

INDUCING RECURRENT FLOWS BY TWISTING ON INFINITE SURFACES WITH UNBOUNDED CUFFS

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ABSTRACT. A Riemann surface X is parabolic if and only if the geodesic flow (for the hyperbolic metric) on the unit tangent bundle of X is ergodic. Consider a Riemann surface X with a single topological end and a sequence α_n of pairwise disjoint, simple closed geodesics converging to the end, called *cuffs*. Basmajian, the first and the third author, proved that when the lengths $\ell(\alpha_n)$ of cuffs are at most $2 \log n$, the surface X is parabolic. One could expect that having arbitrary large cuff lengths $\ell(\alpha_n)$ (think of $\ell(\alpha_n) = n!^{n!}$) would allow the geodesic flow to escape to infinity, thus making X not parabolic.

Contrary to this and motivated by their proof of the Surface Subgroup Theorem, Kahn and Marković conjectured that for every choice of lengths $\ell(\alpha_n)$, there is a choice of twists that would make X parabolic. We show that their conjecture is essentially true. Namely, for any sequence of positive numbers $\{a_n\}$, there is a choice of lengths $\ell(\alpha_n) \geq a_n$ such that the (relative) twists by $1/2$ make X parabolic. This result extends to the surfaces with countably many ends while it does not hold for uncountably many ends.

1. INTRODUCTION

A Riemann surface X is *parabolic* if it does not support a Green's function, which we denote by $X \in O_G$. Then, $X \in O_G$ if and only if the geodesic flow on the unit tangent bundle T^1X is ergodic if and only if the Brownian motion on X recurrent if and only if almost every geodesic on X is recurrent, see [Hop71, Nic80, Sul81, Tsu75]. For an additional (yet partial) list of equivalent conditions to $X \in O_G$ see [BHŠ22, Introduction] as well as [Bis01, Nev50, Nic89, FM01, AZ90, Sul81].

In this paper we study the *type problem for infinite Riemann surfaces*, that is we would like to determine whether a Riemann surface obtained by an explicit construction is parabolic or not. For us the Riemann surface X is equipped with the hyperbolic metric, that is $X = \mathbb{H}/\Gamma$ where Γ is a Fuchsian group. We say that X is *infinite* if Γ is not finitely generated. We assume that Γ is of the first kind, i.e. the limit set $\Lambda(\Gamma) = \mathbb{S}^1$, since otherwise X is easily seen to be not parabolic.

It is known that X can be obtained by gluing countably many geodesic pairs of pants via isometries along their cuffs (i.e. boundary geodesics of pairs of pants), see [AR04] and also [Bas93, BŠ19, PRT12]. The geodesic pairs of pants are uniquely determined by the cuff lengths, and the isometric gluings along two cuffs are determined by a real parameter called the *twist*. The *Fenchel-Nielsen parameters* of X are the pair of sequences $(\{\ell_n\}, \{t_n\})$, where ℓ_n is the length and t_n is the twist of the n -th cuff, see Figure 1 for an example. Therefore, the set of all hyperbolic

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metrics on a topological surface X (modulo isometries and markings) corresponds to all choices of the Fenchel-Nielsen parameters.

The type problem for Riemann surfaces has been studied extensively by many authors when the Riemann surfaces were naturally defined by either gluing construction along the slits or by covering maps or other constructions motivated by complex analysis considerations, see e.g. [AS60, Doy84, LS84, Mil77, Mer08]. In [BHŠ22] the question of deciding when $X \in O_G$ from the data of its Fenchel-Nielsen parameters $(\{\ell_n\}, \{t_n\})$ was studied. A particularly interesting class of examples considered in [BHŠ22] are the Riemann surfaces with one infinite topological end: the (tight) flute surfaces and Loch-Ness monsters.

1.1. Tight flute surfaces. A Riemann surface X is said to be a (*tight*) *flute surface*, if it has countably many punctures that accumulate to a single non-isolated topological end. We will fix the pants decomposition of X with cuffs $\{\alpha_n\}_{n=1}^\infty$ as in Figure 1. Let ℓ_n and t_n be the length and twist parameters of α_n . The twist parameter t_n , for $-1/2 \leq t_n \leq 1/2$, represents the (oriented) relative length of the arc between the feet of γ'_n and γ''_{n-1} along α_n , see Figure 1. By [BHŠ22, Theorem 9.1], a flute (Riemann) surface $X(\{\ell_n\}, \{t_n\})$ is parabolic if

$$\ell_n \leq 2 \log n$$

for all but finitely many n , for any choice of the twists $\{t_n\}$.

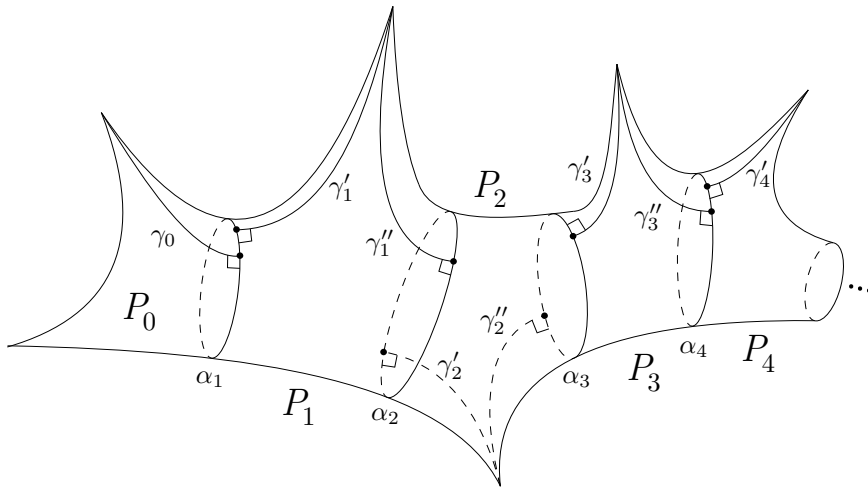


FIGURE 1. A flute surface

Intuitively, the geodesic flow on the unit tangent bundle $T^1 X$ of X is not ergodic if there is a “lot of space” on X for the geodesics to escape towards the end space $\partial_\infty X$. A lot of space could mean that the size of the “openings” ℓ_n converges to ∞ very fast. To support this intuitive reasoning, consider the *zero-twist* flute surface $X = X(\{\ell_n\}, \{0\})$, i.e. $t_n = 0$ for all n . By Theorem 9.4 of [BHŠ22], $X(\{\ell_n\}, \{0\})$ is not parabolic if,

$$\ell_n \geq p \log n,$$

for a fixed $p > 2$ and for all but finitely many n .

The *half-twist* flute surface $X(\{\ell_n\}, \{t_n\})$ is defined by the condition $t_n = 1/2$ for all n . Basmajian, the first and the third author [BHŠ22, Theorem 9.7] established that a half-twist flute surface $X(\{\ell_n\}, \{1/2\})$ with *increasing* and *concave* lengths of cuffs is not parabolic if (and only if) $\ell_n \geq p \log n$ for all but finitely many n and for a fixed $p > 4$. This further supports the intuitive reasoning of large cuffs preventing the geodesic flow from being ergodic. On the other hand, the second and the third author [PŠ23] showed that the condition that the lengths are concave *cannot be removed*. Namely, [PŠ23] finds a class of half-twist flutes that are parabolic with

$$q \log n \geq \ell_n \geq p \log n$$

for any $q > p > 0$.

In two separate discussions with Jeremy Kahn and Vladimir Marković, the following conjecture was made, see also Question 1.9(3) in [BHŠ22].

Conjecture 1.1 (Kahn-Marković). *Given a sequence $\{\ell_n\}$ of non-decreasing positive numbers (possibly $\lim_{n \rightarrow \infty} \ell_n = \infty$), there always exists a choice of twists $\{t_n\}$ such that the geodesic flow on the unit tangent bundle of the flute surface $X(\{\ell_n\}, \{t_n\})$ is ergodic, i.e. $X(\{\ell_n\}, \{t_n\}) \in O_G$.*

The conjecture proposes that the influence of the twists is strong enough to make a flute surface parabolic, even if the lengths of cuffs increase arbitrarily fast. This conjecture was motivated by the proof of the Surface Subgroup Theorem [KM12]. Prior to the conjecture, Basmajian and the third author [BŠ19] showed that for any sequence of lengths $\{\ell_n\}$ there is a choice of twists $\{t_n\}$ such that $X(\{\ell_n\}, \{t_n\})$ has covering group of the first kind (which is a necessary but not sufficient condition for parabolicity).

