TYPE NUMBER FOR ORDERS OF LEVEL (N_1, N_2)

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ABSTRACT. Let $N_1 = p_1^{2u_1+1}...p_w^{2u_w+1}$, where the p_i are distinct primes, $u_1, ..., u_w$ are nonnegative integers and w is an odd integer, and N_2 be a positive integer such that $\gcd(N_1, N_2) = 1$. In this paper we give an explicit formula for the type number, i.e. the number of isomorphism classes, of orders of level (N_1, N_2). The method of proof involves the Siegel-Weil formula for ternary quadratic forms.

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1. Introduction

Let $N_1 = p_1^{2u_1+1}...p_w^{2u_w+1}$, where the p_i are distinct primes, $u_1, ..., u_w$ are nonnegative integers and w is an odd integer, and N_2 is a positive integer such that $\gcd(N_1, N_2) = 1$. The type number stands as a focal point in the theory of quaternion algebras. When N_1 is squarefree, orders of level N_1 are Eichler orders and the type number formula in this case is due to Pizer [10]. In the case $N_1 = p^{2u+1}, p \neq 2$ the type number formula is due to Boyd [2]. Their methods of proof involved an application of the Selberg Trace Formula and the development of the theory of optimal embeddings mod normalizers. In 2021, Li, Skoruppa and Zhou [5] gave a weighted sum of Jacobi theta series associated with Hereditary orders is a Jacobi Eisenstein series which has Fourier coefficients $H^{(N_1,N_2)}(4n - r^2)$, where N_1N_2 is squarefree, and by counting zeros in Hereditary orders they gave the type number of Hereditary orders using $H^{(N_1,N_2)}(D)$. In our contribution, we will give a new proof of type number for orders of level (N_1, N_2) using the Siegel–Weil formula for ternary quadratic forms.

We observe that for orders of level (N_1, N_2), the Fourier coefficients of a weighted sum of theta series align with the Siegel–Weil average of a genus. Our paper is articulated as follows: establishing commutative diagrams between ternary quadratic forms and orders of level (N_1, N_2) (N_1N_2 is not necessarily squarefree) as defined by Pizer [9], employing the Siegel-Weil formula for ternary quadratic forms, further giving the explicit formula for the Fourier coefficients. Once the coefficients have been defined, it is natural to ask if the Fourier coefficients $H^{(N_1,N_2)}(D)$ give the type number formula. We find that counting the normalizers is equivalent to counting representation numbers of ternary quadratic forms under a bijection. By Siegel–Weil formula for ternary quadratic forms, we finally give an explicit formula for the type number of orders of level (N_1, N_2).

Now we give the definition of $H^{(N_1,N_2)}(D)$. Let $N_1 = p_1^{2u_1+1}...p_w^{2u_w+1}$, where the p_i are distinct primes, $u_1, ..., u_w$ are nonnegative integers and w is an odd integer, and N_2 be positive integer such that $\gcd(N_1, N_2) = 1$. For any negative discriminant $-D$, $H(D/f_{N_1,N_2}^2)$ is Hurwitz class number, and f_{N_1,N_2} is the largest positive integer containing only prime factors of N_1N_2 whose square divides D such that $-D/f_{N_1,N_2}^2$ is still a negative discriminant. We use f_p for the exact p-power dividing f_{N_1,N_2}, and $v_p(N_1N_2) = p^u$ if $p^u \parallel N_1N_2$. Define

$$H^{(N_1,N_2)}(D) = H(D/f_{N_1,N_2}^2) \prod_{p|N_1} A_{N_1N_2,p}(D) \prod_{\substack{p|N_2 \\ v_p(N_2) \text{is odd}}} A_{N_1N_2,p,1}(D) \prod_{\substack{p|N_2 \\ v_p(N_2) \text{is even}}} A_{N_1N_2,p,2}(D), \tag{1.1}$$

where in the case $v_p(pf_{N_1,N_2}^2) < v_p(N_1N_2)$ and $p \mid D/f_{N_1,N_2}^2$, then

$$A_{N_1N_2,p}(D) = A_{N_1N_2,p,1}(D) = A_{N_1N_2,p,2}(D) = 0,$$

in the case $v_p(pf_{N_1,N_2}^2) < v_p(N_1N_2)$ and $p \nmid D/f_{N_1,N_2}^2$, then

$$A_{N_1N_2,p}(D) = f_p^2 \left(1 - \left(\frac{-D/f_{N_1,N_2}^2}{p}\right)\right),$$

$$A_{N_1N_2,p,1}(D) = A_{N_1N_2,p,2}(D) = f_p^2 \left(1 + \left(\frac{-D/f_{N_1,N_2}^2}{p}\right)\right),$$

in the case $v_p(pf_{N_1,N_2}^2) \geq v_p(N_1N_2)$ then

$$A_{N_1N_2,p}(D) = p^{v_p(N_1)-1} \left(1 - \left(\frac{-D/f_{N_1,N_2}^2}{p}\right)\right),$$

$$A_{N_1N_2,p,1}(D) = \frac{2p^{\frac{v_p(N_2)+1}{2}} f_p - p^{v_p(N_2)-1}(p+1) - \left(\frac{-D/f_{N_1,N_2}^2}{p}\right)(2p^{\frac{v_p(N_2)-1}{2}} f_p - p^{v_p(N_2)-1}(p+1))}{p-1},$$

$$A_{N_1N_2,p,2}(D) = \frac{(p^{\frac{v_p(N_2)}{2}} f_p - p^{v_p(N_2)-1})(p+1) - \left(\frac{-D/f_{N_1,N_2}^2}{p}\right)(p^{\frac{v_p(N_2)}{2}-1} f_p - p^{v_p(N_2)-1})(p+1)}{p-1}.$$

The products run through all primes p dividing N_1 and N_2, respectively. All $(\frac{\cdot}{p})$ are the Kronecker symbols, and for integer m, Kronecker symbol

$$\left(\frac{m}{2}\right) = \begin{cases} 1 & \text{if} \quad m \equiv \pm 1 \pmod 8, \\ -1 & \text{if} \quad m \equiv \pm 3 \pmod 8, \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$H^{(N_1,N_2)}(0) = \frac{N_1N_2}{12} \prod_{p|N_1} (1 - \frac{1}{p}) \prod_{p|N_2} (1 + \frac{1}{p}),$$

and $H^{(N_1,N_2)}(D) = 0$ for every positive integer $D \equiv 1, 2 \pmod 4$.

Theorem 1.1. *The class number of orders of level* (N_1, N_2) *is*

$$h_{N_1,N_2} = H^{(N_1,N_2)}(4)/2 + H^{(N_1,N_2)}(3) + H^{(N_1,N_2)}(0).$$

The type number of orders of level (N_1, N_2) *is*

$$T_{N_1,N_2} = 2^{-e(N_1N_2)-1} \sum_{n \| N_1N_2} \sum_{\substack{n|r \\ r^2 \leq 4n}} H^{(N_1,N_2)}(4n - r^2) \prod_{p|n} \frac{1 - \left(\frac{\Delta(-4n)}{p}\right)/p}{B_p(n)C_p(n)}.$$

Let $-4n = -n_0n_1^2$ *where* $-n_0$ *is a fundamental discriminant, then*

$$\Delta(-4n) = -n_0,$$

$$B_p(n) = \begin{cases} (p+1)p^{v_p(n)/2-1} & \textit{if} \quad v_p(n) \equiv 0 \pmod 2, \\ p^{(v_p(n)-1)/2} & \textit{if} \quad v_p(n) \equiv 1 \pmod 2, \end{cases}$$

and

$$C_p(n) = \begin{cases} 2 & \textit{if} \quad p = 2, 4 \mid n, \textit{and} \quad \Delta(-4n) \equiv 5 \pmod 8, \\ 1 & \textit{otherwise.} \end{cases}$$

When N_1 and N_2 are squarefree, it is the same as the type number formula in [5]. When N_1 is squarefree, it is equivalent to Pizer's formula of type number in [10]. When $N_1 = p^{2r+1}$, it is equivalent to Boyd's formula of type number in [2]. It is very fallible to compute type number without computer. In this article certain calculations are facilitated by SageMath. In Appendix A, we will present all the class number and type number of orders of level $N_1N_2 \leq 100$, and correct errors ($T_{3^3,5}, T_{5^3,8}, T_{3^7,1}, T_{13^3,2^4}$) in Boyd's table of type numbers[2, p.152]. If $T_{N_1,N_2} = 1$, we can give an exact formula for the representation number of n by ternary quadratic forms. In Appendix B, we will present 157 exact formulas for the representation number of ternary quadratic forms.

2. Ternary quadratic forms

2.1. **Basic knowledge.** We adopt the definition of ternary quadratic forms in [4]. Let f be a ternary quadratic form with integer coefficients, given by the equation

$$f(x,y,z) = ax^2 + by^2 + cz^2 + ryz + sxz + txy.$$

Define f to be primitive that is $\gcd(a,b,c,r,s,t) = 1$. Recall that the matrix associated with f is

$$M = M_f = \begin{pmatrix} 2a & t & s \\ t & 2b & r \\ s & r & 2c \end{pmatrix}.$$

Define the discriminant of f to be

$$d = d_f = \frac{\det(M_f)}{2} = 4abc + rst - ar^2 - bs^2 - ct^2.$$

Define the divisor of f to be the positive integer

$$m = m_f = \gcd(M_{11}, M_{22}, M_{33}, 2M_{23}, 2M_{13}, 2M_{12}),$$

where

$$\begin{aligned} M_{11} = 4bc - r^2, M_{23} = st - 2ar = M_{32}, \\ M_{22} = 4ac - s^2, M_{13} = rt - 2bs = M_{31}, \\ M_{33} = 4ab - t^2, M_{12} = rs - 2ct = M_{21}. \end{aligned}$$

Define the level of f to be the positive integer

$$N = N_f = \frac{4d_f}{m_f}.$$

N is the smallest positive integer such that NM_f^{-1} is even[4, pp.401-402]. A connection exists between the level and discriminant, as outlined below.

Theorem 2.1. [4, Theorem 2] *Let f be a primitive positive definite ternary quadratic form of level N and discriminant d.*

$$N = 2^{n_0} p_1^{n_1} ... p_k^{n_k} \tag{2.1}$$

is the prime factorization of N'. Then $n_0 \geq 2$ and d is of the form

$$d = 2^{d_0} p_1^{d_1} ... p_k^{d_k} \tag{2.2}$$

with the following restrictions on exponents:
(1) $d_0 = n_0 - 2, d_0 = 2n_0$, or $n_0 \leq d_0 \leq 2n_0 - 2$, and
(2) for $1 \leq i \leq k, n_i \leq d_i \leq 2n_i$
Furthermore, if n_i is even for $0 \leq i \leq k$, then either $n_0 \leq d_0 \leq 2n_0 - 2$, or d_i is odd for some $1 \leq i \leq k$.

Ternary quadratic forms f and g are deemed to be equivalent, $f \sim g$, if there is a matrix $M \in GL_3(\mathbb{Z})$ (i.e., M has integer entries and $\det(M) = \pm 1$) such that $M_f = MM_gM^t$. We denote $\mathrm{Aut}(f)$ as the finite group of integral automorphs of f (i.e. $\mathrm{Aut}(f)$ comprises all $M \in GL_3(\mathbb{Z})$ such that $M_f = M^tM_fM$). Equivalent forms are considered part of the same class. In the case $f \sim g$, it follows that $d_f = d_g$ and $N_f = N_g$[4, pp.401-402]. Two integral quadratic forms are termed semi-equivalent if they are equivalent over the p-adic integers for all primes p and equivalent over the real numbers. Semi-equivalent forms are categorized within the same genus of forms. It is noteworthy that equivalent forms are also semi-equivalent, allowing us to refer to a class of forms as belonging to a genus [4, pp.409-410].

Consider f as a ternary quadratic forms over $\mathbb{Q}$, it can be equivalent over $\mathbb{Q}$ to a diagonal form

$$ax^2 + by^2 + cz^2.$$

The Hasse invariant for f at p is defined to be

$$S_p(f) = (a, -1)_p(b, -1)_p(c, -1)_p(a, b)_p(b, c)_p(c, a)_p,$$

which exclusively depends on the equivalence class of f. Here $(a, b)_p$ represents the Hilbert Norm Residue Symbol. We say f_p is isotropic if $f_p(x, y, z) = 0$ for some non-zero elements $x, y, z \in \mathbb{Q}_p$, and it is anisotropic otherwise. In this context f_p denotes the localization of f at p. The modified Hasse symbol S^*_p indicates whether f_p is isotropic or not, that is

$$S^*_p(f) = (-1)^{\delta_{p,2}} S_p(f) = \begin{cases} 1 & \text{if } f_p \text{ is isotropic,} \\ -1 & \text{if } f_p \text{ is anisotropic,} \end{cases}$$

where $\delta_{m,n} = 1$ if $m = n$ and otherwise 0. Recall that Hilbert's reciprocity law says $(a, b)_\infty \prod_p (a, b)_p = 1$. This results in $S_\infty(f) \prod_p S_p(f) = 1$. For a primitive positive definite ternary quadratic forms f, verifying $S_\infty(f) = 1$ is straightforward. Consequently, f is anisotropic only at odd number of primes.

2.2. **Bijections.** We will revisit some commonly employed bijections in ternary quadratic forms, namely Lehman's bijection ϕ_p and Watson transformation λ_4. Prior to delving into the details of these bijections, we must introduce the following lemma.

Lemma 2.2. [4, Lemma 2] *Let f be a primitive positive definite ternary quadratic forms of level N and divisor m. Suppose that $p^i \parallel N$ and $p^j \parallel m$ for some odd prime p and positive integer i. Then f is equivalent to a form (a, b, c, r, s, t) with $p^i \parallel a$, $p^i \mid s$ and t, $p^j \mid b$ and r, and $p \nmid c$. If $0 < j < i$, then we can assume that $p^j \parallel b$.*

Remark 2.3. Let $p^g \parallel N$ and $p^h \parallel d$, it is easy to check $i = g$ and $j = h - g$, i.e. f is equivalent to a form $(p^g a, p^{h-g} b, c, p^{h-g} r, p^g s, p^g t)$, where a, b, c, r, s, t are integers and $p \nmid ac$. We have three cases for $p = 2$. For $h = g - 2$, f is equivalent to a form $(2^{g-2} a, b, c, r, 2^{g-1} s, 2^{g-1} t)$, for $g \le h \le 2g - 2$, f is equivalent to a form $(2^{g-2} a, 2^{g-h} b, c, 2^{h-g+1} r, 2^{g-1} s, 2^{g-1} t)$, and for $h = 2g$, f is equivalent to a form $(2^g a, 2^g b, c, 2^g r, 2^g s, 2^g t)$, where a, b, c, r, s, t are all integers and $2 \nmid ac$.

Remark 2.4. It is evident to extend Lemma 2.2, implying that we can assume f to be equivalent to a form (a, b, c, r, s, t) with $p^i \parallel a$, $p^i \mid s$ and t, $p^j \mid b$ and r, and $p \nmid c$ for all primes $p \mid N$. Refer to [4, Lemma 1, Lemma 2].

Let $C(N, d)$ denote the set of all classes of primitive positive definite ternary quadratic forms of level N and discriminant d.

Theorem 2.5. [4, Theorem 4, Theorem 5] *Let N and d be given by equations (2.1) and (2.2). Suppose that $p^g \parallel N$ and $p^h \parallel d$ for some odd prime p. Write d as $p^h d'$. Then there is a one-to-one correspondence between $C(N, p^h d')$ and $C(N, p^{3g-h} d')$. If $p = 2$, then there is a one-to-one correspondence between $C(N, 2^h d')$ and $C(N, 2^{3g-h-2} d')$.*

Let $f \in C(N', p^h d')$, $p^g \parallel N'$ and $p \nmid d'$ and p is an odd prime. In terms of Lemma 2.2, we can assume that

$$f = (p^g a, p^{h-g} b, c, p^{h-g} r, p^g s, p^g t),$$

with a, b, c, r, s and t integers, $p \nmid ac$. Define the map

$$\begin{aligned} \phi_p : C(N', p^h d') &\to C(N', p^{3g-h} d') \\ f &\mapsto \phi_p(f) \\ M_f &\mapsto P M_f P, \end{aligned}$$

where

$$P = \begin{pmatrix} p^{-g/2} & 0 & 0 \\ 0 & p^{(3g-2h)/2} & 0 \\ 0 & 0 & p^{g/2} \end{pmatrix},$$

$$\phi_p(f) = (a, p^{2g-h}b, p^g c, p^g r, p^g s, p^{2g-h}t).$$

We have three cases for $p = 2$. For $h = g-2$ and $f = (2^{g-2}a, b, c, r, 2^{g-1}s, 2^{g-1}t)$, we have $\phi_2(f) = (a, 2^g b, 2^g c, 2^g r, 2^g s, 2^g t)$. For $g \le h \le 2g-2$ and $f = (2^{g-2}a, 2^{g-h}b, c, 2^{h-g+1}r, 2^{g-1}s, 2^{g-1}t)$, we have $\phi_2(f) = (a, 2^{2g-h-2}b, 2^{g-2}c, 2^{g-1}r, 2^{g-1}s, 2^{2g-h-1}t)$. For $h = 2g$ and $f = (2^g a, 2^g b, c, 2^g r, 2^g s, 2^g t)$, we have $\phi_2(f) = (a, b, 2^{g-2}c, 2^{g-1}r, 2^{g-1}s, t)$, where a, b, c, r, s, t are all integers and $2 \nmid ac$.

Lehman proved that ϕ_p is a one-to-one correspondence. It is clear that $\phi_p^{-1} : C(N', p^{3g-h}d') \to C(N', p^h d')$ is an inverse for ϕ_p. A direct corollary follows.

Corollary 2.6. *Let p be a prime, and $f \in C(N', p^h d')$ where $p \nmid d'$, then we have*
(1) $|\operatorname{Aut}(f)| = |\operatorname{Aut}(\phi_p(f))|$.
(2) if f and g are semi-equivalent, then $\phi_p(f)$ and $\phi_p(g)$ are semi-equivalent,
(3) let q be a prime, then f is isotropic at q if and only if $\phi_p(f)$ is isotropic at q.

These one-to-one correspondences ϕ_p establish a connection among the representation numbers $R_f(n)$.

Proposition 2.7. *Let $f \in C(p^g N', p^{2g}d')$ where p is an odd prime, $p \nmid N'$ and $p \nmid d'$. Suppose*

$$f = (p^g a, p^g b, c, p^g r, p^g s, p^g t),$$

then we have

$$card(\{(x,y,z) \in \mathbb{Z}^3 | f(x,y,z) = p^g n, p^g \mid z\}) = R_{\phi_p(f)}(n).$$

Proof. Let $f \in C(N', p^{2g}d')$. By Lemma 2.2, assume that

$$f = (p^g a, p^g b, c, p^g r, p^g s, p^g t),$$

with a, b, c, r, s, t integers, and $p \nmid ac$. We have

$$\phi_p(f) = (a, b, p^g c, p^g r, p^g s, t).$$

Suppose that there are integers x, y and z satisfying the equation

$$f(x,y,z) = p^g ax^2 + p^g by^2 + cz^2 + p^g ryz + p^g sxz + p^g txy = p^g n.$$

Since all the terms are divisible by p^g expect cz^2, and $p \nmid c$, we have $p^g \mid z^2$. When $p^g \mid z$, let $(x', y', z') = (x, y, z/p^g)$, we have

$$\begin{aligned} p^g n = f(x,y,z) &= p^g ax^2 + p^g by^2 + cz^2 + p^g ryz + p^g sxz + p^g txy \\ &= p^g ax'^2 + p^g by'^2 + c(p^g z')^2 + p^g ry'(p^g z') + p^g sx'(p^g z') + p^g tx'y' \\ &= p^g(ax'^2 + by'^2 + p^g cz'^2 + p^g ry'z' + p^g sx'z' + tx'y') \\ &= p^g \phi_p(f)(x', y', z'). \end{aligned}$$

We have $\phi_p(f)(x', y', z') = n$.

Conversely,

$$\phi_p^{-1}(\phi_p(f)) = (p^g a, p^g b, c, p^g r, p^g s, p^g t) = f.$$

Suppose that there are some integers x, y and z such that

$$\phi_p(f)(x,y,z) = ax^2 + by^2 + p^g cz^2 + p^g ryz + p^g sxz + txy = n.$$

Let $(x', y', z') = (x, y, p^g z)$, we have

$$\begin{aligned} p^g n = p^g \phi_p(f)(x,y,z) &= p^g ax^2 + p^g by^2 + p^{2g} cz^2 + p^{2g} ryz + p^{2g} sxz + p^g txy \\ &= p^g ax'^2 + p^g by'^2 + cz'^2 + p^g ry'z' + p^g sx'z' + p^g tx'y' \\ &= \phi_p^{-1}(\phi_p(f))(x', y', z'). \end{aligned}$$

□

Remark 2.8. When $g = 1$, $p \mid z^2$ implies $p \mid z$, it is clear $R_{\phi_p(f)}(n) = R_f(pn)$.

Proposition 2.9. *Let $f \in C(2^g N', p^{2g} d')$ where $2 \nmid N'$ and $2 \nmid d'$. Suppose*

$$f = (2^g a, 2^g b, c, 2^g r, 2^g s, 2^g t),$$

we have

$$card(\{(x, y, z) \in \mathbb{Z}^3 | f(x, y, z) = 2^g n, 2^{g-1} \mid z\}) = R_{\phi_2(f)}(n).$$

Proof. The proof closely resembles that of Proposition 2.7. □

Remark 2.10. When $g = 2$ (resp. $g = 3$), $2^2 \mid z^2$ (resp. $2^3 \mid z^2$) implies $2 \mid z$ (resp. $2^2 \mid z$), we have $R_{\phi_2(f)}(n) = R_f(4n)$ (resp. $R_{\phi_2(f)}(n) = R_f(8n)$).

Recallling Watson transformation [12], let Λ_f be a 3-dimensional lattice, where f is Λ_f correspondence integral ternary quadratic form. Define $\Lambda_m(f)$ to be the set of all $\boldsymbol{x}$ in Λ_f with

$$f(\boldsymbol{x} + \boldsymbol{z}) \equiv f(\boldsymbol{z}) \pmod{m}, \forall \boldsymbol{z} \in \Lambda_f.$$

$\Lambda_m(f)$ is a 3-dimensional lattice [12, p.578]. Hence we can choose M so that $\boldsymbol{x}$ is in $\Lambda_m(f)$ if and only if $\boldsymbol{x} = M\boldsymbol{y}$, y in Λ_f. Let $g(\boldsymbol{y}) = m^{-1} f(M\boldsymbol{y})$, i.e.

$$\frac{1}{2} \boldsymbol{y}^t M_g \boldsymbol{y} = \frac{1}{2} \cdot m^{-1} (M\boldsymbol{y})^t M_f (M\boldsymbol{y}), \boldsymbol{y} \in \Lambda_f.$$

Theorem 2.11. [12, p.579] *Watson transformation λ_m is a well-define mapping, that is if $f \sim g$, then $\lambda_m(f) \sim \lambda_m(g)$, and if f and g are semi-equivalent, then $\lambda_m(f)$ and $\lambda_m(g)$ are semi-equivalent. Futhermore, the class number does not increase under the Watson transformation, that is, $\lambda_m(f)$ can range over the whole of the genus of $\lambda_m(f)$.*

Typically, the Watson transformation functions is a surjection between two genera. However, for specific levels and discriminants, the Watson transformation λ_4 becomes a bijection.

Proposition 2.12. *Let N' and d' be given by equations (2.1) and (2.2), and suppose that $4 \parallel N'$ (resp. $8 \parallel N'$) and $16 \parallel d'$ (resp. $64 \parallel d'$). Then Watson transformation λ_4 is a bijection between $C(N', d')$ (resp. $C(N', d')$) and $C(N', d'/16)$ (resp. $C(N'/2, d'/16)$) , and f is isotropic at p if and only if $\lambda_4(f)$ is isotropic at p, we also have*

$$|\operatorname{Aut}(f)| = |\operatorname{Aut}(\lambda_4(f))|.$$

Proof. See [7][Proposition 2.7]. □

3. Quaternion algebras

In this section commences with the fundamental knowledge on quaternion algebras. For a more comprehensive definition of quaternion algebras, refer to [11].

3.1. **Basic knowledge.** For a given field F of characteristic 0 and elements $a, b \in F^\times$, we denote by $Q = \left(\frac{a,b}{F}\right)$ for the F-algebra with a basis $1, i, j, k = ij = -ji$ and defining relations $i^2 = a, j^2 = b$. In this paper F specifically represents either $\mathbb{Q}$ or the fields $\mathbb{Q}_p$ of the p-adic numbers where p is a prime. A quaternion algebra Q is a central simple algebra. For the field of p-adic numbers $\mathbb{Q}_p$ or real numbers $\mathbb{Q}_\infty$, there exist only two equivalence classes, namely the algebras $M_2(\mathbb{Q}_p)$ of 2×2 matrices over $\mathbb{Q}_p$, or skew-fields. The standard p-adic Hilbert symbol is denoted by $(a, b)_p$. The quaternion algebra $Q_p = \left(\frac{a,b}{\mathbb{Q}_p}\right) \simeq M_2(\mathbb{Q}_p)$ if and only if $(a, b)_p = 1$. In the case of a quaternion algebra over $\mathbb{Q}$ the algebras $Q = \left(\frac{a,b}{\mathbb{Q}}\right)$ and $Q' = \left(\frac{a',b'}{\mathbb{Q}}\right)$ are isomorphic if and only if $(a, b)_p = (a', b')_p$ for all p and ∞. A quaternion algebra Q over $\mathbb{Q}$ ramifies at p if $\mathbb{Q}_p \otimes Q$ is a skew-field, and splits at p if $\mathbb{Q}_p \otimes Q \simeq M_2(\mathbb{Q}_p)$. If it ramifies at ∞, it is called definite. A quaternion algebra Q ramifies only at finitely many primes and this number is always even. Therefore, given a set S of an even number of primes, we can find a, b in $\mathbb{Q}^\times$ such that $(a, b)_p = -1$ exactly for p in S, and then $Q = \left(\frac{a,b}{F}\right)$ is a (up to isomorphism) quaternion algebra which ramifies exactly at the primes in S. The reduced discriminant of Q,disc(Q), is the product of primes at which Q ramifies.

For any element $\alpha = t+xi+yj+zk$ of quaternion algebra Q, we define an involution $\overline{\alpha} = t-xi-yj-zk$, which satisfies $\overline{\overline{\alpha}} = \alpha$ and $\overline{\alpha\beta} = \bar{\beta}\bar{\alpha}$ for $\beta \in Q$. The reduced trace on Q is $\mathrm{tr}(\alpha) = \alpha + \bar{\alpha}$, and similarly the reduced norm is $\mathrm{n}(\alpha) = \alpha \cdot \bar{\alpha}$. If $Q \simeq M_2(F)$, and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Q,$$

then

$$\bar{A} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Hence $\mathrm{tr}(A) = a + d, \mathrm{n}(A) = ad - bc = \det(A)$. We can see that $a \in Q$ is a root of the polynomial

$$x^2 - tr(a)x + n(a) = 0$$

which is the reduced characteristic polynomial of a.

