n \in \mathbb{N}$  at 0 and is otherwise analytic on  $\mathbb{D}^*$  and continuous on  $\overline{\mathbb{D}^*}$ , then (2.8) is valid when interpreted as an integral on  $\partial\mathbb{D}^*$  with

$$S_{\mathbb{D}^*}(\zeta, z) \equiv S_{\mathbb{D}^*}^{(n)}(\zeta, z) := \frac{1}{2\pi} \frac{1}{(\zeta\overline{z})^n(1 - \zeta\overline{z})}, \quad |\zeta| = 1, \quad 0 < |z| < 1. \quad (2.10)$$

That is, for any  $f$  which has a pole of order  $n$  at 0, we have

$$f(z) = \int_{\partial\mathbb{D}} f(\zeta) \overline{S_{\mathbb{D}^*}^{(n)}(\zeta, z)} d\sigma(\zeta) = \langle f, S_{\mathbb{D}^*}^{(n)}(\cdot, z) \rangle_{L^2(\partial\mathbb{D}, d\sigma)}, \quad z \in \mathbb{D}^*. \quad (2.11)$$

(c) More generally, if  $D$  is a simply-connected domain and  $D^* = D \setminus \{z_0\}$ , for  $z_0 \in D$ , is the punctured domain, and  $f$  has a pole of order  $n$  at  $z_0$ , then

$$f(z) = \int_{\partial D} f(\zeta) \overline{S_{D^*}^{(n)}(\zeta, z)} d\sigma(\zeta) = \langle f, S_{D^*}^{(n)}(\cdot, z) \rangle_{L^2(\partial D, d\sigma)}, \quad z \in D^*, \quad (2.12)$$

with

$$S_{D^*}^{(n)}(\zeta, z) = (M_{z_0} \circ \Phi)^{-n}(z) S_D(\zeta, z) \overline{(M_{z_0} \circ \Phi)^{-n}(\zeta)}, \quad \zeta \in \partial D, \quad z \in D^*, \quad (2.13)$$

where  $S_D(\zeta, z)$  is the Szegő kernel for  $D$ ,  $\Phi : D \rightarrow \mathbb{D}$  is a conformal mapping, and

$$M_{z_0}(z) := \frac{z - z_0}{1 - \overline{z_0}z}. \quad (2.14)$$

The expression (2.10) is a special case of (2.13), with  $D = \mathbb{D}$ ,  $z_0 = 0$  and  $\Phi(z) = z = M_{z_0}(z)$ . Also, note that an analogous formula holds for  $D^* = D \setminus \{z_1, \dots, z_m\}$  (a simply-connected domain with  $m$ -many punctures) [11].

## 2.2.2 Advantages of the Szegő kernel

The advantage of the Szegő kernel is that it enjoys a good transformation law under conformal maps between domains in the complex plane, whereas the Cauchy kernel does not. Namely, if  $\Phi : D_1 \mapsto D_2$  is conformal, then

$$S_{D_1}(\zeta, z) := \overline{\sqrt{\Phi'(z)}} S_{D_2}(\Phi(\zeta), \Phi(z)) \sqrt{\Phi'(\zeta)}, \quad (2.15)$$

which is valid, for example,

- (a) for any  $z \in D_1$  and any  $\zeta \in \partial D_1$ , if  $D_1$  is simply-connected and of class  $C^{1,1}$ , and  $D_2 = \mathbb{D}$  (the classical setting);
- (b) if  $D_1$  is simply-connected and Lipschitz, and  $D_2 = \mathbb{D}$ , then equation (2.15) is valid in the sense that, for any  $z \in D_1$ ,

$$S_{D_1}(\cdot, z) := \overline{\sqrt{\Phi'(z)}} S_{\mathbb{D}}(\Phi(\cdot), \Phi(z)) \sqrt{\Phi'(\cdot)} \quad \text{as functions in } L^2(\partial D, \sigma). \quad (2.16)$$

It follows that, for any  $z \in D_1$  and for  $\sigma$ -almost everywhere  $\zeta \in \partial D_1$ ,

$$S_{D_1}(\zeta, z) \equiv \overline{\sqrt{\Phi'(z)}} S_{\mathbb{D}}(\Phi(\zeta), \Phi(z)) \sqrt{\Phi'(\zeta)}. \quad (2.17)$$

- (c) if  $D_1 = \mathbb{A}_{1,\infty}(0) \equiv \{z \in \mathbb{C} | 1 < |z| < \infty\}$  and  $D_2 = \mathbb{D}^*$ , then (2.15) is valid for any  $\zeta \in \partial \mathbb{A}_{1,\infty}(0)$  and any  $z \in \mathbb{A}_{1,\infty}(0)$  with  $\Phi(z) \equiv 1/z$ .

Returning to (2.15), using (2.4), we can write

$$\overline{S_{D_1}(\zeta, z)} d\sigma(\zeta) = \sqrt{\Phi'(z)} \overline{S_{D_2}(\Phi(\zeta), \Phi(z))} \overline{\sqrt{\Phi'(\zeta)}} \overline{T_{D_1}(\zeta)} d\zeta. \quad (2.18)$$

Furthermore it is known ([1], p. 53) that:

$$T_{D_2}(\Phi(\zeta)) \overline{\sqrt{\Phi'(\zeta)}} = \sqrt{\Phi'(\zeta)} T_{D_1}(\zeta), \quad \zeta \in \partial D. \quad (2.19)$$

Substitution of (2.19) into (2.18) gives:

$$\overline{S_{D_1}(\zeta, z)} d\sigma(\zeta) = \sqrt{\Phi'(z)} \overline{S_{D_2}(\Phi(\zeta), \Phi(z))} \sqrt{\Phi'(\zeta)} \overline{T_{D_2}(\Phi(\zeta))} d\zeta. \quad (2.20)$$

### Note on the regularity of conformal mapping:

- Pommerenke [17] (Theorems 3.5 and 3.6): Let  $D \subset \mathbb{C}$  be a proper simply-connected domain of class  $C^{1,1}$ , and let  $\Phi : D \rightarrow \mathbb{D}$  be a conformal map. Then  $\Phi$  extends to a map:  $\overline{D} \rightarrow \overline{\mathbb{D}}$  with  $\Phi : \partial D \rightarrow \partial \mathbb{D}$ . Furthermore,  $\Phi \in C^1(\overline{D})$ ,  $\Phi'(\zeta) \neq 0$  for any  $\zeta \in \overline{D}$ .
- Duren [8] (Theorem 3.12): Let  $D \subset \mathbb{C}$  be a proper simply-connected domain, let  $\Phi : D \rightarrow \mathbb{D}$  be a conformal map, and let  $\Psi : \mathbb{D} \rightarrow D$  be the inverse map. Then  $\partial D$  is rectifiable if and only if  $\Psi' \in H^1(\partial \mathbb{D})$ . Cauchy's theorem holds on domains that have rectifiable boundary curves: Duren [8] (Theorem 10.4). Lipschitz domains have rectifiable boundary curve, thus Cauchy's theorem may be applied in Lemma 1.

## 2.3 Transform pair for circular domains via the Cauchy kernel function

### 2.3.1 The transform pair

Returning to the Cauchy integral formula (2.3), for the case when  $D = \mathbb{D}$  is the unit disc, it was proved by Crowdy [2] that the expression  $1/(\zeta - z)$  (for  $\zeta \in \partial\mathbb{D}$  and  $z \in \mathbb{D}$ ; see Fig. 1a) admits the spectral representation

$$\frac{1}{\zeta - z} = \int_{L_1} \frac{1}{1 - e^{2\pi ik}} \frac{z^k}{\zeta^{k+1}} dk + \int_{L_2} \frac{z^k}{\zeta^{k+1}} dk + \int_{L_3} \frac{e^{2\pi ik}}{1 - e^{2\pi ik}} \frac{z^k}{\zeta^{k+1}} dk, \quad (2.21)$$

where  $L_j$ ,  $j = 1, 2, 3$  are contours in the spectral  $k$ -plane, as shown in Fig. 1b. The contour  $L_1$  is the union of the negative imaginary axis  $(-i\infty, ir]$  and the arc of the quarter circle  $|k| = r$ ,  $0 < r < 1$ , in the third quadrant traversed in a clockwise sense; the contour  $L_2$  is the real interval  $[-r, \infty)$ ; the contour  $L_3$  is the arc of the quarter circle  $|k| = r$  in the second quadrant traversed in a clockwise sense together with the portion of the positive imaginary axis  $[ir, i\infty)$  [2]. Replacing the function  $1/(\zeta - z)$  with its spectral representation (2.21) in (2.3), we find the following **transform pair for the interior of the unit disc** [2]:

$$\begin{cases} f(z) = \frac{1}{2\pi i} \left[ \int_{L_1} \frac{\rho(k)}{1 - e^{2\pi ik}} z^k dk + \int_{L_2} \rho(k) z^k dk + \int_{L_3} \frac{\rho(k) e^{2\pi ik}}{1 - e^{2\pi ik}} z^k dk \right], \\ \rho(k) := \oint_{\partial\mathbb{D}} \frac{f(\zeta)}{\zeta^{k+1}} d\zeta. \end{cases} \quad (2.22)$$

The **global relation** is given by

$$\rho(k) = 0, \quad k \in \mathbb{Z}^-. \quad (2.23)$$

## 3 A transform pair for simply-connected domains

In this section, we show how to obtain transform pairs for simply-connected domains by exploiting the properties of the Szegő kernel.

