SINGULARITIES OF NORMALIZED R-MATRICES AND E-INVARIANTS FOR DYNKIN QUIVERS

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Abstract. We study the singularities of normalized R-matrices between arbitrary simple modules over the quantum loop algebra of type ADE in Hernandez–Leclerc's level-one subcategory using equivariant perverse sheaves following the previous works by Nakajima [Kyoto J. Math. 51(1), 2011] and Kimura–Qin [Adv. Math. 262, 2014]. We show that the pole orders of these R-matrices coincide with the dimensions of E-invariants between the corresponding decorated representations of Dynkin quivers. This result can be seen as a correspondence of numerical characteristics between additive and monoidal categorifications of cluster algebras of finite ADE type.

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1. Introduction

1.1. The quantum loop algebra $U_q(L\mathfrak{g})$ associated with a complex simple Lie algebra $\mathfrak{g}$ was introduced in mid 80s as the symmetry of certain quantum integrable systems and solvable lattice models in theoretical physics. It is a Hopf algebra deformation of the universal enveloping algebra of the loop Lie algebra $L\mathfrak{g} = \mathfrak{g}[z^{\pm 1}]$. The category $\mathscr{C}$ of finite-dimensional representations of $U_q(L\mathfrak{g})$ exhibits a very interesting monoidal structure and has been studied intensively for several decades.

One of the remarkable features of the monoidal category $\mathscr{C}$ is that it is not braided, in contrast to that of finite-dimensional representations of $U_q(\mathfrak{g})$, but it is "generically braided" in the following sense. Throughout the paper, we assume that the quantum parameter q is generic. For two simple objects L and L' of $\mathscr{C}$, the tensor product $L \otimes L'$ sometimes fails to be isomorphic to the opposite product $L' \otimes L$. However, if we replace L' with its deformation $L'(z)$ with a generic spectral parameter z, there always exists a unique isomorphism

$$R_{L,L'}(z) \colon L \otimes L'(z) \xrightarrow{\simeq} L'(z) \otimes L$$

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called the *normalized R-matrix* between L and L'. It can be seen as a matrix-valued rational function in z, and hence one can talk about its singularities. Let $\mathfrak{o}(L, L')$ denote the pole order of $R_{L,L'}(z)$ at $z = 1$. If both $R_{L,L'}(z)$ and $R_{L',L}(z)$ are regular at $z = 1$, i.e., if $\mathfrak{o}(L, L') = \mathfrak{o}(L', L) = 0$ holds, the objects L and L' commute in $\mathscr{C}$ and the specialization $R_{L,L'}(z)|_{z=1}$ gives an isomorphism $L \otimes L' \simeq L' \otimes L$. Thus, one can think of the pole order $\mathfrak{o}(L, L')$ as a measure of the non-commutativity between L and L'. In fact, it plays a key role in the recent studies on the category $\mathscr{C}$, especially in the theory of monoidal categorification of cluster algebras [27] and in the construction of generalized quantum affine Schur–Weyl duality functors [24].

Despite its importance, computing the pole order $\mathfrak{o}(L, L')$ for general simple objects L and L' seems to be a difficult problem. Explicit computations have been accomplished for fundamental modules and partially for Kirillov–Reshetikhin modules. See [28, Appendix A] and [37] for a list of known computations. Beyond these special classes, no systematic computations are known at this moment as far as the author understands. The purpose of this paper is to provide a description of the pole orders for another class of simple modules in relation with the cluster structure of $\mathscr{C}$.

1.2. To state our main result in a precise manner, we need additional terminologies. From now on, we assume that $\mathfrak{g}$ is of type ADE, and let Q be a Dynkin quiver of the same type as $\mathfrak{g}$. In their seminal works [22, 23], Hernandez–Leclerc introduced a certain monoidal subcategory $\mathscr{C}_1$ of $\mathscr{C}$, depending on (a height function of) the quiver Q, which we call the *level-one subcategory*. They conjectured, and verified in several cases, that it gives a *monoidal categorification* of a cluster algebra A of finite ADE type (the same type as $\mathfrak{g}$), in the sense that there exists a ring isomorphism

$$\tilde{\chi}_q \colon K(\mathscr{C}_1) \xrightarrow{\simeq} A$$

from Grothendieck ring $K(\mathscr{C}_1)$ to the cluster algebra A through which the basis of $K(\mathscr{C}_1)$ formed by the simple isomorphism classes corresponds to the basis of A formed by the cluster monomials. The isomorphism $\tilde{\chi}_q$ is given explicitly by the truncated q-character map of [22]. The conjecture was later verified by Nakajima [36] and Kimura–Qin [29] in full generality using the perverse sheaves on Nakajima's graded quiver varieties.

On the other hand, there is another kind of categorification of A, sometimes called an *additive categorification*. Here, we use the version due to Caldero–Chapoton [5] and Derksen–Weyman–Zelevinsky [11] in terms of *decorated representations* of the Dynkin quiver Q. Recall that a decorated representation of Q is a pair $\mathcal{M} = (M, V)$ of a usual representation M of Q (over $\mathbb{C}$) and a finite-dimensional $\mathtt{I}$-graded $\mathbb{C}$-vector space V, where $\mathtt{I}$ is the set of vertices of Q. For two such pairs $\mathcal{M} = (M, V)$ and $\mathcal{M}' = (M', V')$, the *E-invariant* between them is defined to be

$$E(\mathcal{M}, \mathcal{M}') := \operatorname{Ext}^1_Q(M, M') \oplus \bigoplus_{i \in \mathtt{I}} \operatorname{Hom}_{\mathbb{C}}(M_i, V'_i). \tag{1.1}$$

A decorated representation $\mathcal{M}$ is said to be *rigid* if $E(\mathcal{M}, \mathcal{M}) = 0$. The theory of additive categorification of A tells us that there is a map

$$CC \colon \{\text{decorated representations of } Q\} \to A$$

called the cluster character map (a.k.a. Caldero–Chapoton map) which satisfies $CC(\mathcal{M}\oplus\mathcal{M}') = CC(\mathcal{M})\cdot CC(\mathcal{M}')$ and induces a bijection between the isomorphism classes of rigid decorated representations of Q and the cluster monomials without frozen factors of A.

Now, we are ready to state our main result. It describes the pole orders of the normalized R-matrices in $\mathscr{C}_1$ in terms of the E-invariants for the Dynkin quiver Q.

Theorem 1.1. *For any simple objects L and L' of $\mathscr{C}_1$, we have*

$$\mathfrak{o}(L, L') = \dim E(\mathcal{M}, \mathcal{M}'),$$

where $\mathcal{M}$ and $\mathcal{M}'$ are rigid decorated representations of Q satisfying $\tilde{\chi}_q(L) = CC(\mathcal{M})$ and $\tilde{\chi}_q(L') = CC(\mathcal{M}')$ up to frozen factors.

To obtain the main result, we apply Nakajima's geometric construction of representations of $U_q(L\mathfrak{g})$ [33, 34] and verify a slightly different but equivalent assertion (= Theorem 2.7), where decorated representations are replaced with injective copresentations, following Derksen–Fei [10]. Our proof is based on the key observation by Kimura–Qin [29], generalizing the one by Nakajima [36], that the graded quiver variety relevant to the category $\mathscr{C}_1$ is simply a vector space and its dual is identified with the space X of injective copresentations of the Dynkin quiver Q. In the previous work [29], this fact was crucially used to relate the equivariant perverse sheaves on X to the Grothendieck ring $K(\mathscr{C}_1)$ or rather its quantum deformation. In this paper, we go one more step further to relate the geometry of X directly to representations in $\mathscr{C}_1$. Namely, we interpret the deformed tensor products of simple objects of $\mathscr{C}_1$ and the R-matrices between them (under a certain condition, see §4.6) in terms of canonical operations for the equivariant perverse sheaves on X. The E-invariant in question appears as a transversal slice in X.

1.3. Let $x, x' \in A$ be two non-frozen cluster variables, L, L' prime simple objects of $\mathscr{C}_1$, and $\mathcal{M}, \mathcal{M}'$ rigid indecomposable decorated representations of Q satisfying $x = \tilde{\chi}_q(L) = CC(\mathcal{M})$ and $x' = \tilde{\chi}_q(L') = CC(\mathcal{M}')$. The theory of additive/monoidal categorifications tells us that the following three conditions are mutually equivalent:

(1) x and x' belong to a common cluster (i.e., xx' is a cluster monomial);
(2) $\mathfrak{d}(L, L') \coloneqq \mathfrak{o}(L, L') + \mathfrak{o}(L', L) = 0$;
(3) $\mathfrak{e}(\mathcal{M}, \mathcal{M}') \coloneqq \dim E(\mathcal{M}, \mathcal{M}') + \dim E(\mathcal{M}', \mathcal{M}) = 0$.

The invariant $\mathfrak{d}(L, L')$ was originally introduced by Kashiwara–Kim–Oh–Park [27]. On the other hand, the invariant $\mathfrak{e}(\mathcal{M}, \mathcal{M}')$ was considered by Marsh–Reineke–Zelevinsky [32], which is identical to Fomin–Zelevinsky's compatible degree in [15]. Theorem 1.1 above implies the correspondence of these two numerical characteristics:

$$\mathfrak{d}(L, L') = \mathfrak{e}(\mathcal{M}, \mathcal{M}'), \tag{1.2}$$

which does not seem automatic from the known categorifications results.

It would be interesting to ask if Theorem 1.1 or the equality (1.2) generalize beyond the category $\mathscr{C}_1$ to other monoidal categorifications of cluster algebras. At least for Kirillov–Reshetikhin modules, known computations suggest that such a generalization is plausible [19, §5]. Note that the left-hand side of (1.2) makes sense for graded modules over symmetric quiver Hecke algebras [26] and for the coherent Satake category [6] as well.

1.4. **Organization.** The present paper is organized as follows. In §2, we state the main theorem (= Theorem 2.7) after reviewing some necessary backgrounds. In §2.5, we briefly explain its cluster theoretical interpretation to see that it is equivalent to the above Theorem 1.1. In §3, we summarize Nakajima's geometric construction of representations of $U_q(L\mathfrak{g})$. In the final §4, we apply the materials from §3 to study representations in the category $\mathscr{C}_1$ and discuss the proof of the main theorem.

2. Algebraic preliminaries and main theorem

In this section, some necessary algebraic preliminaries are recalled before we state our main theorem (= Theorem 2.7) in §2.4. In §2.5, we briefly explain a cluster theoretical interpretation of Theorem 2.7 to see that it is indeed equivalent to Theorem 1.1 in Introduction.

2.1. **Representations of quantum loop algebras.** Let $\mathfrak{g}$ be a complex simple Lie algebra and $U_q(L\mathfrak{g})$ the quantum loop algebra associated with $\mathfrak{g}$, which is a Hopf algebra defined over an algebraically closed field $\Bbbk$ of characteristic 0 with $q \in \Bbbk^\times$ being a parameter. In Drinfeld's presentation, it is generated by the elements $x^{\pm}_{i,n}, h_{i,m}, k_i^{\pm 1}$ with $i \in \mathtt{I}, n \in \mathbb{Z}, m \in \mathbb{Z} \setminus \{0\}$, where $\mathtt{I}$ is a labeling set of simple roots of $\mathfrak{g}$. We follow [34] for the convention of the coproduct of $U_q(L\mathfrak{g})$. Throughout this paper, the quantum parameter q is assumed to be algebraically independent over $\mathbb{Q}$. Let $\mathscr{C}$ be the category of finite-dimensional modules over $U_q(L\mathfrak{g})$ of type **1**. This is a $\Bbbk$-linear monoidal abelian category. We often abbreviate $\otimes_\Bbbk$ as $\otimes$.

For objects $M, N \in \mathscr{C}$, we say that M and N *commute* if the tensor product $M \otimes N$ is isomorphic to the opposite product $N \otimes M$. In the category $\mathscr{C}$, there are many pairs of objects in $\mathscr{C}$ which do not commute. Nevertheless, the Grothendieck ring $K(\mathscr{C})$ of $\mathscr{C}$ is known to be a commutative domain [16]. This particularly implies that two simple modules M and N commute if $M \otimes N$ is simple. In this case, we say that M and N *strongly commute*.

The isomorphism classes of simple modules in the category $\mathscr{C}$ are parameterized by the multiplicative monoid $(1 + z\Bbbk[z])^{\mathtt{I}}$ of $\mathtt{I}$-tuples of monic polynomials, called the Drinfeld polynomials [7, Ch. 12]. We denote by $L(\varpi)$ a simple module in $\mathscr{C}$ corresponding to $\varpi \in (1+z\Bbbk[z])^{\mathtt{I}}$. In the terminology of [7], it is an ℓ-highest weight module of ℓ-highest weight ϖ. In particular, $L(\varpi)$ has a distinguished generating vector, called an ℓ-highest weight vector, uniquely up to multiple in $\Bbbk^\times$. Note that the monoid $(1 + z\Bbbk[z])^{\mathtt{I}}$ is generated by the elements $\varpi_{i,a} := ((1 - az)^{\delta_{i,j}})_{j \in \mathtt{I}}$ for $(i, a) \in I \times \Bbbk^\times$, where $\delta_{i,j}$ is the Kronecker delta.

2.2. **Normalized R-matrices.** Let z be an indeterminate. For an object $M \in \mathscr{C}$, we can define a new action of $U_q(L\mathfrak{g})$ on the $\Bbbk[z^{\pm 1}]$-module $M[z^{\pm 1}] := M \otimes \Bbbk[z^{\pm 1}]$ by the formula:

$$x^{\pm}_{i,n}(v \otimes a) := x^{\pm}_{i,n} v \otimes z^n a, \quad k_i(v \otimes a) := k_i v \otimes a, \quad h_{i,m}(v \otimes a) := h_{i,m} v \otimes z^m a,$$

where $v \in M, a \in \Bbbk[z^{\pm 1}]$. The resulting $U_q(L\mathfrak{g})[z^{\pm 1}]$-module $M[z^{\pm 1}]$ is called the *affinization* of M. We set $M(z) \coloneqq M[z^{\pm 1}] \otimes_{\Bbbk[z^{\pm 1}]} \Bbbk(z)$. In what follows, we sometimes identify the subspace $M \otimes 1$ of $M[z^{\pm 1}]$ with M.

