AN ASYMPTOTIC FORMULA WITH POWER-SAVING ERROR TERM FOR COUNTING PRIME SOLUTIONS TO A BINARY ADDITIVE PROBLEM

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1. Introduction

The Waring–Goldbach problem, which is the intersection of two well-known older problems: Waring's problem and Goldbach's conjecture. It asks whether large integers can be written as the sum of powers of primes with a bounded number of terms. Let us consider a prominent case of this problem: the equation of Lagrange

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = N \tag{1.1}$$

with multiplicative restrictions. It is plausible to expect that sufficiently large integers, under certain necessary congruence conditions, can be written as the sum of four squares of primes. Such a result of this strength remains out of reach and is still a conjecture. However, Hua in [Hua38] proved that all large integers congruent to 5 modulo 24 can be represented as the sums of five squares of primes by Vinogradov's method for the ternary Goldbach problem as both the problems are considered to be the ternary additive problems.

In contrast, we are interested in the binary additive problem. It is not possible to distinguish the solutions of such problem in this way $\alpha + \beta + \gamma = N$, where α, β, γ run independently through three sequences and at least two of these are sufficiently densein $[1, N]$. We now briefly survey results related to the solutions of Lagrange's equation as a binary additive problem. In 1994, Brüdern and Fouvry in [BF94] established that for every sufficiently large integer N, congruent to 4 modulo 24, (1.1) admits solutions over integers, each of which is almost prime of order 34 by combining a sieve method with the Hardy-Littlewood-Kloosterman approach. Concerning the Lagrange equation with four almost prime variables, the value 34 due to Brüdern and Fouvry was later sharpened by Heath-Brown and Tolev in [HBT03] to 25, by Tolev in [Tol03] to 21, by Cai in [Cai10] to 13 and by Tsang and Zhao in [TZ17] to 4 .

In 2003, Heath-Brown and Tolev in [HBT03] managed to solve the equation (1.1) with one prime entry and each of the other three x_2, x_3, x_4 having at most 101 prime divisors. Tolev in [Tol03] reduced the number of prime divisors from 101 to 80, and then Cai in [Cai10] showed that 42 is acceptable. The latest improvement has been made in 2017 by Tsang and Zhao in [TZ17]. Combination of the circle method

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with Kloosterman refinement and the square sieve to control the reminder term uniformly for certain level D and then the use of the sieve method to produce the almost primes have enabled them to solve the problem with one prime and each of the three x_2, x_3, x_4 having at most 5 prime factors.

Many other variants of the four squares theorem (see (1.1)) have been studied as an approximation to the conjecture above. Podsypanin in [Pod75] has obtained an asymptotic formula as a function of N for the number of solutions of (1.1) in square-free numbers x_i.

Greaves in [Gre76] applied a sieve method to solve (1.1) for integers x_1, x_2 and primes x_3, x_4 and obtained only a lower bound for the number of representations in this way. Shields in [Shi79], Plaksin in [Pla81] and Kovalchik in [Kov81], by different methods, have obtained an asymptotic formula for such representations where the authors use the multiplicative properties of the arithmetic function $r(n)$ defined as the number of ways n can be represented as the sum of two squares along with the methods of Hooley and Linnik's dispersion method. The best known result today is due to Plaksin in [Pla84] who obtained an error term of size $O\left(N(\log N)^{-2.042}\right)$ by proving the following theorem under some assumptions.

Theorem 1.1. *Let $\kappa(N)$ be the number of solutions of the equation*

$$f(p,q) + \psi(x,y) = N, \tag{1.2}$$

and $\tau(m,N)$ be the number of solutions of the congruence

$$f(u,v) + \psi(x,y) \equiv N \bmod m, \qquad (uv,m) = 1, \tag{1.3}$$

where f and ψ are positive definite quadratic forms with integral coefficients of discriminants $-\Delta$ and $-\tau$ respectively, p and q are prime numbers, and x, y, u and v are integers. We let $p^\beta || (\Delta, 4N)$ and $G = \prod_{p|3\Delta N,\tau} p^{\beta+1}$. Let the congruence (1.3) be solvable for $m = G$. Then we have

$$\kappa(N) = \frac{8\pi}{\sqrt{\tau}} s_f \mathfrak{S}(N) \frac{N}{\log^2 N}\left(1 + \frac{B}{\log^{0.042} N}\right), \tag{1.4}$$

where the singular series has the form

$$\mathfrak{S}(N) = \prod_p \frac{\tau(p^{\beta+1}, N)}{(\phi(p^{\beta+1}))^2 p^{\beta+1}}, \qquad \text{and} \qquad \frac{1}{\sqrt{G}\log\log N} \ll \mathfrak{S}(N) \ll \log\log N, \tag{1.5}$$

and s_f is the area of the region $f(x,y) \leq 1$ such that $x, y \geq 0$ and ϕ is the Euler totient function.

Our goal is here to study a similar type of problem, where we investigate the determinant equation, i.e.,

$$x_1x_2 - x_3x_4 = r, \tag{1.6}$$

with multiplicative constraints imposed on the variables and a non-zero r, instead of the Lagrange equation (1.1). An important thing to note that the solutions of both Lagrange's equation and the determinant equation can not be considered as the ternary problems because the terms in these equations fail to combine in a way that satisfies certain conditions (see [Lin63, Eq. (0.1.6) and (0.1.9)]), whereas they perfectly fit as examples of the binary additive problem which is explained earlier.

We are interested in the number of integer points (a, b, c, d) in an expanding box $[-X, X]^4$ that satisfy (1.6), where we attach an arbitrary weight in one variable, let us say $\alpha(x_1)$, and along with that one of x_3, x_4 is prime. Precisely, we want to have an exact asymptotic formula for this problem with a strong bound on the error term and at the same time we would like to allow r to be as large as possible with respect to X. This seems to be the only result of this type in the literature in this direction.