3.2. **Orders of level** (N_1, N_2)**.** An ideal of quaternion algebra Q is a full $\mathbb{Z}$-lattice of Q, that is, a finitely generated $\mathbb{Z}$-submodule of Q which contains a basis of Q over $\mathbb{Q}$. An order $\mathcal{O}$ in a quaternion algebra Q is an ideal which is also a ring containing $\mathbb{Z}$. For the rest of this paper $\mathcal{O}$ will always denote a quaternion order. If $x \in \mathcal{O}$, then $\mathrm{tr}(x), \mathrm{n}(x) \in \mathbb{Z}$. An order $\mathcal{O}$ is maximal if it is not properly contained in another order. A quaternion division algebra Q_p over $\mathbb{Q}_p$ contains only one maximal order, which is $R_p + R_p j$ with $R_p = \mathbb{Z}_p + i\mathbb{Z}_p$ for odd p and $R_2 = \mathbb{Z}_2 + \frac{i+1}{2}\mathbb{Z}_2$ for $p = 2$, where $i^2 = \epsilon, j^2 = p$ for some $\left(\frac{\epsilon}{p}\right) = -1$, that is

$$Q_p \simeq \{\begin{pmatrix} \alpha & \beta \\ p\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{Q}_p + i\mathbb{Q}_p\},$$

then unique maximal order is

$$\mathcal{O}_p \simeq \{\begin{pmatrix} \alpha & \beta \\ p\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in R_p\}.$$

Let

$$\mathcal{O}_p^{(2u+1)} = \{\begin{pmatrix} \alpha & p^u\beta \\ p^{u+1}\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in R_p\}.$$

where u is a nonnegative integer. One can check $\mathcal{O}_p^{(2u+1)}$ is an order in Q_p, and $[\mathcal{O}_p : \mathcal{O}_p^{(2u+1)}] = p^{2u}$.

Definition 3.1. [8, p.684] Let $N_1 = p_1^{2u_1+1}...p_w^{2u_w+1}$, where the p_i are distinct primes, $u_1, ..., u_w$ are non-negative integers and w is an odd integer, and N_2 be a positive integer such that $\gcd(N_1, N_2) = 1$. An order $\mathcal{O}$ in Q, where Q is only ramifies at p_i for $1 \le i \le w$, is of level (N_1, N_2) if

(1) $\mathcal{O}_{p_i}$ is isomorphic (over $\mathbb{Z}_{p_i}$) to $\mathcal{O}_{p_i}^{(2u_i+1)}$ for $1 \le i \le w$,

(2) $\mathcal{O}_p$ is isomorphic (over $\mathbb{Z}_p$) to $\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ N_2\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$ for $p \nmid N_1$.

Remark 3.2. If N_1 is squarefree and $N_2 = 1$, these correspond to maximal orders. When N_1N_2 is squarefree they represent Hereditary order (or Ecihler orders with squarefree level). In the case of N_1 being squarefree, they are Eichler orders.

For an order $\mathcal{O}$ in Q, the definition of the codifferent $\mathcal{O}^\sharp$ is that

$$\mathcal{O}^\sharp = \{\alpha \in Q : \operatorname{tr}(\alpha\mathcal{O}) \in \mathbb{Z}\}.$$

Definition 3.3. $\mathcal{O}$ is Gorenstein if $\mathcal{O}^\sharp$ is invertible.

The property of being a Gorenstein order is local, i.e. $\mathcal{O}$ is Gorenstein if and only if $\mathcal{O}_p$ is Gorenstein for all primes p. Hence we have the following theorem.

Theorem 3.4. *An order of level* (N_1, N_2) *is Gorenstein.*

Proof. Let $\mathcal{O} \subset Q$ be an order of level (N_1, N_2). If Q ramifies at p, let $p^{2u+1} \parallel N_1$, we have

$$\mathcal{O}_p \cong \mathbb{Z}_p + i\mathbb{Z}_p + p^u j\mathbb{Z}_p + p^u ij\mathbb{Z}_p,$$

where $i^2 = \epsilon, j^2 = p$, $(\frac{\epsilon}{p}) = -1$. It is clear

$$(\mathcal{O}^\sharp)_p \cong \frac{1}{2}\mathbb{Z}_p + \frac{i}{2\epsilon}\mathbb{Z}_p + \frac{j}{2p^{u+1}}\mathbb{Z}_p + \frac{ij}{2\epsilon p^{u+1}}ij\mathbb{Z}_p = \mathbb{Z}_p + i\mathbb{Z}_p + \frac{j}{p^{u+1}}\mathbb{Z}_p + \frac{ij}{p^{u+1}}\mathbb{Z}_p,$$

and

$$\mathbb{Z}_p + i\mathbb{Z}_p + \frac{j}{p^{u+1}}\mathbb{Z}_p + \frac{ij}{p^{u+1}}\mathbb{Z}_p = (\mathbb{Z}_p + i\mathbb{Z}_p + p^u j\mathbb{Z}_p + p^u ij\mathbb{Z}_p) \cdot \frac{j}{p^{u+1}} = \mathcal{O}_p \cdot \frac{j}{p^{u+1}} = \frac{j}{p^{u+1}} \cdot \mathcal{O}_p.$$

If Q ramifies at 2, then

$$\mathcal{O}_2 \cong \mathbb{Z}_2 + \frac{1+i}{2}\mathbb{Z}_2 + 2^u j\mathbb{Z}_2 + 2^u\frac{1+i}{2}j\mathbb{Z}_2,$$

where $i^2 = -3, j^2 = 2$. It is clear

$$(\mathcal{O}^\sharp)_2 \cong (\frac{1}{2} - \frac{i}{6})\mathbb{Z}_2 + \frac{i}{3}\mathbb{Z}_2 + 2^{-u}(\frac{j}{4} + \frac{ij}{12})\mathbb{Z}_2 + 2^{-u}\frac{ij}{6}\mathbb{Z}_2,$$

and

$$(\frac{1}{2} - \frac{i}{6})\mathbb{Z}_2 + \frac{i}{3}\mathbb{Z}_2 + 2^{-u}(\frac{j}{4} + \frac{ij}{12})\mathbb{Z}_2 + 2^{-u}\frac{ij}{6}\mathbb{Z}_2 = \mathcal{O}_2 \cdot 2^{-u}\frac{ij}{6} = 2^{-u}\frac{ij}{6} \cdot \mathcal{O}_2.$$

If Q splits at p, let $p^v \parallel N_2$, we have

$$\mathcal{O}_p \cong \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^v\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}.$$

We have

$$(\mathcal{O}^\sharp)_p \cong \begin{pmatrix} \mathbb{Z}_p & p^{-v}\mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}.$$

It is clear

$$\begin{pmatrix} \mathbb{Z}_p & p^{-v}\mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^v\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \cdot \begin{pmatrix} 0 & p^{-v} \\ 1 & 0 \end{pmatrix} = \mathcal{O}_p \cdot \begin{pmatrix} 0 & p^{-v} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & p^{-v} \\ 1 & 0 \end{pmatrix} \cdot \mathcal{O}_p.$$

It is analogous for $p = 2$. □

For an order $\mathcal{O}$ of level (N_1, N_2), its left $\mathcal{O}$-ideals (resp. right $\mathcal{O}$-ideals) are important.

Definition 3.5. For any order $\mathcal{O}$ of level (N_1, N_2) in a definite quaternion algebra Q over $\mathbb{Q}$, a left $\mathcal{O}$-ideal is a ideal I which satisfies $I_p = \mathcal{O}_p x_p$ for every prime number p, where x_p is in the group $(\mathbb{Q}_p \otimes_{\mathbb{Q}} Q)^\times$ of units of $\mathbb{Q}_p \otimes_{\mathbb{Q}} Q$. One has the analogous definitions for right $\mathcal{O}$-ideal.

We define $\mathcal{O}, \mathcal{O}'$ to be of the same type if there exists $\alpha \in Q^\times$ such that $\mathcal{O}' = \alpha^{-1}\mathcal{O}\alpha$. Two orders $\mathcal{O}, \mathcal{O}'$ are of the same type if and only if they are isomorphic as $\mathbb{Z}$-algebras. Locally, Two orders $\mathcal{O}, \mathcal{O}'$ are locally of the same type or locally isomorphic if $\mathcal{O}_p, \mathcal{O}'_p$ are of the same type for all primes p. The genus of $\mathcal{O}$ is the set of orders in Q locally isomorphic to $\mathcal{O}$. The type set of $\mathcal{O}$ is consists of isomorphism classes of orders in genus of $\mathcal{O}$. The number of the type set of $\mathcal{O}$ is usually called the type number of $\mathcal{O}$. The number of the classes of left $\mathcal{O}$-ideals modulo multiplication with elements of $Q^\times$ from the right is the class number of $\mathcal{O}$, which is only dependent on N_1 and N_2. Both type number T_{N_1,N_2} and class number h_{N_1,N_2} are finite.

To derive calculation formula for them, we investigate two-sided $\mathcal{O}$-ideals and two-sided principal $\mathcal{O}$-ideals.

Definition 3.6. An ideal is called two-sided $\mathcal{O}$-ideal if it is both left $\mathcal{O}$-ideal and right $\mathcal{O}$-ideal.

I is a two-sided $\mathcal{O}$-ideal, if and only if there exist some in the group $(\mathbb{Q}_p \otimes_{\mathbb{Q}} Q)^\times$ such that $I_p = \mathcal{O}_p x_p = y_p \mathcal{O}_p$ for every prime p. Hence if I is a two-sided $\mathcal{O}$-ideal, we can assume $y_p = u_p x_p$ where $u_p \in \mathcal{O}_p$, and $\mathrm{n}(u_p) = \mathrm{n}(y_p)/\mathrm{n}(x_p) \in \mathbb{Z}_p^\times$, hence $u_p \in \mathcal{O}_p^\times$. We have $\mathcal{O}_p = y_p^{-1}\mathcal{O}_p x_p = x_p^{-1}\mathcal{O}_p x_p$. Define the normalizer of $\mathcal{O}_p$ in $\mathbb{Q}_p \otimes_{\mathbb{Q}} Q$ as

$$N(\mathcal{O}_p) = \{x_p \in (\mathbb{Q}_p \otimes_{\mathbb{Q}} Q)^\times | x_p^{-1}\mathcal{O}_p x_p = \mathcal{O}_p\}.$$

Then we can state the above conclusion as follows.

Proposition 3.7. *An ideal is two-sided $\mathcal{O}$-ideal if and only if there exists $x_p \in N(\mathcal{O}_p)$ such that $I_p = \mathcal{O}_p x_p$ for every prime p.*

We introduce the following lemma concerning $N(\mathcal{O}_p)$.

Lemma 3.8. [9, pp.104-105] *If $\mathcal{O}$ is an order of level (N_1, N_2), such that*

$$\mathcal{O}_p = \begin{cases} M_2(\mathbb{Z}_p) & \text{if} \quad p \nmid N_1 N_2, \\ \{\begin{pmatrix} \alpha & p^u \beta \\ p^{u+1}\bar{\beta} & \bar{\alpha} \end{pmatrix} | \alpha, \beta \in R_p\} & \text{if} \quad p^{2u+1} \parallel N_1, \\ \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^v \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} & \text{if} \quad p^v \parallel N_2, \end{cases}$$

then

$$N(\mathcal{O}_p) = \begin{cases} \mathcal{O}_p^\times \mathbb{Q}_p^\times & \text{if} \quad p \nmid N_1 N_2, \\ \mathcal{O}_p^\times \mathbb{Q}_p^\times \cup \begin{pmatrix} 0 & p^u \\ p^{u+1} & 0 \end{pmatrix} \mathcal{O}_p^\times \mathbb{Q}_p^\times & \text{if} \quad p^{2u+1} \parallel N_1, \\ \mathcal{O}_p^\times \mathbb{Q}_p^\times \cup \begin{pmatrix} 0 & 1 \\ p^v & 0 \end{pmatrix} \mathcal{O}_p^\times \mathbb{Q}_p^\times & \text{if} \quad p^v \parallel N_2. \end{cases}$$

We present an equivalent condition for two-sided principal $\mathcal{O}$-ideals.

Lemma 3.9. *Let $\mathcal{O}$ be an order of level (N_1, N_2). For any element $x \in \mathcal{O}$, then $\mathcal{O}x$ is a two-sided principal $\mathcal{O}$-ideal if and only if $n(x) \parallel N_1 N_2, n(x) \mid tr(x)$ and for $p^{2u+1} \parallel N_1$, $p^{2u+1} \parallel n(x)$, we have $p^{2u+1} \mid c_{11}(x_p)$, for $p^v \parallel N_2$, $p^v \parallel n(x)$, we have $p^v \mid c_{11}(x_p)$, where $c_{11}(x_p)$ is the $1,1$ entry of x_p.*

Proof. Let $\mathcal{O}x$ be a two-sided principal $\mathcal{O}$-ideal. We have

$$x \in \begin{cases} \mathcal{O}_p^\times & \text{if} \quad p \nmid N_1N_2, \\ \mathcal{O}_p^\times \cup \begin{pmatrix} 0 & p^u \\ p^{u+1} & 0 \end{pmatrix} \mathcal{O}_p^\times & \text{if} \quad p^{2u+1} \parallel N_1, \\ \mathcal{O}_p^\times \cup \begin{pmatrix} 0 & 1 \\ p^v & 0 \end{pmatrix} \mathcal{O}_p^\times & \text{if} \quad p^v \parallel N_2. \end{cases}$$

Hence

$$\mathrm{n}(x) \in \begin{cases} \mathbb{Z}_p^\times & \text{if} \quad p \nmid N_1N_2, \\ \mathbb{Z}_p^\times \cup p^{2u+1}\mathbb{Z}_p^\times & \text{if} \quad p^{2u+1} \parallel N_1, \\ \mathbb{Z}_p^\times \cup p^v\mathbb{Z}_p^\times & \text{if} \quad p^v \parallel N_2, \end{cases}$$

which implies $\mathrm{n}(x) \parallel N_1N_2$. We will prove $\mathrm{n}(x) \mid \mathrm{tr}(x)$ and for $p^{2u+1} \parallel N_1$, $p^{2u+1} \parallel \mathrm{n}(x)$, we have $p^{2u+1} \mid c_{11}(x_p)$, for $p^v \parallel N_2$, $p^v \parallel \mathrm{n}(x)$ we have $p^v \mid c_{11}(x_p)$ in each $\mathbb{Z}_p$.
In the case $x \in \mathcal{O}_p^\times$, since $\mathrm{n}(x) \in \mathbb{Z}_p^\times$, we have $\mathrm{n}(x) \mid \mathrm{tr}(x)$.
In the case $x \in \begin{pmatrix} 0 & 1 \\ p^v & 0 \end{pmatrix} \mathcal{O}_p^\times$, we have $\mathrm{n}(x) \in p^v\mathbb{Z}_p^\times$. There exists $\begin{pmatrix} a & b \\ p^v & d \end{pmatrix} \in \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^v\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}^\times$ such that

$$\mathrm{tr}(x) = \mathrm{tr}(\begin{pmatrix} 0 & 1 \\ p^v & 0 \end{pmatrix} \begin{pmatrix} a & b \\ p^vc & d \end{pmatrix}) = p^v(b+c) \in p^v\mathbb{Z}_p.$$

It implies $c_{11}(x_p) = p^vc$, we have $p^v \mid c_{11}(x_p)$.
In the case $x \in \begin{pmatrix} 0 & p^u \\ p^{u+1} & 0 \end{pmatrix} \mathcal{O}_p^\times\mathbb{Q}_p^\times$, we have

$$\mathrm{tr}(x) = \mathrm{tr}(\begin{pmatrix} 0 & p^u \\ p^{u+1} & 0 \end{pmatrix} \begin{pmatrix} \alpha & p^u\beta \\ p^{u+1}\bar{\beta} & \bar{\alpha} \end{pmatrix}) = p^{2u+1}\mathrm{tr}(\beta) \in p^{2u+1}\mathbb{Z}_p.$$

It implies $c_{11}(x_p) = p^{2u+1}\bar{\beta}$, we have $p^{2u+1} \mid c_{11}(x_p)$.

Conversely, let $x \in \mathcal{O}$, $\mathrm{n}(x) \parallel N_1N_2$, $\mathrm{n}(x) \mid \mathrm{tr}(x)$, for $p^{2u+1} \parallel N_1$, $p^{2u+1} \parallel \mathrm{n}(x)$ we have $p^{2u+1} \mid c_{11}(x_p)$, and for $p^v \parallel N_2$, $p^v \parallel \mathrm{n}(x)$ we have $p^v \mid c_{11}(x_p)$. Since $\mathcal{O}x$ is a left $\mathcal{O}$-ideal, we will show $\mathcal{O}x$ is a two-sided $\mathcal{O}$-ideal, i.e. for every prime p we have $x \in N(\mathcal{O}_p)$.
In the case $p \nmid \mathrm{n}(x)$, since $\mathrm{n}(x) \in \mathbb{Z}_p^\times$, we have $x \in \mathcal{O}_p^\times \subset N(\mathcal{O}_p)$.
In the case $p \mid \mathrm{n}(x)$, since $\mathrm{n}(x) \parallel N_1N_2$, we have $p \mid N_1N_2$. For $p^{2u+1} \parallel N_1$, we have $p^{2u+1} \parallel \mathrm{n}(x)$. Let $x_p = \begin{pmatrix} \alpha & p^u\beta \\ p^{u+1}\bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathcal{O}_p$. We have $\begin{pmatrix} 0 & p^u \\ p^{u+1} & 0 \end{pmatrix}^{-1} \begin{pmatrix} \alpha & p^u\beta \\ p^{u+1}\bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} \bar{\beta} & p^{-u-1}\bar{\alpha} \\ p^{-u}\bar{\alpha} & \bar{\beta} \end{pmatrix}$. Since $p^{2u+1} \mid c_{11}(x_p)$, it implies $\begin{pmatrix} 0 & p^u \\ p^{u+1} & 0 \end{pmatrix}^{-1} \begin{pmatrix} \alpha & p^u\beta \\ p^{u+1}\bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathcal{O}_p^\times$, i.e. $x \in N(\mathcal{O}_p)$. For $p^v \parallel N_2$, we have $p^v \mid \mathrm{n}(x)$. Suppose $x_p = \begin{pmatrix} a & b \\ p^vc & d \end{pmatrix} \in \mathcal{O}_p$. We have $\begin{pmatrix} 0 & 1 \\ p^v & 0 \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ p^vc & d \end{pmatrix} = \begin{pmatrix} c & p^{-v}d \\ a & b \end{pmatrix}$. Since $\mathrm{n}(x) \mid \mathrm{tr}(x)$ and $p^v \mid c_{11}(x_p)$, it implies $p^v \mid a$ and $p^v \mid d$. Hence $\begin{pmatrix} 0 & 1 \\ p^v & 0 \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ p^vc & d \end{pmatrix} \in \mathcal{O}_p^\times$, i.e. $x \in N(\mathcal{O}_p)$. □

Remark 3.10. For $u = 0$ or $v = 1$, when $p \mid \mathrm{n}(x)$ and $\mathrm{n}(x) \mid \mathrm{tr}(x)$, it implies $p \mid c_{11}(x_p)$. That is for $p \mid \mathrm{tr}(\alpha)$ and $p \mid \mathrm{n}(\alpha)$, it implies $p \mid \alpha$, likewise for $p \mid ad$ and $p \mid (a+d)$, it implies $p \mid a$.

Fix an order $\mathcal{O}$ of level (N_1, N_2), then all two-sided $\mathcal{O}$-ideal form a group, which is denoted by $\mathfrak{I}$. The number of elements of group $\mathfrak{I}(\mathcal{O})/\mathbb{Q}^\times$ is $2^{e(N_1N_2)}$[9, Theorem 2.20]. All two-sided principal $\mathcal{O}$-ideal form a subgroup, which is denoted by $\mathfrak{B}(\mathcal{O})$. We have the following proposition for $\mathrm{card}(\mathfrak{B}(\mathcal{O})/\mathbb{Q}^\times)$ and $\mathrm{Aut}(\mathcal{O})$ which will be used in proving type number formula.

Theorem 3.11. *Let $\mathcal{O}$ be an order of level (N_1, N_2), and $m(\mathcal{O})$ be the number of left $\mathcal{O}$-ideal classes containing a two-sided $\mathcal{O}$-ideal. Then we have*

$$card(\mathfrak{B}(\mathcal{O})/\mathbb{Q}^\times) = \frac{2^{e(N_1N_2)}}{m(\mathcal{O})} \tag{3.1}$$

or

$$card(\mathfrak{B}(\mathcal{O})/\mathbb{Q}^\times) = \sum_{n\|N_1N_2} \sideset{}{'}\sum_{\substack{n|r \\ r^2\le 4n}} \frac{\rho_{\mathcal{O}}(n,r)}{card(\mathcal{O}^\times)}, \tag{3.2}$$

where $\rho_{\mathcal{O}}(n,r)$ be the number of zeros of x^2-rx+n in $\mathcal{O}$, $\sum'$ is taken over all elements x which satisfy for $p^{2u+1} \| N_1$, $p^{2u+1} \| n(x)$, it entails $p^{2u+1} \mid c_{11}(x_p)$, and for $p^v \| N_2$, $p^v \| n(x)$, it entails $p^v \mid c_{11}(x_p)$. We also have

$$card(\mathrm{Aut}(\mathcal{O})) = \frac{2^{e(N_1N_2)}card(\mathcal{O}^\times)}{2m(\mathcal{O})}. \tag{3.3}$$

Proof. For (3.1), see [5, Proposition 2.14].
For (3.2), using Lemma 3.9 we have

$$\mathfrak{B}(\mathcal{O})/\mathbb{Q}^\times = \{\mathcal{O}x\mathbb{Q}^\times | x\in\mathcal{O}, \mathrm{n}(x) \| N_1N_2, \mathrm{n}(x) \mid \mathrm{tr}(x), p^{2u+1} \mid c_{11}(x_p), p^v \mid c_{11}(x_p)\}$$

where the last two conditions means that for $p^{2u+1} \| N_1$, $p^{2u+1} \| \mathrm{n}(x)$ we have $p^{2u+1} \mid c_{11}(x_p)$, and for $p^v \| N_2$, $p^v \| \mathrm{n}(x)$ we have $p^v \mid c_{11}(x_p)$. Since two-sided $\mathcal{O}$-ideal $\mathcal{O}x = \mathcal{O}y$ if and only if $y = ax$, where $a \in \mathcal{O}^\times$, we will prove that there are only $\pm1 \in \mathcal{O}^\times \cap \mathbb{Q}^\times$ which times any $x \in N(\mathcal{O})$ such that satisfy $bx \in N(\mathcal{O})$. For any $q \in \mathbb{Q}$, we have $n(qx) = q^2x$, hence it is clear when N_1N_2 is squarefree. For $p^v \| N_1N_2$, suppose $p \nmid \mathrm{n}(x)$, and $b \neq \pm1, b \in \mathbb{Q}^\times$ such that bx satisfies the conditions, then v is even, $p^v \| N_2$, $p^{v/2} \| b$ and $bx \in \begin{pmatrix} 0 & 1 \\ p^v & 0 \end{pmatrix} \mathcal{O}_p^\times$. Since $x \in \mathcal{O}_p^\times$, this is a contradiction. Hence there are just $a \in \mathcal{O}^\times$ such that $\mathcal{O}x\mathbb{Q}^\times = \mathcal{O}ax\mathbb{Q}^\times$ with ax satisfying the conditions. We have

$$\mathrm{card}(\mathfrak{B}(\mathcal{O})/\mathbb{Q}^\times) = \frac{\sharp\{x \in \mathcal{O} | \mathrm{n}(x) \| N_1N_2, \mathrm{n}(x) \mid \mathrm{tr}(x), p^{2u+1} \mid c_{11}(x_p), p^v \mid c_{11}(x_p)\}}{\mathrm{card}(\mathcal{O}^\times)}.$$

Similarly, to prove (3.3), since

$$\mathrm{Aut}(\mathcal{O}) = \{x\mathbb{Q}^\times | x \in \mathcal{O}, x^{-1}\mathcal{O}x = \mathcal{O}\},$$

using Proposition 3.7 and Lemma 3.9 we have

$$\{x \in \mathcal{O} | x^{-1}\mathcal{O}x = \mathcal{O}\} = \{x \in \mathcal{O} | \mathrm{n}(x) \| N_1N_2, \mathrm{n}(x) \mid \mathrm{tr}(x), p^{2u+1} \mid c_{11}(x_p), p^v \mid c_{11}(x_p)\}$$

and

$$\mathrm{card}(\mathfrak{B}(\mathcal{O})/\mathbb{Q}^\times) = \frac{\sharp\{x \in \mathcal{O} | x^{-1}\mathcal{O}x = \mathcal{O}\}}{\mathrm{card}(\mathcal{O}^\times)}.$$

Moreover $\mathcal{O}^\times \cap \mathbb{Q}^\times = \{\pm1\}$ implies

$$\mathrm{card}(\mathfrak{B}(\mathcal{O})/\mathbb{Q}^\times) = \frac{\sharp\{x\mathbb{Q}^\times | x \in \mathcal{O}, x^{-1}\mathcal{O}x = \mathcal{O}\}}{\mathrm{card}(\mathcal{O}^\times/\{\pm1\})}.$$

Since $\mathrm{card}(\mathfrak{B}(\mathcal{O})/\mathbb{Q}^\times) = 2^{e(N_1N_2)}/m(\mathcal{O})$, we have

$$\mathrm{card}(\mathrm{Aut}(\mathcal{O})) = \sharp\{x\mathbb{Q}^\times | x \in \mathcal{O}, x^{-1}\mathcal{O}x = \mathcal{O}\} = \frac{2^{e(N_1N_2)}\mathrm{card}(\mathcal{O}^\times)}{2m(\mathcal{O})}. \tag{3.4}$$

$\square$

By (3.1), (3.2) and (3.3), we have

$$\begin{aligned} T_{N_1,N_2} &= 2^{-e(N_1N_2)} \sum_{\mu=1}^{T_{N_1,N_2}} 2^{e(N_1N_2)} \\ &= 2^{-e(N_1N_2)} \sum_{\mu=1}^{T_{N_1,N_2}} m(\mathcal{O}_\mu)\mathrm{card}(\mathfrak{B}(\mathcal{O}_\mu)/\mathbb{Q}^\times) \\ &= 2^{-e(N_1N_2)} \sum_{\mu=1}^{T_{N_1,N_2}} m(\mathcal{O}_\mu) \sum_{n\|N_1N_2} \sideset{}{'}\sum_{\substack{n|r \\ r^2\le 4n}} \frac{\rho_{\mathcal{O}}(n,r)}{\mathrm{card}(\mathcal{O}_\mu^\times)} \\ &= 2^{-1} \sum_{n\|N_1N_2} \sideset{}{'}\sum_{\substack{n|r \\ r^2\le 4n}} \sum_{\mu=1}^{T_{N_1,N_2}} \frac{\rho_{\mathcal{O}}(n,r)}{\mathrm{card}(\mathrm{Aut}(\mathcal{O}_\mu))} \end{aligned}$$

In Section 4 and 5, we will prove that caculating

$$\sum_{\mu=1}^{T_{N_1,N_2}} \frac{\rho_{\mathcal{O}}(n,r)}{\mathrm{card}(\mathrm{Aut}(\mathcal{O}_\mu))}$$

is equivalent to

$$\sum_{f\in G_{4N_1N_2,16(N_1N_2)^2,N_1'}} \frac{R_f(4n-r^2)}{|\,\mathrm{Aut}(f)|}.$$

4. Ternary quadratic forms and quaternion orders

In this section, let $N_1 = p_1^{2u_1+1}...p_w^{2u_w+1}$, where the p_i are distinct primes, $u_1, ..., u_w$ are nonnegative integers and w is an odd integer, and N_2 is a positive integer such that $\gcd(N_1, N_2) = 1$. Denote $N_1' = p_1...p_w$.

4.1. **three bijections between orders of level (N_1, N_2) and ternary quadratic forms.** In this section, prior to introducing three bijections between orders of level (N_1, N_2) and ternary quadratic forms for given N_1 and N_2, we first establish the following proposition.