### 3.1 The transform pair

Suppose that  $D$  is a simply-connected domain,  $\Phi : D \mapsto \mathbb{D}$  is a conformal map and  $f : D \mapsto \mathbb{C}$  is analytic and continuous on  $\overline{D}$ . Combining the formula (2.20) for  $D_1 = D$  and  $D_2 = \mathbb{D}$ , with the formula (2.9) for  $S_{\mathbb{D}}(\zeta, z)$  we find

$$\begin{aligned} \overline{S_D(\zeta, z)} d\sigma(\zeta) &= \sqrt{\Phi'(z)} \frac{1}{2\pi i} \frac{T_{\mathbb{D}}(\Phi(\zeta))}{(\Phi(\zeta) - \Phi(z))} \sqrt{\Phi'(\zeta)} T_{\mathbb{D}}(\Phi(\zeta)) d\zeta \\ &= \frac{1}{2\pi i} \sqrt{\Phi'(z)} \frac{\sqrt{\Phi'(\zeta)}}{(\Phi(\zeta) - \Phi(z))} d\zeta. \end{aligned} \quad (3.1)$$

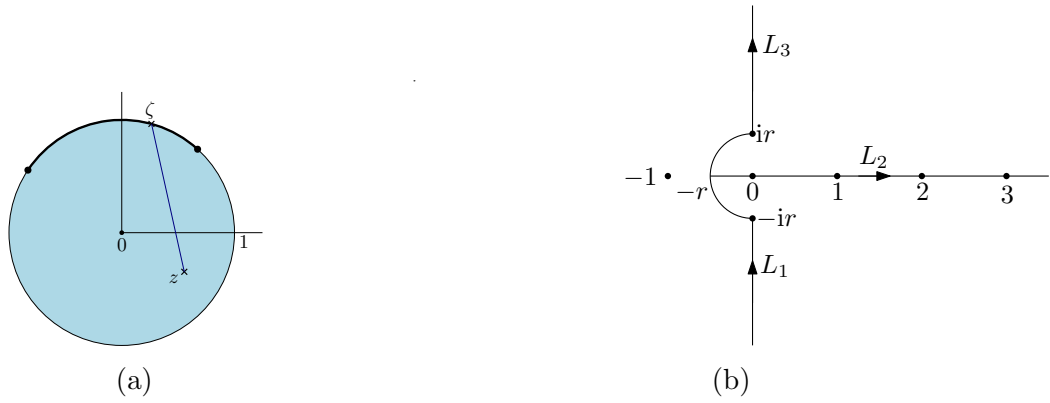


Figure 1: (a) Point  $\zeta$  on the boundary of  $\mathbb{D}$  and point  $z$  in its interior, (b) The fundamental contour for circular arc edges which constitutes of  $L_j$ ,  $j = 1, 2, 3$  [2].

Substitution of this expression in the Szegő formula (2.8) gives

$$\begin{aligned} f(z) &= \int_{\partial D} f(\zeta) \overline{S_D(\zeta, z)} d\sigma(\zeta) \\ &= \frac{1}{2\pi i} \sqrt{\Phi'(z)} \int_{\partial D} f(\zeta) \left[ \frac{\sqrt{\Phi'(\zeta)}}{\Phi(\zeta) - \Phi(z)} \right] d\zeta, \quad z \in D. \end{aligned} \quad (3.2)$$

Next we point out that  $\Phi(\zeta) \in \partial\mathbb{D}$  and  $\Phi(z) \in \mathbb{D}$  by the mapping properties of  $\Phi$ , hence the spectral decomposition (2.21) gives

$$\frac{1}{\Phi(\zeta) - \Phi(z)} = \int_{L_1} \frac{1}{1 - e^{2\pi i k}} \frac{\Phi(z)^k}{\Phi(\zeta)^{k+1}} dk + \int_{L_2} \frac{\Phi(z)^k}{\Phi(\zeta)^{k+1}} dk + \int_{L_3} \frac{e^{2\pi i k}}{1 - e^{2\pi i k}} \frac{\Phi(z)^k}{\Phi(\zeta)^{k+1}} dk, \quad (3.3)$$

with  $\zeta \in \partial D$ ,  $z \in D$  and where the contours  $L_j$ ,  $j = 1, \dots, 3$  are shown in Fig. 1b. Combining (3.3) with (3.2), we obtain a **new transform pair**:

$$\begin{cases} f(z) = \frac{1}{2\pi i} \sqrt{\Phi'(z)} \left[ \int_{L_1} \frac{\rho(k)}{1 - e^{2\pi i k}} \Phi(z)^k dk + \int_{L_2} \rho(k) \Phi(z)^k dk + \int_{L_3} \frac{\rho(k) e^{2\pi i k}}{1 - e^{2\pi i k}} \Phi(z)^k dk \right], \\ \rho(k) := \int_{\partial D} \frac{\sqrt{\Phi'(\zeta)}}{\Phi(\zeta)^{k+1}} f(\zeta) d\zeta. \end{cases} \quad (3.4)$$

The global relation is

$$\rho(k) = 0, \quad k \in \mathbb{Z}^-. \quad (3.5)$$

See the appendix for details concerning switching the order of integration.

### 3.2 Validity of the global relation

Before we prove the equality in equation (3.5), a couple of definitions are introduced. See [8], chapter 10, for the following definitions and a more detailed discussion. Let  $D$  be a simply connected domain with at least two boundary points. An analytic function  $f$  on  $D$  belongs to  $H^p(D)$  if the subharmonic function  $|f|^p$  has a harmonic majorant in  $D$ . The function  $f$  belongs to  $E^p(D)$  if there exists a sequence of rectifiable Jordan Curves  $C_1, C_2, \dots$  in  $D$  converging to the boundary in the sense that  $C_n$  eventually contains each compact subdomain of  $D$ , such that

$$\int_{C_n} |f(z)|^p |dz| \leq M < \infty.$$

It is a well known fact that  $E^p$  and  $H^p$  coincide on the unit disk.

**Lemma 1.** *Let  $D$  be a bounded Lipschitz domain,  $\Phi : D \mapsto \mathbb{D}$  be a conformal map and  $f \in H^2(D)$ . Then*

$$\frac{\sqrt{\phi^j}}{\phi^k} f \in H^2(D), \quad \forall k \in \mathbb{Z}^-. \quad (3.6)$$

Hence

$$\rho(k) := \int_{\partial D} \frac{\sqrt{\Phi'(\zeta)}}{\Phi(\zeta)^{k+1}} f(\zeta) d\zeta = 0, \quad k \in \mathbb{Z}^-, \quad (3.7)$$

satisfies the global relation (3.5) by the Lemma and the Cauchy theorem for  $H^1(D)$  (since  $D$  is bounded, we have that  $H^2(D) \subset H^1(D)$ ).

*Proof.* Let  $k \in \mathbb{Z}^-$  and write  $n := -k - 1 \in \mathbb{Z}_0^+$ . We can then write

$$\tau(n) := \rho(-n - 1) = \int_{\partial D} \sqrt{\Phi'(\zeta)} f(\zeta) \Phi(\zeta)^n d\zeta. \quad (3.8)$$

We want to show that  $\tau(n) = 0$ , for  $n \in \mathbb{Z}_0^+$ . Here  $\Phi : D \rightarrow \mathbb{D}$  is a conformal map and we denote its inverse by  $\Psi := \Phi^{-1} : \mathbb{D} \rightarrow D$ .