For a pair (M, N) of simple modules in $\mathscr{C}$, with fixed ℓ-highest weight vectors $v_M \in M$ and $v_N \in N$, the $U_q(L\mathfrak{g})(z)$-modules $M \otimes N(z)$ and $N(z) \otimes M$ are known to be simple, and therefore we have a unique isomorphism

$$R_{M,N}(z) \colon M \otimes N(z) \to N(z) \otimes M$$

satisfying $R_{M,N}(z)(v_M \otimes v_N) = v_N \otimes v_M$. It is called the *normalized R-matrix* between M and N. Viewing it as a matrix-valued rational function in z, one can talk about the order of its poles, which does not depend on the choice of ℓ-highest weight vectors v_M and v_N. We define a non-negative integer $\mathfrak{o}(M, N)$ to be the pole order of $R_{M,N}(z)$ at $z = 1$, and set

$$\mathfrak{d}(M, N) \coloneqq \mathfrak{o}(M, N) + \mathfrak{o}(N, M).$$

This is the same as the invariant introduced in [27] (cf. [27, Proposition 3.16]). One may understand that the number $\mathfrak{d}(M, N)$ measures the non-commutativity between M and N as the following proposition suggests. We say that a simple module in $\mathscr{C}$ is *real* if it strongly commutes with itself.

Proposition 2.1 ([27, Corollary 3.17])**.** *Let M and N be simple modules in $\mathscr{C}$. Assume that at least one of them is real. Then M and N strongly commute if and only if $\mathfrak{d}(M, N) = 0$.*

The following property is used later in §4.6.

Lemma 2.2 (cf. [25, Corollary 3.11(ii)])**.** *Let M_1, M_2, N be simple modules in $\mathscr{C}$. Assume that M_1 and M_2 strongly commute. Then, we have*

$$\mathfrak{o}(M_1 \otimes M_2, N) = \mathfrak{o}(M_1, N) + \mathfrak{o}(M_2, N),$$
$$\mathfrak{o}(N, M_1 \otimes M_2) = \mathfrak{o}(N, M_1) + \mathfrak{o}(N, M_2).$$

Remark 2.3. For our purpose, it is convenient to have the following characterization of the number $\mathfrak{o}(M, N)$.

Let us introduce another indeterminate u and consider the ring $\Bbbk[\![u]\!]$ of formal power series. Viewing $\Bbbk[z^{\pm 1}]$ as a subring of $\Bbbk[\![u]\!]$ by $z = e^u$, we define the infinitesimal deformation of M to be

$$M[\![u]\!] \coloneqq M[z^{\pm 1}] \otimes_{\Bbbk[z^{\pm 1}]} \Bbbk[\![u]\!].$$

This is a $U_q(L\mathfrak{g})[\![u]\!]$-module. By localization, we also get a $U_q(L\mathfrak{g})(\!(u)\!)$-module $M(\!(u)\!)$. Note that $M[\![u]\!]$ is a $\Bbbk[\![u]\!]$-lattice of $M(\!(u)\!)$.

For simple modules M, N in $\mathscr{C}$, the normalized R-matrix $R_{M,N}(e^u)$ induces an isomorphism

$$\widehat{R}_{M,N} \colon M \otimes N(\!(u)\!) \to N(\!(u)\!) \otimes M$$

of $U_q(L\mathfrak{g})(\!(u)\!)$-modules satisfying $\widehat{R}_{M,N}(v_M \otimes v_N) = v_N \otimes v_M$. Then, the number $\mathfrak{o}(M, N)$ is equal to the non-negative integer d such that we have

$$u^d \widehat{R}_{M,N}(M \otimes N[\![u]\!]) \subset N[\![u]\!] \otimes M$$

and the specialization $(u^d \widehat{R}_{M,N})|_{u=0} \colon M \otimes N \to N \otimes M$ is non-zero.

2.3. **E-invariants.** Let Q be an acyclic quiver. We denote by $\mathsf{rep}\,\mathbb{C}Q$ the category of finite-dimensional representations of Q over $\mathbb{C}$ and by $\mathsf{inj}\,\mathbb{C}Q$ the full subcategory of $\mathsf{rep}\,\mathbb{C}Q$ consisting of injective representations. Let $C^2(\mathsf{inj}\,\mathbb{C}Q)$ be the category of morphisms $\phi\colon I^{(0)} \to I^{(1)}$ in $\mathsf{inj}\,\mathbb{C}Q$. We refer to an object of $C^2(\mathsf{inj}\,\mathbb{C}Q)$ as an injective copresentation of Q. We regard an object of $C^2(\mathsf{inj}\,\mathbb{C}Q)$ as a cochain complex concentrated in the cohomological degrees 0 and 1. For any $\phi, \psi \in C^2(\mathsf{inj}\,\mathbb{C}Q)$, the E-invariant between them is defined to be the vector space

$$E(\phi, \psi) := \mathrm{Hom}_{K^b(\mathsf{inj}\,\mathbb{C}Q)}(\phi, \psi[1]),$$

where $K^b(\mathsf{inj}\,\mathbb{C}Q)$ is the homotopy category of bounded complexes in $\mathsf{inj}\,\mathbb{C}Q$ and $[1]$ is the shift functor. This is a finite-dimensional $\mathbb{C}$-vector space. Note that $K^b(\mathsf{inj}\,\mathbb{C}Q)$ is naturally equivalent to the derived category $D^b(\mathsf{rep}\,\mathbb{C}Q)$ and that ϕ is isomorphic to $\mathrm{Ker}\,\phi[0] \oplus \mathrm{Coker}\,\phi[-1]$ in $D^b(\mathsf{rep}\,\mathbb{C}Q)$. Since $\mathbb{C}Q$ is hereditary, quotients of injective modules are injective. In particular, $\mathrm{Coker}\,\phi$ belongs to $\mathsf{inj}\,\mathbb{C}Q$. Therefore, we have

$$E(\phi, \psi) \simeq \mathrm{Ext}^1_Q(\mathrm{Ker}\,\phi, \mathrm{Ker}\,\psi) \oplus \mathrm{Hom}_Q(\mathrm{Ker}\,\phi, \mathrm{Coker}\,\psi). \tag{2.1}$$

2.4. **Main theorem.** In what follows, we assume that $\mathfrak{g}$ is of simply-laced type (i.e., type ADE). An integer-valued function $\xi\colon \mathtt{I} \to \mathbb{Z}$ is called a *height function* if it satisfies $|\xi(i) - \xi(j)| = 1$ whenever i and j are adjacent in the Dynkin diagram of $\mathfrak{g}$. A height function ξ defines a Dynkin quiver Q_ξ of the same type as $\mathfrak{g}$ in the following way. The set of vertices of Q_ξ is $\mathtt{I}$. For an adjacent pair (i, j) in $\mathtt{I}$, we have an arrow $i \to j$ in Q_ξ if $\xi(i) = \xi(j) + 1$. Note that we have $Q_\xi = Q_{\xi'}$ if and only if the difference $\xi - \xi'$ is constant. Any Dynkin quiver arises from a height function in this way.

Example 2.4. When $\mathfrak{g}$ is of type A_3, the function ξ given by $\xi(i) = i$ under the standard identification $\mathtt{I} = \{1, 2, 3\}$ is a height function. The associated quiver Q_ξ is depicted as $Q_\xi = (\overset{1}{\circ} \longleftarrow \overset{2}{\circ} \longleftarrow \overset{3}{\circ})$.

Throughout this paper, we fix a height function ξ. For each $i \in \mathtt{I}$, let $S_i \in \mathsf{rep}\,\mathbb{C}Q_\xi$ be the simple representation at i, and $I_i \in \mathsf{inj}\,\mathbb{C}Q_\xi$ an injective hull of S_i. Let $\mathbb{N} := \mathbb{Z}_{\ge 0}$. For each pair $w = (w(0), w(1)) \in \mathbb{N}^{\mathtt{I} \sqcup \mathtt{I}} = \mathbb{N}^{\mathtt{I}} \times \mathbb{N}^{\mathtt{I}}$ of I-tuples of non-negative integers, we set

$$I^{w(0)} := \bigoplus_{i \in I} I_i^{\oplus w_i(0)}, \quad I^{w(1)} := \bigoplus_{i \in I} I_i^{\oplus w_i(1)}, \quad X(w) := \mathrm{Hom}_{Q_\xi}(I^{w(0)}, I^{w(1)}),$$

where $w(k) = (w_i(k))_{i \in I} \in \mathbb{N}^I$ for $k = 0, 1$. The automorphism group

$$A(w) := \mathrm{Aut}_{Q_\xi}(I^{w(0)}) \times \mathrm{Aut}_{Q_\xi}(I^{w(1)})$$

acts on the vector space $X(w)$ in the natural way. Since Q_ξ is of finite representation type, there are only finitely many $A(w)$-orbits in $X(w)$ [10, Corollary 2.6]. In particular, there exists a unique open orbit.

Definition 2.5. For each $w \in \mathbb{N}^{\mathtt{I} \sqcup \mathtt{I}}$, we denote by

$$\phi_\xi(w)\colon I^{w(0)} \to I^{w(1)}$$

an injective copresentation in the unique open $A(w)$-orbit in $X(w)$. By definition, it is unique up to isomorphism.

On the other hand, to each $w = (w(0), w(1)) \in \mathbb{N}^{\mathtt{I}\sqcup\mathtt{I}}$, we associate a simple $U_q(L\mathfrak{g})$-module $L_\xi(w)$ in the category $\mathscr{C}$ by

$$L_\xi(w) := L\left(\prod_{i\in I} \varpi_{i,q^{\xi(i)}}^{w_i(0)} \varpi_{i,q^{\xi(i)+2}}^{w_i(1)}\right). \tag{2.2}$$

Definition 2.6 (Hernandez–Leclerc [22, 23])**.** The *level-one subcategory* $\mathscr{C}_{\xi,1}$ is defined to be the Serre subcategory of $\mathscr{C}$ generated by the simple modules $L_\xi(w)$ for $w \in \mathbb{N}^{\mathtt{I}\sqcup\mathtt{I}}$.

By [23, Lemma 3.2], the category $\mathscr{C}_{\xi,1}$ is a monoidal subcategory of $\mathscr{C}$. The main theorem of this paper is the following.

Theorem 2.7. *For any height function ξ and $w, w' \in \mathbb{N}^{\mathtt{I}\sqcup\mathtt{I}}$, we have*

$$\mathfrak{o}(L_\xi(w), L_\xi(w')) = \dim E(\phi_\xi(w), \phi_\xi(w')).$$

A proof is given later in §4.6.

2.5. **Cluster theoretical interpretation.** In this subsection, we briefly explain the equivalence between Theorem 1.1 and Theorem 2.7. It amounts to giving a cluster theoretical interpretation of Theorem 2.7.

2.5.1. *Cluster algebras.* First, we fix our notation around the finite type cluster algebras. Recall that we have fixed a height function $\xi\colon \mathtt{I} \to \mathbb{Z}$. Let $\mathtt{I}' := \{i' \mid i \in \mathtt{I}\}$ be a copy of the set $\mathtt{I}$, which serves the set of frozen indices. Let A_ξ be the cluster algebra of geometric type associated with the exchange matrix $\tilde{B} = (b_{ij})_{i\in\mathtt{I}\sqcup\mathtt{I}', j\in\mathtt{I}}$ given by

$$b_{ij} := n_{ij} - n_{ji}, \quad b_{i'j} = \delta_{i,j} - n_{ij}$$

for $i, j \in \mathtt{I}$, where n_{ij} denotes the number of arrows from i to j in the quiver Q_ξ, and $\delta_{i,j}$ is the Kronecker delta. By the Laurent phenomenon, A_ξ is the subring generated by all the cluster variables inside the ring of Laurent polynomials in the initial cluster variables $\{x_i \mid i \in \mathtt{I} \sqcup \mathtt{I}'\}$. See [13].

Let Δ^+ denote the set of positive roots of $\mathfrak{g}$ and $\alpha_i \in \Delta^+$ the i-th simple root. Since A_ξ is of finite type, the set of non-frozen cluster variables of A_ξ is finite and in bijection with the set of almost positive roots $\Delta_{\geq -1} := \Delta^+ \cup \{-\alpha_i \mid i \in \mathtt{I}\}$. See [14]. Let $x[\alpha]$ denote the cluster variable corresponding to $\alpha \in \Delta_{\geq -1}$. For example, we have $x[-\alpha_i] = x_i$ and $x[\alpha_i] = x_i^{-1}(\prod_{j\in\mathtt{I}} x_j^{n_{ij}} x_{j'}^{n_{ji}} + x_{i'} \prod_{j\in\mathtt{I}} x_j^{n_{ji}})$ for each $i \in \mathtt{I}$. For a positive root $\alpha = \sum_{i\in\mathtt{I}} a_i\alpha_i$, the cluster variable $x[\alpha]$ is the one having $\prod_{i\in\mathtt{I}} x_i^{a_i}$ as its denominator. The cluster variables $x[\alpha]$ ($\alpha \in \Delta_{\geq -1}$), $x_{i'}$ ($i \in \mathtt{I}$) are grouped into several subsets of constant cardinality $2|\mathtt{I}|$, called the clusters. A cluster always contains the frozen variables $\{x_{i'} \mid i \in \mathtt{I}\}$. A monomial of cluster variables of a common cluster is called a cluster monomial. The cluster monomials form a free $\mathbb{Z}$-basis of A_ξ, and equivalently, the cluster monomials without frozen factors form a free $\mathbb{Z}[x_{i'} \mid i \in \mathtt{I}]$-basis of A_ξ.