Proposition 4.1. *Let*

$$\mathcal{O} = \mathbb{Z} + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3 \subset Q$$

be an order of level (N_1, N_2), then there exist $tr(\alpha_1') = tr(\alpha_2') = 0$, and $tr(\alpha_3') = 1$, such that

$$\mathcal{O} = \mathbb{Z} + \mathbb{Z}\alpha_1' + \mathbb{Z}\alpha_2' + \mathbb{Z}\alpha_3'.$$

Proof. See [7][Proposition 4.1]. □

Unless otherwise stated, we assume that for an order of level (N_1, N_2), we have $\mathcal{O} = \mathbb{Z}+\mathbb{Z}\alpha_1+\mathbb{Z}\alpha_2+\mathbb{Z}\alpha_3$ and $\mathrm{tr}(\alpha_1) = \mathrm{tr}(\alpha_2) = 0, \mathrm{tr}(\alpha_3) = 1$. For an order of level (N_1, N_2), where $\mathcal{O} = \mathbb{Z} + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3 \in Q$, lattice $\mathcal{O}^0 = \mathcal{O} \cap Q^0 = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}(2\alpha_3 - 1)$, where $Q^0 = \{\alpha \in Q : \mathrm{tr}(\alpha) = 0\}$, is an even integral positive definite lattice, when equipped with the bilinear form $(x, y) \mapsto \mathrm{tr}(x\overline{y})$. We have the following ternary quadratic form

$$\begin{aligned} f_{\mathcal{O}^0} &= \mathrm{n}(x\alpha_1 + y\alpha_2 + z(2\alpha_3 - 1)) \\ &= \mathrm{n}(\alpha_1)x^2 + \mathrm{n}(\alpha_2)y^2 + (4\mathrm{n}(\alpha_3) - 1)z^2 + 2\mathrm{tr}(\alpha_2\overline{\alpha_3})yz + 2\mathrm{tr}(\alpha_1\overline{\alpha_3})xz + \mathrm{tr}(\alpha_1\overline{\alpha_2})xy, \end{aligned}$$

and the Gram matrix of $f_{\mathcal{O}^0}$

$$M_{f_{\mathcal{O}^0}} = \begin{pmatrix} \mathrm{tr}(\alpha_1\overline{\alpha_1}) & \mathrm{tr}(\alpha_1\overline{\alpha_2}) & 2\mathrm{tr}(\alpha_1\overline{\alpha_3}) \\ \mathrm{tr}(\alpha_2\overline{\alpha_1}) & \mathrm{tr}(\alpha_2\overline{\alpha_2}) & 2\mathrm{tr}(\alpha_2\overline{\alpha_3}) \\ 2\mathrm{tr}(\alpha_3\overline{\alpha_1}) & 2\mathrm{tr}(\alpha_3\overline{\alpha_2}) & 4\mathrm{tr}(\alpha_3\overline{\alpha_3}) - 2 \end{pmatrix}.$$

For an order of level (N_1, N_2) $\mathcal{O} = \mathbb{Z} + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3 \in Q$, define the lattice

$$S = \mathbb{Z} + 2\mathcal{O} = \mathbb{Z} + \mathbb{Z}2\alpha_1 + \mathbb{Z}2\alpha_2 + \mathbb{Z}2\alpha_3,$$

then lattice $S^0 = (\mathbb{Z} + 2\mathcal{O}) \cap Q^0 = \mathbb{Z}2\alpha_1 + \mathbb{Z}2\alpha_2 + \mathbb{Z}(2\alpha_3 - 1)$ is an even integral positive definite lattice, when equipped with the bilinear form $(x, y) \mapsto \mathrm{tr}(x\overline{y})$. We have the following ternary quadratic form

$$\begin{aligned} f_{S^0} &= \mathrm{n}(x(2\alpha_1) + y(2\alpha_2) + z(2\alpha_3 - 1)) \\ &= 4\mathrm{n}(\alpha_1)x^2 + 4\mathrm{n}(\alpha_2)y^2 + (4\mathrm{n}(\alpha_3) - 1)z^2 + 4\mathrm{tr}(\alpha_2\overline{\alpha_3})yz + 4\mathrm{tr}(\alpha_1\overline{\alpha_3})xz + 4\mathrm{tr}(\alpha_1\overline{\alpha_2})xy, \end{aligned}$$

and the Gram matrix of f_{S^0} is

$$M_{f_{S^0}} = \begin{pmatrix} 4\mathrm{tr}(\alpha_1\overline{\alpha_1}) & 4\mathrm{tr}(\alpha_1\overline{\alpha_2}) & 4\mathrm{tr}(\alpha_1\overline{\alpha_3}) \\ 4\mathrm{tr}(\alpha_2\overline{\alpha_1}) & 4\mathrm{tr}(\alpha_2\overline{\alpha_2}) & 4\mathrm{tr}(\alpha_2\overline{\alpha_3}) \\ 4\mathrm{tr}(\alpha_3\overline{\alpha_1}) & 4\mathrm{tr}(\alpha_3\overline{\alpha_2}) & 4\mathrm{tr}(\alpha_3\overline{\alpha_3}) - 2 \end{pmatrix}.$$

Proposition 4.2. *Let $\mathcal{O}, \mathcal{O}' \subset Q$ be orders. Suppose $\mathcal{O}, \mathcal{O}'$ are isomorphic (resp. locally isomorphic), then $f_{\mathcal{O}^0}, f_{\mathcal{O}'^0}$ (or $f_{\mathcal{O}^0}, f_{\mathcal{O}'^0}$) are equivalent (resp. semi-equivalent).*

Proof. See [7][Proposition 4.2]. □

If f is a non-degenerate ternary quadratic form integral over $\mathbb{Z}$, we define $C_0(f)$ to be the even Clifford algebras over $\mathbb{Z}$ associated with f. Subsequently, $C_0(f)$ is an order in a quaternion algebra over $\mathbb{Q}$. For more detailed definition of Clifford algebra, refer to [11, Chapter 22] and [6, Chapter 1]. Conversely, consider that $\mathcal{O} = \mathbb{Z} + \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3$ is an order in a quaternion algebra Q. Then $\Lambda = \mathcal{O}^\sharp \cap Q^0$ represents a 3-dimensional $\mathbb{Z}$-lattice on Q^0. If $\mathcal{O}^\sharp = \langle e_0', e_1', e_2', e_3' \rangle$ where e_i' is the dual basis of e_i, then $\Lambda = \langle e_1', e_2', e_3' \rangle$. Define a ternary quadratic form $f_{\mathcal{O}}$ associated to $\mathcal{O}$ by

$$f_{\mathcal{O}} = \mathrm{discrd}(\mathcal{O}) \cdot n(xe_1' + ye_2' + ze_3').$$

Theorem 4.3. [11, Main Theorem 22.1.1] *Let R be a principal ideal domain. The maps $f \mapsto C_0(f)$ and $\mathcal{O} \mapsto f_{\mathcal{O}}$ are inverses to each other and the discriminants satisfy* $\mathrm{discrd}(\mathcal{O}) = d(f_{\mathcal{O}})$. *Furthermore, the maps give a bijection between analogousity classes of non-degenerate ternary quadratic forms integral over R and isomorphism classes of quaternion R-orders.*

Assume that $f = (a, b, c, r, s, t), d_f = d$. We have

$$\begin{aligned} e_1^2 &= re_1 - bc, e_2e_3 = a\overline{e_1}, \\ e_2^2 &= se_2 - ac, e_3e_1 = b\overline{e_2}, \\ e_3^2 &= te_3 - ab, e_1e_2 = c\overline{e_3}. \end{aligned}$$

By the definetion of dual basis, it follows that $\mathcal{O}^\sharp \supset \mathcal{O}$ and $\mathrm{tr}(e_0') = 1, \mathrm{tr}(e_1') = \mathrm{tr}(e_2') = \mathrm{tr}(e_3') = 0$, where

$$\begin{aligned} de_0' &= d - 2(abc + rst) + (ar + st)e_1 + (bs + rt)e_2 + (ct + rs)e_3, \\ de_1' &= ar + st - 2ae_1 - te_2 - se_3, \\ de_2' &= bs + rt - te_1 - 2be_2 - re_3, \\ de_3' &= ct + rs - se_1 - re_2 - 2ce_3, \end{aligned}$$

and

$$\begin{aligned} \mathrm{n}(Ne_1') &= Na, \mathrm{tr}(Ne_2'\overline{Ne_3'}) = Nr, \\ \mathrm{n}(Ne_2') &= Nb, \mathrm{tr}(Ne_3'\overline{Ne_1'}) = Ns, \end{aligned}$$

$$\mathrm{n}(Ne_3') = Nc, \mathrm{tr}(Ne_1'\overline{Ne_2'}) = Nt.$$

Recalling $\mathcal{O}$ contains a basis of Q over $\mathbb{Q}$, it follows that $\langle e_1', e_2', e_3' \rangle$ form a $\mathbb{Q}-$basis for the trace-zero elements of Q (that is, $\langle i, j, k\rangle$ can be represented by $\langle e_1', e_2', e_3'\rangle$ over $\mathbb{Q}$). Some properties of even Clifford algebras follow.

Theorem 4.4. *(1) $f_{\mathcal{O}}$ is positive definite if and only if $\mathcal{O}$ is positive definite.*

(2) $2\mathrm{card}(\mathrm{Aut}(\mathcal{O})) = |\,\mathrm{Aut}(f_{\mathcal{O}})|$.

(3) $\mathcal{O}$ ramifies at p if and only if $f_{\mathcal{O}}$ is anisotropic at p.

(4) $f_{\mathcal{O}}$ is primitive if and only if $\mathcal{O}$ is Gorenstein.

4.2. **Commutative diagrams for $4 \nmid N_1N_2$.** In this section, we construct commutative diagrams involving M_0, M_1 and C_0 under the condition $4 \nmid N_1N_2$. We begin by providing a detailed characterization of the genera to which $f_{\mathcal{O}^0}$, f_{S^0} and $f_{\mathcal{O}}$ belong.

Proposition 4.5. *For an order $\mathcal{O}$ we denote $f_{\mathcal{O}^0}$ by f.*

(1)Let $\mathcal{O} \subset Q_{N_1'}$ be an order of level (N_1, N_2) where $2 \nmid N_1N_2$, then $d_f = (N_1N_2)^2, N_f = 4N_1N_2$ and f is anisotropic only in the p-adic field for $p \mid N_1'$. The genus which f belongs to is denoted by $G_{4N_1N_2,(N_1N_2)^2,N_1'}$.

(2)Let $\mathcal{O} \subset Q_{N_1'}$ be an order of level (N_1, N_2) where $2 \parallel N_1N_2$, then $d_f = (N_1N_2)^2, N_f = 2N_1N_2$ and f is anisotropic only in the p-adic field for $p \mid N_1'$. The genus which f belongs to is denoted by $G_{2N_1N_2,(N_1N_2)^2,N_1'}$.

Proof. We only prove the second, and the frist case be proved similarly. Let $\mathcal{O} \subset Q_{N_1'}$ be an order of level (N_1, N_2).

In the case $p^{2u+1} \parallel N_1$ where $p \neq 2$, then

$$\mathcal{O}_p \cong \mathbb{Z}_p + i\mathbb{Z}_p + p^u j\mathbb{Z}_p + p^u ij\mathbb{Z}_p,$$

where $i^2 = \epsilon, j^2 = p$, and $(\frac{\epsilon}{p}) = -1$. We have

$$f \underset{p}{\sim} -\epsilon x^2 - p^{2u+1}y^2 + \epsilon p2u + 1z^2 = f_p,$$

where $f \underset{p}{\sim} g$ means that f and g are equivlent over the p-adic integers $\mathbb{Z}_p$. It follows that $d_{f_p} = 4\epsilon^2 p^{4u+2}, m_{f_p} = 4up^{2u+1}$. Hence $v_p(d_{f_p}) = 4u + 2, v_p(m_{f_p}) = 2u + 1$ and $S_p^*(f_p) = -1$. We have f_p is anisotropic. For $2 \parallel N_1$, we have

$$\mathcal{O}_2 \cong \mathbb{Z}_2 + \frac{1+i}{2}\mathbb{Z}_2 + j\mathbb{Z}_2 + \frac{1+i}{2}j\mathbb{Z}_2,$$

where $i^2 = -3, j^2 = 2$, and

$$f \underset{2}{\sim} 3x^2 - 2y^2 - 2z^2 - 2yz \underset{2}{\sim} x^2 - 3y^2 - 3z^2 \underset{2}{\sim} x^2 + 5y^2 + 5z^2 \underset{2}{\sim} x^2 + y^2 + z^2.$$

It follows that $d_{f_2} = 4, m_{f_2} = 4$, and $v_2(d_{f_2}) = 2, v_2(m_{f_2}) = 2$. We can see f_2 is anisotropic.

In the case $p^v \parallel N_2$ and $2 \parallel N_2$, then

$$\mathcal{O}_p \cong \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^v\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$$

and

$$\mathcal{O}_p^0 \cong \mathbb{Z}_p \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \mathbb{Z}_p \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathbb{Z}_p \begin{pmatrix} 0 & 0 \\ p^v & 0 \end{pmatrix},$$

It follows that

$$f \underset{p}{\sim} -x^2 - p^v yz = f_p.$$

We have $d_{f_p} = p^{2v}, m_{f_p} = p^v$, then $v_p(d_{f_p}) = 2v, v_p(m_{f_p}) = v$ and $v_2(d_{f_2}) = 2, v_2(m_{f_2}) = 2$. It is clear f_p is isotropic.
In the case $p \nmid N_1N_2$, then

$$\mathcal{O}_p \cong \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$$

and

$$\mathcal{O}_p^0 \cong \mathbb{Z}_p \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \mathbb{Z}_p \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathbb{Z}_p \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It follows that

$$f \underset{p}{\sim} -x^2 - yz = f_p.$$

We have $d_{f_p} = 1, m_{f_p} = 1$, then $v_p(d_{f_p}) = 0, v_p(m_{f_p}) = 0$ and f_p is isotropic. Since $v_p(d_{f_p}), v_p(m_{f_p})$ are $GL_3(\mathbb{Z}_p)$-invariants of ternary quadratic forms [6, p.4], we have

$$d_f = (N_1N_2)^2, N_f = 2N_1N_2.$$

The genus which f belongs to is denoted by $G_{2N_1N_2,(N_1N_2)^2,N_1'}$. □

For $4 \nmid N_1N_2$, in a positive definite quaternion algebra $Q_{N_1'}$, we choose a complete set of representatives $\{\mathcal{O}_\mu\}_{\mu=1,2\ldots T_{N_1,N_2}}$ for these types of orders of level (N_1, N_2). Then we have the map M_0 as follows.
In the case $2 \nmid N_1N_2$, then

$$\begin{aligned} M_0 : \{\mathcal{O}_\mu\}_{\mu=1,2\ldots T_{N_1,N_2}} &\to G_{4N_1N_2,(N_1N_2)^2,N_1'} \\ \mathcal{O}_\mu \subset Q_{N_1'} &\mapsto f_{\mathcal{O}_\mu^0} \in G_{4N_1N_2,(N_1N_2)^2,N_1'}. \end{aligned}$$

In the case $2 \parallel N_1N_2$, then

$$\begin{aligned} M_0 : \{\mathcal{O}_\mu\}_{\mu=1,2\ldots T_{N_1,N_2}} &\to G_{2N_1N_2,(N_1N_2)^2,N_1'} \\ \mathcal{O}_\mu \subset Q_{N_1'} &\mapsto f_{\mathcal{O}_\mu^0} \in G_{2N_1N_2,(N_1N_2)^2,N_1'}. \end{aligned}$$

The proof of following proposition is analogous.

Proposition 4.6. *For an order $\mathcal{O}$ we denote f_{S^0} by f. Let $\mathcal{O} \subset Q_{N_1'}$ be an order of level (N_1, N_2) where $4 \nmid N_1N_2$, then $d_f = 16(N_1N_2)^2, N_f = 4N_1N_2$ and f is anisotropic only in the p-adic field for $p \mid N_1'$. The genus which f belongs to is denoted by $G_{4N_1N_2,16(N_1N_2)^2,N_1'}$.*

Proof. In the case $2 \nmid N_1N_2$, then

$$f \underset{2}{\sim} -x^2 - 4yz.$$

In the case $2 \parallel N_2$, then

$$f \underset{2}{\sim} -x^2 - 8yz.$$

In the case $2 \parallel N_1$, then

$$f \underset{2}{\sim} 3x^2 - 8(y^2 + z^2 + yz),$$

and the rest of proof is analogous. □

For $4 \nmid N_1N_2$, in a positive definite quaternion algebra Q_N, we choose a complete set of representatives $\{\mathcal{O}_\mu\}_{\mu=1,2\ldots T_{N_1,N_2}}$ for these types of orders of level (N_1, N_2). Then we have the map M_0 as follows.

$$\begin{aligned} M_0 : \{\mathcal{O}_\mu\}_{\mu=1,2\ldots T_{N_1,N_2}} &\to G_{4N_1N_2,16(N_1N_2)^2,N_1'} \\ \mathcal{O}_\mu \subset Q_{N_1'} &\mapsto f_{S_\mu^0} \in G_{4N_1N_2,16(N_1N_2)^2,N_1'}. \end{aligned}$$

Recallling Proposition 2.12, Watson transformation λ_4 is a bijection between $C(4N_1N_2, 16(N_1N_2)^2)$(resp. $C(4N_1N_2, 16(N_1N_2)^2)$) and $C(C(4N_1N_2, 16(N_1N_2)^2))$(resp. $C(2N_1N_2, 16(N_1N_2)^2)$) when $2 \nmid N_1N_2$ (resp. $2 \parallel N_1N_2$). We have the following proposition.

Proposition 4.7. *Let $\mathcal{O}$ be an order of level (N_1, N_2), where $4 \nmid N_1N_2$, then $\lambda_4(f_{S^0}) = f_{\mathcal{O}^0}$.*

Proof. See [7][Proposition 4.5]. □

Proposition 4.8. *For an order $\mathcal{O}$ we denote $C_0(f)$ by f. Let $\mathcal{O} \subset Q_{N_1'}$ be an order of level (N_1, N_2) where $4 \nmid N_1N_2$, then $d_f = (N_1N_2)^2, N_f = 4N_1N_2$ and f is anisotropic only in the p-adic field for $p \mid N_1'$. The genus which f belongs to is denoted by $G_{4N_1N_2,N_1N_2,N_1'}$.*

Proof. In the case $p^{2u+1} \parallel N_1$, then
$$f \underset{p}{\sim} \epsilon p^{2u+1}x^2 - y^2 + \epsilon z^2.$$
In the case $2 \parallel N_1$, then
$$f \underset{2}{\sim} 6x^2 - (y^2 + z^2 - yz).$$
In the case $p^v \parallel N_2$, then
$$f \underset{p}{\sim} -p^v x^2 - yz.$$
In the case $2 \parallel N_2$, then
$$f \underset{p}{\sim} -2x^2 - yz.$$
The rest of proof is analogous. □

Recallling Proposition 2.5, ϕ_p is a bijection between $C(N', p^{2g}d')$ and $C(N', p^g d')$. A proposition follow.

Proposition 4.9. *(1)Let $\mathcal{O}$ be an order of level (N_1, N_2), where $2 \nmid N_1N_2$, i.e. $N_1N_2 = p_1^{l_1}...p_m^{l_m}$, and the p_i are distinct odd primes, then $\phi_{p_1} \circ ... \circ \phi_{p_m}(f_{\mathcal{O}^0}) = f_{\mathcal{O}}$.*
(2)Let $\mathcal{O}$ be an order of level (N_1, N_2), where $2 \parallel N_1N_2$, i.e. $N_1N_2 = 2p_1^{l_1}...p_m^{l_m}$, and the p_i are distinct odd primes, then $\phi_{p_1} \circ ... \circ \phi_{p_m} \circ \phi_2(f_{\mathcal{O}^0}) = f_{\mathcal{O}}$.

Proof. Suppose that $N_1N_2 = p^g$ (Although $N_1N_2 = p^g$ implies g is odd, it will not affect our proof). By Proposition 2.2, assume $f_{\mathcal{O}} = (p^g a, b, c, r, p^g s, p^g t)$ be a primitive, positive definite ternary quadratic form of level $4p^g$ and discriminant p^g. Recalling ϕ_p:
$$\phi_p^{-1}((p^g a, b, c, r, p^g s, p^g t)) = (a, p^g b, p^g c, p^g r, p^g s, p^g t), p \nmid ac.$$
We have $\operatorname{disc}(\mathcal{O}) = p^g$, and
$$\mathcal{O} = \mathbb{Z} + \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3,$$
with
$$e_1^2 = re_1 - bc, \qquad e_2e_3 = p^g a\overline{e_1},$$
$$e_2^2 = p^g se_2 - p^g ac, \qquad e_3e_1 = b\overline{e_2},$$
$$e_3^2 = p^g te_3 - p^g ab, \qquad e_1e_2 = c\overline{e_3}.$$
Then
$$e_0' = 1 - 2(abc + p^g rst) + (ar + p^g st)e_1 + (bs + rt)e_2 + (ct + rs)e_3,$$
$$e_1' = ar + p^g st - 2ae_1 - te_2 - se_3,$$
$$e_2' = bs + rt - te_1 - \tfrac{2b}{p^g}e_2 - \tfrac{r}{p^g}e_3,$$
$$e_3' = ct + rs - se_1 - \tfrac{r}{p^g}e_2 - \tfrac{2c}{p^g}e_3.$$

We have $\mathrm{tr}(e_1') = \mathrm{tr}(e_2') = \mathrm{tr}(e_3') = 0$, and

$$\mathrm{n}(e_1') = a, \mathrm{tr}(p^g e_2' \overline{p^g e_3'}) = p^g r,$$

$$\mathrm{n}(p^g e_2') = p^g b, \mathrm{tr}(p^g e_3' \overline{e_1'}) = p^g s,$$

$$\mathrm{n}(p^g e_3') = p^g c, \mathrm{tr}(e_1' \overline{p^g e_2'}) = p^g t.$$

We will show that

$$\mathcal{O} = \mathbb{Z}e_0' + \mathbb{Z}e_1' + \mathbb{Z}p^g e_2' + \mathbb{Z}p^g e_3'.$$

It is clear

$$\begin{pmatrix} e_0' \\ e_1' \\ p^g e_2' \\ p^g e_3' \end{pmatrix} = \begin{pmatrix} 1-2(abc+p^g rst) & ar+p^g st & bs+rt & ct+rs \\ ar+pst & -2a & -t & -s \\ p^g(bs+rt) & -p^g t & -2b & -r \\ p^g(ct+rs) & -p^g s & -r & -2c \end{pmatrix} \begin{pmatrix} 1 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} = M_p \begin{pmatrix} 1 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

Since $d_{f_{\mathcal{O}}} = p^g$, we have

$$4abc + p^g rst - ar^2 - p^g bs^2 - p^g ct^2 = 1.$$

It is not hard to check that

$$\det(M_p) = -2(4abc + p^g rst - ar^2 - p^g bs^2 - p^g ct^2) + (4abc + p^g rst - ar^2 - p^g bs^2 - p^g ct^2)^2 = -1.$$

Hence

$$\mathcal{O} = \mathbb{Z}e_0' + \mathbb{Z}e_1' + \mathbb{Z}p^g e_2' + \mathbb{Z}p^g e_3',$$

where $\mathrm{tr}(e_1') = \mathrm{tr}(p^g e_2') = \mathrm{tr}(p^g e_3') = 0$, and

$$\begin{aligned} f_{\mathcal{O}^0} &= \mathrm{n}(xe_1' + yp^g e_2' + zp^g e_3') \\ &= \mathrm{n}(e_1')x^2 + \mathrm{n}(p^g e_2')y^2 + \mathrm{n}(p^g e_3')z^2 + \mathrm{tr}(p^g e_2' \overline{p^g e_3'})yz + \mathrm{tr}(e_1' \overline{p^g e_3'})xz + \mathrm{tr}(e_1' \overline{p^g e_2'})xy \\ &= ax^2 + p^g by^2 + p^g cz^2 + p^g ryz + p^g sxz + p^g txy. \end{aligned}$$

It is analogous for $\mathrm{discrd}(\mathcal{O}) = 2$. Let $f_{\mathcal{O}} = (2a, b, c, r, 4s, 4t)$ be a positive definite ternary quadratic forms of level 8 and discriminant 2. Recalling ϕ_2:

$$\phi_2^{-1}((2a, b, c, r, 4s, 4t)) = (a, 8b, 8c, 8r, 8s, 8t), 2 \nmid ac.$$

We have

$$\mathcal{O} = \mathbb{Z}e_0' + \mathbb{Z}e_1' + \mathbb{Z}2e_2' + \mathbb{Z}2e_3',$$

$$\begin{aligned} f_{\mathcal{O}^0} &= \mathrm{n}(xe_1' + ype_2' + zpe_3') \\ &= \mathrm{n}(e_1')x^2 + \mathrm{n}(2e_2')y^2 + \mathrm{n}(2e_3')z^2 + \mathrm{tr}(2e_2' \overline{2e_3'})yz + \mathrm{tr}(e_1' \overline{2e_3'})xz + \mathrm{tr}(e_1' \overline{2e_2'})xy \\ &= ax^2 + 2by^2 + 2cz^2 + 2ryz + 4sxz + 4txy. \end{aligned}$$

It follows

$$f_{S^0} = \lambda_4^{-1}(f_{\mathcal{O}^0}) = ax^2 + 8by^2 + 8cz^2 + 8ryz + 8sxz + 8txy.$$

For $\mathcal{O}$ is an order of level (N_1, N_2), the proof is analogous by Remake 2.4. □

By Proposition 4.7 and Proposition 4.9, we establish the following commutative diagrams among M_0, M_1 and Clifford algebras using λ_4 and ϕ.

Theorem 4.10. *Denote $\phi_{p_1} \circ ... \circ \phi_{p_m}$ (resp. $\phi_{p_1} \circ ... \circ \phi_{p_m} \circ \phi_2$) by $\phi_{N_1N_2}$, where $N_1N_2 = p_1^{l_1}...p_m^{l_m}$ (resp. $N_1N_2 = 2p_1^{l_1}...p_m^{l_m}$) and the p_i are distinct odd primes. In a positive definite quaternion algebra $Q_{N_1'}$, we choose a complete set of representatives $\{\mathcal{O}_\mu\}_{\mu=1,2...T_{N_1,N_2}}$ for these types of orders of level (N_1, N_2). We establish the following commutative diagrams.*
In the case $2 \nmid N_1N_2$, we have

$$\begin{array}{ccc} \{\mathcal{O}_\mu\}_{\mu=1,2...T_{N_1,N_2}} & \xrightarrow{C_0} & G_{4N_1N_2,N_1N_2,N_1'} \\ \downarrow{\scriptstyle M_1} & \searrow^{M_0} & \downarrow{\scriptstyle \phi_{N_1N_2}^{-1}} \\ G_{4N_1N_2,16(N_1N_2)^2,N_1'} & \xrightarrow{\lambda_4} & G_{4N_1N_2,(N_1N_2)^2,N_1'} \end{array}$$

In the case $2 \parallel N_1N_2$, we have

$$\begin{array}{ccc} \{\mathcal{O}_\mu\}_{\mu=1,2...T_{N_1,N_2}} & \xrightarrow{C_0} & G_{4N_1N_2,N_1N_2,N_1'} \\ \downarrow{\scriptstyle M_0} & \searrow^{M_1} & \downarrow{\scriptstyle \phi_{N_1N_2}^{-1}} \\ G_{2N_1N_2,(N_1N_2)^2,N_1'} & \xleftarrow[\lambda_4]{} & G_{4N_1N_2,16(N_1N_2)^2,N_1'} \end{array}$$

By Corollary 2.6, Proposition 2.12 and Theorem 4.4, direct corollarys follow.

Corollary 4.11. *M_0 and M_1 are bijections, and*

$$2\text{card}(\text{Aut}(\mathcal{O})) = |\,\text{Aut}(f_{\mathcal{O}^0})| = |\,\text{Aut}(f_{S^0})|.$$

Corollary 4.12. *Let $|G|$ denote the number of classes in genus G.*
In the case $2 \nmid N_1N_2$, we have

$$|G_{4N_1N_2,16(N_1N_2)^2,N_1'}| = |G_{4N_1N_2,(N_1N_2)^2,N_1'}| = |G_{4N_1N_2,N_1N_2,N_1'}| = T_{N_1,N_2}.$$

In the case $2 \parallel N_1N_2$, we have

$$|G_{2N_1N_2,(N_1N_2)^2,N_1'}| = |G_{4N_1N_2,16(N_1N_2)^2,N_1'}| = |G_{4N_1N_2,N_1N_2,N_1'}| = T_{N_1,N_2}.$$

4.3. **Commutative diagrams for $4 \mid N_1N_2$.** In this section, we construct commutative diagrams involving M_1 and C_0 under the condition $4 \mid N_1N_2$. We begin by providing a detailed characterization of the genera to which $f_{\mathcal{O}^0}$, f_{S^0} and $f_{\mathcal{O}}$ belong.

