MORDELL-TORNHEIM TYPE SERIES OVER LATTICE PARALLELOGRAMS BY TELESCOPIC SUMMATION

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ABSTRACT. We prove a Mordell–Tornheim type formula for a sum over lattice vectors. We obtain yet another simple proof of Zagier's formula for a sum $D_{1,1,1}$ using the same method.

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1. Let $A = \{x, y \in \mathbb{Z}^2_{\geq 0}, \det(x \;\; y) = 1\}$, i.e., the set of all lattice parallelograms in the first quadrant of the oriented area one.

Theorem 1.

$$4\sum_A \frac{1}{|x|^2 \cdot |y|^2 \cdot |x+y|^2} = \pi.$$

Proof. Let us consider $F(x, y) = \frac{x\cdot y}{|x^2|\cdot|y|^2}$. Then,

$$F(x,y) - F(x+y,y) - F(x,x+y) = \frac{-2\det(x \;\; y)^2}{|x|^2 \cdot |y|^2 \cdot |x+y|^2}. \quad (1)$$

Let $A_n = \{x \in \mathbb{Z}^2_{\geq 0} \cap [0,n]^2\}$. We telescope

$$F(x,y) - F(x+y,y) - F(x,x+y)$$

over $\{x, y \in A_n, \det(x \;\; y) = 1\}$ obtaining the sum of $-F(x+y,y) - F(x,x+y)$ over $B_n = \{x, y \in A_n, \det(x \;\; y) = 1, x+y \notin A_n\}$.

The latter sum tends to $-\pi/2$ since the area of the parallelogram spanned by x, y is 1, so $\frac{x\cdot y}{|x^2|\cdot|y|^2}$ is the angle between x and y up to second order terms, and the set of angles at the origin of the parallelograms in B_n partition the angle $\pi/2$ of the first quadrant.

□

The above formula is inspired by the formulae like [5] for $m, n \in \mathbb{Z}_{>0}$

$$\frac{1}{mn} = \frac{1}{(m+n)n} + \frac{1}{m(m+n)}$$

$$G(m,n) - G(m+n,n) - G(m,m+n) = \frac{2}{m^2n^2}, \text{ where}$$

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$$G(m,n)=\frac{2}{m^3n}+\frac{1}{m^2n^2}+\frac{2}{mn^3}.$$

2. The motivation to study such formulae for vectors (and not for numbers) is to get relations for the lattice sums that are similar to the relations for the Mordell-Tornheim series [9], [5]. For $k,n,m\in\mathbb{Z}_{\geq 0}$ define

$$(k,n,m)=\sum_{(b,d)=1,b,d>0}\frac{1}{b^kd^n(b+d)^m}.$$

For example,

$$(2,2,2)=\sum_{(b,d)=1,b,d>0}\frac{1}{b^2d^2(b+d)^2}=1/3.$$

Then $(k,n,m)=(k-1,n,m+1)+(k,n-1,m+1)$ for $k,n\geq 1$ follows from $\frac{1}{bd}=\frac{1}{b(b+d)}+\frac{1}{(b+d)d}$.

Unfortunately, this is no longer true for vectors, as we have

$$\frac{1}{|x||y|}-\frac{1}{|x+y||y|}-\frac{1}{|x||x+y|}=\frac{|x+y|-|x|-|y|}{|x||y||x+y|}.$$

However, telescoping as in Theorem 1 one gets

Theorem 2.

$$\sum_A\frac{|x|+|y|-|x+y|}{|x||y||x+y|}=\pi/2-1.$$

Also it is known [7] that

$$\sum_A(|x|+|y|-|x+y|)=2 \text{ and } \sum_A(|x|+|y|-|x+y|)^2=2-\pi/2.$$

3. One can evaluate the following sum by the same method. Let $B_n=\{x\in\mathbb{Z},y\in\mathbb{Z}_{\geq 0},\det(x\quad y)=n\}$,

Theorem 3.

$$\sum_{(x,y)\in B_n}\frac{n^2}{|x|^2\cdot|y|^2\cdot|x+y|^2}=\frac{1}{2}\frac{\pi}{n}\cdot\sigma_1(n).$$

Proof. We use (1) and follow the proof of Theorem 1. Note that there are $\sigma_1(n)$ (the sum of divisors of n) non-equivalent sublattices of $\mathbb{Z}^2$ of determinant n, each on them is generated by two vectors

$$(\frac{n}{d},0),(k,d),0\leq k<d$$

where d is a divisor of n. For a fixed lattice of this type the vectors $(k+j\frac{n}{d},d),j\in\mathbb{Z}$ partition the angle π of the upper halfplane (c.f. with $\pi/2$ in Theorem 1). Then, $\frac{x\cdot y}{|x^2|\cdot|y|^2}\approx\frac{1}{n}\cdot$ (the angle between x and y), finally, $\frac{1}{2}$ comes from -2 in Eq (1). □

Using our methods, one can give an alternative proof of the following Zagier's formula. Let

$$D_{1,1,1}(z) = \sum_{\omega_1+\omega_2+\omega_3=0}{}' \frac{y^3}{|\omega_1\omega_2\omega_3|^2}, \omega_k \in \mathbb{Z}z+\mathbb{Z}, z = x+i\cdot y,$$

and $\sum'$ is the sum over all non-infinite summands (in this particular case it means that we remove from the summation all terms where one of ω_i is 0).