By Caratheodory's theorem [8],  $\Psi$  has a conformal 1-1 and onto continuous extension to  $\overline{\mathbb{D}}$ . We have  $\Psi'(z) \neq 0$ , for  $|z| < 1$ . Hence, by Duren [8],  $\Psi'(w) \neq 0$ ,  $\forall |w| = 1$ . Moreover,  $\Psi'$  has analytic branch of the square root which is in  $H^2(\mathbb{D})$ . Therefore for such a branch  $\sqrt{\Psi'(w)} = \sqrt{\Psi'(w)}$ ,  $\forall |w| = 1$ .

Now applying the change of variable formula  $\zeta := \Psi(w)$  to  $\tau(n)$ , we find:

$$\begin{aligned} \tau(n) &= \int_{\partial \mathbb{D}} \sqrt{\Phi'(\Psi(w))} (f \circ \Psi)(w) w^n \Psi'(w) dw \\ &= \int_{\partial \mathbb{D}} \sqrt{\Phi'(\Psi(w)) \Psi'(w)} (f \circ \Psi)(w) [\Phi \circ \Psi]^n(w) \sqrt{\Psi'(w)} dw \\ &= \int_{\partial \mathbb{D}} [(f \cdot \Phi^n) \circ \Psi](w) \sqrt{\Psi'(w)} dw, \end{aligned} \quad (3.9)$$

where  $\cdot$  denotes pointwise product. We have also used that  $(\Phi \circ \Psi)'(w) = 1$ ,  $\forall w$ , because  $(\Phi \circ \Psi)(w) = w$ .

Following Duren [8], we have

$$f \cdot \Phi^n \in E^2(D) \quad \Leftrightarrow \quad [f \cdot \Phi^n] \circ \Psi \cdot \sqrt{\Psi'} \in H^2(\mathbb{D}) = E^2(\mathbb{D}), \quad (3.10)$$

in which case  $\tau(n) = 0$ , for  $n \in \mathbb{Z}_0^+$ , by Cauchy theorem for  $H^1(\mathbb{D})$  (and the fact that  $H^2(\mathbb{D}) \subset H^1(\mathbb{D})$ ). Therefore, the question whether  $\tau(n) = 0$  is reduced to whether the following statement is true:

$$f \in H^2(\mathbb{D}) \quad \text{and} \quad \Phi : D \rightarrow \mathbb{D} \text{ conformal} \quad \Rightarrow \quad f \cdot \Phi^n \in H^2(D). \quad (3.11)$$

To this end, we claim that  $\Phi$  extends to  $\Phi \in C(\overline{D}) \cap \vartheta(D)$ . [Proof of claim: Let  $\Psi := \Phi^{-1} : \mathbb{D} \mapsto D$ . By Caratheodory's theorem,  $\Psi$  extends to a homeomorphism  $\Psi : \overline{\mathbb{D}} \mapsto \overline{D}$  (more precisely,  $\Psi \in \vartheta(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ ,  $\Psi : \partial \mathbb{D} \mapsto \partial D$  homeomorphism,  $\Psi|_{\mathbb{D}} = \psi$ ). Hence  $\Psi^{-1} : \overline{D} \mapsto \overline{\mathbb{D}}$  is a homeomorphism and we may take  $\Phi := \Psi^{-1}$ .]

Now it is well known that

$$H^2(D) \cdot \{\vartheta(D) \cap C(\overline{D})\} \subset H^2(D), \quad (3.12)$$

and since  $\Phi \in \vartheta(D) \cap C(\overline{D})$ , that implies the same is true for  $\Phi^n$ ,  $\forall n \in \mathbb{Z}_0^+$ . It follows that  $f \cdot \Phi^n \in H^2(D) \cdot \{\vartheta(D) \cap C(\overline{D})\} \subset H^2(D)$ . Therefore we have shown that the statement (3.11) holds.  $\square$



## 4 A transform pair for punctured domains

The first step is to derive a transform pair for a punctured domain  $D^*$ . We introduce the conformal mapping  $\Phi : D^* \mapsto \mathbb{D}^*$  which maps the punctured domain  $D^*$  to the punctured disc  $\mathbb{D}^*$  and we write  $\eta = \Phi(\zeta)$  and  $w = \Phi(z)$ , for  $\zeta \in \partial D^*$ ,  $z \in D^*$ .

The function  $f$  is defined on  $D^*$  and we assume, without loss of generality, that it has a first-order pole at  $\Phi^{-1}(0)$ . From (2.10), the Szegő kernel for  $\mathbb{D}^*$  with a first-order pole at 0 is given by

$$S_{\mathbb{D}^*}(\eta, w) = \frac{1}{2\pi} \frac{1}{\eta\bar{w}(1-\eta\bar{w})}, \quad |\eta| = 1, \quad 0 < |w| < 1. \quad (4.1)$$

Therefore, we can write:

$$\overline{S_{\mathbb{D}^*}(\eta, w)} = \frac{1}{2\pi} \frac{1}{\bar{\eta}w(1-\bar{\eta}w)} = \frac{1}{2\pi} \frac{\eta^2}{w(\eta-w)}. \quad (4.2)$$

Since for  $D_2 = \mathbb{D}$  we have  $T_{D_2}(\Phi(\zeta)) = i\Phi(\zeta)$ , expression (2.20) can be written as:

$$\overline{S_{D^*}(\zeta, z)} d\sigma(\zeta) = \sqrt{\Phi'(z)} \frac{1}{2\pi} \frac{\Phi(\zeta)^2}{\Phi(z)(\Phi(\zeta) - \Phi(z))} \sqrt{\Phi'(\zeta)} (-i\overline{\Phi(\zeta)}) d\zeta. \quad (4.3)$$

Substitution of (4.3) into (2.8) gives:

$$\begin{aligned} f(z) &= \int_{\partial D^*} f(\zeta) \overline{S_{D^*}(\zeta, z)} d\sigma(\zeta) \\ &= \frac{1}{2\pi i} \frac{\sqrt{\Phi'(z)}}{\Phi(z)} \int_{\partial D^*} f(\zeta) \left[ \frac{\Phi(\zeta) \sqrt{\Phi'(\zeta)}}{\Phi(\zeta) - \Phi(z)} \right] d\zeta, \quad \text{for } z \in D^*. \end{aligned} \quad (4.4)$$

Combining (4.4) with (3.3), we obtain a **transform pair for a punctured domain  $D^*$  at  $\Phi^{-1}(0)$** :

$$\begin{cases} f(z) = \frac{1}{2\pi i} \frac{\sqrt{\Phi'(z)}}{\Phi(z)} \left[ \int_{L_1} \frac{\rho(k)}{1 - e^{2\pi i k}} \Phi(z)^k dk + \int_{L_2} \rho(k) \Phi(z)^k dk + \int_{L_3} \frac{\rho(k) e^{2\pi i k}}{1 - e^{2\pi i k}} \Phi(z)^k dk \right], \\ \rho(k) := \int_{\partial D^*} \frac{\sqrt{\Phi'(\zeta)}}{\Phi(\zeta)^k} f(\zeta) d\zeta. \end{cases} \quad (4.5)$$

The global relation is

$$\rho(k) = 0, \quad k \in \mathbb{Z}^-. \quad (4.6)$$

[Note that the differences between (4.5) and (3.4) are an extra factor  $1/\Phi(z)$  in  $f(z)$  in (4.5) and an extra factor  $1/\Phi(\zeta)$  in the integrand of  $\rho(k)$  in (3.4).]

## 5 Mixed boundary value problem on an elliptical domain and on a Cassini oval

In this section, we present a generalization of the mixed boundary value problem considered by Shepherd [19] on the disc to problems posed on an elliptical domain and Cassini ovals. The mixed boundary value problem on an elliptical domain was also analysed in our previous study [13] and this is used for verification of the transform method developed in this study.

## 5.1 Elliptical domain

Consider an elliptical domain  $D$ , namely

$$D := \left\{ z \in \mathbb{C}, x = \operatorname{Re}[z], y = \operatorname{Im}[z] \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \right\}, \quad (5.1)$$

whose boundary is

$$\partial D = \left\{ z \in \mathbb{C}, x = \operatorname{Re}[z], y = \operatorname{Im}[z] \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}. \quad (5.2)$$

We consider the following incomplete boundary value problem for analytic functions

$$\begin{cases} \bar{\partial}f(z) = 0, & z \in D, \\ \operatorname{Re}f(\zeta) = \operatorname{Re}\bar{\zeta}^m, & \zeta \in C_1 = \{\zeta \in \partial D \text{ and } \operatorname{Re}\zeta > 0\}, \\ \operatorname{Im}f(\zeta) = \operatorname{Im}\bar{\zeta}^m, & \zeta \in C_2 = \{\zeta \in \partial D \text{ and } \operatorname{Re}\zeta < 0\}, \end{cases} \quad (5.3)$$

for a given  $m \in \mathbb{N}$ . (This problem was also considered in our recent study [13].)