2.5.2. *Additive categorification.* For $M \in \mathsf{rep}\,\mathbb{C}Q_\xi$, its dimension vector is $\underline{\dim}\, M := \sum_{i\in\mathtt{I}} (\dim M_i)\alpha_i$. By Gabriel's theorem, for each $\alpha \in \Delta^+$, there exists an indecomposable object $M_\xi[\alpha] \in \mathsf{rep}\,\mathbb{C}Q_\xi$ uniquely up to isomorphism satisfying $\underline{\dim}\, M_\xi[\alpha] = \alpha$, and the set $\{M_\xi[\alpha] \mid \alpha \in \Delta_{\geq -1}\}$ gives a complete system of indecomposable objects of $\mathsf{rep}\,\mathbb{C}Q_\xi$. For $v = (v_i)_{i\in\mathtt{I}} \in \mathbb{N}^{\mathtt{I}}$, we set $\mathbb{C}^v := \bigoplus_{i\in\mathtt{I}} \mathbb{C}^{v_i}$. Recall that a decorated representation of Q_ξ is a pair $\mathcal{M} = (M, V)$ of $M \in \mathsf{rep}\,\mathbb{C}Q_\xi$ and a finite-dimensional $\mathtt{I}$-graded $\mathbb{C}$-vector space V. We set $\mathcal{M}_\xi[\alpha] := (M_\xi[\alpha], 0)$ for $\alpha \in \Delta^+$ and $\mathcal{M}_\xi[-\alpha_i] := (0, \mathbb{C}^{\delta_i})$ for $i \in \mathtt{I}$, where $\delta_i = (\delta_{i,j})_{j\in\mathtt{I}} \in \mathbb{N}^{\mathtt{I}}$ is the delta function at i. Then, the set $\{\mathcal{M}_\xi[\alpha] \mid \alpha \in \Delta_{\geq -1}\}$ gives a complete system of indecomposable decorated representations of Q_ξ.

For a decorated representation $\mathcal{M} = (M, V)$ of Q_ξ, its cluster character $CC(\mathcal{M})$ is defined as in [11]:

$$CC(\mathcal{M}) := \sum_{v\in\mathbb{N}^{\mathtt{I}}} \chi(\mathrm{Gr}_v(M)) \prod_{i\in\mathtt{I}\sqcup\mathtt{I}', j\in\mathtt{I}} x_i^{\tilde{g}_i(\mathcal{M}) - b_{ij}v_j},$$

where $\chi(\mathrm{Gr}_v(M))$ is the Euler characteristic of the submodule Grassmannian $\mathrm{Gr}_v(M)$, i.e., the complex projective variety parametrizing subrepresentations of M of dimension vector $\sum_{i\in\mathtt{I}} v_i\alpha_i$, and $(\tilde{g}_i(\mathcal{M}))_{i\in\mathtt{I}\sqcup\mathtt{I}'}$ is the so-called extended g-vector of $\mathcal{M}$. In our case, it is explicitly written as

$$\tilde{g}_i(\mathcal{M}) = \dim V_i - \dim M_i + \sum_{j\in\mathtt{I}} n_{ij} \dim M_j, \quad \tilde{g}_{i'}(\mathcal{M}) = \dim \bigcap_a \mathrm{Ker}(a|_M)$$

for each $i \in \mathtt{I}$, where a runs over the set of arrows of Q_ξ whose source is i. By [5, 11], we have $CC(\mathcal{M} \oplus \mathcal{M}') = CC(\mathcal{M}) \cdot CC(\mathcal{M}')$ for any decorated representations $\mathcal{M}, \mathcal{M}'$, and $CC(\mathcal{M}_\xi[\alpha]) = x[\alpha]$ for all $\alpha \in \Delta_{\geq -1}$. Recall the E-invariant for decorated representations defined in (1.1). By [32], two cluster variables $x[\alpha]$ and $x[\alpha']$ belong to a common cluster if and only if we have $E(\mathcal{M}_\xi[\alpha], \mathcal{M}_\xi[\alpha']) = E(\mathcal{M}_\xi[\alpha'], \mathcal{M}_\xi[\alpha]) = 0$. The map CC gives a bijection between the isomorphism classes of rigid decorated representations of Q_ξ and the cluster monomials without frozen factors of A_ξ.

Remark 2.8. The following remark is used later in §4.6. It is well known that there is a partial ordering $\leq_\xi$ of the set Δ^+ with the following property: we have $\alpha \leq_\xi \alpha'$ if $\mathrm{Ext}^1_{Q_\xi}(M_\xi[\alpha], M_\xi[\alpha']) \neq 0$. We can extend it to the set $\Delta_{\geq -1}$ so that we have $\alpha \leq_\xi -\alpha_i$ for any $\alpha \in \Delta^+$ and $i \in \mathtt{I}$. Then, for $\alpha, \alpha' \in \Delta_{\geq -1}$, we have $\alpha \leq_\xi \alpha'$ whenever $E(\mathcal{M}_\xi[\alpha], \mathcal{M}_\xi[\alpha']) \neq 0$.

2.5.3. *Interpretation by injective copresentations.* To each $\phi \in C^2(\mathsf{inj}\,\mathbb{C}Q_\xi)$, we assign a decorated representation $\mathcal{M}(\phi)$ of Q_ξ by

$$\mathcal{M}(\phi) := (\mathrm{Ker}\,\phi, \mathbb{C}^c), \quad \text{where } c_i := \dim \mathrm{Hom}_{Q_\xi}(S_i, \mathrm{Coker}\,\phi).$$

Comparing (1.1) and (2.1), we find that

$$E(\phi, \psi) \simeq E(\mathcal{M}(\phi), \mathcal{M}(\psi)) \tag{2.3}$$

holds for any $\phi, \psi \in C^2(\mathsf{inj}\,\mathbb{C}Q_\xi)$. For each $\alpha \in \Delta^+$, let $\phi_\xi[\alpha]$ be a minimal injective resolution of $M_\xi[\alpha]$. For each $i \in \mathtt{I}$, we set $\phi_\xi[-\alpha_i] := (0 \to I_i)$ and $\nu_i := (I_i \xrightarrow{\mathsf{id}} I_i)$. By construction, we have $\mathcal{M}(\phi_\xi[\alpha]) = \mathcal{M}_\xi[\alpha]$ for any $\alpha \in \Delta_{\geq -1}$ and $\mathcal{M}(\nu_i) = 0$ for any $i \in \mathtt{I}$.

The set $\{\phi_\xi[\alpha] \mid \alpha \in \Delta_{\geq -1}\} \sqcup \{\nu_i \mid i \in \mathtt{I}\}$ forms a complete system of indecomposable objects of $C^2(\mathsf{inj}\,\mathbb{C}Q_\xi)$. Since the category $C^2(\mathsf{inj}\,\mathbb{C}Q_\xi)$ is Krull–Schmidt, each object decomposes into a finite direct sum of indecomposable objects in a unique way. For each $\phi \in C^2(\mathsf{inj}\,\mathbb{C}Q_\xi)$, we define

$$CC(\phi) := CC(\mathcal{M}(\phi)) \prod_{i\in\mathtt{I}} x_{i'}^{m_i(\phi)},$$

where $m_i(\phi)$ denotes the multiplicity of the factor ν_i in ϕ.

Lemma 2.9. *The map $w \mapsto CC(\phi_\xi(w))$ gives a bijection from $\mathbb{N}^{\mathtt{I} \sqcup \mathtt{I}}$ to the set of cluster monomials of A_ξ.*

Proof. We say that an injective copresentation ϕ is rigid if $E(\phi, \phi) = 0$. By [10], $\phi \in X(w)$ is rigid if and only if $\phi \simeq \phi_\xi(w)$ (see also §4.5 below). Therefore, the set $\{\phi_\xi(w) \mid w \in \mathbb{N}^{\mathtt{I} \sqcup \mathtt{I}}\}$ gives a complete system of rigid injective copresentations of Q_ξ. On the other hand, we know that ϕ is rigid if and only if $\mathcal{M}(\phi)$ is rigid by (2.3), and that $\mathcal{M}(\phi) \simeq \mathcal{M}(\psi)$ if and only if $[\phi] - [\psi] \in \sum_{i \in \mathtt{I}} \mathbb{Z}[\nu_i]$ in the Grothendieck group of $C^2(\mathsf{inj}\, \mathbb{C}Q_\xi)$. Having these remarks, the assertion now follows from the results explained in the second paragraph of §2.5.2. □

2.5.4. *Monoidal categorification.* The following theorem was originally conjectured by Hernandez–Leclerc [22] when Q_ξ has a sink-source orientation. For this case, it was proved by Hernandez–Leclerc [22] for type A, D_4, and by Nakajima [36] for all type ADE. For general ξ, some results were obtained by Hernandez–Leclerc [23] and Brito–Chari [4]. In full generality, it was proved by Kimura–Qin [29].

Theorem 2.10. *There is a ring isomorphism $\tilde{\chi}_q \colon K(\mathscr{C}_{\xi,1}) \xrightarrow{\simeq} A_\xi$ satisfying*

$$\tilde{\chi}_q(L_\xi(w)) = CC(\phi_\xi(w))$$

for all $w \in \mathbb{N}^{\mathtt{I} \sqcup \mathtt{I}}$. In particular, $\tilde{\chi}_q$ induces a bijection between the simple isomorphism classes of $\mathscr{C}_{\xi,1}$ and the cluster monomials of A_ξ.

2.5.5. *Conclusion.* Now, it is easy to see that Theorem 1.1 and Theorem 2.7 are mutually equivalent by Theorem 2.10 and (2.3).

3. Geometric preliminaries

In this section, we give a brief review of the geometric construction of finite-dimensional representations of $U_q(L\mathfrak{g})$ by means of equivariant constructible sheaves on the graded quiver varieties due to Nakajima. Basic references are [33, 34]. There are no new results in this section.

3.1. **Notation.** Let $\mathscr{V}_\mathbb{C}^\bullet$ denote the category of $\mathbb{Z}$-graded $\mathbb{C}$-vector spaces $V = \bigoplus_{k \in \mathbb{Z}} V^k$ of finite total dimension, i.e., $\sum_{n \in \mathbb{Z}} \dim V^n < \infty$, whose morphisms are homogeneous linear maps. Let t be an indeterminate. For $V \in \mathscr{V}_\mathbb{C}^\bullet$, its graded dimension is defined to be

$$\mathrm{gdim}(V) := \sum_{n \in \mathbb{Z}} (\dim V^n) t^n.$$

This is an element of $\mathbb{N}[t^{\pm 1}]$. For $V, W \in \mathscr{V}_\mathbb{C}^\bullet$ and $l \in \mathbb{Z}$, we denote by $\mathrm{Hom}^l(V, W)$ the space of $\mathbb{C}$-linear maps $f \colon V \to W$ of degree l, i.e., satisfying $f(V^n) \subset W^{n+l}$ for all $n \in \mathbb{Z}$. Let

$$G(V) := \mathrm{Hom}^0(V, V)^\times = \prod_{n \in \mathbb{Z}} GL(V^n).$$

In what follows, a variety always means a complex algebraic variety. When a complex algebraic group G acts on a variety X, we say that X is a G-variety. We set $\mathrm{pt} := \operatorname{Spec} \mathbb{C}$ and view it as a G-variety with the trivial action. Given a field $\Bbbk$ and a G-variety X, we denote by $D^b_G(X,\Bbbk)$ the bounded G-equivariant derived category of constructible $\Bbbk$-complexes on X in the sense of Bernstein–Lunts [2] (see also [1, Ch. 6]). This is a triangulated category equipped with a t-structure whose heart is identical to the category $\mathrm{Perv}_G(X,\Bbbk)$ of G-equivariant perverse sheaves on X. For any objects $\mathcal{F}$, $\mathcal{G} \in D^b_G(X,\Bbbk)$, we set

$$\mathrm{Hom}^\bullet_G(\mathcal{F},\mathcal{G}) \coloneqq \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}^n_G(\mathcal{F},\mathcal{G}), \quad \mathrm{Hom}^\bullet_G(\mathcal{F},\mathcal{G})^\wedge \coloneqq \prod_{n\in\mathbb{Z}} \mathrm{Hom}^n_G(\mathcal{F},\mathcal{G})$$

where $\mathrm{Hom}^n_G(\mathcal{F},\mathcal{G}) \coloneqq \mathrm{Hom}_{D^b_G(X,\Bbbk)}(\mathcal{F},\mathcal{G}[n])$ with $[1]$ being the shift functor. We also use the notations $\mathrm{H}^\bullet_G(\mathcal{F}) \coloneqq \mathrm{Hom}^\bullet_G(\underline{\Bbbk}_X,\mathcal{F})$ and $\widehat{\mathrm{H}}^\bullet_G(\mathcal{F}) \coloneqq \mathrm{Hom}^\bullet_G(\underline{\Bbbk}_X,\mathcal{F})^\wedge$, where $\underline{\Bbbk}_X$ is the constant $\Bbbk$-sheaf (of rank one). When $\mathcal{F} = \underline{\Bbbk}_X$, we recover the G-equivariant cohomology ring of X and hence write $\mathrm{H}^\bullet_G(X,\Bbbk) = \mathrm{H}^\bullet_G(\underline{\Bbbk}_X)$ and $\widehat{\mathrm{H}}^\bullet_G(X,\Bbbk) \coloneqq \widehat{\mathrm{H}}^\bullet_G(\underline{\Bbbk}_X)$.

For a group homomorphism $f\colon H \to G$, we have the associated functor $\mathrm{Res}_f\colon D^b_G(X,\Bbbk) \to D^b_H(X,\Bbbk)$ of equivariance change. When f is the inclusion of a subgroup or the quotient by a normal subgroup, we denote it by For^G_H or Infl^H_G respectively. When f is understood from the context, we often drop Res_f from the notation for the sake of simplicity. For instance, we often denote $\mathrm{Hom}^\bullet_H(\mathrm{Res}_f(\mathcal{F}),\mathrm{Res}_f(\mathcal{G}))$ and $\mathrm{H}^\bullet_H(\mathrm{Res}_f(\mathcal{F}))$ simply by $\mathrm{Hom}^\bullet_H(\mathcal{F},\mathcal{G})$ and $\mathrm{H}^\bullet_H(\mathcal{F})$ respectively. The following fact is used several times below.

Lemma 3.1 ([1, Lemma 6.7.4])**.** *Let $f\colon H \to G$ be a group homomorphism and $\mathcal{F} \in D^b_G(X,\Bbbk)$. If $\mathrm{H}^\bullet_G(\mathcal{F})$ is free over $\mathrm{H}^\bullet_G(\mathrm{pt},\Bbbk)$, we have*

$$\mathrm{H}^\bullet_H(\mathcal{F}) \simeq \mathrm{H}^\bullet_G(\mathcal{F}) \otimes_{\mathrm{H}^\bullet_G(\mathrm{pt},\Bbbk)} \mathrm{H}^\bullet_H(\mathrm{pt},\Bbbk),$$

where $\mathrm{H}^\bullet_G(\mathrm{pt},\Bbbk) \to \mathrm{H}^\bullet_H(\mathrm{pt},\Bbbk)$ is induced from f.