Proposition 4.13. *For an order $\mathcal{O}$ we denote $f_{\mathcal{O}^0}$ by f. Let $\mathcal{O} \subset Q_{N_1'}$ be an order of level (N_1, N_2) where $4 \mid N_1N_2$, then $d_f = (N_1N_2)^2, N_f = N_1N_2$ and f is anisotropic only in the p-adic field for $p \mid N_1'$. The genus which f belongs to is denoted by $G_{N_1N_2,(N_1N_2)^2,N_1'}$.*

Proof. In the case $2^{2u+1} \parallel N_1$, we have

$$f \underset{2}{\sim} 3x^2 - 2^{2u+1}(y^2 + z^2 - yz).$$

In the case $2^v \parallel N_2$, we have

$$f \underset{p}{\sim} -x^2 - 2^v yz.$$

The rest of proof is analogous. ☐

Remark 4.14. For $2 \nmid N_1N_2$, $G_{4N_1N_2,16(N_1N_2)^2,N_1'}$ coincides with $G_{N_1\cdot 4N_2,(N_1\cdot 4N_2)^2,N_1'}$. This implies that the image of map M_1 for orders of level (N_1, N_2) and map M_0 for orders of level $N_1 \cdot 4N_2$ belong to the same genus.

Proposition 4.15. *For an order $\mathcal{O}$ we denote f_{S^0} by f. Let $\mathcal{O} \subset Q_{N_1'}$ be an order of level (N_1, N_2) where $4 \mid N_1N_2$, then $d_f = 16(N_1N_2)^2, N_f = 4N_1N_2$ and f is anisotropic only in the p-adic field for $p \mid N_1'$. The genus which f belongs to is denoted by $G_{4N_1N_2,16(N_1N_2)^2,N_1'}$.*

Proof. In the case $2^{2u+1} \parallel N_1$, we have
$$f \underset{2}{\sim} 3x^2 - 2^{2u+3}(y^2 + z^2 - yz).$$
In the case $2^v \parallel N_2$, we have
$$f \underset{p}{\sim} -x^2 - 2^{v+2}yz.$$
The rest of proof is analogous. □

Remark 4.16. when $4 \mid N_1N_2$, λ_4 does not act as a bijection such that $\lambda_4(f_{S^0}) = f_{\mathcal{O}^0}$. The reason behind this will be elucidated later..

Proposition 4.17. *For an order $\mathcal{O}$ we denote $f_{\mathcal{O}}$ by f. Let $\mathcal{O} \subset Q_{N_1'}$ be an Eichler order of level (N_1, N_2) where $4 \mid N_1N_2$, then $d_f = 16(N_1N_2)^2, N_f = 4N_1N_2$ and f is anisotropic only in the p-adic field for $p \mid N_1'$. The genus which f belongs to is denoted by $G_{4N_1N_2,N_1N_2,N_1'}$.*

Proof. In the case $2^{2u+1} \parallel N_1$, we have
$$f \underset{2}{\sim} 3 \cdot 2^{2u+1}x^2 - (y^2 + z^2 - yz).$$
In the case $2 \parallel N_2$, we have
$$f \underset{p}{\sim} -2^v x^2 - yz.$$
The rest of proof is analogous. □

Recallling Proposition 2.5, ϕ_2 is a bijection between $C(N', 2^{2g}d')$ and $C(N', 2^{g-2}d')$. A proposition follows.

Proposition 4.18. *Let $\mathcal{O}$ be an order of level (N_1, N_2), where $4 \mid N_1N_2$, i.e. $N_1N_2 = 2^{l_0}p_1^{l_1}...p_w^{l_w}$, and the p_i are distinct odd primes, then $\phi_{p_1} \circ ... \circ \phi_{p_w} \circ \phi_2(f_{S^0}) = f_{\mathcal{O}}$.*

Proof. Suppose that $N_1N_2 = 2^{g-2}$ where $g \geq 4$, (Despite the implication that $N_1N_2 = p^{g-2}$ necessitates an odd g, it does not impact our proof). Let $f_{\mathcal{O}} = (2^{g-2}a, b, c, r, 2^{g-1}s, 2^{g-1}t)$ be a positive definite ternary quadratic forms of level 2^g and discriminant 2^{g-2}. Recalling ϕ_2:
$$\phi_2^{-1}((2^{g-2}a, b, c, r, 2^{g-1}s, 2^{g-1}t)) = (a, 2^g b, 2^g c, 2^g r, 2^g s, 2^g t), 2 \nmid ac.$$
We have
$$\mathcal{O} = \mathbb{Z} + \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3,$$
where
$$e_1^2 = re_1 - bc \qquad e_2e_3 = 2^{g-2}a\overline{e_1}$$
$$e_2^2 = 2^{g-1}se_2 - 2^{g-2}ac \qquad e_3e_1 = b\overline{e_2}$$
$$e_3^2 = 2^{g-1}te_3 - 2^{g-2}ab \qquad e_1e_2 = c\overline{e_3}.$$
Hence
$$e_0' = 1 - 2(abc + 2^g rst) + (ar + 2^g st)e_1 + (2bs + 2rt)e_2 + (2ct + 2rs)e_3,$$
$$e_1' = ar + 2^g st - 2ae_1 - 2te_2 - 2se_3,$$
$$e_2' = 2bs + 2rt - 2te_1 - \tfrac{2b}{2^{g-2}}e_2 - \tfrac{r}{2^{g-2}}e_3,$$
$$e_3' = 2ct + 2rs - 2se_1 - \tfrac{r}{2^{g-2}}e_2 - \tfrac{2c}{2^{g-2}}e_3.$$

We have $\mathrm{tr}(e_1') = \mathrm{tr}(e_2') = \mathrm{tr}(e_3') = 0$, and

$$\mathrm{n}(e_1') = a, \mathrm{tr}(2^{g-2}e_2'\overline{2^{g-2}e_3'}) = 2^{g-2}r,$$
$$\mathrm{n}(2^{g-2}e_2') = 2^{g-2}b, \mathrm{tr}(2^{g-2}e_3'\overline{e_1'}) = 2^{g-1}s,$$
$$\mathrm{n}(2^{g-2}e_3') = 2^{g-2}c, \mathrm{tr}(e_1'\overline{2^{g-2}e_2'}) = 2^{g-1}t.$$

It is analogous to check that

$$\mathcal{O} = \mathbb{Z}e_0' + \mathbb{Z}e_1' + \mathbb{Z}2^{g-2}e_2' + \mathbb{Z}2^{g-2}e_3'.$$

We will show

$$\mathbb{Z} + 2\mathcal{O} = \mathbb{Z}2e_0' + \mathbb{Z}e_1' + \mathbb{Z}2^{g-1}e_2' + \mathbb{Z}2^{g-1}e_3',$$

It is not hard to check that

$$2e_0' + re_1' + 2^{g-1}se_2' + 2^{g-1}te_3' = 2 - 4abc - 2^g rst + ar^2 + 2^g bs^2 + 2^g ct^2$$

and

$$d_{f_{\mathcal{O}}} = 4 \cdot 2^{g-2}abc + 2^{2g-2}rst - 2^{g-2}ar^2 - 2^{2g-2}bs^2 - 2^{2g-2}ct^2 = 2^{g-2},$$

we have

$$re_1' = 1 - 2e_0' - 2^{g-1}se_2' - 2^{g-1}te_3'$$

and $2 \nmid r$. Hence

$$\mathbb{Z} + 2\mathcal{O} = \mathbb{Z}2e_0' + \mathbb{Z}e_1' + \mathbb{Z}2^{g-1}e_2' + \mathbb{Z}2^{g-1}e_3',$$

and

$$S^0 = \mathbb{Z}e_1' + \mathbb{Z}2^{g-1}e_2' + \mathbb{Z}2^{g-1}e_3'.$$

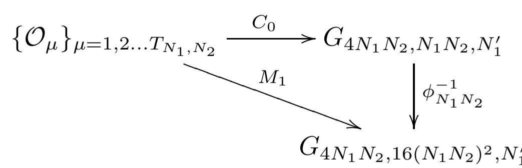
We have

$$\begin{aligned} f_{S^0} &= \mathrm{n}(xe_1' + y2^{g-1}e_2' + z2^{g-1}e_3') \\ &= \mathrm{n}(e_1')x^2 + \mathrm{n}(2^{g-1}e_2')y^2 + \mathrm{n}(2^{g-1}e_3')z^2 + \mathrm{tr}(2^{g-1}e_2'\overline{2^{g-1}e_3'})yz + 2\mathrm{tr}(e_1'\overline{2^{g-1}e_3'})xz + \mathrm{tr}(e_1'\overline{2^{g-1}e_2'})xy \\ &= ax^2 + 2^g by^2 + 2^g cz^2 + 2^g ryz + 2^g sxz + 2^g txy. \end{aligned}$$

If $\mathcal{O}$ is an order of level (N_1, N_2), the proof is analogous to the above by Remake 2.4. □

By Proposition 4.18, we establish the following commutative diagram.

Theorem 4.19. *Denote $\phi_{p_1} \circ ... \circ \phi_{p_w} \circ \phi_2$ by $\phi_{N_1N_2}$, where $N_1N_2 = 2^{l_0}p_1^{l_1}...p_w^{l_w}$ and the p_i are distinct odd primes. In a positive definite quaternion algebra $Q_{N_1'}$, we choose a complete set of representatives $\{\mathcal{O}_\mu\}_{\mu=1,2...T_{N_1,N_2}}$ for these types of orders of level N_1N_2. We establish the following commutative diagram.*



Remark 4.20. In the case $4 \mid N_1N_2$, if λ_4 is a bijection such that $\lambda_4(f_{S^0}) = f_{\mathcal{O}^0}$, it follows $T_{N_1,N_2} = T_{N_1,N_2/4}$ for $4 \parallel N_2$, and it is a contradiction.

By Corollary 2.6 and Theorem 4.4, direct corollarys follow.

Corollary 4.21. *M_1 is a bijection, and*

$$2\mathrm{card}(\mathrm{Aut}(\mathcal{O})) = |\,\mathrm{Aut}(f_{S^0})|.$$

Corollary 4.22. *We have*

$$|G_{4N_1N_2,16(N_1N_2)^2,N_1'}| = |G_{4N_1N_2,N_1N_2,N_1'}| = T_{N_1,N_2}.$$

4.4. **Zeros in orders and representation numbers of ternary quadratic forms.**

Proposition 4.23. *Let $\mathcal{O}$ be an order of level (N_1, N_2), $\rho_{\mathcal{O}}(n,r)$ be the number of zeros of $x^2 - rx + n$ in $\mathcal{O}$, then we have*

$$R_{f_{S^0}}(n) = \rho_{\mathcal{O}}(\frac{n+r^2}{4}, r),$$

that is

$$R_{f_{S^0}}(4n - r^2) = \rho_{\mathcal{O}}(n,r).$$

Proof. Let

$$\begin{aligned} f_{S^0} &= \mathrm{n}(x(2\alpha_1) + y(2\alpha_2) + z(2\alpha_3 - 1)) \\ &= 4\mathrm{n}(\alpha_1)x^2 + 4\mathrm{n}(\alpha_2)y^2 + (4\mathrm{n}(\alpha_3) - 1)z^2 + 4\mathrm{tr}(\alpha_2\overline{\alpha_3})yz + 4\mathrm{tr}(\alpha_1\overline{\alpha_3})xz + 4\mathrm{tr}(\alpha_1\overline{\alpha_2})xy, \end{aligned}$$

In the case $n \equiv 1, 2 \pmod 4$, we have $f_{S^0} \equiv -z^2 \pmod 4$. Hence $R_{f_{S^0}}(n) = 0$
In the case $n \equiv 0 \pmod 4$, we have $2 \mid z$, $2 \mid r$ and

$$n = \mathrm{n}(x(2\alpha_1) + y(2\alpha_2) + z(2\alpha_3 - 1)) = 4\mathrm{n}(x\alpha_1 + y\alpha_2 + \frac{z}{2}(2\alpha_3 - 1)).$$

We have $x\alpha_1 + y\alpha_2 + \frac{z}{2}(2\alpha_3 - 1) \in \mathcal{O}^0$, and $\frac{r}{2} + x\alpha_1 + y\alpha_2 + \frac{z}{2}(2\alpha_3 - 1) \in \mathcal{O}$. Since $\mathrm{tr}(\alpha\overline{\beta}) = \mathrm{n}(\alpha+\beta) - \mathrm{n}(\alpha) - \mathrm{n}(\beta)$, then

$$n = \mathrm{n}(\frac{r}{2} + x(2\alpha_1) + y(2\alpha_2) + z(2\alpha_3 - 1)) = \frac{n+r^2}{4},$$

and

$$\mathrm{tr}(\frac{r}{2} + x(2\alpha_1) + y(2\alpha_2) + z(2\alpha_3 - 1)) = r.$$

Conversely, if

$$\mathrm{n}(x\alpha_1 + y\alpha_2 + z\alpha_3 + t) = \frac{n+r^2}{4}.$$

and

$$\mathrm{tr}(x\alpha_1 + y\alpha_2 + z\alpha_3 + t) = r.$$

We have $\mathrm{tr}(z\alpha_3) + 2t = 2t + z = r$, hence $t = \frac{r-z}{2}$, and

$$n = 4\mathrm{n}(x\alpha_1 + y\alpha_2 + z\alpha_3 + \frac{r-z}{2}) - r^2 = \mathrm{n}(x(2\alpha_1) + y(2\alpha_2) + z(2\alpha_3 - 1)).$$

In the case $n \equiv 3 \pmod 4$, we have $2 \nmid z$ and $2 \nmid r$. Since

$$\mathrm{n}(x(2\alpha_1) + y(2\alpha_2) + z(2\alpha_3 - 1)) = \mathrm{n}(2x\alpha_1 + 2y\alpha_2 + 2z\alpha_3 - z + r - r) = \mathrm{n}(2x\alpha_1 + 2y\alpha_2 + 2z\alpha_3 - z + r) - r.$$

Then $x\alpha_1 + y\alpha_2 + z\alpha_3 - \frac{z-r}{2} \in \mathcal{O}$ and we have

$$\mathrm{n}(x\alpha_1 + y\alpha_2 + z\alpha_3 - \frac{z-r}{2}) = \frac{n+r^2}{4}.$$

and

$$\mathrm{tr}(x\alpha_1 + y\alpha_2 + z\alpha_3 - \frac{z-r}{2}) = r.$$

Conversely, if

$$\mathrm{n}(x\alpha_1 + y\alpha_2 + z\alpha_3 + t) = \frac{n+r^2}{4}.$$

and

$$\mathrm{tr}(x\alpha_1 + y\alpha_2 + z\alpha_3 + t) = r.$$

We have $\mathrm{tr}(z\alpha_3) + 2t = 2t + z = r$. Hence $t = \frac{r-z}{2}$, and

$$n = 4\mathrm{n}(x\alpha_1 + y\alpha_2 + z\alpha_3 - \frac{z-r}{2}) - r^2 = \mathrm{n}(x(2\alpha_1) + y(2\alpha_2) + z(2\alpha_3 - 1)).$$

It follows $R_{f_{S^0}}(n) = \rho_{\mathcal{O}}(\frac{n+r^2}{4}, r)$. □

Let $\mathcal{O}$ be an order of level (N_1, N_2), by Corollary 4.21, Corollary 4.11, and Proposition 4.23, we have

$$\sum_{\mu=1}^{T_{N_1,N_2}} \frac{\rho_{\mathcal{O}}(n,r)}{\mathrm{card}(\mathrm{Aut}(\mathcal{O}_\mu))} = 2^{-1} \sum_{f\in G_{4N_1N_2,16(N_1N_2)^2,N_1'}} \frac{R_f(4n-r^2)}{|\mathrm{Aut}(f)|}.$$

5. The Siegel–Weil formula for ternary quadratic forms

When N_1N_2 is squarefree, Li, Skoruppa and Zhou[5] proved that, for all Eichler orders with a same squarefree level in a definite quaternion algebra over the field of rational numbers, a weighted sum of Jacobi theta series associated with these orders is a Jacobi Eisenstein series which has Fourier coefficients $H^{(N_1,N_2)}(4n-r^2)$.

Theorem 5.1. [5, Main Theorem] *Let N and F be two squarefree positive integers which are coprime, where N has an odd number of prime factors. Use $T_{N,F}$ for the type number of Eichler orders of level F in Q_N, where Q_N ramifies only at the primes which divides N. Choose a complete set of representatives $\mathcal{O}_\mu(\mu = 1, 2, ..., T_{N,F})$ for these types of Eichler orders. Let*

$$\theta_{\mathcal{O}_\mu} = \sum_{\substack{n,r\in\mathbb{Z}\\ 4n-r^2\geq 0}} \rho_{\mathcal{O}_\mu}(n,r) q^n \zeta^r$$

where $\rho_{\mathcal{O}_\mu}(n,r)$ is the number of zeros of $x^2 - rx + n$ in $\mathcal{O}_\mu$. Then we have

$$\sum_{\mu=1}^{T_{N,F}} \frac{\theta_{\mathcal{O}_\mu}}{\mathrm{card}(\mathrm{Aut}(\mathcal{O}_\mu))} = 2^{-e(NF)} \sum_{\substack{n,r\in\mathbb{Z}\\ 4n-r^2\geq 0}} H^{(N,F)}(4n-r^2) q^n \zeta^r, \tag{5.1}$$

where $e(NF)$ is the number of prime factors of NF, and $\mathrm{card}(\mathrm{Aut}(\mathcal{O}_\mu))$ *is the number of elements in the group of automorphisms of $\mathcal{O}_\mu$.*

However, in the general case, it is difficult to explicitly determine the Fourier coefficients of the Jacobi Eisenstein series. It is essential to revisit the Siegel–Weil formula for ternary quadratic forms, considering G a genus of positive ternary forms with the discriminant d_G.

$$\sum_{f\in G} \frac{R_f(n)}{|\mathrm{Aut}(f)|} = 4\pi M(G)\sqrt{\frac{n}{d_G}} \prod_p d_{G,p}(n).$$

This sum should be interpreted as the finite sum resulting from taking a single representative from each equivalence class of forms. The product on the right is over all primes, the mass of the genus $M(G)$ is defined by

$$M(G) = \sum_{f\in G} \frac{1}{|\mathrm{Aut}(f)|}.$$

and $d_{G,p}(n)$ denotes the p-adic local representation density, defined by

$$d_{G,p}(n) = \frac{1}{p^{2t}} |\{(x,y,z)\in\mathbb{Z}^3 : ax^2+by^2+cz^2+ryz+sxz+txy \equiv n \pmod{p^t}\}|$$

for sufficiently large t, and $ax^2 + by^2 + cz^2 + ryz + sxz + txy$ can be chosen to be any form $\in G$. Siegel demonstrated that when $\gcd(2d, p) = 1$, then

$$d_{-x^2-yz,p}(n) = \begin{cases} (\frac{1}{p}+1) + \frac{1}{p^{k+1}}((\frac{-l}{p})-1) & \text{if} \quad n = lp^{2k}, p\nmid l, \\ (\frac{1}{p}+1)(1-\frac{1}{p^{k+1}}) & \text{if} \quad n = lp^{2k+1}, p\nmid l. \end{cases}$$

This section is dedicated to the calculation of local representation densities. Some propositions can be proved by [13, Theorem 3.1], but here we give the proof by directly computing.

5.1. **Computing local representation densities at an odd prime.** In order to compute

$$d_{-\epsilon x^2+p^{2u+1}y^2-\epsilon p^{2u+1}z^2,p}(n)$$

and

$$d_{-x^2-p^v yz,p}(n),$$

we will employ established lemmas.

Lemma 5.2. [1, Lemma 3.2] *Let p be an odd prime not dividing c. For $t \geq 1$, we have*

$$card(\{0 \leq x \leq p^t | x^2 \equiv c \pmod{p^t}\}) = 1 + \left(\frac{c}{p}\right).$$

Lemma 5.3. *Let p be an odd prime not dividing c. We have*

$$\sum_{x=0}^{p-1} \left(\frac{x+a^2}{p}\right) = -1.$$

Proposition 5.4. *Let u be a nonnegative integer, $\left(\frac{\epsilon}{p}\right) = -1$, and $p \nmid m$. We have*

$$d_{-\epsilon x^2+p^{2u+1}y^2-\epsilon p^{2u+1}z^2,p}(n) = \begin{cases} 0 & \text{if} \quad n = mp^{2k+1}, k < u \\ p^k(1-(\frac{-m}{p})) & \text{if} \quad n = mp^{2k}, k \leq u, \\ p^{2u-k}(1+\frac{1}{p}) & \text{if} \quad n = mp^{2k+1}, k \geq u, \\ p^{2u-k}\left(1-\left(\frac{-m}{p}\right)\right) & \text{if} \quad n = mp^{2k}, k > u. \end{cases}$$

Proof. Let $n = p^l m$, where $p \nmid m$. When l is even (resp. odd), let $l = 2k$ (resp. $l = 2k+1$).
In the case $l = 2k+1 < 2u+1$, we have

$$d_{-\epsilon x^2+p^{2u+1}y^2-\epsilon p^{2u+1}z^2,p}(pm) = \frac{1}{p^{2t}} |\{0 \leq x, y, z < p^t : -\epsilon x^2 + p^{2u+1}y^2 - \epsilon p^{2u+1}z^2 \equiv p^{2k+1}m \pmod{p^t}\}|.$$

It implies

$$d_{-\epsilon x^2+p^{2u+1}y^2-\epsilon p^{2u+1}z^2,p}(p^{2k+1}m) = 0.$$

For $u > 0$, we have

$$\begin{aligned} d_{-\epsilon x^2+p^{2u+1}y^2-\epsilon p^{2u+1}z^2,p}(p^l m) &= \frac{1}{p^{2t}} |\{0 \leq x, y, z < p^t : -\epsilon x^2 + p^{2u+1}y^2 - \epsilon p^{2u+1}z^2 \equiv p^l m \pmod{p^t}\}| \\ &= \frac{1}{p^{2t}} |\{0 \leq x < p^{t-1}, 0 \leq y, z < p^t : -\epsilon x^2 + p^{2u-1}y^2 - \epsilon p^{2u-1}z^2 \equiv p^{l-2}m \pmod{p^{t-2}}\}| \\ &= p^5 \frac{1}{p^{2t}} |\{0 \leq x, y, z < p^{t-2} : -\epsilon x^2 + p^{ur-1}y^2 - \epsilon p^{2u-1}z^2 \equiv p^{l-2}m \pmod{p^{t-2}}\}| \\ &= p \frac{1}{p^{2(t-2)}} |\{0 \leq x, y, z < p^{t-2} : -\epsilon x^2 + p^{2u-1}y^2 - \epsilon p^{2u-1}z^2 \equiv p^{l-2}m \pmod{p^{t-2}}\}| \\ &= p d_{-\epsilon x^2+p^{2u-1}y^2-\epsilon p^{2u-1}z^2,p}(p^{l-2}m). \end{aligned}$$

In the case $l = 2k < 2u+1$, when $l = 0$, we have

$$\begin{aligned} d_{-\epsilon x^2+p^{2u+1}y^2-\epsilon p^{2u+1}z^2,p}(m) &= \frac{1}{p^{2t}} |\{0 \leq x, y, z < p^t : -\epsilon x^2 + p^{2u+1}y^2 - \epsilon p^{2u+1}z^2 \equiv m \pmod{p^t}\}| \\ &= \frac{1}{p^{2t}} |\{0 \leq x, y, z < p^t : x^2 \equiv -\epsilon m + \epsilon p^{2u+1}y^2 - \epsilon^2 p^{2u+1}z^2 \pmod{p^t}\}| \end{aligned}$$

$$= \frac{1}{p^{2t}} \sum_{y=0}^{p^t-1} \sum_{z=0}^{p^t-1} (1 + (\frac{-\epsilon m + \epsilon p^{2u+1} y^2 - \epsilon^2 p^{2u+1} z^2}{p}))$$
$$= 1 - (\frac{-m}{p}).$$

Hence for $l = 2k < 2u + 1,$, we have

$$d_{-\epsilon x^2 + p^{2u+1} y^2 - \epsilon p^{2u+1} z^2, p}(p^{2k} m) = p^k d_{-\epsilon x^2 + p^{2u-k-1} y^2 - \epsilon p^{2u-2k-1} z^2, p}(m) = p^k (1 - (\frac{-m}{p})).$$

In the case $l \geq 2u + 1$, we have

$$d_{-\epsilon x^2 + p^{2u+1} y^2 - \epsilon p^{2u+1} z^2, p}(p^l m) = p^u d_{-\epsilon x^2 + p y^2 - \epsilon p z^2, p}(p^{l-2u} m).$$

Since

$$\begin{aligned} d_{-\epsilon x^2 + p y^2 - \epsilon p z^2, p}(p^l m) &= \frac{1}{p^{2t}} |\{0 \leq x, y, z < p^t : -\epsilon x^2 + p y^2 - \epsilon p z^2 \equiv p^l m \pmod{p^t}\}| \\ &= \frac{1}{p^{2t}} |\{0 \leq x, y, z < p^{t-1} : -\epsilon x^2 + p y^2 - \epsilon p z^2 \equiv p^{l-2} m \pmod{p^{t-2}}\}| \\ &= p^3 \frac{1}{p^{2t}} |\{0 \leq x, y, z < p^{t-2} : -\epsilon x^2 + p y^2 - \epsilon p z^2 \equiv p^{l-2} m \pmod{p^{t-2}}\}| \\ &= \frac{1}{p} d_{-\epsilon x^2 + p y^2 - \epsilon p z^2, p}(p^{l-2} m). \end{aligned}$$

Hence for $l = 2k + 1 \geq 2u + 1$, we have

$$d_{-\epsilon x^2 + p^{2u+1} y^2 - \epsilon p^{2u+1} z^2, p}(p^{2k+1} m) = p^u d_{-\epsilon x^2 + p y^2 - \epsilon p z^2, p}(p^{2k+1-2u} m) = p^{2u-k} d_{-\epsilon x^2 + p y^2 - \epsilon p z^2, p}(pm).$$

Since

$$\begin{aligned} d_{-\epsilon x^2 + p y^2 - \epsilon p z^2, p}(pm) &= \frac{1}{p^{2t}} |\{0 \leq x, y, z < p^t : -\epsilon x^2 + p y^2 - \epsilon p z^2 \equiv pm \pmod{p^t}\}| \\ &= p^2 \frac{1}{p^{2t}} |\{0 \leq x, y, z < p^{t-1} : -\epsilon p x^2 + y^2 - \epsilon z^2 \equiv m \pmod{p^t}\}| \\ &= \frac{1}{p^{2t-2}} |\{0 \leq x, y, z < p^{t-1} : y^2 \equiv m + \epsilon p x^2 + \epsilon z^2 \pmod{p^t}\}| \\ &= \frac{1}{p^{2t-2}} \sum_{x=0}^{p^t-1} \sum_{z=0}^{p^t-1} (1 + (\frac{m + \epsilon p x^2 + \epsilon z^2}{p})) \\ &= 1 + \frac{1}{p}. \end{aligned}$$

We get

$$d_{-\epsilon x^2 + p^{2u+1} y^2 - \epsilon p^{2u+1} z^2, p}(p^{2k+1} m) = p^{2u-k} (1 + \frac{1}{p}).$$

For $l = 2k > 2u + 1$, we have

$$d_{-\epsilon x^2 + p^{2u+1} y^2 - \epsilon p^{2u+1} z^2, p}(p^{2k} m) = p^u d_{-\epsilon x^2 + p y^2 - \epsilon p z^2, p}(p^{k-2u} m) = p^{2r-k} d_{-\epsilon x^2 + p y^2 - \epsilon p z^2, p}(m) = p^{2u-k} (1 - (\frac{-m}{p})).$$