Exploiting the special structure of the determinant form, we show the following.

Theorem 1.2. *For an arbitrary sequence of complex numbers $\alpha(n) = O(n^\varepsilon)$, and any non-zero integer r, let us define*

$$\mathcal{S}_r(X) = \sum\sum\sum\sum_{\substack{|a|,|b|,|p|,|d|\leq X\\ ad-pb=r}} \alpha(a), \tag{1.7}$$

where a, b, d vary over the integers and p varies over primes. Then, for large X, we have,

$$\begin{aligned}\mathcal{S}_r(X) = 8 \sum_{1\leq a\leq X} \sum_{1\leq p\leq X} \frac{\alpha(a)}{a}\left[\min\left\{X, \frac{aX-r}{p}\right\} - \max\left\{1, \frac{a-r}{p}\right\}\right]\\ + O\left(X^{1+\frac{3}{4}+\varepsilon} + r^{\frac{1}{5}}X^{1+\frac{11}{20}+\varepsilon}\right),\end{aligned} \tag{1.8}$$

uniformly for all r.

If we take $\alpha(n)$ to be the characteristic function of primes, then we have the following theorem regarding counting lattice points with two prime entries on the determinant surfaces.

Theorem 1.3. *Define*

$$\mathcal{S}(2,\mathbb{Z}) := \left\{\begin{pmatrix} p & b\\ q & d\end{pmatrix} \in M(2,\mathbb{Z}) : p, q \text{ are primes}\right\}. \tag{1.9}$$

Let

$$\sum_{\substack{\gamma\in\mathcal{S}(2,\mathbb{Z})\\ det(\gamma)=r\\ ||\gamma||_\infty\leq X}} 1 = \sum_p\sum_b\sum_q\sum_{\substack{d\\ pd-bq=r\\ |p|,|b|,|q|,|d|\leq X}} 1,$$

where b, d vary over the integers and r is a fixed non-zero integer. Then, for large X, we have

$$\mathcal{S}_r(X) = 8K_r(\mathrm{li}(X))^2 + O\left(X^{1+\frac{3}{4}+\varepsilon} + r^{\frac{1}{5}}X^{1+\frac{11}{20}+\varepsilon}\right), \tag{1.10}$$

where K_r is an explicit constant and the logarithmic integral, denoted by $\mathrm{li}(x)$, is the integral

$$\mathrm{li}(x) := \int_2^x \frac{dt}{\log t}. \tag{1.11}$$

Remark 1.1. *We see that for small r, precisely for $r \leq X$, the error term is bounded by $X^{7/4+\varepsilon}$.*

Remark 1.2. *The important thing to notice here is that even if we have restricted two of entries of the determinant equation (see (1.6)) to be prime, we still have been able to achieve a power saving in the error term, in contrast, for the Lagrange equation (see (1.1)) with two prime entries only the logarithmic saving of size $(\log N)^{2.042}$ in the error term has been obtained.*

Although, when all the variables x_1, x_2, x_3 and x_4 without any restriction run over the integers, Ganguly and Guria in [GG, Theorem 1.1] have established an asymptotic formula for such count with an error term of size $O\left(X^{3/2+\varepsilon}\right)$ as long as $r = O\left(X^{1/3}\right)$. In the same paper, they have also obtained an asymptotic formula for one prime entry with an error term of size $O\left(X^{5/3+\varepsilon}\right)$ as long as $r = O\left(X^{5/3}\right)$ by taking $\alpha(n)$ to be indicator function of primes (see [GG, Corollary 1.5]). In [BC18] and before that in [DFI97b], sums of the form

$$\sum_{\substack{n_1 \\ n_i \in \mathcal{N}_i 1 \leq i \leq 4 \\ n_1 n_4 - n_2 n_3 = r}} \sum_{n_2} \sum_{n_3} \sum_{n_4} \alpha(n_1)\beta(n_2) f(n_3) g(n_4) \tag{1.12}$$

were studied and asymptotic formulae for such sums were obtained. Here, $\mathcal{N}_i = [N_i/2, N_i]$, α and β are arbitrary sequences, and f, g are smooth functions satisfying suitable decay conditions on their derivatives. One may compare our theorems with [BC18, Corollary 1] or [DFI97a, Theorem 1], where, two sequences are arbitrary and the other two are smooth weights of the form $f(n_3)$ and $g(n_4)$ (as in (1.12)). Note that if we apply these results in our situations, we obtain an error term that is rather poor. For example, [BC18, Corollary 1] gives an error term of the order $O\left(X^{1+49/50+\varepsilon}\right)$.

The following two corollaries are direct consequences of Theorem 1.2.

Corollary 1.4. *We have,*

$$\begin{aligned} \sum\sum\sum_{1 \leq a,b,c \leq X} \Lambda\left(\frac{ab \pm r}{c}\right) \alpha(a) = \sum_{p \text{ prime}} \sum_{1 \leq a \leq X} \frac{\alpha(a)}{a} \int_1^X w\left(\frac{xp \mp r}{a}\right) \\ + O\left(X^{1+\frac{3}{4}+\varepsilon} + r^{\frac{1}{5}} X^{1+\frac{11}{20}+\varepsilon}\right), \end{aligned} \tag{1.13}$$

uniformly for all non-zero integer r and a, b, c vary over the integers, and the weight function w satisfies the conditions (3.7)-(3.10).

Corollary 1.5. *For any non-zero integer r, we have*

$$\begin{aligned} \sum\sum\sum_{1 \leq a,b,p \leq X} \alpha\left(\frac{ap \pm r}{b}\right) = \sum_{1 \leq p \leq X} \sum_{n \in \mathbb{Z}} \frac{\alpha(n)}{n} \int_1^X w\left(\frac{xp \pm r}{n}\right) \\ + O\left(X^{1+\frac{3}{4}+\varepsilon} + r^{\frac{1}{5}} X^{1+\frac{11}{20}+\varepsilon}\right), \end{aligned} \tag{1.14}$$

where a, b vary over the integers and p varies over primes, and the function w satisfies the conditions (3.7)-(3.10).

