

A PROJECTIVE TWIST ON THE HASSE NORM THEOREM

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ABSTRACT. A finite extension of global fields L/K satisfies the Hasse norm principle if any nonzero element of K has the property that it is a norm locally if and only if it is a norm globally. In 1931, Hasse proved that any cyclic extension satisfies the Hasse norm principle, providing a novel approach to extending the local-global principle to equations with degree greater than 2. In this paper, we introduce the projective Hasse norm principle, generalizing the Hasse norm principle to multiple fields and asking whether a projective line that contains a norm locally in every field must also contain a norm globally in every field. We show that the projective Hasse norm principle is independent from the conjunction of Hasse norm principles in all of the constituent fields in the general case, but that the latter implies the former when the fields are all Galois and independent. We also prove an analogue of the Hasse norm theorem for the projective Hasse norm theorem, namely that the projective Hasse norm principle holds in all cyclic extensions.

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1. INTRODUCTION

Nearly a century ago, mathematician Helmut Hasse first introduced his concept of the *local-global principle*: the idea that, if a diophantine equation has a solution modulo every positive integer n , i.e. it always has a solution *locally*, then it must have an actual solution *globally* over the integers as well. Since then, the principle has been carefully studied in a variety of spaces, and to great depths. It turns out that in most cases, the principle actually fails: in [Sel51], Selmer's famous counterexample is the equation $3x^3 + 4y^3 + 5z^3 = 0$. It is not hard to show this equation has a solution modulo every integer, and yet it has no solution over the rationals. However, in other instances, the principle holds: for instance, the Hasse-Minkowski theorem proves that the principle is true for all quadratic forms over the rationals: multivariable polynomial expressions whose monomials all have degree two. Selmer's example shows that we cannot extend Hasse-Minkowski's theorem into higher degrees, but there are many approaches to extend the local-global principle into cases involving higher degrees.

One of the most prominent of these approaches is through the use of multiplicative norms. In any field extension, we can assign each element a norm acting as a kind of magnitude. For instance, in the Gaussian rationals over \mathbb{Q} , the norm of an element is the same as the square of its complex magnitude. In 1931, Hasse published his acclaimed Hasse norm theorem [Has31]: given a cyclic extension of the rationals K/\mathbb{Q} , he proved that the norm of the number field K satisfies the local-global principle: given any rational number q , if the equation $N_{K/\mathbb{Q}}(x) = q$ has a solution locally then it has a solution globally.

In our research, we put a projective twist on the problem. Instead of looking at the local-global principle at specific points, we look at lines through the origin instead. Given a space of several field extensions, if a projective line always contains a local solution, when must it also contain a global solution?

To address this question, in §2 we introduce the Hasse Norm Principle (HNP), the Multinorm Problem, as well as the Projective Hasse Norm Principle (PHNP). We fix standardized notations in §3 and introduce the methods used in §4.

In §5 we show a few preliminary results around simple cases of the projective Hasse norm principle and its connection to the Hasse norm principle that do not require cohomology. In §6 we derive a closed form for the character lattices of tori closely related to the projective Hasse norm principle, which we use extensively to analyze our research problem in later section.

In §7, we show that PHNP does not imply HNP by giving an explicit counterexample with a composite quadratic extension. On the other hand, it is often true that when HNP is true in all of the constituent fields, then PHNP holds as well. To concretely show this, we derive a simple sufficient condition for $\text{HNP} \implies \text{PHNP}$ in §8 dependent solely on the Galois group of the composite field extension. We also derive an explicit

construction isomorphic to the Tate-Shafarevich group that encodes the PHNP condition in terms of the decomposition groups of the composite field extension, which allows us to completely reframe the problem of PHNP in terms of discrete group theory. Using these tools, we show that when all of the constituent fields are Galois and independent, then HNP \implies PHNP.

Using these techniques, in §9 we construct an explicit counterexample to HNP \implies PHNP using a non-Galois constituent field. In §10 we further study the implication HNP \longrightarrow PHNP in the Galois case, and show that regardless of the choice of the constituent fields, when the Galois group of the composite group is abelian, dihedral, or of order p^3 for some prime p , then it follows that HNP \longrightarrow PHNP. We hope that the sufficient conditions for this implication and the methods used in this section will shed light on whether HNP implies PHNP in the general Galois case.

2. STATEMENT OF THE PROBLEM AND RELATED QUESTIONS

Let K be a global field, i.e. a finite extension of \mathbb{Q} or the function field of a smooth curve over a finite field. Denote \overline{K} its separable closure. For each place v of K let K_v denote the corresponding completion of K and let \mathbb{A}_K the ring of adèles of K .

Let L/K be a separable extension. For each place v of K and place w of L such that $w|v$, the norm map $N_{L/K} : L^\times \rightarrow K^\times$ matches the norm map $N_{L_w/K_v} : L_w^\times \rightarrow K_v$ and therefore a map $\mathbb{A}_L^\times \rightarrow \mathbb{A}_K^\times$ which we will also denote by $N_{L/K}$.

2.1. The Hasse Norm Principle. The inclusion $L \subset \mathbb{A}_L$ yields

$$N_{L/K}(L^\times) \subseteq K^\times \cap N_{L/K}(\mathbb{A}_L^\times). \quad (2.1)$$

If L is a field, we say the *Hasse Norm Principle* if the inclusion is an equality. We will let $\text{HNP}_K(L)$ be the Boolean equal to True if and only if the Hasse Norm Principle holds.

Theorem 2.1 (Hasse Norm Theorem, 1931). *HNP $_K(L)$ holds whenever L/K is cyclic.*

The modern study of the Hasse Norm principle uses techniques in class field theory to determine when

$$\text{III}(L/K) := (K^\times \cap N_{L/K}(\mathbb{A}_L^\times))/N_{L/K}(L^\times) \cong \text{III}^1(K, \mathbf{T}_{L/K}^1) \quad (2.2)$$

is trivial, where $\mathbf{T}_{L/K}^1 = \mathbf{R}_{L/K}^{(1)}\mathbb{G}_m = \text{Ker}(N_{L/K} : \mathbf{R}_{L/K}\mathbb{G}_m \rightarrow \mathbb{G}_m)$ is norm-one torus corresponding to L/K . For any $\text{Gal}(\overline{K}/K)$ -module M , we use the notation

$$\text{III}^i(K, M) = \text{Ker} \left(H^i(K, M) \rightarrow \prod_v H^i(K_v, M) \right),$$

where v runs over places of K . This group is called the *ith Tate-Shafarevich group*. For ease of notations we write $\text{III}^i(K, \mathbf{Q})$ for $\text{III}^i(K, \mathbf{Q}(\overline{K}))$ for any torus \mathbf{Q} defined over K .

If $\text{Gal}(\overline{K}/L)$ acts trivially on M we will define $\text{III}^i(L/K, M)$ similarly.

The rational points of \mathbf{T}^1 are

$$\mathbf{T}_{L/K}^1(K) = \{x \in L^\times : N_{L/K}(x) = 1\}.$$

The relation (2.2) is made apparent when looking at the Galois cohomology associated to the sequence

$$1 \rightarrow \mathbf{T}_{L/K}^1 \mathbb{G}_m(\overline{K}) \rightarrow \mathbf{R}_{L/K} \mathbb{G}_m(\overline{K}) \xrightarrow{N_{L/K}} \mathbb{G}_m(\overline{K}) \rightarrow 1. \quad (2.3)$$

Moreover, one can compute $\text{III}^1(K, \mathbf{T}_{L/K}^1)$ explicitly via Tate-Nakayama duality, stating that

$$\text{III}^1(K, \mathbf{T}_{L/K}^1) \cong \text{III}^2(L/K, \mathbf{X}^*(\mathbf{T}_{L/K}^1)),$$

where \mathbf{X}^* is the functor mapping a torus to its character lattice.

2.2. The Multinorm Principle. More recently, many have studied the *multinorm principle* which treats eq. (2.1) in the case when L is not a field but an étale algebra $L \cong L_1 \oplus \cdots \oplus L_k$ where each L_i/K is a field extension. We have $N_{L/K} = \prod_{i=1}^k N_{L_i/K}$ and the multinorm principle holds whenever

$$\prod_{i=1}^k N_{L_i/K}(L_i^\times) = K^\times \cap \prod_{i=1}^k N_{L_i/K}(\mathbb{A}_{L_i}^\times).$$

Similarly to the Hasse Norm theorem, one studies this problem by trying to compute the finite group

$$\text{III}(L/K) := \left(K^\times \cap \prod_{i=1}^k N_{L_i/K}(\mathbb{A}_{L_i}^\times) \right) / \left(\prod_{i=1}^k N_{L_i/K}(L_i^\times) \right) \cong \text{III}^1(K, \mathbf{S}).$$

where $\mathbf{S} = \text{Ker}(\prod_i N_{L_i/K} : \prod_i \mathbf{R}_{L_i/K} \mathbb{G}_m \rightarrow \mathbb{G}_m)$. We have

$$\mathbf{S}(K) = \{(\ell_1, \dots, \ell_k) \in L_1^\times \times \cdots \times L_k^\times : N_{L_1/K}(\ell_1) \cdots N_{L_k/K}(\ell_k) = 1\} \supset \mathbf{T}_{\cap_i L_i/K}^1.$$

Let us write a few important results about the multinorm principle.