It implies

$$d_{-\epsilon x^2+p^{2u+1}y^2-\epsilon p^{2u+1}z^2,p}(n) = \begin{cases} 0 & \text{if} \quad n = mp^{2k+1}, k < u \\ p^k(1-(\frac{-m}{p})) & \text{if} \quad n = mp^{2k}, k \leq u, \\ p^{2u-k}(1+\frac{1}{p}) & \text{if} \quad n = mp^{2k+1}, k \geq u, \\ p^{2u-k}\left(1-\left(\frac{-m}{p}\right)\right) & \text{if} \quad n = mp^{2k}, k > u. \end{cases}$$

□

Proposition 5.5. *Let $v_0 \equiv 0 \pmod 2$ be a nonnegative integer, and $p \nmid m$. Then*

$$d_{-x^2-p^{v_0}yz,p}(n) = \begin{cases} 0 & \text{if} \quad n = mp^{2k+1}, 2k+1 < v_0, \\ p^k(1+(\frac{-m}{p})) & \text{if} \quad n = mp^{2k}, 2k < v_0, \\ p^{\frac{v_0}{2}-1}+p^{\frac{v_0}{2}}-p^{v_0-k-2}-p^{v_0-k-1} & \text{if} \quad n = mp^{2k+1}, 2k+1 > v_0, \\ p^{\frac{v_0}{2}}+p^{\frac{v_0}{2}-1}+(\frac{-m}{p})(p^{v_0-k-1})-p^{v_0-k-1} & \text{if} \quad n = mp^{2k}, 2k \geq v_0. \end{cases}$$

Let $v_1 \equiv 1 \pmod 2$ be a nonnegative integer, and $p \nmid m$. Then

$$d_{-x^2-p^{v_1}yz,p}(n) = \begin{cases} 0 & \text{if} \quad n = mp^{2k+1}, 2k+1 < v_1, \\ p^k(1+(\frac{-m}{p})) & \text{if} \quad n = mp^{2k}, 2k < v_1, \\ 2p^{\frac{v_1-1}{2}}-p^{v_1-k-2}-p^{v_1-k-1} & \text{if} \quad n = mp^{2k+1}, 2k+1 \geq v_1, \\ 2p^{\frac{v_1-1}{2}}-p^{v_1-k-1}+(\frac{-m}{p})p^{v_1-k-1} & \text{if} \quad n = mp^{2k}, 2k > v_1. \end{cases}$$

Proof. Let $n = p^l m$, where $p \nmid m$. When l is even (resp. odd), let $l = 2k$ (resp. $l = 2k+1$).
In the case $v_0 = 0$, we have

$$-x^2 - yz \underset{p}{\sim} x^2 + y^2 + z^2.$$

Hence

$$d_{-x^2-yz,p}(n) = \begin{cases} (\frac{1}{p}+1)+\frac{1}{p^{k+1}}((\frac{-l}{p})-1) & \text{if} \quad n = mp^{2k}, \\ (\frac{1}{p}+1)(1-\frac{1}{p^{k+1}}) & \text{if} \quad n = mp^{2k+1}. \end{cases}$$

In the case $v_0 > 0$ and $l = 0$, we have

$$\begin{aligned} d_{-x^2-p^{v_0}yz,p}(m) &= \frac{1}{p^{2t}}|\{0 \leq x,y,z < p^t : -x^2+p^{v_0}y^2-p^{v_0}z^2 \equiv m \pmod{p^t}\}| \\ &= \frac{1}{p^{2t}}|\{0 \leq x,y,z < p^t : x^2 \equiv -m+p^{v_0}y^2-p^{v_0}z^2 \pmod{p^t}\}| \\ &= \frac{1}{p^{2t}}\sum_{y=0}^{p^t-1}\sum_{z=0}^{p^t-1}(1+(\frac{-m+p^{v_0}y^2-p^{v_0}z^2}{p})) \\ &= 1+(\frac{-m}{p}). \end{aligned}$$

In the case $v_0 > 0$ and $l = 2k+1 < v_0$, we have

$$d_{-x^2-p^{v_0}yz,p}(p^{2k+1}m) = \frac{1}{p^{2t}}|\{0 \leq x,y,z < p^t : -x^2+p^{v_0}y^2-p^{v_0}z^2 \equiv p^{2k+1}m \pmod{p^t}\}|.$$

It implies

$$d_{-x^2-p^{v_0}yz,p}(p^{2k+1}m) = 0.$$

For $t \geq 3$ and $v_0 \geq 2$, we have

$$d_{-x^2-p^{v_0}yz,p}(p^l m) = \frac{1}{p^{2t}}|\{0 \leq x,y,z < p^t : -x^2+p^{v_0}y^2-p^{v_0}z^2 \equiv p^l m \pmod{p^t}\}|$$

$$= \frac{1}{p^{2t}}|\{0 \le x < p^{t-1}, 0 \le y, z < p^t : -x^2 + p^{v_0-2}y^2 - p^{v_0-2}z^2 \equiv p^{l-2}m \pmod{p^{t-2}}\}|$$
$$= p^5 \frac{1}{p^{2t}}|\{0 \le x, y, z < p^{t-2} : -x^2 + p^{v_0-2}y^2 - p^{v_0-2}z^2 \equiv p^{l-2}m \pmod{p^{t-2}}\}|$$
$$= p \frac{1}{p^{2t-2}}|\{0 \le x, y, z < p^{t-2} : -x^2 + p^{v_0-2}y^2 - p^{v_0-2}z^2 \equiv p^{l-2}m \pmod{p^{t-2}}\}|$$
$$= pd_{-x^2-p^{v_0-2}yz,p}(p^{l-2}m).$$

In the case $v_0 > 0$ and $l = 2k < v_0$, we have

$$d_{-x^2-p^{v_0}yz,p}(p^{2k}m) = p^k d_{-x^2-p^{v_0-2k}yz,p}(m) = p^k(1 + (\frac{-m}{p})).$$

In the case $v_0 > 0$ and $l \ge v_0$, we have

$$d_{-x^2-p^{v_0}yz,p}(p^l m) = p^{v_0/2} d_{-x^2-yz,p}(p^{l-v_0/2}m).$$

Hence

$$d_{-x^2-p^{v_0}yz,p}(n) = \begin{cases} 0 & \text{if} \quad n = mp^{2k+1}, 2k+1 < v_0, \\ p^k(1 + (\frac{-m}{p})) & \text{if} \quad n = mp^{2k}, 2k < v_0, \\ p^{\frac{v_0}{2}-1} + p^{\frac{v_0}{2}} - p^{v_0-k-2} - p^{v_0-k-1} & \text{if} \quad n = mp^{2k+1}, 2k+1 > v_0, \\ p^{\frac{v_0}{2}} + p^{\frac{v_0}{2}-1} + (\frac{-m}{p})(p^{v_0-k-1}) - p^{v_0-k-1} & \text{if} \quad n = mp^{2k}, 2k \ge v_0. \end{cases}$$

It is analogous when $v_1 \equiv 1 \pmod 2$.
In the case $l = 2k < v_1$, we have

$$d_{-x^2-p^{v_1}yz,p}(p^{2k}m) = p^k d_{-x^2-p^{v_1-2k}yz,p}(m) = p^k(1 + (\frac{-m}{p})).$$

In the case $l = 2k+1 < v_1$, we have

$$d_{-x^2-p^{v_1}yz,p}(p^{2k+1}m) = 0.$$

In the case $l \ge v_1$, we have

$$d_{-x^2-p^{v_1}yz,p}(p^l m) = p^{\frac{v_1-1}{2}} d_{-x^2-pyz,p}(p^{l-v_1+1}m).$$

Since

$$\begin{aligned} d_{-x^2-pyz,p}(p^l m) &= \frac{1}{p^{2t}}|\{0 \le x, y, z < p^t : -x^2 - pyz \equiv p^l m \pmod{p^t}\}| \\ &= 2 \cdot \frac{1}{p^{2(t-2)}}|\{0 \le x, y, z < p^{t-2} : -x^2 - yz \equiv p^{l-2}m \pmod{p^{t-2}}\}| \\ &- \frac{1}{p} \cdot \frac{1}{p^{2(t-2)}}|\{0 \le x, y, z < p^{t-2} : -x^2 - pyz \equiv p^{l-2}m \pmod{p^{t-2}}\}| \\ &= 2d_{-x^2-yz,p}(p^{l-2}m) - \frac{1}{p} d_{-x^2-pyz,p}(p^{l-2}m), \end{aligned}$$

it is essential to calculate $d_{-x^2-pyz,p}(m)$ and $d_{-x^2-pyz,p}(pm)$. We have

$$\begin{aligned} d_{-x^2-pyz,p}(m) &= \frac{1}{p'^{2t}}|\{0 \le x, y, z < p^t : -x^2 + py^2 - pz^2 \equiv m \pmod{p^t}\}| \\ &= \frac{1}{p^{2t}}|\{0 \le x, y, z < p^t : x^2 \equiv -m + py^2 - pz^2 \pmod{p^t}\}| \end{aligned}$$

$$= \frac{1}{p^{2t}} \sum_{y=0}^{p^t-1} \sum_{z=0}^{p^t-1} (1 + (\frac{-m + py^2 - pz^2}{p}))$$
$$= 1 + (\frac{-m}{p}),$$

and

$$\begin{aligned} d_{-x^2-pyz,p}(pm) &= \frac{1}{p^{2t}} |\{0 \le x, y, z < p^t : -x^2 + py^2 - pz^2 \equiv pm \pmod{p^t}\}| \\ &= \frac{1}{p^{2t-2}} |\{0 \le x, y, z < p^{t-1} : px^2 + y^2 - z^2 \equiv m \pmod{p^{t-1}}\}| \\ &= \frac{1}{p^{2t-2}} |\{0 \le x, y, z < p^{t-1} : y^2 \equiv m + z^2 - px^2 \pmod{p^{t-1}}\}| \\ &= \frac{1}{p^{2t-2}} \sum_{x=0}^{p^{t-1}-1} \sum_{z=0}^{p^{t-1}-1} (1 + (\frac{m + px^2 + z^2}{p})) \\ &= 1 - \frac{1}{p^{t-1}} \sum_{z=0}^{p^{t-1}-1} (\frac{m + z^2}{p}) \\ &= 1 - \frac{1}{p}. \end{aligned}$$

For $l = 2k + 1 \ge v_1$, we have

$$d_{-x^2-p^{v_1}yz,p}(p^{2k+1}m) = p^{\frac{v_1-1}{2}}(2 - p^{(v_1-1)/2-k-1} - p^{(v_1-1)/2-k}).$$

For $l = 2k > v_1$, we have

$$d_{-x^2-p^{v_1}yz,p}(p^{2k}m) = p^{\frac{v_1-1}{2}}(2 - p^{(v_1-1)/2-k} + \left(\frac{-m}{p}\right) p^{(v_1-1)/2-k}).$$

Hence we get

$$d_{-x^2-p^{v_1}yz,p}(n) = \begin{cases} 0 & \text{if} \quad n = mp^{2k+1}, 2k+1 < v_1, \\ p^k(1 + (\frac{-m}{p})) & \text{if} \quad n = mp^{2k}, 2k < v_1, \\ 2p^{\frac{v_1-1}{2}} - p^{v_1-k-2} - p^{v_1-k-1} & \text{if} \quad n = mp^{2k+1}, 2k+1 \ge v_1, \\ 2p^{\frac{v_1-1}{2}} - p^{v_1-k-1} + (\frac{-m}{p})p^{v_1-k-1} & \text{if} \quad n = mp^{2k}, 2k > v_1. \end{cases}$$

□

5.2. **Computing local representation densities at 2.** In order to compute

$$d_{3x^2-2^{2u+3}(y^2+z^2+yz),2}(n)$$

and

$$d_{-x^2-2^{v_0+2}yz,2}(n),$$

we will employ established lemmas.

Lemma 5.6. [1, Lemma 2.1] *Let $c \equiv 1 \pmod 2$, and $t \ge 3$. Then*

$$card(\{0 \le x \le 2^t | x^2 \equiv c \pmod{2^t}\}) = 2\left(1 + \left(\frac{c}{2}\right)\right).$$

Lemma 5.7. [1, Theorem 4.1,Theorem 4.2][3, Lemma 2.4] *Let $n = 4^a m$, and $4 \nmid m$. We have*

$$d_{-x^2-yz,2}(n) = \begin{cases} \frac{3}{2} & \text{if} \quad m \equiv 7 \pmod 8, \\ \frac{3}{2} - \frac{1}{2^{a+1}} & \text{if} \quad m \equiv 3 \pmod 8, \\ \frac{3}{2} - \frac{3}{2^{a+2}} & \text{if} \quad m \equiv 1,2 \pmod 4. \end{cases}$$

$$d_{-x^2-2yz,2}(n) = \begin{cases} 2 & \text{if} \quad m \equiv 7 \pmod 8, \\ 2 - \frac{1}{2^{a}} & \text{if} \quad m \equiv 3 \pmod 8, \\ 2 - \frac{3}{2^{a+1}} & \text{if} \quad m \equiv 1,2 \pmod 4. \end{cases}$$

$$d_{-x^2-4yz,2}(n) = \begin{cases} 3 & \text{if} \quad m \equiv 7 \pmod 8, \\ 3 - \frac{1}{2^{a-1}} & \text{if} \quad m \equiv 3 \pmod 8, \\ 3 - \frac{3}{2^{a}} & \text{if} \quad m \equiv 1,2 \pmod 4. \end{cases}$$

$$d_{3x^2-2(y^2+z^2+yz),2}(n) = \begin{cases} \frac{1}{2^a} & \text{if} \quad m \equiv 3 \pmod 8, \\ 0 & \text{if} \quad m \equiv 7 \pmod 8, \\ \frac{3}{2^{a+1}} & \text{if} \quad m \equiv 1,2 \pmod 4. \end{cases}$$

Lemma 5.8. *Let u be a nonnegative integer, we have*

$$d_{3x^2-2^{2u+3}(y^2+z^2+yz),2}(4n) = 2d_{3x^2-2^{2u+1}(y^2+z^2+yz),2}(n).$$

Proof.

$$\begin{aligned} d_{3x^2-2^{2u+3}(y^2+z^2+yz),2} &= \frac{1}{2^{2t}}|\{0 \le x,y,z < 2^t : 3x^2 - 2^{2u+3}(y^2+z^2+yz) \equiv 4n \pmod{2^t}\}| \\ &= \frac{1}{2^{2t}}|\{0 \le x < 2^{t-1}, 0 \le y,z < 2^t : -3x^2 - 2^{2u+1}(y^2+z^2+yz) \equiv n \pmod{2^{t-2}}\}| \\ &= 2 \cdot 4 \cdot 4\frac{1}{2^{2t}}|\{0 \le x,y,z < 2^{t-2} : -x^2 - 2^v yz \equiv n \pmod{2^{t-2}}\}| \\ &= 2d_{-3x^2-2^{2u+1}(y^2+z^2+yz),2}(n) \end{aligned}$$

□

Lemma 5.9. *Let v be a nonnegative integer, we have*

$$d_{-x^2-2^{v+2}yz,2}(4n) = 2d_{-x^2-2^v yz,2}(n).$$

Proof.

$$\begin{aligned} d_{-x^2-2^{v+2}yz,2}(4n) &= \frac{1}{2^{2t}}|\{0 \le x,y,z < 2^t : -x^2 - 2^{v+2}yz \equiv 4n \pmod{2^t}\}| \\ &= \frac{1}{2^{2t}}|\{0 \le x < 2^{t-1}, 0 \le y,z < 2^t : -x^2 - 2^v yz \equiv n \pmod{2^{t-2}}\}| \\ &= 2 \cdot 4 \cdot 4\frac{1}{2^{2t}}|\{0 \le x,y,z < 2^{t-2} : -x^2 - 2^v yz \equiv n \pmod{2^{t-2}}\}| \\ &= 2d_{-x^2-2^v yz,2}(n) \end{aligned}$$

□

Proposition 5.10. *Let u be a nonnegative integer, $n = 4^l m$, and $4 \nmid m$, then we have*

$$d_{3x^2-2^{2u+3}(y^2+z^2+yz),2}(4^l m) = \begin{cases} 2^{l+2} & \textit{if} \quad m \equiv 3 \pmod 8, 2l \le 2u, \\ 0 & \textit{if} \quad m \equiv 1,2,5,6,7 \pmod 8, 2l \le 2u, \\ 2^{2u+2-l} & \textit{if} \quad m \equiv 3 \pmod 8, 2l \ge 2u+2, \\ 0 & \textit{if} \quad m \equiv 7 \pmod 8, 2l \ge 2u+2, \\ 3 \cdot 2^{2u+1-l} & \textit{if} \quad m \equiv 1,2 \pmod 4, 2l \ge 2u+2. \end{cases}$$

Proof. Let $n = 4^l m$, where $4 \nmid m$.
When $2l < 2u+3$, we have

$$d_{3x^2-2^{2u+3}(y^2+z^2+yz),2}(4^l m) = 2^l d_{3x^2-2^{2u+3-2l}(y^2+z^2+yz),2}(m).$$

In the case $2u+3-2l \ge 3$, i.e. $2l \le 2u$, when $m \equiv 1,2,5,6,7 \pmod 8$, we have

$$d_{3x^2-2^{2u+3-2l}(y^2+z^2+yz),2}(m) = 0.$$

When $m \equiv 3 \pmod 8$, we have

$$\begin{aligned} d_{3x^2-2^{2u+3}(y^2+z^2+yz),2}(4^l m) &= 2^l d_{3x^2-2^{2u+3-2l}(y^2+z^2+yz),2}(m) \\ &= \frac{2^l}{2^{2t}} |\{0 \le x, y, z < 2^t : 3x^2 - 2^{2u+3-2l}(y^2+z^2+yz) \equiv m \pmod{2^t}\}| \\ &= \frac{2^l}{2^{2t}} |\{0 \le y, z < 2^t : 3x^2 \equiv 2^{2u+3-2l}(y^2+z^2+yz) + m \pmod{2^t}\}| \\ &= 2^{l+2}. \end{aligned}$$

In the case $2u+3-2l = 1$, i.e. $2l = 2u+2$, we have

$$d_{3x^2-2^{2u+3}(y^2+z^2+yz),2}(4^l m) = 2^l d_{3x^2-2(y^2+z^2+yz),2}(m).$$

In the case $2l > 2u+2$, we have

$$d_{3x^2-2^{2u+3}(y^2+z^2+yz),2}(4^l m) = 2^{u+1} d_{3x^2-2(y^2+z^2+yz),2}(4^{l-u-1} m).$$

Hence

$$d_{3x^2-2^{2u+3}(y^2+z^2+yz),2}(4^l m) = \begin{cases} 2^{l+2} & \text{if} \quad m \equiv 3 \pmod 8, 2l \le 2u, \\ 0 & \text{if} \quad m \equiv 1,2,5,6,7 \pmod 8, 2l \le 2u, \\ 2^{2u+2-l} & \text{if} \quad m \equiv 3 \pmod 8, 2l \ge 2u+2, \\ 0 & \text{if} \quad m \equiv 7 \pmod 8, 2l \ge 2u+2, \\ 3 \cdot 2^{2u+1-l} & \text{if} \quad m \equiv 1,2 \pmod 4, 2l \ge 2u+2. \end{cases}$$

□

Remark 5.11. We note that when n is not a negative discriminant, i.e. $n \equiv 1,2 \pmod 4$, we have $d_{3x^2-2^{2u+3}(y^2+z^2+yz),2}(n) = 0$. Let $n = 4^k m$ be a negative discriminant, where $-m$ is a fundamental discriminant, i.e. $m \equiv 0,3 \pmod 4$ and when $4 \mid m$, $m_0 = m/4$ is not a negative discriminant. We can rewrite Proposition 5.10 as follows.

$$d_{3x^2-2^{2u+3}(y^2+z^2+yz),2}(4^k m) = \begin{cases} 2^{k+1}\left(1 - \left(\frac{-m}{2}\right)\right) & \text{if} \quad m \equiv 3 \pmod 4, 2k < 2u, \\ 0 & \text{if} \quad m_0 \equiv 1,2 \pmod 4, 2k < 2u, \\ 2^{2u+1-k}\left(1 - \left(\frac{-m}{2}\right)\right) & \text{if} \quad m \equiv 3 \pmod 4, 2k \ge 2u, \\ 3 \cdot 2^{2u-k} & \text{if} \quad m_0 \equiv 1,2 \pmod 4, 2k \ge 2u. \end{cases}$$

Proposition 5.12. *Let $v_0 \equiv 0 \pmod 2$ be a nonnegative integer, $n = 4^l m$, and $4 \nmid m$, then we have*

$$d_{-x^2-2^{v_0+2}yz,2}(4^l m) = \begin{cases} 2^{l+2} & \textit{if} \quad m \equiv 7 \pmod 8, 2l \le v_0 - 2, \\ 0 & \textit{if} \quad m \equiv 1,2,3,5,6 \pmod 8, 2l \le v_0 - 2, \\ 3 \cdot 2^l & \textit{if} \quad m \equiv 7 \pmod 8, 2l = v_0, \\ 2^l & \textit{if} \quad m \equiv 3 \pmod 8, 2l = v_0, \\ 0 & \textit{if} \quad m \equiv 1,2 \pmod 4, 2l = v_0, \\ 3 \cdot 2^{v_0/2} & \textit{if} \quad m \equiv 7 \pmod 8, 2l \ge v_0 + 2, \\ 3 \cdot 2^{v_0/2} - 2^{v_0+1-l} & \textit{if} \quad m \equiv 3 \pmod 8, 2l \ge v_0 + 2, \\ 3 \cdot (2^{v_0/2} - 2^{v_0-l}) & \textit{if} \quad m \equiv 1,2 \pmod 4, 2l \ge v_0 + 2. \end{cases}$$

Let $v_1 \equiv 1 \pmod 2$ be a nonnegative integer, $n = 4^l m$, and $4 \nmid m$, then we have

$$d_{-x^2-2^{v_1+2}yz,2}(4^l m) = \begin{cases} 2^{l+2} & \textit{if} \quad m \equiv 7 \pmod 8, 2l < v_1 + 1, \\ 0 & \textit{if} \quad m \equiv 1,2,3,5,6 \pmod 8, 2l < v_1 + 1, \\ 2^{(v_1+3)/2} & \textit{if} \quad m \equiv 7 \pmod 8, 2l \ge v_1 + 1, \\ 2^{(v_1+3)/2} - 2^{v_1+1-l} & \textit{if} \quad m \equiv 3 \pmod 8, 2l \ge v_1 + 1, \\ 2^{(v_1+3)/2} - 3 \cdot 2^{v_1-l} & \textit{if} \quad m \equiv 1,2 \pmod 4, 2l \ge v_1 + 1. \end{cases}$$

Proof. Let $n = 4^l m$, where $4 \nmid m$.
When $2l < v_0 + 2$, we have

$$d_{-x^2-2^{v_0+2}yz,2}(4^l m) = 2^l d_{-x^2-2^{v_0+2-2l}yz,2}(m).$$

In the case $v_0 + 2 - 2l \ge 4$, i.e. $2l \le v_0 - 2$, and $m \equiv 1,2,3,5,6 \pmod 8$, we have

$$d_{-x^2-2^{v_0+2-2l}yz,2}(m) = 0$$

When $m \equiv 7 \pmod 8$, we have

$$\begin{aligned} d_{-x^2-2^{v_0+2}yz,2}(4^l m) &= 2^l d_{-x^2-2^{v_0+2-2l}yz,2}(m) \\ &= \frac{2^l}{2^{2t}} |\{0 \le x,y,z < 2^t : -x^2 - 2^{v_0+2-2l}yz \equiv m \pmod{2^t}\}| \\ &= \frac{2^l}{2^{2t}} |\{0 \le y,z < 2^t : x^2 \equiv -2^{v_0+2-2l}yz - m \pmod{2^t}\}| \\ &= 2^{l+2}. \end{aligned}$$

In the case $v_0 + 2 - 2l = 2$, i.e. $2l = v_0$, we have

$$d_{-x^2-2^{v_0+2}yz,2}(4^l m) = 2^l d_{-x^2-4yz,2}(m).$$

In the case $2l \ge v_0 + 2$, we have

$$d_{-x^2-2^{v_0+2}yz,2}(4^l m) = 2^{v_0/2+1} d_{-x^2-yz,2}(4^{l-1-v_0/2} m).$$

Hence we get

$$d_{-x^2-2^{v_0+2}yz,2}(4^l m) = \begin{cases} 2^{l+2} & \text{if} \quad m \equiv 7 \pmod 8, 2l \le v_0 - 2, \\ 0 & \text{if} \quad m \equiv 1,2,3,5,6 \pmod 8, 2l \le v_0 - 2, \\ 3 \cdot 2^l & \text{if} \quad m \equiv 7 \pmod 8, 2l = v_0, \\ 2^l & \text{if} \quad m \equiv 3 \pmod 8, 2l = v_0, \\ 0 & \text{if} \quad m \equiv 1,2 \pmod 4, 2l = v_0, \\ 3 \cdot 2^{v_0/2} & \text{if} \quad m \equiv 7 \pmod 8, 2l \ge v_0 + 2, \\ 3 \cdot 2^{v_0/2} - 2^{v_0+1-l} & \text{if} \quad m \equiv 3 \pmod 8, 2l \ge v_0 + 2, \\ 3 \cdot (2^{v_0/2} - 2^{v_0-l}) & \text{if} \quad m \equiv 1,2 \pmod 4, 2l \ge v_0 + 2. \end{cases}$$

When $2l < v_1 + 2$, we have

$$d_{-x^2-2^{v_1+2}yz,2}(4^l m) = 2^l d_{-x^2-2^{v_1+2-2l}yz,2}(m).$$

In the case $v_1 + 2 - 2l \ge 3$, i.e. $2l \le v_1 - 1$, and $m \equiv 1,2,3,5,6 \pmod 8$, we have

$$d_{-x^2-2^{v_1+2-2l}yz,2}(m) = 0.$$

When $m \equiv 7 \pmod 8$, we have

$$d_{-x^2-2^{v_1+2-2l}yz,2}(m) = 4.$$

In the case $v_1 + 2 - 2l = 1$, i.e. $2l = v_1 + 1$, we have

$$d_{-x^2-2^{v_1+2}yz,2}(4^l m) = 2^l d_{-x^2-2yz,2}(m).$$

In the case $2l > v_1 + 2$, we have

$$d_{-x^2-2^{v_1+2}yz,2}(4^l m) = 2^{(v_1+1)/2} d_{-x^2-2yz,2}(4^{l-(v_1+1)/2} m).$$

Hence we get

$$d_{-x^2-2^{v_1+2}yz,2}(4^l m) = \begin{cases} 2^{l+2} & \text{if} \quad m \equiv 7 \pmod 8, 2l < v_1 + 1, \\ 0 & \text{if} \quad m \equiv 1,2,3,5,6 \pmod 8, 2l < v_1 + 1, \\ 2^{(v_1+3)/2} & \text{if} \quad m \equiv 7 \pmod 8, 2l \ge v_1 + 1, \\ 2^{(v_1+3)/2} - 2^{v_1+1-l} & \text{if} \quad m \equiv 3 \pmod 8, 2l \ge v_1 + 1, \\ 2^{(v_1+3)/2} - 3 \cdot 2^{v_1-l} & \text{if} \quad m \equiv 1,2 \pmod 4, 2l \ge v_1 + 1. \end{cases}$$

□

Remark 5.13. It is analogous that we can rewrite Proposition 5.12 as follows.

$$d_{-x^2-2^{v_0+2}yz,2}(4^k m) = \begin{cases} 2^{k+1}\left(1 + \left(\frac{-m}{2}\right)\right) & \text{if} \quad m \equiv 3 \pmod 4, 2k \le v_0 - 2, \\ 0 & \text{if} \quad m_0 \equiv 1,2 \pmod 4, 2k \le v_0 - 2, \\ 3 \cdot 2^{v_0/2} & \text{if} \quad m \equiv 7 \pmod 8, 2k \ge v_0, \\ 3 \cdot 2^{v_0/2} - 2^{v_0+1-k} & \text{if} \quad m \equiv 3 \pmod 8, 2k \ge v_0, \\ 3 \cdot (2^{v_0/2} - 2^{v_0+1-k}) & \text{if} \quad m_0 \equiv 1,2 \pmod 4, 2k \ge v_0. \end{cases}$$

$$d_{-x^2-2^{v_1+2}yz,2}(4^k m) = \begin{cases} 2^{k+1}\left(1 + \left(\frac{-m}{2}\right)\right) & \text{if} \quad m \equiv 3 \pmod 4, 2k < v_1 - 1, \\ 0 & \text{if} \quad m_0 \equiv 1,2 \pmod 4, 2k < v_1 - 1, \\ 2^{(v_1+3)/2} & \text{if} \quad m \equiv 7 \pmod 8, 2k \ge v_1 + 1, \\ 2^{(v_1+3)/2} - 2^{v_1+1-k} & \text{if} \quad m \equiv 3 \pmod 8, 2k \ge v_1 + 1, \\ 2^{(v_1+3)/2} - 3 \cdot 2^{v_1+1-k} & \text{if} \quad m_0 \equiv 1,2 \pmod 4, 2k \ge v_1 + 1. \end{cases}$$

5.3. **Computing some other local representation densities.** The following propositions are used in the proof of type number formula in Section 7.