We parametrize the ellipse using polar coordinates  $(\rho, \theta)$ , with  $0 \leq \theta \leq 2\pi$  and

$$\rho(\theta) = \frac{ab}{\sqrt{(b \cos \theta)^2 + (a \sin \theta)^2}} \quad (5.4)$$

and write

$$\zeta(\theta) := \rho(\theta)e^{i\theta} \in D. \quad (5.5)$$

The boundary value problem (5.3) can be written in terms of polar coordinates as

$$\begin{cases} \bar{\partial}f(z) = 0, & z \in D, \\ \operatorname{Re}f = [\rho(\theta)]^m \cos m\theta, & \theta \in (-\pi/2, \pi/2) \quad (\text{or } C_1), \\ \operatorname{Im}f = -[\rho(\theta)]^m \sin m\theta, & \theta \in (\pi/2, 3\pi/2) \quad (\text{or } C_2). \end{cases} \quad (5.6)$$

On  $C_1$ , we write

$$f(\theta, \rho(\theta)) = [\rho(\theta)]^m \cos m\theta + i \left( a_0 + \sum_{n \geq 1} \left[ a_n e^{2in\theta} + \bar{a}_n e^{-2in\theta} \right] \right), \quad (5.7)$$

where coefficients  $\{a_n \in \mathbb{C} | n = 0, 1, 2, \dots\}$  are to be determined.

On  $C_2$ , we write

$$f(\theta, \rho(\theta)) = \left( b_0 + \sum_{n \geq 1} \left[ b_n e^{2in\theta} + \bar{b}_n e^{-2in\theta} \right] \right) - i[\rho(\theta)]^m \sin m\theta, \quad (5.8)$$

where coefficients  $\{b_n \in \mathbb{C} | n = 0, 1, 2, \dots\}$  are to be determined.

On substitution of (5.7) and (5.8) into the global relation (3.5), we obtain a linear system for the unknown coefficients given by

$$\begin{aligned} & a_0 \mathcal{A}(0, k) + \sum_{n \geq 1} \left( a_n \mathcal{A}(2n, k) + \bar{a}_n \mathcal{A}(-2n, k) \right) \\ & + b_0 \mathcal{B}(0, k) + \sum_{n \geq 1} \left( b_n \mathcal{B}(2n, k) + \bar{b}_n \mathcal{B}(-2n, k) \right) = r(k), \quad k \in \mathbb{Z}^-, \\ & a_0 \overline{\mathcal{A}(0, k)} + \sum_{n \geq 1} \left( \bar{a}_n \overline{\mathcal{A}(2n, k)} + a_n \overline{\mathcal{A}(-2n, k)} \right) \\ & + b_0 \overline{\mathcal{B}(0, k)} + \sum_{n \geq 1} \left( \bar{b}_n \overline{\mathcal{B}(2n, k)} + b_n \overline{\mathcal{B}(-2n, k)} \right) = \overline{r(k)}, \quad k \in \mathbb{Z}^-, \end{aligned} \quad (5.9)$$

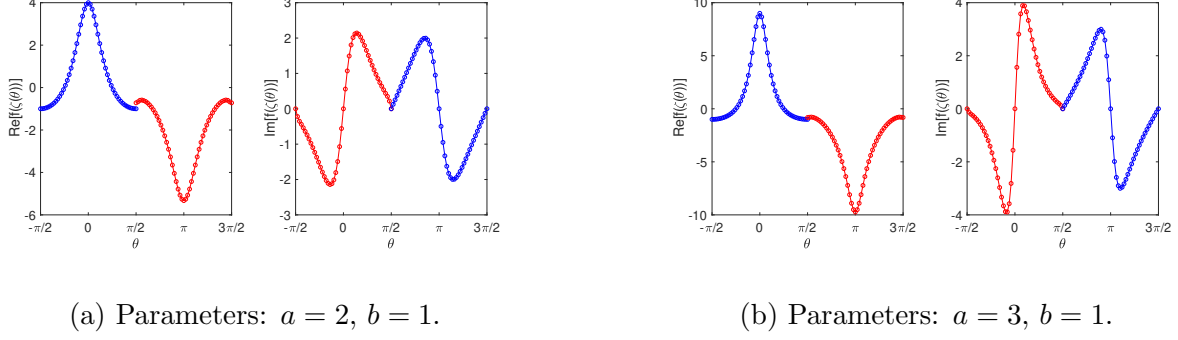


Figure 2: Real and imaginary parts of  $f(\zeta(\theta))$ ,  $\theta \in [-\pi/2, 3\pi/2]$  along the boundary of the ellipse, for  $m = 2$ . Solid lines show the solutions computed via the numerical scheme of our previous study [13] and circles denote the solutions computed via the transform method presented here. The solutions via the transform method and the transform-based technique considered in our previous study [13] are indistinguishable.

where

$$\mathcal{A}(n, k) = i \int_{-\pi/2}^{\pi/2} \frac{\sqrt{\Phi'(\zeta(\theta))}}{\Phi(\zeta(\theta))^{k+1}} e^{in\theta} \zeta'(\theta) d\theta, \quad \mathcal{B}(n, k) = \int_{\pi/2}^{3\pi/2} \frac{\sqrt{\Phi'(\zeta(\theta))}}{\Phi(\zeta(\theta))^{k+1}} e^{in\theta} \zeta'(\theta) d\theta, \quad (5.10)$$

and the function  $r(k)$  is defined by

$$r(k) = - \int_{-\pi/2}^{\pi/2} [\rho(\theta)]^m \cos(m\theta) \frac{\sqrt{\Phi'(\zeta(\theta))}}{\Phi(\zeta(\theta))^{k+1}} \zeta'(\theta) d\theta + i \int_{\pi/2}^{3\pi/2} [\rho(\theta)]^m \sin(m\theta) \frac{\sqrt{\Phi'(\zeta(\theta))}}{\Phi(\zeta(\theta))^{k+1}} \zeta'(\theta) d\theta. \quad (5.11)$$

The conformal mapping  $\Phi$  is defined by (A.2), and maps the interior of the ellipse  $D$  to the unit disc  $\mathbb{D}$ .

To proceed, the sums (5.7) and (5.8) (for  $f(\theta, \rho(\theta))$ ) are truncated to include only terms up to  $n = N$  and we formulate a linear system for the unknown coefficients  $\{a_n, b_n | n = 0, \dots, N\}$ . The linear system comprises of conditions (5.9) evaluated at points  $k \in \mathbb{Z}^-$  which are used to form an overdetermined linear system. This is then solved using least squares. We found that the coefficients  $\{a_n, b_n | n = 0, \dots, N\}$  decay quickly and, therefore, we choose the truncation parameter to be  $N = 16$ . Once the coefficients  $\{a_n, b_n | n = 0, \dots, N\}$  are found, the spectral function  $\rho(k)$  and  $f(z)$  can be computed via the transform pair (3.4). Our results here were verified against the numerical solutions of our previous study [13].

Figure 2 shows the real and imaginary parts of  $f(\zeta(\theta))$ ,  $\theta \in [-\pi/2, 3\pi/2]$  along the boundary of the ellipse, for  $m = 2$  and different parameter choices for  $a$  and  $b$ . Solid lines show the solutions computed via the numerical scheme of our previous study [13] and circles denote the solutions computed via the transform method presented here. We observe that the two solutions are indistinguishable. Note that there appear to be discontinuities in the real part of  $f$  for  $\theta = \pi/2$  and  $\theta = 3\pi/2$  (also corresponding to  $\theta = -\pi/2$ ), values at which the boundary conditions change type. Discontinuities as such are not surprising given the solution is in  $E^1$ .