1.1. **Method of the proofs of Theorem 1.2 and Theorem 1.3.** Let us briefly outline the method of the proof. The main idea is to exploit the special structure of the determinant equation (1.6) that enables us to transform our sum into the average of the sums of the Kloosterman fractions over primes p after using the Poisson Summation formula.

At first, we isolate terms for which $p|a$ and bound them trivially. Now, we are left with the terms which satisfy the condition $(a, p) = 1$. The basic idea is to use Fourier analysis. To do so we first approximate this sum by a weighted sum by introducing a smooth function w in the those variables without any restriction, namely b, d, which runs over integers, that approximates the indicator function $\mathbf{1}_{[1,X]}$ to within an acceptable error term (see §3.2). Now, we replace d by $(r+pb)/a$, using the determinant equation and the congruence condition $b \equiv -r\bar{p} (\text{mod } a)$ arises as we already have the condition (a, p) =1. At this stage, a trivial estimation gives $S_w(X, r) \ll X^{2+\varepsilon}$. Next, we use the Poisson summation formula to evaluate the sum over b as only the variable b runs over integers. We separate the zero frequency and the non-zero frequencies. The zero frequency yields the main term, whereas in regard to the non-zero frequencies, the key observation is that the sum over primes p along with sum over modulus a is precisely the average of the sums of Kloosterman fractions over primes. Next our plan is to implement Lemma 2.2 to get a power-saving error term. It requires some work because of the condition $Q^{4/3} \geq X \geq Q^{1/2}$. We also need a good bound for $\left(\widehat{F_{da_1,t}}\left(\frac{n}{da_1}\right)\right)'$ (see (5.23)) which appears after summation by parts in the sum over p in terms of of t, a_1 and n as the intervals over which the variables run are long and the direct application of integration by parts is not enough.

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2. Preliminaries

In this section we recall some standard formulae and lemma from the literature which will be required in the proofs of our results.

Lemma 2.1. ***(Application of Poisson summation)***
Suppose that both f, $\hat{f}$ are in $L^1(\mathbb{R})$ and have bounded variation. Then

$$\sum_{\substack{n \equiv \alpha (\text{mod } a) \\ n \in \mathbb{Z}}} f(n) = \frac{1}{a} \sum_{n \in \mathbb{Z}} e\left(\frac{\alpha n}{a}\right) \hat{f}\left(\frac{n}{a}\right) \tag{2.1}$$

where both the series converge absolutely.

Proof. See, e.g., [IK04, Theorem 4.4 and Exercise 4]. □

Lemma 2.2. *For any integer $a > 0$ and every $\varepsilon > 0$, we have*

$$\sum_{q\sim Q}\left|\sum_{\substack{1\leq p\leq X\\(p,q)=1}} e\left(\frac{a\bar{p}}{q}\right)\right| \ll \left(1+\frac{a}{XQ}\right)^{\frac{1}{2}}\left(Q^{\frac{1}{2}}X^{\frac{11}{8}}+Q^{\frac{7}{6}}X^{\frac{2}{3}}\right)(aQ)^{\varepsilon}, \tag{2.2}$$

for $Q^{\frac{4}{3}} \geq X \geq Q^{\frac{1}{2}}$.

Proof. See [Irv14, Theorem 1.4]. □

3. The main steps of proof of Theorem 1.3

Let us define

$$S_r(X) := \sum\sum\sum\sum_{\substack{1\leq a,b,p,d\leq X\\ ad-pb=r}} \alpha(a), \tag{3.1}$$

and as noted already in the introduction,

$$\mathcal{S}_r(X) = 8S_r(X). \tag{3.2}$$

To prove Theorem 1.3, it is enough to prove that

$$\begin{aligned} S_r(X) = &\sum_{1\leq a\leq X}\sum_{1\leq p\leq X}\frac{\alpha(a)}{a}\left[\min\left\{X,\frac{aX-r}{p}\right\}-\max\left\{1,\frac{a-r}{p}\right\}\right]\\ &+O\left(X^{1+\frac{3}{4}+\varepsilon}+r^{\frac{1}{5}}X^{1+\frac{11}{20}+\varepsilon}\right). \end{aligned} \tag{3.3}$$

3.1. **Contribution of the terms when $p|a$.** We will now examine the impact of the terms for which $p|a$ so that later we will only focus on the case when $(a,p)=1$. Hence, we have the sum

$$\begin{aligned} \sum\sum\sum\sum_{\substack{1\leq a,b,p,d\leq X\\ ad-pb=r\\ p|a}} \alpha(a) &= \sum\sum\sum_{\substack{1\leq b,p,d\leq X\\ kpd-pb=r\\ p|r}}\sum_{1\leq k\leq \frac{X}{p}}\alpha(pk)\\ &= \sum_{\substack{1\leq p\leq X\\ p|r}}\sum_{1\leq k\leq\frac{X}{p}}\alpha(pk)\sum_{\substack{1\leq b\leq X\\ pk|(bp+r)}}1\\ &\ll X^{\varepsilon}\sum_{\substack{1\leq p\leq X\\ p|r}}\sum_{1\leq k\leq\frac{X}{p}}\frac{X}{k}\\ &\ll X^{1+\varepsilon}, \end{aligned} \tag{3.4}$$

since there are $\omega(r) = O(X^{\varepsilon})$ values of p that contribute in the sum over p and $\alpha(n) = O(X^{\varepsilon})$. Hence,

$$S_r(X) = S_r^{(0)}(X) + O\left(X^{1+\varepsilon}\right), \tag{3.5}$$

where

$$S_r^{(0)}(X) = \sum\sum\sum\sum_{\substack{1\leq a,b,p,d\leq X\\ ad-pb=r\\ (a,p)=1}} \alpha(a) \tag{3.6}$$

We next introduce a nice suitable weight function w to the variables b and d in our sum $S_r^{(0)}(X)$ and we use the same weight function as in [GG, §3]. Let us recall the weight function w.