Theorem 2.2 ([PR13]). *Let L_1, L_2 be separable extensions of K with Galois closure E_1, E_2 respectively. If $E_1 \cap E_2 = K$ then $L = L_1 \oplus L_2$ satisfies the multinorm principle.*

This is proven by studying the surjectivity of the map $\text{III}(L/K) \rightarrow \prod_{i=1}^k \text{III}(L_i/K)$.

Theorem 2.3 ([Pol14]). *If L_1, L_2 are finite abelian field extensions of K then*

$$\text{III}(L_1 \oplus L_2/K) \cong \text{III}(L_1 \cap L_2/K).$$

In particular, the Multinorm Principle corresponding to $L_1 \oplus L_2$ holds if and only if it holds $L_1 \cap L_2$ satisfies the Hasse Norm Principle.

Pollio-Rapinchuk ([PR13]) conjectured that the multinorm principle for $L = L_1 \oplus L_2$ holds whenever every subfield of $L_1 \cap L_2$ satisfies the Hasse Norm Principle but a counterexample appeared in [DW14], where the authors show an isomorphism between distinguished subgroup (weak approximation) of $\text{III}^2(K, \mathbf{X}^*(\mathbf{S}))$ and $\text{III}^2(K, \mathbf{X}^*(\mathbf{T}_{\cap_i L_i/K}^1))$.

2.3. The Projective Hasse Norm Principle. In this paper we offer a modification by introducing an intermediate field. Let $K \subset E \subset L$ be an intermediate separable extension and let $n = [E : K] \geq 2$. As a K -vector space, we have $E \cong K^n$ and therefore we have a projection map

$$\pi : E^\times \rightarrow \mathbb{P}_K(E^\times) = E^\times / K^\times \cong \mathbb{P}^{n-1}(K).$$

We define a similar map $\pi : \mathbb{A}_E \rightarrow \mathbb{P}_{\mathbb{A}_K^\times}(\mathbb{A}_E)$.

We say the *Projective Hasse Norm Principle* holds if

$$\pi(N_{L/E}(L^\times)) = \pi(E^\times) \cap \pi(N_{L/E}(\mathbb{A}_L^\times)).$$

Concretely, the Projective Hasse Norm principle corresponding to (L, E, K) holds if for all $e \in E^\times$, if for all triples (u, v, w) of places of K, E, L respectively such that $u|v$ and $v|w$, there is $\lambda_{u,v,w} \in K_u$ such that $\lambda_{u,v,w}e \in \text{Im}(N_{L_w/E_v})$, then there is $\lambda \in K$ such that $\lambda e \in \text{Im}_{L/E}$.

Let us rephrase this principle in terms of a *projective norm*.

Definition 2.4 (Projective Norm). Let $L \supset E$ be separable extensions of K , we define the *projective norm* $\text{PN}_{L/E/K} : K^\times \times L^\times \rightarrow E^\times$ by

$$(k, \ell) \mapsto k^{-1}N_{L/E}(\ell).$$

We say the Projective Hasse Norm Principle relative to $L/E/K$ holds if

$$\text{PN}_{L/E/K}(K^\times \times L^\times) = E^\times \cap \text{PN}_{L/E/K}(\mathbb{A}_K^\times \times \mathbb{A}_L^\times).$$

We are interested in the case $E = K^{\oplus n}$ and $L = \bigoplus_{i=1}^n L_i$ where each L_i/K is a field extension. We write $\text{PHNP}_K(L_1, \dots, L_n)$ the boolean corresponding to the truth of the corresponding Projective Hasse Norm Principle.

Concretely speaking, $\text{PHNP}_K(L_1, \dots, L_n)$ is true if and only if any line $L \subset K^n$ such that $L \otimes \mathbb{A}_K$ contains a local point in the image of product of the norms $N_{L_1/K} \times \dots \times N_{L_n/K}$, then there is a global point of L such that each coordinate is in the image of $N_{L_1/K} \times \dots \times N_{L_n/K}$.

In a similar fashion to the previous problems, we want to determine when

$$\text{III}(L/E/K) := (E^\times \cap \text{PN}_{L/E/K}(\mathbb{A}_K^\times \times \mathbb{A}_L^\times)) / \text{PN}_{L/E/K}(K^\times \times L^\times) = \text{III}^1(K, \mathbf{T})$$

is trivial, where $\mathbf{T} : \text{Ker} \left(\mathbb{G}_m \times \mathbb{R}_{L/K} \mathbb{G}_m \xrightarrow{\text{PN}_{L/E}} \mathbb{R}_{E/K} \mathbb{G}_m \right)$. The rational points of \mathbf{T} are

$$\begin{aligned} \mathbf{T}(K) &\cong \{(x, y) \in K^\times \times L^\times : N_{L/K}(y) = x\} \\ &\cong \{y \in L^\times : N_{L/E}(y) \in K^\times\}. \end{aligned}$$

In our specific setting, we can write

$$\mathbf{T}(K) \cong \{(y_1, \dots, y_n) \in (L_1 \times \dots \times L_n)^\times : N_{L_i/K}(x_i) = N_{L_j/K}(x_j) \forall 1 \leq i, j \leq n\}$$

We will study this the cohomology of \mathbf{T} with the cohomology of the subtorus $\mathbb{R}_{E/K} \mathbf{T}_{F/E}^1$ which in our case is isomorphic to $\prod_{i=1}^n \mathbf{T}_{L_i/K}^1$.

Firstly, let us note that we can always assume $n > 1$.

Proposition 2.5. *If $E = F$ (i.e. $n = 1$), then the Projective Hasse Norm Principle holds trivially.*

Proof. If $E = K$ then we have $\mathbf{T} \cong \mathbb{R}_{L/K} \mathbb{G}_m$ and hence

$$\text{III}^1(K, \mathbf{T}) \subset H^1(K, \mathbf{T}(\overline{K})) = 1.$$

□

Remark 2.6. If $n = 2$, $\text{PHNP}_K(L_1, L_2)$ can be reformulated as the following statement: for any fixed $y \in K$, the equation $y = \frac{N_{L_1/K}(x_1)}{N_{L_2/K}(x_2)}$ has a global solution if and only if it has a local solution. In particular we get the following result.

Proposition 2.7. *If $n = 2$ and $L_1 = L_2$, we have*

$$\text{PHNP}_K(L_1, L_2) = \text{HNP}_K(L_1).$$

Proof. Following Remark 2.6, the equation reduces to $y = N_{L_1/K}(\frac{x_1}{x_2})$. A local solution implying a global solution is equivalent to $\text{HNP}_K(L_1)$. □

Thusly, we can see that the Projective Hasse Norm Principle is a strict generalization of the Hasse Norm Principle.

In this paper, we compare PHNP_K and HNP_K across many settings, mainly studying whether it is true that $\text{PHNP}_K(L_1, L_2, \dots, L_n)$ implies or is implied by $\bigwedge_i \text{HNP}_K(L_i)$ in each setting of the problem. We will show counter examples for each direction proving that the principles cannot be compared without assumptions on L_1, \dots, L_n . In particular, we show in Theorem 10.3 that $\bigwedge_i \text{HNP}_K(L_i)$ implies $\text{PHNP}_K(L_1, L_2, \dots, L_n)$ when each of the field extensions L_i/K is Galois and abelian, which in turn implies a projective analogue of the Hasse norm theorem. We also show that this result holds when $\text{Gal}((L_1 L_2 \dots L_n)^\sharp)$ is dihedral or has order p^3 for some prime p . We also demonstrate in

Proposition 8.9 that when L_1, L_2, \dots, L_n are each Galois extensions of K and have independent Galois groups, it is also true that $\bigwedge_i \text{HNP}_K(L_i) \implies \text{PHNP}_K(L_1, L_2, \dots, L_n)$.

Remark 2.8. Note that when L/E is quadratic, then the study of the projective Hasse norm principle can be related to computing the Tamgawa number of centralizers of similitude groups, and has connections with counting point on Shimura varieties, as done in [AAgGG23]. We will also mention this application in Remark 5.5.

3. NOTATIONS

Let L_1, L_2, \dots, L_n be finite extensions of a global field K . Let $L = (L_1 L_2 \cdots L_n)^\#$ be the Galois closure of the composite field $L_1 L_2 \cdots L_n$. Let \tilde{L} denote the étale algebra $L_1 \oplus L_2 \oplus \cdots \oplus L_n$. Note that this is different from our previous section where L was the étale algebra. Let G denote $\text{Gal}(L/K)$. For simplicity, we let $H^i(M)$ denote $H^i(G, M)$ for any $\mathbb{Z}[G]$ -module M . For any finite group A , we let $A^\vee = \text{Hom}(A, \mathbb{Q}/\mathbb{Z})$ denote the Pontryagin dual of A and let $A^{\text{ab}} = (A^\vee)^\vee$ denote the abelianization of A . We furthermore let A^{der} denote the derived or commutator subgroup of A .

For each i , we define $G^{(i)} = \text{Gal}(L/L_i)$. If L_i/K is a Galois extension, we define $G_i = \text{Gal}(L_i/K)$ so that $G_i \cong G/G^{(i)}$. If L_1, L_2, \dots, L_n are all independent and Galois, then we have $G \cong \prod_i G_i$ and $G^{(i)} \cong \prod_{j \neq i} G_j$ for all i .