## 5.2 Cassini oval

In this section, we present a generalization of the mixed boundary value problem presented above to a problem posed on a Cassini oval  $D$  parametrized by

$$\zeta(\theta) = p e^{i\theta} \sqrt{1 + \frac{1}{p^2 e^{2i\theta}}}, \quad p > 1, \quad \theta \in [0, 2\pi]. \quad (5.12)$$

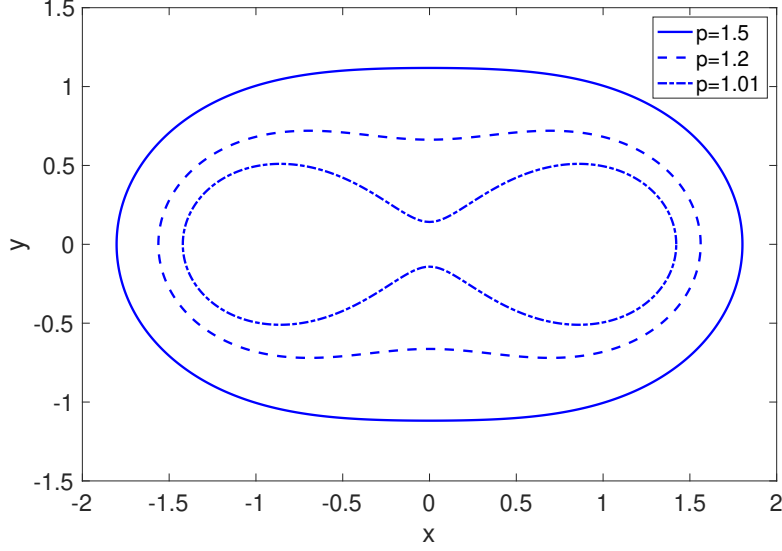


Figure 3: Cassini ovals parametrized by (5.12) for different choices of parameter  $p = 1.01, 1.2, 1.5$ .

Figure 3 shows a schematic of Cassini ovals for different choices of parameter  $p$ .

We consider the two following incomplete boundary value problems for analytic functions; first, we consider

$$\begin{cases} \bar{\partial}f(z) = 0, & z \in D, \\ \operatorname{Re}f(\zeta) = \operatorname{Re} \bar{\zeta}^m, & \zeta \in C_1 = \{\zeta \in \partial D \text{ and } \operatorname{Re} \zeta > 0\}, \\ \operatorname{Im}f(\zeta) = \operatorname{Im} \bar{\zeta}^m, & \zeta \in C_2 = \{\zeta \in \partial D \text{ and } \operatorname{Re} \zeta < 0\}, \end{cases} \quad (5.13)$$

and, secondly,

$$\begin{cases} \bar{\partial}f(z) = 0, & z \in D, \\ \operatorname{Re}f(\zeta) = \operatorname{Re} \bar{\zeta}^m, & \zeta \in C_1 = \{\zeta \in \partial D \text{ and } \operatorname{Im} \zeta > 0\}, \\ \operatorname{Im}f(\zeta) = \operatorname{Im} \bar{\zeta}^m, & \zeta \in C_2 = \{\zeta \in \partial D \text{ and } \operatorname{Im} \zeta < 0\}, \end{cases} \quad (5.14)$$

for a given  $m \in \mathbb{N}$ , and  $\partial D = C_1 \cup C_2$ .

Without loss of generality, we formulate the solution scheme for the mixed boundary value problem (5.13); a similar analysis can be carried out for (5.14):

On  $C_1$ , we write

$$f(\zeta(\theta)) = \operatorname{Re} \left[ \overline{\zeta(\theta)}^m \right] + i \left( a_0 + \sum_{n \geq 1} \left[ a_n e^{2in\theta} + \bar{a}_n e^{-2in\theta} \right] \right), \quad (5.15)$$

where the coefficients  $\{a_n \in \mathbb{C} | n = 0, 1, 2, \dots\}$  are to be determined.

On  $C_2$ , we write

$$f(\zeta(\theta)) = \left( b_0 + \sum_{n \geq 1} \left[ b_n e^{2in\theta} + \bar{b}_n e^{-2in\theta} \right] \right) + i \operatorname{Im} \left[ \overline{\zeta(\theta)}^m \right], \quad (5.16)$$

where the coefficients  $\{b_n \in \mathbb{C} | n = 0, 1, 2, \dots\}$  are to be determined.

On substitution of (5.15) and (5.16) into the global relation (3.5) and evaluation of (3.5) at  $k \in \mathbb{Z}^-$ , we obtain a linear system for the unknown coefficients  $\{a_n, b_n | n = 0, \dots, N\}$  of  $f$ . The linear system is given by (5.9), but now  $A(n, k)$ ,  $B(n, k)$  and  $r(k)$  are defined by

$$\mathcal{A}(n, k) = i \int_{-\pi/2}^{\pi/2} \frac{\sqrt{\overline{\Phi'(\zeta(\theta))}}}{\Phi(\zeta(\theta))^{k+1}} e^{in\theta} \zeta'(\theta) d\theta, \quad \mathcal{B}(n, k) = \int_{\pi/2}^{3\pi/2} \frac{\sqrt{\overline{\Phi'(\zeta(\theta))}}}{\Phi(\zeta(\theta))^{k+1}} e^{in\theta} \zeta'(\theta) d\theta, \quad (5.17)$$

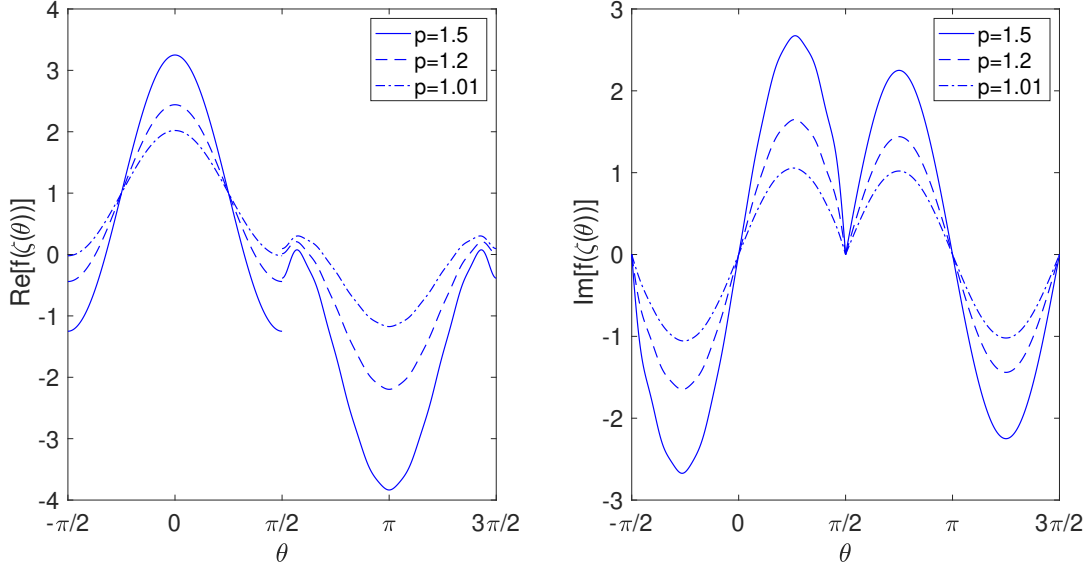


Figure 4: Mixed boundary value problem (5.13) with  $m = 2$ : Real and imaginary parts of  $f(\zeta(\theta))$ ,  $\theta \in [-\pi/2, 3\pi/2]$  along the boundary of the Cassini ovals parametrized by (5.12). Results are shown for different  $p = 1.01, 1.2, 1.5$ .

and

$$r(k) = - \int_{-\pi/2}^{\pi/2} \operatorname{Re} \left[ \overline{\zeta(\theta)}^m \right] \frac{\sqrt{\Phi'(\zeta(\theta))}}{\Phi(\zeta(\theta))^{k+1}} \zeta'(\theta) d\theta - i \int_{\pi/2}^{3\pi/2} \operatorname{Im} \left[ \overline{\zeta(\theta)}^m \right] \frac{\sqrt{\Phi'(\zeta(\theta))}}{\Phi(\zeta(\theta))^{k+1}} \zeta'(\theta) d\theta. \quad (5.18)$$

The conformal mapping  $\Phi$  here is defined by (A.5) (this maps the interior of the Cassini oval  $D$  to the unit disc  $\mathbb{D}$ ).

To proceed, the sums (5.15) and (5.16) are truncated to include only terms up to  $n = N$  and we formulate a linear system for the unknown coefficients  $\{a_n, b_n | n = 0, \dots, N\}$ . The linear system comprises of conditions (5.9) evaluated at points  $k \in \mathbb{Z}^-$  which are used to form an overdetermined linear system. This is then solved using least squares. We found that the coefficients  $\{a_n, b_n | n = 0, \dots, N\}$  decay quickly and, therefore, we choose the truncation parameter to be  $N = 16$ . Once the coefficients  $\{a_n, b_n | n = 0, \dots, N\}$  are found, the spectral function  $\rho(k)$  and  $f(z)$  can be computed via the transform pair (3.4).