3.2. **Introducing a weight function.** Let $w : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying the following conditions:

$$(1)\ w(x) = 0 \text{ if } x \notin (1-H, X+H), \tag{3.7}$$

$$(2)\ w(x) = 1 \text{ if } x \in [1, X], \tag{3.8}$$

$$(3)\ 0 < w(x) < 1 \text{ if } x \in (1-H, 1), \text{ or if } x \in (X, X+H), \tag{3.9}$$

$$(4)\ w^{(j)}(x) \ll_j H^{-j} \text{ for every } j \geq 1, \tag{3.10}$$

where $X > 0$ is a large number and H is a parameter to be fixed later subject to the condition

$$\sqrt{X} \leq H \leq X. \tag{3.11}$$

Remark 3.1. *Finally, our choice will be* $H = \max\left\{X^{\frac{3}{4}}, r^{\frac{1}{5}}X^{\frac{11}{20}}\right\}$.

We define next

$$\begin{aligned} S_w(X,r) &= \sum_{\substack{1\leq a\leq X \\ ad-pb=r \\ (a,p)=1}} \sum_{1\leq p\leq X} \sum_{b\in\mathbb{Z}} \sum_{d\in\mathbb{Z}} \alpha(a)w(b)w(d) \\ &= \sum_{\substack{1\leq a\leq X \\ b\equiv -r\bar{p} (\text{mod } a) \\ (a,p)=1}} \sum_{1\leq p\leq X} \sum_{b\in\mathbb{Z}} \alpha(a)w(b)w\left(\frac{r+pb}{a}\right), \end{aligned} \tag{3.12}$$

where we eliminate d and interpret the equality as a congruence modulo a.

Proposition 3.1. *We have,*

$$S_r^{(0)}(X) = S_w(X,r) + O\left(HX^{1+\varepsilon}\right). \tag{3.13}$$

Proof. We first make the easy observation that the total contribution of the terms for which at least one of the variables b, $\frac{r+pb}{a}$ is zero is $O(\tau(r)X)$.
We will now look into the contribution of the terms for which at least one of the variables b, $\frac{r+pb}{a}$ is in $(1-H,1)\cup(X,X+H)$. Let us first consider the case when $\frac{r+pb}{a} \in (1-H,1)\cup(X,X+H)$. Then we have,

$$\begin{aligned} \sum\sum\sum_{\substack{1\leq a,p,b\leq X \\ \frac{r+pb}{a}\in(1-H,1)\cup(X,X+H) \\ pb\equiv -r (\text{mod } a)}} \alpha(a) &\ll X^{\varepsilon} \sum_{1\leq a\leq X} \sum_{\substack{\frac{n+r}{a}\in(1-H,1)\cup(X,X+H) \\ n\equiv -r (\text{mod } a)}} \tau(n) \\ &\ll X^{\varepsilon} \sum_{1\leq a\leq X} \sum_{k\in(1-H,1)\cup(X,X+H)} 1 \\ &\ll HX^{1+\varepsilon}. \end{aligned} \tag{3.14}$$

The case when b is in $(1-H,1)\cup(X,X+H)$ is similar. Therefore, we have

$$S_r^{(0)}(X)=S_w(X,r)+O\left(HX^{1+\varepsilon}\right). \tag{3.15}$$

□

Our task now is to evaluate $S_w(X,r)$ asymptotically.

3.3. **Application of Poisson summation.** We apply the Poisson summation formula (i.e., Lemma 2.1) to the sub-sum over the variable b and obtain the following expression from (3.12).

$$\begin{aligned} S_w(X,r) &= \sum_{\substack{1\le a\le X\\(a,p)=1}}\sum_{1\le p\le X}\alpha(a)\sum_{\substack{b\in\mathbb{Z}\\ b\equiv -r\bar{p}(\text{mod } a)}} w(b)w\left(\frac{r+pb}{a}\right)\\ &= A_w(X,r)+B_w(X,r), \end{aligned} \tag{3.16}$$

where we separate the zero frequency and define

$$A_w(X,r):=\sum_{\substack{1\le a\le X\\(a,p)=1}}\sum_{1\le p\le X}\frac{\alpha(a)}{a}\widehat{F_{a,p}}(0), \tag{3.17}$$

and the non-zero frequency is defined by

$$B_w(X,r):=\sum_{\substack{1\le a\le X\\(a,p)=1}}\sum_{1\le p\le X}\frac{\alpha(a)}{a}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} e\left(\frac{-nr\bar{p}}{a}\right)\widehat{F_{a,p}}\left(\frac{n}{a}\right), \tag{3.18}$$

where we define for any positive integer a and any integer c

$$F_{a,c}(x):=w(x)w\left(\frac{r+cx}{a}\right). \tag{3.19}$$

3.4. **The main term.** In the following proposition we will see that we can extract a main term of size X^2 from $A_w(X,r)$. This will be proved in the next section.

Proposition 3.2. *We have,*

$$A_w(X,r)=\sum_{1\le a\le X}\sum_{1\le p\le X}\frac{\alpha(a)}{a}\left[\min\left\{X,\frac{aX-r}{p}\right\}-\max\left\{1,\frac{a-r}{p}\right\}\right]+O\left(HX^{1+\varepsilon}\right). \tag{3.20}$$

3.5. **The error term.** The following is the main proposition of this paper where we have used the estimation for the exponential sum over $\bar{p}$ for large moduli, i.e., Lemma 2.2.