Let us state a simple but useful fact about $G^{(i)}$'s.

Lemma 3.1. *We have that $\bigcap_i G^{(i)} = 0$.*

Proof. Suppose a nontrivial element g exists such that $g \in G^{(i)}$ for all i , then let $G' = \langle g \rangle$. Since $|G/G'| < |G|$ is an automorphism group of L fixing each of the subfields L_1, L_2, \dots, L_n and strictly smaller than $\text{Gal}(L/K)$, it follows that L is not the minimal Galois closure of L_1, L_2, \dots, L_n , giving a contradiction. \square

4. METHODOLOGY

Classically, the Hasse norm theorem associated with a Galois field extension L/K is related to the cohomology of norm-one tori as follows. Let \mathbf{R} and $\mathbf{R}^{(1)}$ denote the Weil restriction of scalar and the norm one tori respectively. Let $\mathbf{T}_{L/K}^1 = \mathbf{R}_{L/K}^{(1)} \mathbb{G}_m$. Taking the cohomology of the sequence of $\text{Gal}(L/K)$ modules

$$1 \rightarrow \mathbf{T}_{L/K}^1(L) \rightarrow \mathbf{R}_{L/K} \mathbb{G}_m(L) \cong (L \times L)^\times \rightarrow \mathbb{G}_m(L) \cong L^\times \rightarrow 1,$$

we get

$$\begin{array}{ccccccc}
1 & \longrightarrow & \mathbf{T}_{L/K}^1(K) & \longrightarrow & L^\times & \xrightarrow{N_{L/K}} & K^\times & \longrightarrow & H^1(L/K, \mathbf{T}_{L/K}^1(L)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow r \\
1 & \longrightarrow & \prod_v \mathbf{T}_{L/K}^1(K_v) & \longrightarrow & \prod_{w|v} L_w^\times & \longrightarrow & \prod_v K_v^\times & \longrightarrow & \prod_{v|p} H^1(L_w/K_v, \mathbf{T}_{L/K}^1(L_w)),
\end{array}$$

where v is taken over all places in K and w runs over places of L dividing v . In the diagram, the Hasse Norm Principle is equivalent to injectivity of the map r . We can do something analogous for the projective norm.

Recall that now $\tilde{L} = \bigoplus_{i=1}^n L_i$, where L_i/K is a field extensions and L is the Galois closure of the composite field $L_1 \cdots L_n$. The projective norm can be extended to a map between algebraic tori $\mathbb{G}_m \times \prod R_{L_i/K} \mathbb{G}_m \rightarrow (\mathbb{G}_m)^n$. Let \mathbf{T} be the kernel of this map. We now have an exact sequence

$$1 \rightarrow \mathbf{T}(L) \rightarrow \mathbb{G}_m \times \prod_i R_{L_i/K} \mathbb{G}_m(L) \xrightarrow{PN} L^n \rightarrow 1.$$

The Galois cohomology of that sequence yields

$$1 \longrightarrow \mathbf{T}(K) \longrightarrow K^\times \times \bigoplus_{i=1}^n L_i^\times \xrightarrow{PN} (K^\times)^n \longrightarrow H^1(L/K, \mathbf{T}(L)).$$

Now, taking the injections $K \rightarrow K_v$ yields the commutative diagram

$$\begin{array}{ccccccc}
1 & \longrightarrow & \mathbf{T}(K) & \longrightarrow & K^\times \times \bigoplus_{i=1}^n L_i^\times & \xrightarrow{PN} & (K^\times)^n & \xrightarrow{\delta} & H^1(L/K, \mathbf{T}(L)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \pi & & \downarrow \\
1 & \longrightarrow & \prod_v \mathbf{T}(K_v) & \longrightarrow & \prod_{w|v} K_v^\times \times \bigoplus_{i=1}^n L_{i,w}^\times & \xrightarrow{\gamma} & \prod_v (K_v^\times)^n & \xrightarrow{\eta} & \prod_{w|v} H^1(L_w/K_v, \mathbf{T}(L_w))
\end{array}$$

where the maps $H^1(L/K, \mathbf{T}(L)) \rightarrow H^1(L_w/K_v, \mathbf{T}(L_w))$ are restriction maps.

Proposition 4.1. *PHNP $_K(L_1, L_2, \dots, L_n)$ is true if and only if $H^1(L/K, \mathbf{T}(L)) \rightarrow \prod_{w|v} H^1(L_w/K_v, \mathbf{T}(L_w))$ is injective.*

Proof. One can note that PHNP $_K$ is equivalent to the statement: for all $\alpha \in (K^\times)^n$, $\alpha \in \text{Im}(PN) \iff \pi(\alpha) \in \text{Im}(\gamma)$. Using the exact sequences, this is equivalent to the statement $\delta(\alpha) = 1 \iff \alpha \in \text{Ker}(\delta) \iff \pi(\alpha) \in \text{Ker}(\eta) \iff \eta(\pi(\alpha)) = 1$ for all $\alpha \in (K^\times)^n$. Since the diagram is commutative, this is equivalent to the kernel of the map $H^1(L/K, \mathbf{T}(L)) \rightarrow \prod_{w|v} H^1(L_w/K_v, \mathbf{T}(L_w))$ being trivial, as desired. \square

We have shown that PHNP $_K(L_1, \dots, L_n)$ is true if and only if the first Tate-Shafarevich group $\text{III}^1(L/K, \mathbf{T})$ is trivial. It is known that this group is finite, but it is hard to compute the cohomology of $\mathbf{T}(K)$ directly, so we make use of Tate-Nakayama duality.

Using the functor \mathbf{X}^* mapping a torus to its character lattice, we get an easier way to compute, both mathematically and programmatically, the first Tate-Shafarevich group of an algebraic tori. The structure of these lattices are closely linked to the Galois group of the field extension, and oftentimes can easily be expressed in a closed form in terms of this Galois group.

Theorem 4.2 (Tate-Nakayama theorem, [PR94, Theorem 6.2]). *Let K be a global field, let \mathbf{T} be an algebraic torus defined over K and split over L , and let $\mathbf{X}^*(\mathbf{T})$ be its character lattice. Then $\text{III}^1(L/K, \mathbf{T}) \cong \text{III}^2(L/K, \mathbf{X}^*(\mathbf{T}))$.*

Our approach will therefore be to give an explicit description of the character lattice of \mathbf{T} and compute $\text{III}^2(L/K, \mathbf{X}^*(\mathbf{T}))$.

5. PRELIMINARY RESULTS

Thought closely related, neither HNP nor PHNP directly imply one another, so current results do not extend to the PHNP condition easily. Thus, we aim to draw connections between the conditions. It turns out that in some cases, we can directly construct a global solution from the local solutions of PHNP when HNP holds.

Definition 5.1 (*v*-local solution, scale factor). Fix a place v of K . For a given n -tuple $k_1, k_2, \dots, k_n \in K^n$, if there exists $\ell_i \in L_i \otimes K_v$ and a constant $s_v \in K_v$ such that $N_{L_i/K}(\ell_i) = s_v \cdot k_i$ we call $(\ell_1, \ell_1, \dots, \ell_n; s_v)$ a *v*-local solution (relative to k_1, \dots, k_n) with *scale factor* s_v .

Note that the PHNP is equivalent to the assertion that if there exists a *v*-local solution for every v , then there exists a global solution.

Proposition 5.2. *Suppose that $\text{HNP}_K(L_i)$ holds for $i = 1, \dots, n$ and let $k_1, \dots, k_n \in K^n$. If there there some $s \in K$ that is a scale factor for all local solutions, then there exists a global solution.*

Proof. By definition of scale factors, we get that sk_i is a local norm of L_i/K and the Hasse Norm Principle for L_i/K holds hence there is $x_i \in L_i$ so that $N_{L_i/K}(\ell_i) = sk_i$. The tuple (ℓ_1, \dots, ℓ_n) is therefore a global solution. \square

The proposition below further shows that we can always reduce the projective Hasse norm principle to the case where all extensions $L_1/K, L_2/K, \dots, L_n/K$ have degree greater than 1.

Proposition 5.3. *Let $n \geq 2$. We have:*

$$\text{PHNP}_K(L_1, L_2, \dots, L_n, K) = \text{PHNP}_K(L_1, L_2, \dots, L_n).$$

Proof. (\Leftarrow) Assume $\text{PHNP}_K(L_1, L_2, \dots, L_n)$ holds. Let $k_1, \dots, k_{n+1} \in K^\times$ such that v -local solutions exist for all v , relative to the extensions L_1, \dots, L_n, K . This implies that v -local solutions of (k_1, \dots, k_n) exist for all v , relative to L_1, \dots, L_n . We can use $\text{PHNP}_K(L_1, L_2, \dots, L_n)$ to obtain $(\ell_1, \dots, \ell_n) \in L_1 \times \dots \times L_n$ and $s \in K$ so that $sk_i = N_{L_i/K}(\ell_i) k_i$ for all $1 \leq i \leq n$. Since $N_{K/K} = \text{id}$ we get that $(\ell_1, \dots, \ell_n, sk_{n+1})$ is a global solution for (k_1, \dots, k_{n+1}) , relative to the extensions $(L_1, L_2, \dots, L_n, K)$.