Figures 4–5 show the real and imaginary parts of  $f(\zeta(\theta))$  along the boundary of the Cassini oval, for the mixed boundary value problems defined by (5.13) (for  $m = 2$ ) and (5.14) (for  $m = 2$ ), respectively. Note that there appear to be discontinuities in the real part of  $f$  at values at which the boundary conditions change type (similarly to the mixed boundary value problem for the elliptical domain presented above).

## 6 Application in fluid dynamics: A point vortex inside a Cassini oval

In this section, we present an application of the new transform pairs to a problem in fluid dynamics, in particular within the framework of a two-dimensional, inviscid, incompressible and irrotational (except for point vortices) steady flow. We consider a point vortex (a vortex with infinite vorticity concentrated at a point) in the interior of a Cassini oval and the aim is to find the resulting fluid flow satisfying the imposed impermeability boundary condition.

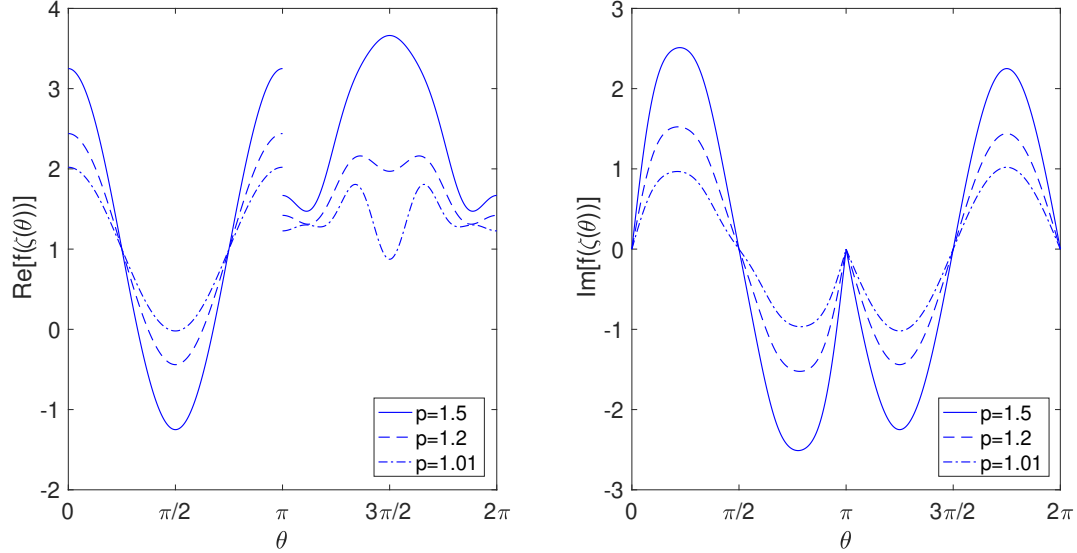


Figure 5: Mixed boundary value problem (5.14) with  $m = 2$ : Real and imaginary parts of  $f(\zeta(\theta))$ ,  $\theta \in [0, 2\pi]$  along the boundary of the Cassini ovals parametrized by (5.12). Results are shown for different  $p = 1.01, 1.2, 1.5$ .

## 6.1 Problem formulation and solution scheme

Consider a point vortex with circulation  $\Gamma$  at point  $z_0$  in the interior of Cassini oval  $D$ . A schematic is shown in Fig. 6. To begin, we introduce a complex potential function  $h(z)$  and write

$$h(z) = f_s(z) + f(z), \quad (6.1)$$

where

$$f_s(z) = \frac{\Gamma}{2\pi i} \log(z - z_0). \quad (6.2)$$

The function  $f(\zeta)$  is analytic in  $D$  and will be found using the transform method.

We impose an impermeability condition on the boundary of  $D$  which can be expressed in terms of the complex potential as:

$$\text{Im}[h(\zeta)] = 0. \quad (6.3)$$

Substitution of (6.1) into (6.3) gives

$$\text{Im}[f(\zeta)] = -\text{Im}[f_s(\zeta)]. \quad (6.4)$$

We represent  $f(\zeta)$  on the boundary of  $D$  using a Fourier expansion:

$$f(\zeta(\theta)) = \left( a_0 + \sum_{n=1}^{\infty} a_n e^{in\theta} + \sum_{n=1}^{\infty} \bar{a}_n e^{-in\theta} \right) - i \text{Im}[f_s(\zeta(\theta))], \quad \text{for } \theta \in [0, 2\pi], \quad (6.5)$$

where coefficients  $a_0 \in \mathbb{R}$  and  $\{a_n \in \mathbb{C} | n = 1, 2, \dots\}$  are to be determined. The parametrization of the Cassini oval is given by

$$\zeta(\theta) = p e^{i\theta} \sqrt{1 + \frac{1}{p^2 e^{2i\theta}}}, \quad \theta \in [0, 2\pi]. \quad (6.6)$$

The interior of the Cassini oval  $D$  is mapped to the unit disc  $\mathbb{D}$  via the mapping  $\Phi$  given by (A.5).

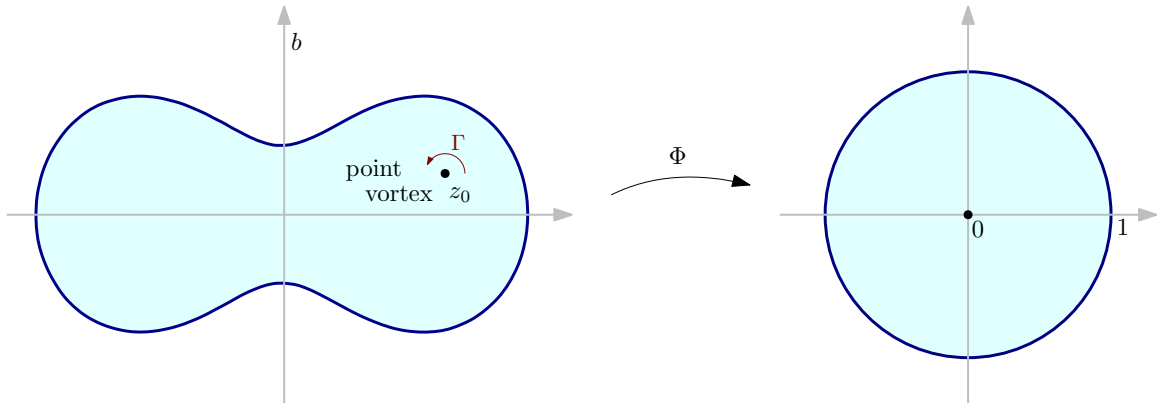


Figure 6: Schematic of the configuration: A point vortex at  $z_0$  in the interior of a Cassini oval  $D$  (left) and the mapped domain  $\mathbb{D}$  (right) via the conformal mapping  $\Phi$ .

On substitution of (6.5) into the global relation (3.5), we obtain (after some algebra and rearrangement) a linear system for the unknown coefficients  $\{a_n | n = 0, \dots, N\}$ . The infinite sums are truncated to include terms up to  $n = N$ . The linear system is given by

$$a_0 P(0, k) + \sum_{n=1}^N \left( a_n P(n, k) + \bar{a}_n P(-n, k) \right) = R(k), \quad \text{for } k \in \mathbb{Z}^-, \quad (6.7)$$

where

$$P(n, k) = \int_0^{2\pi} e^{in\theta} \frac{\sqrt{\Phi'(\zeta(\theta))}}{\Phi(\zeta(\theta))^{k+1}} \zeta'(\theta) d\theta \quad (6.8)$$

and

$$R(k) = i \int_0^{2\pi} \text{Im}[f_s(\zeta(\theta))] \frac{\sqrt{\Phi'(\zeta(\theta))}}{\Phi(\zeta(\theta))^{k+1}} \zeta'(\theta) d\theta, \quad (6.9)$$

with  $\Phi(\zeta)$  defined by (A.5). The linear system comprises (6.7) and its conjugate evaluated at points  $k \in \mathbb{Z}^-$  which are used to form an overdetermined linear system. We found that the coefficients  $\{a_n | n = 0, \dots, N\}$  decay quickly and, therefore, we choose the truncation parameter to be  $N = 16$ . Once the coefficients are found, the spectral functions and  $f(z)$  can be computed via the transform pair (3.4). Note that for the case that  $D$  is an elliptical boundary, our results here were verified against the numerical solutions of our previous study [13] (with  $\Phi(\zeta)$  defined by (A.2)); the latter were checked against an exact solution found using conformal mapping techniques.