Proposition 3.3. *We have*

$$B_w(X,r)\ll X^{\frac{7}{4}+\varepsilon}+\frac{X^{2+\frac{1}{2}+\varepsilon}}{H}+r^{\frac{1}{2}}\left(\frac{X}{H}\right)^{\frac{3}{2}}X^{\frac{7}{8}+\varepsilon}. \tag{3.21}$$

3.6. **Proof of the theorem.** We analyze the error term $B_w(X,r)$ in two cases. We first make an observation that for $r \leq HX^{1/4}$,

$$B_w(X,r) \ll X^{\frac{7}{4}+\varepsilon} + \frac{X^{2+\frac{1}{2}+\varepsilon}}{H}. \tag{3.22}$$

In that case, the optimal choice of H is

$$H = X^{\frac{3}{4}}, \tag{3.23}$$

by comparing (3.22) with $O\left(HX^{1+\varepsilon}\right)$ (see (3.13)). However, in the complimentary situation,

$$B_w(X,r) \ll r^{\frac{1}{2}} \left(\frac{X}{H}\right)^{\frac{3}{2}} X^{\frac{7}{8}+\varepsilon}, \tag{3.24}$$

and we make an optimal choice of H to be

$$H = r^{\frac{1}{5}} X^{\frac{11}{20}}. \tag{3.25}$$

Hence, the theorem follows from Proposition 3.2 and Proposition 3.3, taking into account (3.2), (3.5), (3.13), (3.16), (3.23) and (3.25).

4. The main term and the proof of Proposition 3.2

Using (3.17), we write $A_w(X,r)$ as

$$A_w(X,r) = \sum_{1\leq a\leq X} \sum_{1\leq p\leq X} \frac{\alpha(a)}{a} \widehat{F_{a,p}}(0) - \sum_{1\leq a\leq X} \sum_{\substack{1\leq p\leq X \\ p|a}} \frac{\alpha(a)}{a} \widehat{F_{a,p}}(0) \tag{4.1}$$

Now we trivially bound the second term using the properties of the weight function w. Thus, we have

$$A_w(X,r) = \sum_{1\leq a\leq X} \sum_{1\leq p\leq X} \frac{\alpha(a)}{a} \widehat{F_{a,p}}(0) + O(X^{1+\varepsilon}). \tag{4.2}$$

Using the properties of the weight function w, it follows from (3.17)

$$\begin{aligned} A_w(X,r) &= \sum_{1\leq a\leq X} \sum_{1\leq p\leq X} \frac{\alpha(a)}{a} \int_1^X w\left(\frac{r+px}{a}\right) \mathrm{d}x + O\left(HX^{1+\varepsilon}\right) \\ &= \sum_{1\leq a\leq X} \sum_{1\leq p\leq X} \frac{\alpha(a)}{a} \int_{\max\{1,(a-r)/p\}}^{\min\{X,(aX-r)/p\}} 1\mathrm{d}x + O\left(HX^{1+\varepsilon}\right) \\ &= \sum_{1\leq a\leq X} \sum_{1\leq p\leq X} \frac{\alpha(a)}{a} \left[\max\left\{1, \frac{a-r}{p}\right\} - \min\left\{X, \frac{aX-r}{p}\right\}\right] + O\left(HX^{1+\varepsilon}\right). \end{aligned} \tag{4.3}$$

Thus we conclude Proposition 3.2.

5. The error term and the proof of Proposition 3.3

We begin analyzing the sum $B_w(X,r)$. For that we first prove a simple yet crucial lemma that limits the size of the length of the dual sums over n in $B_w(X,r)$. The lemma below shows that $\widehat{F_{a,p}}\left(\frac{n}{a}\right)$ is negligibly small if $n \gg X^{1+\varepsilon}/H$.

Lemma 5.1. *We have*

$$\widehat{F_{a,c}}(y) \ll_k H^{1-k} y^{-k} \left(1 + \left|\frac{c}{a}\right|^{k-1}\right), \tag{5.1}$$

for every integer $k \geq 1$, where $F_{a,c}(y)$ is defined in (3.19). In particular, by taking $k = 1$, we have the bound

$$\widehat{F_{a,c}}(y) \ll_k y^{-1}. \tag{5.2}$$

Proof. Using (3.19) we integrate by parts k times, getting

$$\widehat{F_{a,c}}(y) = \frac{1}{(-2\pi i y)^k} \sum_{j=0}^{k} \int_{-\infty}^{\infty} w^{(j)}(x) \left(\frac{c}{a}\right)^{k-j} w^{(k-j)}\left(\frac{r+cx}{a}\right) e(-xy)\mathrm{d}x.$$

Note that $w^{(k)}\left(\frac{r+cx}{a}\right) = 0$ unless $\frac{r+cx}{a} \in (1-H, 1) \cup (X, X+H)$, and so $w^{(k)}\left(\frac{cx}{a}\right)$ vanishes outside two intervals of length $O\left(\left|\frac{a}{c}\right| H\right)$ each and by (3.10), we obtain the bound $O(\left|\frac{c}{a}\right|^{k-1} y^{-k} H^{1-k})$ for the term corresponding to $j = 0$. Now we consider the other summands. Since $w^{(j)}(x)$ is non-zero only in the intervals $(1-H, 1)$ and $(X, X+H)$ for every $j \geq 1$, all these terms together contribute only $O(y^{-k} H^{1-k})$, again by (3.10). □

We now focus on $B_w(X,r)$. Interchanging the order of summation,

$$B_w(X,r) := \sum_{1 \leq a \leq X} \frac{\alpha(a)}{a} \sum_{0 \neq n \ll \frac{X^{1+\varepsilon}}{H}} \sum_{\substack{1 \leq p \leq X \\ (a,p)=1}} e\left(\frac{-nr\bar{p}}{a}\right) \widehat{F_{a,p}}\left(\frac{n}{a}\right) + O\left(X^{-100}\right). \tag{5.3}$$

