

# A NOTE ON THE ADMISSIBILITY OF SMOOTH SIMPLE $RG$ -MODULES

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ABSTRACT. Let  $G$  be a  $p$ -adic reductive group and  $R$  be a noetherian Jacobson  $\mathbb{Z}[1/p]$ -algebra. In this note, we show that every smooth irreducible  $R$ -linear representation of  $G$  is admissible using the finiteness result of Dat, Helm, Kurinczuk and Moss for Hecke algebras over  $R$ .

Unless mentioned otherwise, all rings are commutative with unity. A  $p$ -adic reductive group is the group of rational points of a reductive group defined over a non-archimedean local field of residue characteristic  $p > 0$ . Let  $R$  be a ring and  $G$  be a  $p$ -adic reductive group. Let  $RG$  denote the group algebra. An  $RG$ -module  $\pi$  is called *smooth* if every  $v \in \pi$  is fixed by some compact open subgroup in  $G$ , and *admissible* if for every compact open subgroup  $K \subseteq G$ ,  $\pi^K$  is a finitely generated  $R$ -module.

For a compact open subgroup  $K \subseteq G$ , let  $H_R(G, K)$  denote the Hecke algebra of compactly supported  $R$ -valued  $K$ -biinvariant functions on  $G$  equipped with the convolution product and  $Z_R(G, K)$  denote its center. The Hecke algebra  $H_R(G, K)$  is an associative  $R$ -algebra with unity. The following theorem is the joint work [2, Theorem 1.2] of Dat, Helm, Kurinczuk and Moss (see also [1]):

**Theorem 1.** *For any noetherian  $\mathbb{Z}[1/p]$ -algebra  $R$  and any compact open subgroup  $K \subseteq G$ , the Hecke algebra  $H_R(G, K)$  is a finitely generated module over  $Z_R(G, K)$  and  $Z_R(G, K)$  is a finitely generated  $R$ -algebra.*

As an application of Theorem 1, we prove the following result:

**Theorem 2.** *If  $R$  is a noetherian Jacobson  $\mathbb{Z}[1/p]$ -algebra, then any smooth simple  $RG$ -module  $\pi$  is admissible.*

Recall that a ring is called Jacobson if every prime ideal is the intersection of the maximal ideals which contain it. The examples of noetherian Jacobson  $\mathbb{Z}[1/p]$ -algebras include all fields of characteristic not equal to  $p$  as well as finitely generated algebras over such fields such as  $\mathbb{F}_l[T_1, \dots, T_n]$  or finite rings such as  $\mathbb{Z}/l^m\mathbb{Z}$  with  $l \neq p$  for which Theorem 2 is a new result. When  $R = \mathbb{C}$ , Theorem 2 is a classical result in the representation theory of  $p$ -adic groups. The case when  $R$  is any field of characteristic not equal to  $p$  is given in [8, Proposition 4.10]. We remark that Theorem 2 does not hold for representations over a field of characteristic  $p$ , see [4].

Theorem 2 follows from the following Corollary to Theorem 1:

**Corollary to Theorem 1.** *Let  $R$  be a noetherian Jacobson  $\mathbb{Z}[1/p]$ -algebra, and  $M$  be a simple (left) module over  $H_R(G, K)$ . Then  $M$  is a finitely generated  $R$ -module.*

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*Proof.* To ease the notation, let us write  $H = H_R(G, K)$  and  $Z = Z_R(G, K)$ . Choose a surjective map  $H \twoheadrightarrow M$  of  $R$ -modules. Let  $\mathfrak{m}$  be the kernel of the surjection  $H \twoheadrightarrow M$  and  $\mathfrak{m}_Z := \mathfrak{m} \cap Z$ . Note that  $\mathfrak{m}_Z$  is a two-sided ideal of  $Z$  because  $Z$  is the center of  $H$ . We claim that  $\frac{Z}{\mathfrak{m}_Z}$  is a field. Let  $\bar{z} := z + \mathfrak{m}_Z \in \frac{Z}{\mathfrak{m}_Z}$  be a non-zero element. Then  $\bar{z}$  is also non-zero in  $\frac{H}{\mathfrak{m}}$ . So the left  $H$ -submodule  $H\bar{z}$  of  $\frac{H}{\mathfrak{m}}$  generated by  $\bar{z}$  is equal to  $\frac{H}{\mathfrak{m}}$  because  $\frac{H}{\mathfrak{m}}$  is simple. Therefore, there exists  $h \in H$  such that  $\bar{h}\bar{z} = \bar{1}$  in  $\frac{H}{\mathfrak{m}}$ .