3.2. **Graded quiver varieties.** For $V = (V_i)_{i\in\mathtt{I}}$, $W = (W_i)_{i\in\mathtt{I}} \in (\mathscr{V}^\bullet_\mathbb{C})^\mathtt{I}$, we consider the space of linear maps

$$\mathrm{M}^\bullet(V,W) \coloneqq \bigoplus_{i,j\in\mathtt{I}, c_{ij}<0} \mathrm{Hom}^{-1}(V_i,V_j) \oplus \bigoplus_{i\in\mathtt{I}} \left(\mathrm{Hom}^{-1}(V_i,W_i) \oplus \mathrm{Hom}^{-1}(W_i,V_i)\right),$$

where $(c_{ij})_{i,j\in\mathtt{I}}$ is the Cartan matrix of $\mathfrak{g}$. The groups

$$G(V) \coloneqq \prod_{i\in\mathtt{I}} G(V_i), \quad G(W) \coloneqq \prod_{i\in\mathtt{I}} G(W_i)$$

act on $\mathrm{M}^\bullet(V,W)$ by conjugation. Let $\mu\colon \mathrm{M}^\bullet(V,W) \to \bigoplus_{i\in\mathtt{I}} \mathrm{Hom}^{-2}(V_i,V_i)$ be the $G(V)$-equivariant map given by

$$\mu((B_{j,i}),(a_i),(b_i)) \coloneqq (\textstyle\sum_{j\in\mathtt{I},c_{ij}<0} B_{i,j}B_{j,i} + b_i a_i)_{i\in\mathtt{I}},$$

where $B_{j,i} \in \mathrm{Hom}^{-1}(V_i,V_j)$, $a_i \in \mathrm{Hom}^{-1}(V_i,W_i)$, and $b_i \in \mathrm{Hom}^{-1}(W_i,V_i)$. We say that a point $((B_{j,i}),(a_i),(b_i)) \in \mu^{-1}(0)$ is stable if there is no non-zero $\mathbb{Z}$-graded linear subspace $V'_i \subset V_i$ for any $i \in \mathtt{I}$ such that $B_{j,i}(V'_i) = 0$ for any $j \in \mathtt{I}$ with $c_{ij} < 0$. The group $G(V)$ acts freely on the (possibly empty) open subset $\mu^{-1}(0)^{st} \subset \mu^{-1}(0)$ of stable points. The quotient

$$\mathfrak{M}^\bullet(V,W) \coloneqq \mu^{-1}(0)^{st}/G(V)$$

is a smooth quasi-projective $G(W)$-variety. It can be identified with a quotient in the geometric invariant theory. In particular, it comes with a natural projective $G(W)$-equivariant morphism

$$\pi_{V,W} \colon \mathfrak{M}^\bullet(V,W) \to \mathfrak{M}_0^\bullet(V,W) := \operatorname{Spec} \mathbb{C}[\mu^{-1}(0)]^{G(V)}. \tag{3.1}$$

Both varieties $\mathfrak{M}^\bullet(V,W)$ and $\mathfrak{M}_0^\bullet(V,W)$ only depend on the graded dimension vector of V. Therefore, we write $\mathfrak{M}^\bullet(\mathbf{v},W) := \mathfrak{M}^\bullet(V,W)$, $\mathfrak{M}_0^\bullet(\mathbf{v},W) := \mathfrak{M}_0^\bullet(V,W)$, and $\pi_{\mathbf{v},W} := \pi_{V,W}$ when $\mathbf{v} = (\operatorname{gdim}(V_i))_{i\in \mathtt{I}} \in \mathbb{N}[t^{\pm 1}]^{\mathtt{I}}$.

A geometric point of the affine variety $\mathfrak{M}_0^\bullet(\mathbf{v},W)$ corresponds to a closed $G(V)$-orbit in $\mu^{-1}(0)$. Let $\mathfrak{M}_0^\bullet(\mathbf{v},W)^{reg}$ be the smooth open subvariety of $\mathfrak{M}_0^\bullet(\mathbf{v},W)$ corresponding to free orbits. It is non-empty if and only if

$$\mathfrak{M}^\bullet(\mathbf{v},W) \neq \varnothing \quad \text{and} \quad (\operatorname{gdim}(W_i))_{i\in\mathtt{I}} - C(t)\cdot \mathbf{v} \in \mathbb{N}[t^{\pm 1}]^{\mathtt{I}}, \tag{3.2}$$

where $C(t) := (\frac{t^{c_{ij}} - t^{-c_{ij}}}{t - t^{-1}})_{i,j\in\mathtt{I}}$ is the quantum Cartan matrix. The set

$$\Lambda^+(W) := \{\mathbf{v} \in \mathbb{N}[t^{\pm 1}]^{\mathtt{I}} \mid \text{the condition (3.2) is satisfied}\}$$

is finite. For $\mathbf{v} \in \Lambda^+(W)$, the morphism $\pi_{\mathbf{v},W}$ restricts to an isomorphism

$$\pi_{\mathbf{v},W}^{-1}(\mathfrak{M}_0^\bullet(\mathbf{v},W)^{reg}) \xrightarrow{\simeq} \mathfrak{M}_0^\bullet(\mathbf{v},W)^{reg}. \tag{3.3}$$

For $\mathbf{v},\mathbf{v}' \in \mathbb{N}[t^{\pm 1}]^{\mathtt{I}}$, we have a natural closed embedding $\mathfrak{M}_0^\bullet(\mathbf{v},W) \subset \mathfrak{M}_0^\bullet(\mathbf{v}+\mathbf{v}',W)$. Taking the unions over $\mathbf{v} \in \mathbb{N}[t^{\pm 1}]^{\mathtt{I}}$, we define

$$\mathfrak{M}^\bullet(W) := \bigsqcup_{\mathbf{v}} \mathfrak{M}^\bullet(\mathbf{v},W), \quad \mathfrak{M}_0^\bullet(W) := \bigcup_{\mathbf{v}} \mathfrak{M}_0^\bullet(\mathbf{v},W).$$

These unions are essentially finite as $\mathfrak{M}^\bullet(\mathbf{v},W) \neq \varnothing$ only for finitely many $\mathbf{v}$ and $\mathfrak{M}_0^\bullet(\mathbf{v},W)$ stabilizes for sufficiently large $\mathbf{v}$. The morphisms (3.1) are unified into a $G(W)$-equivariant projective morphism

$$\pi_W \colon \mathfrak{M}^\bullet(W) \to \mathfrak{M}_0^\bullet(W).$$

The locally closed subvarieties $\{\mathfrak{M}_0^\bullet(\mathbf{v},W)^{reg}\}_{\mathbf{v}\in\Lambda^+(W)}$ give a finite stratification of $\mathfrak{M}_0^\bullet(W)$. Note that $\mathfrak{M}_0^\bullet(0,W)^{reg} = \{0\}$ is the unique closed stratum.

In what follows, we assume that $\Bbbk$ is an algebraically closed field containing $\mathbb{Q}(q)$ as in §2.1. Consider the proper push-forward $(\pi_W)_! \underline{\Bbbk}_{\mathfrak{M}^\bullet(W)}$ of the constant $\Bbbk$-sheaf on $\mathfrak{M}^\bullet(W)$ and let $\mathcal{L}'_W := {}^p\mathrm{H}^\bullet((\pi_W)_!\underline{\Bbbk}_{\mathfrak{M}^\bullet(W)})$ denote its total perverse cohomology. By the decomposition theorem, this is a semisimple object in $\operatorname{Perv}_{G(W)}(\mathfrak{M}_0^\bullet(W),\Bbbk)$. More precisely, we have

$$\mathcal{L}'_W \simeq \bigoplus_{\mathbf{v}\in\Lambda^+(W)} L_{\mathbf{v},W} \boxtimes \mathrm{IC}_{\mathbf{v},W}, \tag{3.4}$$

where $\mathrm{IC}_{\mathbf{v},W} \in D^b_{G(W)}(\mathfrak{M}_0^\bullet(W),\Bbbk)$ is the intersection cohomology complex of $\overline{\mathfrak{M}_0^\bullet(\mathbf{v},W)^{reg}}$ and $L_{\mathbf{v},W}$ is a non-zero finite-dimensional $\Bbbk$-vector space. Note that $\mathrm{IC}_{0,W}$ is the skyscraper sheaf $\underline{\Bbbk}_{\{0\}}$ at the origin $0 \in \mathfrak{M}_0^\bullet(W)$.

3.3. **Nakajima's homomorphism.** For each $W \in (\mathscr{V}_{\mathbb{C}}^{\bullet})^{\mathtt{I}}$, we consider the completed Yoneda algebra $\mathrm{Hom}^{\bullet}_{G(W)}(\mathcal{L}'_W, \mathcal{L}'_W)^{\wedge}$. Note that this is the completion of an $\mathbb{N}$-graded algebra $\mathrm{Hom}^{\bullet}_{G(W)}(\mathcal{L}'_W, \mathcal{L}'_W)$ with a semisimple 0-th component $\mathrm{Hom}^{0}_{G(W)}(\mathcal{L}'_W, \mathcal{L}'_W) \simeq \prod_{\mathbf{v} \in \Lambda^+(W)} \mathrm{End}_{\Bbbk}(L_{\mathbf{v},W})$. In particular, each $L_{\mathbf{v},W}$ can be regarded as a simple module over $\mathrm{Hom}^{\bullet}_{G(W)}(\mathcal{L}'_W, \mathcal{L}'_W)^{\wedge}$ and the set $\{L_{\mathbf{v},W}\}_{\mathbf{v} \in \Lambda^+(W)}$ gives a complete system of simple modules.

Theorem 3.2 (Nakajima [33])**.** *There is a homomorphism of $\Bbbk$-algebras*

$$\varphi'_W \colon U_q(L\mathfrak{g}) \to \mathrm{Hom}^{\bullet}_{G(W)}(\mathcal{L}'_W, \mathcal{L}'_W)^{\wedge}$$

satisfying the following property. For any $\mathbf{v} \in \Lambda^+(W)$, the pullback $(\varphi'_W)^* L_{\mathbf{v},W}$ is a simple $U_q(L\mathfrak{g})$-module in $\mathscr{C}$ isomorphic to $L(\prod_{i \in \mathtt{I}, n \in \mathbb{Z}} \varpi_{i,q^n}^{m_{i,n}})$, where the multiplicities $m_{i,n} \in \mathbb{N}$ are determined by the formula

$$\left(\textstyle\sum_{n \in \mathbb{Z}} m_{i,n} t^n\right)_{i \in \mathtt{I}} = (\mathrm{gdim}(W_i))_{i \in \mathtt{I}} - C(t) \cdot \mathbf{v}.$$

In particular, when $\mathbf{v} = 0$, we have $(\varphi'_W)^* L_{0,W} \simeq L(\varpi_W)$, where

$$\varpi_W := \prod_{i, \in \mathtt{I}, n \in \mathbb{Z}} \varpi_{i,q^n}^{\dim W_i^n}. \tag{3.5}$$

Proof. The $\Bbbk$-algebra homomorphism φ_W is obtained as the composition of (i) a completion of the homomorphism in [33, Theorem 9.4.1] from $U_q(L\mathfrak{g})$ to the convolution algebra $\widehat{K}^{G(W)}(Z^{\bullet}(W))_{\Bbbk}$ of the completed $G(W)$-equivariant K-theory (see [17, §4.6] for details), where $Z^{\bullet}(W) := \mathfrak{M}^{\bullet}(W) \times_{\mathfrak{M}_0^{\bullet}(W)} \mathfrak{M}^{\bullet}(W)$ is the Steinberg type variety, (ii) the $G(W)$-equivariant Chern character map (suitably twisted by the Todd classes) from $\widehat{K}^{G(W)}(Z^{\bullet}(W))_{\Bbbk}$ to the convolution algebra $\widehat{\mathrm{H}}^{G(W)}_{\bullet}(Z^{\bullet}(W), \Bbbk)$ of the completed $G(W)$-equivariant Borel–Moore homology (equivariant version of [9, Theorem 5.11.11]), and (iii) the completion of an isomorphism between $\mathrm{H}^{G(W)}_{\bullet}(Z^{\bullet}(W), \Bbbk)$ and $\mathrm{Hom}^{\bullet}_{G(W)}(\mathcal{L}'_W, \mathcal{L}'_W)$ (equivariant version of the isomorphism in [9, §8.6]). The desired property is due to [33, Theorem 14.3.2]. □

3.4. **Deformed standard modules.** For each $\varpi \in (1 + z\Bbbk[z])^{\mathtt{I}}$, the *standard module* (also known as the *local Weyl module* in the sense of Chari–Pressley [8]) $M(\varpi)$ is defined. It is the largest ℓ-highest weight module in $\mathscr{C}$ and it has $L(\varpi)$ as a unique simple quotient.

Fix $W \in (\mathscr{V}_{\mathbb{C}}^{\bullet})^{\mathtt{I}}$. Let $T(W) = \mathbb{C}^{\times}\mathsf{id}_W$ denote the one-dimensional torus of non-zero scalar matrices in $G(W)$ and $i_0 \colon \{0\} \to \mathfrak{M}_0^{\bullet}(W)$ the inclusion. The action of $T(W)$ on $\mathfrak{M}_0^{\bullet}(W)$ is trivial. Through the Yoneda product and the functor $\mathrm{For}^{G(W)}_{T(W)}$, the algebra $\mathrm{Hom}^{\bullet}_{G(W)}(\mathcal{L}'_W, \mathcal{L}'_W)^{\wedge}$ acts on $\widehat{\mathrm{H}}^{\bullet}_{T(W)}(i_0^! \mathcal{L}'_W)$. Via φ'_W, this yields a geometric realization of the deformed standard module $M(\varpi_W)[\![u]\!]$ (recall the definition of $M[\![u]\!]$ from Remark 2.3 and ϖ_W from (3.5)) as follows.

Theorem 3.3 (Nakajima [35])**.** *We have*

$$(\varphi'_W)^* \widehat{\mathrm{H}}^{\bullet}_{T(W)}(i_0^! \mathcal{L}'_W) \simeq M(\varpi_W)[\![u]\!]$$

as $U_q(L\mathfrak{g})[\![u]\!]$-modules, where the action of u on the left-hand side is given by the product with a non-zero element of $\mathrm{H}^2_{T(W)}(\mathrm{pt}, \Bbbk)$.