In this paper, we prove Conjecture 1.1 under the additional assumption that the cuff lengths can be taken larger than the assigned lengths $\{\ell_n\}$, and all the twists $\{t_n\}$ are equal to $1/2$. The geometry of X when the lengths converge to infinity is very sensitive to the twists. In fact, uncountably many conditions on the lengths and twists need to hold only to guarantee that the covering group Γ is of the first kind (see [Šar10, Theorem C]). When all the twists are $1/2$, the surface is symmetric, and the uncountably many conditions are reduced to a single condition, which allows us to employ hyperbolic geometry estimates. Surprisingly, there are choices of arbitrarily large cuff lengths that make the covering groups of the *half-twist flutes* of the first kind, which, again, by symmetry, is equivalent to the flutes being parabolic (see Theorem 4.1 and its proof, and Corollary 4.2).

Theorem 1.2. *For every non-decreasing sequence ℓ_n there is a sequence $\ell'_n \geq \ell_n$ such that the half-twist flute surface $X(\{\ell'_n\}, \{1/2\})$ is parabolic.*

The proof of Theorem 1.2 relies on rather precise estimates on the convergence of a nested sequence of geodesics in \mathbb{H} using a characterization in terms of shears from [Šar10, Theorem C] (see also [PŠ23, Proposition A.1.]). We show that ℓ'_n can, in fact, be taken arbitrarily larger than ℓ_n and still satisfy the conclusion of the theorem. We also prove that only an infinite set S of twists needs to be $1/2$ with the rest of twists being 0 in order to obtain the same result (see Corollary 4.1). Note that S can be an extremely sparse subset of \mathbb{N} (like $n_k = k!^{k!}$).

Theorem 1.3. *Let $\{\ell_n\}_{n=1}^\infty$ be a non-decreasing sequence of positive numbers with $\lim_{n \rightarrow \infty} \ell_n = \infty$. Then for every sequence of twists $t_n \in \{0, 1/2\}$ such that $\#\{n :$*

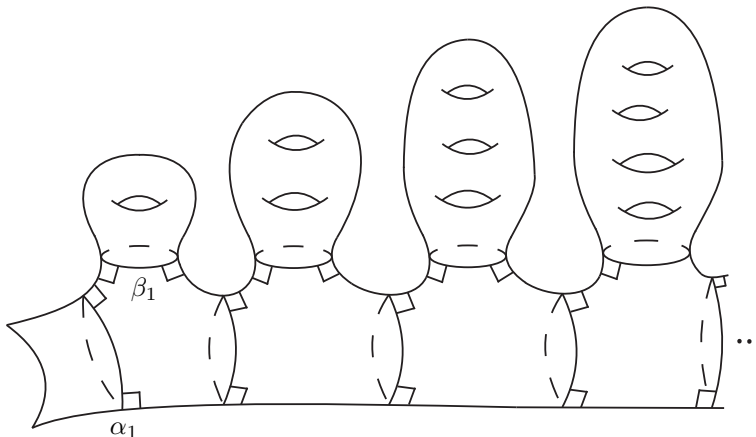


FIGURE 2. The infinite Loch-Ness monster surface

$t_n = 1/2\} = \infty$ there exist non-decreasing sequences ℓ'_n and ℓ''_n satisfying $\ell''_n \leq \ell_n \leq \ell'_n$ such that the surfaces $X(\{\ell'_n\}, \{t_n\})$ and $X(\{\ell''_n\}, \{t_n\})$ are parabolic.

The construction of the sequences $\{\ell'_n\}$ and $\{\ell''_n\}$ is such that it specifically works for surfaces with zero or half twists. There is a delicate balance between the lengths of consecutive cuffs in order to achieve parabolicity. In fact, there are surfaces $X(\{\ell_n\}, \{1/2\})$ and $X(\{\ell'_n\}, \{1/2\})$ with $\lim_{n \rightarrow \infty} (\ell_n - \ell'_n) = 0$ such that the first surface is parabolic and the second is not.

1.2. Non-planar surfaces. Flute surfaces are planar, but the above results hold for (non-planar) surfaces with (infinite) genus. Indeed, let X be an infinite genus surface with a single topological end, called the *infinite Loch-Ness monster surface*. Let α_n be the disjoint cuffs converging to the end such that the part of X between α_n and α_{n+1} is homeomorphic to a torus (or a genus g surface) minus two disks. Let β_n be the geodesic which cuts off the genus between α_n and α_{n+1} . Even more generally, we will allow that β_n cuts off a higher genus but finite surface between α_n and α_{n+1} (see Figure 2 and Corollary 5.2).

Theorem 1.4. *Let X be an infinite Loch-Ness monster surface with cuffs α_n converging to the topological end and cuffs β_n that cut off the genus between α_n and α_{n+1} . We assume that $\ell(\beta_n) \leq M < \infty$ for all n . Given a sequence a_n of positive numbers, there exists a choice of lengths $\ell(\alpha_n) \geq a_n$ such that the infinite Loch-Ness monster surface X with half-twists and chosen lengths is parabolic. Moreover, the lengths can be chosen such that $\ell(\alpha_n) = a_n$ except on an infinite subsequence of \mathbb{N} where the lengths could be larger.*

The above theorem is true for a surface obtained by attaching finitely many infinite Loch-Ness monsters and/or infinite flute surfaces to a finite area bordered surface (see Figure 3). In fact, if a surface X has countably many ends \mathcal{E} then we associate a bordered subsurface X_{e_j} for each end e_j that is not a puncture. The subsurface X_{e_j} accumulates only to e and has countably many closed geodesics on its border. Other subsurfaces corresponding to other ends are attached to these border geodesics. Each X_{e_j} is either a flute surface, a Loch-Ness monster surface,

or a Loch-Ness monster with truncated genus and some punctures in the place of the genus (see §2). Denote by $\alpha_{j,n}$ the cuffs in X_{e_j} that accumulate to e_j .

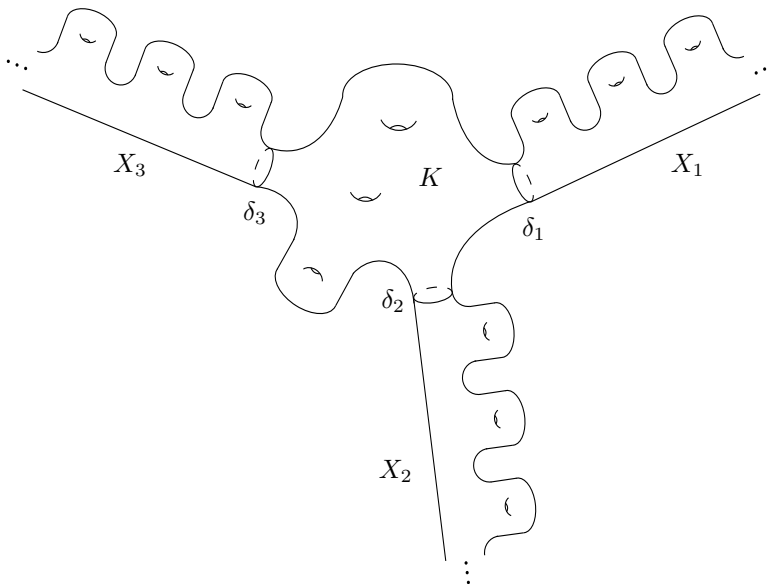


FIGURE 3. A surface with finitely many non-planar ends.

We prove a reduction theorem

Theorem 1.5. *Consider a Riemann surface X with countably many ends \mathcal{E} . Then $X \in O_G$ if and only if, for each $e_j \in \mathcal{E}$, the modulus of the curve family in X_{e_j} that connects $\alpha_{j,1}$ to e_j is zero.*

Using the above reduction theorem, we can extend the examples from single-end surfaces, such as the infinite flute surfaces and the infinite Loch-Ness monster surfaces, to all surfaces with, at most, countably many ends (see Corollary 5.4).

Theorem 1.6. *Let X be a topological surface with countably many ends \mathcal{E} . Given a double sequence $\{a_{j,n}\}$ of positive numbers increasing in n for each fixed j , there exists a Riemann surface structure Y on X such that*

$$\ell(\alpha_{j,n}) \geq a_{j,n}$$

and

$$Y \in O_G.$$

Finally, we remark that surfaces with countably many ends are the largest class of surface for which the above theorem can hold. Indeed, by [Pan24] (see also [Šar24]), if X is a Riemann surface with a Cantor set of ends and $\alpha_{j,n}$, $n = 1, \dots, 2^{j+1}$, are cuffs at level n that, for fixed $r > 1$, satisfy

$$\frac{1}{n^2} \gtrsim \ell(\alpha_{j,n}) \gtrsim \frac{j^r}{2^j}$$

then $X \notin O_G$.

1.3. Overview and open questions. One of the main problems we are interested in is to find (hopefully necessary and sufficient) conditions on the Fenchel-Nielsen parameters of a flute surface which would imply parabolicity. The following table summarizes some of the recent results from [BHŠ22, PŠ23] and the current paper in the case of symmetric surfaces.