## 6.2 Alternative approach via mapping to the punctured disc

Consider the punctured Cassini oval  $D^*$  and introduce the conformal mapping  $\Phi : D^* \mapsto \mathbb{D}^*$  which maps the punctured Cassini oval  $D^*$  to the punctured disc  $\mathbb{D}^*$ . The puncture is located at point  $z_0 = \Phi^{-1}(0)$  and assume we have a point vortex with circulation  $\Gamma$  at point  $z_0$ .

We introduce the function  $w(z)$ :

$$w(z) = \frac{\Gamma}{2\pi i} f(z), \quad (6.10)$$

where  $f(z)$  is given by (4.5). The impermeability boundary condition on the Cassini oval can be written as

$$\text{Im} \left[ h(\zeta) (-i) \frac{d\zeta}{ds} \right] = 0. \quad (6.11)$$

Substitution of (6.10) into (6.11) gives

$$\operatorname{Im} \left[ f(\zeta) \frac{d\zeta}{ds} \right] = 0. \quad (6.12)$$

Next, we write:

$$f(\zeta(\theta(s))) \frac{d\zeta}{ds} = a_0 + \sum_{n=1}^{\infty} a_n e^{in\theta(s)} + \sum_{n=1}^{\infty} \overline{a_n} e^{-in\theta(s)}, \quad (6.13)$$

where coefficients  $a_0 \in \mathbb{R}$  and  $\{a_n \in \mathbb{C} | n = 1, 2, \dots\}$  are to be determined. This can be written as

$$f(\zeta(\theta(s))) = a_0 \left( \frac{d\zeta}{ds} \right)^{-1} + \sum_{n=1}^{\infty} a_n e^{in\theta(s)} \left( \frac{d\zeta}{ds} \right)^{-1} + \sum_{n=1}^{\infty} \overline{a_n} e^{-in\theta(s)} \left( \frac{d\zeta}{ds} \right)^{-1}, \quad \text{for } s \in [-1, 1], \quad (6.14)$$

with

$$\zeta(\theta(s)) = p e^{i\theta(s)} \sqrt{1 + \frac{1}{p^2 e^{2i\theta(s)}}}, \quad \theta(s) = \pi(s+1), \quad s \in [-1, 1]. \quad (6.15)$$

On substitution of (6.14) into the global relation (4.6), we obtain (after some algebra and rearrangement) a linear system for the unknown coefficients  $\{a_n | n = 0, \dots, N\}$ . The infinite sums are truncated to include terms up to  $n = N$ . The linear system is given by

$$a_0 U(0, k) + \sum_{n=1}^N \left( a_n U(n, k) + \overline{a_n} U(-n, k) \right) = 0, \quad \text{for } k \in \mathbb{Z}^-, \quad (6.16)$$

where

$$U(n, k) = \int_{-1}^1 e^{in\theta(s)} \frac{\sqrt{\Phi'(\zeta(\theta(s)))}}{\Phi(\zeta(\theta(s)))^k} ds. \quad (6.17)$$

The linear system comprises of (6.16) and its conjugate evaluated at points  $k \in \mathbb{Z}^-$  which are used to form an overdetermined linear system. We found that the coefficients  $\{a_n | n = 0, \dots, N\}$  decay quickly and, therefore, we choose the truncation parameter to be  $N = 16$ . Once the coefficients are found, the spectral functions and  $f(z)$  can be computed via the transform pair (4.5). Our numerical computations here agree with those obtained in §6.1 above.

## 7 Discussion

In this study, we have expanded the scope of the UTM to address boundary value problems in non-convex planar domains, including shapes beyond circular. Building on the work by Crowdy [2] for circular domains, we developed a novel transform method tailored specifically for non-convex domains. We analysed mixed boundary value problems in elliptical domains and Cassini oval domains with different boundary conditions imposed on different sections of the boundary.

Our work underscores the importance of the Szegő kernel and its connection to the Cauchy kernel in extending the UTM to non-convex and beyond circular domains. The Cauchy kernel lacks favourable transformation properties under conformal maps, while the Szegő kernel has a good transformation law. The applicability of this new transform pair relies on the numerical computations afforded by the conformal mapping  $\Phi$  from domain of interest to the unit disc. This provides a framework that broadens the applicability of the UTM to any simply-connected domain, including non-convex domains, with a conformal map to the unit disc.



The transform method developed in this paper serves as a stepping stone for further research in solving boundary value problems in more complex geometries, including *exterior/unbounded domains* and *multiply-connected domains*. The following questions naturally arise:

- *Exterior/unbounded domains*: Given a bounded domain  $D$ , we denote its complement by  $D_E = \mathbb{C} \setminus \overline{D}$ . Note that  $\partial D_E = \partial D$  and  $d\sigma_{D_E}(z) = -d\sigma_D(z)$ . Is it possible to obtain a Szegő projection/kernel for  $D_E$  via a conformal map  $\Phi : \mathbb{C} \setminus \overline{D} \mapsto \mathbb{C} \setminus \overline{\mathbb{D}}$ ? The Szegő formula still gives

$$f(z) = \int_{\partial D_E} f(\zeta) \overline{S_{D_E}(\zeta, z)} d\sigma_{D_E}(\zeta), \quad z \in D_E = \mathbb{C} \setminus \overline{D}. \quad (7.1)$$

- *Multiply-connected domains*: Consider a multiply-connected domain  $D$ . Without loss of generality, assume that  $D$  is a doubly-connected domain (for example, domain  $D$  could be the region exterior to the ellipse (5.1) and interior to the channel  $-\infty < x < \infty$ ,  $0 < y < h$ , with  $h > 2b$ ). Assume we can find a conformal mapping  $\Phi : D \mapsto \mathbb{A}$ , where  $\mathbb{A}$  is the annulus  $0 < \rho < 1$ , numerically. Given that we have an expression for the Szegő kernel for the annulus given by

$$S_a(z) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(z\bar{a})^n}{1 + p^{2n+1}}, \quad (7.2)$$

is it possible to construct a transform pair for the doubly-connected domain  $D$ ? Is it possible to extend this approach to domains with higher-connectivity?

Finally, we also note that this new approach can be also used to analyse boundary value problems for the biharmonic equation (which will involve solving for two analytic functions; Langlois [15], Luca & Crowdy [16], as well as the complex Helmholtz equation discussed by Hauge & Crowdy [12] and Hulse *et al.* [13]).

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## A Appendix: conformal maps for the ellipse and Cassini ovals

### A.1 Ellipse

Consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad (A.1)$$

with  $a > b$  and foci at distance  $c = \sqrt{a^2 - b^2} > 0$ .

#### A.1.1 Mapping the interior of an ellipse to the disc

The conformal mapping  $\Phi$  from the ellipse (A.1) in the  $z$ -plane to the unit disc  $\mathbb{D}$  is given by (Schwarz [18]):

$$\Phi(z) = \sqrt{k} \operatorname{sn} \left( \frac{2K}{1\sqrt{}} \sin^{-1}(z), m \right), \quad (A.2)$$

where  $\text{sn}(\cdot, \cdot)$  is the Jacobi sn function and definition of parameters  $k, K$  and  $m$  is given in [18, 21].

### A.1.2 Mapping the exterior of an ellipse to the disc

The exterior of the disc is mapped to the exterior of the ellipse via the Joukovskii transformation

$$u(s) = \frac{a+b}{2}s + \frac{a-b}{2s}. \quad (\text{A.3})$$

Therefore the mapping of the exterior of the ellipse to the exterior of the disc can be obtained via  $\Phi(z) = u^{-1}(z)$  and choosing the appropriate branch.

## A.2 Cassini ovals

Consider the Cassini oval

$$\zeta(t) = t\sqrt{1 + \frac{1}{t^2}}, \quad t = pe^{i\theta}, \quad \theta \in [0, 2\pi]. \quad (\text{A.4})$$

If  $p = 1$ , then we have a lemniscata. If  $p > 1$ , then we have a single-contour Cassinian.