Proof. This follows from [35, Theorem 2 and Remark 2.15] and a completion. Note that $\mathrm{H}^\bullet_{G(W)}(i_0^!\mathcal{L}'_W) \simeq \mathrm{H}^{G(W)}_\bullet(\pi^{-1}(0), \Bbbk)$ is free over $\mathrm{H}^\bullet_{G(W)}(\mathrm{pt}, \Bbbk)$ by [33, §7.1] and hence Lemma 3.1 is applicable. □

3.5. **Tensor product.** Let $W, W' \in (\mathscr{V}^\bullet_{\mathbb{C}})^{\mathrm{I}}$. We identify the one-dimensional torus $T(W') \subset G(W')$ with the subtorus $\mathsf{id}_W \oplus \mathbb{C}^\times \mathsf{id}_{W'}$ of $G(W \oplus W')$. By [39, Lemma 3.1], the $T(W')$-fixed locus $\mathfrak{M}^\bullet_0(W \oplus W')^{T(W')}$ is identical to $\mathfrak{M}^\bullet_0(W) \times \mathfrak{M}^\bullet_0(W')$. Consider the attracting locus

$$\mathfrak{A}^\pm(W, W') := \{x \in \mathfrak{M}^\bullet_0(W \oplus W') \mid \lim_{s\to 0}(\mathsf{id}_W \oplus s^{\pm 1}\mathsf{id}_{W'})x \text{ exists.}\},$$

which is Zariski closed by [39, 3.5]. We have $\mathfrak{A}^\pm(W, W') = \mathfrak{A}^\mp(W', W)$. Consider the diagram

$$\mathfrak{M}^\bullet_0(W) \times \mathfrak{M}^\bullet_0(W') \xleftarrow{p'_\pm} \mathfrak{A}^\pm(W, W') \xrightarrow{h'_\pm} \mathfrak{M}^\bullet_0(W \oplus W'), \tag{3.6}$$

where $h'_\pm$ is the inclusion and $p'_\pm(x) := \lim_{s\to 0}(\mathsf{id}_W \oplus s^{\pm 1}\mathsf{id}_{W'})x$, and the hyperbolic localization $(p'_\pm)_!(h'_\pm)^* \simeq (p'_\mp)_*(j'_\mp)^!$ in the sense of Braden [3]. By [39, Lemma 4.1], $(p'_\pm)_!(h'_\pm)^*\mathcal{L}'_{W\oplus W'}$ is a semisimple complex and

$${}^p\mathrm{H}^\bullet((p'_\pm)_!(h'_\pm)^*\mathcal{L}'_{W\oplus W'}) \simeq \mathcal{L}'_W \boxtimes \mathcal{L}'_{W'} \tag{3.7}$$

in $\mathrm{Perv}_{G(W)\times G(W')}(\mathfrak{M}^\bullet_0(W) \times \mathfrak{M}^\bullet_0(W'))$. The isomorphism (3.7) (together with [1, Proposition 6.7.5]) yields the isomorphisms

$$\widehat{\mathrm{H}}^\bullet_{G(W)\times G(W')}(i_0^!(p'_-)_!(h'_-)^*\mathcal{L}'_{W\oplus W'}) \simeq \widehat{\mathrm{H}}^\bullet_{G(W)}(i_0^!\mathcal{L}'_W)\widehat{\otimes}\widehat{\mathrm{H}}^\bullet_{G(W')}(i_0^!\mathcal{L}'_{W'}),$$
$$\widehat{\mathrm{H}}^\bullet_{G(W)\times G(W')}(i_0^!(p'_+)_!(h'_+)^*\mathcal{L}'_{W\oplus W'}) \simeq \widehat{\mathrm{H}}^\bullet_{G(W')}(i_0^!\mathcal{L}'_{W'})\widehat{\otimes}\widehat{\mathrm{H}}^\bullet_{G(W)}(i_0^!\mathcal{L}'_W),$$

where $\widehat{\otimes}$ denotes the completed tensor product and i_0 denotes the inclusions of the origin (into suitable varieties). A sheaf-theoretic interpretation of the results from [34] tells us that these isomorphisms are compatible with the structures of $U_q(L\mathfrak{g})$-modules, given through the homomorphism $\varphi'_{W\oplus W'}$ on the left-hand sides, and through $(\varphi'_W \otimes \varphi'_{W'}) \circ \Delta$ and $(\varphi'_{W'} \otimes \varphi'_W) \circ \Delta$ respectively on the right hand sides, where Δ is the coproduct of $U_q(L\mathfrak{g})$. In particular, applying $\mathrm{For}^{G(W)\times G(W')}_{T(W')}$, we get the following from Theorem 3.3. (We can freely use Lemma 3.1 here, as the freeness assumption is satisfied, see [33, §7.1], [34, Theorem 3.10(1)].)

Theorem 3.4 (Nakajima [34])**.** *We have*

$$(\varphi'_{W\oplus W'})^*\widehat{\mathrm{H}}^\bullet_{T(W')}(i_0^!(p'_-)_!(h'_-)^*\mathcal{L}'_{W\oplus W'}) \simeq M(\varpi_W) \otimes (M(\varpi_{W'})[\![u]\!]),$$
$$(\varphi'_{W\oplus W'})^*\widehat{\mathrm{H}}^\bullet_{T(W')}(i_0^!(p'_+)_!(h'_+)^*\mathcal{L}'_{W\oplus W'}) \simeq (M(\varpi_{W'})[\![u]\!]) \otimes M(\varpi_W)$$

as $U_q(L\mathfrak{g})[\![u]\!]$-modules, where the action of u on the left-hand sides is given by the product with a non-zero element of $\mathrm{H}^2_{T(W')}(\mathrm{pt}, \Bbbk)$.

4. Proof of main theorem

In this section, we prove our main theorem (= Theorem 2.7) applying the geometric construction reviewed in the previous section. In §4.1, we recall the key observation due to Kimura–Qin [29], which enables us to translate the constructions with equivariant perverse sheaves on the graded quiver varieties to those on the space $X(w)$ of injective copresentations of the

Dynkin quiver Q_ξ through the Fourier–Laumon transformation explained in §4.2. Then, we obtain a sheaf theoretic interpretation of deformed simple modules and their tensor products in §§4.3–4.4. In §4.5, we observe that the E-invariant appears as a transversal slice in $X(w)$. The proof ends in §4.6, where we have a sheaf theoretic interpretation of R-matrices in question under a certain condition (4.5).

4.1. **Graded quiver varieties for $\mathscr{C}_{\xi,1}$.** Recall that we have fixed a height function $\xi\colon \mathtt{I} \to \mathbb{Z}$ and the notations from §2.4. For each vertex $i \in \mathtt{I}$, let $P_i \in \mathsf{rep}(\mathbb{C}Q_\xi)$ be a projective cover of S_i. For each $w \in \mathbb{N}^{\mathtt{I}\sqcup\mathtt{I}}$, we set

$$X'(w) := \mathrm{Hom}_{Q_\xi}(P^{w(1)}, I^{w(0)}),$$

where $P^{w(1)} := \bigoplus_{i\in\mathtt{I}} P_i^{\oplus w_i(1)}$. Note that the vector spaces $X(w)$ and $X'(w)$ are dual to each other through the Nakayama functor. Moreover, the group $\mathrm{Aut}_{Q_\xi}(P^{w(1)})$ is naturally identified with $\mathrm{Aut}_{Q_\xi}(I^{w(1)})$ and hence the group $A(w)$ defined in §2.4 acts on $X'(w)$ as well. For $v \in \mathbb{N}^{\mathtt{I}}$, let $\mathrm{Gr}_v(I^{w(0)})$ denote the submodule Grassmannian of $I^{w(0)}$, which is a smooth connected projective variety by [38, Theorem 4.10]. Consider an $A(w)$-variety

$$F(v,w) := \{(N,\psi) \in \mathrm{Gr}_v(I^{w(0)}) \times X'(w) \mid \mathrm{Im}\,\psi \subset N\},$$

together with an $A(w)$-equivariant projective morphism

$$p_{v,w}\colon F(v,w) \to X'(w)$$

given by the second projection $(N,\psi) \mapsto \psi$.

Given $w \in \mathbb{N}^{\mathtt{I}\sqcup\mathtt{I}}$, we choose an $\mathtt{I}$-tuple of $\mathbb{Z}$-graded vector spaces $W_\xi(w) = (W_\xi(w)_i)_{i\in\mathtt{I}} \in (\mathscr{V}_{\mathbb{C}}^\bullet)^{\mathtt{I}}$ satisfying

$$\mathrm{gdim}(W_\xi(w)_i) = w_i(0)t^{\xi(i)} + w_i(1)t^{\xi(i)+2}$$

for all $i \in \mathtt{I}$. Note that $\Lambda^+(W_\xi(w)) \subset (\mathbb{N}t^{\xi(i)+1})^{\mathtt{I}}$ holds. For $v = (v_i)_{i\in\mathtt{I}} \in \mathbb{N}^{\mathtt{I}}$, we put $vt^{\xi+1} := (v_i t^{\xi(i)+1})_{i\in\mathtt{I}}$.

The following observation due to Kimura–Qin [29], generalizing the one by Nakajima [36], is of fundamental importance in our discussion below.

Proposition 4.1 ([29, Propositions 3.1.1 & 3.1.4]). *For any $w \in \mathbb{N}^{\mathtt{I}\sqcup\mathtt{I}}$ and $v \in \mathbb{N}^{\mathtt{I}}$, we have isomorphisms of varieties*

$$\mathfrak{M}_0^\bullet(W_\xi(w)) \simeq X'(w), \quad \mathfrak{M}^\bullet(vt^{\xi+1}, W_\xi(w)) \simeq F(v,w),$$

through which the morphism $\pi_{vt^{\xi+1},W_\xi(w)}$ corresponds to the morphism $p_{v,w}$, and the actions of $G(W_\xi(w))$ correspond to the actions of the standard Levi subgroup $G(w) := \prod_{i\in\mathtt{I},k\in\{0,1\}} \mathrm{Aut}_{Q_\xi}(I_i^{w_i(k)})$ of $A(w)$.

In what follows, we identify the variety $\mathfrak{M}_0^\bullet(W_\xi(w))$ with the variety $X'(w)$ through the isomorphism in Proposition 4.1, and identify $G(w)$ with $G(W_\xi(w))$. Note that the functor $\mathrm{For}_{G(w)}^{A(w)}\colon D^b_{A(w)}(X,\Bbbk) \to D^b_{G(w)}(X,\Bbbk)$ is fully faithful for any $A(w)$-variety X (cf. [1, Theorem 6.6.15]).

Corollary 4.2. *For any $w \in \mathbb{N}^{\mathtt{I}\sqcup\mathtt{I}}$, the object $\mathcal{L}'_{W_\xi(w)}$ is in the essential image of the functor $\mathrm{For}_{G(w)}^{A(w)}\colon \mathrm{Perv}_{A(w)}(X'(w),\Bbbk) \to \mathrm{Perv}_{G(w)}(X'(w),\Bbbk)$.*

Proof. Recall the decomposition (3.4). For any $vt^{\xi+1} \in \Lambda^+(W_\xi(w))$, we know that the simple perverse sheaf $\mathrm{IC}_{vt^{\xi+1},W_\xi(w)}$ appears as a direct summand of a shift of $(p_{v,w})_! \underline{\Bbbk}_{F(v,w)} \in D^b_{A(w)}(X'(w), \Bbbk)$ thanks to Proposition 4.1 and the isomorphism (3.3). Thus $\mathrm{IC}_{vt^{\xi+1},W_\xi(w)}$ is in fact $A(w)$-equivariant and so is $\mathcal{L}'_{W_\xi(w)}$. □

In particular, the functor $\mathrm{For}^{A(w)}_{G(w)}$ gives a $\Bbbk$-algebra isomorphism:

$$\mathrm{Hom}^\bullet_{G(w)}(\mathcal{L}'_{W_\xi(w)}, \mathcal{L}'_{W_\xi(w)})^\wedge \simeq \mathrm{Hom}^\bullet_{A(w)}(\mathcal{L}'_{W_\xi(w)}, \mathcal{L}'_{W_\xi(w)})^\wedge. \tag{4.1}$$

4.2. **Fourier–Laumon transform.** We regard the vector spaces $X'(w)$ as an $(A(w) \times \mathbb{C}^\times)$-variety, where $\mathbb{C}^\times$ acts simply by the scalar multiplication. Note that this action factors through the surjective homomorphism

$$A(w) \times \mathbb{C}^\times \to A(w) \quad \text{given by } (g, s) \mapsto g \cdot (\mathsf{id}_{P^{w(1)}}, s\mathsf{id}_{I^{w(0)}}).$$

The space $X(w) = X'(w)^*$ is also viewed an $(A(w) \times \mathbb{C}^\times)$-variety by the dual action. Consider the $A(w)$-equivariant Fourier–Laumon transform

$$\Phi_{X'(w)} \colon D^b_{A(w)\times\mathbb{C}^\times}(X'(w), \Bbbk) \xrightarrow{\simeq} D^b_{A(w)\times\mathbb{C}^\times}(X(w), \Bbbk)$$

introduced in [30] (see also [1, §6.9]). We define

$$\mathcal{L}_w \coloneqq \Phi_w(\mathcal{L}'_{W_\xi(w)}), \quad \text{where } \Phi_w \coloneqq \mathrm{For}^{A(w)\times\mathbb{C}^\times}_{A(w)} \circ \Phi_{X'(w)} \circ \mathrm{Infl}^{A(w)\times\mathbb{C}^\times}_{A(w)}.$$

Since Φ_w sends $A(w)$-equivariant simple perverse sheaves on $X'(w)$ bijectively to the ones on $X(w)$, $\mathcal{L}_w$ is an $A(w)$-equivariant semisimple perverse sheaf on $X(w)$. Letting $\mathrm{IC}_{v,w} \coloneqq \Phi_w(\mathrm{IC}_{vt^{\xi+1},W_\xi(w)})$ and $L_{v,w} \coloneqq L_{vt^{\xi+1},W_\xi(w)}$, the functor Φ_w translates (3.4) into

$$\mathcal{L}_w \simeq \bigoplus_v L_{v,w} \boxtimes \mathrm{IC}_{v,w}$$

where v runs over the set of elements $v \in \mathbb{N}^{\mathrm{I}}$ satisfying $vt^{\xi+1} \in \Lambda^+(W_\xi(w))$. Since $X(w)$ has finitely many $A(w)$-orbits and the stabilizer $\mathrm{Aut}(\phi)$ of each closed point ϕ is connected, the intersection cohomology complexes of orbit closures exhaust the simple $A(w)$-equivariant perverse sheaves on $X(w)$. When $v = 0$, we have

$$\mathrm{IC}_{0,w} = \Phi_w(\underline{\Bbbk}_{\{0\}}) \simeq \underline{\Bbbk}_{X(w)}[\dim X(w)].$$

We define a $\Bbbk$-algebra homomorphism $\varphi_w \colon U_q(L\mathfrak{g}) \to \mathrm{Hom}^\bullet_{A(w)}(\mathcal{L}_w, \mathcal{L}_w)^\wedge$ to be the following composition:

$$\varphi_w \colon U_q(L\mathfrak{g}) \xrightarrow{\varphi'_{W_\xi(w)}} \mathrm{Hom}^\bullet_{G(w)}(\mathcal{L}'_{W_\xi(w)}, \mathcal{L}'_{W_\xi(w)})^\wedge$$
$$\xrightarrow{(4.1)} \mathrm{Hom}^\bullet_{A(w)}(\mathcal{L}'_{W_\xi(w)}, \mathcal{L}'_{W_\xi(w)})^\wedge \xrightarrow{\Phi_w} \mathrm{Hom}^\bullet_{A(w)}(\mathcal{L}_w, \mathcal{L}_w)^\wedge.$$

4.3. **Deformed simple modules.** For the sake of brevity, we set $T(w) \coloneqq T(W_\xi(w)) \subset G(w) = G(W_\xi(w))$ and $M_\xi(w) \coloneqq M(\varpi_{W_\xi(w)})$. The latter is compatible with the notation (2.2) as we have $L_\xi(w) = L(\varpi_{W_\xi(w)})$ (compare (2.2) with (3.5)). Recall the generic element $\phi_\xi(w) \in X(w)$ from Definition 2.5. For a closed point $\phi \in X(w)$, let $i_\phi \colon \{\phi\} \to X(w)$ denote the inclusion.