Twists	Lengths	Conclusion	Reference
$t_n = 0$	no restrictions	$X \in O_G \iff \sum e^{-\ell_n/2} = \infty$	[BHŠ22]
$t_n = 1/2$	no restrictions	$X \in O_G \iff \sum e^{-\sigma_n/2} = \infty$	[PŠ23]
$t_n = 1/2$	ℓ_n is concave	$X \in O_G \iff \sum e^{-\ell_n/4} = \infty$	[BHŠ22]
$t_n \in \{0, 1/2\}$	no restrictions	$X \in O_G \iff \sum e^{-\tilde{\sigma}_n/2} = \infty$	Thm 4.1

TABLE 1. Conditions for parabolicity of tight flutes.

Thus, a natural question would be the following.

Problem 1.7. *Suppose $X = X(\{\ell_i\}, \{t_i\})$ is a symmetric tight flute surface, i.e., $t_i \in \{0, 1/2\}$. Find necessary and sufficient conditions for $X \in O_G$. What if ℓ_n is concave?*

Below are the results related to the Kahn-Marković conjecture and the effect of twisting (i.e., careful choice of twists) on the type of the surface X .

Twists	Lengths	Conclusion	Reference
$t'_n \in ??$	$\forall \{\ell_n\}$	$X \in O_G \stackrel{??}{\iff} (\ell_n, t_n) \rightsquigarrow (\ell_n, t'_n)$	Conj. 1.1
$t'_n \in \{0, 1/2\}$	$\ell_{n_k} = \ell_{n_k+1}$	$X \in O_G \iff (\ell_n, t_n) \rightsquigarrow (\ell_n, t'_n)$	Cor. 4.2
$t'_n \in \{0, 1/2\}$	$\ell'_n \geq \ell_n$	$X \in O_G \iff (\ell_n, t_n) \rightsquigarrow (\ell'_n, t'_n)$	Thm. 1.2
$t'_n \in \{0, 1/2\}$	$\ell'_n = \ell_n$	$X \in O_G \stackrel{??}{\iff} (\ell_n, t_n) \rightsquigarrow (\ell_n, t'_n)$	Conj. 1.8

TABLE 2. Parabolicity through twisting.

As the second and third lines in the table above suggest one may suspect that the twists can in fact be taken to be in $\{0, 1/2\}$ to guarantee that X is parabolic. This can be thought of as a strong version of the Kahn-Marković conjecture.

Conjecture 1.8 (Basmajian, Hakobyan, Pandazis, Šarić). *For every non-decreasing sequence of lengths $\{\ell_n\}$, there is a choice of twists $t_n \in \{0, 1/2\}$ s.t. $X(\{\ell_n\}, \{t_n\})$ is parabolic.*

Finally, it would be interesting to find conditions on the Fenchel Nielsen parameters which would guarantee that X is complete but is *not parabolic*. That such flute surfaces exist was shown in [Kin11], but no explicit necessary or sufficient condition in terms of the length or twist parameters is known.

2. PRELIMINARIES

A Riemann surface X will always be identified with \mathbb{H}/Γ , where \mathbb{H} is the upper half-plane and Γ is a Fuchsian group. The conformal hyperbolic metric (i.e. the metric of constant curvature -1) is induced by the hyperbolic metric on \mathbb{H} since Γ acts by isometries on \mathbb{H} . The Riemann surface X is said to be *infinite* if Γ is not finitely generated.

A *geodesic pair of pants* is a bordered hyperbolic surface whose interior is homeomorphic to a sphere minus three closed disks and whose boundary components are either simple closed geodesics, called *cuffs*, or punctures. We will assume that at least one boundary component is a cuff. Two geodesic pairs of pants P_1 and P_2 with two cuffs $\alpha_1 \subset \partial P_1$ and $\alpha_2 \subset \partial P_2$ of equal length can be glued by an isometry along the cuffs to form a more complicated hyperbolic surface (that is homeomorphic to \mathbb{S}^2 minus four closed disks). The choice of gluings is determined by a real parameter, called the *twist*. Namely, consider the unique orthogeodesics from α_1 to another boundary component of P_1 and from α_2 to another boundary component of P_2 . The signed distance between the feet $x_1 \in \alpha_1$ and $x_2 \in \alpha_2$ of the orthogeodesics along the identified cuffs $\alpha_1 \equiv \alpha_2$ divided by the common length $\ell_X(\alpha_1) = \ell_X(\alpha_2)$ is the (relative) twist $t(\alpha_1) \in [-1/2, 1/2]$, where the values $-1/2$ and $1/2$ represent the same gluing. Note that if a pair of pants has two cuffs of equal lengths, then they can be glued by an isometry to produce a bordered surface whose interior is homeomorphic to a torus minus a closed disk.

By taking countably many geodesic pairs of pants $\{P_n\}_n$ and gluing them by isometries along the cuffs of equal lengths, we obtain an infinite Riemann surface X . Let $\{\alpha_j\}_j$ be the family of the images of the cuffs in X . The hyperbolic metric (and, by extension, the complex structure) of X is uniquely determined by the lengths $\{\ell(\alpha_j)\}_j$ and the twists $\{t(\alpha_j)\}_j$. In general, the hyperbolic metric on X may be incomplete, and the natural completion is obtained by attaching the hyperbolic funnels to cuffs not identified with other cuffs and half-planes to bi-infinite simple geodesics which are in the completion of the union of the pairs of pants (see Basmajian [Bas93]). In fact, every topological pants decomposition of a Riemann surface can be straightened to a geodesic pants decomposition of the interior of the convex core of X , and the whole surface X is obtained by attaching the funnels and the geodesic half-planes to the boundary components of the convex core (see [AR04], [BS19]).

When we need to add either funnels or the half-planes, the Fuchsian group Γ is of the second kind, and the convex core of X is a proper subset of X . In the case when the convex core is the whole surfaces X , then Γ is of the first kind, and X is the union of countably many pairs of pants. We are mainly interested in the groups of the first kind. The pair of sequences $(\{\ell(\alpha_n) > 0\}, \{t(\alpha_n) \in (-1/2, 1/2]\})$ are called the Fenchel-Nielsen parameters and they determine X .

A Green's function on a Riemann surface X is a harmonic function $g : X \setminus \{z_0\} \rightarrow \mathbb{R}$ with $g(z) = -\log |z - z_0| + o(1)$ for $z \in X$ near the fixed point $z_0 \in X$ and $g(z) \rightarrow 0$ as z leaves every compact subset of X . A Riemann surface X is said to be *parabolic*,

simple ends, then again, we can find a sequence of cuffs that accumulate to the end such that between any two adjacent cuffs, we have a finite surface with genus and punctures. The first cuff cuts out a surface X_{e_3} , which is a Loch-Ness monster with punctures (see Figure 4).

If an isolated point $e_4 \in \mathcal{E}'$ is accumulated by a sequence of non-simple ends $e_4^k \in \mathcal{E} \setminus \mathcal{E}'$, then we cut out along the cuffs β_k the corresponding (Loch-Ness monster) surfaces of the ends e_4^k . We obtained a bordered surface X_{e_4} with one cuff cutting it off from X and with countably many boundary geodesics β_k . Again, there is a sequence of cuffs $\alpha_n \subset X_{e_4}$ accumulating to the end e_4 such that between any two adjacent cuffs, we have a finite subsurface (see Figure 4).

If an isolated end $e_5 \in (\mathcal{E}')'$ is accumulated by a sequence of ends e_5^k that are isolated in \mathcal{E}' , then we can cut off these ends along cuffs $\{\beta_k\}_k$ to obtain the surface X_{e_5} with a sequence of cuffs $\{\alpha_n\}_n$ converging to the end e_5 (see Figure 5).

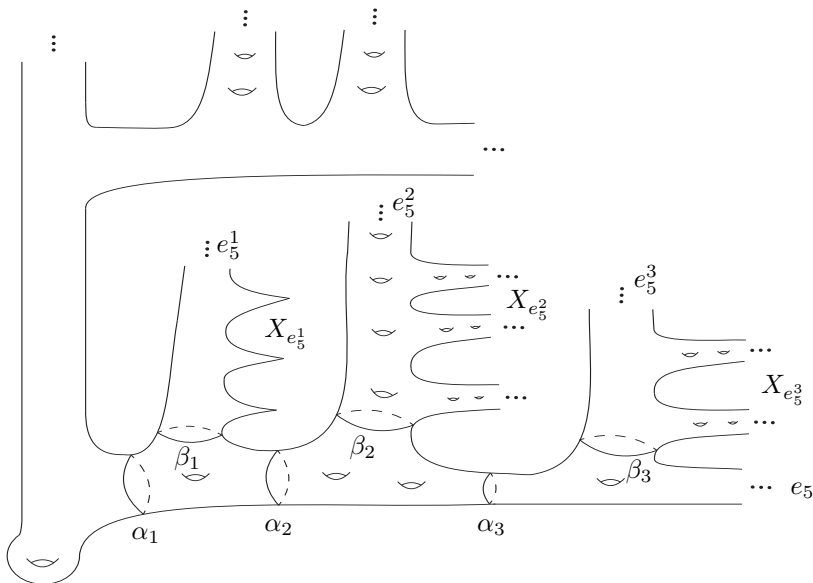


FIGURE 5. A decomposition into subsurfaces.