We first divide the terms in $B_w(X,r)$ according to $(nr, a) = d$ such that $d \geq 1$.

$$\begin{aligned} B_w(X,r) &= \sum_{d \geq 1} \sum_{1 \leq a \leq X} \frac{\alpha(a)}{a} \sum_{\substack{0 \neq n \ll \frac{X^{1+\varepsilon}}{H} \\ (a,nr)=d}} \sum_{\substack{1 \leq p \leq X \\ (a,p)=1}} e\left(\frac{-nr\bar{p}}{a}\right) \widehat{F_{a,p}}\left(\frac{n}{a}\right) + O\left(X^{-100}\right) \\ &= \sum_{d \geq 1} \frac{1}{d} \sum_{\substack{0 \neq n \ll \frac{X^{1+\varepsilon}}{H} \\ d \mid nr}} \sum_{\substack{1 \leq a_1 \leq \frac{X}{d} \\ (a_1, nr/d)=1}} \frac{\alpha(da_1)}{a_1} \sum_{\substack{1 \leq p \leq X \\ (a_1,p)=1}} e\left(\frac{-(nr/d)\bar{p}}{a_1}\right) \widehat{F_{da_1,p}}\left(\frac{n}{da_1}\right) \\ &\quad + O\left(X^{-100}\right) \end{aligned} \tag{5.4}$$

We now divide the range of the variable a_1 into dyadic intervals and we write

$$B_w(X,r) = \sum_{d\geq 1}\frac{1}{d}\sum_{\substack{Q \text{ dyadic}\\ Q\leq \frac{X}{d}}}\sum_{\substack{0\neq n\ll \frac{X^{1+\varepsilon}}{H}\\ d\mid nr}}\sum_{\substack{a_1\sim Q\\ (a_1,nr/d)=1}}\frac{\alpha(da_1)}{a_1}\sum_{\substack{1\leq p\leq X\\ (a_1,p)=1}} e\left(\frac{-(nr/d)\bar{p}}{a_1}\right)\widehat{F_{da_1,p}}\left(\frac{n}{da_1}\right) + O\left(X^{-100}\right). \tag{5.5}$$

Now our plan is to apply Lemma 2.2. For that we separate the terms in $B_w(X,r)$ for which $Q^{4/3}\geq X$. Thus we write $B_w(X,r)$ as

$$B_w(X,r) = B_{1,w}(X,r) + B_{2,w}(X,r) + O\left(X^{-100}\right), \tag{5.6}$$

where

$$B_{1,w}(X,r) := \sum_{d\geq 1}\frac{1}{d}\sum_{\substack{Q \text{ dyadic}\\ Q\leq X^{3/4}}}\sum_{\substack{0\neq n\ll \frac{X^{1+\varepsilon}}{H}\\ d\mid nr}}\sum_{\substack{a_1\sim Q\\ (a_1,nr/d)=1}}\frac{\alpha(da_1)}{a_1}\sum_{\substack{1\leq p\leq X\\ (a_1,p)=1}} e\left(\frac{-(nr/d)\bar{p}}{a_1}\right)\widehat{F_{da_1,p}}\left(\frac{n}{da_1}\right), \tag{5.7}$$

and

$$B_{2,w}(X,r) := \sum_{d\geq 1}\frac{1}{d}\sum_{\substack{Q \text{ dyadic}\\ X^{3/4}<Q\leq \frac{X}{d}}}\sum_{\substack{0\neq n\ll \frac{X^{1+\varepsilon}}{H}\\ d\mid nr}}\sum_{\substack{a_1\sim Q\\ (a_1,nr/d)=1}}\frac{\alpha(da_1)}{a_1}\sum_{\substack{1\leq p\leq X\\ (a_1,p)=1}} e\left(\frac{-(nr/d)\bar{p}}{a_1}\right)\widehat{F_{da_1,p}}\left(\frac{n}{da_1}\right), \tag{5.8}$$

We first bound the sum $B_{1,w}(X,r)$ trivially.

Proposition 5.2. *We have*

$$B_{1,w}(X,r) \ll X^{7/4+\varepsilon}. \tag{5.9}$$

Proof. Using (5.2), we have

$$\begin{aligned} B_{1,w}(X,r) &\ll \sum_{d\geq 1}\sum_{\substack{Q \text{ dyadic}\\ Q\leq X^{3/4}}}\sum_{\substack{0\neq n\ll \frac{X^{1+\varepsilon}}{H}\\ d\mid nr}}\frac{1}{n}\sum_{\substack{a_1\sim Q\\ (a_1,nr/d)=1}}\sum_{\substack{1\leq p\leq X\\ (a_1,p)=1}} 1 \\ &\ll X^{7/4+\varepsilon}, \end{aligned} \tag{5.10}$$

since there are $\tau(nr) = O\left(X^{\varepsilon}\right)$ many values of d that contribute to the sum over d. □

The following proposition is the main result of this section.

Proposition 5.3. *We have*

$$B_{2,w}(X,r) \ll \frac{X^{2+\frac{1}{2}+\varepsilon}}{H} + r^{\frac{1}{2}}\left(\frac{X}{H}\right)^{\frac{3}{2}} X^{\frac{7}{8}+\varepsilon}. \tag{5.11}$$