Proposition 4.3. *We have*

$$\varphi_w^* \widehat{\mathrm{H}}^{\bullet}_{T(w)}(\mathcal{L}_w) \simeq M_\xi(w)[\![u]\!], \quad \varphi_w^* \widehat{\mathrm{H}}^{\bullet}_{T(w)}(i^*_{\phi_\xi(w)}\mathcal{L}_w) \simeq L_\xi(w)[\![u]\!]$$

as $U_q(L\mathfrak{g})[\![u]\!]$-modules, where the action of u on the left-hand sides is given by the product with a non-zero element of $\mathrm{H}^2_{T(w)}(\mathrm{pt}, \Bbbk)$.

Proof. In this proof, we abbreviate $T(w)$, $X(w)$, $i_{\phi_\xi(w)}$ as T, X, i respectively. The first isomorphism follows from Theorem 3.3 through the transform Φ_w. In fact, as $\Phi_w(\underline{\Bbbk}_{\{0\}}) \simeq \underline{\Bbbk}_X[\dim X]$, we have

$$\widehat{\mathrm{H}}^{\bullet}_T(i_0^! \mathcal{L}'_{W_\xi(w)}) = \mathrm{Hom}^{\bullet}_T(\underline{\Bbbk}_{\{0\}}, \mathcal{L}'_{W_\xi(w)})^{\wedge} \overset{\Phi_w}{\simeq} \mathrm{Hom}^{\bullet}_T(\underline{\Bbbk}_X, \mathcal{L}_w)^{\wedge} \simeq \widehat{\mathrm{H}}^{\bullet}_T(\mathcal{L}_w).$$

We shall show the second isomorphism. The functor i^* yields a homomorphism

$$\widehat{\mathrm{H}}^{\bullet}_T(\mathcal{L}_w) = \mathrm{Hom}^{\bullet}_T(\underline{\Bbbk}_X, \mathcal{L}_w)^{\wedge} \xrightarrow{a} \mathrm{Hom}^{\bullet}_T(i^*\underline{\Bbbk}_X, i^*\mathcal{L}_w)^{\wedge} = \widehat{\mathrm{H}}^{\bullet}_T(i^*\mathcal{L}_w)$$

of modules over the Yoneda algebra $\mathrm{Hom}^{\bullet}_{A(w)}(\mathcal{L}_w, \mathcal{L}_w)^{\wedge}$. We recall that $A(w)\phi_\xi(w)$ is the unique open $A(w)$-orbit in $X(w)$ and the stabilizer of $\phi_\xi(w)$ is connected. Therefore, the constant perverse sheaf $\underline{\Bbbk}_X[\dim X]$ is the unique simple object of $\mathrm{Perv}_{A(w)}(X(w))$ whose stalk at $\phi_\xi(w)$ is non-zero. Thus, we have $i^*\mathrm{IC}_{v,w} = 0$ if $v \neq 0$ and hence $i^*\mathcal{L}_w = L_{0,w} \boxtimes i^*\underline{\Bbbk}_X[\dim X]$. Now, we see that $\widehat{\mathrm{H}}^{\bullet}_T(i^*\mathcal{L}_w) \simeq L_{0,w} \otimes \Bbbk[\![u]\!]$ as $\Bbbk[\![u]\!]$-modules, and that a is surjective as its restriction to the summand $L_{0,w} \boxtimes \mathrm{IC}_{0,w} \subset \mathcal{L}_w$ yields an isomorphism. As a $U_q(L\mathfrak{g})$-module, $\widehat{\mathrm{H}}^{\bullet}_T(i^*\mathcal{L}_w)$ is a limit of iterated self-extensions of the simple module $L_\xi(w)$.

Let $N_\xi(w)$ denote the kernel of the quotient homomorphism $M_\xi(w) \to L_\xi(w)$. The $U_q(L\mathfrak{g})$-module $N_\xi(w)$ does not contain $L_\xi(w)$ as its composition factor. Since $M \mapsto M[\![u]\!]$ is an exact functor, we have a short exact sequence

$$0 \to N_\xi(w)[\![u]\!] \xrightarrow{b} M_\xi(w)[\![u]\!] \xrightarrow{c} L_\xi(w)[\![u]\!] \to 0$$

of $U_q(L\mathfrak{g})[\![u]\!]$-modules.

We compare the homomorphisms a and c. For any positive integer n, we consider the base change from $\Bbbk[\![u]\!]$ to the truncated polynomial ring $\Bbbk[\![u]\!]/(u^n)$ to obtain the rigid arrows in the following diagram:

$$\begin{array}{ccccc}
N_\xi(w)[\![u]\!]/(u^n) & \overset{b_n}{\hookrightarrow} & M_\xi(w)[\![u]\!]/(u^n) & \overset{c_n}{\twoheadrightarrow} & L_\xi(w)[\![u]\!]/(u^n) \\
 & & \downarrow \simeq & & \downarrow \theta_n \\
 & & \widehat{\mathrm{H}}^{\bullet}_T(\mathcal{L}_w)/u^n\widehat{\mathrm{H}}^{\bullet}_T(\mathcal{L}_w) & \overset{a_n}{\twoheadrightarrow} & \widehat{\mathrm{H}}^{\bullet}_T(i^*\mathcal{L}_w)/u^n\widehat{\mathrm{H}}^{\bullet}_T(i^*\mathcal{L}_w),
\end{array}$$

where the upper row is exact. We know that both $L_\xi(w)[\![u]\!]/(u^n)$ and $\widehat{\mathrm{H}}^{\bullet}_T(i^*\mathcal{L}_w)/u^n\widehat{\mathrm{H}}^{\bullet}_T(i^*\mathcal{L}_w)$ are iterated self-extensions of $L_\xi(w)$ of the same length n (as $U_q(L\mathfrak{g})$-modules) and that the image of b_n does not contain $L_\xi(w)$ as a composition factor. Therefore, there exists a unique isomorphism θ_n of $U_q(L\mathfrak{g})$-modules represented by the dashed arrow in the diagram. Taking the limit $n \to \infty$, we get the desired isomorphism of $U_q(L\mathfrak{g})[\![u]\!]$-modules. □

4.4. **Tensor product.** Let $w, w' \in \mathbb{N}^{\mathrm{I}\sqcup\mathrm{I}}$. We fix decompositions $I^{(w+w')(k)} = I^{w(k)} \oplus I^{w'(k)}$, $k \in \{0,1\}$, to get

$$X(w+w') = X(w) \oplus X(w') \oplus X(w,w')^+ \oplus X(w,w')^-, \tag{4.2}$$

where $X(w,w')^+ := \mathrm{Hom}_{Q_\xi}(I^{w'(0)}, I^{w(1)})$ and $X(w,w')^- := \mathrm{Hom}_{Q_\xi}(I^{w(0)}, I^{w'(1)})$. Then we have $X(w+w')^{T(w')} = X(w) \oplus X(w')$. Consider the diagram

$$X(w) \oplus X(w') \xleftarrow{p_\pm} X(w) \oplus X(w') \oplus X(w,w')^\pm \xrightarrow{h_\pm} X(w+w'),$$

where $h_\pm$ is the inclusion and $p_\pm$ is the projection.

Proposition 4.4. *Let $\phi := \phi_\xi(w) \oplus \phi_\xi(w') \in X(w) \oplus X(w')$. We have*

$$\varphi^*_{w+w'} \widehat{\mathrm{H}}^\bullet_{T(w')}(i^*_\phi (p_+)_!(h_+)^* \mathcal{L}_{w+w'}) \simeq L_\xi(w) \otimes (L_\xi(w')[\![u]\!]),$$
$$\varphi^*_{w+w'} \widehat{\mathrm{H}}^\bullet_{T(w')}(i^*_\phi (p_-)_!(h_-)^* \mathcal{L}_{w+w'}) \simeq (L_\xi(w')[\![u]\!]) \otimes L_\xi(w).$$

as $U_q(L\mathfrak{g})[\![u]\!]$-modules, where the action of u on the left-hand side is given by the product with a non-zero element of $\mathrm{H}^2_{T(w')}(\mathrm{pt}, \Bbbk)$.

Proof. The assertion follows from Theorem 3.4, Proposition 4.3 and an analog of [31, Proposition 10.1.2]. For completeness, we give some details. Consider the decomposition

$$X'(w+w') = X'(w) \oplus X'(w') \oplus X'(w,w')^+ \oplus X'(w,w')^-,$$

where $X'(w,w')^+ := \mathrm{Hom}_{Q_\xi}(P^{w(1)}, I^{w'(0)})$ and $X'(w,w')^- := \mathrm{Hom}_{Q_\xi}(P^{w'(1)}, I^{w(0)})$. Then we have $X'(w+w')^{T(w')} = X'(w) \oplus X'(w')$. Under the identification $X'(w+w') = \mathfrak{M}^\bullet_0(W_\xi(w+w'))$ in Proposition 4.1, we have

$$\mathfrak{A}^\pm(W_\xi(w), W_\xi(w')) = X'(w) \oplus X'(w') \oplus X'(w,w')^\pm,$$

and $p'_\pm$ in (3.6) is the projection to $X'(w) \oplus X'(w')$. Through the Nakayama functor, $X(w,w')^\pm$ is dual to $X'(w,w')^\pm$. The dual of the diagram (3.6) is identified with

$$X(w) \oplus X(w') \xrightarrow{{}^t p'_\pm} X(w) \oplus X(w') \oplus X(w,w')^\pm \xleftarrow{{}^t h'_\pm} X(w+w')$$

where ${}^t p'_\pm$ is the inclusion and ${}^t h'_\pm$ is the projection with respect to the decomposition (4.2). By [1, Proposition 6.9.13], we have

$$(\Phi_w \boxtimes \Phi_{w'}) \circ (p'_\pm)_! \circ (h'_\pm)^* \simeq ({}^t p'_\pm)^* \circ ({}^t h'_\pm)_! \circ \Phi_{w+w'}[-d_\pm], \tag{4.3}$$

where $d_\pm := \dim X(w,w')^\pm - \dim X(w,w')^\mp$. Since the diagram

$$\begin{array}{ccc} X(w) \oplus X(w') \oplus X(w,w')^\pm & \xrightarrow{h_\pm} & X(w+w') \\ {\scriptstyle p_\pm}\downarrow & & \downarrow{\scriptstyle {}^t h'_\mp} \\ X(w) \oplus X(w') & \xrightarrow{{}^t p'_\mp} & X(w) \oplus X(w') \oplus X(w,w')^\mp \end{array}$$

is cartesian, we have the base change isomorphism

$$({}^t p'_\mp)^* \circ ({}^t h'_\mp)_! \simeq (p_\pm)_! \circ (h_\pm)^*. \tag{4.4}$$

Combining (4.3) with (4.4), we see that the Fourier–Laumon transform induces an isomorphism

$$\widehat{\mathrm{H}}^\bullet_{T(w')}(i^!_0 (p'_\mp)_!(h'_\mp)^* \mathcal{L}'_{W_\xi(w+w')}) \simeq \widehat{\mathrm{H}}^\bullet_{T(w')}((p_\pm)_!(h_\pm)^* \mathcal{L}_{w+w'}).$$

By Theorem 3.4 and the first isomorphism in Proposition 4.3 (or rather, by the discussion before Theorem 3.4), we get an isomorphism

$$\varphi^*_{w+w'}\widehat{\mathrm{H}}^\bullet_{T(w')}((p_+)_!(h_+)^*\mathcal{L}_{w+w'}) \simeq \varphi^*_w \mathrm{H}^\bullet(\mathcal{L}_w) \otimes \varphi^*_{w'}\widehat{\mathrm{H}}^\bullet_{T(w')}(\mathcal{L}_{w'})$$

of $U_q(L\mathfrak{g})[\![u]\!]$-modules. Applying the functor $i^*_\phi \simeq i^*_{\phi_\xi(w)} \boxtimes i^*_{\phi_\xi(w')}$, we obtain

$$\varphi^*_{w+w'}\widehat{\mathrm{H}}^\bullet_{T(w')}(i^*_\phi(p_+)_!(h_+)^*\mathcal{L}_{w+w'}) \simeq \varphi^*_w \mathrm{H}^\bullet(i^*_{\phi_\xi(w)}\mathcal{L}_w) \otimes \varphi^*_{w'}\widehat{\mathrm{H}}^\bullet_{T(w')}(i^*_{\phi_\xi(w')}\mathcal{L}_{w'}).$$

Together with the second isomorphism in Proposition 4.3, we get the first desired isomorphism. The other isomorphism is verified similarly. □

4.5. **Slice and E-invariant.** For any $w \in \mathbb{N}^{\mathrm{I}\sqcup\mathrm{I}}$ and any closed point $\phi \in X(w)$, we have an $A(w)$-equivariant linear map

$$f_\phi\colon \mathrm{End}_{Q_\xi}(I^{w(0)}) \oplus \mathrm{End}_{Q_\xi}(I^{w(1)}) \to X(w) \quad \text{given by } f_\phi(a,b) := b \circ \phi - \phi \circ a.$$

This is equal to the derivation of the action map $A(w) \ni g \mapsto g \cdot \phi \in X(w)$ at $g = 1$. By [10], we have $X(w)/\operatorname{Im} f_\phi \simeq E(\phi,\phi)$ as vector spaces. In particular, ϕ is rigid (i.e., $E(\phi,\phi) = 0$) if and only if the $A(w)$-orbit of ϕ is open in $X(w)$, that is when $\phi \simeq \phi_\xi(w)$.