By continuing this process indefinitely we can make every end to be isolated at a certain stage (over countably many ordinals), see [Kec95, page 34].

Thus to any end e , there corresponds a subsurface X_e , which accumulates to e whose boundaries are cuffs that are used to cut off subsurfaces from the ends that are isolated before e and that has a sequence of disjoint cuffs accumulating to e as in Figures 4 and 5.

3. A REDUCTION FROM COUNTABLY MANY TO A SINGLE END

When X has countably many topological ends, we find a necessary and sufficient condition for $X \in O_G$ expressed in terms of one end at a time.

Let $\{X_n\}_{n=1}^\infty$ be an exhaustion of X by finite area bordered geodesic subsurfaces. Let Γ_n be the curve family in $X_n \setminus X_1$ that connects ∂X_1 and ∂X_n . Recall that $X \in O_G$ if and only if $\text{mod}(\Gamma_n) \rightarrow 0$ as $n \rightarrow \infty$ ([AS60]).

Definition 3.1. *Let Γ_∞ be the family of curves in $X \setminus X_1$ that starts at a single point of ∂X_1 on one side and accumulate to a unique topological end of X on the other side.*

We prove

Theorem 3.2. *Let X be any Riemann surface. Then*

$$X \in O_G \text{ if and only if } \text{mod}(\Gamma_\infty) = 0.$$

Proof. Assume that $X \in O_G$. Then $\text{mod}(\Gamma_n) \rightarrow 0$ as $n \rightarrow \infty$. Since each curve in Γ_n extends to a curve in Γ_∞ , it follows that

$$0 \leq \text{mod}(\Gamma_\infty) \leq \text{mod}(\Gamma_n).$$

By letting $n \rightarrow \infty$, we conclude that $\text{mod}(\Gamma_\infty) = 0$.

Assume that $\text{mod}(\Gamma_\infty) = 0$. If $X \notin O_G$, then there exists a non-constant harmonic function $u : X \setminus \bar{X}_1 \rightarrow \mathbb{R}$ whose boundary values are 0 on ∂X_1 and that satisfies $0 < u < 1$ on $X \setminus \bar{X}_1$ (see [AS60, page 204, Theorem IV.6C]). The function u is the limit of the solutions u_n to the Dirichlet problem in $X_n \setminus X_1$ with boundary values 0 on ∂X_1 and 1 on ∂X_n . If u^* is a local harmonic conjugate of u , then $[d(u + iu^*)]^2$ is a holomorphic quadratic differential on $X \setminus X_1$ whose horizontal trajectories are locally given by $u^* = \text{const}$ (see [Šar24, Theorem 4.2]). The quadratic differential $[d(u + iu^*)]^2$ is integrable since, in the natural parameter, the du -length of each $u^* = \text{const}$ is at most 1, and the du^* -length of ∂X_1 is finite since ∂X_1 is compact.

There is a positive measure set of non-singular horizontal trajectories that start at ∂X_1 and converge to the topological boundary of X in the other direction. Indeed, let e_1 and e_2 be two ends in the accumulation of a horizontal trajectory r . The two ends are separated by a finite set of simple closed (hyperbolic) geodesics on X , and the trajectory r will cross infinitely many times the (hyperbolic) collar neighborhood of a single closed geodesic. The $|d(u + iu^*)|$ -length of each arc of r that connects the two boundary components of the collar is bounded below. Therefore, the length of the trajectory r is infinite. Since there are countably many simple closed geodesics on X and $[d(u + iu^*)]^2$ is integrable, it follows that there can be, at most, a zero-measure set of such trajectories. Since the modulus of the above family of horizontal trajectories is positive (see [Šar24, Theorem 4.2]), it follows by the monotonicity of the modulus that $\text{mod}(\Gamma_\infty) > 0$. This is a contradiction. Thus $X \in O_G$. \square

For a Riemann surface X whose space of ends \mathcal{E} is countable, the above theorem can be used to express the parabolicity condition in terms of conditions on $\{X_e\}_{e \in \mathcal{E}'}$. We prove

Theorem 3.3. *Let X be a Riemann surface whose space of ends \mathcal{E} is countable. Let α_e be the boundary geodesic of the subsurface X_e (that cuts off X_e from the subsurfaces in the previous generations) corresponding to an end $e \in \mathcal{E}'$ and let Γ_e be the family of arcs in X_e connecting α_e to the end e . Then*

$$X \in O_G$$

if and only if

$$(1) \quad \text{mod}(\Gamma_e) = 0, \text{ for all } e \in \mathcal{E}'.$$

Proof. Assume that (1) holds. By Theorem 3.2, $X \in O_G$ if and only if $\text{mod}(\Gamma_\infty) = 0$. Recall that Γ_∞ consists of curves that have one endpoint in a compact subset of X and accumulate to a single topological end in the other direction. This implies that Γ_∞ overflows $\cup_{e \in \mathcal{E}'} \Gamma_e$. It follows that $\text{mod}(\Gamma_\infty) \leq \sum_{e \in \mathcal{E}'} \text{mod}(\Gamma_e) = 0$. Therefore $X \in O_G$.

Assume that $X \in O_G$. Let $e \in \mathcal{E}'$. Let K be a finite subsurface of X that contains cuff α_e that is on the boundary of X_e . Since the curve families $\Gamma_e \subset X_e$ is a subfamily of the curve family Γ_∞ that connects K with the topological ends, it follows that $\text{mod}(\Gamma_e) = 0$. \square

4. PARABOLIC FLUTE SURFACES WITH ARBITRARY LARGE CUFFS

In this section, X is a flute surface with cuffs $\{\alpha_n\}_{n=1}^\infty$ as in Figure 1. We establish that when the twists around α_n are all $1/2$ then the lengths ℓ_n of the cuffs α_n can be chosen to be larger than any prescribed sequence of positive numbers with $X \in O_G$. This answers the original question of Kahn and Marković of whether one can find flute surfaces with arbitrarily large cuffs.

First, we establish a sufficient condition for parabolicity for *arbitrary* flute surfaces with zero or half twists, generalizing the result for half-twist surfaces from [PS23]. We say that a flute surface is a *symmetric flute surface with infinitely many half-twists* if $t_i \in \{0, 1/2\}$ for $i \geq 1$, and $\#\{i : t_i = 1/2\} = \infty$ (see Figure 6). Note that for such a surface, there always exists an increasing sequence of positive integers $\{n_k\}$ such that

$$(2) \quad t_i = \begin{cases} 1/2, & \text{if } i = n_k, \\ 0, & \text{otherwise.} \end{cases}$$

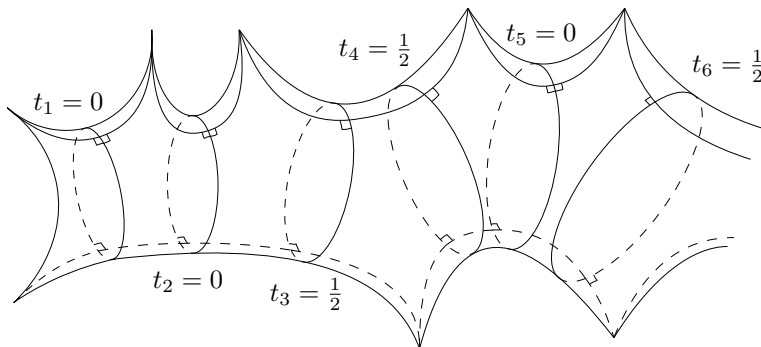


FIGURE 6. The half and zero twists flute surface.

Theorem 4.1. *Suppose $X = X(\{\ell_n\}\{t_n\})$ is a symmetric surface with infinitely many half twists as in (2). Then for every non-decreasing sequence ℓ_n we have that*

$X(\{\ell_n\}, \{t_n\}) \in O_G$, provided

$$(3) \quad \sum_{k=1}^{\infty} e^{-\sigma_k/2} = \infty,$$

where $\sigma_k = \ell_{n_k} - \ell_{n_{k-1}} + \dots + (-1)^{k-1} \ell_1$.

Proof. Let Δ_1 and Δ_2 be ideal geodesic triangles in the upper half-plane \mathbb{H} with disjoint interiors and the common boundary side geodesic g . We choose the orientation of the geodesic g such that Δ_1 is on its left side. Consider the orthogonal projections p_1 and p_2 of the third vertices of Δ_1 and Δ_2 to the common geodesic g . The *shear* $s(g)$ on the geodesic g of the pair (Δ_1, Δ_2) is the signed hyperbolic distance (with respect to the orientation of g) from p_1 to p_2 (see [Pen93], [Šar10]). Note that the shear $s(g)$ is independent of the orientation of g .