Then, as it is proven in [2, 1]

$$D_{1,1,1}(z) = 2E(z,3) + \pi^3\zeta(3), \tag{2}$$

where $E(z,s) = \frac{1}{2}\sum_{m,n\in\mathbb{Z}}{}' \frac{y^s}{|mz+n|^{2s}}$, a non-holomorphic Eisenstein series.

Look at $D_{1,1,1}(i,3) = \sum_{\omega_1+\omega_2+\omega_3=0}{}' \frac{1^3}{|\omega_1\omega_2\omega_3|^2}, \omega_k \in \mathbb{Z}i+\mathbb{Z}$.

Among such triples of vectors, there are collinear triples. If $\omega_1, \omega_2, \omega_3$ are collinear, let ω be the primitive vector in the direction of two of them who point in the same direction. Then the opposite vector can be $\omega_1, \omega_2, \omega_3$ (three choices) and $\sum_{b,d\in\mathbb{Z}_{>0}} \frac{1}{b^2d^2(b+d)^2} = \zeta(6)/3$. So the sum over all such triples is equal to

$\left(\frac{1}{\zeta(6)}\sum_{\omega\in\mathbb{Z}^2}{}' \frac{1}{|\omega|^6}\right)\cdot 3\cdot\frac{\zeta(6)}{3} = 2E(i,3)$ as the first term in Eq (2).

For non-collinear triples, up to central symmetry, we may assume that two (say, ω_1, ω_2) of the vectors ω_i belong to $\mathbb{Z}\times\mathbb{Z}_{\geq 0}\setminus\{\mathbb{Z}_{<0}\times\{0\}\}$. Then, we may assume that the parallelogram generated by ω_1, ω_2 has a positive signed area. All these choices give a factor of 12.

Thus

$$\pi^3\zeta(3) = D_{1,1,1}(i,3) - 2E(i,3) = 12\sum_{n\in\mathbb{Z}_{>0}}\sum_{(x,y)\in B_n}\frac{1}{|x|^2\cdot|y|^2\cdot|x+y|^2} =$$

$$= 12\sum_{n\in\mathbb{Z}_{>0}}\frac{1}{2}\frac{\pi}{n^3}\cdot\sigma_1(n) = \pi\cdot 6\zeta(3)\zeta(2) = \pi^3\zeta(3)$$

since $\sum\frac{\sigma_1(n)}{n^s} = \zeta(s)\zeta(s-1)$ and $\zeta(2) = \pi^2/6$. Thus we reproved Zagier's formula (2) for $z = i, s = 3$.

4. Similarly we get

$$\sum_{\omega_1+\omega_2+\omega_3=0,\ \omega_k\in\mathbb{Z}i+\mathbb{Z}}{}' \frac{|\det(\omega_1\ \ \omega_2)|^{-s}}{|\omega_1\omega_2\omega_3|^2} = 6\pi\cdot\zeta(s+3)\zeta(s+2),$$

By changing the lattice to $\mathbb{Z}\cdot z+\mathbb{Z}$ for $z = x+i\cdot y$ we obtain

Theorem 4.
$$\sum_{\omega_1+\omega_2+\omega_3=0,\ \omega_k\in\mathbb{Z}z+\mathbb{Z}}{}^{\prime} \frac{\det(\omega_1 \quad \omega_2)|^{-s}}{|\omega_1\omega_2\omega_3|^2} = \frac{6\pi}{y^3}\cdot\zeta(s+3)\zeta(s+2),$$

As a corollary of this theorem, by substituting $s = 0$, we obtain (2) for arbitrary values of z.

5. Thanks to users of mathoverflow, I learned that Theorem 1 was used as an intermediate step by Adolf Hurwitz to present class number $h(d)$ as the sum of an infinite series; see his essentially ignored article [6] (§4 equation 8), back in 1905, see also a historical account in [3] (vol. III, p. 167). Hurwits's proof works for any positively defined binary quadratic form (our case corresponds to $q(x) = |x|^2, x \in \mathbb{Z}^2$) and consists of using a rational parametrization of a quadric curve to cut its interior into triangles corresponding to consecutive Farey fractions $(r/s, r+r'/s+s', r'/s')$, and then the area of this triangle is proportional to

$$(q(r,s)\cdot q(r',s')\cdot q(r+r',s+s'))^{-2}.$$

A recent development of Hurwitz's type of formulae [4, 8] is due to the study of Conway's topographs.

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