In what follows, we consider the special case when $\phi = \phi_\xi(w) \oplus \phi_\xi(w')$ as in the previous section. The decomposition (4.2) induces the corresponding decomposition of $E(\phi,\phi)$ again by [10]. Namely, letting

$$\epsilon\colon X(w+w') \to X(w+w')/\operatorname{Im} f_\phi \simeq E(\phi,\phi)$$

be the quotient map, we have

$$\epsilon(X(w)) \simeq E(\phi_\xi(w),\phi_\xi(w)) = 0, \qquad \epsilon(X(w')) \simeq E(\phi_\xi(w'),\phi_\xi(w')) = 0,$$
$$\epsilon(X(w,w')^+) \simeq E(\phi_\xi(w'),\phi_\xi(w)), \quad \epsilon(X(w,w')^-) \simeq E(\phi_\xi(w),\phi_\xi(w')).$$

Choose a linear subspace $E^\pm$ of $X(w,w')^\pm$ stable under the action of the torus $T(w) \times T(w')$ such that the map ϵ restricts to isomorphisms $E^+ \simeq E(\phi_\xi(w),\phi_\xi(w'))$ and $E^- \simeq E(\phi_\xi(w'),\phi_\xi(w))$ respectively. We define

$$S := \phi + (E^+ \oplus E^-), \quad S^\pm := \phi + E^\pm,$$

which are affine subspaces of $X(w+w')$ stable under the action of the torus $T(w) \times T(w')$. Let

$$\{\phi\} \xrightarrow{i_\pm} S^\pm \xrightarrow{j_\pm} S \xrightarrow{i_S} X(w+w')$$

denote the inclusions.

Lemma 4.5. *With the above notation, we have a natural isomorphism*

$$i^*_\phi(p_\pm)_! h^*_\pm \simeq i^!_\pm j^*_\pm i^*_S[e_\pm]$$

of functors from $D^b_{A(w+w')}(X(w+w'),\mathbb{k})$ *to* $D^b_{T(w)\times T(w')}(\{\phi\},\mathbb{k})$*, where*

$$e_\pm := 2(\dim X(w,w)^\pm - \dim E^\pm).$$

Proof. A proof can be parallel to that of [18, Lemma 7.7]. □

For $\mathcal{F} \in D^b_{A(w+w')}(X(w+w'),\mathbb{k})$, we define

$$\mathcal{F}|_S := i^*_S\mathcal{F}[\dim S - \dim X(w+w')].$$

Proposition 4.6. *With the above notation, we have*

$$\varphi^*_{w+w'}\widehat{\mathrm{H}}^\bullet_{T(w')}((i_+)^!(j_+)^*\mathcal{L}_{w+w'}|_S) \simeq L_\xi(w) \otimes (L_\xi(w')[\![u]\!]),$$
$$\varphi^*_{w+w'}\widehat{\mathrm{H}}^\bullet_{T(w')}((i_-)^!(j_-)^*\mathcal{L}_{w+w'}|_S) \simeq (L_\xi(w')[\![u]\!]) \otimes L_\xi(w).$$

as $U_q(L\mathfrak{g})[\![u]\!]$*-modules.*

Proof. The assertion follows from Proposition 4.4 and Lemma 4.5. □

Corollary 4.7. *Let* $w, w' \in \mathbb{N}^{\mathtt{I}\sqcup\mathtt{I}}$*. If we have* $E(\phi_\xi(w), \phi_\xi(w')) = E(\phi_\xi(w'), \phi_\xi(w)) = 0$*, then* $L_\xi(w)$ *and* $L_\xi(w')$ *strongly commute. In particular, any simple module* $L_\xi(w)$ *of the category* $\mathscr{C}_{\xi,1}$ *is real.*

Proof. Under the assumption, we have $S = S^\pm = \{\phi_\xi(w + w')\}$, and therefore $(i_+)^!(j_+)^*\mathcal{L}_{w+w'}|_S$ coincides with $i^*_{\phi_\xi(w+w')}\mathcal{L}_{w+w'}$ up to a shift. Then Propositions 4.3 & 4.6 yield

$$L_\xi(w) \otimes L_\xi(w') \simeq \varphi^*_{w+w'}\mathrm{H}^\bullet(i^*_{\phi_\xi(w+w')}\mathcal{L}_{w+w'}) \simeq L_\xi(w + w'). \qquad \square$$

Note that the above Corollary 4.7 and its converse follow from Theorem 2.10 as well, although we do not rely on it in our proof.

Lemma 4.8. *Assume that we have*

$$E(\phi_\xi(w), \phi_\xi(w')) = 0 \quad \text{or} \quad E(\phi_\xi(w'), \phi_\xi(w)) = 0. \tag{4.5}$$

Then S *meets* $A(w + w')$*-orbits in* $X(w + w')$ *transversally. Moreover, we have* $S \cap A(w + w')\phi = \{\phi\}$ *with* $\phi = \phi_\xi(w) \oplus \phi_\xi(w')$*.*

Proof. Our discussion here mimics that of [20, 2.2] for the Slodowy slice. By the symmetry, we may assume $E(\phi_\xi(w), \phi_\xi(w')) = 0$. Then, we have $E^- = \{0\}$ and $S = S^+$. In particular, the action of the torus $T(w')$ contracts the whole S to the unique fixed point ϕ.

Note that $\mathrm{End}_{Q_\xi}(I^{w(0)}) \oplus \mathrm{End}_{Q_\xi}(I^{w(1)})$ is the Lie algebra of $A(w)$ for any $w \in \mathbb{N}^{\mathtt{I}\sqcup\mathtt{I}}$. In our case, the derivation $\mathrm{d}\alpha$ of the action map $\alpha\colon A(w + w') \times S \to X(w + w')$ at $(1, \phi)$ is identical to the map

$$\mathrm{End}_{Q_\xi}(I^{(w+w')(0)}) \oplus \mathrm{End}_{Q_\xi}(I^{(w+w')(1)}) \oplus E \to X(w + w')$$

given by $(a, b, \psi) \mapsto f_\phi(a, b) + \psi$. Since $\mathrm{Im}\, f_\phi \oplus E = X(w + w')$, this is surjective. Using the contracting action of the torus $T(w')$, we can conclude that the derivation $\mathrm{d}\alpha$ is surjective at $(1, x)$ for any $x \in S$. This implies the first assertion. The last assertion follows from an argument analogous to the proof of [9, Proposition 3.7.15]. □

The following proposition plays a key role in the proof of our main theorem in the next subsection.

Proposition 4.9. *Under the assumption* (4.5)*,* $\mathcal{L}_{w+w'}|_S$ *is a semisimple perverse sheaf on* S *containing both* $\underline{\Bbbk}_S[\dim S]$ *and* $\underline{\Bbbk}_{\{\phi\}}$ *as summands, where* $\phi := \phi_\xi(w) \oplus \phi_\xi(w')$ *as above.*

Proof. For simplicity, we put $\tilde{w} := w + w'$ in this proof. By Lemma 4.8, S is a transversal slice through ϕ. By [21, Theorem 5.4.1], $\mathrm{IC}_{v,\tilde{w}}|_S$ is a simple perverse sheaf for any possible v and hence $\mathcal{L}_{\tilde{w}}$ a semisimple perverse sheaf. It contains $\mathrm{IC}_{0,\tilde{w}}|_S = \underline{\Bbbk}_S[\dim S]$ as a summand. It remains to show that $\mathrm{IC}_{v,\tilde{w}}|_S = \underline{\Bbbk}_{\{\phi\}}$ for some v. To this end, it is enough to show that the intersection cohomology complex $\mathrm{IC}(\bar{O})$ of the closure of the orbit $O := A(\tilde{w})\phi$ coincides with $\mathrm{IC}_{v,\tilde{w}}$ for some v because we know $O \cap S = \{\phi\}$ by the last assertion of Lemma 4.8. In view of Proposition 4.1, it suffices to show that a shift of $\mathrm{IC}(\bar{O})$ appears as a summand of $\Phi_{\tilde{w}}((p_{v,\tilde{w}})_!\underline{\Bbbk}_{F(v,\tilde{w})})$ for a suitable $v \in \mathbb{N}^{\mathtt{I}}$.

By symmetry, we may assume $E(\phi_\xi(w'), \phi_\xi(w)) = 0$. Put $K := \mathrm{Ker}(\phi_\xi(w))$ and $K' := \mathrm{Ker}(\phi_\xi(w'))$. By (2.1), we know that $\mathrm{Ext}^1_{Q_\xi}(K,K)$, $\mathrm{Ext}^1_{Q_\xi}(K',K')$ and $\mathrm{Ext}^1_{Q_\xi}(K',K)$ all vanish. Let $v \in \mathbb{N}^{\mathtt{I}}$ be the dimension vector of K. We shall show that a shift of $\mathrm{IC}(\bar{O})$ appears in $\Phi_{\tilde{w}}((p_{v,\tilde{w}})_!\underline{\Bbbk}_{F(v,\tilde{w})})$ for this v. By definition, $F(v,\tilde{w})$ is a vector subbundle of the trivial bundle $\mathrm{Gr}_v(I^{\tilde{w}(0)}) \times X'(\tilde{w})$ over the quiver Grassmannian $\mathrm{Gr}_v(I^{\tilde{w}(0)})$. Let $F(v,\tilde{w})^\perp$ denote its annihilator subbundle in $\mathrm{Gr}_v(I^{\tilde{w}(0)}) \times X(\tilde{w})$. By [29, Lemma 3.1.7], it is described as

$$F(v,\tilde{w})^\perp = \{(N,\psi) \in \mathrm{Gr}_v(I^{\tilde{w}(0)}) \times X(\tilde{w}) \mid N \subset \mathrm{Ker}\,\psi\}.$$

By [1, Corollary 6.9.14 & Proposition 6.9.15], $\Phi_{\tilde{w}}((p_{v,\tilde{w}})_!\underline{\Bbbk}_{F(v,\tilde{w})})$ is isomorphic to a shift of $(p^\perp_{v,\tilde{w}})_!\underline{\Bbbk}_{F(v',\tilde{w})^\perp}$, where $p^\perp_{v,\tilde{w}} \colon F(v,\tilde{w})^\perp \to X(\tilde{w})$ denotes the second projection $(N,\psi) \mapsto \psi$.

Now, we have to prove that a shift of $\mathrm{IC}(\bar{O})$ occurs in $(p^\perp_{v,\tilde{w}})_!\underline{\Bbbk}_{F(v,\tilde{w})^\perp}$. We need additional notations. For $a, b \in \mathbb{N}^{\mathtt{I}}$ (resp. $\mathbb{N}^{\mathtt{I}\sqcup\mathtt{I}}$), we write $a \le b$ if $a_i \le b_i$ for all $i \in \mathtt{I}$ (resp. $a_i(k) \le b_i(k)$ for all $(i,k) \in \mathtt{I} \times \{0,1\}$). For $M \in \mathsf{rep}\,\mathbb{C}Q_\xi$, we define its Betti vector $b_M = (b_M(0), b_M(1)) \in \mathbb{N}^{\mathtt{I}\sqcup\mathtt{I}}$ by $b_{M,i}(k) := \dim \mathrm{Ext}^k_{Q_\xi}(S_i, M)$ for $k \in \{0,1\}$ and $i \in \mathtt{I}$. Let $\rho_M \in X(b_M)$ denote the minimal injective resolution of M. For any $a = (a(0), a(1)) \in \mathbb{N}^{\mathtt{I}\sqcup\mathtt{I}}$ such that $a(0) \le a(1)$, let $\nu_a \in X(a)$ be an injection $I^{a(0)} \to I^{a(1)}$. With these notations, it is easy to see that any $\psi \in X(w)$ decomposes as $\psi \simeq \rho_M \oplus \nu_{w-b_M}$ with $M = \mathrm{Ker}\,\psi$.

Let us consider the subset U of $F(v,\tilde{w})^\perp$ consisting of pairs (N,ψ) such that (i) $N \simeq K$, (ii) $\mathrm{Ext}^1_{Q_\xi}(\mathrm{Ker}\,\psi/N, K \oplus \mathrm{Ker}\,\psi/N) = 0$, and (iii) $b_{\mathrm{Ker}\,\psi/N} \le b_{K'}$. Since K is a generic representation and the functions mapping $(N,\psi) \in F(v,\tilde{w})^\perp$ to $\dim \mathrm{Ext}^1_{Q_\xi}(\mathrm{Ker}\,\psi/N, K \oplus \mathrm{Ker}\,\psi/N)$ and $b_{\mathrm{Ker}\,\psi/N}$ are upper semi-continuous, U is an open subset. Moreover, it is non-empty as $(K,\phi) \in U$ and hence dense in the smooth connected variety $F(v,\tilde{w})^\perp$. We claim that, for any $(N,\psi) \in U$, there is an isomorphism $\psi \simeq \phi$. Once the claim is verified, we have $p^\perp_{v,\tilde{w}}(U) = O$, which implies $p^\perp_{v,\tilde{w}}(F(v,\tilde{w})^\perp) = \bar{O}$. Therefore a shift of $\mathrm{IC}(\bar{O})$ must contribute to $(p^\perp_{v,\tilde{w}})_!\underline{\Bbbk}_{F(v,\tilde{w})^\perp}$ as desired.