The symmetric flute surface X can be divided into the front and back sides by geodesics connecting the punctures (see Figure 6). The dividing geodesics partition each pair of pants (except the first one) into two isometric regions: the front and the back pentagons with four right angles and one zero angle. There is an orientation reversing isometry of X , which pointwise fixes the dividing geodesics and maps each front pentagon to the corresponding back pentagon. The front side X^* of X is planar, and we fix a single lift of the front side X^* to the universal covering \mathbb{H} . The lift \tilde{X}^* is an infinite ideal polygon (see Figure 7), and the covering map is a conformal map of the polygon onto X^* . Let $\{g_{2n-1}\}$ be the (infinite) geodesics in \mathbb{H} that are lifts of the cuffs α_n and intersect \tilde{X}^* , and $\{g_{2n}\}$ the geodesics which share the initial endpoint with g_{2n-1} and the terminal endpoint with g_{2n+1} .

By the front-to-back symmetry of X , it follows that the family of curves that connects a finite area subsurface of X with the non-simple topological end (in the case of flute surface, the non-simple topological end is accumulated by punctures) has zero modulus if and only if the curve family connecting a finite area sub-polygon of \tilde{X}^* to the boundary at infinity not corresponding to punctures has zero modulus. Indeed, the orientation-reversing conformal map of X , which maps X^* to the back side of X , transfers allowable metrics for X to allowable metrics for X^* with a multiplicative constant 2. Therefore, one family has zero modulus if and only if the other family has zero modulus. The conformal image \tilde{X}^* of X^* is an ideal polygon in \mathbb{H} with countably many ideal vertices corresponding to the punctures. The closure of \tilde{X}^* in $\mathbb{H} \cup \hat{\mathbb{R}}$ in addition to its vertices contains either a single point in $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ (which is the accumulation of the vertices) or an interval. If it is a point, then the modulus of the above family is zero, and X is parabolic. Hence, the covering group is of the first kind. If the closure contains an interval, then X is not parabolic, and the covering group is of the second kind. Thus, the symmetric flute surface is parabolic if and only if its covering group is of the first kind (for more details, see [PŠ23]).

Therefore, the only thing we need to prove is that the infinite polygon \tilde{X}^* accumulates to one point on $\hat{\mathbb{R}}$ in addition to its ideal vertices. The fronts of the geodesic boundaries α_n of X lift to the curves $\tilde{\alpha}_n$ in the universal cover \mathbb{H} . These lifts $\tilde{\alpha}_n$ lay on corresponding geodesics we call g_{2n-1} (see Figure 7). Then define the geodesics g_{2n} to connect the initial point of g_{2n-1} and the terminal point of g_{2n+1} . The result is a nested sequence of geodesics g_n such that g_n and g_{n+1} share

an endpoint and no three geodesics have a common endpoint. We compute the shears $s(g_n)$ for $n \geq 2$ of the geodesics g_n .

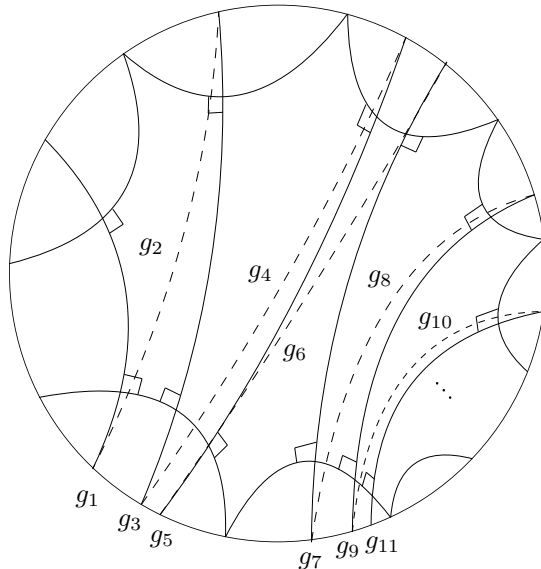


FIGURE 7. Isometric lift of the front of X to \mathbb{D} .

The orthogeodesic arc η_n from g_{2n-1} to g_{2n+1} is on the lift of the front side \tilde{X}^* (see Figure 7). The η_n , $\tilde{\alpha}_{2n-1}$, and $\tilde{\alpha}_{2n+1}$ make up three sides of a geodesic pentagon in \tilde{X}^* that has four right angles and one zero angle. The orthogeodesic ray from the vertex of any of these pentagons with a zero angle to η_n separates it into two Lambert quadrilaterals. For each such pentagon, the sides on g_{2n-1} and g_{2n+1} have lengths $\frac{\ell_n}{2}$ and $\frac{\ell_{n+1}}{2}$. Using a hyperbolic trigonometry formula for Lambert quadrilaterals [Bus10, page 38, Theorem 2.3.1(i)] gives

$$\ell(\eta_n) = \sinh^{-1} \left(\frac{1}{\sinh \frac{\ell_n}{2}} \right) + \sinh^{-1} \left(\frac{1}{\sinh \frac{\ell_{n+1}}{2}} \right).$$

It follows that for large n ,

$$(4) \quad e^{-\frac{\ell_{n+1}}{2}} \lesssim \ell(\eta_n) \lesssim e^{-\frac{\ell_n}{2}}.$$

We define the *cross-ratio* of a quadruple of points (a, b, c, d) in $\hat{\mathbb{R}}$ by

$$cr(a, b, c, d) = \frac{(b-a)(d-c)}{(b-c)(d-a)}.$$

If g is the geodesic with endpoints (a, c) , and if Δ_1 is the ideal triangle with vertices $\{a, c, d\}$ and Δ_2 is the ideal triangle with vertices $\{a, b, c\}$, then the shear along e with respect to Δ_1 and Δ_2 is (see [ŠWW24])

$$s(g) = \log cr(a, b, c, d).$$

Use a Möbius map to map the quadruple of points (a, b, c, d) onto $(-R, -r, r, R)$ for some $0 < r < R$. Then the distance ρ between the geodesic $|z| = r$ and $|z| = R$ is given by

$$\rho = \log \frac{R}{r}.$$

On the other hand, we have

$$e^{s(g)} = cr(-R, -r, r, R) = \frac{(R-r)^2}{4rR}.$$

The above gives

$$e^{s(g)} = \sinh^2 \frac{\rho}{2}.$$

Since $\eta_n \rightarrow 0$ as $n \rightarrow \infty$, it follows that $s(g_{2n}) < 0$ for large enough n and

$$(5) \quad s(g_{2n}) = \log \sinh^2 \frac{\ell(\eta_n)}{2}.$$

We orient all geodesics $\{g_n\}$ to the left as seen from g_1 . Consider g_{2n-1} that contains the lift of a cuff α_n such that $n_k = n$ for some even k and the corresponding quadrilateral as in Figure 8. Let A be the initial point of g_{2n-3} , and D the terminal point of g_{2n+1} . Call P the foot of the orthogedoesic from point A to the geodesic g_{2n-1} and S the foot of the orthogedoesic from point D to g_{2n-1} . Let $B \in g_{2n-3}$ and $Q \in g_{2n-1}$ be the endpoints of orthogedoesic η_{n-1} . Call $R \in g_{2n-1}$ and $C \in g_{2n+1}$ the endpoints of η_n (see Figure 8). Note that η_{n-1} and η_n belong to the boundary sides of the polygon \tilde{X}^* , and that the points (R, P, S, Q) appear in that order for the orientation of g_{2n-1} .

The choice of the half-twists and because the index of g_i has remainder 1 under division by 2 guarantees that the arc PS is contained in arc RQ (see Figures 7 and 8). Note that $s(g_{2n-1})$ is the signed distance from P to S for the orientation of g_{2n-1} . If P comes before S for the orientation of g_{2n-1} , then $s(g_{2n-1}) = \ell(PS)$ and

$$\ell(QR) = \ell(QP) + \ell(RS) - \ell(PS),$$

where $\ell(\cdot)$ is the positive distance. If P comes after S , then $s(g_{2n-1}) = -\ell(PS)$ and

$$\ell(QR) = \ell(QP) + \ell(RS) + \ell(PS).$$

By the above two equalities and $\ell(QR) = \ell_n/2$, we obtain

$$(6) \quad s(g_{2n-1}) = \sinh^{-1} \frac{1}{\sinh \ell(\eta_{n-1})} + \sinh^{-1} \frac{1}{\sinh \ell(\eta_n)} - \frac{\ell_n}{2}.$$

Consider g_{2n-1} that contains the lift of a cuff α_n such that $n_k = n$ for some odd k and the corresponding quadrilateral in Figure 9. Call the vertices of the quadrilateral not on geodesic g_{2n-1} , points A and D to the left and right, respectively. Let P be the foot of the orthogedoesic from point A to geodesic g_{2n-1} , and let S be the foot of the orthogedoesic from point D to the same geodesic. Call B and Q the endpoints of the orthogedoesic η_{n-1} from g_{2n-3} to g_{2n-1} . Then call R and C the endpoints of the orthogedoesic η_n from g_{2n-1} to g_{2n+1} (see Figure 9).