Consider the  $\frac{Z}{\mathfrak{m}_Z}$ -algebra  $\frac{Z}{\mathfrak{m}_Z}[\bar{h}]$  generated by  $\bar{h}$ . It is commutative because  $\frac{Z}{\mathfrak{m}_Z}$  is commutative. Moreover, as  $\frac{H}{\mathfrak{m}}$  is a finitely generated  $\frac{Z}{\mathfrak{m}_Z}$ -module and  $\frac{Z}{\mathfrak{m}_Z}$  is noetherian,  $\frac{Z}{\mathfrak{m}_Z}[\bar{h}]$  is a finitely generated  $\frac{Z}{\mathfrak{m}_Z}$ -module. Hence,  $\bar{h}$  is integral over  $\frac{Z}{\mathfrak{m}_Z}$ , i.e.

$$\bar{h}^n + \bar{a}_{n-1}\bar{h}^{n-1} + \dots + \bar{a}_0 = 0,$$

for some  $n \in \mathbb{N}$  and  $\bar{a}_{n-1}, \bar{a}_{n-2}, \dots, \bar{a}_0 \in \frac{Z}{\mathfrak{m}_Z}$ . Multiplying both sides of the above by  $\bar{z}^{n-1}$  and using that  $\frac{Z}{\mathfrak{m}_Z}$  commutes with  $\bar{h}$ , we obtain that

$$\bar{h} + \bar{a}_{n-1} + \bar{a}_{n-2}\bar{z} + \dots + \bar{a}_0\bar{z}^{n-1} = 0.$$

Hence  $\bar{h} = -(\bar{a}_{n-1} + \bar{a}_{n-2}\bar{z} + \dots + \bar{a}_0\bar{z}^{n-1}) \in \frac{Z}{\mathfrak{m}_Z}$ .

Now the field  $\frac{Z}{\mathfrak{m}_Z}$  is a finitely generated  $R$ -algebra. One of the characterizations of Jacobson rings implies that  $\frac{Z}{\mathfrak{m}_Z}$  is a finitely generated  $R$ -module [3, Theorem 10]. Since  $\frac{H}{\mathfrak{m}}$  is finite over  $\frac{Z}{\mathfrak{m}_Z}$ , we get that  $\frac{H}{\mathfrak{m}} \cong M$  is also a finitely generated  $R$ -module.  $\square$

*Proof of Theorem 2.* Since  $G$  has a fundamental system of neighborhoods of identity consisting of open pro- $p$  subgroups, it is enough to show that  $\pi^K$  is a finitely generated  $R$ -module for  $K \subseteq G$  an open pro- $p$  subgroup. Let  $K \subseteq G$  be an open pro- $p$  subgroup such that  $\pi^K \neq 0$ . Since  $\pi$  is simple and  $p \in R^\times$ ,  $\pi^K$  is a simple  $H_R(G, K)$ -module by [7, I.6.3]. Hence,  $\pi^K$  is a finitely generated  $R$ -module by the Corollary to Theorem 1.  $\square$

**Remark 3.** The requirement for  $R$  to be Jacobson in Corollary to Theorem 1 is necessary. Indeed, if  $R$  is a commutative ring and if all simple modules over  $H_R(G, K)$  are finitely generated  $R$ -modules for all  $p$ -adic reductive groups  $G$  and compact open subgroups  $K$ , then  $R$  is Jacobson. The following proof of this converse statement was communicated to us by M.-F. Vignéras: By Satake [6, §8], if  $G$  is a classical simple group with trivial center and  $K \subseteq G$  a natural maximal compact subgroup, then  $H_R(G, K)$  is a polynomial ring over  $R$  in  $m$  variables where  $m$  is the rank of a maximal split torus in  $G$ . Thus, a finitely generated  $R$ -algebra  $A$  is a quotient of some  $H_R(G, K)$ . If  $A$  is a field, then  $A$  is a simple module over  $H_R(G, K)$ , and hence a finitely generated  $R$ -module by assumption. This means that  $R$  is Jacobson.

**Remark 4.** Let  $R = \mathbb{Z}_l$  with  $l \neq p$  and  $G = \mathrm{GL}_2(\mathbb{Q}_p)$ . As  $R$  is not Jacobson, Remark 3 suggests that  $G$  admits a smooth irreducible  $\mathbb{Z}_l$ -representation that is not admissible. Indeed, let  $K = \mathrm{GL}_2(\mathbb{Z}_p)$ . By [5, Proposition 2.1],  $H = H_R(G, K) \cong R[T_0, T_0^{-1}, T_1]$ . One can make  $M := \mathbb{Q}_l$  into a simple  $H$ -module by defining the action via the surjective map  $H \twoheadrightarrow \mathbb{Q}_l$  which takes  $T_0$  to 1 and  $T_1$  to  $l^{-1}$ . However, note that  $M$  is not a finitely generated  $R$ -module. By choosing a prime  $l$  so that

the pro-order of  $K$  is invertible in  $R$ , there exists a smooth simple  $RG$ -module  $\pi$  such that  $\pi^K \cong M$  as  $H$ -modules [7, I.4.4 and I.6.3]. Since  $\pi^K$  is not a finite  $R$ -module,  $\pi$  is non-admissible.

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