(\Rightarrow) Now assume that $\text{PHNP}_K(L_1, L_2, \dots, L_n, K)$ holds. If $k_1, \dots, k_n \in K^\times$ have v -local solutions for all v , with respect to L_1, \dots, L_n , then $k_1, \dots, k_n, 1$ has v -local solutions for all v with respect to L_1, \dots, L_n, K , then we can use $\text{PHNP}_K(L_1, L_2, \dots, L_n, K)$ to get a global solution and restrict it to its first factors to get a global solution relative to $k_1, \dots, k_n, 1$ and the extensions L_1, \dots, L_n . This finishes the proof. \square

Next, we introduce another reduction of PHNP to HNP when L_1, L_2, \dots, L_n are identical.

Proposition 5.4. *If $L_1 = L_2 = \dots = L_n$ for $n \geq 2$, then we must have*

$$\text{PHNP}_K(L_1, L_2, \dots, L_n) = \text{HNP}_K(L_1).$$

Proof. (\Rightarrow) Suppose $\text{PHNP}_K(L_1, L_2, \dots, L_n)$ holds. Let $k \in K^\times$ such that k belongs in the image of the norm of L_1/K locally. Since 1 is always in the image of the norm, there exists a global solution to $(k, 1, 1, \dots, 1)$ with respect to L_1, \dots, L_n . Therefore, there are $s \in K^\times$ and $(\ell_1, \dots, \ell_n) \in L_1 \times \dots \times L_n$ so that $N_{L_1/K}(\ell_1) = sk$ and $N_{L_i/K}(\ell_i) = N_{L_1/K}(\ell_i) = s \cdot 1$. Therefore, $k = N_{L_1/K}(\ell_1/\ell_2)$. This implies that $\text{HNP}_K(L_1)$ holds.

(\Leftarrow) Assume that $\text{HNP}_K(L_1)$ holds. Assume that $k_1, \dots, k_n \in K^\times$ locally scale to a norm of $L_1 \times \dots \times L_n$. This means that for all place v there are $s_v \in K_v^\times$ and $\ell_1, \dots, \ell_n \in L_1 \otimes K_v$ so that $N_{L_1/K}(\ell_i) = s_v k_i$ for all $i \leq n$. This implies that $k_i/k_n = N_{L_1/K}(\ell_i/\ell_n)$. Using the Hasse norm principle, there are $x_1, \dots, x_{n-1} \in L_1$ so that $N_{L_1/K}(x_i) = k_i/k_n$. Let $x_n = 1$. Then for all $1 \leq i \leq n$ we have $N_{L_1/K}(x_i) = \frac{1}{k_n} k_i$, we can take $\frac{1}{k_n}$ as global scale factor. \square

Remark 5.5. Although the previous proposition examines a very restricted setting, it provides us with a very interesting application. In [AAgGG23], the authors establish a mass formula to count the size of isogeny classes of principally polarized abelian varieties over finite fields, weighted by the size of their respective automorphism groups. This extends to yield a similar formula for elliptic curves. The formula is very concrete, except one constant: the Tamagawa number of a specific global torus, which can be seen locally as the centralizer of the Frobenius element acting on the dual of Tate groups.

This Tamagawa number can be written $\tau(\mathbf{T}) = \frac{|H^1(\mathbb{Q}, \mathbf{X}^*(\mathbf{T}))|}{|\text{III}^1(\mathbf{T})|}$, and the real difficulty in its calculation is to determine the denominator.

One can apply the work done in [AAgGG23] and apply it to a product of elliptic curves, and count either isogenous abelian surfaces, or modify the formula to restrict

the count to other products of elliptic curves. In both cases, the same Tamagawa number computation arises. For one elliptic curve, the torus is a maximal torus of $\mathrm{GL}_2(\mathbb{Q})$ which is either split or a restriction of scalars and has a trivial Tamagawa number. For two elliptic curves, however, we get a maximal torus of $(\mathrm{GL}_2 \times \mathrm{GL}_2)^0(\mathbb{Q}) \subset \mathrm{GSp}_4(\mathbb{Q})$ where $(\mathrm{GL}_2 \times \mathrm{GL}_2)^0$ is the group of pairs of matrices $(g_1, g_2) \in \mathrm{GL}_2$ with matching determinants. The most interesting case is when the torus is compact modulo center (the *elliptic* case), and the corresponding Tate-Shafarevich group is exactly the obstruction to $\mathrm{PHNP}_K(L_1, L_2)$ where L_i is generated by eigenvalues of g_i .

6. DESCRIPTION OF CHARACTER LATTICES

Recall that we have the following exact sequence between algebraic tori by mapping each element of \mathbf{T} to its shared norm under $\prod_i N_{L_i/K}$:

$$1 \longrightarrow \prod_i R_{L_i/K}^{(1)} \mathbb{G}_m \longrightarrow \mathbf{T} \longrightarrow \mathbb{G}_m \longrightarrow 1.$$

Let $\Lambda := \mathbf{X}^*(\mathbf{T})$ and $\Lambda^1 := \mathbf{X}^*(\prod_i R_{L_i/K}^{(1)} \mathbb{G}_m)$. From this, we can obtain a short exact sequence on the character lattice of each tori as in Example 5.1 by taking the dual of the previous exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Lambda \longrightarrow \Lambda^1 \longrightarrow 0. \tag{6.1}$$

We may now describe the character lattice of \mathbf{T} .

Proposition 6.1. *Let S_i denote the left coset space $G/G^{(i)}$ equipped with a left G -action. Let $d_i = \sum_{g \in S_i} g \in \mathbb{Z}[S_i]$ for each i . Then it follows that*

$$\mathbf{X}^*(\mathbf{T}) \cong \left(\prod_i \mathbb{Z}[S_i] \right) / \left\{ (\lambda_1 d_1, \lambda_2 d_2, \dots, \lambda_n d_n) : \lambda_i \in \mathbb{Z}, \sum_i \lambda_i = 0 \right\}.$$

Proof. The embeddings $\mathbf{T}^1(L) \subset \mathbf{T}(L) \subset R_{L/K} \mathbb{G}_m(L)$ yield the surjections

$$\bigoplus_i \mathbb{Z}[S_i] = \mathbf{X}^*(R_{L/K} \mathbb{G}_m) \xrightarrow{\varphi} \mathbf{X}^*(\mathbf{T}) \rightarrow \mathbf{X}^*(\mathbf{T}^1) = \bigoplus_i \mathbb{Z}[S_i] / \langle d_i \rangle.$$

Note that d_i corresponds at the norm map, viewed as a character, and since elements of $\mathbf{T}(L)$ are tuples $\ell = (\ell_1, \dots, \ell_n)$ with matching norm η , we get that an element $\lambda_1 d_1 \oplus \dots \oplus \lambda_n d_n$ maps ℓ to

$$\prod_{i=1}^n N_{L_i/K}(\ell_i)^{\lambda_i} = \eta^{\sum_{i=1}^n \lambda_i}.$$

We just showed that $\{(\lambda_1 d_1, \lambda_2 d_2, \dots, \lambda_n d_n) : \lambda_i \in \mathbb{Z}, \sum_i \lambda_i = 0\}$ is in the kernel of the surjection φ . By rank-checking, we may conclude that this is exactly the kernel of φ and we are done. \square

In the particular case where each extension is Galois, we get the following.

Corollary 6.2. *Suppose L_1, L_2, \dots, L_n are all Galois over K . Let $d_i = \sum_{g \in G_i} g \in \mathbb{Z}[G_i]$ for all i . Then $\mathbf{X}^*(\mathbf{T}) \cong (\prod_i \mathbb{Z}[G_i]) / \{(\lambda_1 d_1, \lambda_2 d_2, \dots, \lambda_n d_n) : \lambda_i \in \mathbb{Z}, \sum_i \lambda_i = 0\}$.*

Proof. When L_i is Galois, then $S_i = G_i$ is a group. □

This description lets us use SageMath to build the character lattice in an elementary way, using the SageMath package for algebraic tori implemented by the second author. Below is an example where L_1, L_2 are independent quadratic extensions of K .

```
def MakeLattice(G, H1, H2):
    L1 = GLattice(H1, 1)
    G = KleinFourGroup()
    H1, H2 = [G.subgroups()[1], G.subgroups()[2]]
    IL1 = GLattice(H1, 1).induced_lattice(G)
    IL2 = GLattice(H2, 1).induced_lattice(G)
    IL = IL1.direct_sum(IL2)
    a, b = IL.fixed_sublattice().basis()
    HNPLattice = IL.quotient_lattice(IL.fixed_sublattice())
    PHNPLattice = IL.quotient_lattice(IL.sublattice([a-b]))
    return [PHNPLattice, HNPLattice]
```

This code returns the pair $(\mathbf{X}^*(\mathbf{T}), \bigoplus_{i=1}^n \mathbf{X}^*(\mathbf{T}_{L_i/K}^1))$.

This description allows us to solve some cases algorithmically.