The choice of the half-twists and because the index of g_i has remainder 1 under division by 2 guarantees that the arc QR is contained in arc PS (see Figures 7 and 9). From Figure 9, analogous to the case above, we obtain

$$(7) \quad s(g_{2n-1}) = \sinh^{-1} \frac{1}{\sinh \ell(\eta_{n-1})} + \sinh^{-1} \frac{1}{\sinh \ell(\eta_n)} + \frac{\ell_n}{2}.$$

Consider g_{2n-1} that contains the lift of a cuff α_n such that $t_n = 0$ and the corresponding quadrilateral in Figure 10. Call the vertices of the quadrilateral not on geodesic g_{2n-1} , points A and D to the left and right, respectively. Let P be the foot of the orthogedoesic from point A to geodesic g_{2n-1} , and let R be the foot of

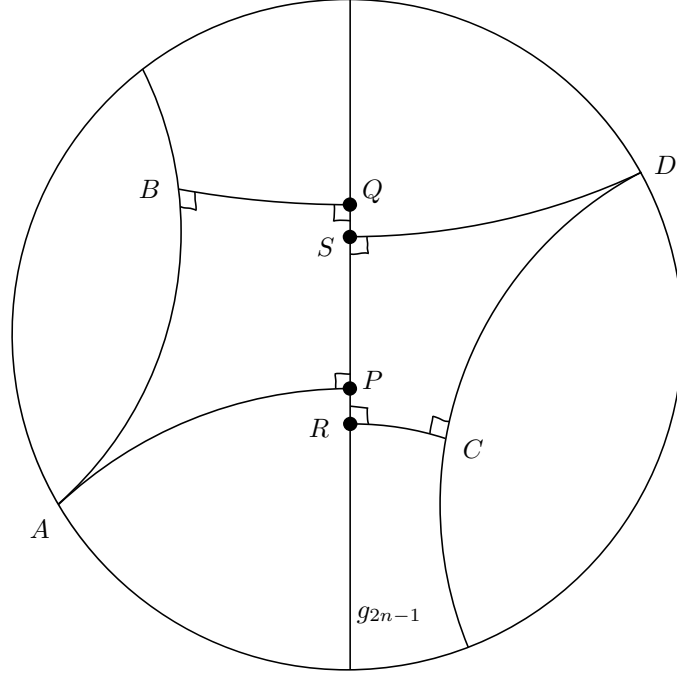


FIGURE 8. $s(g_{2n-1}) = \ell(PQ) + \ell(RS) - \frac{\ell_n}{2}$ if $t_n = \frac{1}{2}$ and $n_k = n$ for some even k .

the orthogeodesic from point D to the same geodesic. Call B and Q the endpoints of the orthogeodesic η_{n-1} from g_{2n-3} to g_{2n-1} . Then call C the endpoint on g_{2n+1} of the orthogeodesic η_n from g_{2n-1} to g_{2n+1} (see Figure 10).

From Figure 10 we obtain

$$(8) \quad s(g_{2n-1}) = \sinh^{-1} \frac{1}{\sinh \ell(\eta_{n-1})} + \sinh^{-1} \frac{1}{\sinh \ell(\eta_n)}.$$

It will be enough to prove that the sequence of nested geodesics $\{g_n\}_{n=1}^{\infty}$ (see Figure 7) does not accumulate in \mathbb{H} . It is immediate that $\sum_{n=1}^{\infty} \ell(\eta_n) = \infty$ implies that \tilde{X}^* has only one point of accumulation on $\hat{\mathbb{R}}$ in addition to its vertices. Therefore we assume that $\sum_{n=1}^{\infty} \ell(\eta_n) < \infty$ in the rest of the proof. This implies that

$$(9) \quad 1 \leq \prod_{n=1}^{\infty} (1 + \ell(\eta_n)) < \infty.$$

By [Šar10], the sequence $\{g_n\}_{n=1}^{\infty}$ does not accumulate in \mathbb{H} if and only if the piecewise horocyclic path obtained by concatenating the horocyclic arcs between the adjacent geodesics g_{n-1} and g_n has infinite length (see also [PŠ23, Proposition A.1]). Denote by $s_n = s(g_n)$ the shear of g_n with respect to the ideal quadrilateral whose vertices are the ideal endpoints of g_{n-1} and g_{n+1} for $n \geq 2$. We do not define the shear of g_1 . We start the piecewise horocyclic path on g_1 such that the part in the wedge between g_1 and g_2 has length $1/e^{s_1}$. By [Šar10] and [PŠ23, Proposition A.3], the length of the part of the piecewise horocyclic path h_n between g_n and

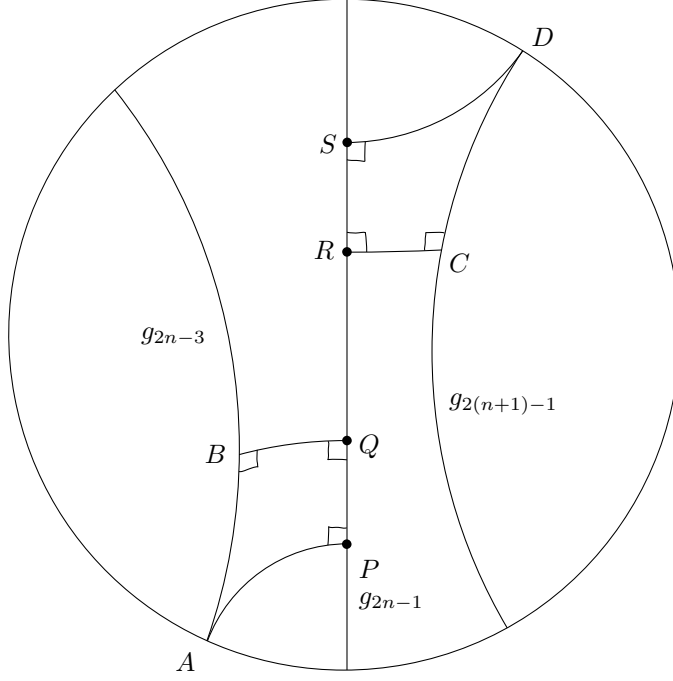


FIGURE 9. $s(g_{2n-1}) = \sinh^{-1} \frac{1}{\sinh \ell(\eta_{n-1})} + \sinh^{-1} \frac{1}{\sinh \ell(\eta_n)} + \frac{\ell_n}{2}$ if $t_n = \frac{1}{2}$ and $n_k = n$ for some odd k .

g_{n+1} , denoted by $\ell(h_n)$, is

$$(10) \quad \ell(h_n) = \begin{cases} e^{-s_1 - s_2 - \dots - s_n}, & \text{if } n \text{ is odd,} \\ e^{s_1 + s_2 + \dots + s_n}, & \text{if } n \text{ is even.} \end{cases}$$

We will use, for $x > 0$, the inequalities $e^{\sinh^{-1} \frac{1}{\sinh x}} > \frac{2}{x}$ and $\sinh x > x$ to get a lower estimate of the length of the piecewise horocyclic path $\ell(h)$.