We prove the claim. Assume $(N,\psi) \in U$. By the conditions (i) and (ii), we have $\mathrm{Ker}\,\psi \simeq K \oplus C$, where $C := \mathrm{Ker}\,\psi/N$. Then, we have

$$\psi \simeq \rho_{K\oplus C} \oplus \nu_{\tilde{w}-b_{K\oplus C}} = \rho_K \oplus \rho_C \oplus \nu_{w-b_K} \oplus \nu_{w'-b_C}.$$

As $\phi_\xi(w) \simeq \rho_K \oplus \nu_{w-b_K}$, we have $\psi \simeq \phi_\xi(w) \oplus \psi'$, where $\psi' := \rho_C \oplus \nu_{w'-b_C} \in X(w')$. Since $\phi_\xi(w')$ is in the unique open orbit in $X(w')$, it follows that $b_C = b_{\mathrm{Ker}\,\psi'} \ge b_{\mathrm{Ker}\,\phi_\xi(w')} = b_{K'}$. The condition (iii) forces $b_C = b_{K'}$, which implies that C shares the same dimension vector as K'. Again by (ii), we have $\mathrm{Ext}^1_{Q_\xi}(C,C)=0$ and hence $C\simeq K'$. This implies $\psi'\simeq \rho_{K'}\oplus\nu_{w'-b_{K'}}\simeq \phi_\xi(w')$. Thus, we obtain $\psi\simeq\phi_\xi(w)\oplus\psi'\simeq\phi_\xi(w)\oplus\phi_\xi(w')=\phi$. □

4.6. **Proof of Theorem 2.7.** Our goal is to show the equality

$$\mathfrak{o}(L_\xi(w),L_\xi(w'))=\dim E(\phi_\xi(w),\phi_\xi(w')) \tag{4.6}$$

for any $w,w'\in\mathbb{N}^{\mathrm{I}\sqcup\mathrm{I}}$. We first prove it under the assumption (4.5), where we obtain a sheaf theoretic interpretation of R-matrices as a byproduct.

Proposition 4.10. *Under the assumption* (4.5)*, the equality* (4.6) *holds.*

Proof. For simplicity, we put $L:=L_\xi(w)$ and $L':=L_\xi(w')$ in this proof. Let $\phi:=\phi_\xi(w)\oplus\phi_\xi(w')$ as before and $i\colon\{\phi\}\to S$ denote the inclusion. We have the following morphisms arising from the adjunction unit/counit:

$$\eta\colon\underline{\Bbbk}_S[d_S]\to i_*i^*\underline{\Bbbk}_S[d_S]=\underline{\Bbbk}_{\{\phi\}}[d_S],\quad \varepsilon\colon\underline{\Bbbk}_{\{\phi\}}=i_!i^!\underline{\Bbbk}_S[2d_S]\to\underline{\Bbbk}_S[2d_S],$$

where $d_S:=\dim S$. We also abbreviate $\mathcal{L}_{w+w'}|_S$ as $\mathcal{L}$, and $T(w')$ as T.

First, we consider the case when $E(\phi_\xi(w),\phi_\xi(w'))=0$. In this case, we have $S^+=\{\phi\}$, $S^-=S$ and hence $(i_+)^!(j_+)^*\mathcal{L}=i^*\mathcal{L}$, $(i_-)^!(j_-)^*\mathcal{L}=i^!\mathcal{L}$. Moreover, we have $i^*\mathcal{L}\simeq p_*\mathcal{L}$ with $p\colon S\to\{\phi\}$ being the obvious morphism (cf. [12, Proposition 2.3]). By Proposition 4.6, we have

$$L\otimes L'[\![u]\!]\simeq\widehat{\mathrm{H}}^\bullet_T(p_*\mathcal{L})\simeq\mathrm{Hom}^\bullet_T(\underline{\Bbbk}_S[d_S],\mathcal{L})^\wedge, \tag{4.7}$$

$$L'[\![u]\!]\otimes L\simeq\widehat{\mathrm{H}}^\bullet_T(i^!\mathcal{L})\simeq\mathrm{Hom}^\bullet_T(\underline{\Bbbk}_{\{0\}},\mathcal{L})^\wedge. \tag{4.8}$$

Choose ℓ-highest weight vectors $v_L\in L$ and $v_{L'}\in L'$. We shall identify the images of $v_L\otimes v_{L'}$ (resp. $v_{L'}\otimes v_L$) under the isomorphism (4.7) (resp. (4.8)). Recall the isomorphism $\varphi^*_{w+w'}L_{0,w+w'}\simeq L_\xi(w+w')$ from Theorem 3.2. Consider the 1-dimensional subspace $(L_{0,w+w'})_0\subset L_{0,w+w'}$ corresponding to the ℓ-highest weight space of $L_\xi(w+w')$. We have the embedding of the corresponding summand $\iota\colon\underline{\Bbbk}_S[d_S]=(L_{0,w+w'})_0\boxtimes\underline{\Bbbk}_S[d_S]\subset\mathcal{L}$. By construction, this contributes to the ℓ-highest weight spaces of $L\otimes L'[\![u]\!]$ and $L'[\![u]\!]\otimes L$. More precisely, we have the following commutative diagrams:

$$\begin{array}{ccc}
L\otimes L'[\![u]\!] & \xrightarrow[\text{(4.7)}]{\simeq} & \mathrm{Hom}^\bullet_T(\underline{\Bbbk}_S[d_S],\mathcal{L})^\wedge \\
\text{inclusion}\uparrow & & \uparrow\iota_* \\
\Bbbk[\![u]\!](v_L\otimes v_{L'}) & \xrightarrow{\simeq} & \mathrm{Hom}^\bullet_T(\underline{\Bbbk}_S[d_S],\underline{\Bbbk}_S[d_S])^\wedge,
\end{array}$$

$$\begin{array}{ccc}
L'[\![u]\!]\otimes L & \xrightarrow[\text{(4.8)}]{\simeq} & \mathrm{Hom}^\bullet_T(\underline{\Bbbk}_{\{\phi\}},\mathcal{L})^\wedge \\
\text{inclusion}\uparrow & & \uparrow\iota_* \\
\Bbbk[\![u]\!](v_{L'}\otimes v_L) & \xrightarrow{\simeq} & \mathrm{Hom}^\bullet_T(\underline{\Bbbk}_{\{\phi\}},\underline{\Bbbk}_S[d_S])^\wedge,
\end{array}$$

where ι_* means the post-composition with ι. Since $\mathrm{Hom}^\bullet_T(\underline{\Bbbk}_S[d_S],\underline{\Bbbk}_S[d_S])^\wedge$ is generated over $\Bbbk[\![u]\!]$ by the identity $\mathsf{id}_{\underline{\Bbbk}_S[d_S]}\in\mathrm{Hom}^0_T(\underline{\Bbbk}_S[d_S],\underline{\Bbbk}_S[d_S])$, we may assume that the isomorphism (4.7) sends the vector $v_L\otimes v_{L'}$ to $\mathsf{id}_{\underline{\Bbbk}_S[d_S]}$.

By the same reason, the isomorphism (4.8) sends the vector $v_{L'} \otimes v_L$ to $\varepsilon \in \mathrm{Hom}_T^{d_S}(\underline{\Bbbk}_{\{\phi\}}, \underline{\Bbbk}_S[d_S])$. Then, the following diagram commutes:

$$\begin{array}{ccccc} L \otimes L'(\!(u)\!) & \hookleftarrow & L \otimes L'[\![u]\!] & \xrightarrow[(4.7)]{\simeq} & \mathrm{Hom}_T^{\bullet}(\underline{\Bbbk}_S[d_S], \mathcal{L})^{\wedge} \\ \Big\downarrow {\scriptstyle \widehat{R}_{L,L'}} & & & & \Big\downarrow {\scriptstyle \varepsilon^*} \\ L'(\!(u)\!) \otimes L & \hookleftarrow & L'[\![u]\!] \otimes L & \xrightarrow[(4.8)]{\simeq} & \mathrm{Hom}_T^{\bullet}(\underline{\Bbbk}_{\{\phi\}}, \mathcal{L})^{\wedge}, \end{array}$$

where ε^* denotes the pre-composition with ε. Indeed, the homomorphism ε^* intertwines the $U_q(L\mathfrak{g})[\![u]\!]$-actions (given through $\varphi_{w+w'}$), and sends $\iota_* \mathsf{id}_{\underline{\Bbbk}_S[d_S]}$ (= the image of $v_L \otimes v_{L'}$) to $\iota_* \varepsilon$ (= the image of $v_{L'} \otimes v_L$). The above commutative diagram implies $\widehat{R}_{L,L'}(L \otimes L'[\![u]\!]) \subset L'[\![u]\!] \otimes L$. By Remark 2.3, we obtain $\mathfrak{o}(L, L') = 0 = E(\phi_\xi(w), \phi_\xi(w'))$ as desired.

Next, we consider the remaining case when $E(\phi_\xi(w'), \phi_\xi(w)) = 0$. Then we have $S^+ = S$ and $S^- = \{\phi\}$. Similarly, there are isomorphisms

$$L \otimes L'[\![u]\!] \simeq \mathrm{Hom}_T^{\bullet}(\underline{\Bbbk}_{\{\phi\}}, \mathcal{L})^{\wedge}, \tag{4.9}$$

$$L'[\![u]\!] \otimes L \simeq \mathrm{Hom}_T^{\bullet}(\underline{\Bbbk}_S[d_S], \mathcal{L})^{\wedge}, \tag{4.10}$$

under which the vector $v_L \otimes v_{L'}$ corresponds to $\iota_* \varepsilon$, and the vector $v_{L'} \otimes v_L$ corresponds to $\iota_* \mathsf{id}_{\underline{\Bbbk}_S[d_S]}$. Let $c \in \Bbbk$ be a scalar determined by the equation

$$\varepsilon \circ \eta = c u^{d_S} \mathsf{id}_{\underline{\Bbbk}_S[d_S]}$$

in $\mathrm{Hom}_T^{2d_S}(\underline{\Bbbk}_S[d_S], \underline{\Bbbk}_S[d_S])$. Then, the diagram

$$\begin{array}{ccccc} L \otimes L'(\!(u)\!) & \hookleftarrow & L \otimes L'[\![u]\!] & \xrightarrow[(4.9)]{\simeq} & \mathrm{Hom}_T^{\bullet}(\underline{\Bbbk}_{\{\phi\}}, \mathcal{L})^{\wedge} \\ \Big\downarrow {\scriptstyle c u^{d_S} \widehat{R}_{L,L'}} & & & & \Big\downarrow {\scriptstyle \eta^*} \\ L'(\!(u)\!) \otimes L & \hookleftarrow & L'[\![u]\!] \otimes L & \xrightarrow[(4.10)]{\simeq} & \mathrm{Hom}_T^{\bullet}(\underline{\Bbbk}_S[d_S], \mathcal{L})^{\wedge} \end{array}$$

commutes because η^* intertwines the $U_q(L\mathfrak{g})[\![u]\!]$-actions and sends $\iota_* \varepsilon$ (= the image of $v_L \otimes v_{L'}$) to $\iota_*(c u^{d_S} \mathsf{id}_{\underline{\Bbbk}_S[d_S]})$ (= the image of $c u^{d_S} v_L \otimes v_{L'}$). The specialization of η^* at $u = 0$ is equal to its non-equivariant version $\eta^* \colon \mathrm{Hom}^{\bullet}(\underline{\Bbbk}_{\{\phi\}}, \mathcal{L}) \to \mathrm{Hom}^{\bullet}(\underline{\Bbbk}_S[d_S], \mathcal{L})$ (use Lemma 3.1), which is non-zero as $\mathcal{L}$ contains $\underline{\Bbbk}_{\{\phi\}}$ as a summand by Proposition 4.9. Therefore, we have $c \neq 0$ and hence $u^{d_S} \widehat{R}_{L,L'}(L \otimes L'[\![u]\!]) \subset L'[\![u]\!] \otimes L$ with $(u^{d_S} \widehat{R}_{L,L'})|_{u=0} \neq 0$. By Remark 2.3, we obtain $\mathfrak{o}(L, L') = d_S = \dim E(\phi_\xi(w), \phi_\xi(w'))$. $\square$

Finally, we treat the general case. Let $\phi_\xi(w) = \phi_\xi(w^{(1)}) \oplus \cdots \oplus \phi_\xi(w^{(l)})$ and $\phi_\xi(w') = \phi_\xi(w'^{(1)}) \oplus \cdots \oplus \phi_\xi(w'^{(l')})$ be decompositions in $C^2(\mathsf{inj}\, \mathbb{C}Q_\xi)$ with all the summands indecomposable. Then, by Corollary 4.7, we have the corresponding factorizations $L_\xi(w) \simeq L_\xi(w^{(1)}) \otimes \cdots \otimes L_\xi(w^{(l)})$ and $L_\xi(w') \simeq L_\xi(w'^{(1)}) \otimes \cdots \otimes L_\xi(w'^{(l')})$. By Lemma 2.2, we have

$$\mathfrak{o}(L_\xi(w), L_\xi(w')) = \sum_{1 \le k \le l} \sum_{1 \le k' \le l'} \mathfrak{o}(L_\xi(w^{(k)}), L_\xi(w'^{(k')})). \tag{4.11}$$

On the other hand, we have an obvious isomorphism

$$E(\phi_\xi(w), \phi_\xi(w')) \simeq \bigoplus_{1 \le k \le l} \bigoplus_{1 \le k' \le l'} E(\phi_\xi(w^{(k)}), \phi_\xi(w'^{(k')})). \tag{4.12}$$

For indecomposables, the assumption (4.5) is always satisfied by Remark 2.8 and (2.3). Therefore, by Proposition 4.10, we have

$$\mathfrak{o}(L_\xi(w^{(k)}), L_\xi(w'^{(k')})) = \dim E(\phi_\xi(w^{(k)}), \phi_\xi(w'^{(k')}))$$

for any $1 \le k \le l$ and $1 \le k' \le l'$. Thus, together with (4.11) and (4.12), we get (4.6) for general $w, w' \in \mathbb{N}^{\mathtt{I} \sqcup \mathtt{I}}$, completing the proof of Theorem 2.7.

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