Proposition 6.3. *If L_1, L_2 are quadratic extensions, then $\text{PHNP}_K(L_1, L_2)$ holds.*

Proof. For the case where $L_1 \neq L_2$, we run the following SageMath code:

```
sage: G = KleinFourGroup()
sage: H1, H2 = [G.subgroups()[1], G.subgroups()[2]]
sage: PHNP = MakeLattice(G, H1, H2)[0]
sage: PHNP.Tate_Shafarevich_lattice(2)
[]
```

This demonstrates that PHNP holds for any pair of distinct quadratic extensions. We will see in Proposition 5.4 that both PHNP also holds if both quadratic extensions are the same. □

Notably in the above proposition, all quadratic extensions are Galois and have abelian Galois groups. In later sections, we generalize this result.

7. COUNTEREXAMPLE TO PHNP \implies HNP

Though the two conditions may appear equivalent for lower degree choices of L_i , once we choose extensions of degree 4 or higher, counterexamples appear quite often. To find our counterexamples as well as calculate cohomology groups, we used SageMath to compute the results using the code shown in Appendix A.

In this section, we assert that L_1, L_2 are Galois. Using our result and notations from 6.2 and denoting the character lattice $\mathbf{X}^*(\mathbf{T})$ as Λ and the lattice $\{(\lambda_i d_i) \mid \lambda_i \in \mathbb{Z}, \sum_i \lambda_i = 0\}$ as Λ^1 , we have the following short exact sequence:

$$1 \longrightarrow \Lambda^1 \longrightarrow \prod_i \mathbb{Z}[G_i] \longrightarrow \Lambda \longrightarrow 1.$$

Using Galois cohomology, we can obtain the long exact sequence

$$1 \longrightarrow H^0(G, \Lambda^1) \longrightarrow H^0(G, \prod_i \mathbb{Z}[G_i]) \longrightarrow H^0(G, \Lambda) \longrightarrow H^1(G, \Lambda^1) \longrightarrow \dots$$

Example 7.1. In particular, we investigate the case where $L_1 \subset L_2$ and each extension in the tower $L_2/L_1/K$ is quadratic. This boils down to picking $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, choosing $n = 2$, $G_1 = \mathbb{Z}/2\mathbb{Z}$, and $G_2 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Furthermore, define $G = G_2$ and $N = \text{Gal}(L_2/L_1)$. We have that $\Lambda^1 \cong \mathbb{Z}$, and $\prod_i \mathbb{Z}[G_i] = \mathbb{Z}[G] \times \mathbb{Z}[N]$. We can then compute the cohomology groups in the sequence near $H^2(G, \Lambda)$ using Shapiro's lemma (see [Mil97], p.62):

$$H^i(G, \mathbb{Z}[G] \times \mathbb{Z}[N]) = H^i(G, \mathbb{Z}[G]) \times H^i(G, \mathbb{Z}[N]) = H^i(G/N, \mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & 2 \mid i \\ 1 & 2 \nmid i \end{cases}.$$

It is well known that $H^2(G, \mathbb{Z}) \cong G^{\text{ab}}$ and $H^3(G, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ by Lyndon's formula (see [Lyn48]), giving us the following exact sequence:

$$1 \longrightarrow H^1(G, \Lambda) \longrightarrow (\mathbb{Z}/2\mathbb{Z})^2 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow H^2(G, \Lambda) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.$$

From this sequence, we know that we must have $H^1(G, \Lambda) \in \{\mathbb{Z}/2\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^2\}$ and $H^2(G, \Lambda) \in \{\mathbb{Z}/2\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^2, \mathbb{Z}/4\mathbb{Z}\}$. By computing these cohomology groups explicitly in SageMath, we obtain that $H^1(G, \Lambda) \cong H^2(G, \Lambda) \cong \mathbb{Z}/2\mathbb{Z}$.

We can now repeat the same process to compute these cohomology groups for two nontrivial subgroups N, H where H is a complement of N in G . For N , we obtain

$$1 \longrightarrow H^1(N, \Lambda) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1 \longrightarrow H^2(N, \Lambda) \longrightarrow 1 \longrightarrow 1.$$

So we have that $H^1(N, \Lambda) \cong \mathbb{Z}/2\mathbb{Z}$ and $H^2(N, \Lambda) \cong 1$. Similarly, for H , we obtain

$$1 \longrightarrow H^1(H, \Lambda) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow (\mathbb{Z}/2\mathbb{Z})^2 \longrightarrow H^2(H, \Lambda) \longrightarrow 1 \longrightarrow 1.$$

Computing these cohomology groups in SageMath gives $H^1(H, \Lambda) \cong 1$, $H^2(H, \Lambda) \cong \mathbb{Z}/2\mathbb{Z}$. Furthermore, the program tell us that $\text{Ker}(H^2(G, \Lambda) \rightarrow H^2(H, \Lambda)) \cong 1$. Hence $\text{III}^2(G, \Lambda) \cong 1$, so $\text{III}^1(\mathbf{T})$ is trivial in this example if none of the decomposition groups are equal to G . (It is well known that this can only happen over ramified primes in K .)

Now, choosing $L_1 = \mathbb{Q}(\sqrt{-3})$, $L_2 = \mathbb{Q}(\sqrt{-3}, \sqrt{13})$ gives exactly the desired selection of Galois groups, so it follows that $\text{PHNP}_{\mathbb{Q}}(L_1, L_2)$ holds in this case. However, it is shown in [Has31] that $\mathbb{Q}(\sqrt{-3}, \sqrt{13})$ does not satisfy the Hasse norm principle, hence this example demonstrates that PHNP does not imply HNP in each of the constituent fields.

Similarly, we can take $L_1 = \mathbb{Q}(\sqrt{p})$ and $L = L_2 = \mathbb{Q}(\sqrt{p}, \sqrt{q})$, where p, q are prime numbers such that $\left(\frac{p}{q}\right) = 1$ and $p, q \equiv 1 \pmod{4}$, which ensures that the decomposition groups of $\text{Gal}(L/K)$ are always cyclic. Such a pair verifies $\text{PHNP}_{\mathbb{Q}}(L_1, L_2)$ however it is well known that $\text{III}^1(L_2/\mathbb{Q}, R_{L_2/\mathbb{Q}}^{(1)}\mathbb{G}_m) \cong \mathbb{Z}/2\mathbb{Z}$ hence $\text{HNP}_{\mathbb{Q}}(L_2)$, and by extension $\text{HNP}_{\mathbb{Q}}(L_1) \wedge \text{HNP}_{\mathbb{Q}}(L_2)$, does not hold.

8. A CONDITION FOR $\text{HNP} \implies \text{PHNP}$

In contrast to $\text{PHNP} \implies \text{HNP}$, counterexamples to $\text{HNP} \implies \text{PHNP}$ are far more sparse. In this section, we find sufficient conditions for identifying cases where HNP in all of the constituent fields implies PHNP based off work in [Rüd22] and [LOY24].

Proposition 8.1. *Suppose $\bigwedge_i \text{HNP}_K(L_i)$ holds. Let x be any element of $\text{III}^2(\Lambda) \subseteq H^2(\Lambda)$, and let ι be the canonical map from $H^2(\Lambda)$ to $H^2(\Lambda^1)$ defined from equation 6.1. Then $x \in \text{Ker}(\phi)$.*

Proof. We begin by taking the cohomology of the exact sequence in equation (6.1). We get

$$\dots \longrightarrow H^2(\mathbb{Z}) \longrightarrow H^2(\Lambda) \longrightarrow H^2(\Lambda^1) \longrightarrow \dots$$

Consider the following induced diagram.

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \text{III}^2(\mathbb{Z}) & & \text{III}^2(\Lambda) & & \text{III}^2(\Lambda^1) \\ & & \downarrow & & \downarrow \iota & & \downarrow \\ \dots & \longrightarrow & H^2(\mathbb{Z}) & \xrightarrow{e} & H^2(\Lambda) & \xrightarrow{\phi} & H^2(\Lambda^1) \longrightarrow \dots \\ & & \downarrow & & \downarrow a & & \downarrow d \\ \dots & \longrightarrow & \prod H^2(D, \mathbb{Z}) & \longrightarrow & \prod H^2(D, \Lambda) & \xrightarrow{b} & \prod H^2(D, \Lambda^1) \longrightarrow \dots \end{array}$$

Now, since we suppose that $\bigwedge_i \text{HNP}_K(L_i)$ holds, it follows that $\text{III}^2(\Lambda^1) = 0$. Hence d is injective. Since $\text{Ker } a = \text{Im } \iota$, we have $b(a(\iota(x))) = 0$, so $d(c(\iota(x))) = 0$. Since ι, d are both injective, it follows that $\iota(x) \in \text{Ker } \phi$ as desired. \square

Notice furthermore that $\text{Im } e = \text{Ker } \phi$. Therefore if HNP holds, equivalently if d is injective, we have that $\text{Im } \iota \subseteq \text{Im } e$. Note that if e is trivial, it follows that $\text{III}^2(\Lambda)$ is trivial, so PHNP must hold with the assumption that $\bigwedge_i \text{HNP}_K(L_i)$ is true. Following the notations of [LOY24], let $H^2(\mathbb{Z})' := \{x \in H^2(\mathbb{Z}) : e(x) \in \text{Im } \iota\}$. We can now extract a set of sufficient conditions for when HNP implies PHNP using decomposition groups.