We first estimate $\ell(h_{2n})$. By the above, we have

$$\ell(h_{2n}) = e^{s_{2n} + \dots + s_1}$$

and we partition the sum of the first $2n$ shears into n consecutive groups $s_{2j} + s_{2j-1}$ for $j = 1, \dots, n$. For $x > 0$, we have the inequalities $e^{\sinh^{-1} \frac{1}{\sinh x}} > \frac{2}{x}$ and $\sinh x > x$. Together with (5)–(8), this gives

$$e^{s_{2n}} = \sinh^2 \frac{\ell(\eta_n)}{2} > \frac{[\ell(\eta_n)]^2}{4}$$

and

$$e^{s_{2n-1}} > \frac{2}{\ell(\eta_{n-1})} \frac{2}{\ell(\eta_n)} e^{a_n},$$

where

$$a_n = \begin{cases} 0, & \text{if } n \neq n_k \\ \ell_n/2, & \text{if } n = n_k, k \text{ odd} \\ -\ell_n/2, & \text{if } n = n_k, k \text{ even} \end{cases}$$

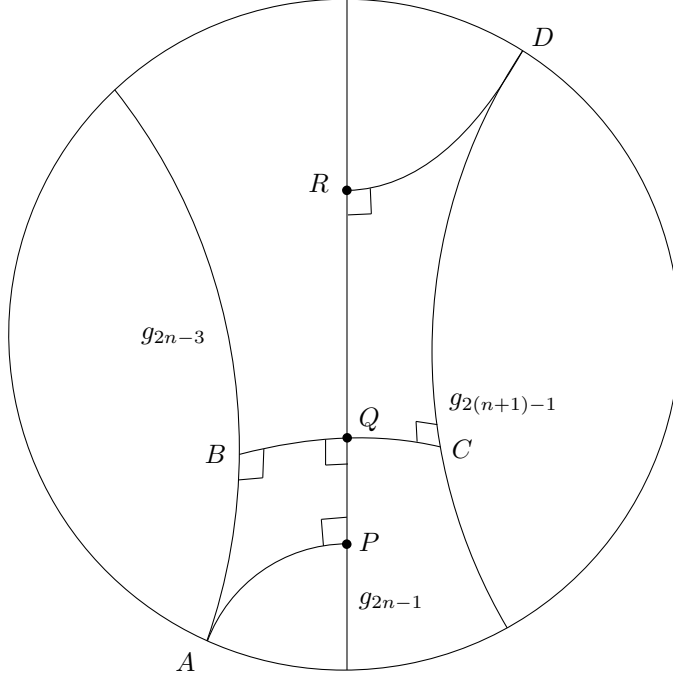


FIGURE 10. $s(g_{2n-1}) = \sinh^{-1} \frac{1}{\sinh \ell(\eta_{n-1})} + \sinh^{-1} \frac{1}{\sinh \ell(\eta_n)}$ if $t_n = 0$.

Then we have

$$e^{s_{2n}+s_{2n-1}} > \frac{\ell(\eta_n)}{\ell(\eta_{n-1})} e^{a_n}$$

which gives

$$e^{s_{2n}+\dots+s_1} > \frac{\ell(\eta_n)}{\ell(\eta_1)} e^{a_n+\dots+a_1}.$$

By summing over all $n = n_k$ with k odd and using the inequality $\ell(\eta_n) > e^{-\frac{\ell_n}{2}}$, we get

$$(11) \quad \sum_{\text{odd } k} e^{s_{2n_k}+\dots+s_1} > C \sum_{\text{odd } k} e^{-\frac{\sigma_{k-1}}{2}}.$$

By (5)–(8), the inequalities $e^{-\sinh^{-1} \frac{1}{\sinh x}} > \frac{x}{5}$ and $\frac{e^{-\sinh^{-1} \frac{1}{\sinh x}}}{\sinh \frac{x}{2}} > \frac{1}{1+x}$ for small $x > 0$, and $\ell(\eta_{n_k}) > e^{-\frac{\ell_{n_k+1}}{2}} \geq e^{-\frac{\ell_{n_k+1}}{2}}$ using an argument similar to the above, we obtain for even k

$$(12) \quad \sum_{\text{even } k} e^{-s_{2n_k-1}-\dots-s_1} > C \sum_{\text{even } k} e^{-\frac{\sigma_{k+1}}{2}}.$$

By summing (11) and (12) we obtain for some constant $C > 0$ that the piecewise horocyclic path has length $\ell(h)$ greater than

$$C \sum_{k=1}^{\infty} e^{-\frac{\sigma_k}{2}}$$

and the assumption of the theorem implies that it is of infinite length. Thus \tilde{X}^* accumulates to exactly one point in addition to its vertices. This implies that X is parabolic. \square

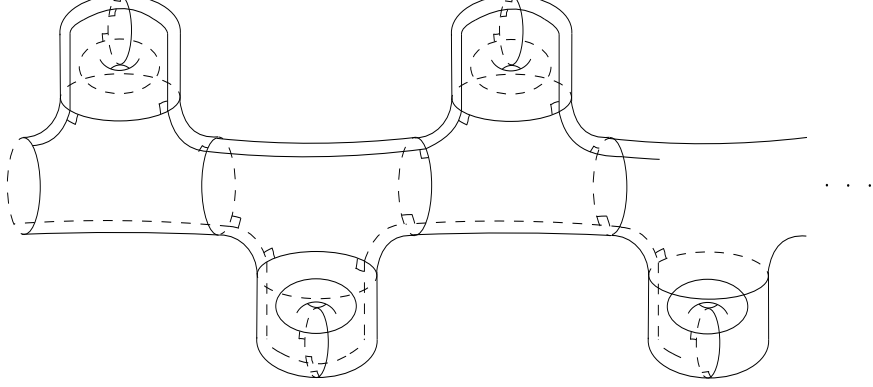


FIGURE 11. A symmetric end surface with half twists.

Theorem 4.1 implies that Kahn-Marković conjecture holds for every $\{\ell_n\}$ satisfying (15). In particular, we have the following.

Corollary 4.2. *Suppose $\{\ell_n\}$ is a non-decreasing sequence which has a subsequence $\{\ell_{n_k}\}$ s.t. for $k \geq 1$ we have*

$$(13) \quad \ell_{n_{2k-1}} = \ell_{n_{2k}}.$$

Then for the twists $\{t_n\}$ given by (2) the flute surface $X(\{\ell_n\}, \{t_n\})$ is parabolic.

Proof. By (13) we have $\sigma_{2k} = \ell_{n_{2k}} - \ell_{n_{2k-1}} + \dots + \ell_{n_2} - \ell_{n_1} = 0$, and $e^{-\sigma_{2k}} = 1$ for $k \geq 1$. Hence (15) holds and $X(\{\ell_n\}, \{t_n\}) \in O_G$. \square

5. LOCH-NESS MONSTER SURFACES AND BASIC END SURFACES

In this section, we consider the Loch-Ness monster Riemann surface X with cuffs α_n converging to the infinite end such that the subsurface $Y_n \subset X$ with boundaries α_n and α_{n+1} has genus one. Let β_n be the cuff in Y_n that cuts off the genus of Y_n . In other words, α_n , α_{n+1} , and β_n are on the boundary of a pair of pants.

We can form the Loch-Ness monster surface by starting from the bordered surface X_0 , which has one end at infinity, a sequence of cuffs α_n converging to infinity, and between any two cuffs α_n and α_{n+1} there is a closed boundary geodesic β_n . Then, to X_0 , we can attach a genus one surface to each β_n . The surface X_0 can have a puncture instead of β_n at arbitrary places. We call this surface X_0 the *basic end surface*.

It is also possible to attach a higher genus surface to a border β_n of X_0 and even an infinite surface. In fact, in Section 1 we associated to each topological end e of X a basic end surface X_e .

Theorem 5.1. *Let X_0 be a basic end surface with cuffs $\{\alpha_n\}_n$ converging to the end with increasing lengths ℓ_n and border closed geodesics $\{\beta_n\}_n$ with lengths bounded above, where some β_n can be punctures. Assume that the twists of X_0 around the*

cuffs α_n with respect to the orthogeodesics from β_n are in $\{0, 1/2\}$ with infinitely many twists $t_{n_k} = 1/2$. Then the modulus of the curve family in X_0 connecting α_1 to the point at infinity is zero under the condition that

$$(14) \quad \sum_{k=1}^{\infty} e^{-\sigma_k/2} = \infty,$$

where $\sigma_k = \ell_{n_k} - \ell_{n_{k-1}} + \dots + (-1)^{k-1} \ell_1$.

Proof. Since the surface X_0 is symmetric, it is enough to prove that the curve family in the front half X_0^* has zero modulus. The front half X_0^* is planar and simply connected (since the genus is cut off by β_n). Therefore X_0^* has a conformal image \tilde{X}_0^* in \mathbb{H} , which is an infinite polygon. It is enough to prove that the accumulation of \tilde{X}_0^* on $\hat{\mathbb{R}}$ is a single point. This follows if the lifts of α_n do not accumulate in \mathbb{H} . We form the same zig-zag pattern as in the case of the flute surface by drawing extra geodesics between two lifts of the cuffs. The geodesics $\{g_n\}_n$ have asymptotically the same distance $\ell(\eta_n)$ as in the case of the flute surface because $\ell(\beta_n)$ is bounded above. Therefore, the accumulation set of \tilde{X}_0^* is a single point on $\hat{\mathbb{R}}$. The modulus of the curve family is zero. \square

The above theorem has an immediate corollary for the Loch-Ness monster surfaces.

Corollary 5.2. *Let X_0 be a basic end surface as above with $\ell_n = \ell(\alpha_n)$ increasing and closed boundary geodesics β_n with $\ell(\beta_n)$ bounded above. Assume that the twists $t_n \in \{0, 1/2\}$ with infinitely many twists $t_{n_k} = 1/2$. Let X be a Loch-Ness monster surface obtained by attaching to each β_n an arbitrary surface with a finite area (and no boundary). Then*

$$X \in O_G$$

provided that

$$(15) \quad \sum_{k=1}^{\infty} e^{-\sigma_k/2} = \infty,$$

where $\sigma_k = \ell_{n_k} - \ell_{n_{k-1}} + \dots + (-1)^{k-1} \ell_1$.