Proof. To bound $B_{2,w}(X,r)$, we first use the summation by parts in the sum over p to separate $\widehat{F_{da_1,p}}\left(\frac{n}{da_1}\right)$. Therefore,

$$B_{2,w}(X,r) = \sum_{d\geq 1}\frac{1}{d}\sum_{\substack{Q \text{ dyadic}\\ X^{3/4}<Q\leq \frac{X}{d}}}\sum_{\substack{0\neq n\ll \frac{X^{1+\varepsilon}}{H}\\ d|nr}}\sum_{\substack{a_1\sim Q\\ (a_1,nr/d)=1}}\frac{\alpha(da_1)}{a_1}\widehat{F_{da_1,X}}\left(\frac{n}{da_1}\right)\sum_{\substack{1\leq p\leq X\\ (a_1,p)=1}} e\left(\frac{-(nr/d)\bar{p}}{a_1}\right)$$

$$+\sum_{d\geq 1}\frac{1}{d}\sum_{\substack{Q \text{ dyadic}\\ X^{3/4}<Q\leq \frac{X}{d}}}\sum_{\substack{0\neq n\ll \frac{X^{1+\varepsilon}}{H}\\ d|nr}}\sum_{\substack{a_1\sim Q\\ (a_1,nr/d)=1}}\frac{\alpha(da_1)}{a_1}\int_1^X\left(\widehat{F_{da_1,t}}\left(\frac{n}{da_1}\right)\right)'\sum_{\substack{1\leq p\leq t\\ (a_1,p)=1}} e\left(\frac{-(nr/d)\bar{p}}{a_1}\right)\mathrm{d}t. \tag{5.12}$$

Next we estimate the first term. Applying (5.2) and Lemma 2.2,

$$\sum_{d\geq 1}\frac{1}{d}\sum_{\substack{Q \text{ dyadic}\\ X^{3/4}<Q\leq \frac{X}{d}}}\sum_{\substack{0\neq n\ll \frac{X^{1+\varepsilon}}{H}\\ d|nr}}\sum_{\substack{a_1\sim Q\\ (a_1,nr/d)=1}}\left|\frac{\alpha(da_1)}{a_1}\right|\left|\widehat{F_{da_1,X}}\left(\frac{n}{da_1}\right)\right|\left|\sum_{\substack{1\leq p\leq X\\ (a_1,p)=1}} e\left(\frac{-(nr/d)\bar{p}}{a_1}\right)\right|$$

$$\ll r^{\frac{1}{2}}\left(\frac{X}{H}\right)^{\frac{1}{2}}X^{\frac{7}{8}+\varepsilon}. \tag{5.13}$$

Now we focus on the second term of (5.12). First we obtain a bound for $\left(\widehat{F_{da_1,t}}\left(\frac{n}{p}\right)\right)'$. We have

$$\left(\widehat{F_{da_1,t}}\left(\frac{n}{da_1}\right)\right)' = \int\frac{x}{da_1}w(x)w'\left(\frac{r+tx}{da_1}\right)e\left(-\frac{nx}{da_1}\right)\mathrm{d}x. \tag{5.14}$$

Using integration by parts

$$\begin{aligned}\left(\widehat{F_{da_1,t}}\left(\frac{n}{da_1}\right)\right)' &= -\frac{1}{2\pi i}\frac{da_1}{n}\int\frac{1}{da_1}w(x)w'\left(\frac{r+tx}{da_1}\right)e\left(-\frac{nx}{da_1}\right)\mathrm{d}x\\ &\quad-\frac{1}{2\pi i}\frac{da_1}{n}\int\frac{x}{da_1}w'(x)w'\left(\frac{r+tx}{da_1}\right)e\left(-\frac{nx}{da_1}\right)\mathrm{d}x\\ &\quad-\frac{1}{2\pi i}\frac{da_1}{n}\int\frac{x}{da_1}\frac{t}{da_1}w(x)w''\left(\frac{r+tx}{da_1}\right)e\left(-\frac{nx}{da_1}\right)\mathrm{d}x.\end{aligned} \tag{5.15}$$

Now we bound the three terms separately. By the properties of the weight function w (see (3.9), (3.10)), we know that

$$w'\left(\frac{r+tx}{da_1}\right)\neq 0 \iff \frac{r+tx}{da_1}\in(1-H,1)\cup(X,X+H), \tag{5.16}$$

and the length of the integral is $\frac{Hda_1}{t}$. Hence, we have the bound for the first term.

$$\begin{aligned}
&-\frac{1}{2\pi i}\frac{da_1}{n}\int \frac{1}{da_1}w(x)w'\left(\frac{r+tx}{da_1}\right)e\left(-\frac{nx}{da_1}\right)\mathrm{d}x\\
&\ll \frac{da_1}{n}.\frac{1}{da_1}.\frac{Hda_1}{t}.\frac{1}{H}\\
&\ll \frac{da_1}{nt}.
\end{aligned} \tag{5.17}$$

Next, to bound the second term of (5.15), we use that fact (see (3.10)) that the length of the integral is H, since

$$w'(x) \neq 0 \iff x \in (1-H, 1) \cup (X, X+H), \tag{5.18}$$

and

$$\frac{x}{da_1} \ll \frac{X}{t} \tag{5.19}$$

from the support of $w'\left(\frac{r+tx}{da_1}\right)$ (see (3.9)). Therefore, we have

$$\begin{aligned}
&-\frac{1}{2\pi i}\frac{da_1}{n}\int \frac{x}{da_1}w'(x)w'\left(\frac{r+tx}{da_1}\right)e\left(-\frac{nx}{da_1}\right)\mathrm{d}x\\
&\ll \frac{da_1}{n}.\frac{X}{t}.H.\frac{1}{H^2}\\
&\ll \frac{da_1X}{ntH}.
\end{aligned} \tag{5.20}$$

From the support of $w''\left(\frac{r+tx}{da_1}\right)$ (see (3.9), (3.10)), we see that

$$\frac{tx}{da_1} \ll X, \tag{5.21}$$

and the length of the integral is $\frac{Hda_1}{t}$. Hence, we have the bound for the last term of (5.15).

$$\begin{aligned}
&-\frac{1}{2\pi i}\frac{da_1}{n}\int \frac{x}{da_1}\frac{t}{da_1}w(x)w''\left(\frac{r+tx}{da_1}\right)e\left(-\frac{nx}{da_1}\right)\mathrm{d}x\\
&\ll \frac{da_1}{n}\frac{X}{da_1}\frac{Hda_1}{t}\frac{1}{H^2}\\
&\ll \frac{da_1X}{ntH}.
\end{aligned} \tag{5.22}$$