Proposition 8.2. *Suppose $\bigwedge_i \text{HNP}_K(L_i)$ holds. Let ψ be the map from $\prod H^1(D, \Lambda^1)$ to $\prod H^2(D, \mathbb{Z})$ in the commutative diagram from Proposition 8.1. Then $|\text{III}^2(\Lambda)| = \frac{|H^2(\mathbb{Z})'|}{|\text{Ker } e|}$. In particular, if $|\text{Im } \psi| = |\text{Ker } e|$ then PHNP holds.*

Proof. First, we show that $\text{III}^2(\mathbb{Z}) = 0$. Suppose otherwise, then there exists a nonzero map $f \in H^2(G, \mathbb{Z}) \cong G^\vee$ in $\text{III}^2(\mathbb{Z})$. In particular, there exists some $g \in G$ such that $f(g) \neq 0$. Then $\langle g \rangle \subset G$ is a cyclic subgroup of G , so by the Chebotarev density theorem (see [Mil97], p.164), it follows that $\langle g \rangle$ is a decomposition group. However $f|_{\langle g \rangle} \neq 0$, giving contradiction.

Now, we can redraw the commutative diagram from Proposition 8.1, shifting the sequence one term to the left and labeling functions e, f, g, a, ξ as shown.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \text{III}^1(\Lambda^1) & & 0 & & \text{III}^2(\Lambda) \\
 & & \downarrow & & \downarrow & & \downarrow \iota \\
 \dots & \longrightarrow & H^1(\Lambda^1) & \xrightarrow{\xi} & H^2(\mathbb{Z}) & \xrightarrow{e} & H^2(\Lambda) \longrightarrow \dots \\
 & & \downarrow & & \downarrow f & & \downarrow a \\
 \dots & \longrightarrow & \prod H^1(D, \Lambda^1) & \xrightarrow{\psi} & \prod H^2(D, \mathbb{Z}) & \xrightarrow{g} & \prod H^2(D, \Lambda) \longrightarrow \dots
 \end{array}$$

In particular, since ι is injective, the image of $H^2(\mathbb{Z})'$ under e is exactly $\text{Im } \iota$. Thus $|\text{III}^2(\Lambda)| = |\text{Im } \iota| = \frac{|H^2(\mathbb{Z})'|}{|\text{Ker } e|}$.

Furthermore, for any element x of $H^2(\mathbb{Z})'$, $e(x) \in \text{Im } \iota$ so $a(e(x)) = 0$. Thus $f(x) \in \text{Ker } g$ so $f(x) \in \text{Im } \psi$. Thus since f is injective, it follows that $|H^2(\mathbb{Z})'| \leq |\text{Im } \psi|$ so if $|\text{Im } \psi| = |\text{Ker } e|$ then $1 = \frac{|\text{Im } \psi|}{|\text{Ker } e|} \geq \frac{|H^2(\mathbb{Z})'|}{|\text{Ker } e|} = |\text{III}^2(\Lambda)|$ as desired. \square

Corollary 8.3. *If $\text{Ker } e = H^2(\mathbb{Z}) \cong G^\vee$, then $\bigwedge_i \text{HNP}_K(L_i) \implies \text{PHNP}_K(L_1, L_2, \dots, L_n)$.*

Proof. It follows from Proposition 8.2 that $1 \geq \frac{|H^2(\mathbb{Z})'|}{H^2(\mathbb{Z})} = |\text{III}^2(\Lambda)|$, so $\text{III}^2(\Lambda) = 0$ as desired. \square

Now that we have a path to determining when PHNP holds in a given collection of field extensions, we hope to understand the map e so that we can explicitly compute its kernel and $H^2(\mathbb{Z})'$. We proceed by analyzing the map ξ .

Proposition 8.4. *Let $\Lambda_i^1 = \text{Ind}_{G^{(i)}}^G \mathbb{Z}/\langle d_i \rangle$ for any i . It follows that $H^1(G, \Lambda_i^1) \cong \{f : G \rightarrow \mathbb{Q}/\mathbb{Z} : f|_{G^{(i)}} \equiv 0\}$.*

Proof. Recall from Proposition 6.1 that

$$\begin{aligned} \Lambda^1 &= \bigoplus_{i=1}^n \text{Ind}_{G^{(i)}}^G \mathbb{Z}/\langle d_i \rangle \\ &\cong \bigoplus_{i=1}^n \mathbb{Z}[G/G^{(i)}] / \left\langle \sum_{g \in G/G^{(i)}} g \right\rangle. \end{aligned}$$

Thus we have $H^i(G, \Lambda^1) = \bigoplus_i H^i(G, \Lambda_i^1)$.

Letting diag_i denote the map from $k \in \mathbb{Z}$ to kd_i , we obtain the following short exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\text{diag}_i} \text{Ind}_{G^{(i)}}^G \mathbb{Z} \longrightarrow \Lambda_i^1 \longrightarrow 0.$$

Computing the cohomology, we obtain:

$$\dots \rightarrow H^1(G, \text{Ind}_{G^{(i)}}^G \mathbb{Z}) \rightarrow H^1(G, \Lambda_i^1) \rightarrow H^2(G, \mathbb{Z}) \rightarrow H^2(G, \text{Ind}_{G^{(i)}}^G \mathbb{Z}) \rightarrow \dots$$

Note that $H^1(G, \text{Ind}_{G^{(i)}}^G \mathbb{Z}) \cong H^1(G^{(i)}, \mathbb{Z}) = 0$ by Shapiro's Lemma and Hilbert 90 (see [Mil97], p.67), $H^2(G, \mathbb{Z}) = G^\vee$, and $H^2(G, \text{Ind}_{G^{(i)}}^G \mathbb{Z}) \cong H^2(G^{(i)}, \mathbb{Z}) = (G^{(i)})^\vee$. Rewriting the cohomology, we have

$$0 \longrightarrow H^1(G, \Lambda_i^1) \longrightarrow G^\vee \xrightarrow{r_i} (G^{(i)})^\vee \longrightarrow \dots$$

It follows that $H^1(G, \Lambda_i^1) \cong \text{Ker } r_i = \{f : G \rightarrow \mathbb{Q}/\mathbb{Z} : f|_{G^{(i)}} \equiv 0\}$ as desired. \square

In particular, analogously to [LOY24], it follows the map ξ is the sum map, obtained by taking the f_i such that $H^1(\Lambda^1) = \bigoplus_i H^1(\Lambda_i^1) = (f_1, f_2, \dots, f_n)$ and summing them to obtain $f = f_1 + f_2 + \dots + f_n \in G^\vee = H^2(G, \mathbb{Z})$. Hence we obtain the following proposition.

Proposition 8.5. *We have*

$$\text{Ker } e = \text{Im } \xi = \left\{ f \in G^\vee : \forall i, \exists f_i \in G^\vee, f_i|_{G^{(i)}} \equiv 0, f = \sum_i f_i \right\}$$

where $f_1, f_2, \dots, f_n \in G^\vee$.

Remark 8.6. Both of Propositions 8.4 and 8.5 can be adapted for a different choice of Galois group $D \subseteq G$. It suffices to replace $G^{(i)}$ with $D^{(i)} = G^{(i)} \cap D$ and redefine $\Lambda_i^1 = \text{Ind}_{D^{(i)}}^D$ throughout the proof. These changes yield the following proposition.

Proposition 8.7. *Let e_D denote the map $H^2(D, \mathbb{Z}) \rightarrow H^2(D, \Lambda)$ given in the commutative diagram. We have*

$$\text{Ker } e_D = \left\{ f \in D^\vee : \forall i, \exists f_i \in D^\vee, f_i|_{D^{(i)}} \equiv 0, f = \sum_i f_i \right\}$$

where $f_1, f_2, \dots, f_n \in D^\vee$.

Using the above result, we can also deduce an explicit formulation for $H^2(\mathbb{Z})'$ analogous to that of Proposition 8.5 for $\text{Ker } e$.

Proposition 8.8. *The set of $H^2(\mathbb{Z})' \subseteq H^2(\mathbb{Z}) \cong G^\vee$ is exactly $\{f : G \rightarrow \mathbb{Q}/\mathbb{Z} : \forall D, f|_D \in \text{Ker } e_D\}$ where D is chosen over all decomposition groups of G .*

Proof. Recall the following commutative diagram from the proof of Proposition 8.2.

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \text{III}^1(\Lambda^1) & & 0 & & \text{III}^2(\Lambda) \\ & & \downarrow & & \downarrow & & \downarrow \iota \\ \dots & \longrightarrow & H^1(\Lambda^1) & \xrightarrow{\xi} & H^2(\mathbb{Z}) & \xrightarrow{e} & H^2(\Lambda) & \longrightarrow \dots \\ & & \downarrow & & \downarrow f & & \downarrow a \\ \dots & \longrightarrow & \prod H^1(D, \Lambda^1) & \xrightarrow{\psi} & \prod H^2(D, \mathbb{Z}) & \xrightarrow{g} & \prod H^2(D, \Lambda) & \longrightarrow \dots \end{array}$$

By definition, $H^2(\mathbb{Z})'$ is exactly the set of elements x of $H^2(\mathbb{Z})$ such that $a(e(x)) = 0$. Since this diagram is commutative, it follows that $g(f(x)) = 0$. However f is injective and given by the restriction map to each decomposition group D . Thus $H^2(\mathbb{Z})'$ is exactly the set of maps $f \in G^\vee$ such that $f|_D \in \text{Ker } e_D$ for every decomposition group D , since g can be decomposed as a direct product of e_D for each decomposition group D of G . \square

These tools gives us an explicit way to prove that PHNP holds in a given choice of L_1, L_2, \dots, L_n by combining our formulation in Proposition 8.5 and the result in Proposition 8.2. We apply these tools to the simple case when L_1, L_2, \dots, L_n are all Galois and independent:

Proposition 8.9. *If L_1, L_2, \dots, L_n are all Galois over \mathbb{Q} and $G \cong \prod_i G_i$, then*

$$\bigwedge_i \text{HNP}_K(L_i) \implies \text{PHNP}_K(L_1, L_2, \dots, L_n).$$

Equivalently, $G^\vee \cong \prod_i G_i^\vee$.