Proof. The basic end surface satisfies the condition that the modulus of the curve family going to its end is zero. The attached surface can have a genus going to infinity, but each individual attached surface does not have an infinite end. Therefore, the curve families in the attached surfaces going to infinity have zero modulus as well. By Theorem 3.3, the surface X is parabolic. \square

In fact, the proof of the above theorem only requires that the cuffs α_n lift to geodesics that do not accumulate in \mathbb{H} . It is possible that $\ell(\beta_n)$ are unbounded, and this still to be true.

We are also in a position to have a general theorem regarding the parabolicity of surfaces with countably many ends that have arbitrarily large sequences of cuffs going to every topological end.

Theorem 5.3. *Let X be a Riemann surface with at most countably many ends \mathcal{E} . Assume that each subsurface X_{e_j} for $e_j \in \mathcal{E}'$ has a sequence of cuffs $\alpha_{j,n}$ converging to e_j with increasing lengths $\ell_{j,n} = \ell(\alpha_{j,n})$ and twist in $\{0, 1/2\}$ with infinitely many twists equal to $1/2$. If $\ell_{j,n}$ satisfy (15) for each j then X is parabolic.*

Proof. By Theorem 3.3, it is enough to prove that the family of curves starting on the boundary geodesic of X_{e_j} that is attached to an end surface in the previous level and going to the end e_j has zero modulus. This follows from the condition (15) by Theorem 5.1. \square

We note that the condition (15) is satisfied for arbitrarily large cuffs in each end. One example of accomplishing this is to make infinitely many pairs of adjacent cuffs equal to each other while keeping the other lengths as assigned by some double sequence of positive numbers $\{a_{j,n}\}$ increasing in n for each fixed j .

Corollary 5.4. *Let X be a topological surface with countably many ends \mathcal{E} . Let X_{e_j} be the end surface corresponding to $e_j \in \mathcal{E}'$ and let $\{\alpha_{j,n}\}_{n=1}^\infty$ be the cuffs in X_{e_j} accumulating to e_j . Given a double sequence $\{a_{j,n}\}$ of positive numbers increasing in n for each j , there exists a Riemann surface structure on X such that $\ell(\alpha_{j,n}) \geq a_{j,n}$ and $X \in O_G$.*

Moreover, if $n_{j,k}$ is an infinite subsequence and if we choose $\ell(\alpha_{j,n}) = a_{j,n}$ for all $n \neq n_{j,k} - 1$, and $\ell(\alpha_{j,n_{j,k}-1}) = \ell(\alpha_{j,n_{j,k}})$ then $X \in O_G$.

REFERENCES

- [AS60] L. Ahlfors and L. Sario, *Riemann surfaces*, Princeton Mathematical Series, No. 26 Princeton University Press, Princeton, N.J. 1960.
- [AR04] V. Álvarez and J. Rodríguez, *Structure theorems for Riemann and topological surfaces*, J. London Math. Soc. (2) 69 (2004), no. 1, 153-168.
- [AZ90] K. Astala and M. Zinsmeister, *Mostow rigidity and Fuchsian groups*. C. R. Acad. Sci. Paris Sér. I Math. 311 (1990), no. 6, 301-306.
- [Bas93] A. Basmajian, *Hyperbolic structures for surfaces of infinite type*, Trans. Amer. Math. Soc. 336 (1993), no. 1, 421-444.
- [BHŠ22] A. Basmajian, H. Hakobyan and D. Šarić, *The type problem for Riemann surfaces via Fenchel-Nielsen parameters*, Proc. Lond. Math. Soc. (3) **125** (2022), no. 3, 568–625.
- [BŠ19] A. Basmajian and D. Šarić, *Geodesically complete hyperbolic structures*, Math. Proc. Cambridge Philos. Soc. 166 (2019), no. 2, 219-242.
- [Bear83] A. Beardon, *The Geometry of Discrete Groups*, Springer-Verlag, New York, 1983.
- [Bis01] C. Bishop, *Divergence groups have the Bowen property*, Ann. of Math. (2) 154 (2001), no. 1, 205-217.
- [Bis02] C. Bishop, *Quasiconformal mappings of Y-pieces*, Rev. Mat. Iberoamericana 18 (3) 627 - 652, October, 2002.
- [Bis03] C. Bishop, *Big deformations near infinity*, Illinois J. Math. 47 (2003), no.4, 977-996.
- [Bus10] P. Buser, *Geometry and Spectra of Compact Riemann Surfaces*, Birkhäuser Boston, Ltd., Boston, MA; 2010.
- [Doy84] P. Doyle, *Random walk on the Speiser graph of a Riemann surface*, Bull. Amer. Math. Soc. (N.S.) 11 (1984), no. 2, 371-377.
- [FM01] J. L. Fernández and M. V. Melián, *Escaping geodesics of Riemannian surfaces*. Acta Math. 187 (2001), no. 2, 213-236.
- [Hop71] E. Hopf, *Ergodic theory and the geodesic flow on surfaces of constant negative curvature*, Bull. Amer. Math. Soc. 77 (1971), 863–877.
- [KM12] J. Kahn and V. Markovic, *Immersing almost geodesic surfaces in a closed hyperbolic three manifold*. Ann. of Math. (2) 175 (2012), no. 3, 1127-1190.
- [Kec95] A. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, 156. Springer-Verlag, New York, 1995.
- [Ker23] B. Kerékjártó, *Vorlesungen über Topologie. I*, Springer, Berlin, 1923.
- [Kin11] E. Kinjo, *On Teichmüller metric and the length spectrums of topologically infinite Riemann surfaces*, Kodai Math. J. 34 (2011), no. 2, 179-190.
- [LS84] T. Lyons and D. Sullivan, *Function theory, random paths and covering spaces*, J. Differential Geom. 19 (1984), no. 2, 299-323.

- [McM98] C. McMullen, *Hausdorff dimension and conformal dynamics. III. Computation of dimension*, Amer. J. Math. 120 (1998), no. 4, 691-721.
- [Mil77] J. Milnor, *On deciding whether a surface is parabolic or hyperbolic*, Amer. Math. Monthly 84 (1977), no. 1, 43-46.
- [Mer08] S. Merenkov, *Rapidly growing entire functions with three singular values*, Illinois J. Math. 52 (2008), no. 2, 473-491.
- [Nev50] Nevanlinna, R. *Über die Existenz von beschränkten Potentialfunktionen auf Flächen von unendlichem Geschlecht.* (German) Math. Z. 52 (1950), 599-604.
- [Nic80] P. Nicholls, *Fundamental regions and the type problem for a Riemann surface*, Math. Z. 174 (1980), no. 2, 187-196.
- [Nic89] P. Nicholls, *The ergodic theory of discrete groups*, London Mathematical Society Lecture Note Series, 143. Cambridge University Press, Cambridge, 1989.
- [Pan24] M. Pandazis, *Nonergodicity of the geodesic flow on a special class of Cantor tree surfaces*, Proc. Amer. Math. Soc. Ser. B 11 (2024), 315-329.
- [PŠ23] M. Pandazis and D. Šarić, *Ergodicity of the geodesic flow on symmetric surfaces*, Trans. Amer. Math. Soc. 376 (2023), no. 10, 7013-7043.
- [Pen93] R. Penner, *Universal constructions in Teichmüller theory*, Adv. Math. 98 (1993), no. 2, 143-215.
- [PRT12] A. Portilla, J. M. Rodríguez and E. Tourís, *Structure theorem for Riemannian surfaces with arbitrary curvature*, Math. Z. 271, 45–62 (2012).
- [Ric63] I. Richards, *On the classification of noncompact surfaces*, Trans. Amer. Math. Soc. 106 (1963), 259-269.
- [Šar10] D. Šarić, *Circle homeomorphisms and shears*, Geom. Topol. 14 (2010), no. 4, 2405-2430.
- [Šar24] D. Šarić, *Quadratic differentials and foliations on infinite Riemann surfaces*, Duke Math. J. 173 (2024), no. 10, 1883-1930.
- [ŠWW24] D. Šarić, Y. Wang and C. Wolfram, *Circle homeomorphisms with square summable diamond shears*, Int. Math. Res. Not. IMRN 2024, no. 17, 12219-12268.
- [Sul81] D. Sullivan, *On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions*. Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), pp. 465-496, Ann. of Math. Stud., 97, Princeton Univ. Press, Princeton, N.J., 1981.
- [Tsu75] M. Tsuji, *Potential theory in modern function theory*. Reprinting of the 1959 original. Chelsea Publishing Co., New York, 1975.

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