### A.2.1 Mapping interior of a Cassini oval to the disc

The interior of the Cassini oval is mapped to the disc via the mapping [14]:

$$\Phi(z) = \pm \frac{1}{p}\sqrt{z^2 - 1}, \quad p > 1. \quad (\text{A.5})$$

## B Appendix: Switching the order of integration in the transform pairs

For the new transform pair, the order of integration may be switched if

$$\int_{L_1} \int_{\partial D} \left| \frac{\sqrt{\Phi'(\zeta)}}{\Phi(\zeta)^{k+1}} \frac{f(\zeta)}{1 - e^{2\pi ik}} \Phi(z)^k \right| d|k| d|\zeta| < \infty, \quad (\text{B.1})$$

where  $d|\zeta|$  and  $d|k|$  are the arc-length measures. We also need the corresponding inequalities to hold for  $L_2$  and  $L_3$  in the transform pair to switch the order of integration. Further, we only need to check that the modulus of the integrand decays sufficiently as  $|k| \rightarrow \infty$ . Hence it suffices to check the integrals above are finite with  $L_j$  replaced with  $L_j^+ := L_j \cap \{|k| > r\}$  for  $1 \leq j \leq 3$ . Let  $\text{Arg}(\zeta)$  denote the principle argument of  $\zeta$ . Choose a  $2\pi$ -width interval for  $\text{Arg}(\zeta)$ , say  $0 \leq \text{Arg}(\zeta) < 2\pi$  or  $-\pi < \text{Arg}(\zeta) \leq \pi$ . The complex exponential is multi-valued, in particular when  $|w| = 1$ , we have that

$$w^{2k+1} = e^{2ik(\text{Arg}(w)+2\pi m)}, \quad m \in \mathbb{Z}.$$

Choose  $m = 0$ . Further, recall that  $k$  is assumed to be on the imaginary axis,  $k = ix$  with  $x$  real. Denote  $\theta := \text{Arg}(w)$ . Then we have

$$\left| \frac{1}{w^{2(k+1)}} \right| = \left| \frac{1}{e^{2i(ix)\text{Arg}(w)}} \right| = |e^{2x\text{Arg}(w)}| = e^{2x\theta} \quad (\text{B.2})$$

Applying a change of variables, we find:

$$\int_{\partial D} \frac{\sqrt{\Phi'(\zeta)} f(\zeta)}{\Phi(\zeta)^{k+1}} d\zeta = \int_{\partial \mathbb{D}} \frac{f(\Psi(w))}{w^{k+1}} \sqrt{\Psi'(w)} dw.$$

Given  $\Psi' \in H^1(\mathbb{D})$ , we may apply Holder's inequality to arrive at

$$\begin{aligned} \int_{\partial \mathbb{D}} \left| \frac{f(\Psi(w))}{w^{k+1}} \sqrt{\Psi'(w)} \right| |dw| &\leq \sqrt{\int_{\partial \mathbb{D}} \frac{f^2(\Psi(w))}{w^{2(k+1)}} |dw|} \sqrt{\int_{\partial \mathbb{D}} |\Psi'(w)| |dw|} \\ &\leq M \sqrt{\int_{\partial \mathbb{D}} \left| \frac{1}{w^{2(k+1)}} \right| |dw|} = M \sqrt{\left[ \frac{1}{2x} e^{2\theta x} \right]_a^b}. \end{aligned} \quad (\text{B.3})$$

Let  $a \leq \text{Arg}(w) = \theta < b$  where  $b - a = 2\pi$ . If  $a = -\pi$  and  $b = \pi$ , then

$$\oint_{\partial \mathbb{D}} \left| \frac{f(\Psi(w))}{w^{k+1}} \sqrt{\Psi'(w)} \right| |dw| \leq \frac{M}{\sqrt{2|x|}} \frac{\sqrt{|1 - e^{-4\pi x}|}}{e^{-\pi x}}. \quad (\text{B.4})$$

On  $L_1^+$ , we have  $k = ix$  with  $x < -r$ . Then

$$\begin{aligned} \int_{L_1^+} \int_{\partial D} \left| \frac{\Phi(z)^k}{1 - e^{2i\pi k}} \frac{\sqrt{\Phi'(\zeta)} f(\zeta)}{\Phi(\zeta)^{k+1}} \right| |d\zeta| |dk| &= \int_{L_1^+} \left| \frac{\rho(k)}{1 - e^{2\pi i k}} \Phi(z)^k \right| |dk| \\ &\leq \int_{-\infty}^{-r} \left| \frac{\Phi(z)^{ix}}{1 - e^{-2\pi x}} \left[ \frac{M}{\sqrt{2|x|}} \frac{\sqrt{|1 - e^{-4\pi x}|}}{e^{-\pi x}} \right] \right| dx \\ &= \int_{-\infty}^{-r} \frac{M}{\sqrt{2|x|}} \left| \frac{\sqrt{|1 - e^{-4\pi x}|}}{1 - e^{-2\pi x}} \right| e^{x(\pi - \arg(\Phi(z)))} dx. \end{aligned} \quad (\text{B.5})$$

For the integral at the end of (B.5) to be finite, we need

$$x(\pi - \arg(\Phi(z))) < 0.$$

Given that  $x < 0$ , this holds when  $\arg(\Phi(z)) < \pi$ . On  $L_3^+ = L_3 \cap \{|k| > r\}$ , we have  $k = ix$  with  $x \geq r$ . Then

$$\begin{aligned} \int_{L_3^+} \int_{\partial D} \left| \frac{\Phi(z)^k e^{2\pi i k}}{1 - e^{2i\pi k}} \frac{\sqrt{\Phi'(\zeta)} f(\zeta)}{\Phi(\zeta)^{k+1}} \right| |d\zeta| |dk| &= \int_{L_3^+} \left| \frac{\rho(k)}{1 - e^{2\pi i k}} e^{2\pi i k} \Phi(z)^k \right| |dk| \\ &\leq \int_r^\infty \left| \frac{\Phi(z)^{ix} e^{-2\pi x}}{1 - e^{-2\pi x}} \left[ \frac{M}{\sqrt{2|x|}} \frac{\sqrt{|1 - e^{-4\pi x}|}}{e^{-\pi x}} \right] \right| dx \\ &= \int_r^\infty \frac{M}{\sqrt{2|x|}} \left| \frac{\sqrt{|1 - e^{-4\pi x}|}}{1 - e^{-2\pi x}} \right| e^{-x(\pi + \arg(\Phi(z)))} dx. \end{aligned} \quad (\text{B.6})$$

In order for the integral at the end of (B.6) to be finite, we need

$$-x(\pi + \arg(\Phi(z))) < 0.$$

Given that  $x > 0$ , this holds when  $\arg(\Phi(z)) > -\pi$ . If we choose  $-\pi < \arg(\Phi(z)) < \pi$ , then both integrals will be finite and one may switch the order of integration. If instead  $0 \leq \text{Arg}(\zeta) < 2\pi$ , then repeating this process shows that the order of integration may be switched when  $0 < \arg(\Phi(z)) < 2\pi$ .

Switching the order of integration on  $L_2$  is more straightforward. Indeed, consider  $k \in L_2$ ,  $k > 0$ . Since  $k$  is real, then  $|w^{k+1}| = 1$  for  $|w| = 1$ . Given that  $f$  is continuous on  $\overline{D}$  and  $\Psi' \in H^1(\mathbb{D})$ , then

$$\begin{aligned} \oint_{\partial\mathbb{D}} \left| \frac{f(\Psi(w))}{w^{k+1}} \sqrt{\Psi'(w)} \right| |dw| &= \oint_{\partial\mathbb{D}} \left| f(\Psi(w)) \sqrt{\Psi'(w)} \right| |dw| \\ &\leq \sqrt{\int_{\partial\mathbb{D}} |f^2(\Psi(w))| |dw|} \sqrt{\int_{\partial\mathbb{D}} |\Psi'(w)| |dw|} = M < \infty. \end{aligned} \tag{B.7}$$

Next, observe

$$|\Phi(z)^k| = |e^{k \ln |\Phi(z)|} e^{ki \arg(\Phi(z))}| = e^{k \ln |\Phi(z)|}. \tag{B.8}$$

Thus

$$\int_{L_2} |\Phi(z)^k \rho(k)| dk \leq M \int_{L_2} |z^k| |dk| = M \int_r^\infty e^{k \ln |\Phi(z)|} dk. \tag{B.9}$$

Given  $|\Phi(z)| < 1$ , then  $\ln |\Phi(z)| < 0$  and the integral above is finite. Thus by choosing the  $2\pi$ -width interval for  $\text{Arg}(w)$  appropriately, one may switch the order of integration in the new transform pair.

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