Proposition 5.14. *Let u be a nonnegative integer, and $\left(\frac{\epsilon}{p}\right) = -1$, where p is an odd prime, then we have*

$$d_{-\epsilon p^{2u+1}x^2+y^2-\epsilon z^2,p}(1) = d_{-\epsilon p^{2u+1}x^2+y^2-\epsilon z^2,p}(4) = 1 + \frac{1}{p}.$$

Let v be a nonnegative integer, then we have

$$d_{-p^v x^2-yz,p}(1) = d_{-p^v x^2-yz,p}(4) = 1 - \frac{1}{p}.$$

Proof. We have

$$\begin{aligned} d_{-\epsilon p^{2u+1}x^2+y^2-\epsilon z^2,p}(1) &= \frac{1}{p^{2t}}|\{0 \le x, y, z < p^t : -\epsilon p^{2u+1}x^2 + y^2 - \epsilon z^2 \equiv 1 \pmod{p^t}\}| \\ &= \frac{1}{p^{2t}}|\{0 \le x, y, z < p^t : y^2 \equiv 1 + \epsilon p^{2u+1}x^2 + \epsilon z^2 \pmod{p^t}\}| \\ &= \frac{1}{p^{2t}} \sum_{y=0}^{p^t-1} \sum_{z=0}^{p^t-1} (1 + (\frac{1 + \epsilon p^{2u+1}x^2 + \epsilon z^2}{p})) \\ &= 1 + \frac{1}{p} \end{aligned}$$

and

$$\begin{aligned} d_{-p^v x^2-yz,p}(1) &= \frac{1}{p^{2t}}|\{0 \le x, y, z < p^t : -p^v x^2 + y^2 - z^2 \equiv 1 \pmod{p^t}\}| \\ &= \frac{1}{p^{2t}}|\{0 \le x, y, z < p^t : y^2 \equiv 1 + p^v x^2 + z^2 (mod p^t)\}| \\ &= \frac{1}{p^{2t}} \sum_{z=0}^{p^t-1} \sum_{z=0}^{p^t-1} (1 + (\frac{1 + p^{v_0} x^2 + z^2}{p})) \\ &= 1 - \frac{1}{p}. \end{aligned}$$

The rest is analogous. $\square$

Proposition 5.15. *Let u be a nonnegative integer, then we have*

$$d_{3\cdot 2^{2u+3}x^2-(y^2+z^2+yz),2}(1) = \frac{3}{2}.$$

Let v be a nonnegative integer, then we have

$$d_{-2^{v+2}x^2-yz,2}(1) = \frac{1}{2}.$$

Proof. It is not hard to check that when c is odd, we have

$$|\{0 \le y, z < 2^t : y^2 + z^2 + yz \equiv c \pmod{2^t}\}| = 3 \cdot 2^{t-1}$$

and

$$|\{0 \le y, z < 2^t : yz \equiv c \pmod{2^t}\}| = 2^{t-1}.$$

Hence we get

$$d_{3\cdot 2^{2u+3}x^2-(y^2+z^2+yz),2}(1) = d_{3\cdot 2x^2-(y^2+z^2+yz),2}(1) = d_{3x^2-2(y^2+z^2+yz),2}(2) = \frac{3}{2},$$

and

$$d_{-2^{v+2}x^2-yz,2}(1) = d_{-2x^2-yz,2}(1) = d_{-x^2-2yz,2}(2) = \frac{1}{2}.$$

□

6. The Siegel–Weil type formula

This section present the proof of the following theorem.

Theorem 6.1. *Let $N_1 = p_1^{2u_1+1} ... p_w^{2u_w+1}$, where the p_i are distinct primes, $u_1, ..., u_w$ are nonnegative integers and w is an odd integer, and N_2 be a positive integer such that $\gcd(N_1, N_2) = 1$. Use T_{N_1,N_2} for the type number of orders of level (N_1, N_2). Choose a complete set of representatives $\mathcal{O}_\mu(\mu = 1, 2, ..., T_{N_1,N_2})$ for these types of orders, and $\rho_{\mathcal{O}_\mu}(n, r)$ is the number of zeros of $x^2 - rx + n$ in $\mathcal{O}_\mu$. Then we have*

$$\sum_{\mu=1}^{T_{N_1,N_2}} \frac{\rho_{\mathcal{O}_\mu}(n, r)}{\mathrm{card}(\mathrm{Aut}(\mathcal{O}_\mu))} = 2^{-e(N_1N_2)} H^{(N_1,N_2)}(4n - r^2),$$

where $e(N_1N_2)$ is the number of prime factors of N_1N_2, and $\mathrm{card}(\mathrm{Aut}(\mathcal{O}_\mu))$ is the number of elements in the group of automorphisms of $\mathcal{O}_\mu$.

Before proving this theorem, we possess the explicit formula for the mass of orders of level (N_1, N_2).

Proposition 6.2. [8, Proposition 25]

$$\sum_{i=1}^{h_{N_1,N_2}} \frac{1}{card(\mathcal{O}_i^\times)} = \frac{N_1N_2}{24} \prod_{p|N_1} \left(1 - \frac{1}{p}\right) \prod_{p|N_2} \left(1 + \frac{1}{p}\right).$$

Then by Theorem 3.11, we have

$$\sum_{\mu=1}^{T_{N_1,N_2}} \frac{1}{\mathrm{card}(\mathrm{Aut}(\mathcal{O}_\mu))} = 2^{1-e(N_1N_2)} \sum_{i=1}^{h_{N_1,N_2}} \frac{1}{\mathrm{card}(\mathcal{O}_i^\times)},$$

and

$$M(G_{4N_1N_2,16(N_1N_2)^2,N_1'}) = \frac{1}{2} \sum_{\mu=1}^{T_{N_1,N_2}} \frac{1}{\mathrm{card}(\mathrm{Aut}(\mathcal{O}_\mu))} = 2^{-e(N_1N_2)-1} \frac{N_1N_2}{12} \prod_{p|N_1} \left(1 - \frac{1}{p}\right) \prod_{p|N_2} \left(1 + \frac{1}{p}\right).$$

By Proposition 4.23, we only prove that

$$\sum_{f \in G_{4N_1N_2,16(N_1N_2)^2,N_1'}} \frac{R_f(D)}{|\mathrm{Aut}(f)|} = 2^{-e(N_1N_2)-1} H^{(N_1,N_2)}(D),$$

where $D = 4n - r^2$. When N_1N_2 is squarefree, it is clear by [5, Theorem 1.2], and recalling that

$$H^{(N_1,N_2)}(D) = H(D/f_{N_1,N_2}^2) \prod_{p|N_1} \left(1 - \left(\frac{-D/f_{N_1,N_2}^2}{p}\right)\right) \prod_{p|N_2} \frac{2pf_p - p - 1 - \left(\frac{-D/f_{N_1,N_2}^2}{p}\right)(2f_p - p - 1)}{p - 1}.$$

If $p^{2u+1} \parallel N_1$, we have

$$d_{G_{4N_1N_2,16(N_1N_2)^2,N_1'},q}(n) = d_{G_{4N_1N_2,16(N_1N_2/p^{2u})^2,N_1'},q}(n)$$

for all prime $q \neq p$, and

$$\frac{M(G_{4N_1N_2,16(N_1N_2)^2,N_1'})}{\sqrt{d_{G_{4N,16(N_1N_2)^2,N_1'}}}} = \frac{M(G_{4N_1N_2,16(N_1N_2/p^{2u})^2,N_1'})}{\sqrt{d_{G_{4N,16(N_1N_2/p^{2u})^2,N_1'}}}},$$

hence we only compare $d_{G,p}(n)$, and it is analogous when $p^v \parallel N_2$.
Let $p^{2u+1} \parallel N_1$, where $p \neq 2$. By Proposition 5.4, in the case $v_p(pf^2_{N_1,N_2}) < v_p(N_1N_2)$ and $p \mid D/f^2_{N_1N_2}$, we have

$$d_{-\epsilon x^2+p^{2u+1}y^2-\epsilon p^{2u+1}z^2,p}(D) = 0.$$

In the case $v_p(pf^2_{N_1,N_2}) < v_p(N_1N_2)$ and $p \nmid D/f^2_{N_1N_2}$, we have

$$d_{-\epsilon x^2+p^{2u+1}y^2-\epsilon p^{2u+1}z^2,p}(D) = f_p^2 d_{-\epsilon x^2+py^2-\epsilon pz^2,p}(D).$$

In the case $v_p(pf^2_{N_1N_2}) \geq v_p(N_1N_2)$, we have

$$d_{-\epsilon x^2+p^{2u+1}y^2-\epsilon p^{2u+1}z^2,p}(D) = p^{2u} d_{-\epsilon x^2+py^2-\epsilon pz^2,p}(D).$$

It is analogous when $p^v \parallel N_2$. Let $p^{v_0} \parallel N_2$, where $p \neq 2$, and $v_0 \equiv 0 \pmod 2$. By Proposition 5.5, in the case $v_p(pf^2_{N_1,N_2}) < v_p(N_1N_2)$ and $p \mid D/f^2_{N_1N_2}$, we have

$$d_{-x^2-p^{v_0}yz,p}(D) = 0.$$

In the case $v_p(pf^2_{N_1,N_2}) < v_p(N_1N_2)$ and $p \nmid D/f^2_{N_1N_2}$, we have

$$d_{-x^2-p^{v_0}yz,p}(D) = \frac{f_p^2\left(1+\left(\frac{-D/f^2_{N_1N_2}}{p}\right)\right)}{2f_p+\left(\frac{-D/f^2_{N_1N_2}}{p}\right)-1} d_{-x^2-pyz,p}(D).$$

Since

$$2pf_p - p - 1 - \left(\frac{-D/f^2_{N_1N_2}}{p}\right)(2f_p - p - 1) = \left(2f_p + \left(\frac{-D/f^2_{N_1N_2}}{p}\right) - 1\right)\left(p - \left(\frac{-D/f^2_{N_1N_2}}{p}\right)\right),$$

we have

$$\frac{f_p^2\left(1+\left(\frac{-D/f^2_{N_1N_2}}{p}\right)\right)}{2f_p+\left(\frac{-D/f^2_{N_1N_2}}{p}\right)-1} \cdot \frac{\left(2f_p+\left(\frac{-D/f^2_{N_1N_2}}{p}\right)-1\right)\left(p-\left(\frac{-D/f^2_{N_1N_2}}{p}\right)\right)}{p-1} = f_p^2\left(1+\left(\frac{-D/f^2_{N_1N_2}}{p}\right)\right)$$

In the case $v_p(pf^2_{N_1N_2}) \geq v_p(N_1N_2)$ and $\left(\frac{-D/f^2_{N_1N_2}}{p}\right) = 0$, we have

$$d_{-x^2-p^{v_0}yz,p}(D) = \frac{p^{\frac{v_0}{2}} f_p(p+1) - p^{v_0-1}(p+1)}{2pf_p - p - 1} d_{-x^2-pyz,p}(D).$$

We have

$$\frac{2pf_p - p - 1}{p-1} \cdot \frac{p^{\frac{v_0}{2}} f_p(p+1) - p^{v_0-1}(p+1)}{2pf_p - p - 1} = \frac{(p^{\frac{v_0}{2}} f_p - p^{v_0-1})(p+1)}{p-1}.$$

When $\left(\frac{-D/f^2_{N_1N_2}}{p}\right) \neq 0$, we have

$$d_{-x^2-p^{v_0}yz,p}(D) = \frac{p^{\frac{v_0}{2}-1} f_p(p+1) - p^{v_0-1}\left(1-\left(\frac{-D/f^2_{N_1N_2}}{p}\right)\right)}{2f_p+\left(\frac{-D/f^2_{N_1N_2}}{p}\right)-1} d_{-x^2-pyz,p}(D).$$

we have

$$\frac{2pf_p - p - 1 - \left(\frac{-D/f_{N_1N_2}^2}{p}\right)(2f_p - p - 1)}{p-1} \cdot \frac{p^{\frac{v_0}{2}-1}f_p(p+1) - p^{v_0-1}\left(1 - \left(\frac{-D/f_{N_1N_2}^2}{p}\right)\right)}{2f_p + \left(\frac{-D/f_{N_1N_2}^2}{p}\right) - 1}$$

$$= \frac{(p^{\frac{v_0}{2}}f_p - p^{v_0-1})(p+1) - \left(\frac{-D/f_{N_1,N_2}^2}{p}\right)(p^{\frac{v_0}{2}-1}f_p - p^{v_0-1})(p+1)}{p-1}.$$

Let $p^{v_1} \parallel N_2, p \neq 2$, and $v_1 \equiv 1 \pmod 2$. By Proposition 5.5, in the case $v_p(pf_{N_1,N_2}^2) < v_p(N_1N_2)$ and $p \mid D/f_{N_1N_2}^2$, we have

$$d_{-x^2-p^{v_1}yz,p}(D) = 0,$$

In the case $v_p(pf_{N_1,N_2}^2) < v_p(N_1N_2)$ and $p \nmid D/f_{N_1N_2}^2$, we have

$$d_{-x^2-p^{v_1}yz,p}(D) = \frac{f_p^2\left(1 + \left(\frac{-D/f_{N_1N_2}^2}{p}\right)\right)}{2f_p + \left(\frac{-D/f_{N_1N_2}^2}{p}\right) - 1} d_{-x^2-pyz,p}(D),$$

and it is analogous. In the case $v_p(pf_{N_1N_2}^2) \geq v_p(N_1N_2)$ and $\left(\frac{-D/f_{N_1N_2}^2}{p}\right) = 0$, we have

$$d_{-x^2-p^{v_1}yz,p}(D) = \frac{2p^{\frac{v_1+1}{2}}f_p - p^{v_1+1} - p^{v_1}}{2pf_p - p - 1} d_{-x^2-pyz,p}(D).$$

We have

$$\frac{2pf_p - p - 1}{p-1} \cdot \frac{2p^{\frac{v_1+1}{2}}f_p - p^{v_1+1} - p^{v_1}}{2pf_p - p - 1} = \frac{2p^{\frac{v_1+1}{2}}f_p - p^{v_1}(p+1)}{p-1}.$$

When $\left(\frac{-D/f_{N_1N_2}^2}{p}\right) \neq 0$, we have

$$d_{-x^2-p^{v_1}yz,p}(D) = \frac{2p^{\frac{v_1-1}{2}}f_p - p^{v_1-1}\left(1 - \left(\frac{-D/f_{N_1N_2}^2}{p}\right)\right)}{2f_p + \left(\frac{-D/f_{N_1N_2}^2}{p}\right) - 1} d_{-x^2-pyz,p}(D).$$

Hence

$$\frac{2pf_p - p - 1 - \left(\frac{-D/f_{N_1N_2}^2}{p}\right)(2f_p - p - 1)}{p-1} \cdot \frac{2p^{\frac{v_1-1}{2}}f_p - p^{v_1-1}\left(1 - \left(\frac{-D/f_{N_1N_2}^2}{p}\right)\right)}{2f_p + \left(\frac{-D/f_{N_1N_2}^2}{p}\right) - 1}$$

$$= \frac{2p^{\frac{v_1+1}{2}}f_p - p^{v_1-1}(p+1) - \left(\frac{-D/f_{N_1,N_2}^2}{p}\right)(2p^{\frac{v_1-1}{2}}f_p - p^{v_1-1}(p+1))}{p-1}.$$

The rest of proof for $p = 2$ is analogous by Remark 5.11 and Remark 5.13.

7. Proof of type number formula

We will give the proof of the formula for type number. By (3.1), (3.2), (3.3), Corollary 4.11, Corollary 4.21 and Proposition 4.23, we have

$$\begin{aligned}
T_{N_1,N_2} &= 2^{-e(N_1N_2)} \sum_{\mu=1}^{T_{N_1,N_2}} 2^{e(N_1N_2)} \\
&= 2^{-e(N_1N_2)} \sum_{\mu=1}^{T_{N_1,N_2}} m(\mathcal{O}_\mu)\mathrm{card}(\mathfrak{B}(\mathcal{O}_\mu)/\mathbb{Q}^\times) \\
&= 2^{-e(N_1N_2)} \sum_{\mu=1}^{T_{N_1,N_2}} m(\mathcal{O}_\mu) \sum_{n\|N_1N_2} \sideset{}{'}\sum_{\substack{n|r\\ r^2\le 4n}} \frac{\rho_{\mathcal{O}}(n,r)}{\mathrm{card}(\mathcal{O}_\mu^\times)} \\
&= 2^{-1} \sum_{\substack{n\|N_1N_2\\ n\le 3}} \sum_{\substack{n|r\\ r^2\le 4n}} \sum_{\mu=1}^{T_{N_1,N_2}} \frac{\rho_{\mathcal{O}}(n,r)}{\mathrm{card}(\mathrm{Aut}(\mathcal{O}_\mu))} + 2^{-e(N_1N_2)} \sum_{\mu=1}^{T_{N_1,N_2}} m(\mathcal{O}_\mu) \sum_{\substack{n\|N_1N_2\\ n\ge 4}} \sideset{}{'}\sum_{\substack{n|r\\ r^2\le 4n}} \frac{\rho_{\mathcal{O}}(n,r)}{\mathrm{card}(\mathcal{O}_\mu^\times)} \\
&= 2^{-e(N_1N_2)-1} \sum_{\substack{n\|N_1N_2\\ n\le 3}} \sum_{\substack{n|r\\ r^2\le 4n}} H^{(N_1,N_2)}(4n-r^2) + \sideset{}{'}\sum_{\substack{n\|N_1N_2\\ n\ge 4}} \sum_{\mu=1}^{T_{N_1,N_2}} \frac{R_{f_{S_\mu^0}}(4n)}{|\mathrm{Aut}(f_{S_\mu^0})|}
\end{aligned}$$

here when $n \ge 4$, if $\mathrm{n}(x) \mid \mathrm{tr}(x)$ and $4\mathrm{n}(x) \ge \mathrm{tr}(x)^2$, it necessitates that $\mathrm{tr}(x) = 0$. The remaining task is to compute

$$\sideset{}{'}\sum_{\substack{n\|N_1N_2\\ n\ge 4}} \sum_{\mu=1}^{T_{N_1,N_2}} \frac{R_{f_{S_\mu^0}}(n)}{|\mathrm{Aut}(f_{S_\mu^0})|}.$$

In order to compute this sum, the following two lemmas are required.

Lemma 7.1. *Let $\mathcal{O}$ be an order of level (N_1, N_2), $n \parallel N_1N_2$, where $n \ge 4$. Here $n = p_1^{l_1}...p_m^{l_m}$ and the p_i are distinct primes. Denote*

$$R_{\mathcal{O}}(n) = \{x \in \mathcal{O} | x \in N(\mathcal{O}), n(x) = n, n(x) \mid tr(x)\}.$$

Then we have

$$card(R_{\mathcal{O}}(n)) = R_{\phi_{p_1}\circ...\circ\phi_{p_m}(f_{S^0})}(4^{\delta_{v_2(n),0}}),$$

where $\delta_{v_2(n),0} = 1$ if $v_2(n) = 0$ and otherwise $\delta_{v_2(n),0} = 0$.

Proof. For $n \ge 4$, if $\mathrm{n}(x) \mid \mathrm{tr}(x)$ and $4\mathrm{n}(x) \ge \mathrm{tr}(x)^2$, it necessitates that $\mathrm{tr}(x) = 0$. Let $p^{2u+1} \parallel N_1$, and $p^{2u+1} \parallel n$, where $p \ne 2$. By Proposition 4.9, we have

$$\mathcal{O} = \mathbb{Z}e_0' + \mathbb{Z}e_1' + \mathbb{Z}e_2' + \mathbb{Z}e_3',$$

where $\mathrm{tr}(e_1') = \mathrm{tr}(e_2') = \mathrm{tr}(e_3') = 0$ and

$$\mathrm{n}(e_1') = p^{2u+1}a, \mathrm{tr}(e_2'\overline{e_3'}) = p^{2u+1}r,$$

$$\mathrm{n}(e_2') = p^{2u+1}b, \mathrm{tr}(e_3'\overline{e_1'}) = p^{2u+1}s,$$

$$\mathrm{n}(e_3') = c, \mathrm{tr}(e_1'\overline{e_2'}) = p^{2u+1}t,$$

here $p \nmid ac$, $2 \nmid c$. It is not to check

$$S = \mathbb{Z} + \mathbb{Z}2e_1' + \mathbb{Z}2e_2' + \mathbb{Z}e_3',$$

$$f_{S^0}(x,y,z) = p^{2u+1}ax^2 + p^{2u+1}y^2 + cz^2 + p^{2u+1}ryz + p^{2u+1}sxz + p^{2u+1}txy,$$

$$\phi_p(f_{S^0})(x,y,z) = ax^2 + by^2 + p^{2u+1}cz^2 + ryz + p^{2u+1}sxz + p^{2u+1}txy.$$

For simplicity, here e_3' is equivalent to e_1' in Proposition 4.9, and e_1' (resp.e_2') is equivalent to $p^g e_3'$ (resp. $p^g e_2'$) in Proposition 4.9. Consider that

$$\mathcal{O}_p = \mathbb{Z}_p e_0' + \mathbb{Z}_p e_1' + \mathbb{Z}_p e_2' + \mathbb{Z}_p e_3' \simeq \{\begin{pmatrix} \alpha & p^u\beta \\ p^{u+1}\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in R_p\}.$$

Since $\mathrm{n}(\begin{pmatrix} \alpha & p^u\beta \\ p^{u+1}\bar{\beta} & \bar{\alpha} \end{pmatrix}) = \mathrm{n}(\alpha) - p^{2u+1}\mathrm{n}(\beta)$ and $p \nmid \mathrm{n}(e_3')$, we have $p \nmid c_{11}(e_3')$. Since $p^{2u+1} \mid \mathrm{tr}(e_2'\bar{e_3'}), p^{2u+1} \mid \mathrm{tr}(e_3'\bar{e_1'})$, and $\mathrm{tr}(e_1') = \mathrm{tr}(e_2') = 0$, we have $p^{2u+1} \mid c_{11}(e_1')$ and $p^{2u+1} \mid c_{11}(e_2')$.

By Proposition 2.7, we have

$$\mathrm{card}(\{(x,y,z) \in \mathbb{Z}^3 | f_{S^0}(x,y,z) = 4n, p^{2u+1} \mid z\}) = R_{\phi_p(f_{S^0})}(4n/p^{2u+1}).$$

We will show

$$\mathrm{card}(\{(x,y,z) \in \mathbb{Z}^3 | f_{S^0}(x,y,z) = 4n, p^{2u+1} \mid z\}) = \mathrm{card}(\{x \in \mathcal{O} | x \in N(\mathcal{O}_p), \mathrm{n}(x) = n, \mathrm{n}(x) \mid \mathrm{tr}(x)\}).$$

Suppose there exists (x,y,z) such that $f_{S^0}(x,y,z) = 4n$ and $p^{2u+1} \mid z$, it is not hard to check $2 \mid z$, $xe_1' + ye_2' + (z/2)e_3' \in \mathcal{O}$, $\mathrm{tr}(xe_1' + ye_2' + (z/2)e_3') = 0$, and $\mathrm{n}(xe_1' + ye_2' + (z/2)e_3') = n$. Since $p^{2u+1} \mid z$ implies $p^{2u+1} \mid c_{11}(xe_1' + ye_2' + (z/2)e_3')$, we have $xe_1' + ye_2' + (z/2)e_3' \in N(\mathcal{O}_p)$.

Conversely, if $xe_1' + ye_2' + ze_3' \in N(\mathcal{O}_p), \mathrm{n}(xe_1' + ye_2' + ze_3') = n, \mathrm{tr}(xe_1' + ye_2' + ze_3') = 0$, and $p^{2u+1} \mid c_{11}(xe_1' + ye_2' + ze_3')$, we have $p^{2u+1} \mid c_{11}(ze_3')$. Since $p \nmid c_{11}(e_3')$, it implies $p^{2u+1} \mid z$. Hence $\mathrm{n}(2xe_1' + 2ye_2' + 2ze_3') = 4n$, i.e. there exists $(2x, 2y, z)$ such that $f_{S^0}(2x, 2y, z) = 4n$ and $p^{2u+1} \mid z$.

Similarly, when $p^v \parallel N_2$ and $p^v \parallel n$, where $p \neq 2$, we have

$$\mathcal{O}_p = \mathbb{Z}_p e_0' + \mathbb{Z}_p e_1' + \mathbb{Z}_p e_2' + \mathbb{Z}_p e_3' \simeq \{\begin{pmatrix} a & b \\ p^v c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}_p\}.$$

Since $\mathrm{n}(\begin{pmatrix} a & b \\ p^v c & d \end{pmatrix}) = ad - p^v bc$, and $p \nmid \mathrm{n}(e_3')$, we have $p \nmid c_{11}(e_3')$. Since $p^v \mid \mathrm{tr}(e_2'\bar{e_3'})$, $p^v \mid \mathrm{tr}(e_3'\bar{e_1'})$, and $\mathrm{tr}(e_1') = \mathrm{tr}(e_2') = 0$, we have $p^v \mid c_{11}(e_1')$ and $p^v \mid c_{11}(e_2')$. The rest is analogous.

When $p = 2$, it is analogous. Let $2^{2u+1} \parallel N_1$ and $2^{2u+1} \parallel n$, we have $2^{2u+3} \parallel N_{f_{S^0}}$, $2^{4u+6} \parallel N_{f_{S^0}}$ and

$$\mathcal{O} = \mathbb{Z}e_0' + \mathbb{Z}e_1' + \mathbb{Z}e_2' + \mathbb{Z}e_3',$$

$$S = \mathbb{Z} + \mathbb{Z}2e_1' + \mathbb{Z}2e_2' + \mathbb{Z}e_3',$$

$$f_{S^0}(x,y,z) = 2^{2u+3}ax^2 + 2^{2u+3}by^2 + cz^2 + 2^{2u+3}ryz + 2^{2u+3}sxz + 2^{2u+3}txy,$$

$$\phi_2(f_{S^0})(x,y,z) = ax^2 + by^2 + 2^{2u+1}cz^2 + ryz + 2^{2u+2}sxz + 2^{2u+2}txy,$$

where $2 \nmid a, c$, $\mathrm{tr}(e_1') = \mathrm{tr}(e_2') = \mathrm{tr}(e_3') = 0$, and

$$\mathrm{n}(e_1') = 2^{2u+1}a, \mathrm{tr}(e_2'\overline{e_3'}) = 2^{2u+2}r,$$

$$\mathrm{n}(e_2') = 2^{2u+1}b, \mathrm{tr}(e_3'\overline{e_1'}) = 2^{2u+2}s,$$

$$\mathrm{n}(e_3') = c, \mathrm{tr}(e_1'\overline{e_2'}) = 2^{2u+1}t.$$

Consider that
$$\mathcal{O}_2 = \mathbb{Z}_2 e_0' + \mathbb{Z}_2 e_1' + \mathbb{Z}_2 e_2' + \mathbb{Z}_2 e_3' \simeq \{\begin{pmatrix} \alpha & 2^u\beta \\ 2^{u+1}\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in R_p\}.$$
Since $\mathrm{n}(\begin{pmatrix} \alpha & 2^u\beta \\ 2^{u+1}\bar{\beta} & \bar{\alpha} \end{pmatrix}) = \mathrm{n}(\alpha) - 2^{2u+1}\mathrm{n}(\beta)$, $2 \nmid \mathrm{n}(e_3')$ and $\mathrm{tr}(e_3') = 0$, we have $2 \nmid c_{11}(e_3')$. Since $2^{2u+2} \mid \mathrm{tr}(e_2'\bar{e_3'}), 2^{2u+2} \mid \mathrm{tr}(e_3'\bar{e_1'})$, and $\mathrm{tr}(e_1') = \mathrm{tr}(e_2') = 0$, we have $2^{2u+1} \mid c_{11}(e_1')$ and $2^{2u+1} \mid c_{11}(e_2')$.