Proof. Since $G = \prod_i G_i$, each element of G can be represented as (g_1, g_2, \dots, g_n) such that $g_i \in G_i$ for each i and $G_i = \{(g_1, g_2, \dots, g_n) : \forall j \neq i, g_j = 0\}$. For any choice of $f \in G^\vee$, choosing f_i such that $f_i(g_1, g_2, \dots, g_n) = f(0, \dots, 0, g_i, 0, \dots, 0)$ satisfies the conditions on f_i and $f = \sum_i f_i$ given in Proposition 8.5.

Thus $f \in \text{Ker } e$. Hence $H^2(G, \mathbb{Z}) \subseteq \text{Ker } e \subseteq H^2(G, \mathbb{Z})$, so $\text{Ker } e = H^2(G, \mathbb{Z})$. Applying Proposition 8.2, it follows that PHNP holds. \square

9. COUNTEREXAMPLE TO HNP \implies PHNP

In this section, we construct a counterexample to the claim that $\bigwedge_i \text{HNP}_K(L_i) \implies \text{PHNP}_K(L_1, L_2, \dots, L_n)$, showing that neither PHNP or HNP directly implies the other. We provide an explicit example.

Example 9.1. Let L be the extension of $K = \mathbb{Q}$ generated by the roots of $x^4 - x + 1$. It is known that $\text{Gal}(L/\mathbb{Q}) \cong S_4$ (from [LMF24]). Let $H_1 = \langle (12) \rangle$ and $H_2 = \langle (1234) \rangle$ be subgroups of $\text{Gal}(L/\mathbb{Q})$ and let L_1, L_2 be the subfields of L fixed by these two subgroups respectively. From LMFDB, we know that the only ramified prime of L is 229, whose decomposition group is $\mathbb{Z}/2\mathbb{Z}$.

Thus all decomposition groups of $\text{Gal}(L/\mathbb{Q})$ are cyclic. It can be furthermore verified that HNP is satisfied in L_1 and L_2 respectively.

Now, consider Proposition 8.2, which states that $|\text{III}^2(\Lambda)| = \frac{|H^2(\mathbb{Z})'|}{|\text{Ker } e|}$. The denominator $\text{Ker } e$ can be computed through Proposition 8.5: for any $g = aba^{-1}b^{-1} \in G^{\text{der}}$, we have $f(g) = f(a)f(b)f(a^{-1})f(b^{-1}) = 0$ for $f : G \rightarrow \mathbb{Q}/\mathbb{Z}$.

Since the commutator subgroup of S_4 is A_4 , it follows that the only two maps f in $H^2(\mathbb{Z})$ are $f \equiv 0$ and

$$f = \begin{cases} 0 & g \in A_4 \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

However, since $G^{(1)} = \text{Gal}(L/L_1) = H_1$ and $G^{(2)} = \text{Gal}(L/L_2) = H_2$, the latter map is nonzero over both H_1 and H_2 , so it follows that the only choice for f_1 and f_2 is the trivial map. Thus, $\text{Ker } e$ is trivial.

Next, we show that $H^2(\mathbb{Z})'$ contains at least two elements, which would imply that $\text{III}^2(\Lambda)$ is nontrivial and thus that $\text{PHNP}_{\mathbb{Q}}(L_1, L_2)$ is false.

It suffices to show that f satisfies the conditions of Proposition 8.8. First, notice that no cyclic subgroup of S_4 intersects both $\langle(1234)\rangle$ and $\langle(12)\rangle$, so for each decomposition group D , it follows that we can solve $f|_D = f_1 + f_2$ by choosing $\{f_1, f_2\} = \{0, f|_D\}$ depending on whether $D^{(1)}$ or $D^{(2)}$ is trivial. Hence $f \in \text{Ker } e_D$ for all decomposition groups D so $f \in H^2(\mathbb{Z})'$ as desired.

We now know that $\text{HNP} \implies \text{PHNP}$ holds for independent Galois field extensions but does not hold in the non-Galois case. It remains to be shown whether it is true that HNP implies PHNP when L_1, L_2, \dots, L_n are non-independent Galois field extensions. We give a few partial results in this direction.

10. STUDYING $\text{HNP} \implies \text{PHNP}$ IN THE GALOIS CASE

We proceed to derive a set of sufficient conditions that allow us to prove $\text{HNP} \implies \text{PHNP}$ using only the Galois groups of the fields L_1, L_2, \dots, L_n . In this section, we exclusively consider the case where each L_i/K is a Galois extension.

Proposition 10.1. *If for any $f \in G^\vee$, there exists functions $f_i \in G^\vee$ for each i such that $f_i|_{G^{(i)}} \equiv 0$ and such that $f = \sum_i f_i$, then $\bigwedge_i \text{HNP}_K(L_i) \implies \text{PHNP}_K(L_1, L_2, \dots, L_n)$.*

Proof. First, recall from Proposition 8.5 that

$$\text{Ker } e = \{f \in G^\vee : \forall i, \exists f_i \in G^\vee, f_i|_{G^{(i)}} \equiv 0, f = \sum_i f_i\}.$$

If it is true that $\text{Ker } e \cong H^2(\mathbb{Z})$, then Proposition 8.2 implies that $|\text{III}^2(\Lambda)| = \frac{|H^2(\mathbb{Z})'|}{|\text{Ker } e|} \leq \frac{|H^2(\mathbb{Z})|}{|\text{Ker } e|} = 1$, so PHNP holds. \square

This weaker condition for PHNP allows us to study $\text{HNP} \implies \text{PHNP}$ using only the structure of the Galois group G . The following proposition gives us an algorithmic approach to verify the conditions of Proposition 10.1 in G .

Proposition 10.2. *If for any $g \in G$ and $1 \leq i \leq n$, we have that $gG^{(i)} \subseteq G^{\text{der}}G^{(i)} \implies g \in G^{\text{der}}$, then it follows that $\bigwedge_i \text{HNP}_K(L_i) \implies \text{PHNP}_K(L_1, L_2, \dots, L_n)$. Equivalently, if $\bigcap_i G^{\text{der}}G^{(i)} = G^{\text{der}}$, then it follows that $\bigwedge_i \text{HNP}_K(L_i) \implies \text{PHNP}_K(L_1, L_2, \dots, L_n)$.*

Proof. Note that $G^{(i)} \trianglelefteq G$ for every i from the Galois condition. We have the equality $(G/G^{(i)})^{\text{der}} = G^{\text{der}}G^{(i)}/G^{(i)}$ since for any $g_1, g_2 \in G$,

$$g_1G^{(i)}g_2G^{(i)}(g_1G^{(i)})^{-1}(g_2G^{(i)})^{-1} = (g_1g_2g_1^{-1}g_2^{-1})G^{(i)}.$$

Let α be the homomorphism from G^{ab} to $(G/G^{(1)})^{\text{ab}} \times (G/G^{(2)})^{\text{ab}} \times \dots \times (G/G^{(n)})^{\text{ab}}$ obtained by mapping each $g \in G^{\text{ab}}$ to the coset of $G^{\text{ab}}/G^{(i)} \cong (G/G^{(i)})^{\text{ab}}$ containing it. Here, $G^{\text{der}}G^{(i)}$ is exactly the kernel of the abelianization map from $G/G^{(i)}$ to $(G/G^{(i)})^{\text{ab}}$.

Now, α is injective if and only if there does not exist a nonzero element $g \in G^{\text{ab}}$ such that

$$\alpha(g) = 0 \iff \forall i, gG^{(i)} \subseteq G^{\text{der}}G^{(i)}.$$

Note that the existence of such an element is equivalent to the conditions given in the proposition, so α is injective.

Since $(G^{\text{ab}})^{\vee} \cong ((G^{\vee})^{\vee})^{\vee} \cong G^{\vee}$, consider the dual map $\chi : (G/G^{(1)} \times G/G^{(2)} \times \cdots \times G/G^{(n)})^{\vee} \cong (G/G^{(1)})^{\vee} \times (G/G^{(2)})^{\vee} \times \cdots \times (G/G^{(n)})^{\vee} \rightarrow G^{\vee}$ of α , where the isomorphism is defined by mapping $f(x_1, x_2, \dots, x_n) \in (G/G^{(1)} \times G/G^{(2)} \times \cdots \times G/G^{(n)})^{\vee}$ to (f_1, f_2, \dots, f_n) such that $f_i = f(1, 1, \dots, x_i, \dots, 1, 1) \in (G/G^{(i)})^{\vee}$ for each i . Since $(G^{\text{ab}})^{\text{der}} = 0$, χ is surjective if and only if α is injective.