Combining (5.15), (5.17), (5.20) and (5.22), we obtain

$$\left(\widehat{F_{da_1,t}}\left(\frac{n}{da_1}\right)\right)' \ll \frac{da_1X}{ntH}. \tag{5.23}$$

Now, in order to apply Lemma 2.2, we first take out the integral over t outside the a_1-sum and we decompose the t integral into two parts.

$$\sum_{d\geq 1}\frac{1}{d}\sum_{\substack{Q \text{ dyadic}\\ X^{3/4}<Q\leq \frac{X}{d}}}\sum_{\substack{0\neq n\ll \frac{X^{1+\varepsilon}}{H}\\ d|nr}}\int_{1}^{X}\sum_{\substack{a_1\sim Q\\ (a_1,nr/d)=1}}\frac{\alpha(da_1)}{a_1}\left(\widehat{F_{da_1,t}}\left(\frac{n}{da_1}\right)\right)'\sum_{\substack{1\leq p\leq t\\ (a_1,p)=1}} e\left(\frac{-(nr/d)\bar{p}}{a_1}\right)\mathrm{d}t$$

$$=\sum_{d\geq 1}\frac{1}{d}\sum_{\substack{Q \text{ dyadic}\\ X^{3/4}<Q\leq \frac{X}{d}}}\sum_{\substack{0\neq n\ll \frac{X^{1+\varepsilon}}{H}\\ d|nr}}\int_{1}^{X^{1/2}}\sum_{\substack{a_1\sim Q\\ (a_1,nr/d)=1}}\frac{\alpha(da_1)}{a_1}\left(\widehat{F_{da_1,t}}\left(\frac{n}{da_1}\right)\right)'\sum_{\substack{1\leq p\leq t\\ (a_1,p)=1}} e\left(\frac{-(nr/d)\bar{p}}{a_1}\right)\mathrm{d}t$$

$$+\sum_{d\geq 1}\frac{1}{d}\sum_{\substack{Q \text{ dyadic}\\ X^{3/4}<Q\leq \frac{X}{d}}}\sum_{\substack{0\neq n\ll \frac{X^{1+\varepsilon}}{H}\\ d|nr}}\int_{X^{1/2}}^{X}\sum_{\substack{a_1\sim Q\\ (a_1,nr/d)=1}}\frac{\alpha(da_1)}{a_1}\left(\widehat{F_{da_1,t}}\left(\frac{n}{da_1}\right)\right)'\sum_{\substack{1\leq p\leq t\\ (a_1,p)=1}} e\left(\frac{-(nr/d)\bar{p}}{a_1}\right)\mathrm{d}t. \tag{5.24}$$

Trivially bounding the sum over p in the first term, we have by (5.23)

$$\sum_{d\geq 1}\frac{1}{d}\sum_{\substack{Q \text{ dyadic}\\ X^{3/4}<Q\leq \frac{X}{d}}}\sum_{\substack{0\neq n\ll \frac{X^{1+\varepsilon}}{H}\\ d|nr}}\int_{1}^{X^{1/2}}\sum_{\substack{a_1\sim Q\\ (a_1,nr/d)=1}}\frac{\alpha(da_1)}{a_1}\left(\widehat{F_{da_1,t}}\left(\frac{n}{da_1}\right)\right)'\sum_{\substack{1\leq p\leq t\\ (a_1,p)=1}} e\left(\frac{-(nr/d)\bar{p}}{a_1}\right)\mathrm{d}t$$
$$\ll \frac{X^{2+1/2+\varepsilon}}{H}. \tag{5.25}$$

At last we focus on the second term of (5.24) and the main contribution will come from this term. We are ready to apply Lemma 2.2 to get a cancellation in our sum which now satisfies the condition $Q^{4/3}\geq t\geq Q^{1/2}$.

$$\sum_{d\geq 1}\frac{1}{d}\sum_{\substack{Q \text{ dyadic}\\ X^{3/4}<Q\leq \frac{X}{d}}}\sum_{\substack{0\neq n\ll \frac{X^{1+\varepsilon}}{H}\\ d|nr}}\int_{X^{1/2}}^{X}\sum_{\substack{a_1\sim Q\\ (a_1,nr/d)=1}}\frac{\alpha(da_1)}{a_1}\left(\widehat{F_{da_1,t}}\left(\frac{n}{da_1}\right)\right)'\sum_{\substack{1\leq p\leq t\\ (a_1,p)=1}} e\left(\frac{-(nr/d)\bar{p}}{a_1}\right)\mathrm{d}t$$
$$\ll \sum_{d\geq 1}\frac{1}{d}\sum_{\substack{Q \text{ dyadic}\\ X^{3/4}<Q\leq X}}\sum_{\substack{0\neq n\ll \frac{X^{1+\varepsilon}}{H}\\ d|nr}}\int_{X^{1/2}}^{X}\sum_{\substack{a_1\sim Q\\ (a_1,nr/d)=1}}\left|\frac{\alpha(da_1)}{a_1}\right|\left|\left(\widehat{F_{da_1,t}}\left(\frac{n}{da_1}\right)\right)'\right|$$
$$\left|\sum_{\substack{1\leq p\leq t\\ (a_1,p)=1}} e\left(\frac{-(nr/d)\bar{p}}{a_1}\right)\right|\mathrm{d}t$$
$$\ll r^{\frac{1}{2}}\left(\frac{X}{H}\right)^{\frac{3}{2}}X^{\frac{7}{8}+\varepsilon}. \tag{5.26}$$

Combining (5.12), (5.13), (5.24), (5.25) and (5.26) we arrive at the proposition. □

5.1. **Proof of Proposition 3.3.** Proposition 3.3 directly follows by bringing together Proposition 5.2 and Proposition 5.3.

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