By Proposition 2.9, we have
$$\mathrm{card}(\{(x, y, z) \in \mathbb{Z}^3 | f_{S^0}(x, y, z) = 4n, 2^{2u+2} \mid z\}) = R_{\phi_2(f_{S^0})}(n/2^{2u+1}).$$
We will prove
$$\mathrm{card}(\{(x, y, z) \in \mathbb{Z}^3 | f_{S^0}(x, y, z) = 4n, 2^{2u+2} \mid z\}) = \mathrm{card}(\{x \in \mathcal{O} | x \in N(\mathcal{O}_2), \mathrm{n}(x) = n, \mathrm{n}(x) \mid \mathrm{tr}(x)\}).$$
Suppose there exists (x, y, z) such that $f_{S^0}(x, y, z) = 4n$ and $2^{2u+2} \mid z$, it is not hard to check $xe_1' + ye_2' + ze_3'/2 \in \mathcal{O}$ and $\mathrm{tr}(xe_1' + ye_2' + ze_3'/2) = 0, \mathrm{n}(xe_1' + ye_2' + ze_3'/2) = \mathrm{n}(2xe_1' + 2ye_2' + ze_3')/4 = n$. Since $2^{2u+1} \mid \frac{z}{2}$, we have $2^{2u+1} \mid c_{11}(xe_1' + ye_2' + ze_3'/2)$, i.e. $xe_1' + ye_2' + ze_3'/2 \in N(\mathcal{O}_p)$.

Conversely, if $xe_1' + ye_2' + ze_3' \in N(\mathcal{O}_2), \mathrm{n}(xe_1' + ye_2' + ze_3') = n, \mathrm{tr}(xe_1' + ye_2' + ze_3') = 0$, and $2^{2u+1} \mid c_{11}(xe_1' + ye_2' + ze_3')$, we have $2^{2u+1} \mid c_{11}(ze_3')$. Since $2 \nmid c_{11}(e_3')$, it implies $2^{2u+1} \mid z$. Hence $\mathrm{n}(2xe_1' + 2ye_2' + 2ze_3') = 4\mathrm{n}(xe_1' + ye_2' + ze_3') = 4n$, i.e. there exists $(x, y, 2z)$ such that $f_{S^0}(x, y, 2z) = 4n$ and $2^{2u+2} \mid (2z)$.

Similarly, when $2^v \parallel N_2$ and $2^v \parallel n$, we have
$$\mathcal{O}_p = \mathbb{Z}_2 e_0' + \mathbb{Z}_2 e_1' + \mathbb{Z}_2 e_2' + \mathbb{Z}_2 e_3' \simeq \{\begin{pmatrix} a & b \\ 2^v c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}_2\}.$$
Since $\mathrm{n}(\begin{pmatrix} a & b \\ 2^v c & d \end{pmatrix}) = ad - 2^v bc$, $2 \nmid \mathrm{n}(e_3')$, and $\mathrm{tr}(e_3') = 0$, we have $2 \nmid c_{11}(e_3')$. Since $2^{v+1} \mid \mathrm{tr}(e_2'\bar{e_3'})$, $2^{v+1} \mid \mathrm{tr}(e_3'\bar{e_1'})$, and $\mathrm{tr}(e_1') = \mathrm{tr}(e_2') = 0$, we have $2^v \mid c_{11}(e_1')$ and $2^v \mid c_{11}(e_2')$. The rest is analogous.

By induction we have
$$R_{\phi_{p_1} \circ \ldots \circ \phi_{p_w}(f_{S^0})}(4^{\delta_{v_2(n),0}}) = \{x \in \mathcal{O} | x \in N(\mathcal{O}_{p_1}), ..., x \in N(\mathcal{O}_{p_w}), \mathrm{n}(x) = n, \mathrm{n}(x) \mid \mathrm{tr}(x)\}.$$
It is clear
$$\mathrm{card}(\{x \in \mathcal{O} | x \in N(\mathcal{O}_{p_1}), ..., x \in N(\mathcal{O}_{p_w}), \mathrm{n}(x) = n, \mathrm{n}(x) \mid \mathrm{tr}(x)\}) = R_{\mathcal{O}}(n).$$
□

Lemma 7.2. *Let $\mathcal{O}$ be an order of level (N_1, N_2), $n \parallel N_1N_2$, where $n \geq 4$. Here $n = p_1^{l_1} ... p_m^{l_m}$ and the p_i are distinct primes, we have*
$$\sum_{\mu=1}^{T_{N_1,N_2}} \frac{R_{\phi_{p_1} \circ \ldots \circ \phi_{p_m}(f_{S_\mu^0})}(4^{\delta v_2(n),0)})}{|\operatorname{Aut}(\phi_{p_1} \circ \ldots \circ \phi_{p_m}(f_{S_\mu^0}))|} = 2^{-e(N_1N_2)-1} H^{(N_1,N_2)}(4n) \prod_{p|n} \frac{1 - \left(\frac{\Delta(-4n)}{p}\right)/p}{B_p(n)C_p(n)}.$$

Proof. Recalling Siegel-Weil formula:
$$\sum_{f \in G} \frac{R_f(n)}{|\operatorname{Aut}(f)|} = 4\pi M(G) \sqrt{\frac{n}{d}} \prod_p d_{G,p}(n).$$
According to Corollary 2.6, it can be verified that when $2 \nmid n$, $M(G)\sqrt{\frac{n}{d}}$ remains constant. Consequently, our focus is solely on local representation densities. In the case $p^{2u+1} \parallel n$, where $p \neq 2$, by Proposition 5.14 and Proposition 5.4 we have
$$d_{-\epsilon x^2 + p^{2u+1}y^2 - \epsilon p^{2u+1}z^2, p}(4n) = p^u(1 + \frac{1}{p})$$

and

$$d_{-\epsilon p^{2u+1}x^2+y^2-\epsilon z^2,p}(4^l) = 1 + \frac{1}{p},$$

where $l = 0, 1$. Hence

$$d_{-\epsilon p^{2u+1}x^2+y^2-\epsilon z^2,p}(4^l) = p^{-u} d_{-\epsilon x^2+p^{2u+1}y^2-\epsilon p^{2u+1}z^2,p}(4n).$$

In the case $p^{v_0} \parallel n$, where $p \neq 2$. Here $n = p^{v_0}m$. By Proposition 5.14 and Proposition 5.5 we have

$$d_{-x^2-p^{v_0}yz,p}(4n) = p^{v_0/2}\left(1 + \left(\frac{-m}{p}\right)/p\right)$$

and

$$d_{-p^{v_0}x^2-yz,p}(4^l) = 1 - \frac{1}{p}.$$

Hence

$$d_{-p^{v_0}x^2-yz,p}(4^l) = \frac{1-\left(\frac{-m}{p}\right)/p}{p^{v_0/2-1}(p+1)} d_{-x^2-p^{v_0}yz,p}(4p^{v_0}m)$$

In the case $p^{v_1} \parallel n$, where $p \neq 2$. Here $n = p^{v_1}m$. By Proposition 5.14 and Proposition 5.5 we have

$$d_{-x^2-p^{v_1}yz,p}(4p^{v_1}m) = p^{(v_1-1)/2}(1 - \frac{1}{p})$$

and

$$d_{-p^{v_1}x^2-yz,p}(4^l) = 1 - \frac{1}{p}.$$

Hence

$$d_{-p^{v_0}x^2-yz,p}(4^l) = p^{-(v_1-1)/2} d_{-x^2-p^{v_1}yz,p}(4p^{v_1}m).$$

In the case $2^{2u+1} \parallel n$, here $4n = 4^{u+1} \cdot 2m$. By Proposition 5.15 and Proposition 5.10 we have

$$d_{3x^2-2^{2u+3}(y^2+z^2+yz),2}(4^{u+1} \cdot 2m) = 3 \cdot 2^u$$

and

$$d_{3\cdot 2^{2u+3}x^2-(y^2+z^2+yz),2}(1) = \frac{3}{2}.$$

Since $2^{2u+3} \parallel 4n$, $2^{4u+6} \parallel d_f$ and $2^{2u+1} \parallel d_{\phi_n(f)}$, we have $\sqrt{\frac{1}{d_{\phi_n(f)}}} = 2\sqrt{\frac{4n}{d_f}}$. Hence

$$2d_{3\cdot 2^{2u+3}x^2-(y^2+z^2+yz),2}(1) = 2^{-u} d_{3x^2-2^{2u+3}(y^2+z^2+yz),2}(4n).$$

In the case $2^{v_0} \parallel n$, here $4n = 4^{v_0/2+1}m$. By Proposition 5.15 and Proposition 5.12 we have

$$d_{-x^2-2^{v_0+2}yz,2}(4^{v_0/2+1}m) = \begin{cases} 3 \cdot 2^{v_0/2} & \text{if} \quad m \equiv 7 \pmod 8, \\ 2^{v_0/2+1} & \text{if} \quad m \equiv 3 \pmod 8, \\ 3 \cdot 2^{v_0/2-1} & \text{if} \quad m \equiv 1, 2 \pmod 4, \end{cases}$$

and

$$d_{-2^{v_0+2}x^2-yz,2}(1) = \frac{1}{2}.$$

Since $2^{v_0+2} \parallel 4n$, $2^{2v_0+4} \parallel d_f$, and $2^{v_0} \parallel d_{\phi_n(f)}$, we have $\sqrt{\frac{1}{d_{\phi_n(f)}}} = 2\sqrt{\frac{4n}{d_f}}$. Hence

$$2d_{-2^{v_0+2}x^2-yz,2}(1) = \begin{cases} \frac{1}{3} \cdot 2^{-v_0/2} d_{-x^2-2^{v_0+2}yz,2}(4^{v_0/2+1}m) & \text{if} \quad m \equiv 7 \pmod 8, \\ 2^{-v_0/2-1} d_{-x^2-2^{v_0+2}yz,2}(4^{v_0/2+1}m) & \text{if} \quad m \equiv 3 \pmod 8, \\ \frac{1}{3} \cdot 2^{-v_0/2+1} d_{-x^2-2^{v_0+2}yz,2}(4^{v_0/2+1}m) & \text{if} \quad m \equiv 1, 2 \pmod 4. \end{cases}$$

In the case $2^{v_1} \parallel n$, here $4n = 4^{(v_1+1)/2} \cdot 2m$. By Proposition 5.15 and Proposition 5.12 we have

$$d_{-x^2-2^{v_0+2}yz,2}(4^{(v_1+1)/2} \cdot 2m) = 2^{(v_1-1)/2}$$

and

$$d_{-2^{v_0+2}x^2-yz,2}(1) = \frac{1}{2}.$$

Since $2^{v_1+2} \parallel 4n$, $2^{2v_1+4} \parallel d_f$, and $2^{v_1} \parallel d_{\phi_n(f)}$, we have $\sqrt{\frac{1}{d_{\phi_n(f)}}} = 2\sqrt{\frac{4n}{d_f}}$. Hence

$$2d_{-2^{v_0+2}x^2-yz,2}(1) = 2^{-(v_1-1)/2} d_{-x^2-2^{v_0+2}yz,2}(4^{(v_1+1)/2} \cdot 2m).$$

If $p^{2u+1} \parallel n$ or $p^{v_1} \parallel n$, it becomes evident to verify condition

$$1 - \left(\frac{\Delta(-4n)}{p}\right)/p = 1.$$

When $2^{v_0} \parallel n$, it is clear to check

$$2d_{-2^{v_0+2}x^2-yz,2}(1) = \begin{cases} \frac{1-\left(\frac{-m}{2}\right)/2}{2^{v_0/2-1}(2+1)} d_{-x^2-2^{v_0+2}yz,2}(4^{v_0/2+1}m) & \text{if} \quad m \equiv 7 \pmod 8, \\ \frac{1-\left(\frac{-m}{2}\right)/2}{2^{v_0/2-1}(2+1)\cdot 2} d_{-x^2-2^{v_0+2}yz,2}(4^{v_0/2+1}m) & \text{if} \quad m \equiv 3 \pmod 8, \\ \frac{1-\left(\frac{-4m}{2}\right)/2}{2^{v_0/2-1}(2+1)} d_{-x^2-2^{v_0+2}yz,2}(4^{v_0/2+1}m) & \text{if} \quad m \equiv 1, 2 \pmod 4. \end{cases}$$

That is

$$2d_{-2^{v_0+2}x^2-yz,2}(1) = \begin{cases} \frac{1-\left(\frac{-m}{2}\right)/2}{2^{v_0/2-1}(2+1)C_2(2)} d_{-x^2-2^{v_0+2}yz,2}(4^{v_0/2+1}m) & \text{if} \quad m \equiv 3 \pmod 4, \\ \frac{1-\left(\frac{-4m}{2}\right)/2}{2^{v_0/2-1}(2+1)C_2(2)} d_{-x^2-2^{v_0+2}yz,2}(4^{v_0/2+1}m) & \text{if} \quad m \equiv 1, 2 \pmod 4. \end{cases}$$

□

We will give the complete proof of the formula for type number. By (3.1), (3.2), (3.3), Corollary 4.11, Corollary 4.21, Proposition 4.23, Lemma 7.1 and Lemma 7.2, we have

$$\begin{aligned} T_{N_1,N_2} &= 2^{-e(N_1N_2)} \sum_{\mu=1}^{T_{N_1,N_2}} 2^{e(N_1N_2)} \\ &= 2^{-e(N_1N_2)} \sum_{\mu=1}^{T_{N_1,N_2}} m(\mathcal{O}_\mu) \mathrm{card}(\mathfrak{B}(\mathcal{O}_\mu)/\mathbb{Q}^\times) \\ &= 2^{-e(N_1N_2)} \sum_{\mu=1}^{T_{N_1,N_2}} m(\mathcal{O}_\mu) \sum_{n \parallel N_1N_2} \sideset{}{'}\sum_{\substack{n|r \\ r^2 \le 4n}} \frac{\rho_{\mathcal{O}}(n,r)}{\mathrm{card}(\mathcal{O}_\mu^\times)} \\ &= 2^{-1} \sum_{\substack{n \parallel N_1N_2 \\ n \le 3}} \sum_{\substack{n|r \\ r^2 \le 4n}} \sum_{\mu=1}^{T_{N_1,N_2}} \frac{\rho_{\mathcal{O}}(n,r)}{\mathrm{card}(\mathrm{Aut}(\mathcal{O}_\mu))} + 2^{-e(N_1N_2)} \sum_{\mu=1}^{T_{N_1,N_2}} m(\mathcal{O}_\mu) \sum_{\substack{n \parallel N_1N_2 \\ n \ge 4}} \sideset{}{'}\sum_{\substack{n|r \\ r^2 \le 4n}} \frac{\rho_{\mathcal{O}}(n,r)}{\mathrm{card}(\mathcal{O}_\mu^\times)} \\ &= 2^{-e(N_1N_2)-1} \sum_{\substack{n \parallel N_1N_2 \\ n \le 3}} \sum_{\substack{n|r \\ r^2 \le 4n}} H^{(N_1,N_2)}(4n - r^2) + \sideset{}{'}\sum_{\substack{n \parallel N_1N_2 \\ n \ge 4}} \sum_{\mu=1}^{T_{N_1,N_2}} \frac{R_{f_{S_\mu^0}}(4n)}{|\mathrm{Aut}(f_{S_\mu^0})|} \end{aligned}$$

$$= 2^{-e(N_1N_2)-1} \sum_{\substack{n \| N_1N_2 \\ n \leq 3}} \sum_{\substack{n|r \\ r^2 \leq 4n}} H^{(N_1,N_2)}(4n - r^2) + \sum_{\substack{n \| N_1N_2 \\ n \geq 4}} \sum_{\mu=1}^{T_{N_1,N_2}} \frac{R_{\phi_n(f_{S_\mu^0})}(4^{\delta_{v_2(n),0}})}{|\operatorname{Aut}(\phi_n(f_{S_\mu^0}))|}$$

$$= 2^{-e(N_1N_2)-1} \sum_{\substack{n \| N_1N_2 \\ n \leq 3}} \sum_{\substack{n|r \\ r^2 \leq 4n}} H^{(N_1,N_2)}(4n - r^2) + 2^{-e(N_1N_2)-1} \sum_{\substack{n \| N_1N_2 \\ n \geq 4}} H^{(N_1,N_2)}(4n) \prod_{p|n} \frac{1 - \left(\frac{\Delta(-4n)}{p}\right)/p}{B_p(n)C_p(n)}$$

$$= 2^{-e(N_1N_2)-1} \sum_{n \| N_1N_2} \sum_{\substack{n|r \\ r^2 \leq 4n}} H^{(N_1,N_2)}(4n - r^2) \prod_{p|n} \frac{1 - \left(\frac{\Delta(-4n)}{p}\right)/p}{B_p(n)C_p(n)}.$$

Appendix A. h_{N_1,N_2} and T_{N_1,N_2} for orders of level N_1N_2

The following results were calculated by SageMath employing the type number formula. We correct errors ($T_{3^3,5}, T_{5^3,8}, T_{3^7,1}, T_{13^3,2^4}$) in Boyd's table of type numbers[2, p.152].

Table 1: h_{N_1,N_2} and T_{N_1,N_2} for orders of level (N_1, N_2) where $N_1N_2 \leq 100$

Level	N_1	N_2	h_{N_1,N_2}	T_{N_1,N_2}	Level	N_1	N_2	h_{N_1,N_2}	T_{N_1,N_2}
2	2	1	1	1	56	7	8	6	4
3	3	1	1	1	56	8	7	4	2
5	5	1	1	1	57	3	19	4	3
6	2	3	1	1	57	19	3	6	2
6	3	2	1	1	58	2	29	3	2
7	7	1	1	1	58	29	2	7	3
8	8	1	1	1	59	59	1	6	6
10	2	5	1	1	60	3	20	6	2
10	5	2	1	1	60	5	12	8	3
11	11	1	2	2	61	61	1	5	4
12	3	4	1	1	62	2	31	4	2
13	13	1	1	1	62	31	2	8	5
14	2	7	2	1	63	7	9	6	3
14	7	2	2	2	65	5	13	6	3
15	3	5	2	1	65	13	5	6	3
15	5	3	2	2	66	2	33	4	2
17	17	1	2	2	66	3	22	6	2
18	2	9	1	1	66	11	6	10	3
19	19	1	2	2	66	66	1	4	2
20	5	4	2	1	67	67	1	6	4
21	3	7	2	2	68	17	4	8	3
21	7	3	2	1	69	3	23	4	2
22	2	11	1	1	69	23	3	8	5
22	11	2	3	2	70	2	35	4	2
23	23	1	3	3	70	5	14	8	3
24	3	8	2	1	70	7	10	10	3
24	8	3	2	2	70	70	1	2	1
26	2	13	3	2	71	71	1	7	7
26	13	2	3	2	72	8	9	4	2
27	27	1	2	2	73	73	1	6	4
28	7	4	3	2	74	2	37	5	3
29	29	1	3	3	74	37	2	9	4
30	2	15	2	1	75	3	25	6	3
30	3	10	4	2	76	19	4	9	4
30	5	6	4	2	77	7	11	6	4
30	30	1	2	1	77	11	7	8	3
31	31	1	3	3	78	2	39	6	2
32	32	1	2	2	78	3	26	8	3
33	3	11	2	1	78	13	6	12	4

33	11	3	4	3	78	78	1	2	1
34	2	17	2	2	79	79	1	7	6
34	17	2	4	2	80	5	16	8	3
35	5	7	4	3	82	2	41	4	3
35	7	5	4	2	82	41	2	10	4
37	37	1	3	2	83	83	1	8	7
38	2	19	3	2	84	3	28	8	3
38	19	2	5	3	84	7	12	12	2
39	3	13	4	3	85	5	17	6	3
39	13	3	4	2	85	17	5	8	3
40	5	8	4	2	86	2	43	5	3
40	8	5	2	1	86	43	2	11	5
41	41	1	4	4	87	3	29	6	3
42	2	21	4	2	87	29	3	10	6
42	3	14	4	2	88	8	11	4	3
42	7	6	6	2	88	11	8	10	3
42	42	1	2	1	89	89	1	8	7
43	43	1	4	3	90	2	45	6	2
44	11	4	5	3	90	5	18	12	3
45	5	9	4	2	91	7	13	8	3
46	2	23	2	1	91	13	7	8	4
46	23	2	6	4	92	23	4	11	6
47	47	1	5	5	93	3	31	6	4
48	3	16	4	2	93	31	3	10	3
50	2	25	3	2	94	2	47	4	2
51	3	17	4	2	94	47	2	12	7
51	17	3	6	4	95	5	19	8	4
52	13	4	6	2	95	19	5	10	6
53	53	1	5	4	96	3	32	8	3
54	2	27	3	2	96	32	3	6	4
54	27	2	5	3	97	97	1	8	5
55	5	11	4	2	98	2	49	6	3
55	11	5	6	4	99	11	9	10	5

Table 2: h_{N_1,N_2} and T_{N_1,N_2} for orders of level $p^{2r+1}N_2$

Level	N_1	N_2	h_{N_1,N_2}	T_{N_1,N_2}
5	5	1	1	1
15	3	5	2	1
27	3^3	1	2	2
35	5	7	4	3
125	5^3	1	9	7
135	3^3	5	10	4
189	3^3	7	12	6
243	3^5	1	14	10
250	5^3	2	25	9
343	7^3	1	25	16
405	5	3^4	36	11
750	5^3	6	100	18
972	3^5	4	81	25
1000	5^3	8	100	28
1331	11^3	1	102	54
2187	3^7	1	122	70
3125	5^5	1	209	117
4116	7^3	12	588	77
16807	7^5	1	1201	625
35152	13^3	2^4	4056	1027
78125	5^7	1	5209	2667
322102	11^5	2	36603	9272
823543	7^7	1	58825	29584

Appendix B. Representation numbers of ternary quadratic forms

If the class number of a genus is one, an exact formula for the representation number of n by ternary quadratic forms can be provided. The type number is 1 if its level (N_1, N_2) belongs to the following set: (2, 1), (2, 3), (2, 5), (2, 7), (2, 9), (2, 11), (2, 15), (2, 23), (3, 1), (3, 2), (3, 4), (3, 5), (3, 8), (3, 11), (5, 1), (5, 2), (5, 4), (7, 1), (7, 3), (8, 1), (8, 5), (13, 1), (30, 1), (42, 1), (70, 1), (78, 1). Herein, we provide the explicit formulas for the representation numbers of 157 ternary quadratic forms.

Table 3: Genera with one class

Genus	N_f	d_f	$R_f(n)$
$G_{8,64,2}$	$4\cdot 2$	$16\cdot 2^2$	$R_{(3,3,3,-2,-2,-2)}(n) = 12H^{(2,1)}(n)$
$G_{8,2,2}$	$4\cdot 2$	2	$R_{(1,1,1,1,1,1)}(n) = 12H^{(2,1)}(8n)$
$G_{4,4,2}$	$2\cdot 2$	2^2	$R_{(1,1,1,0,0,0)}(n) = 12H^{(2,1)}(4n)$
$G_{24,576,2}$	$4\cdot 2\cdot 3$	$16\cdot (2\cdot 3)^2$	$R_{(3,8,8,-8,0,0)}(n) = 3H^{(2,3)}(n)$
$G_{24,192,2}$	$4\cdot 2\cdot 3$	$16\cdot 2^2\cdot 3$	$R_{(1,8,8,-8,0,0)}(n) = 3H^{(2,3)}(3n)$
$G_{24,18,2}$	$4\cdot 2\cdot 3$	$2\cdot 3^2$	$R_{(1,1,6,0,0,-1)}(n) = 3H^{(2,3)}(8n)$
$G_{24,6,2}$	$4\cdot 2\cdot 3$	$2\cdot 3$	$R_{(1,1,2,0,0,-1)}(n) = 3H^{(2,3)}(24n)$
$G_{12,36,2}$	$2\cdot 2\cdot 3$	$(2\cdot 3)^2$	$R_{(2,2,3,0,0,-2)}(n) = 3H^{(2,3)}(4n)$
$G_{12,12,2}$	$2\cdot 2\cdot 3$	$2^2\cdot 3$	$R_{(1,2,2,-2,0,0)}(n) = 3H^{(2,3)}(12n)$
$G_{40,1600,2}$	$4\cdot 2\cdot 5$	$16\cdot (2\cdot 5)^2$	$R_{(4,11,11,2,4,4)}(n) = 2H^{(2,5)}(n)$
$G_{40,320,2}$	$4\cdot 2\cdot 5$	$16\cdot 2^2\cdot 5$	$R_{(4,4,7,-4,-4,0)}(n) = 2H^{(2,5)}(5n)$
$G_{40,50,2}$	$4\cdot 2\cdot 5$	$2\cdot 5^2$	$R_{(2,3,3,1,2,2)}(n) = 2H^{(2,5)}(8n)$
$G_{40,10,2}$	$4\cdot 2\cdot 5$	$2\cdot 5$	$R_{(1,1,3,-1,-1,0)}(n) = 2H^{(2,5)}(40n)$
$G_{20,100,2}$	$2\cdot 2\cdot 5$	$(2\cdot 5)^2$	$R_{(1,5,5,0,0,0)}(n) = 2H^{(2,5)}(4n)$
$G_{20,20,2}$	$2\cdot 2\cdot 5$	$2^2\cdot 5$	$R_{(1,1,5,0,0,0)}(n) = 2H^{(2,5)}(20n)$
$G_{56,3136,2}$	$4\cdot 2\cdot 7$	$16\cdot (2\cdot 7)^2$	$2R_{(3,19,19,-18,-2,-2)}(n) = 3H^{(2,7)}(n)$
$G_{56,448,2}$	$4\cdot 2\cdot 7$	$16\cdot 2^2\cdot 7$	$2R_{(5,5,5,2,2,2)}(n) = 3H^{(2,7)}(7n)$
$G_{56,98,2}$	$4\cdot 2\cdot 7$	$2\cdot 7^2$	$2R_{(3,3,3,-1,-1,-1)}(n) = 3H^{(2,7)}(8n)$
$G_{56,14,2}$	$4\cdot 2\cdot 7$	$2\cdot 7$	$2R_{(1,1,5,1,1,1)}(n) = 3H^{(2,7)}(56n)$
$G_{28,196,2}$	$2\cdot 2\cdot 7$	$(2\cdot 7)^2$	$2R_{(3,5,5,-4,-2,-2)}(n) = 3H^{(2,7)}(4n)$
$G_{28,28,2}$	$2\cdot 2\cdot 7$	$2^2\cdot 7$	$2R_{(2,2,3,2,2,2)}(n) = 3H^{(2,7)}(28n)$
$G_{72,5184,2}$	$4\cdot 2\cdot 3^2$	$16\cdot (2\cdot 3^2)^2$	$R_{(8,11,20,4,8,8)}(n) = H^{(2,9)}(n)$
$G_{72,162,2}$	$4\cdot 2\cdot 3^2$	$2\cdot (3^2)^2$	$R_{([1,7,7,5,1,1)}(n) = H^{(2,9)}(8n)$
$G_{36,324,2}$	$2\cdot 2\cdot 3^2$	$(2\cdot 3^2)^2$	$R_{(2,5,9,0,0,-2)}(n) = H^{(2,9)}(4n)$
$G_{88,7744,2}$	$4\cdot 2\cdot 11$	$16\cdot (2\cdot 11)^2$	$R_{(8,11,24,0,-8,0)}(n) = H^{(2,11)}(n)$
$G_{88,704,2}$	$4\cdot 2\cdot 11$	$16\cdot 2^2\cdot 11$	$R_{(1,8,24,-8,0,0)}(n) = H^{(2,11)}(11n)$
$G_{88,242,2}$	$4\cdot 2\cdot 11$	$2\cdot 11^2$	$R_{(1,3,22,0,0,-1)}(n) = H^{(2,11)}(8n)$
$G_{88,22,2}$	$4\cdot 2\cdot 11$	$2\cdot 11$	$R_{(1,2,3,0,-1,0)}(n) = H^{(2,11)}(88n)$
$G_{44,484,2}$	$2\cdot 2\cdot 11$	$(2\cdot 11)^2$	$R_{(2,6,11,0,0,-2)}(n) = H^{(2,11)}(4n)$
$G_{44,44,2}$	$2\cdot 2\cdot 11$	$2^2\cdot 11$	$R_{(1,2,6,-2,0,0)}(n) = H^{(2,11)}(44n)$
$G_{120,14400,2}$	$4\cdot 2\cdot 3\cdot 5$	$16\cdot (2\cdot 3\cdot 5)^2$	$2R_{(11,11,35,-10,-10,2)}(n) = H^{(2,15)}(n)$
$G_{120,4800,2}$	$4\cdot 2\cdot 3\cdot 5$	$16\cdot 2^2\cdot 3\cdot 5^2$	$2R_{(8,12,17,4,8,8)}(n) = H^{(2,15)}(3n)$
$G_{120,2880,2}$	$4\cdot 2\cdot 3\cdot 5$	$16\cdot 2^2\cdot 3^2\cdot 5$	$2R_{(4,7,31,2,4,4)}(n) = H^{(2,15)}(5n)$
$G_{120,960,2}$	$4\cdot 2\cdot 3\cdot 5$	$16\cdot 2^2\cdot 3\cdot 5$	$2R_{(5,5,12,-4,-4,-2)}(n) = H^{(2,15)}(15n)$
$G_{120,450,2}$	$4\cdot 2\cdot 3\cdot 5$	$2\cdot (3\cdot 5)^2$	$2R_{(3,7,7,4,3,3)}(n) = H^{(2,15)}(8n)$
$G_{120,150,2}$	$4\cdot 2\cdot 3\cdot 5$	$2\cdot 3\cdot 5^2$	$2R_{(1,5,9,-5,1,0)}(n) = H^{(2,15)}(24n)$