Now, it follows for our description of the isomorphism that χ is the sum map, defined by mapping $(f_1, f_2, \dots, f_n) \in (G/G^{(1)})^{\vee} \times (G/G^{(2)})^{\vee} \times \cdots \times (G/G^{(n)})^{\vee}$ to $f \in G^{\vee}$ such that for every $g \in G$, we define $f(g) = \sum f_i(g_i)$ where $(g_1, g_2, \dots, g_n) = \alpha(g)$.

Thus, it follows that $\text{Im } \chi$ is exactly $\text{Ker } e$, so since χ is surjective into $G^{\vee} \cong H^2(\mathbb{Z})$, it follows from Proposition 10.1 that $\bigwedge_i \text{HNP}_K(L_i) \implies \text{PHNP}_K(L_1, L_2, \dots, L_n)$ as desired. \square

We use this result to prove that $\text{HNP} \implies \text{PHNP}$ in several classes of G , including when G is abelian or dihedral.

Theorem 10.3. *If L_i/K is Galois and abelian for each i , then $\bigwedge_i \text{HNP}_K(L_i) \implies \text{PHNP}_K(L_1, L_2, \dots, L_n)$.*

Proof. For any abelian group A , we have $A^{\text{der}} = 0$ and $A^{\text{ab}} \cong A$. Since G directly embeds into $\prod_i G/G^{(i)}$, which must be abelian, it follows that G is abelian and thus $G^{\text{der}} = 0$. Therefore by Proposition 10.2, it suffices to show that there is no nontrivial $g \in G$ such that $g \in \bigcap_i G^{(i)}$. This follows from Lemma 3.1. \square

This result immediately gives us an analogue of the Hasse norm theorem for the projective Hasse norm principle:

Corollary 10.4. *If L_1, L_2, \dots, L_n are cyclic Galois field extensions, then it follows that $\text{PHNP}_K(L_1, L_2, \dots, L_n)$ holds.*

Proposition 10.2 can also be used to tackle many nonabelian cases. To demonstrate this, we show below that this proposition is sufficient to address the case where G is dihedral.

Theorem 10.5. *If G is dihedral and L_i/K is Galois for each i , then $\bigwedge_i \text{HNP}_K(L_i) \implies \text{PHNP}_K(L_1, L_2, \dots, L_n)$.*

Proof. Let $G = D_n = \langle r, s : r^n = s^2 = (sr)^2 = 1 \rangle$, then we have that $G^{\text{der}} = \langle r^2 \rangle$. By Proposition 10.2, it suffices to show that there does not exist $g \in \bigcap_i G^{\text{der}}G^{(i)}$ such that $g \notin G^{\text{der}}$.

Suppose such a g exists. If $g = r^m$ for some m then n is even and m is odd. Furthermore, it follows that $r = gr^{1-m} \in G^{(i)}$ for all m , however this gives contradiction by Lemma 3.1.

Otherwise, if $g = r^m s$ for some s . Then for each $G^{(i)}$, there exists k such that $r^k s \in G^{(i)}$. Since each $G^{(i)} \leq G$, it follows that $r^{k+2} s = r r^k s r^{-1} \in G^{(i)}$, so $r^2 \in G^{(i)}$ for all i . This gives contradiction by Lemma 3.1 for all $n \geq 3$. Otherwise, if $n = 1, 2$ then G is abelian, so the result follows from Theorem 10.3 as desired. \square

By running the Oscar code in Appendix B, we found that in the groups we investigated, the conditions of Proposition 10.2 held in all groups G whose order is quarticfree. We prove two theorems supporting this observation:

Theorem 10.6. *If G has order p^3 for some prime p , then we have $\bigwedge_i \text{HNP}_K(L_i) \implies \text{PHNP}_K(L_1, L_2, \dots, L_n)$.*

Proof. If G is abelian, then the result follows immediately from Theorem 10.3. Otherwise, it follows that G^{der} is a nontrivial normal subgroup of G . If any of $G^{(i)}$ are trivial, it follows that $L_i = K$ and can be removed using Proposition 5.3. Now for each i , we have that $G^{(i)}$ is a normal subgroup of G , so since $|G/G^{(i)}| \in \{1, p, p^2\}$ for all i , so it follows that $G/G^{(i)}$ is an abelian group. However by definition, G^{der} is the minimal subgroup G' of G such that G/G' is abelian. Thus, it follows that $G^{\text{der}} \subseteq G^{(i)}$ for all i . However, since G^{der} is nontrivial, this contradicts Lemma 3.1, finishing. \square

Proposition 10.7. *Suppose $G \cong \mathbb{Z}/p\mathbb{Z} \times H$ where H is a nonabelian group of order p^3 . Then there exists a choice of $G^{(1)}, G^{(2)}$ such that $G^{\text{der}}G^{(1)} \cap G^{\text{der}}G^{(2)} \neq G^{\text{der}}$.*

Proof. Any p -group has a nontrivial center (by the class equation), so H must have a normal subgroup H' of order p . Since H/H' is abelian, it follows that, $H^{\text{der}} = H' \cong \mathbb{Z}/p\mathbb{Z}$. Suppose G is isomorphic to $H \times \langle \alpha \rangle$ where α has order p , and let β be a generator of H^{der} . Now, choose $G^{(1)} = \langle \alpha \rangle, G^{(2)} = \langle \alpha \beta \rangle$. We have that $G^{(1)} \cap G^{(2)}$ is trivial and $G^{\text{der}}G^{(1)} = G^{\text{der}}G^{(2)} = \langle \alpha, \beta \rangle$. However, $G^{\text{der}} = \langle \beta \rangle \not\cong \langle \alpha, \beta \rangle$, as desired. \square

The author hopes that determining the exact groups for which the conditions of Proposition 10.2 hold may give further insight on whether HNP implies PHNP in the general Galois case. In addition, a more general set of conditions utilizing Proposition 8.8 and Proposition 8.5 may suffice to show the implication.

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APPENDIX A. SAGEMATH CODE TO TEST FOR HNP AND PHNP

```

def TestAll(G, check):
    subg = G.normal_subgroups()
    subg.pop()
    for i in range(len(subg)):
        H1 = subg[i]
        if H1.order() == G.order():
            continue
        L1 = GLattice(H1, 1)
        IL1 = L1.induced_lattice(G)
        for j in range(i, len(subg)):
            H2 = subg[j]
            if H2.order() == G.order():
                continue
            L2 = GLattice(H2, 1)
            IL2 = L2.induced_lattice(G)

            if check:
                P1 = False
                P2 = False
                if len(prime_factors(IL1.rank())) <= 1:
                    P1 = True
                if len(prime_factors(IL2.rank())) <= 1:
                    P2 = True
                if P1 and P2:
                    continue

            if H1.group_id() == H2.group_id():
                continue

            IL = IL1.direct_sum(IL2)
            SL = IL.fixed_sublattice()
            a, b = SL.basis()
            SSL = SL.sublattice([a-b])

            QL1 = IL.quotient_lattice(SSL)
            QL2 = IL.quotient_lattice(SL)

            TS1 = QL1.Tate_Shafarevich_lattice(2)
            TS2 = QL2.Tate_Shafarevich_lattice(2)

            if TS1 != TS2:
                print('FOUND!!')
                print(G)
                print(H1)
                print(H2)
                print(TS1)
                print(TS2)
                print('-----')

def HuntHNP(depth):
    for i in range(1, depth):
        print(i)
        for j in range(1, TransitiveGroups(i).cardinality()+1):
            print(TransitiveGroup(i, j))
            TestAll(TransitiveGroup(i, j), True)

```

APPENDIX B. OSCAR CODE TO TEST FOR THE GALOIS CASE

```

function test(g,h1,h2)
  d = derived_subgroup(g)[1]
  h1d = sub(g,[gens(h1);gens(d)])[1]
  h2d = sub(g,[gens(h2);gens(d)])[1]
  i = intersect([h1d,h2d])[1]
  if order(i)>order(d)
    true
  else
    false
  end
end
end

function testall(g)
  n = normal_subgroups(g)
  for h1 in n
    for h2 in n
      if order(intersect([h1,h2])[1]) == 1
        if test(g,h1,h2)
          print("FOUND")
        end
      end
    end
  end
end

function looptestall(n)
  numb = number_small_groups(n)
  for i in 1:numb
    println(i)
    testall(small_group(n,i))
    println(" ")
  end
end

function testv2(g,h1,h2)
  d = derived_subgroup(g)[1]
  q1, f1 = quo(g,h1)
  q2, f2 = quo(g,h2)
  d1 = derived_subgroup(q1)[1]
  d2 = derived_subgroup(q2)[1]
  for i in g
    if !(i in d)
      if (f1(i) in d1)&(f2(i) in d2)
        return true
      end
    end
  end
  return false
end

function testallv2(g)

```

```

n = normal_subgroups(g)
l = []
for h1 in n
  for h2 in n
    if order(intersect([h1,h2])[1]) == 1
      if testv2(g,h1,h2)
        print("FOUND ")
        append!(l, [[g,h1,h2]])
      end
    end
  end
end
return l
end

function looptestallv2(n)
  numb = number_small_groups(n)
  for i in 1:numb
    println(i)
    s = testallv2(small_group(n,i))
    for r in s
      println(r)
    end
    println(" ")
  end
end
end

```

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