$G_{120,90,2}$	$4\cdot2\cdot3\cdot5$	$2\cdot3^2\cdot5$	$2R_{(2,3,5,-3,2,0)}(n) = H^{(2,15)}(40n)$
$G_{120,30,2}$	$4\cdot2\cdot3\cdot5$	$2\cdot3\cdot5$	$2R_{(1,3,3,1,1,1)}(n) = H^{(2,15)}(120n)$
$G_{60,900,2}$	$2\cdot2\cdot3\cdot5$	$(2\cdot3\cdot5)^2$	$2R_{(5,6,9,-6,0,0)}(n) = H^{(2,15)}(4n)$
$G_{60,300,2}$	$2\cdot2\cdot3\cdot5$	$2^2\cdot3\cdot5^2$	$2R_{(2,3,15,0,0,-2)}(n) = H^{(2,15)}(12n)$
$G_{60,180,2}$	$2\cdot2\cdot3\cdot5$	$2^2\cdot3^2\cdot5$	$2R_{(1,6,9,-6,0,0)}(n) = H^{(2,15)}(20n)$
$G_{60,60,2}$	$2\cdot2\cdot3\cdot5$	$2^2\cdot3\cdot5$	$2R_{(2,3,3,0,0,-2)}(n) = H^{(2,15)}(60n)$
$G_{184,33856,2}$	$4\cdot2\cdot23$	$16\cdot(2\cdot23)^2$	$2R_{(11,19,51,-14,-6,-10)}(n) = H^{(2,23)}(n)$
$G_{184,1472,2}$	$4\cdot2\cdot23$	$16\cdot2^2\cdot23$	$2R_{(5,8,12,-8,-4,0)}(n) = H^{(2,23)}(23n)$
$G_{184,1058,2}$	$4\cdot2\cdot23$	$2\cdot23^2$	$2R_{([5,7,10,2,4,5)}(n) = H^{(2,23)}(8n)$
$G_{184,46,2}$	$4\cdot2\cdot23$	$2\cdot23$	$2R_{(1,3,5,3,1,1)}(n) = H^{(2,23)}(184n)$
$G_{92,2116,2}$	$2\cdot2\cdot23$	$(2\cdot23)^2$	$2R_{(5,10,14,10,2,4)}(n) = H^{(2,23)}(4n)$
$G_{92,92,2}$	$2\cdot2\cdot23$	$2^2\cdot23$	$2R_{(2,3,5,-2,0,-2)}(n) = H^{(2,23)}(92n)$
$G_{12,144,3}$	$4\cdot3$	$16\cdot3^2$	$R_{(3,4,4,-4,0,0)}(n) = 6H^{(3,1)}(n)$
$G_{12,48,3}$	$4\cdot3$	$16\cdot3$	$R_{(1,4,4,-4,0,0)}(n) = 6H^{(3,1)}(3n)$
$G_{12,9,3}$	$4\cdot3$	3^2	$R_{(1,1,3,0,0,-1)}(n) = 6H^{(3,1)}(4n)$
$G_{12,3,3}$	$4\cdot3$	3	$R_{(1,1,1,0,0,-1)}(n) = 6H^{(3,1)}(12n)$
$G_{24,576,3}$	$4\cdot2\cdot3$	$16\cdot(2\cdot3)^2$	$R_{(4,7,7,2,4,4)}(n) = 2H^{(3,2)}(n)$
$G_{24,192,3}$	$4\cdot2\cdot3$	$16\cdot2^2\cdot3$	$R_{(4,4,5,-4,-4,0)}(n) = 2H^{(3,2)}(3n)$
$G_{24,18,3}$	$4\cdot2\cdot3$	$2\cdot3^2$	$R_{(2,2,2,1,2,2)}(n) = 2H^{(3,2)}(8n)$
$G_{24,6,3}$	$4\cdot2\cdot3$	$2\cdot3$	$R_{1,1,2,-1,-1,0)}(n) = 2H^{(3,2)}(24n)$
$G_{12,36,3}$	$2\cdot2\cdot3$	$(2\cdot3)^2$	$R_{(1,3,3,0,0,0)}(n) = 2H^{(3,2)}(4n)$
$G_{12,12,3}$	$2\cdot2\cdot3$	$2^2\cdot3$	$R_{(1,1,3,0,0,0)}(n) = 2H^{(3,2)}(12n)$
$G_{48,2304,3}$	$4\cdot2^2\cdot3$	$16\cdot(2^2\cdot3)^2$	$R_{(7,7,15,-6,-6,-2)}(n) = H^{(3,4)}(n)$
$G_{48,768,3}$	$4\cdot2^2\cdot3$	$16\cdot2^2\cdot3$	$R_{(4,5,13,2,4,4)}(n) = H^{(3,4)}(3n)$
$G_{60,3600,3}$	$4\cdot3\cdot5$	$16\cdot(3\cdot5)^2$	$R_{(4,15,16,0,-4,0)}(n) = H^{(3,5)}(n)$
$G_{60,1200,3}$	$4\cdot3\cdot5$	$16\cdot3\cdot5^2$	$R_{(5,8,8,-4,0,0)}(n) = H^{(3,5)}(3n)$
$G_{60,720,3}$	$4\cdot3\cdot5$	$16\cdot3^2\cdot5$	$R_{(3,8,8,-4,0,0)}(n) = H^{(3,5)}(5n)$
$G_{60,240,3}$	$4\cdot3\cdot5$	$16\cdot3\cdot5$	$R_{(1,4,16,-4,0,0)}(n) = H^{(3,5)}(15n)$
$G_{60,225,3}$	$4\cdot3\cdot5$	$(3\cdot5)^2$	$R_{(1,4,15,0,0,-1)}(n) = H^{(3,5)}(4n)$
$G_{60,75,3}$	$4\cdot3\cdot5$	$3\cdot5^2$	$R_{(2,2,5,0,0,-1)}(n) = H^{(3,5)}(12n)$
$G_{60,45,3}$	$4\cdot3\cdot5$	$3^2\cdot5$	$R_{(2,2,3,0,0,-1)}(n) = H^{(3,5)}(20n)$
$G_{60,15,3}$	$4\cdot3\cdot5$	$3\cdot5$	$R_{(1,1,4,0,-1,0)}(n) = H^{(3,5)}(60n)$
$G_{96,9216,3}$	$4\cdot2^3\cdot3$	$16\cdot(2^3\cdot3)^2$	$2R_{(7,15,28,-12,-4,-6)}(n) = H^{(3,8)}(n)$
$G_{96,3072,3}$	$4\cdot2^3\cdot3$	$6\cdot(2^3)^2\cdot3$	$2R_{(5,13,13,-6,-2,-2)}(n) = H^{(3,8)}(3n)$
$G_{132,17424,3}$	$4\cdot3\cdot11$	$16(\cdot3\cdot11)^2$	$2R_{(7,19,39,-18,-6,-2)}(n) = H^{(3,11)}(n)$
$G_{132,5808,3}$	$4\cdot3\cdot11$	$16\cdot3\cdot11^2$	$2R_{(8,13,17,2,4,8)}(n) = H^{(3,11)}(3n)$
$G_{132,1584,3}$	$4\cdot3\cdot11$	$16\cdot3^2\cdot11$	$2R_{(5,5,17,-2,-2,-2)}(n) = H^{(3,11)}(11n)$
$G_{132,528,3}$	$4\cdot3\cdot11$	$16\cdot3\cdot11$	$2R_{(4,7,7,-6,0,-4)}(n) = H^{(3,11)}(33n)$
$G_{132,1089,3}$	$4\cdot3\cdot11$	$(3\cdot11)^2$	$2R_{(6,7,10,7,3,6)}(n) = H^{(3,11)}(4n)$
$G_{132,363,3}$	$4\cdot3\cdot11$	$3\cdot11^2$	$2R_{(2,7,7,3,1,1)}(n) = H^{(3,11)}(12n)$
$G_{132,99,3}$	$4\cdot3\cdot11$	$3^2\cdot11$	$2R_{(2,3,5,-3,-1,0)}(n) = H^{(3,11)}(44n)$
$G_{132,33,3}$	$4\cdot3\cdot11$	$3\cdot11$	$2R_{(1,2,5,1,1,1)}(n) = H^{(3,11)}(132n)$
$G_{20,400,5}$	$4\cdot5$	$16\cdot5^2$	$R_{(3,7,7,-6,-2,-2)}(n) = 3H^{(5,1)}(n)$
$G_{20,80,5}$	$4\cdot5$	$16\cdot5$	$R_{(3,3,3,2,2,2)}(n) = 3H^{(5,1)}(5n)$
$G_{20,25,5}$	$4\cdot5$	5^2	$R_{(2,2,2,-1,-1,-1)}(n) = 3H^{(5,1)}(4n)$
$G_{20,5,5}$	$4\cdot5$	5	$R_{(1,1,2,1,1,1)}(n) = 3H^{(5,1)}(20n)$

$G_{40,1600,5}$	$4\cdot 2\cdot 5$	$16\cdot(2\cdot 5)^2$	$R_{(7,7,12,-4,-4,-6)}(n)=2H^{(5,2)}(n)$
$G_{40,320,5}$	$4\cdot 2\cdot 5$	$16\cdot 2^2\cdot 5$	$R_{(3,3,11,-2,-2,-2)}(n)=2H^{(5,2)}(20n)$
$G_{40,50,5}$	$4\cdot 2\cdot 5$	$2\cdot 5^2$	$R_{(1,4,4,3,1,1)}(n)=2H^{(5,2)}(8n)$
$G_{40,25,5}$	$4\cdot 2\cdot 5$	$2\cdot 5$	$R_{(1,2,2,2,1,1)}(n)=2H^{(5,2)}(40n)$
$G_{20,100,5}$	$2\cdot 2\cdot 5$	$(2\cdot 5)^2$	$R_{(2,3,5,0,0,-2)}(n)=2H^{(5,2)}(4n)$
$G_{20,20,5}$	$2\cdot 2\cdot 5$	$2^2\cdot 5$	$R_{(1,2,3,-2,0,0)}(n)=2H^{(5,2)}(20n)$
$G_{80,6400,5}$	$4\cdot 2^2\cdot 5$	$16\cdot(2^2\cdot 5)^2$	$2R_{(7,12,23,12,2,4)}(n)=H^{(5,4)}(n)$
$G_{80,1280,5}$	$4\cdot 2^2\cdot 5$	$16\cdot 2^2\cdot 5$	$2R_{(3,11,11,6,2,2)}(n)=H^{(5,4)}(5n)$
$G_{28,784,7}$	$4\cdot 7$	$16\cdot 7^2$	$R_{(4,7,8,0,-4,0)}(n)=2H^{(7,1)}(n)$
$G_{28,112,7}$	$4\cdot 7$	$16\cdot 7$	$R_{(1,4,8,-4,0,0)}(n)=2H^{(7,1)}(7n)$
$G_{28,49,7}$	$4\cdot 7$	7^2	$R_{(1,2,7,0,0,-1)}(n)=2H^{(7,1)}(4n)$
$G_{28,7,7}$	$4\cdot 7$	7	$R_{(1,1,2,0,-1,0)}(n)=2H^{(7,1)}(28n)$
$G_{84,7056,7}$	$4\cdot 3\cdot 7$	$16\cdot(3\cdot 7)^2$	$2R_{(8,11,23,2,8,4)}(n)=H^{(7,3)}(n)$
$G_{84,2352,7}$	$4\cdot 3\cdot 7$	$16\cdot 3\cdot 7^2$	$2R_{(5,12,12,-4,-4,-4)}(n)=H^{(7,3)}(3n)$
$G_{84,1008,7}$	$4\cdot 3\cdot 7$	$16\cdot 3^2\cdot 7$	$2R_{(5,8,8,4,4,4)}(n)=H^{(7,3)}(7n)$
$G_{84,336,7}$	$4\cdot 3\cdot 7$	$16\cdot 3\cdot 7$	$2R_{(3,3,11,2,2,2)}(n)=H^{(7,3)}(21n)$
$G_{84,441,7}$	$4\cdot 3\cdot 7$	$(3\cdot 7)^2$	$2R_{(2,8,8,-5,-1,-1)}(n)=H^{(7,3)}(4n)$
$G_{84,147,7}$	$4\cdot 3\cdot 7$	$3\cdot 7^2$	$2R_{(3,3,5,-2,-2,-1)}(n)=H^{(7,3)}(12n)$
$G_{84,63,7}$	$4\cdot 3\cdot 7$	$3^2\cdot 7$	$2R_{(2,2,5,2,2,1)}(n)=H^{(7,3)}(28n)$
$G_{84,21,7}$	$4\cdot 3\cdot 7$	$3\cdot 7$	$2R_{(1,2,3,-1,-1,0)}(n)=H^{(7,3)}(84n)$
$G_{32,1024,2}$	$4\cdot 2^3$	$16\cdot(2^3)^2$	$R_{(3,11,11,-10,-2,-2)}(n)=3H^{(8,1)}(n)$
$G_{160,25600,2}$	$4\cdot 2^3\cdot 5$	$16\cdot(2^3\cdot 5)^2$	$2R_{(11,16,44,16,4,8)}(n)=H^{(8,5)}(n)$
$G_{160,5120,2}$	$4\cdot 2^3\cdot 5$	$16\cdot(2^3)^2\cdot 5$	$2R_{(7,15,15,-2,-6,-6)}(n)=H^{(8,5)}(5n)$
$G_{52,2704,13}$	$4\cdot 13$	$16\cdot 13^2$	$R_{(7,8,15,8,2,4)}(n)=H^{(13,1)}(n)$
$G_{52,208,13}$	$4\cdot 13$	$16\cdot 13$	$R_{(3,3,7,2,2,2)}(n)=H^{(13,1)}(13n)$
$G_{52,169,13}$	$4\cdot 13$	13^2	$R_{(2,5,5,-3,-1,-1)}(n)=H^{(13,1)}(4n)$
$G_{52,13,13}$	$4\cdot 13$	13	$R_{(1,2,2,-1,0,-1)}(n)=H^{(13,1)}(52n)$
$G_{120,3600,30}$	$4\cdot 2\cdot 3\cdot 5$	$16\cdot(2\cdot 3\cdot 5)^2$	$2R_{(3,40,40,-40,0,0)}(n)=3H^{(30,1)}(n)$
$G_{120,1200,30}$	$4\cdot 2\cdot 3\cdot 5$	$16\cdot 2^2\cdot 3\cdot 5^2$	$2R_{(1,40,40,-40,0,0)}(n)=3H^{(30,1)}(3n)$
$G_{120,720,30}$	$4\cdot 2\cdot 3\cdot 5$	$16\cdot 2^2\cdot 3^2\cdot 5$	$2R_{(8,8,15,0,0,-8)}(n)=3H^{(30,1)}(5n)$
$G_{120,240,30}$	$4\cdot 2\cdot 3\cdot 5$	$16\cdot 2^2\cdot 3\cdot 5$	$2R_{(5,8,8,-8,0,0)}(n)=3H^{(30,1)}(15n)$
$G_{120,450,30}$	$4\cdot 2\cdot 3\cdot 5$	$2\cdot(3\cdot 5)^2$	$2R_{(5,6,6,0,0,-5)}(n)=3H^{(30,1)}(8n)$
$G_{120,150,30}$	$4\cdot 2\cdot 3\cdot 5$	$2\cdot 3\cdot 5^2$	$2R_{(2,5,5,-5,0,0)}(n)=3H^{(30,1)}(24n)$
$G_{120,90,30}$	$4\cdot 2\cdot 3\cdot 5$	$2\cdot 3^2\cdot 5$	$2R_{(1,1,30,0,0,-1)}(n)=3H^{(30,1)}(40n)$
$G_{120,30,30}$	$4\cdot 2\cdot 3\cdot 5$	$2\cdot 3\cdot 5$	$2R_{(1,1,10,0,0,-1)}(n)=3H^{(30,1)}(120n)$
$G_{60,900,30}$	$2\cdot 2\cdot 3\cdot 5$	$4\cdot(3\cdot 5)^2$	$2R_{(3,10,10,-10,0,0)}(n)=3H^{(30,1)}(4n)$
$G_{60,300,30}$	$2\cdot 2\cdot 3\cdot 5$	$4\cdot 3\cdot 5^2$	$2R_{(1,10,10,-10,0,0)}(n)=3H^{(30,1)}(12n)$
$G_{60,180,30}$	$2\cdot 2\cdot 3\cdot 5$	$4\cdot 3^2\cdot 5$	$2R_{(2,2,15,0,0,-2)}(n)=3H^{(30,1)}(20n)$
$G_{60,60,30}$	$2\cdot 2\cdot 3\cdot 5$	$4\cdot 3\cdot 5$	$2R_{(2,2,5,0,0,-2)}(n)=3H^{(30,1)}(60n)$
$G_{168,28224,42}$	$4\cdot 2\cdot 3\cdot 7$	$16\cdot(2\cdot 3\cdot 7)^2$	$R_{(4,43,43,2,4,4)}(n)=H^{(42,1)}(n)$
$G_{168,9408,42}$	$4\cdot 2\cdot 3\cdot 7$	$16\cdot 2^2\cdot 3\cdot 7^2$	$R_{(12,17,17,6,12,12)}(n)=H^{(42,1)}(3n)$
$G_{168,4032,42}$	$4\cdot 2\cdot 3\cdot 7$	$16\cdot 2^2\cdot 3^2\cdot 7$	$R_{(12,12,13,-12,-12,0)}(n)=H^{(42,1)}(7n)$
$G_{168,1344,42}$	$4\cdot 2\cdot 3\cdot 7$	$16\cdot 2^2\cdot 3\cdot 7$	$R_{(4,4,23,-4,-4,0)}(n)=H^{(42,1)}(21n)$
$G_{168,882,42}$	$4\cdot 2\cdot 3\cdot 7$	$2\cdot(3\cdot 7)^2$	$R_{(2,11,11,1,2,2)}(n)=H^{(42,1)}(8n)$
$G_{168,294,42}$	$4\cdot 2\cdot 3\cdot 7$	$2\cdot 3\cdot 7^2$	$R_{(5,5,5,-3,-3,-4)}(n)=H^{(42,1)}(24n)$
$G_{168,126,42}$	$4\cdot 2\cdot 3\cdot 7$	$2\cdot 3^2\cdot 7$	$R_{(3,3,5,-3,-3,0)}(n)=H^{(42,1)}(56n)$

$G_{168,42,42}$	$4 \cdot 2 \cdot 3 \cdot 7$	$2 \cdot 3 \cdot 7$	$R_{(1,1,11,-1,-1,0)}(n) = H^{(42,1)}(168n)$
$G_{84,1764,42}$	$2 \cdot 2 \cdot 3 \cdot 7$	$(2 \cdot 3 \cdot 7)^2$	$R_{(1,21,21,0,0,0)}(n) = H^{(42,1)}(4n)$
$G_{84,588,42}$	$2 \cdot 2 \cdot 3 \cdot 7$	$2^2 \cdot 3 \cdot 7^2$	$R_{(3,7,7,0,0,0)}(n) = H^{(42,1)}(12n)$
$G_{84,252,42}$	$2 \cdot 2 \cdot 3 \cdot 7$	$2^2 \cdot 3^2 \cdot 7$	$R_{(3,3,7,0,0,0)}(n) = H^{(42,1)}(28n)$
$G_{84,84,42}$	$2 \cdot 2 \cdot 3 \cdot 7$	$2^2 \cdot 3 \cdot 7$	$R_{(1,1,21,0,0,0)}(n) = H^{(42,1)}(84n)$
$G_{280,78400,70}$	$4 \cdot 2 \cdot 5 \cdot 7$	$16 \cdot (2 \cdot 5 \cdot 7)^2$	$2R_{(8,35,72,0,-8,0)}(n) = H^{(70,1)}(n)$
$G_{280,15680,70}$	$4 \cdot 2 \cdot 5 \cdot 7$	$16 \cdot 2^2 \cdot 5 \cdot 7^2$	$2R_{(7,24,24,-8,0,0)}(n) = H^{(70,1)}(5n)$
$G_{280,11200,70}$	$4 \cdot 2 \cdot 5 \cdot 7$	$16 \cdot 2^2 \cdot 5^2 \cdot 7$	$2R_{(5,24,24,-8,0,0)}(n) = H^{(70,1)}(7n)$
$G_{280,2240,70}$	$4 \cdot 2 \cdot 5 \cdot 7$	$16 \cdot 2^2 \cdot 5 \cdot 7$	$2R_{(1,8,72,-8,0,0)}(n) = H^{(70,1)}(35n)$
$G_{280,2450,70}$	$4 \cdot 2 \cdot 5 \cdot 7$	$2 \cdot (5 \cdot 7)^2$	$2R_{(1,9,70,0,0,-1)}(n) = H^{(70,1)}(8n)$
$G_{280,490,70}$	$4 \cdot 2 \cdot 5 \cdot 7$	$2 \cdot 5 \cdot 7^2$	$2R_{(3,3,14,0,0,-1)}(n) = H^{(70,1)}(40n)$
$G_{280,350,70}$	$4 \cdot 2 \cdot 5 \cdot 7$	$2 \cdot 5^2 \cdot 7$	$2R_{(3,3,10,0,0,-1)}(n) = H^{(70,1)}(56n)$
$G_{280,70,70}$	$4 \cdot 2 \cdot 5 \cdot 7$	$2 \cdot 5 \cdot 7$	$2R_{(1,2,9,0,-1,0)}(n) = H^{(70,1)}(280n)$
$G_{140,4900,70}$	$2 \cdot 2 \cdot 5 \cdot 7$	$(2 \cdot 5 \cdot 7)^2$	$2R_{(2,18,35,0,0,-2)}(n) = H^{(70,1)}(4n)$
$G_{140,980,70}$	$2 \cdot 2 \cdot 5 \cdot 7$	$2^2 \cdot 5 \cdot 7^2$	$2R_{(6,6,7,0,0,-2)}(n) = H^{(70,1)}(20n)$
$G_{140,700,70}$	$2 \cdot 2 \cdot 5 \cdot 7$	$2^2 \cdot 5^2 \cdot 7$	$2R_{(5,6,6,-2,0,0)}(n) = H^{(70,1)}(28n)$
$G_{140,140,70}$	$2 \cdot 2 \cdot 5 \cdot 7$	$2^2 \cdot 5 \cdot 7$	$2R_{(1,2,18,-2,0,0)}(n) = H^{(70,1)}(140n)$
$G_{312,97344,78}$	$4 \cdot 2 \cdot 3 \cdot 13$	$16 \cdot (2 \cdot 3 \cdot 13)^2$	$2R_{19,19,84,-12,-12,-14)}(n) = H^{(78,1)}(n)$
$G_{312,32448,78}$	$4 \cdot 2 \cdot 3 \cdot 13$	$16 \cdot 2^2 \cdot 3 \cdot 13^2$	$2R_{8,28,41,4,8,8)}(n) = H^{(78,1)}(3n)$
$G_{312,7488,78}$	$4 \cdot 2 \cdot 3 \cdot 13$	$16 \cdot 2^2 \cdot 3^2 \cdot 13$	$2R_{4,7,79,2,4,4)}(n) = H^{(78,1)}(13n)$
$G_{312,2496,78}$	$4 \cdot 2 \cdot 3 \cdot 13$	$16 \cdot 2^2 \cdot 3 \cdot 13$	$2R_{5,5,28,-4,-4,-2)}(n) = H^{(78,1)}(39n)$
$G_{312,3042,78}$	$4 \cdot 2 \cdot 3 \cdot 13$	$2 \cdot (3 \cdot 13)^2$	$2R_{3,17,17,8,3,3)}(n) = H^{(78,1)}(8n)$
$G_{312,1014,78}$	$4 \cdot 2 \cdot 3 \cdot 13$	$2 \cdot 3 \cdot 13^2$	$2R_{1,13,23,-13,-1,0)}(n) = H^{(78,1)}(24n)$
$G_{312,234,78}$	$4 \cdot 2 \cdot 3 \cdot 13$	$2 \cdot 3^2 \cdot 13$	$2R_{2,3,11,-3,-2,0)}(n) = H^{(78,1)}(104n)$
$G_{312,78,78}$	$4 \cdot 2 \cdot 3 \cdot 13$	$2 \cdot 3^2 \cdot 13^2$	$2R_{1,5,5,4,1,1)}(n) = H^{(78,1)}(312n)$
$G_{156,6084,78}$	$2 \cdot 2 \cdot 3 \cdot 13$	$(2 \cdot 3 \cdot 13)^2$	$2R_{(6,13,21,0,-6,0)}(n) = H^{(78,1)}(4n)$
$G_{156,2028,78}$	$2 \cdot 2 \cdot 3 \cdot 13$	$2^2 \cdot 3 \cdot 13^2$	$2R_{(2,7,39,0,0,-2)}(n) = H^{(78,1)}(12n)$
$G_{156,468,78}$	$2 \cdot 2 \cdot 3 \cdot 13$	$2^2 \cdot 3^2 \cdot 13$	$2R_{(1,6,21,-6,0,0)}(n) = H^{(78,1)}(52n)$
$G_{156,156,78}$	$2 \cdot 2 \cdot 3 \cdot 13$	$2^2 \cdot 3 \cdot 13$	$2R_{(2,33,7,0,-2,0)}(n) = H^{(78,1)}(156n)$

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